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Sudev Naduvath, N Sudev, K Chithra, K Germina

Research & Reviews: Discrete Mathematical Structures
ISSN: 2394-1979 (Online)
Volume 4, Issue 3
www.stmjournals.com

Topological Integer Additive Set-Valued Graphs: A Review

N.K. Sudev[1],*, K.P. Chithra[2], K.A. Germina[3]
[1]Centre for Studies in Discrete Mathematics, Vidya Academy of Science & Technology, Thalakkottukara, Thrissur, Kerala, India
[2]Naduvath Mana, Nandikkara, Thrissur, Kerala, India
[3]Department of Mathematical Sciences, Central University of Kerala Kasaragod, Kerala, India

Abstract
Let X denote a set of non-negative integers and $P_0(X)$ be the collection of all non-empty subsets of X. An integer additive set-labeling (IASL) of a graph G is an injective set-valued function $f: V(G) \to P_0(X)$ where induced function $f^+: E(G) \to P_0(X)$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$. A set-labeling $f: V(G) \to P_0(X)$ is said to be a topological set-labeling if $f(V(G)) \cup \{\emptyset\}$ is a topology on the ground set X and a set-labeling $f: V(G) \to P_0(X)$ is said to be a topogenic set-labeling if $f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\}$ is a topology on X. In this article, we critically review some interesting studies on the properties and characteristics of different topological and topogenic integer additive set-labeling of certain graphs.

Keywords: *Integer additive set-labeled graphs, topological additive set-labeled graphs, topogenic integer additive set-labeled graphs, integer additive set-filter graphs*

Mathematics Subject Classification 2010: *05C78.*

****Author for Correspondence*** E-mail: sudevnk@gmail.com

INTRODUCTION
For all terms and definitions, not defined specifically in this paper, we refer to [1–4]. Unless mentioned otherwise, all graphs considered here are simple, finite, non-trivial and connected.

The concepts of number valuations of graphs were first introduced by Rosa [5]. The notion of β-valuation of graphs is popularly called *graceful labeling* of graphs in later studies. Since then, the studies on graph labeling problems attracted wide interest. Several types of number valuations of graphs were introduced and numerous researches have been conducted on many of these types of graph labelings and related fields. The studies on different types of graph labeling have contributed significantly to the researches in graph theory. Graph labeling problems have so many theoretical and practical applications. Many types of graph labelings are surveyed and listed by Gallian [6].

Analogous to the number valuation of graphs, the notion of the *set-valuation* or *set-labeling* of graphs was introduced by Acharya [7] as an injective set assignment of G in which the vertices of G are labeled by the subsets of a ground set X according certain rules. A graph with a set-valuation is called a *set-valued graph* or a *set-labeled graph*. Several studies on different types of set-valued graphs have also been taken place since then.

A set-labeling f is a *topological set-labeling* if $f(V(G))$ is a topology on the ground set X. A *graceful set-labeling* of a graph G is a set-labeling f such that $f(V(G)) = X$ and a *sequential set-labeling* of G is a set-labeling f such that $f(V(G) \cup E(G)) = X$.

Different types of set-labeling of graphs have been effectively and efficiently used for modelling many challenging problems in social networks and social interactions and in many other practical situations.

Basics of Integer Additive Set-Labeled Graphs

The *sumset* of two sets A and B of integers, denoted by $A+B$, is defined as $A+B=\{a+b: a \in A, b \in B\}$ [8]. If $C=A+B$, then A and B are said to be the *summands* of C. Note that $A+\{0\}=A$. Hence, A and $\{0\}$ are said to be the *trivial summands* of the set A and A is called the *trivial sumset* of the sets A and $\{0\}$. The summands of a set A which are not trivial summands of A may be called *non-trivial summands*. If A and B are non-trivial summands of a set C, then C may be called the non-trivial sumset of the sets A and B. We also note that $A+\{\emptyset\}=\emptyset$.

If A or B is countably infinite, then their sumset $A+B$ will also be countably infinite. Hence, all sets we consider here are non-empty, finite sets of non-negative integers.

Invoking the concepts of sumsets of the sets of non-negative integers, the notion of an integer additive set-labeling of a graph has been introduced as follows:

Definition 1.1. Let X be a non-empty finite set of non-negative integers and let $P(X)$ be its power set [9,10]. An *integer additive set-valuation* or an *integer additive set-labeling* (IASL) of a graph G is an injective function $f: V(G) \to P(X)-\{\emptyset\}$ such that the induced function $f^+: E(G) \to P(X)-\{\emptyset\}$ is defined by $f^+uv = f(u)+f(v)\ \forall uv \in E(G)$. A graph G which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

Definition 1.2. An *integer additive set-labeling* of a graph G is said to be an *integer additive set-indexer* (IASI) if the induced function f^+ is also an injective [9,10]. A graph G which admits an IASI is called an *integer additive set-indexed graph* (IASI-graph).

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-labeling number* of that element.

For fundamental terminology on graph labeling, number theory and sumset theory, relevant in this discussion, please see [23-39]. Studies on certain integer additive set-labelings, under which the collections of the set-labels of the elements of given graphs have certain algebraic or topological properties, have attracted much interest. Certain studies in this direction have been done by Mehra et al. and Sudev et al. [11–17]. In this write-up, we review the studies in which the set-labels of the elements of a graph form topology of the ground set X.

TOPOLOGICAL IASL-GRAPHS

Let X be a non-empty finite set and let $P(X)$ be the power sets of X. A set-labeling $f: V(G) \to P(X)$ is said to be a *topological set-labeling* if $f(V(G))$ is a topology on the ground set X. Analogous to topological set-labeling of a graph, the notion of topological integer additive set-labeling of a graph G was introduced by Sudev and Germina [18] as follows:

Definition 2.1. Let G be a graph and let X be a non-empty set of non-negative integers [18]. An integer additive set-indexer $f: V(G) \to P_0(X)$ is called a *topological integer additive set-labeling* (TIASL, in short) of G if $f(V(G)) \cup \{\emptyset\}$ is a topology on X. A graph G which admits a topological integer additive set-labeling is called a *topological integer additive set-labeled graph* (in short, TIASL-graph).

Definition 2.2. A topological IASL f of a graph G is said to be a *topological IASI* of G if the induced function $f^+: E(G) \to P_0(X)$ is also an injective function. A graph which admits a topological IASI may be called *topological IASI-graph*.

A graph which admits a topological IASL is illustrated in Figure 1.

Remark 2.1. For a finite set X of non-negative integers, let the given function $f: V(G) \to P_0(X)$ be an integer additive set-labeling on a graph G. Since the set-label of every edge uv is the sumset of the sets $f(u)$ and $f(v)$, it can be observed that $\{0\}$ cannot be the set-label of any edge of G.
Moreover, since f is a topological integer additive set-labeling defined on G, X must be the set-label of some vertex, say u, of G and hence the set $\{0\}$ will be the set-label of a vertex, say v, and the vertices u and v are adjacent in G.

