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Kaitong Hu, Zhenjie Ren, Junjian Yang

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Abstract

We consider a moral hazard problem with multiple principals in a continuous-time model. The agent can only work exclusively for one principal at a given time, so faces an optimal switching problem. Using a randomized formulation, we manage to represent the agent's value function and his optimal effort by an Itô process. This representation further helps to solve the principals' problem in case we have infinite number of principals in the sense of mean field game. Finally the mean field formulation is justified by an argument of propagation of chaos.

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1 Introduction

The principal-agent problem is a study of optimizing the incentives, so central in economics. In particular, the optimal contracting between the two parties, principal and agent(s), is called moral hazard, when the agent's effort is not observable by the principal. It has been widely applied in many areas of economics and finance, for example in corporate finance (see [BD05]). More recently, we also witness the works [APT18, AEE$^+$17] using the principal-agent formulation to study how to encourage people to embrace the energy transition.

While the research on the discrete-time model can be dated back further, the first paper on continuous-time principal-agent problem is the seminal work by Holmström and Milgrom [HM87], who study a simple continuous-time model in which the agent gets paid at the end of a finite time interval. They show that optimal contracts are linear in aggregate output when the agent has exponential utility with a monetary cost of effort. The advantage of the continuous-time model is further explored by Sannikov [San08]. Not only he considers a new model which allows the agent to retire, but also (and more importantly in the mathematical perspective) he introduces new dynamic insights to the principal-agent problem, and it leads to simple computational procedure to find the optimal contract by solving an ordinary differential equation in his case.

Later, the idea of Sannikov is interpreted and introduced to the mathematical finance community by Cvitanić, Possamaï and Touzi [CPT18]. Let us illustrate their contribution with a

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toy model. Denote by ξ the contract paid at the end of a finite time interval $[0,T]$. Assume the agent faces the following optimization:

$$\max_{\alpha} \mathbb{E}\left[\xi(X^\alpha) - \int_0^T c(\alpha_t)dt\right], \quad \text{where} \quad dX_t^\alpha = dW_t - \alpha_t dt.$$

The crucial observation in [CPT18] is that both the contract ξ and the Agent's best response $\alpha^*[\xi]$ can be characterized by the following backward stochastic differential equation (in short, BSDE, for readers not familiar with BSDE we refer to [PP90, EPQ97], and in particular to [CZ13] for the applications on the contract theory):

$$dY_t = -c^*(Z_t)dt + Z_t dW_t, \quad Y_T = \xi, \quad \text{where} \quad c^*(z) = \max_a \big\{az - c(a)\big\}, \tag{1.1}$$

namely, $\xi = Y_T$ and $\alpha_t^*[Z] = \arg\max_a \big\{aZ_t - c(a)\big\}$ for all $t \in [0,T]$. This induces a natural (forward) representation of the couple $(\xi, \alpha^*[\xi])$:

$$\begin{cases} \xi = Y_T^{Y_0,Z} := Y_0 - \int_0^T c^*(Z_t)dt + \int_0^T Z_t dW_t \\ \alpha_t^*[\xi] := \alpha_t^*[Z] = \arg\max\limits_a \big\{aZ_t - c(a)\big\}, \quad \text{for all } t \in [0,T] \end{cases} \quad \text{for some } (Y_0, Z),$$

and this neat representation transforms the once puzzling principal's problem to be a classical dynamic programming problem, namely,

$$\max_{\xi} \mathbb{E}\Big[U\left(X_T^{\alpha^*[\xi]} - \xi\right)\Big] = \max_{Y_0,Z} \mathbb{E}\Big[U\left(X_T^{\alpha^*[Z]} - Y_T^{Y_0,Z}\right)\Big].$$

This applaudable idea of representation is further applied to study the case where the principal can hire multiple agents [EPar]. Eventually, in [EMPar] the authors follow the same machinery to study the model where the principal hires infinite number of agents using the formulation of mean field games (as for the mean field game we refer to the seminal paper [LL07] and the recent books [CD18a] and [CD18b]).

There are fewer existing literature considering the model with multiple principals. In [MR18] the authors consider the case where the several principals hire one common agent *at the same time* (i.e. the agent simultaneously works on different projects for different principals). However, to the best of our knowledge, no one has yet considered a n-principal/1-agent model where the agent can only exclusively work for one principal at a given time. In such a model, the agent is facing an optimal switching (among the principals) problem. According to the classic literature of optimal switching, see e.g. [PVZ09, HZ10, HT10, CEK11], the counterpart of the BSDE characterization (1.1) for the agent's problem would be a system of reflected BSDE in the form:

$$dY_t = -f_t(Y_t, Z_t)dt + Z_t dW_t - dK_t, \quad Y_T = \xi,$$

where K is an increasing process satisfying some (backward) Skorokhod condition. The presence of the process K and its constraint make it difficult to find a representation of the couple $(\xi, \alpha^*[\xi])$ as in [CPT18]. In this paper, we propose an alternative facing this difficulty. Instead of letting the agent choose stopping times to change his employers, we study a randomized optimal switching problem where the switching time is modelled by a random time characterized by a Poisson point process and the agent influences the random time by controlling the intensity of the Poisson point process. It is fair to note that similar randomized formulations of switching have been discussed in the literature, see e.g. [Bou09, EK10]. In such framework, we may characterize $(\xi, \alpha^*[\xi])$ by the solution to a system of BSDEs (without reflection).

Unfortunately, this characterization of the agent's problem and the corresponding representation do not help us to solve simply the principals' problem. We observe that the optimizations the principals face are time-inconsistent, and thus cannot be solved by dynamic programming. However, we also note that this time-inconsistency disappears once the number of principals tends to infinity. In the setting of infinite number of principals, it is natural to adopt the formulation of mean field game. We prove the existence of mean field equilibrium largely based on the recipes in Lacker [Lac15]. Further, in order to justify our mean field game formulation, we introduce a machinery which we name 'backward propagation of chaos' which may carry independent interest itself.

The rest of the paper is organized as follows. In Section 2, we state our moral hazard problem with n-principal and one agent, we shall solve the agent's problem under the previously mentioned randomized optimal switching formulation, and observe the time-inconsistency of the n-principal problem. Then in section 3, we shall derive the mean field game among the principals, prove its existence, and justify it using the "backward propagation of chaos" technique.

2 Moral hazard problem : n principals and one agent

In this paper we consider the principal-agent problem on the a finite time horizon $[0, T]$ for some $T > 0$. The main novelty is to introduce the new model and method to allow the agent to choose working for different employers. In this section, we set up the model in which one agent switches among n different principals.

For the agent, the set of possible regimes is $\mathbb{I}_n := \{1, 2, \cdots, n\}$. Denote by $C([0, T]; \mathbb{R}^n)$ the set of continuous functions from $[0, T]$ to $\mathbb{R}^n$, endowed with the supremum norm $\|\cdot\|_T$, where

$$\|x\|_t := \sup_{0 \le s \le t} |x_s|, \quad t \in [0, T], \quad x \in C([0, T]; \mathbb{R}^n).$$

Denote by $D([0, T], \mathbb{I}_n)$ the set of càdlàg functions from $[0, T]$ to $\mathbb{I}_n$. We introduce the canonical space $\Omega := C([0, T]; \mathbb{R}^n) \times D([0, T]; \mathbb{I}_n)$, denote the canonical process by (X, I), and the canonical filtration by $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$. The process X_t represents the outputs of the principals and I_t records for which principal the agent is working at time t. Denote by $\mathbb{F}^X$ the filtration generated by the process X alone. We also define $\mathbb{P}_0$ as the Wiener measure on $C([0, T], \mathbb{R}^n)$.

