THE DISCRETE LAPLACIAN ACTING ON 3–FORMS

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ABSTRACT. In the current paper, we study the discrete Laplacian acting on 3-forms. We establish a new criterion of essential self-adjointness using the Nelson lemma. Moreover, we give an upper bound on the infimum of the essential spectrum.

CONTENTS

1. INTRODUCTION

Spectral graph theory represents an active area of research. In the last few years, the questions of the essential self-adjointness of discrete Laplacian operators on infinite graphs have attracted a lot of interest, see [11, 18, 22,17]. There exist other definitions of the discrete Laplacian, e.g., [24, 16, 4, 2]. The one we are studying here is the discrete Laplacian acting on 3-forms and denoted by $L_{3,skew}$, where skew stands for skew-symmetric. This operator was introduced by us in [7]. We have showed the relation between the $\chi-$completeness geometric hypothesis for the graph and essentially self-adjointness for the discrete Laplacian $L_{3,skew}$. More specifically, we have proved that $L_{3,skew}$ is essential self-adjoint, when the 3-simplicial complex is $\chi-$complete. The current study has two major aims. It first aims to discuss the question of essential self-adjointness for $L_{3,skew}$. It is worth noting that this operator depends on the weight t on oriented tetrahedrons, on the weight s on triangular oriented faces and the weight r on oriented edges. In the setting of electrical networks, the weight r correspond to the conductance. We establish a hypothesis on the weights and involve essential self-adjointness by using

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the Nelson commutator theorem. The technique of the proof are inspired from [3]. Moreover, we give an upper bound on the infimum of the essential spectrum $\sigma_{ess}(L^F_{2,skew})$, where $L^F_{3,skew}$ is the Friedreichs extension of $L_{3,skew}$. Note that this discrete Laplacian was introduced on a skew-symmetric statistic on the space of 3-forms. We can define the discrete Laplacian $L_{3,sym}$ in the symmetric case by the same expression of $L_{3,skew}$. In the case of a four-partite graph, we prove that the two operators are unitarily equivalent. We recall that the spectral theory of adjacency matrix acting on graphs is useful for the study of some gelling polymers, of some electrical networks, and in number theory, see [14, 13, 23]. As for the rest of this paper, it is structured as follows: The next section is devoted to some definitions and notations for graph. We find the definitions of two different Hilbert structures on the set of tetrahedrons, in Section 2. Both definitions have their own interest. This permits to define two different types of discrete Laplacian associated to tetrahedrons. The relation between these two operators is clearly presented in Section 3. In Section 4, we discuss the question of essential self-adjointness for the discrete Laplacian $L_{3,skew}$. We establish a new criterion of essential self-adjointness using the Nelson lemma. In Section 5, we give an upper bound on the infimum of the essential spectrum.

2. Preliminaries

2.1. **The basic concepts.** A graph K is a pair (V, E), where V is the countable set of vertices and E the set of oriented edges, considered as a subset of $V \times V$. When two vertices x and y are connected by an edge e, we say they are neighbors. We denote $x \sim y$ and $e = (x, y) \in E$. We assume that E is symmetric, ie. $(x, y) \in E \implies (y, x) \in E$. An oriented graph K is given by a partition of E:

$E = E^- \cup E^+ (x, y) \in E^- \Leftrightarrow (y, x) \in E^+$. In this case for $e = (x, y) \in E^-$, we define the origin $e^- = x$, the termination $e^+ = y$ and the opposite edge $-e = (y, x) \in E^+$. Let $c : V \to (0, \infty)$ the weight on the vertices. We also have $r : E^- \to (0, \infty)$ the weight on the oriented edges with $\forall e \in E, r(-e) = r(e)$. A path between two vertices $x, y \in V$ is a finite set of oriented edges $e^1, ..., e^n, n \geq 1$ such that $e_1^- = x$, $e_n^+ = y$ and, if $n \geq 2$, $\forall j, 1 \leq j \leq n-1 \implies e+j = e-j+1$. The path is called a cycle or closed when the origin and the end are identical, ie. $e_1^- = e_n^+$, with $n \geq 3$. If no cycles appear more than once in a path, the path is called a simple path. The graph K is connected if any two vertices x and y can be connected by a path with $e_1^- = x$ and $e_n^+ = y$. We say that the graph K is locally finite if each vertex belongs to a finite number of edges. The graph K is without loops if there is not the type of edges (x, x), ie.$\forall x \in V \implies (x, x) \notin E$. The set of neighbors of $x \in V$ is denoted by $V(x) = \{y \in V : y \sim x\}$. The degree of $x \in V$ is by definition $deg(x)$, the number of neighbors of x. In the sequel, we assume that K is without loops, connected, locally finite and oriented. *An oriented triangular face* of K is a surface limited by a simple closed path of lenght equals 3, considered as an element of E^3, i. e ϖ is an oriented triangular face $\Rightarrow \varpi = (e_1, e_2, e_3) \in E^3$ such that $\{e_i\}_{1 \leq i \leq 3} \subseteq E$ is a simple closed path. Let F be the set of all oriented faces of K. In the sequel we will represent the oriented faces by their vertices For a face $\varpi = [(x, y, z)] \in F$. Let us set $\varpi = (x, y, z) = (y, z, x) = (z, x, y) \in F \Rightarrow -\varpi = (y, x, z) = (x, z, y) = (z, y, x) \in F$. A triangulation T is a 2-simplicial complex such that all the faces are triangular. To define weighted triangulations we need weights, let us give $s : F \to (0, \infty)$

