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Abstract

The Coble cubics were discovered more than a century ago in connection with genus two Riemann surfaces and theta functions. They have attracted renewed interest ever since. Recently, they were reinterpreted in terms of alternating trivectors in nine variables. Exploring this relation further, we show how the Hilbert scheme of pairs of points on an abelian surface, and also its Kummer fourfold, a very remarkable hyper-Kähler manifold, can very naturally be constructed in this context. Moreover, we explain how this perspective allows us to describe the group law of an abelian surface, in a strikingly similar way to how the group structure of a plane cubic can be defined in terms of its intersection with lines.

1 Introduction

The Coble hypersurfaces are very remarkable cubics and quartics in complex projective spaces, discovered by Coble more than a century ago. They can be characterized as the unique hypersurfaces whose singular locus is the Jacobian of a genus two curve embedded in $\mathbf{P}^8$, or the associated Kummer variety of a genus three curve embedded in $\mathbf{P}^7$, respectively.

The Coble hypersurfaces have been revisited several times. In the eighties, Narasimhan and Ramanan interpreted them in terms of moduli spaces of vector bundles with fixed determinant on a curve of genus two or three [NR87]. This perspective has been explored by a number of authors, see [Bea03] and the references therein.

More recently, the Coble hypersurfaces have been given interpretations coming from Lie theory, more precisely from the Kac–Vinberg theory of so-called θ-groups [GSW13]. It seems to be just a coincidence that θ-groups were coined this way at a time where there was no apparent relation with theta functions, but the fact is that there is a very rich interplay between the invariant theory of θ-representations and certain moduli spaces of polarized abelian varieties.

From this point of view, genus two curves are naturally related to alternating trivectors, that is, elements of $\wedge^3 V_9$ for V_9 a nine-dimensional vector space. Over the complex numbers, it was explained in [GSW13] how to associate to a general

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such trivector an abelian surface A in $\mathbf{P}(V_9^\vee)$, the projective space of hyperplanes in V_9, and a cubic hypersurface which is singular exactly along A; to be precise, A is only a torsor over an abelian surface: to make it an abelian surface stricto sensu, one needs to fix an origin. Of course this cubic hypersurface has to be the same as the one discovered by Coble. This point of view was further explored in [GS15] and, over an arbitrary field, in [RS18].

In this paper we shall enrich the picture by passing to the dual projective space $\mathbf{P}(V_9)$, where our trivector defines a wealth of interesting subvarieties, that we interpret as orbital degeneracy loci in Section 4, following the terminology of [BFMT17, BFMT18]. Already well-known were the projective dual to the cubic, which is a special sextic hypersurface, and a (singular) subvariety Σ of its singular locus which can be identified with the triples of degree zero line bundles on the genus two curve whose product is trivial. A nice ingredient from the theory of orbital degeneracy loci is that they usually come with simple resolutions of singularities, just like the usual degeneracy loci of morphisms between vector bundles. We show that our natural resolution of Σ is nothing else than the Kummer fourfold of A (Theorem 5.1). Moreover, we observe that from the orbital degeneracy loci point of view, one can define two natural smooth covers of Σ, generically finite of degree three. We identify these covers, one with the Hilbert scheme of length two subschemes of A (Theorem 5.14), the other one with the nested Kummer fourfold (Theorem 5.11). More precisely, the main results can be summarized in the following theorem, where we denoted by $\mathcal{U}_{r_i}$ the tautological bundle of rank r_i over the flag variety $F(r_1, \dots, r_i, \dots, V_9)$.

Theorem. *Let $\omega \in \wedge^3 V_9$ be a general alternating trivector and let A be its associated abelian surface in $\mathbf{P}(V_9^\vee)$. If we regard ω as a general section of the trivial vector bundle $\wedge^3 V_9$ over $\mathbf{P}(V_9)$, then:*

- *the zero locus of the section induced by ω of the vector bundle*

$$\pi^*(\wedge^3 V_9)/(\mathcal{U}_1 \wedge \wedge^2 V_9 + \mathcal{U}_3 \wedge \mathcal{U}_6 \wedge V_9 + \wedge^3 \mathcal{U}_6)$$

  over the flag variety $F(1,3,6,V_9) \xrightarrow{\pi} \mathbf{P}(V_9)$ is isomorphic to the generalized Kummer fourfold $\mathrm{Kum}^2(A)$;

- *the zero locus of the section induced by ω of the vector bundle*

$$\pi^*(\wedge^3 V_9)/(\mathcal{U}_1 \wedge \wedge^2 V_9 + \wedge^2 \mathcal{U}_5 \wedge V_9 + \wedge^3 \mathcal{U}_7)$$

  over the flag variety $F(1,5,7,V_9) \xrightarrow{\pi} \mathbf{P}(V_9)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^2(A)$;

- *the zero locus of the section induced by ω of the vector bundle*

$$\pi^*(\wedge^3 V_9)/(\mathcal{U}_1 \wedge \wedge^2 V_9 + \mathcal{U}_3 \wedge \mathcal{U}_5 \wedge V_9 + \mathcal{U}_3 \wedge \mathcal{U}_6 \wedge \mathcal{U}_7 + \wedge^3 \mathcal{U}_6)$$

  over the flag variety $F(1,3,5,6,7,V_9) \xrightarrow{\pi} \mathbf{P}(V_9)$ is isomorphic to the nested Kummer fourfold $\mathrm{Kum}^{2,3}(A)$.

On the one hand, the above constructions provide to interesting and well-studied objects such as $\mathrm{Kum}^2(A)$ and $\mathrm{Hilb}^2(A)$ an interpretation as zero loci of sections of suitable vector bundles over some flag varieties. On the other hand, this perspective allows us to give the following nice description of the group structure on A (defined once an origin O has been fixed):

Proposition (Proposition 6.1)**.** *Let $A \subset \mathbf{P}(V_9^\vee)$ be the abelian surface associated to a general alternating trivector $\omega \in \wedge^3 V_9$. Then we can fix the origin O in A in such a way that the following holds: for any three general points $P, Q, R \in A \subset \mathbf{P}(V_9^\vee)$, $P+Q+R = O$ if and only if contracting ω with any two of the three points yields the same line in V_9, i.e., if and only if*

$$[\omega(P, Q, \cdot)] = [\omega(P, R, \cdot)] = [\omega(Q, R, \cdot)] \in \mathbf{P}(V_9).$$

This description is, at least formally, completely similar to the classical description of the group structure over a plane cubic, from its intersection with lines. The main difference is that the space of "lines", rather than the dual projective plane, is now the Kummer fourfold itself.

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2 Classical facts

2.1 The Coble cubic

Let A be an abelian surface, and Θ a principal polarization. Then 3Θ defines an embedding of A inside $|3\Theta|^\vee = \mathbf{P}(V_9^\vee) \simeq \mathbf{P}^8$, where $V_9 := \mathrm{H}^0(A, 3\Theta)$. The following result is essentially due to Coble [Cob17]:

Theorem 2.1. *There exists a unique cubic hypersurface $\mathcal{C}_3$ in $\mathbf{P}(V_9^\vee)$ which is singular along A.*

Proof. See, e.g., [Bea03, Proposition 3.1]. □

We will refer to $\mathcal{C}_3$ as the Coble cubic. Note that $A[3]$, the finite group of three-torsion points in A, fixes 3Θ. Therefore it acts on $\mathbf{P}(V_9^\vee)$ by leaving A invariant.

2.2 Moduli of vector bundles on genus two curves

Let C be a genus two curve whose Jacobian $JC \cong A$. Let $\mathrm{SU}_C(r)$ denote the moduli space of semistable rank r vector bundles on C with trivial determinant. There is a natural morphism from $\mathrm{SU}_C(r)$ to the linear system $|r\Theta|^\vee$ (see [Ort05]):

- if $r = 2$, we get an isomorphism $\mathrm{SU}_C(2) \cong \mathbf{P}^3$;
- if $r = 3$, we get a finite morphism of degree two $\mathrm{SU}_C(3) \to |3\Theta|^\vee = \mathbf{P}(V_9)$, branched along a sextic hypersurface $\mathcal{C}_6 \subset \mathbf{P}(V_9)$.

The following result was conjectured by Dolgachev, and proved in [Ort05] and [Ngu07]:

Theorem 2.2. *The sextic hypersurface $\mathcal{C}_6$ is the projective dual of the Coble cubic $\mathcal{C}_3$.*

Remark 2.3. The singular locus of $\mathcal{C}_6$ is the same as $\mathrm{Sing}(\mathrm{SU}_C(3))$, and can be identified with the set of strictly semistable vector bundles on C. Its dimension is five. Let $A^{(3)} := \mathrm{Sym}^3 A$, let $\sigma : A^{(3)} \to A$ be the sum morphism, and Σ the zero fiber. Then

$$\Sigma \cong \{E \in \mathrm{SU}_C(3) \text{ s.t. } E = L_1 \oplus L_2 \oplus L_3 \text{ with } L_1, L_2, L_3 \in JC\}$$

is contained inside $\mathrm{Sing}(\mathrm{SU}_C(3)) \cong \mathrm{Sing}(\mathcal{C}_6) \subset \mathcal{C}_6 \subset \mathbf{P}(V_9)$.

2.3 Alternating trivectors

Following [GSW13], one can give another description of the embedding of A in $\mathbf{P}(V_9^\vee)$, starting from an alternating trivector (or three-form).