In our model, for simplicity, the output process follows the dynamic

$$X_t = X_0 + \int_0^t e_{I_s} ds + W_t, \quad \text{where} \quad e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^\top \in \mathbb{R}^n, \tag{2.1}$$

that is, the principal employing the agent has a constant-1 drift while outputs of the others just follow the Brownian motions. The agent controls the process I, that is, he switches from one employer to another so as to maximize his utility:

$$V_0^A(\xi, w) = \sup_I \mathbb{E}\left[\sum_{i=1}^n \xi^i \mathbf{1}_{\{I_T = i\}} + \int_0^T w_t^i \mathbf{1}_{\{I_t = i\}} du - \text{"cost of switching"}\right], \tag{2.2}$$

where ξ^i is the reward at the terminal time T, w^i is the wage payed by principal i, and the cost of switching will be defined later when we analyze the agent's problem in detail. Naturally, the payment ξ^i should be an $\mathcal{F}_T^X$-measurable random variable and w^i should be an $\mathbb{F}^X$-adapted process.

Remark 2.1. The upcoming method can easily handle the output process:

$$dX_t = b(t, X_t, I_t)dt + \sigma(t, X_t)dW_t,$$

where b, σ are uniformly Lipschitz in x. However, for the simplification of notations, we prefer to focus on the form of (2.1).

It is the n principals that provide the contracts $\{(\xi^i, w^i)\}_{i\in\mathbb{I}_n}$. Among them, they search for a Nash equilibrium so as to maximize their profits:

$$V_0^{P,i} = \max_{(\xi^i, w^i)} \mathbb{E}\left[X_T^{i,\xi^i} - U(\xi^i)\mathbf{1}_{\{I_T=i\}} - \int_0^T U(w_t^i)\mathbf{1}_{\{I_t=i\}}dt\right], \tag{2.3}$$

where X^{i,ξ^i} denotes the optimal output process controlled by the agent.

As in [San08] and [CPT18], we are going to provide a representation of the value function of the agent's problem so as to solve the principals' problem by dynamic programming.

2.1 Agent's problem

In our model, instead of allowing the agent to control the process I in a "singluar" way, we assume that I is driven by a Poisson random measure (p.r.m.) and allow the agent to control its intensity. More precisely, in the weak formulation of stochastic control, the agent aims at choosing his optimal control among:

$$\Big\{\mathbb{P}^\alpha \in \mathcal{P}(\Omega): \quad \mathbb{P}^\alpha\text{-a.s. (2.1) holds true and } dI_t = \int_{\mathbb{I}_n}(k - I_{t-})\mu(dt, dk),$$
$$\text{where } \mu \text{ is a p.r.m. with intensity } \alpha(dt,\cdot) = \sum_{k\in\mathbb{I}_n}\alpha_t^k\delta_k dt,$$
$$\text{for some nonnegative } \mathbb{F}\text{-adapted } (\alpha^k)_{k\in\mathbb{I}_n},\ W \text{ and } \mu \text{ are independent}\Big\},$$

where $\mathcal{P}(\Omega)$ is the set of all probability measures on Ω.

Remark 2.2. Define the first jump time of the process I:

$$\tau_t := \inf\{s \geq t : I_s \neq I_t\}. \tag{2.4}$$

It follows from the Girsanov theorem for multivariate point processes (see e.g. [Jac75]) that for any $\mathbb{F}^X$-adapted processes $\{\theta(i)\}_{i\in\mathbb{I}_n}$ we have

$$\mathbb{E}_t^{\mathbb{P}^\alpha}\left[\theta_{\tau_t}(I_{\tau_t})\mathbf{1}_{\{\tau_t\leq s,\ I_{\tau_t}=j\}}\right] = \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\int_t^s \beta^{I_t,\alpha}(t,u)\alpha_u^j\theta_u(j)du\right], \quad \text{for all } j \in \mathbb{I}_n \text{ and } j \neq I_t,$$

$$\text{as well as}\quad \mathbb{P}^\alpha\left[\tau_t > s\middle|\mathcal{F}_s\right] = \beta^{I_t,\alpha}(t,s), \quad \text{where}\quad \beta^{i,\alpha}(t,s) := \exp\left(-\int_t^s \sum_{j\neq i}\alpha_u^j du\right).$$

These results will be useful in the upcoming calculus.

With the intensity α introduced as above, we can now make it precise how to define the cost of switching in (2.2). Given n contracts $\{(\xi^i, w^i)\}_{i\in\mathbb{I}_n}$, the agent solves the following optimal switching problem

$$V_0^A = \sup_\alpha \mathbb{E}^{\mathbb{P}^\alpha}\left[\sum_{i=1}^n \xi^i\mathbf{1}_{\{I_T=i\}} + \int_0^T\left(\sum_{i=1}^n w_u^i\mathbf{1}_{\{I_u=i\}} - \frac{1}{n-1}\sum_{i\neq I_u}c\big((n-1)\alpha_u^i\big)\right)du\right]. \tag{2.5}$$

Intuitively, the intensity process α describes the hesitation of changing employer. The bigger α is, the less hesitation the agent has to change his employer. The normalization $\frac{1}{n-1}c\big((n-1)\cdot\big)$ is for later use, as $n \to \infty$.

As in the classical literature of the optimal switching problems, we shall use the dynamic programming principle to obtain the system of equations characterizing the value function. First, define the dynamic version of (2.5):

$$V_t^A = \operatorname*{ess\,sup}_{\alpha} \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\sum_{i=1}^n \xi^i \mathbf{1}_{\{I_T=i\}} + \int_t^T \left(\sum_{i=1}^n w_u^i \mathbf{1}_{\{I_u=i\}} - \frac{1}{n-1}\sum_{i\neq I_u} c\big((n-1)\alpha_u^i\big)\right)du\right].$$

Recall τ_t defined in (2.4). By the dynamic programming, we have

$$V_t^A = \operatorname*{ess\,sup}_{\alpha} \mathbb{E}_t^{\mathbb{P}^\alpha}\Bigg[\sum_{i=1}^n \xi^i \mathbf{1}_{\{I_T=i,\ \tau_t>T\}} + V_{\tau_t}^A \mathbf{1}_{\{\tau_t\le T\}} + \int_t^{\tau_t\wedge T} \left(\sum_{i=1}^n w_u^i \mathbf{1}_{\{I_u=i\}} - \frac{1}{n-1}\sum_{i\neq I_u} c\big((n-1)\alpha_u^i\big)\right)du\Bigg].$$

Further, by defining $V_t^{A,i} := V_t^A|_{I_t=i}$, we obtain

$$\begin{aligned} V_t^{A,i} &= \operatorname*{ess\,sup}_{\alpha} \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\xi^i \mathbf{1}_{\{\tau_t^i>T\}} + \sum_{j\neq i}\left(V_{\tau_t}^{A,j}\mathbf{1}_{\{\tau_t\le T,\ I_{\tau_t}=j\}} + \int_t^{\tau_t\wedge T} \frac{w_u^i - c\big((n-1)\alpha_u^j\big)}{n-1}du\right)\right] \\ &= \operatorname*{ess\,sup}_{\alpha} \mathbb{E}_t^{\mathbb{P}^\alpha}\left[\xi^i \beta^{i,\alpha}(t,T) + \int_t^T \beta^{i,\alpha}(t,u)\sum_{j\neq i}\left(\alpha_u^j V_u^{A,j} + \frac{w_u^i - c\big((n-1)\alpha_u^j\big)}{n-1}\right)du\right]. \quad (2.6) \end{aligned}$$

The second equality above is due to the results in Remark 2.2. In view of (2.6), it becomes a classical stochastic control problem. In particular, the value function of this control problem can be characterized by the BSDE.

Assumption 2.3. *Assume that the cost function c is convex and lower semicontinuous, and c takes value of $+\infty$ out of a compact set K. We also assume that there exists a unique maximizer*

$$a^*(y) := \arg\max_{a\ge 0}\big\{ay - c(a)\big\} \in K, \quad \textit{for all } y \in \mathbb{R}.$$

Define the convex conjugate of c

$$c^*(y) := \sup_{a\ge 0}\big\{ay - c(a)\big\}, \quad \textit{for all } y \in \mathbb{R}.$$

Further assume that c^* is Lipschitz continuous.