the weight on oriented faces such that for all $\varpi \in F$, $s(-\varpi) = s(\varpi)$. The weighted triangulation (T, c, r, s) is given by the triangulation $T = (V, E, F)$. We say that T is simple if the weights of the vertices, the edges and faces equals 1. For an edge $e \in E$, we also denote the oriented face (e^-, e^+, x) by (e, x), with $x \in V(e^-) \cap V(e^+)$.The set of vertices belonging to the edge $e \in E$ is given by $F_e = \{x \in V, (e, x) \in F\} = V(e^-) \cap V(e^+)$. *An oriented tetrahedron* of K is a volume limited by four oriented triangular faces of F, considered as an element of V^4 , i. e $\bar{\Omega}$ is an oriented tetrahedron $\Rightarrow \bar{\Omega} = (x, y, z, v) \in V^4$. Let Δ_3 be the set of all oriented tetrahedron of K. We consider the pair (T, Δ_3) as a 3-simplicial complex, we denote it by Δ. We can denote also $\Delta = (V, E, F, \Delta_3)$. Odd permutation means we exchange the position of any two vertices an odd number of times. Even permutation means we exchange the position of any two vertices an even number of times. For a $\left(\bar{\Omega}, \bar{\Pi}\right) \in \Delta_3^2$ we have :

$$\bar{\Omega} = \bar{\Pi} \iff \bar{\Pi} \text{ is obtened from } \bar{\Omega} \text{ by an even permutation.}$$
$$\bar{\Omega} = -\bar{\Pi} \iff \bar{\Pi} \text{ is obtened from } \bar{\Omega} \text{ by an odd permutation.}$$

To define weighted 3-simplicial complex we need weights, let us give $t : \Delta_3 \to (0, \infty)$ the weight on oriented tetrahedrons such that for all $\bar{\Omega} \in \Delta_3$, $t(-\bar{\Omega}) = t(\bar{\Omega})$.The weighted 3-simplicial complex (V, c, r, s, t) is giving by the 3-simplicial complex $\Delta = (V, E, F, \Delta_3)$. We say that Δ is simple if the weights of the vertices, the edges, the triangular faces and tetrahedrons equals 1. The set of vertices belonging to the tetrahedron where $(x, y, z) \in F$ is an oriented triangular face si giving by: $\Delta_3(x, y, z) = \{v \in V, (x, y, z, v) \in \Delta_3\} = V(x) \cap V(y) \cap V(z)$. When Δ is simple, the degre of faces is $d_F(x, y, z) = \#\Delta_3(x, y, z)$.

2.2. **Functions spaces.** We denote the set of $0-$cochains or functions on V by:

$$C(V) = \{f : V \to \mathbb{C}\}$$

and the set of functions of finite support by $C_c(V)$. Similarly, we denote the set of $1-$cochains or $1-$forms on E by:

$$C(E) = \{\varphi : E \to \mathbb{C}, \varphi(-e) = -\varphi(e)\}$$

and the set of $1-$forms of finite support by $C^c(E)$. Moreover, we denote the set of $2-$cochains or 2-forms on F by:

$$C_{skew}(F) = \{\phi : F \to \mathbb{C}, \phi(-\varpi) = -\phi(\varpi)\}$$

and the set of $2-$forms of finite support by $C^c_{skew}(F)$.

$$C_{sym}(F) = \{\phi : F \to \mathbb{C}, \phi(-\varpi) = \phi(\varpi)\}$$

and the set of $2-$forms of finite support by $C^c_{sym}(F)$. Further, we denote the set of $3-$cochains or $3-$forms on E by:

$$C_{skew}(\Delta_3) = \{\rho : \Delta_3 \to \mathbb{C}, \rho(-\bar{\Omega}) = -\rho(\bar{\Omega})\}$$

and the set of $3-$forms of finite support by $C^c_{skew}(\Delta_3)$.