Let $\omega \in \wedge^3 V_9$ be a general alternating trivector. Let H denote the hyperplane bundle on $\mathbf{P}(V_9^\vee)$. Then $\wedge^3 H$ is a subbundle of the trivial bundle with fiber $\wedge^3 V_9$, and the quotient is $\wedge^2 H(1)$. So ω defines a section of $\wedge^2 H(1)$ over $\mathbf{P}(V_9^\vee)$, and the latter can be stratified by the rank of this two-form. We denote

$$D_{Y_r^{Sp}} := \{P \in \mathbf{P}(V_9^\vee) \text{ s.t. } \mathrm{rank}(\omega(P, \cdot, \cdot)) \leq r\}.$$

These loci are nothing more than the degeneracy loci (or Pfaffian loci) of the skew-symmetric morphism $H^\vee \to H(1)$ corresponding to ω. For dimensional reasons, $D_{Y_2^{Sp}}$ is empty, and therefore $D_{Y_4^{Sp}}$ is a smooth surface. By [GSW13, Theorem 5.5], $D_{Y_4^{Sp}}$ is a torsor, that we denote by A, over an abelian surface. By [GSW13, Proposition 5.6], the restriction of the ambient polarization is of type $(3,3)$. Moreover, the surface A is the singular locus of $D_{Y_6^{Sp}}$, the Pfaffian cubic hypersurface. By Theorem 2.1, this hypersurface must be the Coble cubic $\mathcal{C}_3$. Of course all these loci depend on ω, but we will omit this dependence in our notation.

The geometry of the pair $(A, \mathcal{C}_3)$ was described in more details in [GS15]. For example, it can be proved that A parametrizes the family of $\mathbf{P}^4$'s contained in $\mathcal{C}_3$. Moreover, each such $\mathbf{P}^4$ cuts A along a theta-divisor [GS15, Theorem 3.6].

By varying ω, one gets a locally complete family of $(3,3)$-polarized abelian surfaces [GSW13].

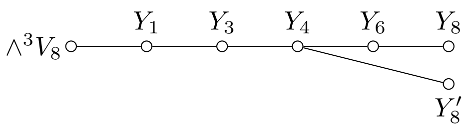
On the dual space $\mathbf{P}(V_9)$, notice that $\omega \in \wedge^3 V_9 \cong \mathrm{H}^0(\mathbf{P}(V_9), \wedge^3 \mathcal{Q})$, where $\mathcal{Q}$ denotes the tautological quotient bundle (of rank eight). The fiber of $\wedge^3 \mathcal{Q}$ is isomorphic to $\wedge^3 \mathbb{C}^8$. Just as we did on $\mathbf{P}(V_9^\vee)$ when we defined the Pfaffian loci, we can define subvarieties of $\mathbf{P}(V_9)$ as loci where the trivector that we obtain on $\mathcal{Q}$ has some special behavior, in the sense that it belongs to some proper GL_8-orbit (or rather, orbit closure) in $\wedge^3 \mathbb{C}^8$. This is precisely the idea behind the notion of orbital degeneracy loci introduced in [BFMT17, BFMT18]. The next section will be devoted to the study of the relevant orbits in $\wedge^3 \mathbb{C}^8$; in the last sections, we will provide geometric interpretations for the corresponding orbital degeneracy loci in $\mathbf{P}(V_9)$.

3 The affine model: trivectors in eight variables

Our model will be the GL_8-representation $\wedge^3 V_8$. This is a classical example of a representation with a finite number of orbits. The properties we will need in the following regarding this space can be found in [Gur64, KW].

We denote by Y_i a codimension i orbit closure inside $\wedge^3 V_8$. As we want to construct some orbital degeneracy loci (see Section 4.1) inside $\mathbf{P}(V_9)$, whose dimension is eight, we will focus on the varieties Y_i for $i \leq 8$. As it turns out, there is exactly one orbit closure of codimension i for $i = 1, 3, 4, 6$ and two distinct orbit closures Y_8 and Y_8' of codimension eight. The inclusion diagram is given in Figure 1.

Figure 1: Inclusions of orbit closures up to codimension eight in $\wedge^3 V_8$



3.1 Kempf collapsings

In [KW], the geometry of these orbit closures has been studied with the help of birational Kempf collapsings. These are particular resolutions of singularities given by total spaces of homogeneous vector bundles on some auxiliary flag manifolds.

Let $F = G/P$ such a flag manifold, for P a parabolic subgroup of an algebraic group G. Then a homogeneous bundle on F is of the form $\mathcal{E}_U = G \times_P U$ for some P-module U. If U is a P-submodule of a G-module V, then $\mathcal{E}_U$ is a sub-vector bundle of $\mathcal{E}_V$, which is the trivial bundle on F with fiber V. In particular, if we denote by W the total space of $\mathcal{E}_U$, this construction induces a proper G-equivariant map $\pi_U : W \to V$, called a Kempf collapsing. When V has only finitely many G-orbits (e.g., the GL_8-representation $\wedge^3 V_8$), the image of π_U must be some orbit closure Y. In many cases, Y being given, we can always find a parabolic P and a P-module U such that the image of π_U is Y.

Partially following [KW], in Table 2 we provide a few finite Kempf collapsings for the biggest orbit closures in $V = \wedge^3 V_8$, together with their degrees. We denote by $F(r_1, \dots, r_i, \dots, V_8)$ the variety parametrizing flags of subspaces of V_8 of dimensions $r_1, \dots, r_i$. On this flag manifold, we denote by $\mathcal{U}_{r_i}$ the tautological bundle of rank r_i.

Table 2: Some $d : 1$ Kempf collapsings for the orbit closures in $\wedge^8 V_8$ of codimension up to 8.

Y	F	$\mathcal{E}_U$	d
Y_1	$\mathrm{Gr}(5, V_8)$	$\wedge^2 \mathcal{U}_5 \wedge V_8$	1
Y_3	$F(1, 4, V_8)$	$\mathcal{U}_1 \wedge (\wedge^2 V_8) + (\wedge^2 \mathcal{U}_4) \wedge V_8$	2
Y_4	$F(2, 5, V_8)$	$\wedge^3 \mathcal{U}_5 + \mathcal{U}_2 \wedge \mathcal{U}_5 \wedge V_8$	1
Y_4	$F(4, 6, V_8)$	$\wedge^2 \mathcal{U}_4 \wedge V_8 + \wedge^3 \mathcal{U}_6$	3
Y_4	$F(2, 4, 5, 6, V_8)$	$\mathcal{U}_2 \wedge (\mathcal{U}_4 \wedge V_8 + \mathcal{U}_5 \wedge \mathcal{U}_6) + \wedge^3 \mathcal{U}_5$	3
Y_6	$F(1, 3, 4, 6, 7, V_8)$	$\mathcal{U}_1 \wedge (\mathcal{U}_4 \wedge V_8 + \wedge^2 \mathcal{U}_6) + \mathcal{U}_3 \wedge (\mathcal{U}_3 \wedge \mathcal{U}_7 + \mathcal{U}_4 \wedge \mathcal{U}_6)$	1
Y_8	$F(2, 5, 7, V_8)$	$\mathcal{U}_2 \wedge (\mathcal{U}_2 \wedge V_8 + \mathcal{U}_5 \wedge \mathcal{U}_7) + \wedge^3 \mathcal{U}_5$	1
Y_8'	$\mathrm{Gr}(2, V_8)$	$\mathcal{U}_2 \wedge (\wedge^2 V_8)$	1

Remark 3.1. The Kempf collapsing corresponding to Y_3 is finite by a dimension count. Its degree is at least 2: indeed, by [Gur64], a general element of Y_3 is $y_3 = v_{123}+v_{456}+v_{147}+v_{268}+v_{358}$ and at least the two flags $\langle v_1\rangle \subset \langle v_1, v_5, v_6, v_8\rangle$ and $\langle v_4\rangle \subset \langle v_2, v_3, v_4, v_8\rangle$ are in the preimage of y_3 in the total space of $\mathcal{U}_1 \wedge (\wedge^2 V_8) + (\wedge^2 \mathcal{U}_4) \wedge V_8$ over $F(1, 4, V_8)$. A direct computation in the proof of Proposition 5.4 will show that it is exactly 2.

In Proposition 3.2 we will show that the second Kempf collapsing for Y_4 appearing in Table 2 is indeed of degree 3. The third one will appear as the fiber product of the first two at the end of Section 3.2.

Notation. For a flag $V_a \subset V_b \subset V_c$, we will write $V_{abc} := V_a \wedge V_b \wedge V_c$ for short. For instance, for $y \in \wedge^3 V_8$, it turns out that $y \in Y_3$ if and only if $y \in V_{188} + V_{448}$ for some flag $V_1 \subset V_4 \subset V_8$ (see Remark 3.1), meaning that y can be decomposed into a sum of elements of V_{188} and V_{448}.

A birational Kempf collapsing should be interpreted in the following way: for each point ω of the open orbit in Y, there exists a unique flag $V_\bullet \in F$ such that ω belongs to the fiber of $\mathcal{E}_U$ over $V_\bullet$. In some sense this yields a normal form for ω. It is mainly from this perspective that in the sequel we will make use of Kempf collapsings.

Let us mention some of the properties of the orbit closures $Y_i \subset \wedge^8 V_8$ that will be useful to us in the sequel.