Proposition 2.4. *Under Assumption 2.3, given $\xi^i \in \mathbb{L}^2(\mathbb{P}_0), w^i \in \mathbb{H}^2(\mathbb{P}_0)$[1] for all $i \in \mathbb{I}_n$, the following system of BSDEs has a unique solution $(Y^i, Z^i)_{i\in\mathbb{I}_n}$:*

$$Y_t^i = \xi^i + \int_t^T \left(\frac{1}{n-1}\sum_{j\neq i} c^*(Y_u^j - Y_u^i) + w_u^i + e_i Z_u^i\right)du - \int_t^T Z_u^i \cdot dX_u, \quad i \in \mathbb{I}_n, \quad \mathbb{P}_0\text{-}a.s. \ (2.7)$$

[1]We denote by $\mathbb{H}^2(\mathbb{P}_0)$ the space of progressively measurable processes w such that

$$\|w\|^2_{\mathbb{H}^2(\mathbb{P}_0)} := \mathbb{E}^{\mathbb{P}_0}\left[\left(\int_0^T |w_u|^2 du\right)^{\frac{1}{2}}\right] < \infty.$$

Moreover, we have $Y^i = V^{A,i}$, $\mathbb{P}_0$-a.s. In particular, the optimal intensity satisfies

$$\alpha_t^{j,*} = \frac{1}{n-1} a^*\big(Y_t^j - Y_t^{I_t}\big), \quad j \neq I_t, \quad \textit{for all} \;\; t \in [0,T], \quad \mathbb{P}_0\textit{-a.s.} \tag{2.8}$$

Proof. In view of the control problem (2.6), the corresponding BSDE reads

$$\begin{aligned} Y_t^i = \xi^i + \int_t^T & \left(\frac{1}{n-1} \sum_{j\neq i} \sup_{a^j \geq 0} \Big\{ (n-1)a^j\big(Y_u^j - Y_u^i\big) - c\big((n-1)a^j\big) \Big\} + w_u^i + e_i Z_u^i \right) du \\ & - \int_t^T Z_u^i \cdot dX_u. \end{aligned}$$

Then, (2.7) follows from the definition of c^*. Since all the coefficients are Lipschitz continuous, the wellposedness of the BSDE system and the verification for the control problem is classical, see e.g. [PP90]. □

2.2 Principals' problem: time inconsistency

In the previous section, we managed to represent the value function of the agent by an Itô process (Proposition 2.4). As in [San08, CPT18], we expect this representation would help us solve the principals' problem by dynamic programming. However, in this model, this approach does not work. We shall explain in the case $n = 2$ for the simplification of notation.

Consider the set of all contracts

$$\Xi := \Big\{ \big\{(\xi^i, w^i)\big\}_{i=1,2} : \; \xi^i \in \mathbb{L}^2(\mathbb{P}_0), \; w^i \in \mathbb{H}^2(\mathbb{P}_0) \text{ and } V^A(\xi, w) \geq R \Big\},$$

where R is the reservation value of the agent for whom only the contracts such that $V^A(\xi, w) \geq R$ are acceptable. Now define

$$\mathcal{V} := \Big\{ \big\{(Y_0^i, Z^i)\big\}_{i=1,2} : \; Y_0^i \geq R, \; Z^i \in \mathbb{H}^2(\mathbb{P}_0), \; i = 1,2 \Big\}.$$

It follows from Proposititon 2.4 that

$$\begin{aligned} \Xi = \Big\{ \big\{(\xi^i, w^i)\big\}_{i=1,2} : \; & w^i \in \mathbb{H}^2(\mathbb{P}_0) \text{ and } \xi^i = Y_T^{i,Y_0^i,Z^i,w^i}, \text{ where } Y^{i,Y_0^i,Z^i,w^i} \text{ satisfies} \\ & Y_T^i = Y_0^i + \int_0^T \Big(c^*\big(Y_t^j - Y_t^i\big) + w_t^i + e_i Z_t^i \Big)\, dt - \int_0^T Z_t^i \cdot dX_t, \; \mathbb{P}_0\text{-a.s.}, \\ & \text{with } \; j \neq i \; \text{ and } \; \big\{(Y_0^i, Z^i)\big\}_{i=1,2} \in \mathcal{V} \Big\}. \end{aligned}$$

For simplicity of notation, if there is no ambiguity, we write simply Y^i instead of Y^{i,Y_0^i,Z^i,w^i}. But just keep in mind that the process Y^i is controlled by Z^i and w^i with initial value Y_0^i. The corresponding optimal intensity reads $\alpha_t^* = \big(a^*(Y_t^2 - Y_t^1)\mathbf{1}_{\{I_t=1\}}, a^*(Y_t^1 - Y_t^2)\mathbf{1}_{\{I_t=2\}}\big)$. Therefore, the principals are searching for a Nash equilibrium so as to maximize:

$$\begin{aligned} V_0^{P,i} &= \sup_{(\xi^i, w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[X_T^i - U(\xi^i)\mathbf{1}_{\{I_T=i\}} - \int_0^T U(w_t^i)\mathbf{1}_{\{I_t=i\}} dt \right] \\ &= \sup_{(Y_0^i, Z^i, w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[X_T^i - U(Y_T^i)\mathbf{1}_{\{I_T=i\}} - \int_0^T U(w_t^i)\mathbf{1}_{\{I_t=i\}} dt \right] = \sup_{Y_0^i \geq R} J_0^i(Y_0^i), \end{aligned}$$

where

$$J_0^i(Y_0) := \sup_{(Z^i,w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[X_T^i - U\left(Y_T^{i,Y_0^i,Z^i,w^i}\right)\mathbf{1}_{\{I_T=i\}} - \int_0^T U(w_t^i)\mathbf{1}_{\{I_t=i\}}dt\right].$$

Up to now, we are applying the same strategy as in [CPT18]. Assume the control problem is time-consistent, that is, admits dynamic programming. Fix Y_0 and define the dynamic version of J_0^i:

$$J_t^i(Y_t^i) := \operatorname*{ess\,sup}_{(Z^i,w^i)} \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[X_T^i - U(Y_T^i)\mathbf{1}_{\{I_T=i\}} - \int_t^T U(w_u^i)\mathbf{1}_{\{I_u=i\}}du\right].$$

Defining $J_t^{i,1} := J_t^i(Y_t^i)|_{I_t=1}$ and $J_t^{i,2} = J_t^i(Y_t^i)|_{I_t=2}$ according to the different regimes, we expect to have for $i=1$

$$\begin{aligned} J_t^{1,1} &= \operatorname*{ess\,sup}_{(Z^1,w^1)} \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[\left(X_T^1 - U(Y_T^1)\right)\mathbf{1}_{\{\tau_t>T\}} + J_{\tau_t}^{1,2}\mathbf{1}_{\{\tau_t\le T\}} - \int_t^{\tau_t\wedge T} U(w_u^1)du\right] \\ &= \operatorname*{ess\,sup}_{(Z^1,w^1)} \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\Big[\beta^{1,*}(t,T)\big(X_T^1 - U(Y_T^1)\big) \\ &\qquad + \int_t^T \beta^{1,*}(t,u)a^*\big(Y_u^2 - Y_u^1\big)\big(J_u^{1,2} - U(w_u^1)\big)du\Big] \end{aligned} \tag{2.9}$$

where

$$dX_t^1 = dt + dW_t^1, \quad \beta^{1,*}(t,s) := \exp\left(-\int_t^s a^*(Y_u^2 - Y_u^1)du\right),$$

and similarly

$$\begin{aligned} J_t^{1,2} &= \operatorname*{ess\,sup}_{(Z^1,w^1)} \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[X_T^1\mathbf{1}_{\{\tau_t>T\}} + J_{\tau_t}^{1,1}\mathbf{1}_{\{\tau_t\le T\}}\right] \\ &= \operatorname*{ess\,sup}_{(Z^1,w^1)} \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[\beta^{2,*}(t,T)X_T^1 + \int_0^T \beta^{2,*}(t,u)a^*(Y_u^1 - Y_u^2)J_u^{1,1}du\right] \end{aligned} \tag{2.10}$$

where

$$dX_t^1 = dW_t^1, \quad \beta^{2,*}(t,s) := \exp\left(-\int_t^s a^*(Y_t^1 - Y_t^2)du\right).$$

Although it seems promising to solve the system of value functions $(J^{i,1}, J^{i,2})$ as in the agent problem, note that the admissible controls (Z^i, w^i) are constrained to be $\mathbb{F}^X$-adapted, in particular, (Z^i, w^i) cannot depend on the regimes. Since the optimizations in (2.9) and (2.10) are not symmetric, we cannot expect that the optimal controls (Z^i, w^i) coincide on the different regimes. Therefore, it is a contradiction to the alleged time-consistency.