$$C_{sym}(\Delta_3) = \{\rho : \Delta_3 \to \mathbb{C}, \rho(-\bar{\Omega}) = \rho(\bar{\Omega})\}$$

and the set of 3−forms of finite support by $C^c_{sym}(\Delta_3)$. Let us define the Hilbert spaces $l^2(V)$, $l^2(E)$, $l^2_{skew}(F)$, $l^2_{sym}(F)$, $l^2_{skew}(\Delta_3)$ and $l^2_{sym}(\Delta_3)$ as the sets of cochains with finite norm, we have

$$l^2(V) = \left\{ f \in C(V); \sum_{x\in V} c(x)|f(x)|^2 < \infty \right\},$$

with the inner product

$$\langle f, g\rangle_{l_2(V)} = \sum_{x\in V} c(x) f(x)\bar{g}(x).$$

$$l^2(E) = \left\{ \varphi \in C(E); \sum_{e\in E} r(e)|\varphi(e)|^2 < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{l^2(E)} = \frac{1}{2}\sum_{e\in E} r(e)\varphi(e)\bar{\psi}(e).$$

$$l^2_{skew}(F) = \left\{ \phi \in C_{skew}(F); \sum_{\varpi\in F} s(\varpi)|\phi(\varpi)|^2 < \infty \right\},$$

with the inner product

$$\langle \phi_1, \phi_2\rangle_{l^2_{skew}(F)} = \frac{1}{6}\sum_{(x,y,z)\in F} s(x,y,z)\phi_1(x,y,z)\bar{\phi}_2(x,y,z).$$

$$l^2_{sym}(F) = \left\{ \phi \in C_{sym}(F); \sum_{\varpi\in F} s(\varpi)|\phi(\varpi)|^2 < \infty \right\},$$

with the inner product

$$\langle \phi_1, \phi_2\rangle_{l^2_{sym}(F)} = \frac{1}{6}\sum_{(x,y,z)\in F} s(x,y,z)\phi_1(x,y,z)\bar{\phi}_2(x,y,z).$$

$$l^2_{skew}(\Delta_3) = \left\{ \rho \in C_{skew}(\Delta_3); \sum_{\bar{\Omega}\in \Delta_3} t(\bar{\Omega})|\rho(\bar{\Omega})|^2 < \infty \right\},$$

with the inner product

$$\langle \rho_1, \rho_2\rangle_{l^2_{skew}(\Delta_3)} = \frac{1}{24}\sum_{(x,y,z,v)\in \Delta_3} t(x,y,z,v)\rho_1(x,y,z,v)\bar{\rho}_2(x,y,z,v).$$

$$l^2_{sym}(\Delta_3) = \left\{ \rho \in C_{sym}(\Delta_3); \sum_{\bar{\Omega}\in \Delta_3} t(\bar{\Omega})|\rho(\bar{\Omega})|^2 < \infty \right\},$$

with the inner product

$$\langle \rho_1, \rho_2\rangle_{l^2_{sym}(\Delta_3)} = \frac{1}{24}\sum_{(x,y,z,v)\in \Delta_3} t(x,y,z,v)\rho_1(x,y,z,v)\bar{\rho}_2(x,y,z,v).$$

3. Operators

In this section, we recall the concept of exterior derivative operator associated to a tetrahedrons space, we refer to [7] for more details. This permits to define the discrete Laplacian acting on 3-forms.

3.1. Skew-symmetric case.

We start with defining the operators in the skew-symmetric case. The skew-symmetric exterior operator is the operator

$$d^2_{skew} : C^c_{skew}(F) \longrightarrow C^c_{skew}(\Delta_3),$$

given by $\forall \varphi \in C^c_{skew}(F)$

$$d^2_{skew}(\varphi)(x,y,z,v) = \varphi(y,z,v) + \varphi(v,z,x) + \varphi(x,y,v) + \varphi(z,y,x).$$

The formal adjoint of d^2_{skew}, denoted δ^2_{skew}.

$$\delta^2_{skew} : C^c_{skew}(\Delta_3) \longrightarrow C^c_{skew}(F)$$

satisfies

$$\langle d^2_{skew}\varphi, \psi\rangle_{l^2_{skew}(\Delta_3)} = \langle \varphi, \delta^2_{skew}\psi\rangle_{l^2_{skew}(F)}, \forall(\varphi,\psi) \in C^c_{skew}(F) \times C^c_{skew}(\Delta_3).$$

Lemma 1. *The formal adjoint*

$$\delta^2_{skew} : C^c_{skew}(\Delta_3) \longrightarrow C^c_{skew}(F)$$

is given by

$$\forall \psi \in C^c_{skew}(\Delta_3), \delta^2_{skew}(\psi)(y,z,v) = \frac{1}{s(y,z,v)} \sum_{x\in\Delta_3(y,z,v)} t(x,y,z,v)\psi(x,y,z,v).$$

Proof. Let $(\varphi,\psi) \in C^c_{skew}(F) \times C^c_{skew}(\Delta_3)$

We have

$$\left\langle d^2_{skew}\varphi, \psi\right\rangle_{l^2_{skew}(\Delta_3)} = \left\langle \varphi, \delta^2_{skew}\psi\right\rangle_{l^2_{skew}(F)}$$