- The orbit closures Y_1, Y_3, Y_4, Y_8 are normal, Cohen–Macaulay, and have rational singularities; only Y_1 and Y_4 are Gorenstein. Y_6 and Y_8' are neither normal nor Cohen–Macaulay.
- Any other orbit of codimension higher than eight is contained in $Y_8 \cup Y_8'$.
- Y_1 is the hypersurface defined by the hyperdeterminant, the unique SL_8-invariant polynomial of degree 16 over $\wedge^3 V_8$.
- Y_3 is the singular locus of Y_1.
- The structure sheaf of Y_4 admits the following self-dual resolution:

$$0 \to \det(V_8^\vee)^9 \to \mathrm{Sym}^2 V_8^\vee \otimes \det(V_8^\vee)^5 \to \wedge^4 V_8^\vee \otimes \det(V_8^\vee)^4 \to$$
$$\to \mathrm{Sym}^2 V_8 \otimes \det(V_8^\vee)^4 \to \mathcal{O}_{\wedge^3 V_8} \to \mathcal{O}_{Y_4} \to 0.$$

3.2 Two triple covers

A rather delicate but very interesting point is that there are a priori more Kempf collapsings than orbit closures. It can happen that some orbit closures have several resolutions of singularities by different Kempf collapsings. It can also happen that a Kempf collapsing is not birational onto its image, either because the dimension drops or, more scarcely, because it has positive degree. Although the latter phenomenon cannot happen for the Kempf collapsing of a completely reducible homogeneous vector bundle ([Kem76, Proposition 2 (c)]), we already met an instance of it in Remark 3.1. The next example will be essential in the sequel:

Proposition 3.2. *On the flag manifold $F_2 := F(4,6,V_8)$, consider the homogeneous vector bundle $\mathcal{E}_2 = \wedge^2\mathcal{U}_4 \wedge V_8 + \wedge^3\mathcal{U}_6$. Then the Kempf collapsing π_2 of its total bundle W_2 is a generically $3:1$ cover of Y_4.*

Proof. Recall from Table 2 that Y_4 admits a resolution of singularities $\pi_1 : W_1 \to Y_4$, where W_1 is the total space of the vector bundle $\wedge^3\mathcal{U}_5 + \mathcal{U}_2 \wedge \mathcal{U}_5 \wedge V_8$ on $F_1 := F(2,5,V_8)$. Being equivariant, it must be an isomorphism on the open orbit $\mathcal{O}_4$. We will show that

i. if $y \in \pi_2(F_2)$ (or, equivalently, if there exist $U_4 \subset U_6 \subset V_8$ such that $y \in \wedge^2 U_4 \wedge V_8 + \wedge^3 U_6$), then there exists a flag $V_2 \subset V_5$ with $V_2 \subset U_4 \subset V_5 \subset U_6$ such that $y \in V_{555} + V_{258}$. In particular, $\pi_2(F_2) \subset Y_4$;

ii. if $y \in \mathcal{O}_4$, then there exist exactly three flags $U_4 \subset U_6$ such that $y \in U_{448} + U_{666}$. In particular, $\mathcal{O}_4 \subset \pi_2(F_2)$.

These two claims will imply that $\pi_2(F_2) = Y_4$ and that π_2 is a $3:1$ over $\mathcal{O}_4$.

To prove i., we fix $U_4 \subset U_6$ and we consider the space of parameters for $V_2 \subset V_5$ with $V_2 \subset U_4 \subset V_5 \subset U_6$, i.e., $\mathrm{Gr}(2,U_4) \times \mathbf{P}(U_6/U_4)$. Over this space, the point y can be regarded as a section of the trivial vector bundle $\wedge^2 U_4 \wedge V_8 + \wedge^3 U_6$; on a point $V_2 \subset V_5$, $y \in V_2 \wedge U_4 \wedge U_8 + \wedge^3 U_6$ if and only if the induced section $\bar{y}$ on the quotient bundle $V_8/V_6 \otimes \wedge^2(U_4/V_2)$ vanishes. On the points where $\bar{y}$ vanishes, y induces a section of $V_{258} + \wedge^3 U_6$; it will belong to $V_{258} + \wedge^3 V_5$ if and only the induced section on $\wedge^3 U_6/(V_{258} + \wedge^3 U_5) = (U_6/V_5) \otimes \wedge^2(V_5/U_2)$ vanishes. In other words, the points in $\mathrm{Gr}(2,U_4) \times \mathbf{P}(U_6/U_4)$ in the zero locus of the section induced by y of the vector bundle

$$E = \wedge^2(U_4/V_2)^{\oplus 2} \oplus \wedge^2(U_5/V_2)$$

will give the flags we are looking for. A straightforward computation shows that $c_5(E) = 1$, so there exists at least one solution to our problem, as claimed.

To prove ii., we start with a point $y \in \mathcal{O}_4$, for which there exists a unique flag $V_2 \subset V_5$ inside F_1 such that $y \in V_{258} + V_{555}$. By i., any flag $U_4 \subset U_6$ such that $y \in U_{448} + U_{666}$ has to satisfy $V_2 \subset U_4 \subset V_5 \subset U_6$. This can be reformulated as the condition that contracting y with any element $\eta \in U_6^\perp$, we get an element of $\wedge^2 U_4$. Note that since $y \in V_2 \wedge V_5 \wedge V_8 + \wedge^3 V_5$, such a contraction will belong to $V_2 \wedge V_5$. So it belongs to $\wedge^2 U_4$ if and only if its image in $V_2 \otimes (V_5/V_2)$ is contained in $V_2 \otimes (U_4/V_2)$. Dualizing, we need that the induced morphism from $U_6^\perp \otimes V_2^\vee$ to V_5/V_2 has rank two, which occurs in codimension two, that is, for a finite number of spaces U_6. Then U_4/V_2 (hence U_4) is determined as the image of the previous morphism. We conclude that the number of solutions to our problem can be computed on $\mathbf{P}(V_8/V_5)$, by the Thom–Porteous formula, as the class

$$c_2(U_6^\perp \otimes V_2^\vee - V_5/V_2) = c_2(U_6^\perp \otimes V_2^\vee) = 3.$$

Being equivariant, π_2 must be finite and étale over the open orbit $\mathcal{O}_4$ in Y_4, hence there exist exactly three flags over any point in the orbit. □

Example 3.3. Let us denote again by W_1 the (total space of the) vector bundle $\wedge^3\mathcal{U}_5 + \mathcal{U}_2 \wedge \mathcal{U}_5 \wedge \mathcal{U}_8$ over $F(2,5,V_8)$, which yields a Kempf collapsing resolving Y_4. Consider the point $y_4 = v_{456} + v_{147} + v_{257} + v_{268} + v_{358} + v_{367}$ of $\mathcal{O}_4$, see [Gur64]. Its unique preimage in W_1 is given by the flag

$$V_2 = \langle v_7, v_8 \rangle \quad \subset \quad V_5 = \langle v_4, v_5, v_6, v_7, v_8 \rangle.$$

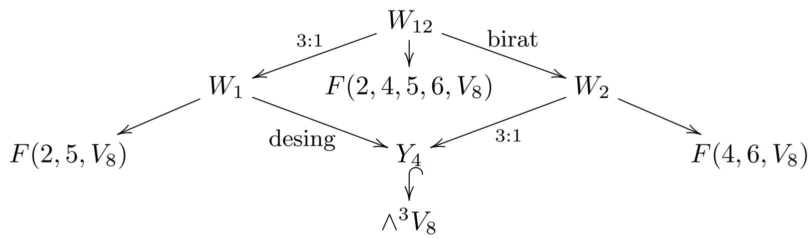
Its three preimages in W_2 are given by

$$\begin{array}{rcl} U_4 = \langle v_5, v_6, v_7, v_8\rangle & \subset & U_6 = \langle v_1, v_4, v_5, v_6, v_7, v_8\rangle, \\ U_4 = \langle v_4, v_5 - v_6, v_7, v_8\rangle & \subset & U_6 = \langle v_2 + v_3, v_4, v_5, v_6, v_7, v_8\rangle, \\ U_4 = \langle v_4, v_5 + v_6, v_7, v_8\rangle & \subset & U_6 = \langle v_2 - v_3, v_4, v_5, v_6, v_7, v_8\rangle. \end{array}$$

Let y be a point in Y_4, and consider a flag $V_2 \subset V_4 \subset V_5 \subset V_6$ defining points of W_1 and W_2 over y. This means that y belongs to the intersection of $\wedge^3 V_5 + V_2 \wedge V_5 \wedge V_8$ with $\wedge^3 V_6 + \wedge^2 V_4 \wedge V_8$, that is, to $V_{248} + V_{256} + V_{555}$. This suggests to consider, on the flag manifold $F(2,4,5,6,V_8)$, the vector bundle $\mathcal{U}_{248} + \mathcal{U}_{256} + \mathcal{U}_{555}$, and to denote its total space by W_{12}. Note that the latter bundle has rank 27 on a 25-dimensional flag manifold, so that W_{12} has dimension 52, as expected.

We get the following diagram of morphisms:

(3.4)



where $W_{12} \to W_1$ is generically 3:1.

4 The degeneracy loci

4.1 Orbital degeneracy loci: generalities

Let us briefly recall how orbital degeneracy loci are constructed [BFMT17, BFMT18]. One starts with a model, that we will choose to be a representation V of some algebraic group G. Inside V, we consider a closed G-stable subvariety Y, usually the closure of a G-orbit. For any G-principal bundle $\mathcal{E}$ over a variety X, we can consider its associated vector bundle $\mathcal{E}_V$ on X. By construction, each fiber of this bundle gets identified with V, not canonically, but the ambiguity only comes from the action of G. This allows us to define, for any global section s of $\mathcal{E}_V$, the orbital degeneracy locus

$$D_Y(s) = \{x \in X \text{ s.t. } s(x) \in Y \subset V \simeq \mathcal{E}_{V,x}\}.$$

In the algebraic context, there is a natural scheme structure induced on $D_Y(s)$ that we will not consider. In the usual situation where s is general in a finite-dimensional space of global sections that generates $\mathcal{E}_V$ everywhere, the orbital degeneracy loci are well-behaved, in the sense that their properties faithfully reflect the properties of Y. In particular, $\mathrm{Sing}(D_Y(s)) = D_{\mathrm{Sing}(Y)}(s)$ and

$$\mathrm{codim}_X D_Y(s) = \mathrm{codim}_V Y, \quad \mathrm{codim}_{D_Y(s)} \mathrm{Sing}(D_Y(s)) = \mathrm{codim}_Y \mathrm{Sing}(Y).$$

A remarkable feature of an orbital degeneracy locus $D_Y(s)$ associated to a subvariety Y admitting a Kempf collapsing is that it is easy to relativize such a collapsing to get a surjective map $\mathscr{Z} \to D_Y(s)$. $\mathscr{Z}$ turns out to be the zero locus of an induced section of a vector bundle on a manifold, both determined by the collapsing. Moreover, if the Kempf collapsing is finite, such a map will be finite as well. We refer to [BFMT17, BFMT18] for more details.

