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Abstract. This paper presents a global stabilization for the two and three-dimensional Navier-Stokes equations in a bounded domain Ω around a given unstable equilibrium state, by means of a boundary normal feedback control. The control is expressed in terms of the velocity field by using a non-linear feedback law. In order to determine the feedback control law, we consider an extended system coupling the equations governing the perturbation with an equation satisfied by the control on the domain boundary. By using the Faedo-Galerkin method and a priori estimation techniques, a stabilizing boundary control is built. This control law ensures a decrease of the energy of the controlled discrete system. A compactness result then allows us to pass to the limit in the system satisfied by the approximated solutions.

1. **Introduction.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ ($d = 2, 3$) with a boundary Γ and let $\Gamma_i \subset \Gamma$, $i = 0, 1, 2, \cdots, N$, be open boundary and nonzero surface measure such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $\Gamma = \cup_{i=0}^N \overline{\Gamma}_i$. Further, we denoted by $\Gamma_l = \Gamma_0$ and $\Gamma_b = \cup_{i=1}^N \Gamma_i$. In particular, the boundary Γ_b is the part of Γ, where a Dirichlet boundary control in feedback form has to be determined. The usual function spaces $L^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$, are used and we let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^1(\Omega) = (H^1(\Omega))^d$ and $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$. Negative ordered Sobolev space $\mathbf{H}^{-1}(\Omega)$ is defined as the dual space, i.e., $\mathbf{H}^{-1}(\Omega) = \{\mathbf{H}_0^1(\Omega)\}'$. We denote by $\langle \cdot | \cdot \rangle$ and $\| \cdot \| = \| \cdot \|_{\mathbf{L}^2(\Omega)}$, the scalar product and the norm in $\mathbf{L}^2(\Omega)$, respectively.

Further, if $\mathbf{u} \in \mathbf{L}^2(\Omega)$ is such that $\nabla \cdot \mathbf{u} \in L^2(\Omega)$, we denote the normal trace of $\mathbf{u}$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ by $\mathbf{u} \cdot \mathbf{n}$, where $\mathbf{n}$ denotes the unit outer normal vector to Γ.

We consider a stationary motion of an incompressible fluid described by the velocity and pressure couple $(\mathbf{v}_s, q_s)$, which is the solution to the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\mathbf{v}_s + (\mathbf{v}_s.\nabla)\mathbf{v}_s + \nabla q_s = \mathbf{f}_s & \text{in } \Omega, \\ \nabla \cdot \mathbf{v}_s = 0 & \text{in } \Omega, \\ \mathbf{v}_s = \boldsymbol{\psi} & \text{on } \Gamma. \end{cases} \tag{1}$$

In this setting, $\nu > 0$ is the viscosity, $\mathbf{f}_s$ is a function in $\mathbf{L}^2(\Omega)$, $\boldsymbol{\psi}$ belongs to $V^{\frac{1}{2}}(\Gamma)$ defined as $V^{\frac{1}{2}}(\Gamma) = \left\{ \mathbf{u} \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma \mathbf{u} \cdot \mathbf{n}\, d\zeta = 0 \right\}$. In [16], it is shown that a solution $(\mathbf{v}_s, q_s)$ to (1) exists in $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ where

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega), \int_\Omega p(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

For $T > 0$ fixed, let $Q = [0, T[\times\Omega, \Sigma_l = [0, T[\times\Gamma_l$ and $\Sigma_b = [0, T[\times\Gamma_b$ and consider a trajectory $(\mathbf{u}, q)$ solution of the non stationary Navier-Stokes equations

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla q = \mathbf{f}_s & \text{in } Q, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u} = \boldsymbol{\psi}|_{\Gamma_b} + \mathbf{u}_b & \text{on } \Sigma_b, \\ \mathbf{u} = \boldsymbol{\psi}|_{\Gamma_l} & \text{on } \Sigma_l, \\ \mathbf{u}_0(\mathbf{x}) = \mathbf{v}_s(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{2}$$

with $\mathbf{x} = (x, y, z)$ if $d = 3$. Consequently, the couple $(\mathbf{v} = \mathbf{u} - \mathbf{v}_s, p = q - q_s)$ satisfies the following non stationary system

$$\begin{cases} a. & \dfrac{\partial\mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\ b. & \nabla \cdot \mathbf{v} = 0 & \text{in } Q, \\ c. & \mathbf{v} = \mathbf{u}_b & \text{on } \Sigma_b, \\ d. & \mathbf{v} = 0 & \text{on } \Sigma_l, \\ e. & \mathbf{v}(t = 0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3}$$

The control $\mathbf{u}_b(t)$ is called a feedback if there exists a mapping $\mathcal{M} : \mathbf{X}(\Omega) \to \mathbf{U}(\Gamma_b)$ such that

$$\mathbf{u}_b(t) = \mathcal{M}(\mathbf{v}(t)), \quad t \in (0, \infty), \tag{4}$$

where the spaces $\mathbf{X}(\Omega)$ and $\mathbf{U}(\Gamma_b)$ will be defined in the sequel. Our goal is the following: for a prescribed rate of decrease $\sigma > 0$, we need to find a feedback control $\mathbf{u}_b$ on Σ_b such that the velocity $\mathbf{v}$ in (3) satisfies the exponential decay

$$\|\mathbf{v}(t)\|_{\mathbf{X}(\Omega)} \le C\, e^{-\sigma t}, \quad t \in (0, \infty). \tag{5}$$

The theoretical setting of the stabilization procedure, for the non stationary incompressible Navier-Stokes equations using a feedback control, has been studied by a number of authors, e.g. A.V. Fursikov et al. [13, 14], V. Barbu et al. [3, 7,

8, 9, 10], J.-P. Raymond et al. [25, 26, 27] and M. Badra et al. [1, 2]. In these papers, the linear feedback law $\mathcal{M}$ is first determined by solving a linear control problem for the linearized system of equations (for example the Oseen system) and then this linear feedback is used in order to stabilize the original non linear system (for example the Navier-Stokes system).

By employing the extension operator, A.V. Fursikov [13, 14] addressed the stabilization of the $2D$ and $3D$ Navier-Stokes equations. In [2, 6, 7, 8, 26, 27], the feedback control laws are determined by solving a Riccati equation in a space of infinite dimension. In such a case, an optimal control problem has to be solved, involving the minimization of an objective functional. In practice, the control is calculated through approximation via the solution of an algebraic Riccati equation, which may be computationally expensive. The use of finite-dimensional controllers may be more appropriate to stabilize the Navier-Stokes equations. Such an approach is performed in [9], in the case of an internal control, and in [1, 6, 7, 8, 25], in the case of a boundary control. In [1, 9, 25], the authors search for a boundary control $\mathbf{u}_b$ of finite dimension of the form

$$\mathbf{u}_b = \sum_{j=1}^{N} u_j(t)\boldsymbol{\psi}_j(\mathbf{x}), \qquad t \geq 0,\ x \in \Gamma,$$

where $(\boldsymbol{\psi}_j)_{j=1,2,3,\ldots,N}$ is a finite-dimensional basis obtained from the eigenfunctions of some operator and $\overline{\mathbf{u}} = (u_1, u_2, u_3, \ldots, u_N)$ is a control function expressed with a feedback formulation. In [25] and [1], where $d = 2$, and $d = 3$, respectively, the feedback control is obtained from the solution of a finite-dimensional Riccati equation while a stochastic-based stabilization technique is employed in [5], in the case of an internal control, which avoids the difficult computation problems related to infinite-dimensional Riccati equations. The procedure employed in [3] for a boundary control resembles the form of stabilizing noise controllers designed in [5].

A linear feedback law is first determined by solving a linear control problem in all the papers cited above, and this linear feedback is then used in order to stabilize the original non linear system. Such a procedure leads to choose the initial velocity small enough, limiting the generality of the result. Moreover, it usually requires to search for the initial condition and the control $\mathbf{u}_b$ in sufficiently regular spaces. The choice of the control profile is also very critical. Indeed, the case of a normal profile is very useful in many applications [15, 20, 23], but the control laws built in all the papers cited above does not guarantee $\mathbf{u}_b \cdot \mathbf{n} \not\equiv 0$ on Γ, since $\mathbf{u}_b \in \{\mathbf{u} \in \mathbf{L}^2(\Gamma) : \int_\Gamma \mathbf{u} \cdot \mathbf{n} = 0\}$.