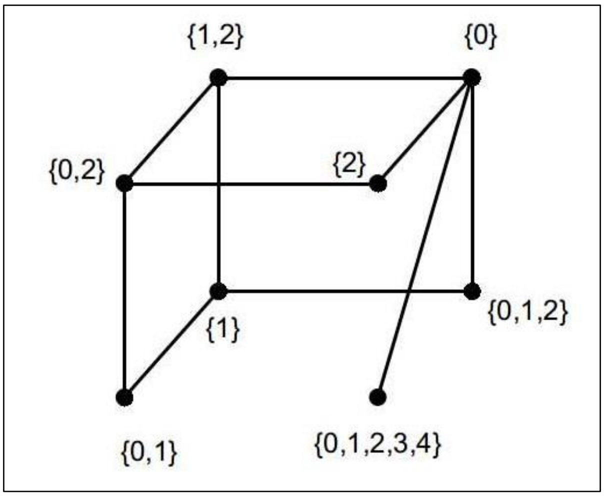

Fig. 1: *An Illustration to a Topological IASL-graph.*

Let f be a topological integer additive set-labeling of a given graph G with respect to a non-empty finite ground set X. Then, $T = f(V(G)) \cup \{\emptyset\}$ is a topology on X. Then, the graph G is said to be a f*-graphical realisation* (or simply f*-realisation*) of T [18]. The elements of the sets $f(V)$ are called f*-open sets* in G.

The existence of an f-graphical realisations for different topologies T of a given non-empty set X has been established by Sudev and Germina [18] as follows:

Theorem 2.1. Let X be a non-empty finite set of non-negative integers. A topology T of X which consists of the set $\{0\}$, is graphically realizable [18].

The above result can trivially be verified by labelling the central vertex of a star graph $K_{1,|T|-2}$ by $\{0\}$ and label the other vertices by the remaining $|T| - 2$ non-empty open sets in T. If we join two non-adjacent vertices u and v of the TIASL-graph $K_{1,|T|-2}$, subject to the condition that $f(u) + f(v) \subseteq X$, the resultant graph will also be a graphical realisation of T. If $f: V(G) \to P_0(X)$ is a topological integer additive set-labeling of a graph G, then there exists some vertex $v \in V(G), f(v) = X$ and hence v is adjacent to one and only vertex, say u such that $f(u) = \{0\}$. Therefore, we have the following result.

Proposition 2.2. If a graph G admits a TIASL, then G has at least one pendant vertex [18].
Let X be the ground set and T be a topology on X and let $f: V(G) \to P_0(X)$ is a topological integer additive set-labeling of a graph G. Regarding the structural properties of a topological IASL-graph, the following observations have been noted by Sudev and Germina [18].

1. An element x_r in X can be an element of the set-label $f(v)$ of a vertex v of G if and only if $x_r + x_s \leq l$, where x_s is any element of the set-label of an adjacent vertex u of v in G and l is the maximal element in X.
2. The vertices whose set-labels containing the maximal element of the ground set X are pendant vertices which are adjacent to the vertex having the set-label $\{0\}$.
3. If G has only one pendant vertex and if G admits a TIASL, then X is the only vertex set-label in G, which contains the maximal element of X.

Let G be a graph on n vertices and let P_m be a path on m vertices which has no common vertex with G. The graph obtained by identifying one vertex of G and one end vertex of P_m is called an $(n; m)$-ladle. If G is a cycle C_n, then this ladle graph is called an (n, m)-*tadpole graph* or a *dragon graph*. If $m = 1$ in a tadpole graph, then it is called an n*-pan graph*. If G is a complete graph K_n, then the corresponding $(n; m)$-ladle graph is called an $(n; m)$-*shovel*.

The existence of topological IASL for n-pan, $(n;m)$-tadpole, and (n,m)-shovel have been established by Sudev and Germina [18]. Figures 2a, 2b and 2c illustrate the existence of topological IASLs for pan, tadpole and shovel graphs, respectively.