4.2 Loci associated to an alternate trivector

An example of orbital degeneracy locus is the abelian surface $A = D_{Y_4^{Sp}}$ constructed in Section 2.3. Recall that A is in fact only a torsor over an abelian surface, as it has no fixed origin; in order to simplify the terminology, from now on we will abusively call it the abelian surface A. Here the model V is the space of alternating bivectors $\wedge^2 V_8$, on which the group GL_8 acts. The stable closed subvarieties are the loci Y_r^{Sp} where the rank is bounded above by r. The trivector $\omega \in \wedge^3 V_9$ defines a section of $\wedge^2 H(1)$ over $\mathbf{P}(V_9^\vee)$, where H denotes the tautological bundle on $\mathbf{P}(V_9^\vee)$, and the corresponding orbital degeneracy loci are the Pfaffian loci $D_{Y_r^{Sp}} \subset \mathbf{P}(V_9^\vee)$.

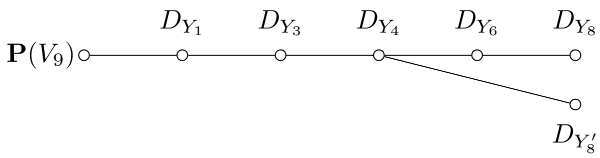
On the dual projective space $\mathbf{P}(V_9)$, the trivector $\omega \in \wedge^3 V_9$ can be seen as a section of $\wedge^3 \mathcal{Q}$, $\mathcal{Q}$ being the tautological quotient bundle on $\mathbf{P}(V_9)$. For Y_i the orbit closures inside $\wedge^3 V_8$ introduced in Section 3, the associated degeneracy loci are

$$D_{Y_i} := \{[V_1] \in \mathbf{P}(V_9) \text{ s.t. } \omega \,(\mathrm{mod}\, V_1) \in Y_i \subset \wedge^3(V_9/V_1)\},$$

where we omit, for simplicity, the dependence on ω. We will often write ω/V_1 instead of $\omega \,(\mathrm{mod}\, V_1)$ where no confusion can arise.

For a generic ω, D_{Y_i} has codimension i inside $\mathbf{P}(V_9)$ and $\mathrm{Sing}(D_{Y_i}) = D_{\mathrm{Sing}(Y_i)}$ (see Figure 3 for the inclusion graph). For example, D_{Y_1} is a sextic hypersurface inside $\mathbf{P}(V_9)$, singular along D_{Y_3}, which is five-dimensional.

Figure 3: Inclusions of degeneraci locy inside $\mathbf{P}(V_9)$



Proposition 4.1. *D_{Y_1} is the Coble sextic $\mathcal{C}_6$.*

Proof. Let us prove that D_{Y_1} is the dual hypersurface to the Coble cubic $\mathcal{C}_3 = D_{Y_6^{Sp}}$. The conclusion will then follow from Theorem 2.2.

A general point of the cubic hypersurface $D_{Y_6^{Sp}}$ is a hyperplane H such that we can decompose $\omega = e \wedge \theta + \sigma$ with $e \notin H$, $\sigma \in \wedge^3 H$, and $\theta \in \wedge^2 H$ is degenerate, i.e. $\theta^4 := \theta \wedge \theta \wedge \theta \wedge \theta = 0$. At a smooth point of this hypersurface, θ has rank six. Let us analyze how H can be deformed inside $D_{Y_6^{Sp}}$.

We choose a basis $e_1, \dots, e_9$ of V_9 such that H is generated by $e_1, \dots, e_8$, and $e_9 = e$. On a neighborhood of H in $\mathbf{P}(V_9^\vee)$, a hyperplane H_z has a basis $f_1 = e_1 + z_1 e_9, \dots, f_8 = e_8 + z_8 e_9$. In the basis $f_1, \dots, f_8, e_9$ of V_9, our ω decomposes as $e_9 \wedge (\theta + z \lrcorner \sigma)(f) + \sigma(f)$, where $z = z_1 e_1^* + \cdots + z_8 e_8^*$ and $\theta(f), \sigma(f)$ are obtained by expressing θ, σ in the basis $e_1, \dots, e_8$, and replacing formally each e_i by f_i. So H_z remains inside $D_{Y_6^{Sp}}$ if and only if $(\theta + z \lrcorner \sigma)^4 = 0$. In particular, the tangent hyperplane to $D_{Y_6^{Sp}}$ at H is given by the condition that $\theta^3 \wedge (z \lrcorner \sigma) = 0$.

Suppose our basis has been chosen so that $\theta = e_{12}+e_{34}+e_{56}$, and decompose further our tensor as

$$\omega = e_9 \wedge (e_{12} + e_{34} + e_{56}) + e_7 \wedge \sigma_7 + e_8 \wedge \sigma_8 + e_7 \wedge e_8 \wedge \sigma_{78},$$

where $\sigma_7, \sigma_8, \sigma_{78}$ only involve $V_6 := \langle e_1, \dots, e_6 \rangle$. The condition $\theta^3 \wedge (z \lrcorner \sigma) = 0$ simply becomes $z \lrcorner \sigma_{78} = 0$, or equivalently, that σ_{78} belongs to H_z. In other words, the tangent hyperplane to $D_{Y_6^{Sp}}$ at $[H]$ is the orthogonal hyperplane to the vector σ_{78}.

Finally, we claim that $[\sigma_{78}]$ belongs to D_{Y_1}. Indeed, when we mod out by σ_{78}, we get

$$\bar{\omega} = e_9 \wedge (e_{12} + e_{34} + e_{56}) + e_7 \wedge \sigma_7 + e_8 \wedge \sigma_8,$$

now in $\wedge^3(V_9/\langle \sigma_{78} \rangle)$ (with some abuse of notation), and the factors of e_9, e_7, e_8 now live in $\wedge^2(V_6/\langle \sigma_{78} \rangle)$. If $\bar{V}_5 := V_6/\langle \sigma_{78} \rangle$ and $\bar{V}_8 := V_9/\langle \sigma_{78} \rangle$, we conclude that $\bar{\omega}$ belongs to $\wedge^2 \bar{V}_5 \wedge \bar{V}_8$. But this is precisely the condition that defines D_{Y_1}. We have thus proved that the dual of $\mathcal{C}_3 = D_{Y_6^{Sp}}$ is contained in D_{Y_1}.

Conversely, let us consider a general point $[V_1]$ of D_{Y_1}, generated by e_1. From the Kempf collapsing resolving Y_1, we see that this means that there must exist a unique V_6, with $V_1 \subset V_6 \subset V_9$, such that ω belongs to $V_{199} + V_{669}$. This is equivalent to the fact that the contraction by ω sends $\wedge^2 V_6^\perp$ to V_1.

Let us consider a basis $e_1, \dots, e_6$ of V_6. Note that $(V_{199} + V_{669})/V_{669}$ is isomorphic to $V_1 \otimes \wedge^2(V_9/V_6)$. Since V_9/V_6 is three-dimensional, every bivector in $\wedge^2(V_9/V_6)$ is decomposable. This allows us to complete our basis of V_6 in a basis of V_9 with three vectors e_7, e_8, e_9 in V_9 such that

$$\omega = e_{178} + \phi_7 \wedge e_7 + \phi_8 \wedge e_8 + \phi_9 \wedge e_9 + \psi,$$

where ϕ_7, ϕ_8, ϕ_9 belong to $\wedge^2 V_6$ and ψ to to $\wedge^3 V_6$. We claim that the tangent space to D_{Y_1} at $[V_1]$ is the hyperplane defined by the linear form e_9^* from the dual basis. Indeed, we describe points $[U_1]$ in D_{Y_1} locally around $[V_1]$ by moving V_6 to spaces U_6 such that the contraction by ω from $\wedge^2 U_6^\perp$ to V_9 keeps rank one, and defining U_1 as the image. Locally around V_6, such a space U_6 is defined by linear forms $f_i^* = e_i^* + t_i$, for $i = 7, 8, 9$, where t_i is a linear combination of $e_1^*, \dots, e_6^*$. Clearly the contraction $\omega(f_7^*, f_8^*, \cdot)$ is a non-zero vector, which must therefore generate U_1, and its coefficient on e_9 is $\phi_9(t_7, t_8)$, which has order two. Modding out order two deformations, U_1 is thus contained in $\langle e_1, \dots, e_8 \rangle$, which implies the claim.

In order to conclude the proof, we just need to check that this tangent hyperplane belongs to $\mathcal{C}_3 = D_{Y_6^{Sp}}$, or equivalently, that the contraction $\omega(e_9^*, \cdot, \cdot)$ has rank at most six. But that is clear, since this contraction is ϕ_9, an element of $\wedge^2 V_6$. □

Corollary 4.2. *D_{Y_3} is the singular locus of the sextic $\mathcal{C}_6$.*

Proposition 4.3. *There exists a natural birational map $D_{Y_6} \dashrightarrow A$.*

Proof. Let $[e_0]$ be a point in D_{Y_6}. By definition, this means that we can decompose ω with respect to a decomposition $V_9 = \mathbb{C}e_0 \oplus V_8$ as $\omega = e_0 \wedge \alpha + \eta$, where η belongs to $Y_6 \subset \wedge^3 V_8$. By Table 2, for $[e_0]$ outside D_{Y_8}, this implies that there exists a unique flag $V_1 \subset V_4 \subset V_7 \subset V_8$ such that η belongs to $V_{148} + \wedge^3 V_7$. Let

$W_8 = \mathbb{C}e_0 \oplus V_7$. Let us also choose a generator e_1 of V_1, and some $e_8 \notin W_8$. We can then write ω as

$$\omega = e_0 \wedge u \wedge e_8 + e_1 \wedge v \wedge e_8 + \xi,$$

for some $u \in V_7$, $v \in V_4$, $\xi \in \wedge^3 W_8$. Since the two-form $e_0 \wedge u + e_1 \wedge v$ has rank at most four, this implies that W_8 is a point of A. We have thus a rational map sending $[e_0]$ to $V_7([e_0]) \oplus \mathbb{C}e_0$.