In the above mentioned studies, for a prescribed rate of decay $\sigma > 0$, an exponential decay of the following form is obtained

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega)} \leq C\|\mathbf{v}_0\|_{\mathbf{X}(\Omega)}e^{-\sigma t}, \quad t > 0, \tag{6}$$

where $\mathbf{X}(\Omega)$ is the adequate space and the constant $C \geq 1$. In practice, it is preferable to have $C = 1$, yielding an immediate exponential decay.

Another approach for stabilizing fluid dynamics equations is proposed in [12, 17, 18, 22, 28]. The method was first published with application on a 1D shallow water equation in [28]. It consists on establishing an equation involving the derivative of energy with respect to time, and the boundary conditions. Then, by utilizing adequate feedback boundary conditions, the authors manage to get the energy's exponential decrease. So far, the method has been applied to stabilize irrigation

channel networks [17, 18], coupled shallow-water and erosion-sedimentation equations [12], and the Navier-Stokes system around a steady-state [22]. Note that in [22], an extended system is considered with an additional equation satisfied by the control on the domain boundary, and the boundary feedback control is constructed via a Galerkin method. Thereby, the authors stabilize the Navier-Stokes equations in a bounded domain Ω around a given steady-state which satisfies the stationary Navier-Stokes equations. However, in [22], the result of stabilization is obtained with a rate of decay σ depending on the viscosity ν and the steady-state $\mathbf{v}_s$. Consequently, the problem of stabilization remains uncontrolled for unstable equilibrium states.

In this paper, the approach of [22], using an extended system is followed, and σ not only depends on the viscosity ν and the steady-state $\mathbf{v}_s$, but also on the size N of the control. Thus, for any fixed ν, we can find N which is greater or equal to the number of unstable mode, such that the problem of stabilization is controllable. This allows the stabilization rate sigma to be arbitrarily large.

The boundary control $\mathbf{u}_b$ in (3) is rewritten on the form $\mathbf{u}_b = \alpha_i(t)\mathbf{g}_i(\mathbf{x})$ on $\Sigma_i = [0, T[\times\Gamma_i$, where the quantity α_i is a priori unknown and the fixed profile $\mathbf{g}_i$ is such that $\mathbf{g}_i \in \mathbf{H}^{1/2}(\Gamma)$, $\mathbf{g}_i = 0$ on $\Gamma_j \cup \Gamma_l$ for $j \neq i$, $\mathbf{g}_i \cdot \mathbf{n} \not\equiv 0$ on Γ_i and $\int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{n} = 0$. In order to stabilize (3), with $\mathbf{u}_b = \alpha_i(t)\mathbf{g}_i(\mathbf{x})$ on Σ_i, by employing energy a priori estimation technics, the quantity α_i is found to satisfy the relation

$$\int_{\Gamma_i} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g}_i \, d\zeta = f_i(\mathbf{v}), \ i = 1, 2, 3, \cdots, N, \tag{7}$$

where f_i is a polynomial in α_i of degree 2 and will be defined later in (28). The quantity α_i depends nonlinearly on $\mathbf{v}$ in (7), and hence α_i satisfies a nonlinear feedback law of the form (4). System (3) is then extended by adding (7), and the extended system, namely (3) and (7), with $\mathbf{u}_b = \alpha_i(t)\mathbf{g}_i(\mathbf{x})$ on Σ_i, is the stabilization problem considered in this paper, i.e.

$$\begin{cases} a. & \dfrac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } Q, \\[2mm] b. & \nabla \cdot \mathbf{v} = 0 & \text{in } Q, \\[2mm] c. & \mathbf{v} = \alpha_i(t)\mathbf{g}_i(\mathbf{x}), \ i = 1, 2, 3, \cdots, N & \text{on } \Sigma_i, \\[2mm] d. & \mathbf{v} = 0 & \text{on } \Sigma_l, \\[2mm] e. & \mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \\[2mm] f. & \displaystyle\int_{\Gamma_i} [\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n}] \cdot \mathbf{g}_i \, d\zeta = f_i(\mathbf{v}), \ \ i = 1, 2, 3, \cdots, N. \end{cases} \tag{8}$$

In order to determine α_i, leading to the determination of the boundary control $\mathbf{u}_b$, system (8) is solved via a Galerkin procedure which consists of building a sequence of approximated solutions using an adequate Galerkin basis.

The paper is organized as follows. In section 2, the notations and mathematical preliminaries are given. In section 3, stabilization is proved and, thanks to technics developed in [19] (which are not related specifically to a stabilization problem), the existence of at least one weak solution of the non-linear Navier-Stokes system is established by applying the Galerkin method.

2. **Notation and Preliminaries.**

2.1. **Function Spaces.** Some spaces of free divergence functions are introduced:

$$\mathbf{V}(\Omega) \quad = \quad \big\{\mathbf{u} \in \mathbf{H}^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = 0 \text{ on } \Gamma_l, \int_{\Gamma_b} \mathbf{u} \cdot \mathbf{n} \, d\zeta = 0\big\}, \quad (9)$$

$$\mathbf{V}_0(\Omega) \quad = \quad \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}, \tag{10}$$

$$\mathbf{H}(\Omega) \quad = \quad \big\{\mathbf{u} \in \mathbf{L}^2(\Omega) \ : \nabla \cdot \mathbf{u} = 0, \ \ \mathbf{u} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma_l\big\}. \tag{11}$$

Since $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$, we have, by definition $\|\cdot\|_{\mathbf{V}(\Omega)} = \|\cdot\|_{\mathbf{H}^1(\Omega)}$.

2.2. **Linear Forms.** In order to define a weak form of the Navier-Stokes equations, we introduce the continuous bilinear forms

$$a(\mathbf{v}_1, \mathbf{v}_2) = \int_\Omega \nabla \mathbf{v}_1 : \nabla \mathbf{v}_2 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega),$$

and the trilinear form:

$$b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \int_\Omega (\mathbf{v}_1 \nabla)\mathbf{v}_2 \cdot \mathbf{v}_3 \, d\mathbf{x}, \quad \forall (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega).$$

In this respect, by integration by parts, we obtain, respectively

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \frac{1}{2} \int_{\Gamma_b} |\mathbf{v}|^2 (\mathbf{u} \cdot \mathbf{n}) \, d\zeta, \ \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}(\Omega). \tag{12}$$

Thanks to Hölder inequality, we obtain

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \le \|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \|\nabla \mathbf{v}_2\| \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)}, \quad \forall \mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3 \in \mathbf{H}^1(\Omega).$$

Further, due to the generalized Sobolev's inequality, there exists a positive constant C such that

$$\|\mathbf{v}_1\|_{\mathbf{L}^3(\Omega)} \le C\|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}_3\|_{\mathbf{L}^6(\Omega)} \le C\|\nabla \mathbf{v}_3\|, \quad \text{for } d = 2, 3,$$

hence,

$$|b(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)| \le C\|\mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_1\|^{\frac{1}{2}} \|\nabla \mathbf{v}_2\| \|\nabla \mathbf{v}_3\|. \tag{13}$$

We now built a hilbertian basis and a control law.