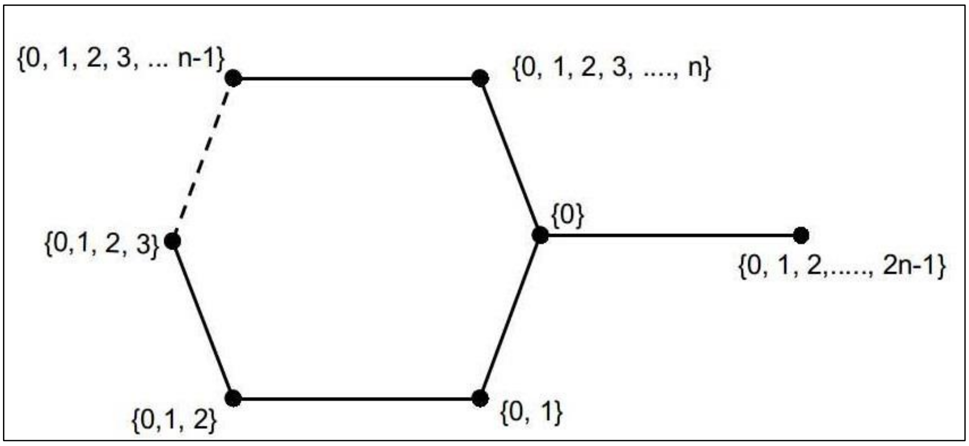

Fig. 2(a): *Existence of Topological IASL for Pan.*

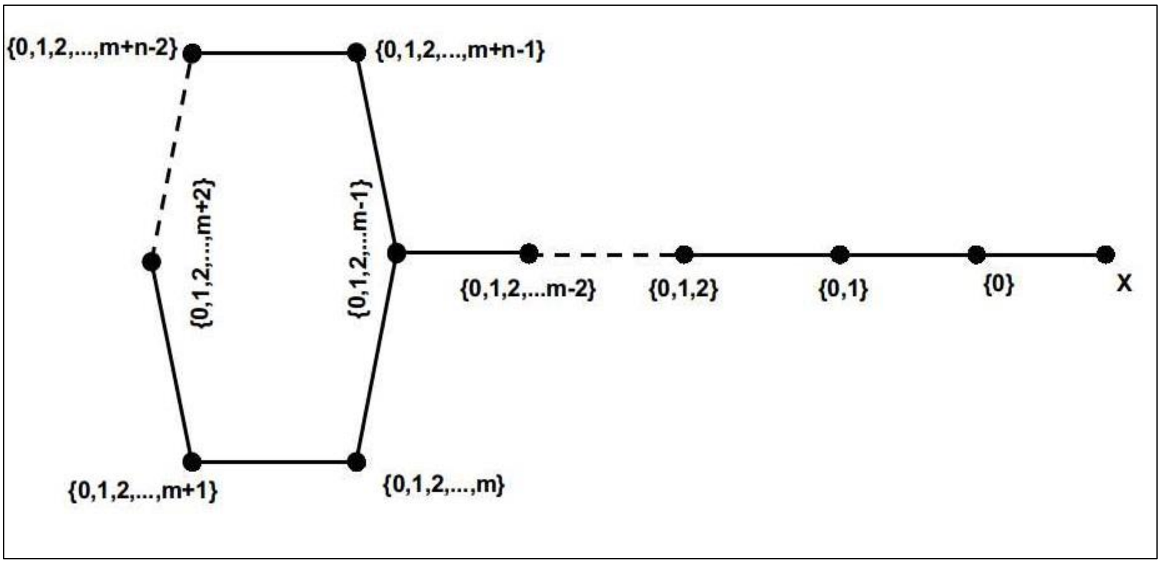

Fig. 2(b): *Existence of Topological IASL for Tadpole.*

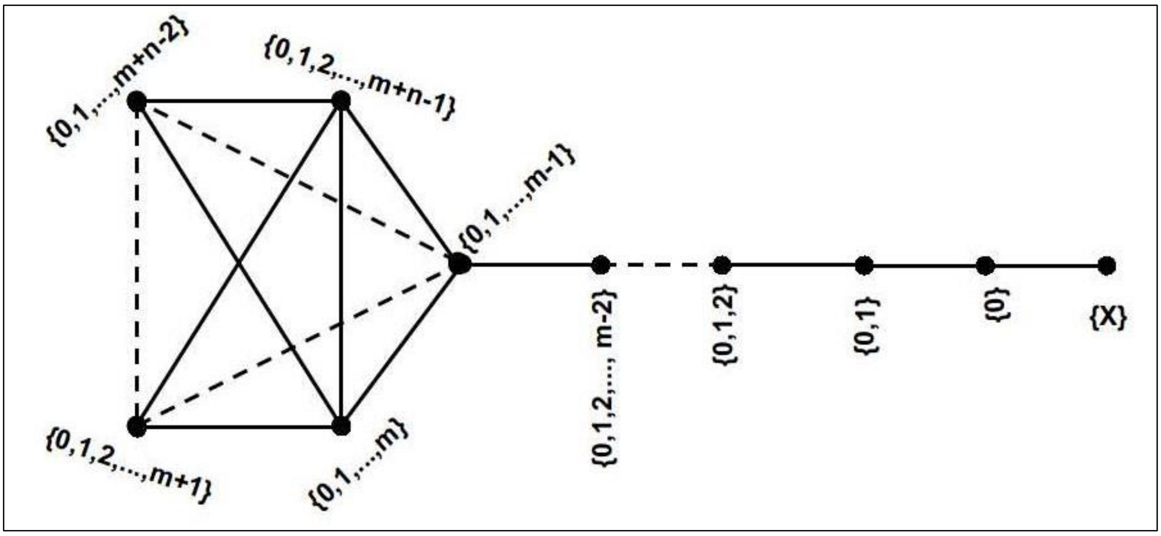

Fig. 2(c): *Existence of Topological IASL for Shovel.*

A much interesting question regarding the minimum number of pendant vertices required for a graph to admit a topological integer additive set-labeling with respect to a given topology T of the ground set X has been addressed by Sudev and Germina [18] as follows:

1. The minimum number of pendant edges incident on a particular vertex of a TIASL-graph is equal to the number of f-open sets in $f(V(G))$ containing the maximal element of the ground set X,
2. The minimum number of pendant vertices of a TIASL-graph G is the number of f-open sets in T, each of which is the non-trivial summand of at most one f-open set in T.

In view of the above results, a necessary and sufficient condition for a given graph to admit a topological integer additive set-labeling has been established by Sudev and Germina [18] as follows:

Theorem 2.3. A graph G admits a topological integer additive set-labeling if and only if G has at least one pendant vertex [18].

In view of Theorem 4, one can note the following facts.

1. All path graphs and trees admit topological IASLs;
2. For $n \geq 3$, no cycles C_n admit topological IASLs;
3. For $n \geq 3$, no complete graphs K_n admit topological IASLs;
4. For $k \geq 2$, no k-connected graph admits a TIASL; and
5. Also, for $m, n \geq 2$, no complete bipartite graph $K_{m,n}$ admits a TIASL.

The following result is an interesting result on the subgraph of a TIASL-graph obtained by a vertex deletion.

Theorem 2.4. Let G be a graph with a pendant vertex v which admits a TIASL, say f, with respect to a ground set X. Let f_1 be the restriction of f to the graph $G - v$. Then, there exists a collection B of proper subsets of X which, together with $\{\emptyset\}$, form a topology on the union of all elements of B [18].

Remark 2.2. If v is the only pendant vertex of a given graph G, then the collection $B = f(V(G - v))$, chosen as explained in Theorem 5 does not induce a topological IASL on the graph $G - v$, since $f^+(uw) \neq f(u) + f(w)$, for some edge $uw \in E(G - v)$ [18].

TIASLS WITH RESPECT TO STANDARD TOPOLOGIES

In this section, we review the results, provided by Sudev and Germina [18], on the existence of topological IASLs with respect to some standard topologies like indiscrete topologies and discrete topologies on the given ground set X.

1. A topology T is said to be an indiscrete topology on X if $T = \{\emptyset, X\}$. The only graph G which admits a topological integer additive set-labeling with respect to the indiscrete topology is the trivial graph K_1.
2. If X is a two-point set, say $X = \{0,1\}$, then the topology $T_1 = \{\emptyset, \{0\}, X\}$ and $T_2 = \{\emptyset, \{1\}, X\}$ are the Sierpenski's topologies. Then, the only graph G which admits a topological integer additive set-labelingwith respect to a Sierpenski's topology is the graph K_2.

The following theorem establishes a necessary and sufficient condition required for G to admit a topological integer additive set-labeling with respect to the discrete topology on X.

Theorem 3.1. A graph G, on n vertices, admits a topological integer additive set-labeling with respect to the discrete topology on the ground set X if and only if G has at least $2^{|X|-1}$ pendant vertices which are adjacent to a single vertex of G [18].

The proof of the above theorem follows directly from the fact that the discrete topology on a finite set is its power set itself and hence the number of non-empty subsets $P_0(X)$ of X is $2^{|X|}-1$ and there are least $2^{|X|-1}$ elements in $P_0(X)$ containing the maximal element of X. All such vertices can be adjacent to the vertex having the set-label $\{0\}$.

In view of Theorem 1, we can observe the following facts:
1. No paths P_n with $n \geq 3$, can have a topological integer additive set-labeling with respect to discrete topology on X;
2. No cycle admits a topological integer additive set-labeling with respect to discrete topology on X;
3. No complete graph admits a topological integer additive set-labeling with respect to discrete topology on X; and
4. No complete bipartite graph admits a topological integer additive set-labeling with respect to the discrete topology on X.

The following result is an important condition for different graphs to have a topological IASL with respect to the discrete topology on the ground set X.

Corollary 3.2. A graph on even number of vertices does not admit a topological integer additive set-labeling with respect to the discrete topology on the ground set X [18].

Corollary 3.3. A star graph $K_{1,r}$ admits a topological integer additive set-labeling with respect to the discrete topology on the ground set X, if and only if $r = 2^{|X|} - 2$.

