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Further Studies on the Sparing Number of Graphs

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Abstract: Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u)+f(v)$ is also injective, where $f(u)+f(v)$ is the sum set of $f(u)$ and $f(v)$. If $f^+(uv) = k \ \forall\ uv \in E(G)$, then f is said to be a k-uniform integer additive set-indexer. An integer additive set-indexer f is said to be a weak integer additive set-indexer if $|f^+(uv)| = \max(|f(u)|, |f(v)|)\ \forall\ uv \in E(G)$. In this paper, we study the admissibility of weak integer additive set-indexer by certain graphs and graph operations.

Keywords: Integer additive set-indexers, weak integer additive set-indexers, weakly uniform integer additive set-indexers, mono-indexed elements of a graph, sparing number of a graph.

AMS Subject Classification: 05C78

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [12], [3] and [6] and for different graph classes, we further refer to [4] and [8]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

For two non-empty sets A and B, the *sum set* of A and B is denoted by $A + B$ and is defined by $A + B = \{a + b : a \in A, b \in B\}$. Using the concepts of sum sets, an integer additive set-indexer is defined as follows.

Definition 1.1. [9] Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An *integer additive set-indexer* (IASI, in short) is defined as an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A Graph which admits an IASI is called an *integer additive set-indexed graph* (IASI graph).

Definition 1.2. [10] The cardinality of the labeling set of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element.

Definition 1.3. [9] An IASI is said to be *k-uniform* if $|f^+(e)| = k$ for all $e \in E(G)$. That is, a connected graph G is said to have a k-uniform IASI if all of its edges have the same set-indexing number k. In particular, we say that a graph G has an *arbitrarily k-uniform IASI* if G has a k-uniform IASI for every positive integer k.

The characteristics of a special type of k-uniform IASI graphs, called *weakly k-uniform IASI graphs*, has been studied in [10]. A characterisation of weak IASI graphs has been done in [16]. The following are the major notions and results established in these papers.

Lemma 1.4. *[10] For an integer additive set-indexer f of a graph G, we have*

$$\max(|f(u)|, |f(v)|) \leq |f^+(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)|,$$

where $u, v \in V(G)$.

Definition 1.5. [10] An IASI f is said to be a *weak IASI* if

$$|f^+(uv)| = \max(|f(u)|, |f(v)|) \text{ for all } u, v \in V(G).$$

An IASI f is said to be a *strong IASI* if

$$|f^+(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)| \text{ for all } u, v \in V(G).$$

A weak IASI is said to be *weakly uniform IASI* if

$$|f^+(uv)| = k \text{ for all } u, v \in V(G) \text{ and for some positive integer } k.$$

A graph which admits a weak IASI may be called a *weak IASI graph*.

The following result provides a necessary and sufficient condition for a graph G to admit a weak IASI.

Lemma 1.6. *[10] A graph G admits a weak IASI if and only if at least one end vertex of every edge of G has a singleton set-label.*

Definition 1.7. [16] An element (a vertex or an edge) of graph which has the set-indexing number 1 is called a *mono-indexed element* of that graph. The *sparing number* of a graph G is defined to be the minimum number of mono-indexed edges required for G to admit a weak IASI and is denoted by $\varphi(G)$.

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Theorem 1.8. *[16] If a graph G is a weak (or weakly uniform) IASI graph, then any subgraph H of G is also a weak (or weakly uniform) IASI graph.*

Theorem 1.9. *[16] If a connected graph G admits a weak IASI, then G is bipartite or G has at least one mono-indexed edge.*

From the above theorem, it is clear that all paths, trees and even cycles admit a weak IASI. We observe that the sparing number of bipartite graphs is 0.

Theorem 1.10. *[16] The complete graph K_n admits a weak IASI if and only if the number of edges of K_n that have set-indexing number 1 is $\frac{1}{2}(n-1)(n-2)$.*

We can also infer that a complete graph K_n admits a weak IASI if and only if at most one vertex (and hence at most $(n-1)$ edges) of K_n can have a non-singleton set-label.