And

$$\left\langle \varphi, \delta^2_{skew}\psi\right\rangle_{l^2_{skew}(F)} = \frac{1}{6} \sum_{(y,z,v)\in F} s(y,z,v)\varphi(y,z,v)\overline{\delta^2_{skew}\psi}(y,z,v)$$

Moreover,

$$\begin{aligned}\langle d^2_{skew}\varphi, \psi\rangle_{l^2_{skew}(\Delta_3)} &= \frac{1}{24} \sum_{(x,y,z,v)\in\Delta_3} t(x,y,z,v) d^2\varphi(x,y,z,v)\bar{\psi}(x,y,z,v) \\ &= \frac{1}{24} \sum_{(x,y,z,v)\in\Delta_3} t(x,y,z,v)\left[\varphi(y,z,v) + \varphi(v,z,x) + \varphi(x,y,v)\right. \\ &\quad \left. + \varphi(z,y,x)\right]\bar{\psi}(x,y,z,v)\end{aligned}$$

The expression of d^2_{skew} contributing to the first sum is divided into four similar parts. So we obtain

$$\begin{aligned}\frac{1}{6}\sum_{(x,y,z,v)\in\Delta_3} t(x,y,z,v)\varphi(y,z,v)\bar{\psi}(x,y,z,v) &= \frac{1}{6}\sum_{(y,z,v)\in F} s(y,z,v)\varphi(y,z,v)\\ &\quad \overline{\delta^2_{skew}\psi}(y,z,v).\\ \frac{1}{6}\sum_{(y,z,v)\in F}\left[\varphi(y,z,v)\sum_{x\in\Delta_3(y,z,v)} t(x,y,z,v)\bar{\psi}(x,y,z,v)\right] &= \frac{1}{6}\sum_{(y,z,v)\in F} s(y,z,v)\varphi(y,z,v)\\ &\quad \overline{\delta^2_{skew}\psi}(y,z,v)\end{aligned}$$

Then

$$\delta^2_{skew}(\psi)(y,z,v) = \frac{1}{s(y,z,v)}\sum_{x\in\Delta_3(y,z,v)} t(x,y,z,v)\psi(x,y,z,v).$$

□

Both operators are closable (see [7, Lemme 6]). We denote their closure by the same symbol. The skew-symmetric discrete Laplacian operator acting on 3-forms is given by

$$\begin{aligned}L_{3,skew}\psi\,(x,y,z,v) &= d^2_{skew}\delta^2_{skew}\psi\,(x,y,z,v)\\ &= \frac{1}{s\,(y,z,v)}\sum_{u\in\Delta_3(y,z,v)} t(u,y,z,v)\psi\,(u,y,z,v) + \frac{1}{s\,(v,z,x)}\\ &\quad \sum_{w\in\Delta_3(v,z,x)} t(w,v,z,x)\psi\,(w,v,z,x) + \frac{1}{s\,(x,y,v)}\\ &\quad \sum_{b\in\Delta_3(x,y,v)} t(b,x,y,v)\psi\,(b,x,y,v) + \frac{1}{s\,(z,y,x)}\\ &\quad \sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x)\psi\,(a,z,y,x)\,.\end{aligned}$$

with $\psi \in C^c_{skew}(\Delta_3)$.

3.2. **Symmetric case.** We turn to the symmetric case. The symmetric exterior operator is the operator

$$d^2_{sym} : C^c_{sym}(F) \longrightarrow C^c_{sym}(\Delta_3)$$

given by $\forall\varphi \in C^c_{sym}(F)$

$$d^2_{sym}(\varphi)(x,y,z,v) = \varphi(y,z,v) + \varphi(v,z,x) + \varphi(x,y,v) + \varphi(z,y,x).$$

The formal adjoint of d^2_{sym}, denoted δ^2_{sym}.

$$\delta^2_{sym} : C^c_{sym}(\Delta_3) \longrightarrow C^c_{sym}(F)$$

satisfies

$$\langle d^2\varphi,\psi\rangle_{l^2_{sym}(\Delta_3)} = \left\langle \varphi,\delta^2\psi\right\rangle_{l^2_{sym}(F)}, \forall(\varphi,\psi) \in C^c_{sym}(F)\times C^c_{sym}(\Delta_3).$$

Lemma 2. *The formal adjoint*

$$\delta^2_{sym} : C^c_{skew}(\Delta_3) \longrightarrow C^c_{sym}(F)$$

is given by

$$\forall \psi \in C^c_{sym}(\Delta_3), \delta^2_{sym}(\psi)(y,z,v) = \frac{1}{s(y,z,v)} \sum_{x \in \Delta_3(y,z,v)} t(x,y,z,v)\psi(x,y,z,v).$$

The operators d^2_{sym} and δ^2_{sym} are closable. Indeed, since $\delta^2_{sym} : l^2_{sym}(\Delta_3) \longrightarrow l_{3,sym}(F)$ (resp. $d^2_{sym} : l^2_{sym}(F) \longrightarrow l^2_{sym}(\Delta_3)$) is with dense domain then δ^2_{sym} (resp. d^2_{sym}) is closable. We denote their closure by the same symbol. The symmetric discrete Laplacian operator acting on 3-forms is the operator $L_{3,sym} = d^2_{sym}\delta^2_{sym}$, given by the same expression of $L_{3,skew}$.