Conversely, we claim that for a general $P \in A$, there exists a unique line $l_P \subset P$ such that $(\omega/l_P) \in Y_6 \subset \wedge^3(V_9/l_P)$ and $V_7((\omega/l_P)) = P/l_P$. By hypothesis there exist $\sigma \in \wedge^2 P$ of rank four and $\sigma' \in \wedge^3 P$, $v_8 \notin P$ such that $\omega = v_8 \wedge \sigma + \sigma'$.

We want to show that there exists in general exactly one flag $V_1 \subset V_3 \subset V_4 \subset V_6 \subset V_7 \subset (V_9/l_P)$ such that $\omega/l_P \in V_{148} + V_{166} + V_{337} + V_{346}$. The space $V_7 = P/l_P$ is determined by l_P. Denote by W the four-dimensional space defined by σ; then, $l_P \subset W$ and therefore $\mathbf{P}(W)$ is the right parameter space for l_P. Indeed, the contraction of ω/l_P by any element of $(P/l_P)^{\perp}$ has to belong to $V_1 \wedge V_4$, hence it has rank at most two, but this means that also the rank of σ/l_P can be at most two. Moreover, the two-dimensional space $l_P^{\perp_\sigma}/l_P$ defined by σ/l_P must contain V_1 and be contained inside V_4. Therefore V_1 is parametrized by $\mathbf{P}(l_P^{\perp_\sigma}/l_P)$, while $V_4/l_P^{\perp_\sigma} \subset V_6/l_P^{\perp_\sigma} \subset P/l_P^{\perp_\sigma}$ is parametrized by $F(2,4,P/l_P^{\perp_\sigma})$ and $V_3/V_1 \subset V_4/V_1$ is parametrized by $\mathrm{Gr}(2, V_4/V_1)$.

Inside this fourteen-dimensional parameter space, $(v_8 \wedge \sigma)/l_P$ belongs by construction to V_{148}. If we interpret σ'/l_P as a section of the bundle $\wedge^3(P/l_P)$, the flags we are looking for are defined by the vanishing of the section that σ'/l_P induces on the quotient bundle $\wedge^3 V_7/(V_{147} + V_{337} + V_{346} + V_{166})$. To determine the number of such flags, we must compute the top Chern class of this quotient bundle. In order to do this, we first filter our bundle by homogeneous subbundles such that the associated graded bundle is completely reducible. Explicitely, the associated bundle we get is

$$(V_7/V_6) \otimes \big(V_4 \otimes (V_6/V_4) \oplus \wedge^2(V_6/V_4) \oplus (V_4/V_3) \otimes (V_3/V_1)\big) \oplus \\ \oplus \wedge^2(V_6/V_4) \otimes (V_4/V_1).$$

A computation with [GS] shows that the top Chern class of the latter bundle is 1, hence also of the original one, and the claim follows.

We have thus defined two rational maps $D_{Y_6} \dashrightarrow A$ and $A \dashrightarrow D_{Y_6}$, inverse one to the other. This implies the claim. $\square$

We will now focus on the four-dimensional orbital degeneracy locus $D_{Y_4} \subset \mathbf{P}(V_9)$. A priori, it contains D_{Y_6} (dimension two), D_{Y_8} and $D_{Y_8'}$ (dimension 0). Following [BFMT17, BFMT18], we can relativize the three Kempf collapsings of Table 2 and Section 3.2. We will denote by Z the desingularization of D_{Y_4} which is a zero locus inside the flag bundle $F(2,5,\mathcal{Q}) \cong F(1,3,6,V_9)$ and which relativizes the first Kempf collapsing; more precisely, it is the zero locus of a section of the vector bundle

$$\wedge^3 V_9/(\mathcal{U}_{199} + \mathcal{U}_{369} + \mathcal{U}_{666})$$

induced by ω. Similarly, the relativization of the second one yields a generically $3:1$ morphism to D_{Y_4} from a variety T which is a zero locus of a section of the vector bundle

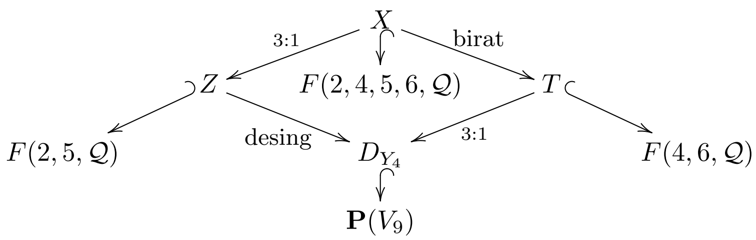
$$\wedge^3 V_9/(\mathcal{U}_{199}+\mathcal{U}_{559}+\mathcal{U}_{777})$$

inside $F(4,6,\mathcal{Q}) \cong F(1,5,7,V_9)$. Finally, we have also an associated zero locus X of a section of the vector bundle

$$\wedge^3 V_9/(\mathcal{U}_{199}+\mathcal{U}_{359}+\mathcal{U}_{367}+\mathcal{U}_{666})$$

inside the flag bundle $F(2,4,5,6,\mathcal{Q}) \cong F(1,3,5,6,7,V_9)$ which relativizes the fiber product $W_1 \times_{Y_4} W_2$, see (3.4). The situation is described in (4.4).

(4.4)



Remark 4.5. The varieties Z, T, X depend on ω. Once again, for the sake of lightness of notation, we will omit to write this dependence explicitly.

By using the description of the model Y_4 given in the previous section, the following facts can be checked.

Proposition 4.6. *Let $\omega \in \wedge^3 V_9$ be a generic trivector. Then:*

1. *$D_{Y_8} = \{p_1, \cdots, p_{81}\}$ consists in 81 reduced points, while $D_{Y_8'} = \emptyset$.*
2. *The surface D_{Y_6} is smooth outside D_{Y_8}.*
3. *D_{Y_4} is a normal, linearly non-degenerate fourfold with $h^i(\mathcal{O}_{D_{Y_4}}) = 1$ for $i = 0, 2, 4$, and 0 otherwise.*
4. *Z has trivial canonical bundle, $\chi(\mathcal{O}_Z) = 3$, $\chi(\Omega_Z^1) = -6$ and $\chi(\Omega_Z^2) = 90$.*
5. *T also has trivial canonical bundle, but $\chi(\mathcal{O}_T) = \chi(\Omega_T^1) = 0$.*

Proof. Statements 1., 3., 4., 5. can be proved by using the desingularization of the loci given by the Kempf collapsings of the affine model, and by computing the corresponding Chern classes with [GS]. Statement 2. is a consequence of the fact that $\mathrm{Sing}(Y_6) \subset Y_8 \cup Y_8'$. □

Corollary 4.7. *Z is a hyper-Kähler fourfold.*

Proof. This follows from 4. and the Beauville–Bogomolov decomposition. □

5 The Kummer geometry of a trivector

This section relates the loci we have constructed with the geometry of the abelian surface $A = D_{Y_4^{Sp}}$. More precisely, in the four theorems of this section we will identify the varieties D_{Y_4}, Z, T, X with four "classical" fourfolds that can be constructed from A.

5.1 The Kummer fourfold

We will denote by $\mathrm{Kum}^2(A)$ the generalized Kummer hyper-Kähler fourfold associated to A. It is a subvariety of the Hilbert scheme $\mathrm{Hilb}^3(A)$ of three points over A. Recall that there is a well-defined natural morphism $HC : \mathrm{Hilb}^3(A) \to A^{(3)}$, called the Hilbert–Chow morphism. Composing with the sum map $A^{(3)} \to A$, we get a morphism $\mathrm{Hilb}^3(A) \to A$ whose fibers are all copies of $\mathrm{Kum}^2(A)$ (note that this ensures that the Kummer fourfold is not affected by the choice of the origin in our torsor A). The Kummer fourfold is a resolution of the singularities of the fiber $\Sigma \subset A^{(3)}$ of the sum morphism.

Theorem 5.1. *Z is isomorphic to the generalized Kummer fourfold* $\mathrm{Kum}^2(A)$.

Proof. We will construct a finite flat morphism $Z' \to Z$ of degree three. It will induce a morphism from Z to $\mathrm{Hilb}^3(A)$. Since Z is hyper-Kähler, the composition $Z \to A$ with the sum map must be constant, otherwise we would get non-trivial one-forms on Z. Therefore our morphism factorizes through $\mathrm{Kum}^2(A)$. Finally, it will turn out to be birational. Since Z is a minimal model, such a birational morphism must be an isomorphism.