2.3. **Hilbertian basis and control law.** Let $\{\mathbf{z}_j, \lambda_j, j = 1, 2, 3, \cdots\}$ be the eigenfunctions and eigenvalues of the following spectral problem for the Stokes operator:

$$-\Delta \mathbf{z}_j + \nabla p_j = \lambda_j \mathbf{z}_j, \qquad \nabla \cdot \mathbf{z}_j = 0 \quad \text{in } \Omega; \qquad \mathbf{z}_j|_\Gamma = 0. \tag{14}$$

As shown in [11], $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$ as $j \to \infty$, and $\{\mathbf{z}_j\}$ forms an orthonormal basis in $\mathbf{V}_0(\Omega)$:

$$\langle \mathbf{z}_j, \mathbf{z}_k \rangle = \delta_{jk}, \qquad a(\mathbf{z}_j, \mathbf{z}_k) = \lambda_j \delta_{jk}, \quad \forall j, k. \tag{15}$$

Since λ_j in (14) goes to ∞ as $j \to \infty$, for a prescribed rate of decrease $\sigma > 0$, we always find N_σ belonging to $\mathbb{N}^*$, such that

$$\sigma \le \sigma_{N_\sigma} = \frac{\nu}{4}\lambda_{N_\sigma+1} - \left(\frac{2}{3\nu}\right)^3 C^4 \|\nabla \mathbf{v}_s\|^4, \tag{16}$$

where C is defined in (13). Still denoting N_σ by N in the remaining of this paper, we define $I = \{1, 2, 3, \cdots, N\}$ and we let $\mathbf{V}^{1/2}(\Gamma_i)$, $i \in I$, the space of trace functions

whose extended by zero over Γ belongs to $\mathbf{H}^{1/2}(\Gamma)$. In order to built the control law, for all $i \in I$, the profile $\mathbf{g}_i$ must satisfies

$$\mathbf{g}_i \in \mathbf{V}^{1/2}(\Gamma_i), \tag{17}$$

$$\mathbf{g}_i \cdot \mathbf{n} \not\equiv 0 \ \text{ on } \Gamma_i, \tag{18}$$

$$\int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{n} \, d\zeta = 0. \tag{19}$$

Further, in the eventual purpose of ensuring a normal profile for the control, the condition

$$\mathbf{g}_i(\zeta) \cdot \boldsymbol{\tau}(\zeta) = 0, \ \forall \zeta \in \Gamma_i \text{ and } \forall \boldsymbol{\tau} \in T_\zeta(\Gamma_i), \tag{20}$$

might be added to (17)-(19), where the tangent space of Γ_i at ζ, denoted by $T_\zeta(\Gamma_i)$, is defined as the set of all tangent vectors at ζ. For example, assuming that the boundary Γ is regular enough for $\mathbf{n}$ to be in $\mathbf{H}^{1/2}(\Gamma)$, we consider $\mathbf{h}_j$, $j = 0, 1$, defined on Γ_i, $i \in I$ as

$$\mathbf{h}_j(\mathbf{x}) = \begin{cases} \exp\left(-\dfrac{\|\mathbf{x} - \mathbf{x}_j\|^2}{r_j^2 - \|\mathbf{x} - \mathbf{x}_j\|^2}\right) \mathbf{n} & \text{if } \mathbf{x} \in B(\mathbf{x}_j, r_j) \cap \Gamma_i, \\ \mathbf{0} & \text{else,} \end{cases} \tag{21}$$

where $B(\mathbf{x}_j, r_j)$ is the open ball centered at $\mathbf{x}_j \in \Gamma_i$, with radius r_j, such that $B(\mathbf{x}_0, r_0) \cap B(\mathbf{x}_1, r_1) = \emptyset$. By choosing r_0 and r_1 appropriately, one can satisfy (21). If β_j, $j = 0, 1$, is defined as

$$\beta_j = \int_{B(\mathbf{x}_j, r_j) \cap \Gamma_i} \mathbf{h}_j \cdot \mathbf{n} \, d\zeta,$$

the control profile $\mathbf{g}_i = \beta_0 \mathbf{h}_1 - \beta_1 \mathbf{h}_0$ satisfies not only (17)-(19) but also (20). However, note that condition (20) is not necessary to construct the control law in the present study. One might also consider $\mathbf{g}_i(\zeta) = \beta_0 \mathbf{h}_1(\zeta) - \beta_1 \mathbf{h}_0(\zeta) + A(\zeta)\boldsymbol{\tau}(\zeta)$, where $A(\zeta)$ is a sufficiently regular function.

Let us consider the following Stokes problem

$$\begin{cases} a. & -\Delta \boldsymbol{\psi}_i + \nabla q_i = 0, & \text{in } \Omega, \\ b. & \nabla \cdot \boldsymbol{\psi}_i = 0 & \text{in } \Omega, \\ c. & \boldsymbol{\psi}_i = 0 & \text{on } \Gamma_l \cup \Gamma_j, \ j \neq i, \\ d. & \boldsymbol{\psi}_i = \mathbf{g}_i & \text{on } \Gamma_i. \end{cases} \tag{22}$$

First, thanks to (17) and (19), the Stokes problem (22) admits a unique solution $(\boldsymbol{\psi}_i, q_i)$ belonging to $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ (See [11, Proposition III.4.1]). Therefore, the sequence $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3, \cdots, \boldsymbol{\psi}_N, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \cdots$, is linearly independent. Hence, we choose to search the solution $\mathbf{v}$ of (8) in

$$\mathbf{W}(\Omega) = \text{span}(\boldsymbol{\psi}_n + \mathbf{z}_n)_{\{n \in I\}} \oplus \text{span}(\mathbf{z}_n)_{\{n > N\}}. \tag{23}$$

We do not propose any particular norm for the space $\mathbf{W}(\Omega)$ as it is not needed in the manuscript. Further, the solution $\mathbf{v}$ can be expressed as:

$$\mathbf{v} = \mathbf{w} + \mathbf{z}, \ \text{ where } \ \mathbf{w} = \sum_{i=1}^{N} \alpha_i \mathbf{w}_i \quad \text{and} \quad \mathbf{z} = \sum_{i=N+1}^{\infty} \theta_i \mathbf{z}_i, \tag{24}$$

with $\mathbf{w}_i = \boldsymbol{\psi}_i + \mathbf{z}_i$, for $i \in I$.

Secondly, conditions (17) and (18) lead to Lemma 2.1 (see proof in [19]), which is used in the sequel.

Lemma 2.1. *There exists a constant $C_i > 0$ such that, for all $\mathbf{v} \in \mathbf{W}(\Omega)$,*

$$|\alpha_i| \leq C_i \|\mathbf{v}\|, \quad i \in I. \tag{25}$$

Further, using (25), Lemma 2.2 is obtained.

Lemma 2.2. *For all $\mathbf{v} = \mathbf{w} + \mathbf{z} \in \mathbf{W}(\Omega)$, $\mathbf{w}$ satisfies*

$$\|\mathbf{w}\| \leq C_N \|\mathbf{v}\|, \tag{26}$$

where $C_N = \left(\sum_{i=1}^{N} C_i^2 \|\mathbf{w}_i\|^2 + 2 \sum_{1 \leq i < j \leq N} |C_i||C_j||\langle \mathbf{w}_i, \mathbf{w}_j \rangle| \right)^{\frac{1}{2}}, \quad N \in \mathbb{N}^*.$

Proof. Developing $\|\mathbf{w}\|^2$ from (24) and using (25) yields

$$
\begin{aligned}
\|\mathbf{w}\|^2 &= \sum_{i=1}^{N} \alpha_i^2 \|\mathbf{w}_i\|^2 + 2 \sum_{1 \leq i < j \leq N} \alpha_i \alpha_j \langle \mathbf{w}_i, \mathbf{w}_j \rangle \\
&\leq \sum_{i=1}^{N} \alpha_i^2 \|\mathbf{w}_i\|^2 + 2 \sum_{1 \leq i < j \leq N} |\alpha_i||\alpha_j||\langle \mathbf{w}_i, \mathbf{w}_j \rangle| \\
&\leq \left(\sum_{i=1}^{N} C_i^2 \|\mathbf{w}_i\|^2 + 2 \sum_{1 \leq i < j \leq N} |C_i||C_j||\langle \mathbf{w}_i, \mathbf{w}_j \rangle| \right) \|\mathbf{v}\|^2. \tag{27}
\end{aligned}
$$

$\square$

Finally, for $i \in I$, the control law is defined as

$$f_i(\mathbf{v}) = a_i \alpha_i^2 + b_i \alpha_i - (\nu - \epsilon) \lambda_{N+1} \left(\alpha_i \|\mathbf{w}_i\|^2 + 2\langle \mathbf{w}_i, \mathbf{z} \rangle + \sum_{j=1; \ j \neq i}^{N} \alpha_j \langle \mathbf{w}_i, \mathbf{w}_j \rangle \right) \tag{28}$$

where the positive constant $\epsilon < \nu$ is defined in (39) and for $i \in I$,

$$a_i = \frac{1}{2} \int_{\Gamma_i} |\mathbf{g}_i|^2 (\mathbf{g}_i \cdot \mathbf{n}) \, d\zeta \quad \text{and} \quad b_i = \frac{1}{2} \int_{\Gamma_i} |\mathbf{g}_i|^2 (\mathbf{v}_s \cdot \mathbf{n}) \, d\zeta.$$

The dependence of f on $\mathbf{v}$ is realized in the right side of (28) with help of parameters α_i, $i \in I$, and vector $\mathbf{z}$ (see definition (24) of $\mathbf{v}$).