Theorem 1.11. *[16] Let C_n be a cycle of length n which admits a weak IASI, for a positive integer n. Then, C_n has an odd number of mono-indexed edges when it is an odd cycle and has even number of mono-indexed edges, when it is an even cycle.*

Theorem 1.12. *[16] The sparing number of C_n is 0 if n is an even number and is 1 if n is an odd integer.*

The admissibility of weak IASIs by the union of weak IASI graphs has been established in [17] and hence proposed the following theorem.

Theorem 1.13. *[17] Let G_1 and G_2 be two cycles. Then, $G_1 \cup C_2$ admits a weak IASI if and only if both G_1 and G_2 are weak IASI graphs.*

The above theorem is proved by defining a weak IASI for the union G of two graphs G_1 and G_2, which is a combination of the weak IASIs of G_1 and G_2. The sparing number of the union of two weak IASI graphs is provided in the following theorem.

Theorem 1.14. *[17] Let G_1 and G_2 be two weak IASI graphs. Then, $\varphi(G_1 \cup G_2) = \varphi(G_1) + \varphi(G_2) - \varphi(G_1 \cap G_2)$. More over, if G_1 and G_2 are edge disjoint graphs, then $\varphi(G_1 \cup G_2) = \varphi(G_1) + \varphi(G_2)$.*

2 New results on graph joins

The join of two graphs is defined as follows.

Definition 2.1. [12] Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs. Then, their *join* (or *sum*), denoted by $G_1 + G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup E_{ij}$, where $E_{ij} = \{u_iv_j : u_i \in G_1, v_j \in G_2\}$.

Certain studies about the admissibility of weak IASIs by the join of graphs have been done in [17]. In this section, we verify the admissibility of a weak IASI by certain graphs which are the joins of some other graphs.

A major result regarding the admissibility of weak IASI by graph joins, proved in [17], is the following.

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Theorem 2.2. *[17] Let $G_1, G_2, G_3, \ldots, G_n$ be weak IASI graphs. Then, the graph $\sum_{i=1}^{n} G_i$ is a weak IASI graph if and only if all given graphs G_i, except one, are 1-uniform IASI graphs.*

Invoking Theorem 2.2, we study about the sparing number of certain graph classes which are the joins of some weak IASI graphs. First, recall the definition of a fan graph.

Definition 2.3. [8] The graph $P_n + \overline{K}_m$ is called an (m, n)-*fan graph* and is denoted by $F_{m,n}$.

The following result estimates the sparing number of a fan graph $F_{m,n}$.

Theorem 2.4. *For two non-negative integers $m, n > 1$, the sparing number of an (m, n)-fan graph $F_{m,n} = P_n + \overline{K}_m$ is n, the length of the path P_n.*

Proof. By Theorem 2.2, $F_{m,n}$ admits a weak IASI if and only if either P_n or $\overline{K}_m$ is 1-uniform.

If P_n is not 1-uniform, then no vertex of $\overline{K}_m$ can have a non-singleton set-label. In this case, the number of mono-indexed edges is $m\lfloor \frac{n+1}{2} \rfloor$.

If P_n is 1-uniform, since no two vertices in $\overline{K}_m$ are adjacent, all vertices of $\overline{K}_m$ can be labeled by non-singleton set-labels. Therefore, no edge between P_n and $\overline{K}_m$ is mono-indexed in G. That is, in this case, the number of mono-indexed edges in $F_{m,n}$ is n.

Since $m > 1$, we have $n < m\lfloor \frac{n+1}{2} \rfloor$. Hence, the sparing number of $F_{m,n}$ is n. □

The above theorem raises a natural question about the sparing number of a graph which is the join of a cycle and a trivial graph. Let us recall the definition of an (m, n)-cone.

Definition 2.5. [8] The graph $C_n + \overline{K}_m$ is called an (m, n)-*cone*.

The following theorem establishes the sparing number of an (m, n)-cone $G = C_n + \overline{K}_m$.

Theorem 2.6. *For two non-negative integers $m, n > 1$, the sparing number of an (m, n)-cone $C_n + \overline{K}_m$ is n.*

Proof. Let $G = C_n + \overline{K}_m$. Then, by Theorem 2.2, G admits a weak IASI if and only if either C_n or $\overline{K}_m$ is i-uniform. If some vertex of C_n has a non-singleton set-label, then all vertices in $\overline{K}_m$ must be mono-indexed. In this case, the number of mono-indexed vertices is $m\lceil n \rceil$. If C_n is 1-uniform, then all vertices in $\overline{K}_m$ can be labeled by distinct non-singleton sets, as they are non-adjacent among themselves. In this case, the number of mono-indexed edges is n. Since $m > 1$, we have $n \leq m\lceil n \rceil$. Therefore, $\varphi(C_n + \overline{K}_m)$ is n. □

Now, let us consider some graphs which are the join of more than two graphs. Consider an (m, n)-tent graph which is defined as follows.