A subsequent study on topological IASI graphs has been done by Mehra and Puneet [11]. Mehra and Puneet [11] have proved, the following results:
1. A path graph P_n admits a Top-IASL.
2. Every n-pan graph has a Top-IASI.
3. Every $(n;2)$-tadpole graph has a Top-IASI.
4. Every tadpole graph has a Top-IASI.
5. Every $(n;m)$-shovel graph has Top-IASI.

TOPOGENIC IASL-GRAPHS

The notion of topogenic set-labeling was introduced by Acharya et al. [19] and the structural properties and characteristics of graphs which admit the topogenic set-labeling have been studied by Acharya et al. and Joy et al. [19,20]. Motivated by these studies, a sumset analogue of the topogenic set-labeling of graphs has been introduced by Sudev and Germina [13] as follows:

Definition 4.1. Let X be a finite non-empty set of non-negative integers [13]. An integer additive set-indexer f of a given graph G, defined by $f: V(G) \to P_0(X)$, is said to be a *topogenic integer additive set-labeling* (topogenic IASL) of G, with respect to the ground set X, if $T = f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\}$ is a topology on X. A graph G that admits a topogenic IASL is said to be a *topogenic IASL-graph*.

Definition 4.2. A topogenic integer additive set-indexer $f: V(G) \to P_0(X)$ of a graph G, is a topogenic integer additive set-labeling of G such that the induced function $f^+: E(G) \to P_0(X)$ is also an injective function [13]. A graph G that admits a topogenic IASI is said to be a *topogenic IASI-graph*.

Consider the function f^* defined on the given graph G as follows:

$$f^*(G) = \begin{cases} f(x) & \text{if } x \in V(G) \\ f^+(x) & \text{if } x \in E(G) \end{cases}$$

Therefore, $f^*(G) = f(V(G)) \cup f^+(E(G)) \cup \{\emptyset\}$. Hence, f is a topogenic IASL of G if $f^*(G) \cup \{\emptyset\}$ is a topology on the ground set X.

Figure 3 depicts a graph which admits a topogenic IASL-graphs.

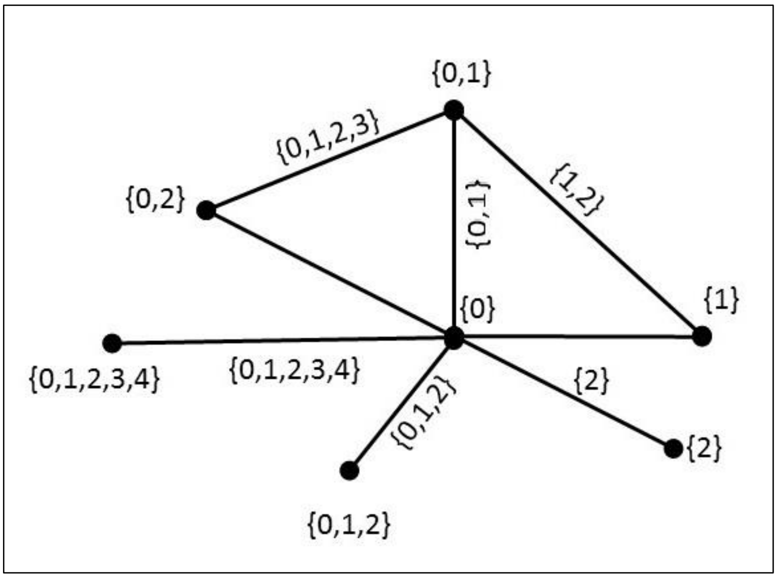

Fig. 2: *An Illustration to Topogenic IASL-graphs.*

For a non-empty finite set X of non-negative integers, the trivial graph K_1 admits a topogenic IASL with respect to the indiscrete topology on X. Let X be a two point set of non-negative integers. Let T be a topology on X, which is neitherthe indiscrete topology nor the discrete topology on X. Then, $T_1 = \{\emptyset, \{a\}, X\}$ where $a \in X$. Then, the graph K_2 admits a topogenic IASL with respect to the set X if and only if $a = 0$.

If $f: V(G) \to P_0(X)$ is a topogenic IASL of G, then it is possible for a vertex $v \in V(G)$ and an edge $e \in E(G)$, that $f(v) = f^+(e)$. The minimum number of elements in a topogenic IASL-graph G; having the same set-labels have been determined in the following lemma:

Lemma 4.1. Let f be a topogenic IASL defined on a given graph G with respect to a topology T of a ground set X. Then, there are at least ρ' elements in T common to $f(V)$ and $f^+(E)$, where ρ' is the number of elements in sT which are neither the non-trivial summands nor the non-trivial sumsets of some other sets in T [13].

If A is not a non-trivial sumset of any two subsets of X (and non-trivial summand of any set $B \subseteq X$), then it can be the set-label of a vertex which is adjacent to the vertex having set-label $\{0\}$. Then, the set-label of the corresponding edge is also A and hence the validity of the above lemma can be verified.

Throughout this section, the sumsets and summands we mention are non-trivial sumsets and non-trivial summands of the subsets of the ground set X.

As an immediate consequence of Lemma 1, the following results have been proved by Sudev and Germina [13].

Lemma 4.2. Let X be a non-empty finite ground set [13]. If X is not a sumset of two subsets of X, then a graph G admits a topogenic IASL f if

1. $0 \in X$ and $\{0\} \in f(V)$.
2. G has at least one pendant vertex.

In view of the above results, a necessary condition for a graph to admit a topogenic IASL with respect to a given topology of the ground set X, which is not a sumset of its subsets.

Theorem 4.3. Let X be non-empty finite ground set and let T be a topology on X such that X is not a sumset of any elements in T. If a graph G admits a topogenic IASL f with respect to T, then $\{0\} \in T$ and G has at least one pendant vertex [13].

As a consequence of the above theorem, a necessary condition for a topogenic IASL-graph to have pendant vertices has been established by Sudev and Germina [13] as follows.

Theorem 4.4. Let f be a topogenic IASL of a graph G with respect to a topology T of the ground set X. If T, consists of ρ' sets which are neither the sumsets nor the summands of some other sets in T, then G should necessarily have at least ρ' pendant vertices [13].

Invoking the above results, one can note that if $f: V(G) \to P_0(X)$ is a topogenic IASL of a graph G with respect to a ground set X, then the set X is the set-label of an edge or a pendant vertex of G. Hence, the following theorem has been proved by Sudev and Germina [13] as an immediate consequence of the above results.

Theorem 4.5. Let f be a topogenic IASL defined on a graph G with respect to a non-empty finite set X. Then, every set in T_f is either a summand of some sets in T_f or the sumset of two sets in T_f [13].

Analogous the topological graphical realisations mentioned in the previous section, the notion of topogenic graphical realisations has been introduced by Sudev and Germina [13] as given below.