This time-inconsistency appears in our model, if we consider a finite number of principals. In the following section, we shall bypass the difficulty in the case of infinite number of principals, using the mean field formulation.

Remark 2.5. Besides, the approach for solving the n-principal problem has another drawback. Note that each contract ξ^i for $i \in \mathbb{I}_n$ is a function of the entire vector X, that is, the principal need to know the other companies' performance in order to design his own contract. This problem can be also avoided in the mean field approach, where ξ^i will be a function of X^i.

3 Infinite number of principals: mean field approach

3.1 Heuristic analysis

Heuristically, as $n \to \infty$ the equation (2.7), which characterizes the agent's value function, converges to

$$dY_t^i = -\left(\int c^*(y - Y_t^i)p_t(dy) + w_t^i + Z_t^i\right)dt + Z_t^i dX_t^i, \quad p_t = \mathcal{L}(Y_t^i). \tag{3.1}$$

Besides, it follows from (2.8) and the definition of the discount factor β that the optimal α^* and β^* converge to

$$\alpha_t^{j,*} \to 0 \text{ for } j \neq i, \qquad \sum_{j \neq i} \alpha_t^{j,*} \to \alpha_t^* := \int a^*(y - Y_t^i)p_t(dy), \tag{3.2}$$

and $\beta_t^{n,*} \to \beta_t^* := e^{-\int_0^t \alpha_s^* \mathrm{d}s}$.

Remark 3.1.

- To be fair, the form of the limit equation (3.1) does not entirely follow the intuition. Note that we removed the e_i in front of the term $Z_t^i dt$ and replaced the stochastic integrator dX_t by dX_t^i. At this stage, let us admit that once in the drift of the BSDE there is no longer dependence on other Y^j $(j \neq i)$, the system would be decoupled, that is, the equation of Y^i no longer needs the information of X^j $(j \neq i)$, and it leads to the limit form (3.1). We will justify the mean field formulation in Section 3.3.
- The first observation in (3.2) implies that in the limit case once the agent leaves the company i, he will have null probability to come back. Therefore, the principal should be only interested in the optimization for the regime where the agent works for her. Remember that in the n-principal problem, the time-inconsistency is due to the optimizations in the different regimes. Therefore, it is reasonable that the mean field formulation would bypass this difficulty.

Recall the principal's problem (2.3). It follows from the observation in Remark 3.1 that

$$V_0^{P,i} = \begin{cases} \mathbb{E}^{\mathbb{P}^{\alpha^*}}[X_0^i + W_T] = X_0^i, & \text{as } I_0 \neq i, \\ \max_{(\xi^i, w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[X_\tau^i 1_{\{\tau \le T\}} + \left(X_T^i - U(\xi^i)\right)\mathbf{1}_{\{\tau > T\}} - \int_0^{\tau \wedge T} U(w_u^i)du\right], & \text{as } I_0 = i. \end{cases}$$

Therefore, in the upcoming mean field game, we should only keep the nontrivial regime $(I_0 = i)$, i.e., denote

$$\begin{aligned} V_0^{P,i} &:= \max_{(\xi^i, w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[X_\tau^i 1_{\{\tau \le T\}} + \left(X_T^i - U(\xi^i)\right)\mathbf{1}_{\{\tau > T\}} - \int_0^{\tau \wedge T} U(w_u^i)du\right] \\ &= \max_{(\xi^i, w^i)} \mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[\int_0^T \alpha_u^* \beta_u^* \left(X_u^i - U(w_u^i)\right)du + \beta_T^*\left(X_T^i - U(\xi^i)\right)\right]. \end{aligned} \tag{3.3}$$

Though the rigorous definition of the mean field game will be introduced in the next section, we are ready to give a simple description of the mean field equilibrium we are searching for. In the mean field formulation, we remove the superscript i, and instead we use the notations $\overline{X}, \overline{Y}, \overline{Z}$. Also note that $(\overline{X}_t - t)_{t \in [0,T]}$ is a $\mathbb{P}^{\alpha^*}$-Brownian motion, denoted by $\overline{W}$. Following the dynamic programming approach in [San08, CPT18], given $(p_t)_{t \in [0,T]}$, consider the contracts in the form:

$$\xi \in \Xi(p) = \left\{\overline{Y}_T : \overline{Y}_T = \overline{Y}_0 - \int_0^T \left(\int c^*(y - \overline{Y}_u)p_u(dy) + w_u\right)dt + \int_0^T \overline{Z}_u d\overline{W}_u\right\}.$$

It follows from (3.3) that each principal faces the optimization:

$$V_0^P(p) = \sup_{(\overline{Y}_0,\overline{Z},w)} \mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[\int_0^T \alpha_u^*\beta_u^*\big(\overline{X}_u - U(w_u)\big)du + \beta_T^*\big(\overline{X}_T - U(\overline{Y}_T)\big)\right]. \tag{3.4}$$

The maximizer $\big(\overline{Y}_0^*, \overline{Z}^*, w^*\big)$ would define a process $\overline{Y}^*$. The law $(p_t)_{t\in[0,T]}$ is a mean field equilibrium, if $p_t = \mathcal{L}\big(\overline{Y}_t^*\big)$ for all $t \in [0,T]$.

3.2 Mean field game: existence

Our rigorous definition of the mean field game and the argument to prove its existence rely largely on the framework in Lacker [Lac15].

In this part of the paper, we denote the canonical space by $\overline{\Omega} = C([0,T],\mathbb{R}^2)$, the canonical process by $\Theta := (\overline{X},\overline{Y})$ and the canonical filtration by $\overline{\mathbb{F}}$. Given $p \in \mathcal{P}(\overline{\Omega})$, define $p_t := p\circ Y_t^{-1} \in \mathcal{P}(\mathbb{R})$ and

$$\begin{aligned}\mathcal{W}(p) := \Big\{\mathbb{P}^{\lambda,\eta} \in \mathcal{P}(\overline{\Omega}) :\ & \mathbb{P}^{\lambda,\eta}\text{-a.s.}\quad d\overline{X}_t = dt + dW_t, \quad X_0 = x_0,\\ & d\overline{Y}_t = -\int \big(c^*(y-\overline{Y}_t)p_t(dy) + w_t\big)dt + \sqrt{\eta_t}dW_t, \quad \mathcal{L}(\overline{Y}_0) = \lambda,\\ & \text{for some}\ \ \lambda \in \mathcal{I},\quad \eta \in \mathcal{U}\Big\}\end{aligned}$$

where, for technical reasons, we define

$$\begin{aligned} & \mathcal{I} := \big\{\lambda \in \mathcal{P}(\mathbb{R}) :\ \lambda \text{ with a compact support } K \text{ in } [R,\infty)\big\}\\ \text{and}\quad & \mathcal{U} := \big\{\eta\ \overline{\mathbb{F}}\text{-adapted} :\ \eta \text{ takes values in a compact set } \Sigma \text{ in } \mathbb{R}^+\big\}.\end{aligned}$$

In other words, we will consider a mean field game in which the choice of the initial value and the volatility is constrained in compact sets. Further define

$$J(\mathbb{P},w;\,p) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T \alpha_u^*\beta_u^*\big(\overline{X}_u - U(w_u)\big)du + \beta_T^*\big(\overline{X}_T - U(\overline{Y}_T)\big)\right],$$

where we recall $\alpha_t^* = \int a^*(y-\overline{Y}_t)p_t(dy)$ and $\beta_t^* = \exp\big(-\int_0^t \alpha_s^* ds\big)$.