3. **Stability Result.**

3.1. **The variational formulation.** We first consider the variational formulation of the extended Navier-Stokes system.

Definition 3.1. Let $T > 0$ an arbitrary real number and $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, we shall say that $\mathbf{v}$ is a weak solution of (8) on $[0, T]$ if

- $\mathbf{v} \in [L^\infty(0, T; \mathbf{H}(\Omega)) \cap L^2(0, T; \mathbf{V}(\Omega))]$,

- $\exists \alpha_i \in L^\infty(0, T)$ such that $\mathbf{v} = \alpha_i \mathbf{g}_i$ on Γ_i, $i \in I$,

- $\forall \widetilde{\mathbf{v}} = \sum_{i=1}^{N} \widetilde{\alpha}_i \mathbf{w}_i + \sum_{i=N+1}^{\infty} \widetilde{\theta}_i \mathbf{z}_i \in \mathbf{W}(\Omega)$, we have

$$
\begin{cases}
a. \quad \dfrac{d}{dt} \displaystyle\int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} + \nu a(\mathbf{v}, \widetilde{\mathbf{v}}) + b(\mathbf{v}, \mathbf{v}_s, \widetilde{\mathbf{v}}) + b(\mathbf{v}_s, \mathbf{v}, \widetilde{\mathbf{v}}) \\
\hspace{5cm} + b(\mathbf{v}, \mathbf{v}, \widetilde{\mathbf{v}}) = \displaystyle\sum_{i=1}^{N} \widetilde{\alpha}_i f_i(\mathbf{v}), \\
b. \quad \left(\displaystyle\int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \, d\mathbf{x} \right)(0) = \displaystyle\int_\Omega \mathbf{v}_0 \cdot \widetilde{\mathbf{v}} \, d\mathbf{x}.
\end{cases}
\tag{29}
$$

Note that the initial condition (29-b) makes sense because for any solution $\mathbf{v}$ of (29-a), we see that $t \longrightarrow \int_\Omega \mathbf{v}(t) \cdot \widetilde{\mathbf{v}} \, d\mathbf{x}$ is continuous in time (see [11] Corollaire II.4.2).

The main achievement of this paper is the following boundary stabilization result.

Theorem 3.2. *Let $\sigma > 0$ a prescribed rate of decrease, assume that (16) is satisfied and let $\mathbf{g}_i$, $i \in I$, such that*

$$
\mathbf{g}_i \in \mathbf{V}^{1/2}(\Gamma_i), \quad \mathbf{g}_i \cdot \mathbf{n} \not\equiv 0 \ \ on \ \Gamma_i, \quad \int_{\Gamma_i} \mathbf{g}_i \cdot \mathbf{n} \, d\zeta = 0.
$$

For arbitrary initial data $\mathbf{v}_0 \in \mathbf{H}(\Omega)$, there exists a weak solution $\mathbf{v}$ of (8) in the sense of definition 3.1. Moreover, $\mathbf{v}$ satisfies the following estimates:

$$
\|\mathbf{v}(t)\| \ \leq \ \|\mathbf{v}_0\| \, e^{-\sigma t}, \ \forall t > 0,
\tag{30}
$$

$$
\int_0^T \|\nabla \mathbf{v}(s)\|^2 \, ds \ \leq \ C,
\tag{31}
$$

where C is a constant.

Proof. Let us begin with the proof of the stability estimates (Section 3.2) followed by the existence result (Section 3.3).

3.2. A priori estimates.

3.2.1. *A priori estimate for (30).* We take $\widetilde{\mathbf{v}} = \mathbf{v} = \sum_{i=1}^{N} \alpha_i \mathbf{w}_i + \sum_{i=N+1}^{\infty} \alpha_i \mathbf{z}_i$ in (29-a), we have

$$
\frac{1}{2} \frac{d}{dt} \|\mathbf{v}\|^2 + \nu \|\nabla \mathbf{v}\|^2 + b(\mathbf{v}, \mathbf{v}, \mathbf{v}) + b(\mathbf{v}_s, \mathbf{v}, \mathbf{v})
$$

$$
+ b(\mathbf{v}, \mathbf{v}_s, \mathbf{v}) = \sum_{i=1}^{N} \alpha_i f_i(\mathbf{v}).
\tag{32}
$$

Due to (12), we obtain respectively

$$
b(\mathbf{v}, \mathbf{v}, \mathbf{v}) \ = \ \frac{1}{2} \int_{\Gamma_b} |\mathbf{v}|^2 (\mathbf{v} \cdot \mathbf{n}) \, d\zeta = \sum_{i=1}^{N} a_i \alpha_i^3,
\tag{33}
$$

$$
b(\mathbf{v}_s, \mathbf{v}, \mathbf{v}) \ = \ \frac{1}{2} \int_{\Gamma_b} |\mathbf{v}|^2 (\mathbf{v}_s \cdot \mathbf{n}) \, d\zeta = \sum_{i=1}^{N} b_i \alpha_i^2.
\tag{34}
$$

Using (13) and the Young's inequality leads to

$$
\begin{aligned}
|b(\mathbf{v}, \mathbf{v}_s, \mathbf{v})| &\leq C\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{v}_s\|\|\nabla\mathbf{v}\| \\[2mm]
&\leq \frac{\epsilon_1}{2}\|\nabla\mathbf{v}\|^2 + \frac{C^2}{2\epsilon_1}\|\nabla\mathbf{v}_s\|^2\|\mathbf{v}\|\|\nabla\mathbf{v}\| \\[2mm]
&\leq \frac{\epsilon_1}{2}\|\nabla\mathbf{v}\|^2 + \frac{\epsilon_2}{2}\|\nabla\mathbf{v}\|^2 + \frac{1}{2\epsilon_2}\left(\frac{C^2}{2\epsilon_1}\|\nabla\mathbf{v}_s\|^2\right)^2\|\mathbf{v}\|^2.
\end{aligned}
$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon$, we deduce

$$
|b(\mathbf{v}, \mathbf{v}_s, \mathbf{v})| \leq \epsilon\|\nabla\mathbf{v}\|^2 + \left(\frac{C^4}{8\epsilon^3}\|\nabla\mathbf{v}_s\|^4\right)\|\mathbf{v}\|^2. \tag{35}
$$

Definition (28) leads to

$$
\sum_{i=1}^{N}\alpha_i f_i(\mathbf{v}) = \sum_{i=1}^{N}a_i\alpha_i^3 + \sum_{i=1}^{N}b_i\alpha_i^2 - (\nu-\epsilon)\lambda_{N+1}(\|\mathbf{v}\|^2 - \|\mathbf{z}\|^2). \tag{36}
$$

Using (33)-(36) in (32) leads to

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + (\nu-\epsilon)\|\nabla\mathbf{v}\|^2 + C_\epsilon\|\mathbf{v}\|^2 \leq (\nu-\epsilon)\lambda_{N+1}\|\mathbf{z}\|^2. \tag{37}
$$

where

$$
C_\epsilon = (\nu-\epsilon)\lambda_{N+1} - \frac{C^4}{8\epsilon^3}\|\nabla\mathbf{v}_s\|^4
$$

with λ_{N+1} such that $C_\epsilon > 0$, namely

$$
h(\epsilon) = \frac{C^4}{8(\nu-\epsilon)\epsilon^3}\|\nabla\mathbf{v}_s\|^4 < \lambda_{N+1}. \tag{38}
$$

To optimize the choice of λ_{N+1} in (38), we search for $\epsilon \in\,]0,\nu[$ which minimizes h. Consequently, ϵ is such that $h'(\epsilon) = 0$, i.e.