Definition 4.3. Let X be a non-empty finite ground set and let T be a topology on X. Then, a graph G is said to be a *topogenic graphical realisation* of T if there exists a topogenic IASL f on G such that $f(V(G)) \cup f^+(E(G)) cup\{\emptyset\} = T$ [13].

The existence of topogenic graphical realisation for a given topology on the ground set X has been established in the following theorem.

Theorem 4.6. Let X be a ground set which contains the element 0 and T be any topology on X which contains the set $\{0\}$. Then, there exists a topogenic graphical realisation for the topology T [13].

A necessary and sufficient condition for the existence of a topogenic graphical realisation for a given topology on the ground set X has been described as,

Theorem 4.7. Let X be a non-empty finite ground set and let T be a topology on X. Then, a graph G is a topogenic graphical realisation of T if and only if every non-empty set of T is either a summand of some other elements of T or the sumset of two elements of T [13].

The following theorem has discussed the number of elements in a graph which admits a topogenic IASL with respect to a given ground set X.

Theorem 4.8. Let G be a topogenic IASL-graph with respect to a topology T of a given ground set X and let T contains ρ sets which are not the sumsets of any other sets in T. Then, $|V(G)| \geq \rho$ and $|E(G)| \geq \tau - \rho$, where$\tau = |T|$ [13].

TOPOLOGICAL INTEGER ADDITIVE SET-GRACEFUL LABELINGS

An *integer additive set-graceful labeling* (IASGL) of a graph G is defined by Sudev and Germina [14] as an integer additive set-labeling $f: V(G) \to \mathcal{P}(X) - \{\emptyset\}$ such that the induced function $f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$. Agraph G which admits an integer additive set-graceful labeling is called an *integer additive set-graceful graph* (in short, IASG-graph).

Sometimes, under certain IASLs, the collection of the set-labels of vertices of a graph can be both graceful collection and topology of the ground set X. Hence, the notion of topological integer additive set-graceful labeling has been introduced by Sudev et al. [16] as follows.

Definition 5.1. An integer additive set-graceful labeling f of a graph G, with respect to a finite set X, is said to be a *topological integer additive set-graceful labeling* (Top-IASGL) if $f(V(G)) \cup \{\emptyset\}$ is a topology on X and the induced edge-function $f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{0\}\}$ [16].

It is proved by Sudev and Germina [14] that a tree which admits an integer additive set-graceful labeling with respect to a ground set X is a star graph $K_{1,2^{|X|}-2}$. Invoking this result, the following result is an obvious result regarding the existence of a topological integer additive set-graceful labeling for a tree.
Theorem 5.1. An integer additive set-graceful labeling of a tree, if exists, is also a topological integer additive set-graceful labeling [16].

The above theorem follows immediately from the fact that for tree having an integer additive set-graceful labeling f, $f(V(G)) = P(X) - \{\emptyset\}$, $f(E(G)) = P(X) - \{\emptyset, \{0\}\}$ and $P(X)$ is a topology on X. Hence, the following result is a straight forward one.

Theorem 5.2. If G is an acyclic graph which admits a topological integer additive set-graceful labeling, with respect to a ground set X, then G is a star $K_{1,2^{|X|}-2}$ [16].

The following theorem ruled out the existence of a topological integer additive set-graceful labelings for regular graphs.

Theorem 5.3. No connected regular graphs admit a topological integer additive set-graceful labeling [16].

The above results follow from the facts that every integer additive set-graceful labeled graph G must have some pendant vertices. The only regular graph with pendant vertices is K_2, which is not a star graph of the form $K_{1,2^{|X|}-2}$.

In view of Theorem 3, note that the cycles C_n, paths P_n, complete graph K_n and complete bipartite graph $K_{n,n}$ (other than star graphs of the form $K_{1,2^{|X|}-2}$) do not admit topological integer additive set-graceful labeling.

Note that all graphs, in general, do not admit Top-IASLs. Also, note that all topological integer additive set-labelings on a given graph G do not induce integer additive set-graceful labelings on G. Hence, the studies on characteristics and structural properties of graphs which admit topological integer additive set-labelings created much interest.

Note that a trivial graph does not admit a topological integer additive set-graceful labeling. A necessary and sufficient condition for a graph to admit a topological integer additive set-graceful labeling has been established in the following theorem.

Theorem 5.4. Let X be a non-empty finite set. Then, a graph G admits a Top-IASGL if and only if the following conditions hold [16].

1. G has $2^{|X|} - 2$ edges and at least $2^{|X|} - (\rho + 1)$ vertices, where ρ is the number of subsets of X, which can be expressed as the sumsets of two subsets of X.
2. One vertex, say v, of G has degree ρ'', which is the number of subsets of X which are neither the non-trivial summands of any subsets of X nor the sumsets of any elements of $\mathcal{P}(X)$.
3. G has at least ρ' pendant vertices if X is not a sumset of the subsets of it and has at least $1 + \rho'$ pendant vertices if X is a sumset of some subsets of it, where ρ' is the number of subsets of X, which are not the sumsets of any subsets of X and not a summand of any sub set of X.

Figure 4 depicts a graph which admits a topological integer additive set- graceful labeling.

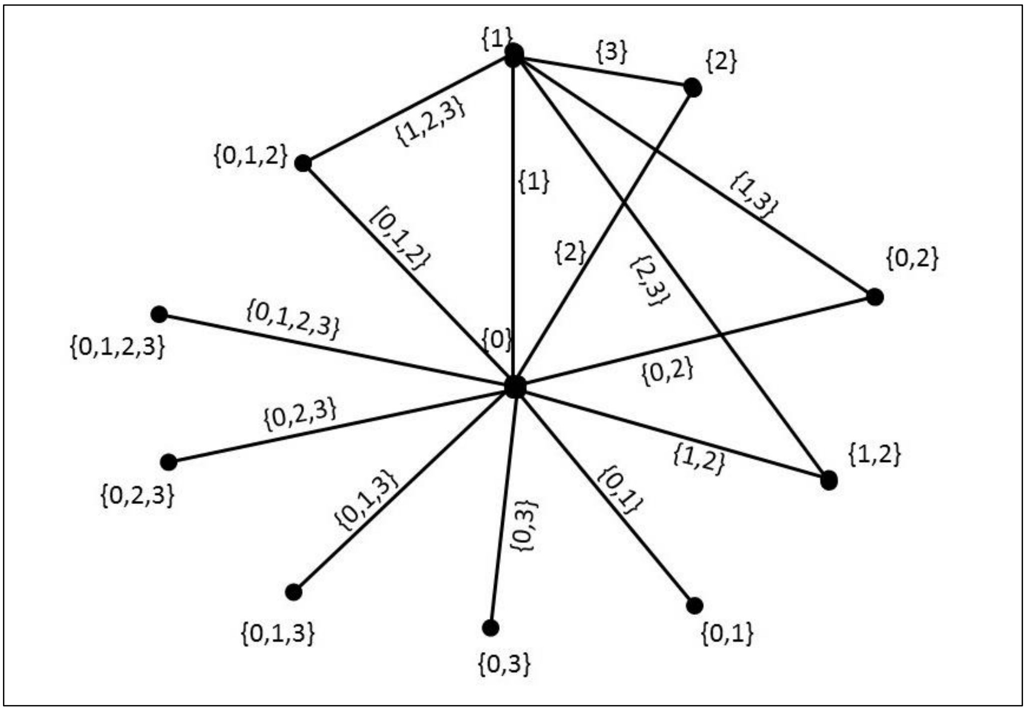

***Fig. 4:** An Illustration to a Topological Integer Additive Set-graceful Labeled Graph.*

The following theorem discussed the admissibility of a topological integer additive set-graceful labeling with respect to the discrete topology of the ground set X.