3.3. **Relationship between $L_{3,skew}$ and $L_{3,sym}$.** The two operators $L_{3,skew}$ and $L_{3,sym}$ have the same expression. However, they do not act on the same spaces. Namely, when Δ is four-partite, we shall prove that the two operators are unitarily equivalent. *A four-partite graph* is a graph whose vertices can be partitioned into 4 disjoint sets so that there are no two vertices within the same set are adjacent. A four-partite 3-simplicial complex is a 3-simplicial complex $\Delta = (V, E, F, \Delta_3)$ such that $G = (V, E)$ is four-partite.

Theorem 1. *Let $\Delta = (V, E, F, \Delta_3)$ be a four-partite weighted 3-simplicial complex. Then, $L_{3,skew}$ and $L_{3,sym}$ are unitarily equivalent.*

Proof. We consider the four-partite decomposition $\{V_1, V_2, V_3, V_4\}$. Set

$$\circlearrowleft V_1 \times V_2 \times V_3 \times V_4 = \{\circlearrowleft (x,y,z,v) \mid (x,y,z,v) \in V_1 \times V_2 \times V_3 \times V_4\}.$$

Let

$$h : l^2_{skew}(\Delta_3) \longrightarrow l^2_{sym}(\Delta_3)$$

be the unitary map given by

$$h(\psi)(x,y,z,v) = t(x,y,z,v)\psi(x,y,z,v),$$

Where

$$t(x,y,z,v) = \begin{cases} 1, \text{ if}(x,y,z,v) \in \circlearrowleft V_1 \times V_2 \times V_3 \times V_4 \ , \\ -1, \text{ if}(x,y,z,v) \in \circlearrowleft V_4 \times V_2 \times V_3 \times V_1. \end{cases}$$

Let ρ be the following mapping from $l^2_{sym}(\Delta_3)$ into $l^2_{skew}(\Delta_3)$ such that $\rho(\psi)(x,y,z,v) = t(x,y,z,v)\psi(x,y,z,v)$. Then

$$\langle h\psi, \phi \rangle_{l^2_{sym}(\Delta_3)} = \langle \psi, \rho\phi \rangle_{l^2_{skew}(\Delta_3)},$$

and

$$\rho(h(\psi)) = \psi$$

for all $\psi \in l^2_{skew}(\Delta_3)$ and $\phi \in l^2_{sym}(\Delta_3)$.
So we have

$$\rho(\psi) = h^{-1}(\psi) = h^*(\psi)$$

for all $\psi \in l^2_{sym}(\Delta_3)$.
Therefore,

$$hL_{3,skew}h^{-1}(\psi)(x,y,z,v) = L_{3,sym}(\psi)(x,y,z,v)$$

for all $\psi \in C^c_{sym}(\Delta_3)$.
Then $L_{3,skew}$ and $L_{3,sym}$ are unitarily equivalent. □

4. A Nelson criterium

For the general theory of unbounded Hermitian operators and their extensions, we refer the reader to [25, 20, 27]. Let L_{skew} be the following mapping from $C_{skew}(\Delta_3)$ into itself:

$$\begin{aligned} L_{skew}\psi(x,y,z,v) \quad = \quad & \frac{1}{s(y,z,v)} \sum_{u\in\Delta_3(y,z,v)} t(u,y,z,v)\psi(u,y,z,v) + \frac{1}{s(v,z,x)} \\ & \sum_{w\in\Delta_3(v,z,x)} t(w,v,z,x)\psi(w,v,z,x) + \frac{1}{s(x,y,v)} \\ & \sum_{b\in\Delta_3(x,y,v)} t(b,x,y,v)\psi(b,x,y,v) + \frac{1}{s(z,y,x)} \\ & \sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x)\psi(a,z,y,x). \end{aligned}$$

Let $L_{3,max,skew}$ be the restrictions of L_{skew} to

$$D(L_{3,max,skew}) = \left\{\psi \in l^2_{skew}(\Delta_3) \text{ such that } L_{skew}\psi \in l^2_{skew}(\Delta_3)\right\}.$$

Lemma 3. $L^*_{3,skew} = L_{3,max,skew}$.