$$
\epsilon = \frac{3\nu}{4}, \tag{39}
$$

which is unique. Due to (22), for all $i \in I$ and $j > N$, we have $\langle\nabla\mathbf{w}_i, \nabla\mathbf{z}_j\rangle = 0$ and hence we deduce,

$$
\|\nabla\mathbf{v}\|^2 = \|\nabla\mathbf{w}\|^2 + \|\nabla\mathbf{z}\|^2. \tag{40}
$$

Using (39) and (40) in (37), it follows

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{w}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{z}\|^2 + \sigma_N\|\mathbf{v}\|^2 \leq \frac{\nu}{4}\lambda_{N+1}\|\mathbf{z}\|^2 \tag{41}
$$

where $\sigma_N = \dfrac{\nu}{4}\lambda_{N+1} - \left(\dfrac{2}{3\nu}\right)^3 C^4\|\nabla\mathbf{v}_s\|^4$ has been used in (16). Since

$$
\lambda_{N+1}\|\mathbf{z}\|^2 = \lambda_{N+1}\sum_{i=N+1}^{\infty}\alpha_i^2 \leq \sum_{i=N+1}^{\infty}\lambda_i\alpha_i^2 = \|\nabla\mathbf{z}\|^2,
$$

according to (41) we obtain

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{\nu}{4}\|\nabla\mathbf{w}\|^2 + \sigma_N\|\mathbf{v}\|^2 \leq 0. \tag{42}
$$

Consequently, for all σ such that $0 < \sigma \leq \sigma_N$, we have

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \sigma\|\mathbf{v}\|^2 \leq 0 \tag{43}
$$

and hence $\mathbf{v}$ satisfies

$$\|\mathbf{v}\| \le \|\mathbf{v}(0)\| e^{-\sigma t}. \tag{44}$$

Moreover, taking $\widetilde{\mathbf{v}} = \mathbf{v}(0)$ in (29-b), leads to

$$\int_\Omega |\mathbf{v}(0)|^2 = \int_\Omega \mathbf{v}_0 \cdot \mathbf{v}(0) \le \frac{1}{2} \int_\Omega |\mathbf{v}_0|^2 + \frac{1}{2} \int_\Omega |\mathbf{v}(0)|^2$$

hence

$$\|\mathbf{v}(0)\|^2 \le \|\mathbf{v}_0\|^2, \tag{45}$$

and according to (44), we obtain

$$\|\mathbf{v}\| \le \|\mathbf{v}_0\| e^{-\sigma t}. \tag{46}$$

3.2.2. *A priori estimate for (31).* Since $\sigma_N > 0$, from (37) with ϵ in (39), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{\nu}{4}\|\nabla \mathbf{v}\|^2 &\le \frac{\nu}{4}\lambda_{N+1}\|\mathbf{z}\|^2 = \frac{\nu}{4}\lambda_{N+1}\|\mathbf{v}-\mathbf{w}\|^2 \\
&\le \frac{\nu}{2}\lambda_{N+1}\|\mathbf{v}\|^2 + \frac{\nu}{2}\lambda_{N+1}\|\mathbf{w}\|^2.
\end{aligned} \tag{47}$$

Using Lemma 2.2 in (47), yields

$$\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{\nu}{2}\|\nabla \mathbf{v}\|^2 \le M_N \|\mathbf{v}\|^2 \tag{48}$$

where $M_N = \nu \lambda_{N+1}(1 + C_N^2)$. Integrating (48) over $(0, t)$ and using the stability estimate (46), we obtain

$$\begin{aligned}
\|\mathbf{v}\|^2 + \frac{\nu}{2}\int_0^t \|\nabla \mathbf{v}\|^2 \, ds &\le \|\mathbf{v}_0\|^2 + M_N \int_0^t \|\mathbf{v}\|^2 \, ds \\
&\le \left(1 + M_N \int_0^t e^{-2\sigma s} \, ds\right)\|\mathbf{v}_0\|^2 \\
&\le \left(1 + \frac{M_N}{2\sigma}\right)\|\mathbf{v}_0\|^2.
\end{aligned} \tag{49}$$

Therefore, we obtain the a priori estimate

$$\int_0^T \|\nabla \mathbf{v}\|^2 \, ds \le \left(\frac{2}{\nu} + \frac{1 + C_N^2}{\sigma}\lambda_{N+1}\right)\|\mathbf{v}_0\|^2. \tag{50}$$

3.3. **Existence.** The proof of the existence follows a standard procedure. In a first step a sequence of approximate solutions using a Galerkin method is built. A compactness result from [21] allows us to pass to the limit in the system satisfied by the approximated solutions.

3.3.1. *The Galerkin Method.* For all $m > N$, we define the space W_m as:

$$W_m = span(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_m\}),$$

where

$$\mathbf{w}_i = \begin{cases} \mathbf{w}_i & \text{if } 1 \le i \le N, \\ \mathbf{z}_i & \text{if } N+1 \le i \le m. \end{cases} \tag{51}$$

Then for $\mathbf{v}_m \in W_m$, $l \in I$, we write $\mathbf{v}_m = \sum_{i=1}^{m} \phi_{im} \mathbf{w}_i$ and we define the following finite-dimensional problem

$$
\begin{cases}
(a) & \langle \partial_t \mathbf{v}_m, \mathbf{w}_j \rangle + \nu a(\mathbf{v}_m, \mathbf{w}_j) + b(\mathbf{v}_m, \mathbf{v}_s, \mathbf{w}_j) + b(\mathbf{v}_s, \mathbf{v}_m, \mathbf{w}_j) \\
& \quad + b(\mathbf{v}_m, \mathbf{v}_m, \mathbf{w}_j) = \sum_{l=1}^{N} \delta_{lj} f_l(\mathbf{v}_m), \\
(b) & \langle \mathbf{v}_m(0) - \mathbf{v}_0, \mathbf{w}_j \rangle = 0, \ \text{for } j = 1, 2, 3, \cdots, m,
\end{cases}
\tag{52}
$$

where δ_{lj} defines the Kronecker symbol and for $\mathbf{z}_m = \sum_{i=N+1}^{m} \phi_{im} \mathbf{w}_i$ and $l \in I$,

$$
f_l(\mathbf{v}_m) = a_l \phi_{lm}^2 + b_l \phi_{lm} - \frac{\nu}{4} \lambda_{N+1} \phi_{lm} \|\mathbf{w}_l\|^2 - \frac{\nu}{2} \lambda_{N+1} \langle \mathbf{w}_l, \mathbf{z}_m \rangle
$$

$$
- \frac{\nu}{4} \lambda_{N+1} \sum_{i=1,\ i\neq l}^{N} \phi_{im} \langle \mathbf{w}_l, \mathbf{w}_i \rangle,
\tag{53}
$$

Recall that ϕ_{im}, $i \in I$, is a priori unknown and thanks to (53), it satisfies a nonlinear feedback law leading to search for $\phi_{im}(\mathbf{v}_m(t))$. Because (53) is independent of $\mathbf{x}$, $\phi_{im}(\mathbf{v}_m(t))$ is a function of t only. For the sake of simplicity, $\phi_{im}(\mathbf{v}_m(t))$ is written ϕ_{im} in the sequel.

Lemma 3.3. *The discrete problem* (52) *has a unique solution* $\mathbf{v}_m \in W^{1,\infty}(0, T; W_m)$. *Moreover this solution satisfies :*

$$
\|\mathbf{v}_m\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\nabla \mathbf{v}_m\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C,
\tag{54}
$$

where C *is a positive constant independent of* m.