Definition 2.7. A graph $C_n + K_1 + \overline{K}_m$ is called an (m, n)-*tent*. A tent graph can also be considered as the join of a wheel graph $W_{n+1} = C_n + K_1$ and the trivial graph $\overline{K}_m$.

We now proceed to find out the sparing number of an (m,n)-tent graph.

Theorem 2.8. *For two non-negative integers $m,n>1$, the sparing number of an (m,n)-tent $C_n+K_1+\overline{K}_m$ is $2n$.*

Proof. Let $G=C_n+K_1+\overline{K}_m$. By Theorem 2.2, only one among C_n, K_1 and $\overline{K}_m$ can have non-singleton set-label at a time.

If the vertex in K_1 has a non-singleton set label, then both C_n and $\overline{K}_m$ are 1-uniform and hence the number of mono-indexed edges in G is $(m+1)n$.

If some of the vertices in C_n are not 1-uniform, then it has (at least) $\lceil\frac{n}{2}\rceil$ mono-indexed vertices. In this case, all vertices in $K_1\cup\overline{K}_m$ are 1-uniform. Therefore, G has a minimum of $(m+1)\frac{n}{2}$ mono-indexed edges if C_n is an even cycle and has a minimum of $(m+1)\frac{n+1}{2}+1$ mono-indexed edges if C_n is an odd cycle.

If both K_1 and C_n are 1-uniform, then all vertices of $\overline{K}_m$ can be labeled by non-singleton sets. Then, the number of mono-indexed edges in G is $2n$.

Since m and n are positive integers, $2n$ is the minimum among these numbers of mono-indexed edges. Hence, the sparing number of an (m,n)-tent is $2n$. $\square$

Another class of graphs, common in many literature, is the class of friendship graphs which is defined as

Definition 2.9. A graph K_1+mK_2 is called an *m-friendship graph*, where mK_2 is the disjoint union of m copies of K_2. A generalised friendship graph is the graph K_1+mP_n which is usually called by an *(m,n)-friendship graph*. A graph K_1+mC_n is called an *(m,n)-closed friendship graph*. A graph K_1+mK_n is called an *(m,n)-complete friendship graph* or a *windmill graph*.

Theorem 2.10. *The sparing number of an m-friendship graph is n, the sparing number of an (m,n)-friendship graph is $m\lfloor\frac{n+1}{2}\rfloor$, the sparing number of an (m,n)-closed friendship graph is $m\lceil\frac{n}{2}\rceil$ and the sparing number of an (m,n)-complete friendship graph is $\frac{1}{2}mn(n-1)$.*

Proof. Let $G=K_1+mK_2$. If K_1 is not mono-indexed, then no vertex in the copies of K_2 can be labeled by a non-singleton set. Therefore, all edges of mK_2 are mono-indexed. That is, the number of mono-indexed edges in G is m. If K_1 is mono-indexed, then, in each copy K_2, one vertex can be labeled by a singleton set and the other vertex can be labeled by a non-singleton set. Then, one of the two edges between K_1 and each copy of K_2 is mono-indexed. That is, the number of mono-indexed edges in this case is also m. Hence, $\varphi(K_1+mK_2)=m$.

Let us next consider the graph $G=K_1+mP_n$. If K_1 is not mono-indexed, then no vertex in the copies of P_n can be labeled by a non-singleton set. Therefore, the number of mono-indexed edges of G is mn. If K_1 is mono-indexed, then the vertices in each copy of P_n can be labeled alternately by distinct non-singleton sets and distinct singleton sets. Then, the number of mono-indexed vertices in each copy ofP_n is $\lfloor\frac{n+1}{2}\rfloor$. Also, the edges connecting K_1 and these mono-indexed vertices are also mono-indexed. Hence, the total number of mono-indexed edges in G is $m\lfloor\frac{n+1}{2}\rfloor$. Since $m,n > 1$, $m\lfloor\frac{n+1}{2}\rfloor < mn$. Therefore, the sparing number of an (m,n)-friendship graph is $m\lfloor\frac{n+1}{2}\rfloor$.