Theorem 5.5. A graph G admits a topological integer additive set-graceful labeling f with respect to the discrete topology of a non-empty finite set X if and only if $G \cong K_{1,2^{|X|}-2}$ [16].

TOPOLOGICAL INTEGER ADDITIVE SET- SEQUENTIAL LABELING OF GRAPHS

An integer additive set-labeling f of a graph G, with respect to a finite set X, is said to be an *integer additive set-sequential labeling* (IASSL) of G if there exists an extension function $f^*: G \to \mathcal{P}(X)$ of f defined by

$$f^*(G) = \begin{cases} f(x) & \text{if } x \in V(G) \\ f^+(x) & \text{if } x \in E(G) \end{cases}$$

such that $f^*(G) \cup \{\emptyset\} = \mathcal{P}(X)$.

A graph which admits an integer additive set-sequential labeling is called an *integer additive set-sequential* graph (IASS-graph) [15] (for the terminology).

Analogous to the definition of topological integer additive set-graceful labeling, the notion of topological integer additive set-sequential labeling of graphs has been introduced by Sudev et al. [17] as follows:

Definition 6.1. An integer additive set-labeling f of a graph G, with respect to a finite set X, is said to be a *topological integer additive set-sequential labeling* (TIASSL) if $f(V) \cup \{\emptyset\}$ is a topology on X and $f^*(G) = \mathcal{P}(X) - \{\emptyset\}$, where $f^*(G) = f(V(G)) \cup f^+(E(G))$, as defined in the previous definition [17].

That is, a topological integer additive set-sequential labeling of a graph G is an IASI of G which is both a topological integer additive set-labeling and an integer additive set-sequential labeling of G. Figure 5 illustrates a topological integer additive set-sequential graph.

A topological integer additive set-sequential labeling $f: V(G) \to \mathcal{P}(X)$ is said to be a *topological integer additive set-sequential indexer* (TIASSI) if the induced function f^* is also injective.

If $f: V(G) \to P(X) - \{\emptyset\}$ be a topological integer additive set-sequential labeling of a given connected graph G, with respect to a given ground set X, then $X \in f(V)$ as $f(V(G)) \cup \{\emptyset\}$ is a topology of X. Therefore, $\{0\}$must be an element in $f(V)$ and the vertex having the set-label X can be adjacent only to the vertex having $\{0\}$. That is, the vertex having the set label X is a pendant vertex of G adjacent to the vertex having set-label $\{0\}$.

A relation between a topological integer additive set-graceful labeling and a topological integer additive set-sequential labeling of a graph G has been described by Sudev et al. [17] as given below.

Theorem 6.1. Every topological integer additive set-graceful labeling of a given graph G is also a topological integer additive set-sequential labeling of G [17].

The result follows from the fact that $f^+(E) = \mathcal{P}(X) - \{\emptyset\}$ implies $f(V) \cup f^+(E) = \mathcal{P}(X) - \{\emptyset\}$. But, since $f(V) \cup f^+(E) = \mathcal{P}(X) - \{\emptyset\}$ does not necessarily imply that $f^+(E) = \mathcal{P}(X) - \{mptyset, \{0\}\}$, the converse of the above theorem does not necessarily hold.

Figure 6 depicts a graph which admits a topological integer additive set-sequential labeling, which is not a topological integer additive set-graceful labeling.

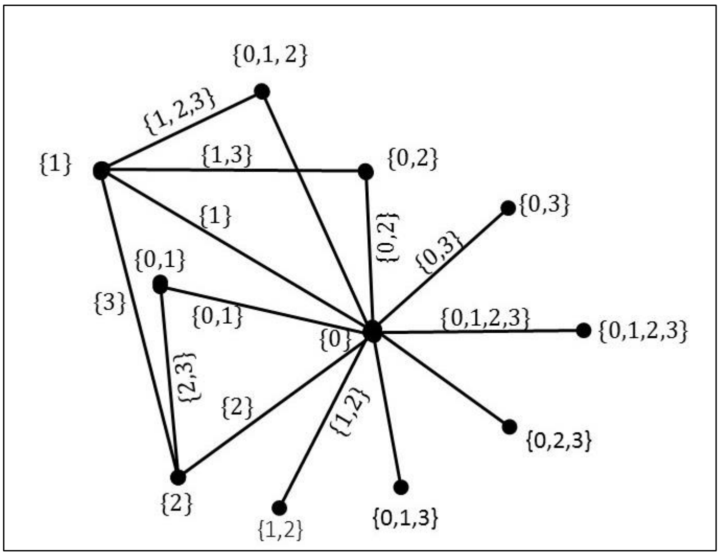

Fig. 5: *An Illustration to a Topological Integer Additive Set-sequential Graphs.*

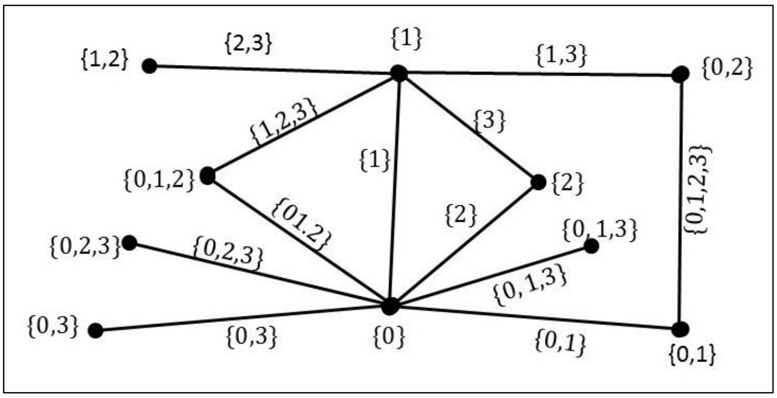

Fig. 6: *An Illustration to a Topological Integer Additive Set-sequential Graphs.*

A necessary and sufficient condition for a graph to admit a topological integer additive set-sequential labeling with respect to the discrete topology of a given ground set X has been explained in the following theorem.

Theorem 6.2. A graph G admits a TIASSL f with respect to the discrete topology of a non-empty finite set X if and only if $G \cong K_{1,2^{|X|}-2}$ [17].

For a topological integer additive set-sequential labeling f of a graph G, we have $|f(V(G))| = 2^{|X|} - 1$ and $|f^+|(E(G))| = 2^{|X} - 2$. Therefore, G is a tree on $2^{|X|} - 1$ vertices. Then the above result follows immediately as $f(V) \cup \{t\}$ is a topology on X.