Recall that Z is embedded inside $F(1,3,6,V_9)$. Denote by F_Z the restriction to Z of the $\mathbf{P}^2$-bundle defined by the natural projection $F(1,3,6,8,V_9) \to F(1,3,6,V_9)$. By definition, for any flag $V_1 \subset V_3 \subset V_6 \subset V_9$ in Z, ω belongs to $V_{199} + V_{369} + V_{666}$. Modding out the latter bundle by $V_{199} + V_{339} + V_{666}$, we get the vector bundle $A_2 \otimes B_3 \otimes C_3$, where $A_2 = V_3/V_1$, $B_3 = V_6/V_3$ and $C_3 = V_9/V_6$. Since $F_Z = \mathbf{P}_Z(V_9/V_6)^\vee$, the class $\bar{\omega}$ of ω in $A_2 \otimes B_3 \otimes C_3$ defines a morphism

$$\hat{\omega} : A_2^\vee(-1) \longrightarrow B_3$$

of vector bundles over F_Z. We define Z' as the first degeneracy locus $D_1(\hat{\omega})$. We will prove that $Z' \to Z$ is finite of degree three, i.e., the fibers always have expected codimension. As they are determinantal, this implies that they are Cohen–Macaulay, hence the projection $Z' \to Z$ is flat. It will further induce a morphism to $\mathrm{Hilb}^3(A)$ because the points in the fibers of $Z' \to Z$ are defined by hyperplanes V_8 that must belong to A. Indeed, since V_8 contains V_6, when we mod out by $\wedge^3 V_8$ the class of ω belongs to $(V_{18} + V_{36}) \otimes (V_9/V_8)$. When V_8 defines a point of $D_1(\hat{\omega})$, the term from V_{36} has rank two modulo V_1, so that the image of ω in $V_{18} + V_{36}$ has rank at most four, which is exactly the condition for V_8 to belong to A.

There remains to check that the projection $Z' \to Z$ is indeed finite of degree three. First note that on $\mathbf{P}(C_3^\vee)$, the morphism $\hat{\omega} : A_2^\vee(-1) \longrightarrow B_3$ is expected to drop rank in codimension two, hence on a finite scheme of length $c_2(B_3 - A_2^\vee(-1)) = s_2(A_2^\vee(-1)) = 3$.

Let us prove that $D_1(\hat{\omega})$ cannot be positive dimensional. By what has been said before, $D_1(\hat{\omega})$ is contained in A. Moreover, it is defined as a subscheme of a projective plane by three quadrics, the 2×2-minors of the matrix $\hat{\omega}$. If it is not the whole plane, this immediately implies that $D_1(\hat{\omega})$ is contained in a conic. In any case, if $D_1(\hat{\omega})$ has positive dimension, it must contain a rational curve. Since A does not contain any rational curve, we get a contradiction.

Finally, we have to prove that the morphism is birational; for this sake, we provide an explicit description of the image of the morphism on a general point and show that it is generically injective. Let $[e_0] \in D_{Y_4}$ be a general point. Let

$V_9 = \mathbb{C}e_0 \oplus V_8$. Then $\omega = e_0 \wedge \sigma + \sigma'$, where $\sigma \in \wedge^2 V_8$ and

$$\sigma' = v_{456} + v_{147} + v_{257} + v_{268} + v_{358} + v_{367}$$

in a suitable basis of V_8. One readily checks that the hyperplanes $[v_1^*]^\perp, [v_2^* + v_3^*]^\perp, [v_2^* - v_3^*]^\perp$ belong to A and contain V_6, hence they must be the three points in A which correspond to $[e_0]$ via the morphism. If we contract any two linear forms among $v_1^*, v_2^* + v_3^*, v_2^* - v_3^*$ with σ' we get zero, so the same contraction with ω yields a multiple of e_0 (non-zero, since σ is general). This means that we can generically recover $[e_0]$ from its image in $\mathrm{Hilb}^3(A)$. □

Theorem 5.2. *D_{Y_4} is projectively equivalent to Σ.*

Proof. We want to compare the vertical projections in the diagram

$$\begin{array}{ccccc} & Z & \simeq & \mathrm{Kum}^2(A) & \\ & \downarrow p & & \downarrow q & \\ \mathbf{P}(V_9) \longleftarrow & D_{Y_4} & & \Sigma & \longrightarrow |3\Theta| \end{array}$$

Recall that since $\mathrm{Kum}^2(A)$ has no holomorphic one-forms, its Picard group and its Néron-Severi group are the same. Moreover, by [Bea83, Proposition 8], the Néron-Severi group of the Kummer fourfold is

$$NS(\mathrm{Kum}^2(A)) = \mathrm{Pic}(\mathrm{Kum}^2(A)) = \iota(NS(A)) \oplus \mathbb{Z}E,$$

where the map ι is injective and E is the exceptional divisor of the projection from $\mathrm{Kum}^2(A)$ to Σ. We will denote by $L^{[2]}$ the image of $L \in NS(A)$ by ι. For ω generic the abelian surface A is generic, so $NS(A) = \mathbb{Z}\Theta$. The projection q is then defined by the full linear system $|\Theta^{[2]}|$.

We will show below that p also contracts E, so that the pull-back $\mathcal{L}$ of the dual tautological line bundle on $\mathbf{P}(V_9)$ must be of the form $L^{[2]}$ for some $L \in \mathrm{Pic}(A)$, hence $\mathcal{L} = \ell\Theta^{[2]}$. By [BN01, Lemma 5.2],

$$\chi(\mathrm{Kum}^2(A), \ell\Theta^{[2]}) = 3\binom{\ell+2}{2}.$$

A computation with [GS] yields that $\chi(Z, \mathcal{L}) = 9$, hence $\ell = 1$. Since D_{Y_4} is linearly non-degenerate by Proposition 4.6, p must be defined by the full linear system $|\mathcal{L}| \simeq \mathbf{P}^8$. So p and q are the same maps, and the conclusion follows.

It remains to show that p contracts E. Recall from the proof of Theorem 5.1 how we constructed an isomorphism from Z to $\mathrm{Kum}^2(A)$: for any V_1 in D_{Y_4} and any flag $V_1 \subset V_3 \subset V_6 \subset V_9$ such that ω belongs to $V_{199} + V_{369} + V_{666}$ (hence defining a point of Z above V_1), we deduced an element $\bar{\omega}$ of $A_2 \otimes B_3 \otimes C_3$, where $A_2 = V_3/V_1$, $B_3 = V_6/V_3$ and $C_3 = V_9/V_6$. Then we proved that the first degeneracy locus of the induced morphism $A_2^\vee(-1) \to B_3$ over $\mathbf{P}(C_3^\vee)$ defines a length three subscheme of A.

It will sufficient to show that the preimage $p^{-1}(D_{Y_6})$ is a three-dimensional subscheme of E. Since E is irreducible, the two are in facts equal and their image through p is 2-dimensional. If V_1 is a general point of D_{Y_6}, then we can write ω/V_1 as $v_{456}+v_{147}+v_{257}+v_{268}+v_{358}$ for a suitable choice of a basis of V_9/V_1. This determines the unique flag

$$(5.3)\quad \langle v_8\rangle \subset \langle v_4,v_7,v_8\rangle \subset \langle v_4,v_5,v_7,v_8\rangle \subset \langle v_2,v_4,v_5,v_6,v_7,v_8\rangle \subset \\ \subset \langle v_1,v_2,v_4,v_5,v_6,v_7,v_8\rangle \subset V_9/V_1.$$

given by the desingularization of Y_6. As it turns out, any flag in the rational normal curve

$$V_3/V_1 = \langle v_8, \alpha v_4+\beta v_7\rangle \subset V_6/V_1 = \langle v_4,v_5,v_7,v_8,\alpha v_2+\beta v_6\rangle$$

is contained in $p^{-1}(V_1)$ since $\omega \in V_{199}+V_{369}+V_{666}$, hence the conclusion follows if we can prove that it is contained in E. On any such flag, (5.3) induces flags $A_1\subset A_2$, $B_1\subset B_2\subset B_3$ and $C_1\subset C_2\subset C_3$ such that

$$\bar\omega \in A_1\otimes B_2\otimes C_3 + A_2\otimes B_1\otimes C_2 + (A_1\otimes B_3 + A_2\otimes B_2)\otimes C_1.$$

Then it is easy to see that the length three subscheme we get in $\mathbf{P}(C_3^\vee)$ has multiplicity two at the point defined by the hyperplane C_2 (note that this point is exactly the hyperplane V_8, uniquely defined by $V_1\in D_{Y_6}$). Since E is precisely the locus of non-reduced schemes, we are done. □

Let $P\in A$ be a hyperplane in $\mathbf{P}(V_9)$, and let P_4 be the four-dimensional subspace of P such that $\omega(P,\cdot,\cdot)\in\wedge^2 P_4$. Then:

Proposition 5.4. *D_{Y_3} is covered by a family of $\mathbf{P}^3$ parametrized by A. More precisely, for any point $P\in A$, we have that $\mathbf{P}(P_4)\subset D_{Y_3}$.*

Proof. As $P\in A$ and by the definition of P_4, we know that

$$\omega \in \wedge^3 P + V_9\wedge(\wedge^2 P_4).$$

In order to show that $\mathbf{P}(P_4)\subset D_{Y_3}$, we need to prove that for any $V_1\subset P_4$, there exist $U_2\subset U_5\subset V_9$ such that $V_1\subset U_2$ and $\omega\in U_2\wedge(\wedge^2V_9)+V_9\wedge(\wedge^2U_5)$. Indeed, if this happens, then ω modulo V_1 belongs to the total space of the vector bundle which gives a Kempf collapsing of Y_3 inside V_9/V_1 (see Table 2), and therefore $V_1\in D_{Y_3}$.

Let $V_1\subset P_4$. We construct U_5 as a subspace of P. Moreover, $\omega(P,\cdot,\cdot)$ is a two-form on P_4, and therefore we can consider the orthogonal $V_1^{\perp_\omega}\subset P_4$ of V_1 inside P_4 with respect to this two-form. We construct U_2 as a subspace of $V_1^{\perp_\omega}$ containing V_1. The parameter space for U_2 is then $\mathbf{P}(V_1^{\perp_\omega}/V_1)$ and the parameter space for the pair $U_2\subset U_5$ is the Grassmannian bundle $\mathrm{Gr}(3,P/U_2)$ over $\mathbf{P}(V_1^{\perp_\omega}/V_1)$, a variety of dimension ten.