Assumption 3.2. *We assume that $a^* : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. U is convex and of q-polynomial growth, i.e. there are constants $C < C'$ and $q > 1$ such that*

$$C(|w|^q - 1) \le U(w) \le C'(|w|^q + 1).$$

Theorem 3.3. *Under Assumption 2.3 and 3.2, there exists $p \in \mathcal{P}(\overline{\Omega})$, $\big(\widehat{\lambda},\widehat{\eta}\big) \in \mathcal{I}\times\mathcal{U}$ and an $\overline{\mathbb{F}}$-adapted process $\widehat{w}$ such that*

$$\big(\mathbb{P}^{\widehat{\lambda},\widehat{\eta}}, \widehat{w}\big) \in \operatorname*{arg\,min}_{\mathbb{P}\in\mathcal{W}(p),w} J(\mathbb{P},w;\,p) \quad \textit{and} \quad p = \mathbb{P}^{\widehat{\lambda},\widehat{\eta}}.$$

Remark 3.4.
- This definition of mean field game via the weak formulation of stochastic control follows the same spirit as those in Carmona & Lacker [CL15] and Lacker [Lac15]. However, here we also include the control of the initial distribution λ of $\overline{Y}$.
- It is noteworthy that among the triple of control (λ,η,w) of this mean field game, λ takes values of measures, that is, the principals are allowed to play a mixed strategy.

- If we constrain w to take values in a compact set in $\mathbb{R}$, then given a bounded function U which is convex on this compact set, the mean field game exists.

We realize that Theorem 3.3 can be proved by largely the same argument as in [Lac15]. In the rest of the section, we shall outline the strategy of the proof and refer the readers for details to the well-written paper [Lac15].

First, we shall linearize the functional J using the so-called relaxed control. Denote by $\mathcal{D}$ the set of measures q on $[0,T]\times\Sigma\times\mathbb{R}$. Instead of controlling via the processes η (taking values in Σ) and w (taking values in $\mathbb{R}$), we shall control through the measure q in the relaxed formulation. The canonical space for the relaxed control becomes $\widehat{\Omega} := \overline{\Omega}\times\mathcal{D}$. Denote the canonical process by $(\overline{X},\overline{Y},\Lambda)$, and the canonical filtration by $\widehat{\mathbb{F}}$. Note that one may define a $\mathcal{P}(\Sigma\times\mathbb{R})$-valued $\widehat{\mathbb{F}}$-predictable process $(\Lambda_t)_{0\le t\le T}$ such that $\Lambda(dt,d\eta,dw)=\Lambda_t(d\eta,dw)dt$. Here we abuse the notations using η, w to represent points in $(\Sigma,\mathbb{R})$.

Denote by $\mathcal{C}_0^\infty(\mathbb{R}^2)$ denote the set of infinitely differentiable functions $\phi:\mathbb{R}^2\mapsto\mathbb{R}$ with compact support. Define the generator L on $\mathcal{C}_0^\infty(\mathbb{R}^2)$ by

$$L^{p,\eta}\phi(t,x,y)=\left(1,-\int c^*(y'-y)p_t(dy')-w_t\right)\nabla\phi+\frac{1}{2}\big(\partial_{xx}\phi+\eta\partial_{yy}\phi+2\sqrt{\eta}\partial_{xy}\phi\big),$$

for $(t,x,y,p,\eta)\in[0,T]\times\mathbb{R}^2\times\mathcal{P}(\overline{\Omega})\times\Sigma$. Further define

$$M_t^{p,\phi} := \phi(\overline{X}_t,\overline{Y}_t)-\int_0^t\int_{\Sigma\times\mathbb{R}} L^{p,\eta}\phi(s,\overline{X}_s,\overline{Y}_s)\Lambda_s(d\eta)ds$$

Definition 3.5. *Given $p\in\mathcal{P}(\overline{\Omega})$, define the set of the controlled martingale problems:*

$$\begin{aligned}\mathcal{R}(p)=\Big\{\widehat{\mathbb{P}}\in\mathcal{P}\big(\widehat{\Omega}\big):\ &\widehat{\mathbb{P}}\text{-a.s. }\overline{X}_0=x_0,\ \overline{Y}_0\sim\lambda,\ \text{for some }\lambda\in\mathcal{I},\\ &\mathbb{E}^{\widehat{\mathbb{P}}}\left[\int_0^T|\Lambda_t|^q dt\right]<\infty,\\ &M^{p,\phi}\text{ is a }\widehat{\mathbb{P}}\text{-martingale for each }\phi\in\mathcal{C}_0^\infty(\mathbb{R}^2)\Big\},\end{aligned}\tag{3.5}$$

where $|\Lambda_t|^q := \int_{\Sigma\times\mathbb{R}}|(\eta,w)|^q\Lambda_t(d\eta,dw)$.

Further, in the relaxed formulation the object function of the principals reads:

$$\hat{J}(\widehat{\mathbb{P}};\,p):=\mathbb{E}^{\widehat{\mathbb{P}}}\left[\int_0^T\int_{\mathbb{R}}\alpha_u^*\beta_u^*\big(\overline{X}_u-U(w)\big)\Lambda_u(dw)\mathrm{d}u+\beta_T^*\big(\overline{X}_T-U(\overline{Y}_T)\big)\right],$$

and define the set of the optimal control:

$$\mathcal{R}^*(p):=\underset{\widehat{\mathbb{P}}\in\mathcal{R}(p)}{\arg\max}\,\hat{J}\big(\widehat{\mathbb{P}};\,p\big).$$

We say $\widehat{\mathbb{P}}\in\mathcal{P}(\widehat{\Omega})$ is a relaxed mean field game if $\widehat{\mathbb{P}}\in\mathcal{R}^*\big(\widehat{\mathbb{P}}\circ(\overline{X},\overline{Y})^{-1}\big)$.

Theorem 3.6. *Under Assumption 2.3 and 3.2, there exists a relaxed mean field game.*

Proof. Our setting slightly distinguishes from the one in [Lac15], because we allow to control the initial law λ of $\overline{Y}$. However, since we constrain the choice of λ among the distribution in $\mathcal{P}(K)$ where K is a compact set in $\mathbb{R}$, we are still able to prove the tightness of $\mathcal{R}(p)$ (in the case Λ is truncated), and the rest of the argument would follow the same lines in [Lac15, Section 4 and 5]. □

It remains to construct a strict mean field game (as in Theorem 3.3) based on a relaxed one. Again we can follow the classical argument. We shall only provide the sketch of the proof, for more details we refer the readers to [Lac15, Proof of Theorem 3.7]

Proof of Theorem 3.3. Let $\widehat{\mathbb{P}}$ be a relaxed mean field game. First, we may find an $\overline{\mathbb{F}}$-adapted process $\widehat{q} : [0,T] \times \overline{\Omega} \to \mathcal{P}(\Sigma \times \mathbb{R})$ such that

$$\widehat{q}(t, \overline{X}, \overline{Y}) = \mathbb{E}^{\widehat{\mathbb{P}}}[\Lambda_t | \overline{\mathcal{F}}_t], \quad \widehat{\mathbb{P}}\text{-a.s.} \quad t \in [0,T]. \tag{3.6}$$