Proof. We rewrite (52) in terms of the unknown ϕ_{im}, $i = 1, 2, 3 \cdots m$, and we obtain

$$
\begin{cases}
\sum_{i=1}^{m} \frac{d\phi_{im}}{dt} \langle \mathbf{w}_i, \mathbf{w}_j \rangle + \sum_{i=1}^{m} \phi_{im} \left(\nu\, a(\mathbf{w}_i, \mathbf{w}_j) + b(\mathbf{v}_s, \mathbf{w}_i, \mathbf{w}_j) + b(\mathbf{w}_i, \mathbf{v}_s, \mathbf{w}_j) \right) \\
\quad + \sum_{i,k=1}^{m} \phi_{km} \phi_{im} b(\mathbf{w}_i, \mathbf{w}_k, \mathbf{w}_j) = \sum_{l=1}^{N} \delta_{lj} f_l(\mathbf{v}_m), \\
\sum_{i=1}^{m} \phi_{im}(0) \langle \mathbf{w}_i, \mathbf{w}_j \rangle = \langle \mathbf{v}_0, \mathbf{w}_j \rangle, \ \text{for } j = 1, 2, 3 \cdots, m.
\end{cases}
\tag{55}
$$

Since the matrix with elements $\langle \mathbf{w}_i, \mathbf{w}_j \rangle$ $(1 \leq i, j \leq m)$ is nonsingular, (55) reduces to a nonlinear system with constant coefficients

$$
\begin{cases}
\frac{d\phi_{im}}{dt} + \sum_{j=1}^{m} \phi_{jm} X_{ij} + \sum_{j,k=1}^{m} \phi_{km} \phi_{jm} Y_{ijk} = \sum_{j=1}^{m} \delta_{lj} f_l(\mathbf{v}_m) Z_{ij}, \\
\phi_{im}(0) = \sum_{j=1}^{m} \langle \mathbf{v}_0, \mathbf{w}_j \rangle Z_{ij},
\end{cases}
\tag{56}
$$

where $X_{ij}, Y_{ijk}, Z_{ij}, \in \mathbb{R}$. Then, there exists T_m $(0 < T_m \leq T)$ such that the nonlinear differential system (56) has a maximal solution defined on some interval $[0, T_m]$. In order to show that T_m is independent of m, it is sufficient to verify the boundedness of ϕ_{im}, and hence the boundedness of the L^2-norm of $\mathbf{v}_m$ independently of

m. Following the same procedure as for the derivation of the a priori estimates (46) and (50), yields

$$\begin{cases} a. & \|\mathbf{v}_m(t)\| \leq \|\mathbf{v}_0\| \, e^{-\sigma t}, \ \forall t > 0 \\[2mm] b. & \displaystyle\int_0^T \|\nabla \mathbf{v}_m(s)\| \, ds \leq C. \end{cases} \tag{57}$$

Consequently, according to (57-a), we obtain $T_m = T$. □

Moreover, a consequence of the a priori estimates (57) is that $(\mathbf{v}_m)_m$ is bounded in $L^2(0,T;\mathbf{V}(\Omega))$ and $L^\infty(0,T;\mathbf{H}(\Omega))$. Therefore, for a subsequence of $\mathbf{v}_m$ (still denoted by $\mathbf{v}_m$), the estimates in (57) yield the following weak convergences as m tends to ∞ :

$$\begin{cases} \mathbf{v}_m \rightharpoonup \mathbf{v} \ \text{weakly* in } L^\infty(0,T;\mathbf{H}(\Omega)), \\[2mm] \mathbf{v}_m \rightharpoonup \mathbf{v} \ \text{weakly in } L^2(0,T;\mathbf{V}(\Omega)). \end{cases} \tag{58}$$

Nevertheless, the convergences in (58) are not sufficient to pass to the limit in the weak formulation (52), because of the presence of the convection term. Consequently, we need to obtain additional bounds in order to utilize the compactness theory on the sequence of approximated solution $(\mathbf{v}_m)_m$.

3.3.2. *Additional bounds.* As in [21], let us assume that B_0, B and B_1 are three Hilbert spaces such that $B_0 \subset B \subset B_1$. If $\mathbf{v} : \mathbb{R} \to B_1$ is a function, we denote by $\widehat{\mathbf{v}}$ its Fourier transform

$$\widehat{\mathbf{v}}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} \mathbf{v}(t) dt.$$

Let us recall the following identity about the Fourier transform of differential operators:

$$\widehat{D_t^\gamma \mathbf{v}(\tau)} = (2i\pi\tau)^\gamma \widehat{\mathbf{v}}(\tau),$$

for a given $\gamma > 0$, and let us define the space

$$H^\gamma(\mathbb{R}; B_0, B_1) = \{\mathbf{u} \in L^2(\mathbb{R}, B_0), D_t^\gamma \mathbf{u} \in L^2(\mathbb{R}, B_1)\}.$$

The space $H^\gamma(\mathbb{R}; B_0, B_1)$ is endowed with the norm

$$\|\mathbf{v}\|_{H^\gamma(\mathbb{R};B_0,B_1)} = (\|\mathbf{v}\|_{L^2(\mathbb{R};B_0)}^2 + \||\tau|^\gamma \widehat{\mathbf{v}}\|_{L^2(\mathbb{R};B_1)}^2)^{\frac{1}{2}}.$$

We also define $H^\gamma(0,T; B_0, B_1)$, as the space of functions obtained by restriction to $[0,T]$ of functions of $H^\gamma(\mathbb{R}; B_0, B_1)$. Further, we recall the following result [21]:

Lemma 3.4. *Let B_0, B and B_1 be three Hilbert spaces such that $B_0 \subset B \subset B_1$ and B_0 is compactly embedded in B. Then for all $\gamma > 0$, the injection $H^\gamma(0,T; B_0, B_1) \to L^2(0,T;B)$ is compact.*

For small enough ε, this lemma is used later with

$$B_0 = \mathbf{V}(\Omega), \qquad B = \mathbf{H}(\Omega), \qquad B_1 = \mathbf{H}(\Omega), \qquad \gamma = \frac{1}{4} - \epsilon.$$

The main result of the present section, based on utilizing Lemma 3.4, is furnished by the following lemma:

Lemma 3.5. *The sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0,T;\mathbf{V}(\Omega),\mathbf{H}(\Omega))$ for $0 \leq \gamma \leq \frac{1}{4} - \epsilon$.*

Proof. We denote by $\overline{\mathbf{v}}_m$ the extension of $\mathbf{v}_m$ by zero 0 for $t < 0$ and $t > T$, and $\widehat{\mathbf{v}}_m$ the Fourier transform with respect to time of $\overline{\mathbf{v}}_m$. It is classical that since $\overline{\mathbf{v}}_m$ has two discontinuities at 0 and T, in the distributional sense, the derivative of $\overline{\mathbf{v}}_m$ is given by

$$\frac{d}{dt}\overline{\mathbf{v}}_m = \overline{\mathbf{u}}_m + \mathbf{v}_m(0)\delta_0 - \mathbf{v}_m(T)\delta_T, \tag{59}$$

where δ_0, δ_T are Dirac distributions at 0 and T, and

$$\overline{\mathbf{u}}_m = \mathbf{v}'_m = \text{the derivative of } \mathbf{v}_m \text{ on } [0, T].$$

After a Fourier transformation, (59) gives

$$2i\pi\tau\widehat{\mathbf{v}}_m(\tau) = \widehat{\mathbf{u}}_m(\tau) + \mathbf{v}_m(0) - \mathbf{v}_m(T)e^{-2i\pi\tau T},$$

where $\widehat{\mathbf{v}}_m$ and $\widehat{\mathbf{u}}_m$ denote the Fourier transforms of $\overline{\mathbf{v}}_m$ and $\overline{\mathbf{u}}_m$ respectively. Since we already know that $\mathbf{v}_m$ is uniformly bounded in $L^2(0, T, \mathbf{V}(\Omega))$, it remains to prove that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\widehat{\mathbf{v}}_m(\tau)\|d\tau \leq C. \tag{60}$$