Now, we shall consider the graph $G = K_1 + mC_n$. If K_1 is not mono-indexed, then no vertex in the copies of C_n can be labeled by a non-singleton set. That is, the number of mono-indexed edges of G is mn. If K_1 is mono-indexed, then the vertices in each copy of C_n can be labeled alternately by distinct non-singleton sets and distinct singleton sets. Then, the number of mono-indexed vertices in each copy ofC_n is $\lceil\frac{n}{2}\rceil$. Hence, the total number of mono-indexed edges in G is $m\lceil\frac{n}{2}\rceil$. Since $m,n > 1$, $m\lceil\frac{n}{2}\rceil < mn$. Therefore, the sparing number of an (m,n)-closed friendship graph is $m\lceil\frac{n}{2}\rceil$.

We now consider the graph$G = K_1 + mK_n$. If K_1 is not mono-indexed, then no vertex in the copies of K_n can be labeled by a non-singleton set. That is, all copies of K_n are 1-uniform. Therefore, the total number of mono-indexed edges in G is $\frac{1}{2}mn(n-1)$. If K_1 is mono-indexed, then exactly one vertex of each copy of K_n can have non-singleton set-label. Therefore, there exist $n-1$ mono-indexed edges between K_1 and each copy of K_n. By Theorem 1.10, each copy of K_n has $\frac{1}{2}(n-1)(n-2)$ mono-indexed edges. Therefore, the total number of mono-indexed edges is $m(n-1) + m\frac{1}{2}(n-1)(n-2) = m\frac{1}{2}n(n-1)$. Note that the number of mono-indexed edges is the same in both cases. Hence, $\varphi(G) = \frac{1}{2}mn(n-1)$.

This completes the proof. □

3 Weak IASI of the Ring sum of Graphs

Analogous to the symmetric difference of sets, we have the definition of the symmetric difference or ring sum of two graphs as follows.

Definition 3.1. [6] Let G_1 and G_2 be two graphs. Then the *ring sum* (or *symmetric difference*) of these graphs, denoted by $G_1 \oplus G_2$, is defined as the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \oplus E_2$, leaving all isolated vertices, where $E_1 \oplus E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$.

The following theorem establishes the admissibility of a weak IASI by the ring sum of two graphs.

Theorem 3.2. *The ring sum of two weak IASI graphs admits a (an induced) weak IASI.*

Proof. Let G_1 and G_2 be two graphs which admit weak IASIs f_1 and f_2 respectively. Choose the functions f_1 and f_2 in such way that no set $A_i \subset \mathbb{N}_0$ is the set-label of a vertex u_i in G_1 and a vertex v_j in G_2 simultaneously.

Let $H_1 = G_1 - G_1 \cap G_2$ and $H_2 = G_2 - G_1 \cap G_2$. Then, by Theorem 1.8, the restriction $f'_1 = f_1|_{H_1}$ is an induced weak IASI for H_1 and the restriction $f'_2 = f_2|_{H_2}$ is an induced weak IASI for H_2. Also, $G_1 \oplus G_2 = H_1 \cup H_2$.

Now, define a function $f : V(G_1 \oplus G_2) \to \mathcal{P}(\mathbb{N}_0)$ such that

$$f(v) = \begin{cases} f_1'(v) & \text{if } v \in V(H_1) \\ f_2'(v) & \text{if } v \in V(H_2). \end{cases} \tag{1}$$

Since f_1' and f_2' are weak IASIs of H_1 and H_2, which are edge disjoint graphs such that $H_1 \cup H_2 = G_1 \oplus G_2$, f is a weak IASI on $G_1 \oplus G_2$. □

The most interesting and relevant question that arises here is about the sparing number of the ring sum of two weak IASI graphs. The following theorem estimates the sparing number of the join of two weak IASI graphs.