Further, for each $(t,x,y,p) \in [0,T] \times \overline{\Omega} \times \mathcal{P}(\overline{\Omega})$, since α^*, β^* are always nonnegative and function U is convex, the following subset

$$K(t,x,y,p) := \big\{(\eta, w, l) : \ \eta \in \Sigma, \ l \le f(t,x,y,p,w) := \alpha_t^*(y,p)\beta_t^*(y,p)\big(x_t - U(w)\big)\big\}$$

is convex. This verifies the "convexity" assumption in [Lac15, Assumption (Convex)]. Therefore, using the measurable selection result in [HL90, Lemma 3.1], there exist $\overline{\mathbb{F}}$-adapted processes $\widehat{\eta}, \widehat{w}$ and $\hat{l} \ge 0$ such that

$$\int \big(\eta, f(t,x,y,p,w)\big)\hat{q}(t,x,y)(d\eta, dw) = \Big(\widehat{\eta}(t,x,y), f\big(t,x,y,p,\widehat{w}(t,x,y)\big) - \hat{l}(t,x,y)\Big).$$

Further, it would be easy to verify that $p := \widehat{\mathbb{P}} \circ (\overline{X}, \overline{Y})^{-1} = \mathbb{P}^{\lambda, \widehat{\eta}}$ as well as $J(\mathbb{P}^{\lambda,\widehat{\eta}}, \widehat{w}; p) \ge \hat{J}(\widehat{\mathbb{P}}; p)$. Therefore, we find a mean field game in sense of Theorem 3.3. □

From the sketch of proof, we may observe the following.

Corollary 3.7. *Under Assumption 2.3 and 3.2, there exists a mean field game such that*

$$\mathbb{E}^{\mathbb{P}^{\widehat{\lambda},\widehat{\eta}}}\left[\int_0^T |\widehat{w}_u|^q du\right] < \infty.$$

Proof. Note that in the proof of Theorem 3.3, the $\widehat{w}$ we constructed satisfies

$$C(|\widehat{w}_t|^q - 1) \le U(\widehat{w}_t) \le \int_{\mathbb{R}} U(w)\widehat{q}_t(dw),$$

where $\widehat{q}$ is defined in (3.6). Therefore,

$$\begin{aligned} C\mathbb{E}^{\mathbb{P}^{\widehat{\lambda},\widehat{\eta}}}\left[\int_0^T |\widehat{w}_u|^q du\right] &\le C + \mathbb{E}^{\widehat{\mathbb{P}}}\left[\int_0^T \int_{\mathbb{R}} U(w)\Lambda_u(dw)du\right] \\ &\le C + \mathbb{E}^{\widehat{\mathbb{P}}}\left[\int_0^T \int_{\mathbb{R}} C'(|w|^q + 1)\Lambda_u(dw)du\right]. \end{aligned}$$

Then, the desired result follows from (3.5). □

3.3 Mean field game: approximation

In this section, we shall justify our mean field formulation, that is, answer the question why the principals would apply the mean field game we studied in the previous section. Unlike the classical cases, recall that in our n-player problem the stochastic control problem is time-inconsistent. Therefore, it would be difficult to verify whether a contract provided by the mean field game would be ε-optimal for the n-player game. Instead, we shall verify that using a contract provided by the mean field game, the principals would receive, in a n-player game, a utility ε-close to the value function calculated in the mean field game.

3.3.1 Agent problem: backward propagation of chaos

Here we shall analyze the agent's behavior, once he is given n contracts provided by the mean field game. Note that the mean field contract has the form:

$$\xi = \overline{Y}_0 - \int_0^T \left(\int_{\mathbb{R}} c^*(y - \overline{Y}_u) p_u^*(dy) + w_u + \sqrt{\eta_u} \right) du + \int_0^T \sqrt{\eta_u} d\overline{X}_u. \tag{3.7}$$

In this section, we shall assume that the contract (ξ, w) is $\mathcal{F}_T^{\overline{X},\overline{Y}_0}$-measurable, that is, we may write $\xi(\overline{X}, \overline{Y}_0)$ and $w(\overline{X}, \overline{Y}_0)$. Now, recall the canonical space Ω in the setting of n-principal problem, and the canonical process $X = \{X^i\}_{i\in\mathbb{I}_n}$ representing the outputs of all principals. Further, let $\{(Y^{*,i}, \xi^i, w^i, \eta^i)\}_{i\in\mathbb{I}_n}$ be the n independent copies of $(\overline{Y}, \xi, w, \eta)$ such that

$$\begin{aligned} Y_t^{*,i} &= \xi^i + \int_t^T \left(\int_{\mathbb{R}} c^*(y - Y_u^{*,i}) p_u^*(dy) + w_u^i + \sqrt{\eta_u^i} \right) du - \int_t^T \sqrt{\eta_u^i} dX_u^i \\ &= \xi^i + \int_t^T \left(\int_{\mathbb{R}} c^*(y - Y_u^{*,i}) p_u^*(dy) + w_u^i + e_i Z_u^{*,i} \right) du - \int_t^T Z_u^{*,i} \cdot dX_u, \end{aligned} \tag{3.8}$$

with $Z^{*,i} := e_i \sqrt{\eta^i} = \left(0, \cdots, 0, \sqrt{\eta^i}, 0, \cdots, 0\right)^\top$.

In this section, we shall simply use the notation $\mathbb{E}$ instead of $\mathbb{E}^{\mathbb{P}^{\hat{\lambda},\hat{\eta}}}$. The following estimate follows directly from Corollary 3.7.

Lemma 3.8. *Assume $q \geq 2$ in Assumption 3.2. Then we have*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left| Y_t^{*,i} \right|^2 \right] < \infty$$

In particular $\mathbb{E}[|\xi^i|^2] < \infty$.

Given such contracts $\{(\xi^i, w^i)\}_{i\in\mathbb{I}_n}$, the agent would solve the system of BSDE (2.7), namely,

$$\begin{aligned} Y_t^{n,i} &= \xi^i + \int_t^T \left(\tfrac{1}{n-1} \sum_{\substack{j=1 \\ j\neq i}}^n c^*(Y_u^{n,j} - Y_u^{n,i}) + w_u^i + e_i Z_u^{n,i} \right) du - \int_t^T Z_u^{n,i} \cdot dX_u \\ &= \xi^i + \int_t^T \left(\tfrac{n}{n-1} \int_{\mathbb{R}} c^*(y - Y_u^{n,i}) p_u^n(dy) - \tfrac{c^*(0)}{n-1} + w_u^i + e_i Z_u^{n,i} \right) du - \int_t^T Z_u^{n,i} \cdot dX_u, \end{aligned} \tag{3.9}$$

where p^n is the empirical measure $\frac{1}{n}\sum_{j=1}^n \delta_{Y^j}$ and $p_u^n := \frac{1}{n}\sum_{j=1}^n \delta_{Y_u^j}$. Define $\Delta Y^i := Y^{n,i} - Y^{*,i}$, $\Delta Z^i := Z^{n,i} - Z^{*,i}$. Let $\mathcal{C} := C([0,T];\mathbb{R})$. Denote the square of the Wasserstein-2 distance on $\mathcal{P}^2(\mathcal{C})$ by

$$d_t^2(\mu, \nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{C}\times\mathcal{C}} \sup_{t\leq u\leq T} |x_u - y_u|^2 \pi(dx, dy).$$

Here is the main result concerning the agent's problem.

Proposition 3.9. *Assume that the contract (ξ, w) is $\mathcal{F}_T^{\overline{X},\overline{Y}_0}$-measurable and Assumption 2.3 and 3.2 hold true for some $q \geq 2$. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[d_0^2(p^n, p^*) \right] = 0. \tag{3.10}$$

Remark 3.10. • There is a certain similarity between the previous result and the one in [BDLP09] where the authors also study the convergence from a n-player BSDE to the mean field limit equation in the form of (3.8). To our understanding of their paper, the solution to the n-player BSDE there is a fixed point of the following map:

$$p \mapsto Y(p) \mapsto \frac{1}{n}\sum_{i=1}^{n} \delta_{Y^i(p)},$$

where $Y(p)$ is the solution of the BSDE

$$Y_t = \xi + \int_t^T \left(\int_{\mathbb{R}} c^*(y - Y_u) p_u(dy) + w_u + Z_u \right) du - \int_t^T Z_u d\overline{X}_u, \tag{3.11}$$

and $\{Y^i(p)\}_{i\in\mathbb{I}_n}$ are the independent copies of $Y(p)$. In other words, their formulation of the n-player problem is via the open loop while ours is via the closed loop. Another crucial difference between the two limit results is that the solution (Y, Z) to (3.11) is adapted in the filtration generated by $(p, \overline{X})$, in particular, note that $\overline{X}$ is of 1-dimension, while the solution $\{(Y^{n,i}, Z^{n,i})\}_{i\in\mathbb{I}_n}$ to our n-player BSDE system is adapted in the filtration generated by X, the n-dimensional process.