In view of the above theorem, the result given below has been proved by Sudev et al. [17].

Theorem 6.3. Let X be a non-empty finite set of non-negative integers, which contains 0. Then, for a star graph $K_{1,2^{|X|}-2}$, the existence of a topological integer additive set-sequential labeling is equivalent to the existence of a topological integer additive set-graceful labeling, with respect to the ground set X [17].

Note that there exists topological integer additive set-sequential labeling with respect to the indiscrete topology of a given non-empty ground set X for any graph.

The existence of a graph which admits a topological integer additive set- sequential labeling with respect to a given ground set having certain properties has been established in the following result.

Theorem 6.4. Let X be a finite set of non-negative integers containing 0. Then, there exists a graph G which admits a topological integer additive set-sequential labeling with respect to the ground set X [17].

In general, all graphs need not necessarily admit a topological integer additive set-sequential labeling. This fact created much interest in studying the structural properties of topological integer additive set-sequential graphs. The minimum number of vertices required in a graph to admit a topological integer additive set-sequential graph has been determined in the following result.

Proposition 6.5. Let the graph G admit a TIASSL f with respect to a given ground set X and let $\mathcal{A}$ be the set of subsets of X, which are not non-trivial sumsets of any subsets of X [17]. If $|\mathcal{A}| = \rho$, then G must have

1. at least ρ vertices if $X \in \mathcal{A}$.
2. at least $1 + \rho$ vertices if $X \notin \mathcal{A}$.

In view of the above results, one can note that the number of vertices adjacent to the vertex having the set-label $\{0\}$ is an important factor in the study of topological integer additive set-sequential graphs. The following result described the number of vertices that are adjacent to the vertex having the set-label $\{0\}$.

Proposition 6.6. Let the graph G admit a TIASSL f with respect to a given ground set X. Then, G has at least ρ' vertices adjacent to the unique vertex of G, labeled by $\{0\}$, where ρ' is the number of subsets of X which are not thesummands of any subsets of X [17].

As f is a topological IASL, G must have some pendant vertices. The minimum number of pendant vertices required for a graph G to admit a TIASSL has been determined in the following result.

Proposition 6.7. Assume that the graph G admits a TIASSL f with respect to a given ground set X and let $\mathcal{D}$ be the set of subsets of X which are neither the non-trivial sumsets of two subsets of X nor

the summands of any subsets of X [17]. If $|\mathcal{D}| = \rho''$, then G must have
1. at least ρ'' pendant vertices if $X \in \mathcal{D}$.
2. at least $1 + \rho''$ pendant vertices if $X \notin \mathcal{D}$.

The number of edges in the given graph G is also important in determining the admissibility of a topological integer additive set-sequential labeling by G. The minimum number of vertices required in a topological integer additive set-sequential graph G have already been determined and in view of this result, the maximum number of edges required in a G has also been determined in Sudev et al. [17] as given below:

Proposition 6.8. Let the graph G admit a topological integer additive set-labeling f with respect to a given ground set X. Let $\mathcal{A}$ be the set of subsets of X, which are not the sumsets of any subsets of X and $\mathcal{B}$ be the collection of all subsets of X which are not the summands of any subsets of X. Also, let $|\mathcal{A}| = \rho$ and $|\mathcal{B}| = \rho'$. Then, G has at most $2^{|X|} + \rho' - \rho - 1$ edges [17].

The following theorem is a summary of the above-mentioned results and gives overall structural characteristics of a topological integer additive set-sequential graph.

Theorem 6.9. Let f be a topological integer additive set-sequential labeling defined on a graph with respect to a given ground set X containing 0. Let $\mathcal{A}$ be the set of all subsets of X which are not the non-trivial sumsets of any two subsets of X and $\mathcal{B}$ be the the set of all subsets of X, which are non-trivial summands in some subsets of X [17]. Then, we have:
1. The number of vertices in G is at least $|\mathcal{A}|$ or $|\mathcal{A}| + 1$ in accordance with whether X is in $\mathcal{A}$ or not,
2. The maximum number of edges in G is $|\mathcal{B}| + |\mathcal{A}'|$, where $\mathcal{A}' = \mathcal{P}(X) - (\mathcal{A} \cup \{\emptyset\})$,
3. The minimum degree of the vertex that has the set-label $\{0\}$ in G is $|\mathcal{B}|$,
4. The minimum number of pendant vertices is $|\mathcal{A} \cap \mathcal{B}|$ or $|\mathcal{A} \cap \mathcal{B}| + 1$ in accordance with whether or not X is in $\mathcal{A}$.

The following result ruled out the existence of topological integer additive set-sequential indexer with respect to a given non-empty finite set X for connected graphs.

Proposition 6.10. No connected graph admits a topological integer additive set-sequential indexer [17].

Analogous to the Theorem 9, the characterisation of a (disconnected) graph that admits a topological integer additive set-sequential indexer with respect to a given ground set X has been established as given in the following theorem.

Theorem 6.11. Let f be a topological integer additive set-sequential indexer defined on a graph with respect to a given ground set X containing 0. Let $\mathcal{A}$ be the set of all subsets of X which are not the non-trivial sumsets of any twosubsets of X and $\mathcal{B}$ be the the set of all subsets of X, which are non-trivial summands of any subsets of X [17]. Then,
1. The minimum number of vertices in G is $|\mathcal{A}|$ or $|\mathcal{A}| + 1$ in accordance with whether X is in $\mathcal{A}$ or not,
2. The minimum number of isolated vertices in G is $|\mathcal{A} \cap \mathcal{B}|$ or $|\mathcal{A} \cap \mathcal{B}| + 1$ in accordance with whether X is in $\mathcal{A}$ or not,
3. The maximum number of edges in G is $|\mathcal{A}'|$, where $\mathcal{A}' = \mathcal{P}(X) - (\mathcal{A} \cup \{\emptyset\})$.

INTEGER ADDITIVE SET-FILTER GRAPHS

Another interesting study was done on certain types of integer additive set-labelings under which the collection set-labels of the vertices of G forms a filter on the ground set X. First, recall the definition of filters on a set as follows:

Given a set X, a partial ordering $\subseteq$ can be defined on the power set $P(X)$ by subset inclusion, turning

$(P(X), \subseteq)$ into a lattice. A *filter* on X [21,22], denoted by F, is a non-empty subset of the power set $P(X)$ of X which has the following properties.

1. $\emptyset \notin F, X \in F$.
2. $A, B \in FA \cap B \in F$.
3. $A \in F, A \subset B, B \in F$ where, B is a non-empty subset of X.

In view of the above definition of filters, the notion of an *integer additive set-filter labeling* of a graph has been defined by Sudev and Germina [15] as given below:

Definition 7.1. Let X be a finite set of non-negative integers. Then, an integer additive set-labeling $f: V(G) \to P_0(X)$ is said to be an *integer additive set-filter labeling* (IASFL, in short) of G if $F = f(V)$ is a proper filter on X. A graph G which admits an IASFL is called an *integer additive set-filter graph* (IASF-graph) [16].