For all $\widetilde{\mathbf{v}} \in W_m$ with $\widetilde{\mathbf{v}} = \widetilde{\alpha}_i \mathbf{g}_i$ on Γ_i, we have that $\overline{\mathbf{v}}_m$ satisfies

$$\int_\Omega \frac{\partial\overline{\mathbf{v}}_m}{\partial t} \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \nu\int_\Omega \nabla\overline{\mathbf{v}}_m : \nabla\widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega G_m \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega G_m^0 \cdot \widetilde{\mathbf{v}}\, d\mathbf{x} + \int_\Omega G_m^1 \cdot \widetilde{\mathbf{v}}\, d\mathbf{x}$$

$$= -\int_\Omega \overline{\mathbf{v}}_m(T) \cdot \widetilde{\mathbf{v}}\delta_T\, d\mathbf{x} + \int_\Omega \overline{\mathbf{v}}_m(0) \cdot \widetilde{\mathbf{v}}\delta_0\, d\mathbf{x} + \sum_{i=1}^N \widetilde{\alpha}_i H_{im}, \tag{61}$$

where $G_m = (\overline{\mathbf{v}}_m\nabla)\overline{\mathbf{v}}_m$, $G_m^0 = (\overline{\mathbf{v}}_m\nabla)\mathbf{v}_s$, $G_m^1 = (\mathbf{v}_s\nabla)\overline{\mathbf{v}}_m$ and $H_{im} = f_i(\overline{\mathbf{v}}_m)$. We now apply the Fourier transform to the equation (61) and take $\widehat{\mathbf{v}}_m$ as a test function, it yields

$$2i\pi\tau\int_\Omega |\widehat{\mathbf{v}}_m(\tau)|^2\, d\mathbf{x} + \nu\int_\Omega \nabla\widehat{\mathbf{v}}_m(\tau) : \nabla\widehat{\mathbf{v}}_m(\tau)\, d\mathbf{x} + \int_\Omega \widehat{G}_m(\tau) \cdot \widehat{\mathbf{v}}_m(\tau)\, d\mathbf{x}$$

$$+ \int_\Omega \widehat{G}_m^0(\tau) \cdot \widehat{\mathbf{v}}_m(\tau)\, d\mathbf{x} + \int_\Omega \widehat{G}_m^1(\tau) \cdot \widehat{\mathbf{v}}_m(\tau)\, d\mathbf{x} = \int_\Omega \overline{\mathbf{v}}_m(0) \cdot \widehat{\mathbf{v}}_m(\tau)\, d\mathbf{x}$$

$$- \int_\Omega \overline{\mathbf{v}}_m(T) \cdot \widehat{\mathbf{v}}_m(\tau)e^{-2i\pi\tau T}\, d\mathbf{x} + \sum_{i=1}^N \widehat{\phi}_{im}\widehat{H}_{im}, \tag{62}$$

where $\widehat{G}_m$, $\widehat{G}_m^0$, $\widehat{G}_m^1$ and $\widehat{H}_{im}$ are respectively the Fourier transform with respect to time of G_m, G_m^0, G_m^1 and H_{im}. Note that

$$\widehat{\phi}_{im}\widehat{H}_{im} = -\frac{\nu}{4}\lambda_{N+1}(\widehat{\phi}_{im})^2\|\mathbf{w}_i\|^2 - \frac{\nu}{2}\lambda_{N+1}\widehat{\phi}_{im}\langle\mathbf{w}_i, \widehat{\mathbf{z}}_m\rangle$$

$$- \frac{\nu}{4}\lambda_{N+1}\sum_{j=1,\, j\neq i}^N \widehat{\phi}_{im}\widehat{\phi}_{jm}\langle\mathbf{w}_i, \mathbf{w}_j\rangle + a_i\widehat{\phi}_{im}\widehat{(\phi_{im}^2)} + b_i(\widehat{\phi}_{im})^2$$

hence

$$\sum_{i=1}^N \widehat{\phi}_{im}\widehat{H}_{im} = -\frac{\nu}{4}\lambda_{N+1}\big[\|\widehat{\mathbf{v}}_m\|^2 - \|\widehat{\mathbf{z}}_m\|^2\big] + \sum_{i=1}^N a_i\widehat{\phi}_{im}\widehat{(\phi_{im}^2)} + \sum_{i=1}^N b_i(\widehat{\phi}_{im})^2. \tag{63}$$

Thanks to Lemma 2.1, we have

$$|\widehat{\phi}_{im}(\tau)| \leq C_i\|\widehat{\mathbf{v}}_m(\tau)\|,$$

using (63) in (62) and taking the imaginary part of (62) leads to

$$|\tau| \|\widehat{\mathbf{v}}_m(\tau)\|^2 \le C \|\widehat{\mathbf{v}}_m(\tau)\| \big(\sup_{\tau \in \mathbb{R}} \sum_{i=1}^{N} a_i C_i \widehat{(\phi_{im}^2)} + \|\overline{\mathbf{v}}_m(T)\| + \|\overline{\mathbf{v}}_m(0)\| \big)$$
$$+ C \|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{V}(\Omega)} \Big(\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^0(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^1(\tau)\|_{\mathbf{V}'(\Omega)} \Big). \quad (64)$$

Note that in the sequel, C stands for different positive constants. We now prove that each term lying in the right hand side of (64) is bounded.

First, we have

$$\|G_m\|_{\mathbf{V}'(\Omega)} \le c_1 \|\mathbf{v}_m\|_{\mathbf{H}^1(\Omega)}^2, \qquad \|G_m^s\|_{\mathbf{V}'(\Omega)} \le c_2 \|\mathbf{v}_m\|_{\mathbf{H}^1(\Omega)}, \quad s = 0, 1,$$

and thanks to the energy estimate (57) satisfied by $\mathbf{v}_m$, G_m and G_m^s remain bounded in $L^1(\mathbb{R}; \mathbf{V}'(\Omega))$ and the functions $\widehat{G}_m$, $\widehat{G}_m^s$ are bounded in $L^\infty(\mathbb{R}; \mathbf{V}'(\Omega))$. Consequently, we have

$$\sup_{\tau \in \mathbb{R}} (\|\widehat{G}_m(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^0(\tau)\|_{\mathbf{V}'(\Omega)} + \|\widehat{G}_m^1(\tau)\|_{\mathbf{V}'(\Omega)}) \le C.$$

Finally, we show that the three last terms in the right hand side of (64) are bounded. Thanks to lemma 2.1, we show that ϕ_{im}^2 is bounded in $L^1(\mathbb{R})$, and the function $\widehat{\phi_{im}^2}$ is bounded in $L^\infty(\mathbb{R})$ with:

$$\sup_{\tau \in \mathbb{R}} \sum_{i=1}^{N} a_i C_i \widehat{(\phi_{im}^2)} \le C.$$

Thanks to the energy estimate (57-a) satisfied by $\mathbf{v}_m$, we have $\|\mathbf{v}_m(T)\| \le C$ and $\|\mathbf{v}_m(0)\| \le C$. Then, inequation (64) finally reduces to

$$|\tau| \|\widehat{\mathbf{v}}_m(\tau)\|^2 \le C(\|\widehat{\mathbf{v}}_m(\tau)\| + \widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}) \le C \|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)},$$

where C stands for different positive constants.

For $0 < \gamma < \frac{1}{4}$, we now estimate the norm

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau.$$

Note that, (see [21])

$$|\tau|^{2\gamma} \le c(\gamma) \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \ \forall \tau \in \mathbb{R}.$$

Consequently, we deduce

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\widehat{\mathbf{v}}_m(\tau)\|^2 d\tau$$
$$\le c(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau + c(\gamma) \int_{-\infty}^{+\infty} \frac{|\tau| \|\widehat{\mathbf{v}}_m(\tau)\|^2}{1 + |\tau|^{1-2\gamma}} d\tau$$
$$\le c_3(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}^2}{1 + |\tau|^{1-2\gamma}} d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau$$
$$\le c_3(\gamma) \int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}^2 d\tau + c_4(\gamma) \int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1 + |\tau|^{1-2\gamma}} d\tau. \quad (65)$$

The last integral in the right hand side of (65) satisfies

$$\int_{-\infty}^{+\infty} \frac{\|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}}{1+|\tau|^{1-2\gamma}} d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\gamma})^2} \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \quad (66)$$

and the first integral in the right hand side of (66) is convergent for any $0 < \gamma < \frac{1}{4}$. On the other hand, using the Parseval equality leads to

$$\int_{-\infty}^{+\infty} \|\widehat{\mathbf{v}}_m(\tau)\|_{\mathbf{H}^1(\Omega)}^2 d\tau = \int_0^T \|\mathbf{v}_m(t)\|_{\mathbf{H}^1(\Omega)}^2 dt \leq C.$$

Then, the sequence $\mathbf{v}_m$ is bounded in $H^\gamma(0,T;\mathbf{V}(\Omega),\mathbf{H}(\Omega))$, for $0 \leq \gamma \leq \frac{1}{4} - \epsilon$. $\qquad \square$

Now, applying Lemmas 3.4 and 3.5, there is a subsequence of $(\mathbf{v}_m)_{m \in \mathbb{N}}$ which converges strongly in $L^2(0,T,\mathbf{H}(\Omega))$.