Proof. Let $\psi \in C^c_{skew}(\Delta_3)$ and $\phi \in C^c_{sym}(\Delta_3)$. Let T_{00} the support of ψ and set

$$T_0 = \{(x,y,z,v) \in \Delta_3 \mid \exists m \in V, \{ (m,y,z,v)+(m,v,z,x)+(m,x,y,v)+(m,z,y,x)\} \cap T_{00} \neq \emptyset\}$$

which is a finite set. Then, $supp(L_{3,skew}) \subset T_0$ and the following relation holds:

$$
\begin{aligned}
& \frac{1}{24}\sum_{(x,y,z,v)\in T_0} t(x,y,z,v)L_{3,skew}(\psi)(x,y,z,v)\bar{\phi}(x,y,z,v) \\
= \; & \frac{1}{24}\sum_{(x,y,z,v)\in T_0} t(x,y,z,v)(\frac{1}{s(y,z,v)}\sum_{u\in\Delta_3(y,z,v)} t(u,y,z,v)\psi(u,y,z,v) + \\
& \frac{1}{s(v,z,x)}\sum_{w\in\Delta_3(v,z,x)} t(w,v,z,x)\psi(w,v,z,x) + \frac{1}{s(x,y,v)}\sum_{b\in\Delta_3(x,y,v)} t(b,x,y,v) \\
& \psi(b,x,y,v) + \frac{1}{s(z,y,x)}\sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x)\psi(a,z,y,x))\bar{\phi}(x,y,z,v) \\
= \; & \frac{1}{24}\sum_{(x,y,z,v)\in T_{00}} t(x,y,z,v)\psi(x,y,z,v)\left[\overline{\frac{1}{s(y,z,v)}\sum_{m\in V} t(m,y,z,v)\phi(m,y,z,v)}\right. \\
& +\overline{\frac{1}{s(v,z,x)}\sum_{m\in V} t(m,v,z,x)\phi(m,v,z,x)} + \overline{\frac{1}{s(x,y,v)}\sum_{m\in V} t(m,x,y,v)\phi(m,x,y,v)} \\
& \left. + \overline{\frac{1}{s(z,y,x)}\sum_{m\in V} t(m,z,y,x)\phi(m,z,y,x)}\right] \\
= \; & \frac{1}{24}\sum_{x,y,z,v\in V} t(x,y,z,v)\psi(x,y,z,v)\,\overline{L_3\phi(x,y,z,v)}. \quad (1)
\end{aligned}
$$

Let $\phi \in D(L_{3,max,skew})$. It follows from (1) that

$$\langle L_{3,skew}\psi, \phi\rangle = \langle \psi, L_{3,max,skew}\phi\rangle$$

for all $\psi \in C^c_{skew}(\Delta_3)$, which implies that $\phi \in D\left(L^*_{3,skew}\right)$. Now let $\phi \in D\left(L^*_{3,skew}\right)$. Let $(x,y,z,v) \in \Delta_3$ and let

$$\psi = \frac{1}{t(x,y,z,v)}\left(1_{\circlearrowleft(x,y,z,v)} - 1_{\circlearrowright(x,y,z,v)}\right).$$

Then, $\psi \in C^c_{skew}(\Delta_3)$ and we obtain from (1):

$$
\begin{aligned}
\overline{\left(L^*_{3,skew}\phi\right)(x,y,z,v)} &= \left\langle \psi, L^*_{3,skew}\phi\right\rangle \\
&= \langle L_{3,skew}\psi, \phi\rangle \\
&= \frac{1}{24}\sum_{(m,\beta,\lambda,\mu)\in\Delta_3} t(m,\beta,\lambda,\mu)\,L_{3,skew}(\psi)(m,\beta,\lambda,\mu)\,\bar{\phi}(m,\beta,\lambda,\mu) \\
&= \frac{1}{24}\sum_{(m,\beta,\lambda,\mu)\in\Delta_3} t(m,\beta,\lambda,\mu)\,\psi(m,\beta,\lambda,\mu)\,\overline{L_{3,max,skew}\phi(m,\beta,\lambda,\mu)} \\
&= \overline{(L_{3,max,skew}\phi)(x,y,z,v)}.
\end{aligned}
$$

which implies that $L_{3,skew}\phi = L_{3,max,skew}\phi \in l^2_{skew}(\Delta_3)$ by the definition of the adjoint, it follows that $\phi \in D(L_{3,max,skew})$. Hence

$$L^*_{3,skew} = L_{3,max,skew}.$$

□

Remark 1. *Let L_{sym} be the mapping from $C_{sym}(\Delta_3)$ into itself given by the same expression of $L_{3,sym}$. Then, $L^*_{3,sym} = L_{3,max,sym}$ where $L_{3,max,sym}$ is the restrictions of L_{sym} to*

$$D(L_{3,max,sym}) = \left\{\psi \in l^2_{sym}(\Delta_3) \text{ such that } L_{sym}\psi \in l^2_{sym}(\Delta_3)\right\}.$$

Using the Nelson commutator theorem, we prove the criterium of essential self-adjointness for $L_{3,skew}$ and $L_{3,sym}$.