Figure 7 illustrates a graph which admits an integer additive set-filter labeling.

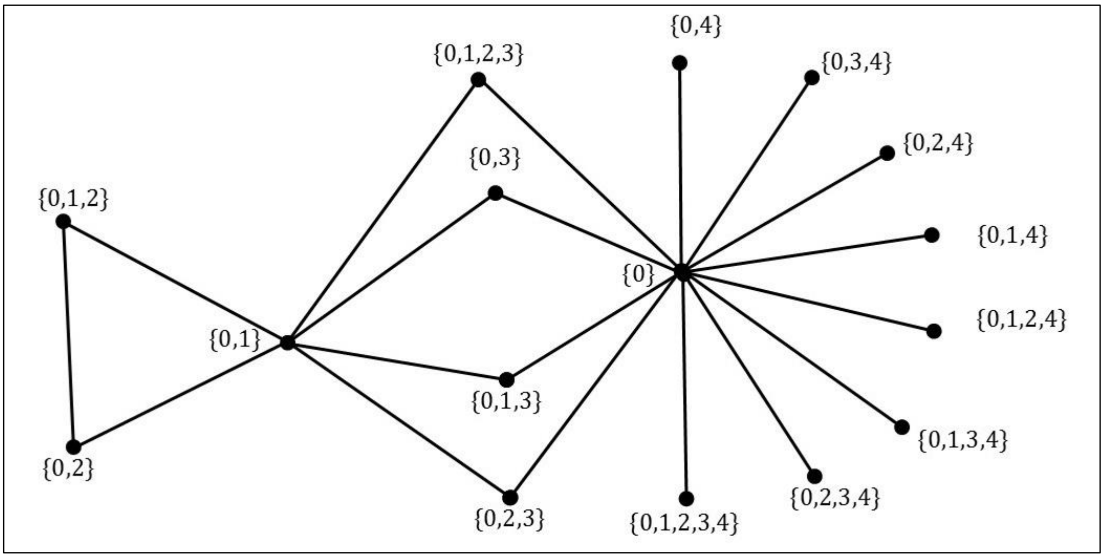

Fig. 7: *An Illustration to an Integer Additive Set-filter Graph.*

It can be noted that the null set cannot be the set-label of any element of the graph G, with respect to an IASL defined on it.

A necessary and sufficient condition for an IASL f of a given graph G to be an integer additive set-filter labeling of G is described in [15] as given below:

Theorem 7.1. An IASL f defined on a given graph G with respect to a non-empty ground set X is an integer additive set-filter labeling of G if and only if the following conditions hold [15].

1. $0 \in X$.
2. every subset of X containing 0 is the set-label of some vertex in G.
3. 0 is an element of the set-label of every vertex in G.

From the above result, we noticed that all graphs do not possess IASFLs, in general. Hence, a characterization of the graphs that admit IASFLs created much interest. In view of Theorem 1, the characteristics and properties of the graphs which admit IASFLs have been found out as explained in the following results.

Corollary 7.2. If a graph G admits an IASFL, then G has $2^{|X|-1}$ vertice [16].

Corollary 7.3. If a given graph G admits an IASFL f, then only one vertex of G can have a singleton set-label [15].

It can be noted that the above results are immediate consequences of Theorem 1. The relation between the collection of the set-labels of vertices and the collection of the set-labels of the edges in an IASF-graph G has been determined in the following result.

Proposition 7.4. If f is an IASFL of a graph G, then $f^+(E(G)) \subseteq f(V(G))$.

The following theorem is a consequence of Theorem 1 [15].

Theorem 7.5. If a given graph G admits an integer additive set-filter labeling f, then every element of the collection $F = f(V(G))$ belongs to some finite chain of sets in F of the form $\{0\} = f(v_1) \subset f(v_2) \subset f(v_3) \subset ts \ldots \subset f(v_r) = X$ [15].

Certain structural properties of an IASFL-graph are discussed in the following result.

Theorem 7.6. If a graph G admits an integer additive set-filter labeling, with respect to a non-empty ground set X, then G must have at least $2^{|X|-2}$ pendant vertices that are incident on a single vertex of G [17].

The following are some of the immediate observations based on the discussions made above.
1. The existence of an IASFL is not a hereditary property. That is, an IASFL of a graph need not induce an IASFL for all its subgraphs.
2. For $n \geq 3$, no paths P_n admit an IASFL. No cycles admit IASFLs and thus neither Eulerian graphs nor Hamiltonian graphs admit IASFLs. Neither complete graphs nor complete bipartite graphs admit IASFLs. For $r > 2$, complete r-partite graphs also do not admit IASFLs.
3. Graphs having an odd number of vertices never admit an IASFL.
4. Removing any non-leaf edge of IASF-graph preserves the IASFL of that graph. This property is known as *monotone property*.

CONCLUSION
In this write-up, we have done an extensive review of certain studies on the integer additive set-labelings of graphs with respect to which the collections of set-labels of the elements of a given graphs form topologies of the ground set used for labeling. During the preparation of this article, we have identified certain problems which are still to be settled. We are still to establish necessary and sufficient conditions for a graph to admit topogenic IASL and integer additive set-filter labeling.

The *topological set-indexing number* of a graph G is the minimum cardinality of required for the ground set X such that G admits a topological integer additive set-labeling.

Problem 1. Determine the topological set-indexing number of different graphs and graph classes which admit topological integer additive set-labelings.

In a similar way, we can define the topogenic set-indexing number and filter set-indexing number of graphs and determine these parameters are also open.

Problem 2. Establish the necessary and sufficient conditions for a graph to admit topological integer additive set-indexer.

Analogous to topological IASI, we can define the notion of integer additive set-filter indexer (IASFI) and determine the conditions for a graph to admit topological integer additive set-filter indexer are open.

Problem 3. For a given ground set X, Find the number of topologies on X with respect to which a given graph G admits a topological (or topogenic) IASL.

We note that the join and certain products of topological (or topogenic) IASL-graphs do not admit topological or topogenic IASLs. Verifying whether other operations of graphs admit these types of IASLs seems to be promising for further studies.

Problem 4. Verify whether the graphs such as line graphs, total graphs, subdivisions etc. associated with given topological (or topogenic) IASL-graphs admit topological (or topogenic) IASL.

There are more open problems in this area. The algebraic properties of the set-labels of the elements of given graphs can also be studied.

ACKNOWLEDGEMENT

Authors of this article would like to dedicate this article to the memory of the renowned mathematician and teacher Professor (Dr.) Belamannu Devadas Acharya, who introduced the notion of integer additive set-indexers of graphs.

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Cite this Article
Sudev NK, Chithra KP, Germina KA. Topological Integer Additive Set-Valued Graphs: A Review. *Research & Reviews: Discrete Mathematical Structures*. 2017; 4(3): 1–17p.