3.3.3. *Passage to the limit.* The compactness result obtained in the previous section implies the following strong convergence (at least for a subsequence of $\mathbf{v}_m$ still denoted $\mathbf{v}_m$)

$$\mathbf{v}_m \to \mathbf{v} \text{ strongly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

This convergence result together with (58) enable us to pass to the limit in the following weak formulation, obtained from (52) by multiplication by $\varphi \in \mathcal{D}(]0,T[)$ and integration by parts with respect to time

$$-\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\, d\mathbf{x}dt - \int_\Omega \mathbf{v}_m(0)\widetilde{\mathbf{v}}_j\varphi(0)\, d\mathbf{x} + \nu \int_0^T \int_\Omega \nabla\mathbf{v}_m : \nabla\widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt$$

$$+ \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla\mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla\mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt$$

$$+ \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla\mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt = \int_0^T \widetilde{\alpha}_j\delta_{lj}f_l(\mathbf{v}_m)\varphi(t)\, dt \quad (67)$$

for all $\widetilde{\mathbf{v}}_j = \widetilde{\alpha}_j\mathbf{w}_j$. As a first step the integrals in the left hand side of (67) are examined. Using the weak estimates (58) leads to

$$\int_0^T \int_\Omega \mathbf{v}_m \cdot \widetilde{\mathbf{v}}_j\varphi'(t)\, d\mathbf{x}dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j\varphi'(t)\, d\mathbf{x}dt,$$

$$\int_0^T \int_\Omega \nabla\mathbf{v}_m : \nabla\widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt,$$

$$\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla\mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla\mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt,$$

$$\int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla\mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla\mathbf{v}) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt,$$

for the linear terms. Further, since $\mathbf{v}_m$ converges to $\mathbf{v}$ in $L^2(0,T;\mathbf{V}(\Omega))$ weakly, and in $L^2(0,T;\mathbf{L}^2(\Omega))$ strongly, we can pass to the limit in the nonlinear term to obtain

$$\int_0^T \int_\Omega (\mathbf{v}_m \cdot \nabla\mathbf{v}_m) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt \quad \xrightarrow[m \to +\infty]{} \quad \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla\mathbf{v}) \cdot \widetilde{\mathbf{v}}_j\varphi(t)\, d\mathbf{x}dt. \quad (68)$$

As far as the right hand side of (67) is concerned. Using Lemma 2.1 and according to (57-a), we have $\phi_{im} \in L^\infty(0,T)$. Then for a subsequence of ϕ_{im} (still denoted by ϕ_{im}):

$$\phi_{im} \rightharpoonup \alpha_i \text{ weakly}^* \text{ in } L^\infty(0,T). \tag{69}$$

Let us notice that the convergence of $\mathbf{v}_m$ in $L^2([0,T] \times \Omega)$ implies its convergence in $L^1(0,T;\mathbf{L}^2(\Omega))$. Hence

$$\|\mathbf{v}_m\| \qquad \longrightarrow \qquad \|\mathbf{v}\| \text{ in } L^1(0,T). \tag{70}$$

Due to lemma 2.1, for all $i \in I$, we have

$$|\phi_{ip} - \phi_{iq}| \le C_b \|\mathbf{v}_p - \mathbf{v}_q\|, \quad \forall \mathbf{v}_p, \mathbf{v}_q \in W_m,$$

and ϕ_{im} is then a Cauchy sequence in $L^1(0,T)$ and

$$\phi_{im} \longrightarrow \phi_i \text{ in } L^1(0,T). \tag{71}$$

Further, according to (69) we have $\phi_i = \alpha_i \in L^\infty(0,T)$ from [11, Proposition II.1.26] and since ϕ_{im} is bounded in $L^\infty(0,T)$, using (71) we obtain

$$\phi_{im} \longrightarrow \alpha_i \text{ in } L^p(0,T),$$

from [11, Corollaire II.1.24], for all $p \in]1,+\infty[$.

Now we can pass to the limit in the following term:

$$\int_0^T \widetilde{\alpha}_i \phi_{im}^2 \varphi(t)\,dt \qquad \xrightarrow[m \to +\infty]{} \qquad \int_0^T \widetilde{\alpha}_i \alpha_i^2 \varphi(t)\,dt, \tag{72}$$

and since $\mathbf{z}_m = \mathbf{v}_m - \sum_{i=1}^N \phi_{im}\mathbf{w}_i$, we have

$$\int_0^T \widetilde{\alpha}_i \langle \mathbf{w}_i, \mathbf{z}_m \rangle \varphi(t)\,dt \qquad \xrightarrow[m \to +\infty]{} \qquad \int_0^T \widetilde{\alpha}_i \langle \mathbf{w}_i, \mathbf{z} \rangle \varphi(t)\,dt. \tag{73}$$

Consequently

$$\int_0^T \widetilde{\alpha}_i f(\mathbf{v}_m) \varphi(t)\,dt \qquad \xrightarrow[m \to +\infty]{} \qquad \int_0^T \widetilde{\alpha}_i f_i(\mathbf{v}) \varphi(t)\,dt,$$

where

$$f_i(\mathbf{v}) = a_i \alpha_i^2 + b_i \alpha_i - \frac{\nu}{4}\lambda_{N+1}\left(\alpha_i\|\mathbf{w}_i\|^2 + 2\langle \mathbf{w}_i, \mathbf{z}\rangle + \sum_{j=1;\ j \ne i}^N \alpha_j \langle \mathbf{w}_i, \mathbf{w}_j \rangle \right).$$

Thus, passing to the limit in (67) gives

$$-\int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}}_j \varphi'(t)\,d\mathbf{x}dt - \int_\Omega \mathbf{v}_0 \widetilde{\mathbf{v}}_j \varphi(0)\,d\mathbf{x} + \nu \int_0^T \int_\Omega \nabla\mathbf{v} : \nabla\widetilde{\mathbf{v}}_j \varphi(t)\,d\mathbf{x}dt$$

$$+\int_0^T \int_\Omega (\mathbf{v} \cdot \nabla\mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\,d\mathbf{x}dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla\mathbf{v}_s) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\,d\mathbf{x}dt$$

$$+\int_0^T \int_\Omega (\mathbf{v}_s \cdot \nabla\mathbf{v}) \cdot \widetilde{\mathbf{v}}_j \varphi(t)\,d\mathbf{x}dt = \int_0^T \widetilde{\alpha}_j \delta_{ij} f_i(\mathbf{v}) \varphi(t)\,dt. \tag{74}$$

for all $\widetilde{\mathbf{v}}_j = \widetilde{\alpha}_j \mathbf{w}_j$, $j \in \mathbb{N}^*$. By linearity, equation (74) holds true for all $\widetilde{\mathbf{v}}$ combination of finite $\widetilde{\mathbf{v}}_j$ and by density, for any element of $\mathbf{W}(\Omega)$. $\qquad\square$

4. **Concluding remarks.** In this work the global exponential stabilization of the two and three-dimensional Navier-Stokes equations in a bounded domain is studied around a given unstable equilibrium state, using a boundary feedback control. In order to determine a feedback law, an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the domain boundary is considered. We first assume that on Σ_i, $i \in I$, the i-th part of the domain boundary Σ_b, the trace of the fluid velocity is proportional to a given normal velocity profile $\mathbf{g}_i$. The proportionality coefficient α_i measures the velocity flux at the interface, it is an unknown of the problem and it is written in feedback form. By using the Galerkin method, α_i is determined such that the Dirichlet boundary control $\mathbf{u}_b = \alpha_i \mathbf{g}_i$ is satisfied on Σ_i, and the stabilizing boundary control is built. The resulting nonlinear feedback control is proven to be globally exponentially stabilizing the unstable equilibrium state of the two and three-dimensional weak Navier-Stokes equations in the $\mathbf{L}^2$-norm.

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