Theorem 2. *Let $\Delta = (V, E, F, \Delta_3)$ be a weighted 3-simplicial complex. Set*

$$N(x,y,z,v) = 1 + d_F(y,z,v) + d_F(v,z,x) + d_F(x,y,v) + d_F(z,y,x).$$

Suppose that

$$\sup_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} \sum_{r\in\Delta_3(x,y,z)} \frac{1}{s(x,y,z)} t(x,y,z,r) \mid N(x,y,z,r) - N(x,y,z,v) \mid^2 < \infty.$$

Then

$L_{3,skew}$ is essentially self-adjoint on $C^c_{skew}(\Delta_3)$ and $L_{3,sym}$ is essentially self-adjoint on $C^c_{skew}(\Delta_3)$.

Proof. Let N be the operator of multiplication by $N(.,.,.,.)$ ant take $f \in C^c_{skew}(\Delta_3)$. Going over the same techniques of the proof of [3, Theorem 5.13], we obtain:

$$\begin{aligned}
&\| L_{3,skew} f \|^2 \leq \tfrac{2}{3} \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \Bigg(\tfrac{1}{s^2(y,z,v)} \mid \sum_{u\in\Delta_3(y,z,v)} t(u,y,z,v) f(u,y,z,v) \mid^2 + \\
&\tfrac{1}{s^2(v,z,x)} \mid \sum_{w\in\Delta_3(v,z,x)} t(w,v,z,x) f(w,v,z,x) \mid^2 + \tfrac{1}{s^2(x,y,v)} \\
&\mid \sum_{b\in\Delta_3(x,y,v)} t(b,x,y,v) f(b,x,y,v) \mid^2 + \tfrac{1}{s^2(z,y,x)} \mid \sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x) f(a,z,y,x) \mid^2 \Bigg) \\
&\leq 2 \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \tfrac{1}{s^2(z,y,x)} \mid \sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x) f(a,z,y,x) \mid^2 \\
&\leq 2 \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \tfrac{1}{s^2(z,y,x)} \Bigg(\sum_{r\in\Delta_3(z,y,x)} t(r,y,z,v) \Bigg) \\
&\Bigg(\sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x) \mid f(a,z,y,x) \mid^2 \Bigg) \\
&= 2 \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \Bigg(\tfrac{1}{s(z,y,x)} \sum_{a\in\Delta_3(z,y,x)} t(a,z,y,x) \Bigg)^2 \mid f(v,z,y,x) \mid^2 \\
&\leq 12 \| N(f) \|^2.
\end{aligned}$$

Moreover, we notice that $N(.,.,.,.)$ is symmetric and $f(x,y,z,v) = -f(v,y,z,x)$ and let $J = |\langle f, [L_{3,skew}, N]\, f \rangle|$. We get:

$$\begin{aligned}
&J \leq \tfrac{1}{12} \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \left(|f(x,y,z,v)|^2 + |[L_{3,skew}, N](f)(x,y,z,v)|^2 \right) \\
&\leq \tfrac{1}{2} \left\| N^{\frac{1}{2}}(f) \right\|^2 + \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \ \Big| \sum_{a\in\Delta_3(z,y,x)} \tfrac{1}{s(z,y,x)} t(a,z,y,x) \\
&(N(a,z,y,x) - N(v,z,y,x)) f(a,z,y,x) \Big|^2 \\
&\leq \tfrac{1}{2} \| N^{\frac{1}{2}}(f) \|^2 + \sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)} t(x,y,z,v) \Bigg(\sum_{a\in\Delta_3(z,y,x)} \tfrac{1}{s(z,y,x)} t(a,z,y,x) \Bigg) \times
\end{aligned}$$

$$\left(\sum_{r\in\Delta_3(z,y,x)}\tfrac{1}{s(z,y,x)}t\left(r,z,y,x\right)\mid N\left(r,z,y,x\right)-N\left(v,z,y,x\right)\mid^2\mid f\left(r,z,y,x\right)\mid^2\right)$$

$$=\tfrac{1}{2}\parallel N^{\frac{1}{2}}\left(f\right)\parallel^2+\sum_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)}t(x,y,z,v)\left(\sum_{a\in\Delta_3(z,y,x)}\tfrac{1}{s(z,y,x)}t\left(a,z,y,x\right)\right)\times$$

$$\underbrace{\sum_{r\in V}\frac{1}{s\left(z,y,x\right)}t\left(y,z,y,x\right)\mid N\left(r,z,y,x\right)-N\left(v,z,y,x\right)\mid^2}_{\le\ C}\mid f\left(v,z,y,x\right)\mid^2$$

$$\le\left(\tfrac{1+12C}{2}\right)\parallel N^{\frac{1}{2}}\left(f\right)\parallel.$$

Applying [26, Theorem X.37], the result follows. The proof of $L_{3,sym}$ may be checked in the same way as the proof of $L_{3,skew}$. □

Corollary 1. *Let $\Delta=(V,E,F,\Delta_3)$ be a simple 3-simplicial complex. Assume that*

$$\sup_{x\sim y,\ x\sim z,\ y\sim z,\ v\in\Delta_3(x,y,z)}\sum_{r\in\Delta_3(z,y,x)}\mid\#\Delta_3(y,z,v)+\#\Delta_3(v,z,x)+\#\Delta_3(x,y,v)-\#\Delta_3(y,z,r)-\#\Delta_3(r,z,x)-\#\Delta_3(x,y,r))\mid^2<\infty.$$

Then $L_{3,skew}$ is essentially self-adjoint on $C^c_{skew}\left(\Delta_3\right)$ and $L_{3,sym}$ is essentially self-adjoint on $C^c_{sym}\left(\Delta_3\right)$.