Theorem 3.3. *[16] Let G_1 and G_2 be two weak IASI graphs. Then, $\varphi(G_1 \oplus G_2) = \varphi(G_1) + \varphi(G_2) - 2\varphi(G_1 \cap G_2)$.*

Proof. Let G_1 and G_2 be two weak IASI graphs and let $H_1 = G_1 - G_1 \cap G_2$ and $H_2 = G_2 - G_1 \cap G_2$. Then, $G_1 \oplus G_2 = H_1 \cup H_2$. Therefore, $\varphi(H_1) = \varphi(G_1 - G_1 \cap G_2) = \varphi(G_1) - \varphi(G_1 \cap G_2)$. Similarly, $\varphi(H_2) = \varphi(G_1) - \varphi(G_1 \cap G_2)$.

Since $H_1 \cup H_2 = G_{\oplus} G_2$ and H_1 and H_2 are edge disjoint, $\varphi(G_1 \oplus G_2) = \varphi(H_1 \cup H_2)$. Then, we have

$$\begin{aligned} \varphi(G_1 \oplus G_2) &= \varphi(H_1) + \varphi(H_2) \\ &= \varphi(H_1) + \varphi(H_2) \quad \text{(by Theorem 1.14)} \\ &= [\varphi(G_1) - \varphi(G_1 \cap G_2)] + [\varphi(G_2) - \varphi(G_1 \cap G_2)] \\ &= \varphi(G_1) + \varphi(G_2) - 2\varphi(G_1 \cap G_2). \end{aligned}$$

This completes the proof. □

Remark 3.4. It is to be noted that if G_1 and G_2 are edge disjoint graphs, then their ring sum and union are the same. In this case, $\varphi(G_1 \oplus G_2) = \varphi(G_1 \cup G_2)$.

Now, we proceed to discuss the sparing number of certain graphs which are the joins of path and cycles.

Remark 3.5. Let P_m and P_n be two paths in a given graph G. Then, $P_m \oplus P_n$ is a path or edge disjoint union of paths or a cycle or union of edge disjoint cycles. Hence, the sparing number of $P_m \oplus P_n$ is zero if P_m and P_n have at most one vertex in common after the removal of all common edges. If P_m and P_n have two or more common vertices after the removal of all common edges, then the $P_m \oplus P_n$ is a cycle or the union of cycles. Then, the sparing number of $P_m \oplus P_n$ is the sum of the sparing numbers of all these individual cycles.

Remark 3.6. Let P_m be a path and C_n be a cycle in a given graph G. If P_m and C_n are edge disjoint, then $P_m \oplus C_n = P_m \cup C_n$. Therefore, $\varphi(P_m \oplus C_n) = \varphi(C_n)$. If P_m and C_n have some edges in common, then $P_m \oplus C_n$ is a path or edge disjoint union of a cycle and a path. Hence, the sparing number of $P_m \oplus C_n$ is the sum of individual edge-disjoint subgraphs obtained after the removal of common edges of P_m and C_n.

The following theorem establishes the admissibility of weak IASI by the ring sum of two cycles.

Theorem 3.7. *If C_m and C_n are two cycles which admit weak IASIs, and $C_m \oplus C_n$ be the ring sum of C_m and C_n. Then,*

1. *if C_m and C_n are edge disjoint, $\varphi(C_m \oplus C_n) = \varphi(C_m) + \varphi(C_n)$.*
2. *if C_m and C_n are of same parity, then $(C_m \oplus C_n)$ contains even number of mono-indexed edges. Moreover, $\varphi(C_m \oplus C_n) = 0$.*
3. *if C_m and C_n are of different parities, then $(C_m \oplus C_n)$ contains odd number of mono-indexed edges. Moreover, $\varphi(C_m \oplus C_n) = 1$.*

Proof. Let C_m and C_n be two cycles which admit weak IASIs. If C_m and C_n have no common edges then, $C_m \oplus C_n = C_m \cup C_n$. By Theorem 1.13, the union of two weak IASI graphs admits a weak IASI and $\varphi(C_m \oplus C_n) = \varphi(C_m) + \varphi(C_n)$.