Asking that $V_1\subset U_2\subset V_1^{\perp_\omega}$ implies that $\omega(P,\cdot,\cdot)\in U_2\wedge P_4$. Therefore we have that $\omega\in\wedge^3P+U_2\wedge(\wedge^2V_9)$. Let us consider the element $\tilde\omega\in\wedge^3(P/U_2)$ induced by ω. Then $\omega\in U_2\wedge(\wedge^2V_9)+V_9\wedge(\wedge^2U_5)$ if and only if $\tilde\omega\in\wedge^2(U_5/U_2)\wedge P/U_2$. Over our parameter space, U_5/U_2 is parametrized by the rank three tautological bundle $\mathcal{U}$ over $\mathrm{Gr}(3,P/U_2)$. As a consequence, requiring that $\tilde\omega\in\wedge^2\mathcal{U}\wedge(P/U_2)$ is the same as asking that the induced section $\bar\omega$ of the vector bundle $F:=\wedge^3(P/U_2)/(\wedge^2\mathcal{U}\wedge(P/U_2))$ vanishes. F is a rank ten vector bundle over the ten-dimensional parameter space, and the zero locus of its general section $\bar{\omega}$ parametrizes the pairs U_2, U_5 such that $\omega \in U_2 \wedge (\wedge^2 V_9) + V_9 \wedge (\wedge^2 U_5)$. This zero locus consists in general of

$$c_{10}(F) = 2$$

points, as a computation with [GS] shows; as it is nonempty, there exist $U_2 \subset U_5$ with the required properties, and $V_1 \in D_{Y_3}$. This concludes the proof. (Note that the existence of different flags comes from the fact that the Kempf collapsing we used has degree $d > 1$, see Table 2: the above computation actually shows that $d = 2$, as stated in Remark 3.1.) □

Remark 5.5. The singular locus of $SU_C(3)$ can be identified with D_{Y_3}. This singular locus is known to coincide with the set of strictly semistable rank three vector bundles with trivial determinant. A generic point of this set is a bundle $L \oplus E$, where L is a line bundle (or a point of A) and E is a rank two vector bundle such that $\det(E) = L^{-1}$. Therefore, having fixed L, this set contains

$$\{E \text{ of rank 2 s.t. } \det(E) = L^{-1}\} \cong SU_C(2),$$

which is a $\mathbf{P}^3$, see Section 2.2. This gives the family of $\mathbf{P}^3$'s parametrized by A covering D_{Y_3} exhibited in Proposition 5.4. Moreover, each $\mathbf{P}^3$ contains a copy of $\mathrm{Kum}^1(A)$, the Kummer surface associated to A: this gives the family of $\mathrm{Kum}^1(A)$ parametrized by A covering $D_{Y_4} \cong \Sigma$.

Corollary 5.6. *D_{Y_6} is not normal, and A is its normalization.*

Proof. We know that D_{Y_6} is the singular locus of D_{Y_4}, so by Theorem 5.2 it coincides with the set of triples of the form $(P, P, -2P)$ in Σ. In particular there is a bijective morphism $A \to D_{Y_6}$, which implies that the normalization of D_{Y_6} is isomorphic to A.

There just remains to prove that the singular locus of Σ is not normal. Let us consider the following commutative diagram

$$\begin{array}{ccc} \Sigma \times A & \xrightarrow{\ \alpha\ } & A^{(3)} \\ \uparrow & & \uparrow \\ \alpha^{-1}(\mathrm{Sing}(A^{(3)})) & \longrightarrow & \mathrm{Sing}(A^{(3)}) \end{array}$$

where $\alpha : ([P, Q, R], t) \mapsto [P + t, Q + t, R + t]$. The preimage of a point is

$$\alpha^{-1}([P', Q', R']) = \{([P' - s, Q' - s, R' - s], s) \text{ s.t. } 3s = P' + Q' + R'\};$$

in particular, α is a $81 : 1$ étale cover of $A^{(3)}$, and induces an étale cover of $\mathrm{Sing}(A^{(3)})$ by the singular locus of $\Sigma \times A$. Consider the following diagram:

$$\begin{array}{ccc} \Delta_{12} \cup \Delta_{13} \cup \Delta_{23} & \hookrightarrow & A^3 \\ \downarrow & & \downarrow \\ \mathrm{Sing}(A^{(3)}) & \hookrightarrow & A^{(3)} \end{array}$$

where, e.g., $\Delta_{12} = \{(P, P, Q) \in A^3\}$ is the first diagonal. The map on the left is generically $3 : 1$, while the map on the right is generically $6 : 1$. The restriction

$\Delta_{12} \to \mathrm{Sing}(A^{(3)})$ is a birational finite morphism, hence it is an isomorphism if $\mathrm{Sing}(A^{(3)})$ is normal.

Locally, we have $A^3 \cong (\mathbb{C}^2)^3$ and $\mathcal{O}_{A^3} \cong R := \mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3]$. The above diagram induces the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\Delta_{12}} & \twoheadleftarrow & R \\ \beta \uparrow & & \uparrow \\ \mathcal{O}_{\mathrm{Sing}(A^{(3)})} & \twoheadleftarrow & R^{S_3} \end{array}$$

To conclude, it is enough to show that β is not an isomorphism. Locally, $\mathcal{O}_{\Delta_{12}} \cong R/(x_1 - x_2, y_1 - y_2)$, while by construction $\mathcal{O}_{\mathrm{Sing}(A^{(3)})}$ is the quotient of R^{S_3} by the homogeneous ideal $(x_1 - x_2, y_1 - y_2)^{S_3}$. Since for instance $x_3 \in \mathcal{O}_{\Delta_{12}}$ but it is not in the image of β, the conclusion follows. □

Corollary 5.7. *The orbit closure $Y_4 \subset \wedge^3 V_8$ is singular along Y_6. The orbit closure $Y_6 \subset \wedge^3 V_8$ is non-normal along Y_8.*

We observe that the last statement agrees with and specifies the claim in [KW] about the non-normality of Y_6.

Remark 5.8. One can observe that the isomorphism $D_{Y_4} \to \Sigma$ constructed in Theorem 5.2 restricts to the birational map $D_{Y_6} \dashrightarrow A$ described in Proposition 4.3. A point x of $D_{Y_6} \setminus D_{Y_8}$ corresponds to $[P, P, Q]$, where $P \in \mathbf{P}(V_9^\vee)$ is the hyperplane defined by the preimage of x in the desingularization of D_{Y_6}. Similarly, a point $x' \in D_{Y_8}$ corresponds to $[P', P', P']$, where $P' \in \mathbf{P}(V_9^\vee)$ is the hyperplane defined by the preimage of x' in the desingularization of D_{Y_8}.

5.2 The nested Kummer fourfold and the Hilbert scheme

Let us consider now the nested Hilbert scheme $\mathrm{Hilb}^{2,3}(A)$ parametrizing pairs (S, S'), where S is a length two subscheme of A, and $S' \supset S$ a length three subscheme. Such a nested Hilbert scheme is known to be smooth. Moreover, it admits an action of A by translation, compatible with the sum map. So all the fibers of the latter are equivalent, and smooth. We denote them by $\mathrm{Kum}^{2,3}(A)$, the nested Kummer fourfold. By restriction from the Hilbert schemes, we get a triple cover $\mathrm{Kum}^{2,3}(A) \to \mathrm{Kum}^2(A)$, branched over the exceptional divisor, and also a morphism $\mathrm{Kum}^{2,3}(A) \to A$ defined by taking the residual scheme.

In our situation, T is a triple cover of D_{Y_4}, which is birational to $Z \simeq \mathrm{Kum}^2(A)$. So the fiber product of Z with T over D_{Y_4} will be a triple cover of $\mathrm{Kum}^2(A)$, and we can expect it to be isomorphic to $\mathrm{Kum}^{2,3}(A)$. Rather than taking formally the direct product, we define $X \subset F(1, 3, 5, 6, 7, V_9)$ as parametrizing the flags $V_1 \subset V_3 \subset V_5 \subset V_6 \subset V_7 \subset V_9$ such that ω belongs to $V_{199} + V_{359} + V_{377} + V_{666}$. Just like T and Z, for ω generic this is a smooth fourfold. Since $V_{199} + V_{359} + V_{377} + V_{666}$ is exactly the intersection of $V_{199} + V_{369} + V_{666}$ and $V_{199} + V_{559} + V_{777}$, X admits natural projections to Z and T:

$$\begin{array}{ccccc} & & X & & \\ & \overset{3:1}{\swarrow} & & \overset{\text{birat}}{\searrow} & \\ Z & & & & T \\ & \underset{\text{desing}}{\searrow} & & \underset{3:1}{\swarrow} & \\ & & \Sigma & & \end{array} \qquad (5.9)$$

Beware that the degree three morphisms in this diagram are only generically finite. In view of Theorem 5.11, we can give a precise statement concerning $X \to Z$:

Lemma 5.10. *The positive dimensional fibers of $X \to Z$ are 81 projective lines.*

Proof. We will prove that the positive dimensional fibers are 81 projective lines which are contracted to 81 points in Z, whose image via the desingularization $Z \to \Sigma$ is precisely D_{Y_8}. The reason behind this phenomenon is further clarified in Remark 5.13 below.