5. Essential spectrum

Let A be a closed, densely defined linear operator on a Banach space X, and let $\sigma\left(A\right)$ denote the spectrum of A. We denote by $\mathcal{K}\left(X\right)$ the set of compact operators on X to itself. We define the essential spectrum of the operator A by

$$\sigma_{ess}\left(A\right)=\bigcap_{k\in\mathcal{K}(X)}\sigma\left(A+k\right).$$

It is well known that if A is a self-adjoint operator on a Hilbert space, the essential spectrum of A is the set of limit points of the spectrum of A, i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity, see [30]. Let $\Delta=(V,E,F,\Delta_3)$ be a weighted 3-simplicial complex. Note that $L_{3,skew}$ is non-negative symmetric operator on $C^c_{skew}\left(\Delta_3\right)$. We consider

$$q\left(f,g\right)=\langle f,L_{3,skew}g\rangle+\langle f,g\rangle$$

On $C^c_{skew}\left(\Delta_3\right)\times C^c_{skew}\left(\Delta_3\right)$. Let H_1 be the completion of $C^c_{skew}\left(\Delta_3\right)$ under the norm

$$\parallel f\parallel_q=\sqrt{\langle L_{3,skew}f,f\rangle+\parallel f\parallel^2}.$$

We define the Friedrichs extension $L^F_{3,skew}$ of $L_{3,skew}$ by:

i) A vector f is in domain $D\left(L^F_{3,skew}\right)$ if and only if $f\in H_1$ and $C^c_{skew}\left(\Delta_3\right)\ni g\longrightarrow\langle f,L_{3,skew}g\rangle+\langle f,g\rangle$ extends to a norm continuous function on $l^2_{skew}\left(\Delta_3\right)$.

ii) For each $f\in D\left(L^F_{3,skew}\right)$, there is a unique u_f such that $\langle f,L_{3,skew}g\rangle+\langle f,g\rangle=\langle u_f,g\rangle$ by Riesz' Theorem. The Friedrichs extension of $L_{3,skew}$, is given by $L^F_{3,skew}f=u_f-f$. It is a self-adjoint extension of $L_{3,skew}$, e.g.

see [26, Theorem $X.23$]. Note that $L^F_{3,skew}$ is bounded if and only if $d_F(.)$ is bounded, e.g. see [8].

Theorem 3. *Let $\Delta = (V, E, F, \Delta_3)$ be a weighted 3-simplicial complex and let*

$$T_0 = \{\mathcal{K} \subset \Delta_3 \mid \mathcal{K} \textit{ finite}\}$$

Then,

$$\inf \sigma\left(L^F_{3,skew}\right) \leq \inf_{(x,y,z,v)\in\Delta_3} t\left(x,y,z,v\right)\left(\frac{1}{s(y,z,v)} + \frac{1}{s(v,z,x)} + \frac{1}{s(x,y,v)} + \frac{1}{s(z,y,x)}\right).$$

and

$$\inf\ \sigma_{ess}\left(L^F_{3,skew}\right) \leq \sup_{\mathcal{K}\subset T_0} \inf_{(x,y,z,v)\in\mathcal{K}^c} t\left(x,y,z,v\right)\left(\frac{1}{s(y,z,v)} + \frac{1}{s(v,z,x)} + \frac{1}{s(x,y,v)} + \frac{1}{s(z,y,x)}\right).$$

In particular, if Δ is a simple 3-simplicial complex then $L^F_{3,skew}$ is not with compact resolvent.

Proof. Let $(x_0, y_0, z_0, v_0) \in \Delta_3$ and let

$$f = \frac{1_{\circlearrowleft(x_0,y_0,z_0,v_0)} - 1_{\circlearrowleft(x_0,y_0,z_0,v_0)}}{\sqrt{t\left(x_0,y_0,z_0,v_0\right)}}$$

Where $1_{\circlearrowleft(x_0,y_0,z_0,v_0)}$ denotes the indicator function of $_{\circlearrowleft(x_0,y_0,z_0,v_0)}$. Then $\parallel f \parallel = 1$ and

$$\langle f, L_{3,skew} f\rangle = t\left(x_0,y_0,z_0,v_0\right)\left(\frac{1}{s(y_0,z_0,v_0)} + \frac{1}{s(v_0,z_0,x_0)} + \frac{1}{s(x_0,y_0,v_0)} + \frac{1}{s(z_0,y_0,x_0)}\right)$$

Applying [21, Proposition 3], the result follows. □

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