- As we will show, the technique involved to prove Proposition 3.9 is a combination of the BSDE estimates and the argument for proving the propagation of chaos. That is why we would name this section the backward propagation of chaos.
- Indeed the result of Proposition 3.9 and the upcoming analysis hold valid for the broader class of BSDE systems in the form of

$$Y_t^{n,i} = \xi^i + \int_t^T F(t, Y^{n,i}, Z^{n,i}, p_t^n) dt - Z_t^{n,i} dX_t, \quad i \in \mathbb{I}_n,$$

where F is Lipschitz continuous in (y, z, p). Apparently, this result could have an independent interest, once one want to look into the asymptotic behavior of solutions to such BSDE systems.

Before proving this main result, we first obtain the following estimates through some classical BSDE arguments.

Lemma 3.11. *We have*

$$\mathbb{E}\left[\sup_{t\le s\le T} |\Delta Y_s^i|^2 \right] \le C\mathbb{E}\left[\int_t^T d_u^2(p^n, p^*) du \right], \tag{3.12}$$

$$\text{and} \quad \mathbb{E}\left[\int_t^T |\Delta Z_u^i|^2 du \right] \le C\left(\mathbb{E}\left[\int_t^T d_u^2(p^n, p^*) du \right] + \frac{1}{n^2} \right). \tag{3.13}$$

Proof. Comparing (3.8) and (3.9) we obtain

$$\begin{aligned} |\Delta Y_t^i| \le{}& \frac{n}{n-1} \int_t^T \left| \int_{\mathbb{R}} c^*(y - Y_u^{n,i}) p_u^n(dy) - \int_{\mathbb{R}} c^*(y - Y_u^{*,i}) p_u^*(dy) \right| du + \frac{|c^*(0)|}{n-1} T \\ &+ \frac{1}{n-1} \int_t^T \int_{\mathbb{R}} c^*(y - Y_u^{*,i}) p_u^*(dy) du + \left| \int_t^T \Delta Z_u^i \cdot (dX_u - e_i du) \right|. \end{aligned} \tag{3.14}$$

Using the Kontorovich duality and Lipschitz continuity of c^*, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} c^*(y-Y_u^{n,i})p_u^n(dy) - \int_{\mathbb{R}} c^*(y-Y_u^{*,i})p_u^*(dy)\Big| \\
&\leq \Big|\int_{\mathbb{R}} c^*(y-Y_u^{n,i})p_u^n(dy) - \int_{\mathbb{R}} c^*(y-Y_u^{n,i})p_u^*(dy)\Big| \\
&\quad + \Big|\int_{\mathbb{R}} c^*(y-Y_u^{n,i})p_u^*(dy) - \int_{\mathbb{R}} c^*(y-Y_u^{*})p_u^*(dy)\Big| \\
&\leq \mathcal{W}_1(p_u^n, p_u^*) + L|\Delta Y_u^i| \\
&\leq \mathcal{W}_2(p_u^n, p_u^*) + L|\Delta Y_u^i|. \qquad (3.15)
\end{aligned}$$

Squaring and taking supremum and expectation on both sides of (3.14), and using Jensen and BDG-inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leq s\leq T}|\Delta Y_s^i|^2\right] &\leq C_T\mathbb{E}\left[\int_t^T \mathcal{W}_2^2(p_u^n,p_u^*)du\right] + C_{L,T}\mathbb{E}\left[\int_t^T |\Delta Y_u^i|^2 du\right] + C_T|c^*(0)|^2 \\
&\quad + C_T\mathbb{E}\left[\sup_{t\leq s\leq T}\left|\int_t^s \Delta Z_u^i\cdot(dX_u - e_i du)\right|^2\right] \\
&\quad + C_T\mathbb{E}\left[\int_t^T\left|\int_{\mathbb{R}} c^*(y-Y^{*,i})p_u^*(dy)\right|^2 du\right] \\
&\leq C_T\mathbb{E}\left[\int_t^T \mathcal{W}_2^2(p_u^n,p_u^*)du\right] + C_{L,T}\mathbb{E}\left[\int_t^T |\Delta Y_u^i|^2 du\right] + C_T|c^*(0)|^2 \qquad (3.16) \\
&\quad + C_{T,BDG}\mathbb{E}\left[\int_t^T|\Delta Z_u^i|^2 du\right] + C_T\mathbb{E}\left[\int_t^T\left|\int_{\mathbb{R}} c^*(y-Y^{*,i})p_u^*(dy)\right|^2 du\right].
\end{aligned}$$

Further we shall estimate $\mathbb{E}\left[\int_t^T|\Delta Z_u^i|^2 du\right]$. By Itô's formula,

$$\begin{aligned}
|\Delta Y_t^i|^2 + \int_t^T|\Delta Z_u^i|^2 du &= \frac{2n}{n-1}\int_t^T \Delta Y_u^i\left(\int_{\mathbb{R}} c^*(y-Y_u^{n,i})p_u^n(dy) - \int_{\mathbb{R}} c^*(y-Y_u^{*,i})p_u^*(dy)\right)du \\
&\quad - \frac{2c^*(0)}{n-1}\int_t^T \Delta Y_u^i du - 2\int_t^T \Delta Y_u^i \Delta Z_u^i\cdot(dX_u - e_i du) \\
&\quad + \frac{2}{n-1}\int_t^T \Delta Y_u^i\int_{\mathbb{R}} c^*(y-Y_u^{*,i})p_u^*(dy)du.
\end{aligned}$$

Together with (3.15) and Young's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_t^T|\Delta Z_u^i|^2 du\right] &\leq 2\mathbb{E}\left[\int_t^T|\Delta Y_u^i|\left(\mathcal{W}_2(p_u^n,p_u^*) + C_L|\Delta Y_u^i|\right)du\right] + \frac{c^*(0)^2}{(n-1)^2}T \\
&\quad + 2\mathbb{E}\left[\int_t^T|\Delta Y_u^i|^2 du\right] + \frac{1}{(n-1)^2}\mathbb{E}\left[\int_t^T\left|\int_{\mathbb{R}} c^*(y-Y_u^{*,i})p_u^*(dy)\right|^2 du\right] \\
&\leq (3+2C_L)\mathbb{E}\left[\int_t^T|\Delta Y_u^i|^2 du\right] + \mathbb{E}\left[\int_t^T \mathcal{W}_2^2(p_u^n,p_u^*)du\right] \\
&\quad + \frac{c^*(0)^2}{(n-1)^2}T + \frac{1}{(n-1)^2}\mathbb{E}\left[\int_t^T\left|\int_{\mathbb{R}} c^*(y-Y_u^{*,i})p_u^*(dy)\right|^2 du\right]. \qquad (3.17)
\end{aligned}$$