Let us now assume that C_m and C_n have some common edges. Let v_i and v_j be the end vertices of the path common to C_m and C_n. Let $P_r, r < m$ be the (v_i, v_j)-section of C_m and $P_s, s < n$ be the (v_i, v_j)-section of C_n, which have no common elements other than v_i and v_j. Hence, we have $C_m \oplus C_n = P_r \cup P_s$ is a cycle. Then, we have the following cases.
Case 1: Let C_m and C_n be of same parity. Then we need to consider the following subcases.
Subcase-1.1: Let C_m and C_n are odd cycles. If C_m and C_n have an odd number of common edges, then both P_r and P_s are paths of even length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, $C_m \oplus C_n$ has a weak IASI with sparing number 0. If C_m and C_n have an even number of common edges, then both P_r and P_s are paths of odd length. Therefore, the cycle $P_r \cup P_s$ is an even cycle. Hence, by Theorem 1.11, $C_m \oplus C_n$ has even number of mono-indexed edges and by Theorem 1.9, the sparing number of $C_m \oplus C_n = 0$.
Subcase-1.2: Let C_m and C_n be even cycles. If C_m and C_n have an odd number of common edges, then both P_r and P_s are paths of odd length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, $C_m \oplus C_n$ has a weak IASI. If C_m and C_n have an even number of common edges, then both P_r and P_s are paths of even length. Hence, the cycle $P_r \cup P_s$ is an even cycle. Therefore, by Theorem 1.11, $C_m \oplus C_n$ has even number of mono-indexed edges and by Theorem 1.9, the sparing number of $C_m \oplus C_n = 0$.
Case 2: Let C_m and C_n be two cycles of different parities. Without loss of generality, assume that C_m is an odd cycle and C_n is an even cycle. Then, we have the following subcases.
Subcase-2.1: Let C_m and C_n have an odd number of common edges. Then, the path P_r has even length and the path P_s has odd length. Hence, the cycle $P_r \cup P_s$ is an odd cycle. Therefore, by Theorem 1.11, $C_m \oplus C_n$ has odd number of mono-indexed edges. More over, by Theorem 1.12, $\varphi(C_m \oplus C_n) = 1$.
Subcase-2.2: Let C_m and C_n have an even number of common edges. Then, P_r has odd length and P_s has even length. Hence, the cycle $P_r \cup P_s$ is an odd cycle. therefore, by Theorem 1.11, $C_m \oplus C_n$ has odd number of mono-indexed edges and by Theorem 1.12, $\varphi(C_m \oplus C_n) = 1$.

This completes the proof. □

To discuss the next result we need the following notion.

Definition 3.8. Let H be a subgraph of the given graph G, then the graph $G - H$ is called *complement* of H in G.

Theorem 3.9. *Let G be a weak IASI graph and H be a subgraph of G. Then, $\varphi(G \oplus H) = \varphi(G) - \varphi(H)$.*

Proof. Let H be a subgraph of the graph G. The complement of H in G, $G \oplus H = G - H$, is a subgraph of G. Hence, as G is a weak IASI graph, by Theorem 1.8, the restriction of this IASI to $G - H$ is a weak IASI for $G - H$. Therefore, $\varphi(G \oplus H) = \varphi(G - H) = \varphi(G) - \varphi(H)$. □

4 Conclusion

In this paper, we have discussed the existence of weak IASI for different graph classes which are the joins of some other graphs and for the ring sum of two graphs. Certain problems in this area are still open. We have not addressed the problem of finding a weak IASI for the join of two arbitrary graphs. Uncertainty in adjacency and incidence patterns among the elements of arbitrary graphs make our studies complex. There are several other factors such as degree of the vertices of the gien graphs, order and size of the graphs etc. which influence the set-labeling of graphs and hence graph joins.

The problems regarding the admissibility of weak IASIs by the join of certain graph classes, whose adjacency and incidence relations are well-known, are yet to be settled. Some of the most promising problems among them are the following.

Problem 4.1. Estimate the sparing number of the join of two regular graphs.

Problem 4.2. Estimate the sparing number of the join of two generalised Petersen graphs.

Problem 4.3. Estimate the sparing number of the join of two bipartite (and complete bipartite) graphs.

Problem 4.4. Estimate the sparing number of the join of two graphs, one of which is a complete graph.

We have already formulated some conditions for some graph classes and graph operations to admit weak IASIs. More properties and characteristics of weak IASIs, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain other types IASIs are still open. A study about those IASIs which assign sets having specific properties, to the elements of a given graph is also noteworthy. All these facts highlight a wide scope for further studies in this area.

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