We now estimate the last term above,

$$\mathbb{E}\left[\int_t^T \left|\int_{\mathbb{R}} c^*(y - Y_u^{*,i}) p_u^*(dy)\right|^2 du\right] \le \mathbb{E}\left[\int_t^T \left(\int_{\mathbb{R}} \left(L|y| + L|Y_u^{*,i}| + |c^*(0)|\right) p_u^*(dy)\right)^2 du\right]$$
$$\le \mathbb{E}\left[\int_t^T \left(L\mathbb{E}[|Y_u^{*,i}|] + L|Y_u^{*,i}| + |c^*(0)|\right)^2 du\right]$$
$$\le 6L^2 T\mathbb{E}\left[\sup_{0\le u\le T} |Y_u^{*,i}|^2\right] + 3T|c^*(0)|^2,$$

which is bounded from above by a constant C_0, independent of t and i, by the a priori estimate for $Y^{*,i}$ in Lemma 3.8. Together with (3.16) we obtain

$$\mathbb{E}\left[\sup_{t\le s\le T} |\Delta Y_s^i|^2\right] \le (C_T + 2C_{T,BDG})\mathbb{E}\left[\int_t^T \mathcal{W}_2^2(p_u^n, p_u^*) du\right]$$
$$+ \left(C_{L,T} + (4C_L + 10)C_{T,BDG}\right) \int_t^T \mathbb{E}\left[\sup_{u\le s\le T} |\Delta Y_s^i|^2\right] du$$
$$+ (C_T + 4C_{T,BDG}T)c^*(0)^2 + (C_T + 2C_{T,BDG})C_0.$$

Applying Grönwall inequality, we get

$$\mathbb{E}\left[\sup_{t\le s\le T} |\Delta Y_s^i|^2\right] \le C\mathbb{E}\left[\int_t^T \mathcal{W}_2^2(p_u^n, p_u^*) du\right] \le C\mathbb{E}\left[\int_t^T d_u^2(p^n, p^*) du\right],$$

that is, the estimate (3.12), for some constant depending on T, L, C_0 and the constant from BDG inequality. Finally, the estimate (3.13) follows from (3.17) and (3.12). □

Proof of Proposition 3.9 : Define the empirical measure,

$$\nu^n := \frac{1}{n}\sum_{i=1}^n \delta_{Y^{*,i}}.$$

The empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{(Y^{n,i}, Y^{*,i})}$ is a coupling of the empirical measures p^n and ν^n, so

$$d_t^2(p^n, \nu^n) \le \frac{1}{n}\sum_{i=1}^n \sup_{t\le u\le T} |\Delta Y_u^i|, \quad a.s.$$

Together with (3.12), we obtain that

$$\mathbb{E}\left[d_t^2(p^n, \nu^n)\right] \le C\mathbb{E}\left[\int_t^T d_u^2(p^n, p^*) du\right].$$

Apply the triangle inequality and the previous inequality to obtain

$$\mathbb{E}\big[d_t^2(p^n, p^*)\big] \le 2\mathbb{E}\big[d_t^2(p^n, \nu^n)\big] + 2\mathbb{E}\big[d_t^2(\nu^n, p^*)\big] \le 2C\mathbb{E}\left[\int_t^T d_u^2(p^n, p^*) du\right] + 2\mathbb{E}\big[d_t^2(\nu^n, p^*)\big].$$

Using Grönwall's inequality we obtain

$$\mathbb{E}\big[d_0^2(p^n, p^*)\big] \le 2e^{2CT}\mathbb{E}\big[d_0^2(\nu^n, p^*)\big].$$

Each ν^n is the empirical measures of i.i.d. samples from the law p^*, (3.10) follows from the law of large numbers. □

3.3.2 Principals' problem

Recall the third point in Remark 3.4. For technical reasons, in this section we would consider the case where U is bounded.

Proposition 3.12. *Let all the n principals offer the contract provided by a mean field game, for principal i the reward becomes*

$$V_0^{n,i} = \int_K \lambda(dy_0)\mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[X_T^i - U(\xi^i)\mathbf{1}_{\{I_T=i\}} - \int_0^T U(w_u^i)\mathbf{1}_{\{I_u=i\}}du\right], \tag{3.18}$$

where α^* is the optimal intensity of the agent satisfying (2.8). Then, as $n\to\infty$, $V_0^{n,i}$ converges to the value of the mean field game.

Proof. The dynamic version of (3.18) reads

$$V_t^{n,i} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[X_T^i - U(\xi^i)\mathbf{1}_{\{I_T=i\}} - \int_t^T U(w_u^i)\mathbf{1}_{\{I_u=i\}}du\right], \quad \text{for } 0<t\le T.$$

Since U is bounded

$$\mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[\sup_{0\le t\le T}\left|V_t^{n,i}\right|^2\right] < \infty. \tag{3.19}$$

We denote $V_t^{n,i,0} := V_t^{n,i}\big|_{I_t\neq i}$ and $V_t^{n,i,1} := V_t^{n,i}\big|_{I_t=i}$ on different regimes. Recall $\tau_t = \inf\{s\ge t : I_s \neq I_t\}$ and define $\overline{\tau}_t := \inf\{s\ge t : I_s = i\}$. By the tower property of the conditional expectation, we have

$$\begin{cases} V_t^{n,i,1} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[V_{\tau_t}^{n,i,0}\mathbf{1}_{\{\tau_t\le T\}} + \left(X_T^i - U(\xi^i)\right)\mathbf{1}_{\{\tau_t>T\}} - \int_t^{\tau_t\wedge T} U(w_u^i)du\right] \\ V_t^{n,i,0} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[V_{\overline{\tau}_t}^{n,i,1}\mathbf{1}_{\{\overline{\tau}_t\le T\}} + X_T^i\mathbf{1}_{\{\overline{\tau}_t>T\}}\right] = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}}\left[\left(V_{\overline{\tau}_t}^{n,i,1} - X_T^i\right)\mathbf{1}_{\{\overline{\tau}_t\le T\}}\right] + X_t^i. \end{cases} \tag{3.20}$$

It follows from (2.8) that $\lim_{n\to\infty}\mathbb{P}^{\alpha^*}[\overline{\tau}_t\le T] = 0$. Together with (3.19), we obtain

$$\lim_{n\to\infty}\mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[\left|V_t^{n,i,0} - X_t^i\right|^2\right] = 0. \tag{3.21}$$

Define

$$\widetilde{V}_0^{n,i,1} = \int_K \lambda(dy_0)\mathbb{E}^{\mathbb{P}^{\alpha^*}}\left[\int_0^T \alpha_u^n\beta_u^n\left(X_u^i - U(w_u^i)\right)\mathrm{d}u + \beta_T^n\left(X_T^i - U(\xi^i)\right)\right],$$

where $\alpha_t^n = \frac{1}{n-1}\sum_{j\neq i} a^*(Y_t^{n,j} - Y_t^{n,i})$ and $\beta_t^n := \exp\left(-\int_0^t \alpha_u^n du\right)$. Note that α^*, β^* are bounded. By (3.21), we have

$$\lim_{n\to\infty}\left|\widetilde{V}_0^{n,i,1} - V_0^{n,i,1}\right| = 0.$$

Further, on the regime $I=i$, the expectation $\int_K \widehat{\lambda}(dy_0)\mathbb{E}^{\alpha^*}[\cdot]$ coincides with $\mathbb{E}(=\mathbb{E}^{\widehat{\lambda},\widehat{\eta}})$, so

$$\widetilde{V}_0^{n,i,1} = \mathbb{E}\left[\int_0^T \alpha_u^n\beta_u^n\left(X_u^i - U(w_u^i)\right)du + \beta_T^n\left(X_T^i - U(\xi^i)\right)\right].$$

Since the function a^* is bounded and Lipschitz continuous, we have

$$|\beta_t^n - \beta_t^*|^2 + |\alpha_t^n\beta_t^n - \alpha_t^*\beta_t^*|^2 \le C\left(d_0^2(p^n,p^*) + \left|Y_t^{n,i} - Y_t^{*,i}\right|^2\right).$$

Finally, the convergence result (3.10) and the estimate (3.12) imply that

$$\lim_{n\to\infty} \widetilde{V}_0^{n,i,1} = \mathbb{E}\left[\int_0^T \alpha_u^* \beta_u^* \big(X_u^i - U(w_u^i)\big) du + \beta_T^* \big(X_T^i - U(Y_T^{*,i})\big)\right],$$

and the latter is the value of the mean field game. □

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