A point z of Z is a flag $V_1 \subset V_3 \subset V_6 \subset V_9$ such that ω belongs to $M = V_{199} + V_{369} + V_{666}$. A point p of X above z is a pair of subspaces (V_5, V_7), with $V_1 \subset V_3 \subset V_5 \subset V_6 \subset V_7 \subset V_9$, such that ω belongs to $N = V_{199} + V_{359} + V_{377} + V_{666}$. This is a subspace of M, and

$$M/N \simeq V_{369}/(V_{359} + V_{169} + V_{367}) = V_3/V_1 \otimes V_6/V_5 \otimes V_9/V_7.$$

Let $A_2 = V_3/V_1$. The pairs (V_5, V_7) are parametrized by the product of Grassmannians $\mathrm{Gr}(2, V_6/V_3) \times \mathrm{Gr}(1, V_9/V_6) \simeq \mathbf{P}^2 \times \mathbf{P}^2$. On this variety, ω defines a global section of the rank four vector bundle $A_2 \otimes V_6/V_5 \otimes V_9/V_7 = A_2 \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2$, and the fiber over z identifies with the zero locus of this section. Here we have denoted by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ the quotient bundles, of rank one and two, over $\mathrm{Gr}(2, V_6/V_3)$ and $\mathrm{Gr}(1, V_9/V_6)$. An easy computation shows that $c_4(A_2 \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2) = c_2(\mathcal{Q}_1 \otimes \mathcal{Q}_2)^2 = 3$, confirming that the general fiber consists in three points.

There remains to identify the infinite fibers. For this we need to analyze when a section of $A_2 \otimes \mathcal{Q}_1 \otimes \mathcal{Q}_2$ on $\mathrm{Gr}(2, B_3) \times \mathrm{Gr}(1, C_3)$ vanishes in positive dimension, where $B_3 = V_6/V_3$ and $C_3 = V_9/V_6$. Such a global section is an element γ of $A_2 \otimes B_3 \otimes C_3$; as $A_2 \otimes B_3 \otimes C_3 = M/V_{199} + V_{339} + V_{666}$, we have

$$\gamma = \omega \,(\mathrm{mod}\, V_{199} + V_{339} + V_{666}).$$

We consider γ as a family Γ of linear maps from $B_3^\vee$ to C_3. This section vanishes at (U_2, U_1), where $U_2 \subset B_3$ and $U_1 \subset C_3$, if and only if γ belongs to $A_2 \otimes (U_2 \otimes C_3 + B_3 \otimes U_1)$. In other words, all the linear maps in Γ send the line $U_2^\perp$ to the line U_1.

The classification of pencils of 3×3-matrices is well-known: there are exactly seventeen orbits (see e.g. [KW12, 5.4]). A straightforward check shows that the maximal orbits such that γ vanishes in infinitely many pairs (U_2, U_1) are those named as $\mathcal{O}_{14}$, $\mathcal{O}_{13}$, $\mathcal{O}_{11}$, $\mathcal{O}_{10}$, whose elements can be written respectively as follows:

i. $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_3$ for some $a_i \in A_2$, $b_j \in B_3$, $c_k \in C_3$;

ii. $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_3 \otimes c_1$ for some $a_i \in A_2$, $b_j \in B_3$, $c_k \in C_3$.

iii. $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_1 \otimes c_3$ for some $a_i \in A_2$, $b_j \in B_3$, $c_k \in C_3$;

iv. $a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_3 \otimes c_1$ for some $a_i \in A_2$, $b_j \in B_3$, $c_k \in C_3$.

We will show that ω being generic, γ will never be of any of the first three types. By contradiction, we will prove that γ being of those special types would force the class of ω modulo V_1 to belong to some orbit closure Y of codimension bigger than eight in $\wedge^3(V_9/V_1)$. In other words, we would get a point in an ODL $D_Y \subset \mathbf{P}(V_9)$, which we know to be empty for a generic ω.

i. In case i., we can find $A_1 \subset A_2$, $B_1 \subset B_3$, $C_1 \subset C_3$ such that γ belongs to $A_1 \otimes B_3 \otimes C_3 + A_2 \otimes B_1 \otimes C_1$. This means that we can find V_2, V_4, V_7, with $V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_6 \subset V_7 \subset V_9$ such that ω belongs to $V_{199} + V_{269} + V_{347} + V_{666}$. Modding out by V_1 and letting $U_i = V_{i+1}/V_1$, we get a point in the total space of the vector bundle $U_{158} + U_{236} + U_{555}$ over $F(1, 2, 3, 5, 6, U_8)$. Note that this vector bundle is a subbundle of $U_{168} + U_{666}$, which has rank 30 over the 17-dimensional flag manifold $F(1, 6, U_8)$. So it will collapse to an orbit closure of dimension at most $30 + 17 < 48$ in $\wedge^3 U_8$.

ii. In case ii., we observe that we can find $B_1 \subset B_3$ and $C_1 \subset C_3$ such that γ belongs to $A_2 \otimes B_1 \otimes C_3 + A_2 \otimes B_3 \otimes C_1$. This means that we can find V_4, V_7, with $V_3 \subset V_4 \subset V_6 \subset V_7 \subset V_9$ such that ω belongs to $V_{199} + V_{349} + V_{367} + V_{666}$. Modding out by V_1, we get a point in the total space of the vector bundle $U_{238} + U_{256} + U_{555}$ over $F(2, 3, 5, 6, U_8)$. The latter flag manifold has dimension 25, and the vector bundle has rank 23. But note that $U_{238} + U_{256} + U_{555} \subset U_{338} + U_{666}$, and that the vector bundle $U_{338} + U_{666}$ has rank 26 over the 21-dimensional flag manifold $F(3, 6, U_8)$. So again it will collapse to an orbit closure of dimension at most $26 + 21 < 48$ in $\wedge^3 U_8$.

iii. In case iii., there exists $B_2 \subset B_3$ such that γ belongs to $A_2 \otimes B_2 \otimes C_3$. So there exists V_5, with $V_3 \subset V_5 \subset V_6$, such that ω belongs to $V_{199} + V_{359} + V_{666}$. Modding out by V_1 as before, we get a point in the total space of the vector bundle $U_{248} + U_{555}$ over $F(2, 4, 5, U_8)$. The latter flag manifold has dimension 23, and the vector bundle has rank 25, so it would seem to collapse to a codimension 8 orbit closure. But notice that $U_{248} + U_{555} \subset U_{448} + U_{555}$, and that now $U_{448} + U_{555}$ is a rank 28 vector bundle over $F(4, 5, U_8)$, whose dimension is 19. So the collapsing will have for image an orbit closure of dimension at most $28 + 19 < 48$.

So we only remain with case iv.. Observe that it occurs if and only if there exists $C_2 \subset C_3$ such that γ belongs to $A_2 \otimes B_3 \otimes C_2$. So there exists V_8, with $V_6 \subset V_8 \subset V_9$, such that ω belongs to $V_{199} + V_{339} + V_{368} + V_{666}$, and its class modulo V_1 is contained in some $U_{228} + V_{257} + U_{555}$. This is the vector bundle on $F(2, 5, 7, U_8)$ that desingularizes Y_8. In particular V_1 defines one of the 81 points of $D_{Y_8} \subset \mathbf{P}(V_9)$, the flag $V_3 \subset V_6 \subset V_8$ is uniquely defined by V_1 and there is a uniquely defined lift of V_1 in Z. An easy computation shows that the fiber in X of this lift is a projective line, as claimed, and no further degeneration of ω can occur for ω generic. □

Theorem 5.11. *X is isomorphic to the nested Kummer fourfold* $\mathrm{Kum}^{2,3}(A)$.

Proof. We would like to lift the isomorphism between Z and $\mathrm{Kum}^2(A)$:

$$
\begin{array}{ccc}
X & \overset{?}{\dashrightarrow} & \mathrm{Kum}^{2,3}(A) \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\simeq} & \mathrm{Kum}^2(A)
\end{array}
\tag{5.12}
$$

By definition, X parametrizes the flags $V_1 \subset V_3 \subset V_5 \subset V_6 \subset V_7 \subset V_9$ such that

$$\omega \in V_{199} + V_{359} + V_{377} + V_{666}.$$

Consider a hyperplane $V_8 = \mathrm{Ker}(\phi)$ containing V_7. It defines a point in A if the contraction of ω by ϕ has rank four. Let v_8, v_9 be vectors in V_9, independent modulo V_7. Modulo V_{777}, which is killed by ϕ, we can write $\omega = v_1 \wedge \alpha + \beta_8 \wedge v_8 + \beta_9 \wedge v_9$, for some $\alpha \in \wedge^2 V_9$ and $\beta_8, \beta_9 \in V_3 \wedge V_5$. Therefore

$$\phi \lrcorner \omega = -v_1 \wedge (\phi \lrcorner \alpha) + \phi(v_8)\beta_8 + \phi(v_9)\beta_9.$$

Consider the pencil $\langle \beta_8, \beta_9 \rangle$. If we mod out by $V_1 \wedge V_5$, we get $\langle \bar\beta_8, \bar\beta_9 \rangle$ in $\wedge^2(V_5/V_1)$, hence in general a pencil that cuts the Grassmannian of rank two tensors along a length two subscheme. Substracting $v_1 \wedge (\phi \lrcorner \alpha)$ to a rank two tensor yields a tensor of rank at most four. So we get a rational map $X \to \mathrm{Hilb}^2(A)$.

What could prevent it to be regular? First, it could happen that when we mod out by $V_1 \wedge V_5$, the pencil $\langle \beta_8, \beta_9 \rangle$ collapses. In other words, β_8 and β_9 could be proportional up to $V_1 \wedge V_5$. Then we may suppose that $v_9 = 0$. But this would mean that modulo V_1, ω depends only on seven variables, a condition that inside $\wedge^3 \mathbb{C}^8$ defines an orbit of codimension $14 > 8$. So this cannot happen.

Second, the projected pencil $\langle \bar\beta_8, \bar\beta_9 \rangle$ could be contained in the Grassmannian of rank two tensors. But then the pencil of hyperplanes that contain V_7 would be contained in A. Since an abelian surface cannot contain any line, this cannot happen either.

Combining the regular map $X \to \mathrm{Hilb}^2(A)$ with the projection $X \to Z \simeq \mathrm{Kum}^2(A)$, we get a morphism $X \to \mathrm{Kum}^2(A) \times \mathrm{Hilb}^2(A)$ whose image is by construction contained in, hence equal to, $\mathrm{Kum}^{2,3}(A)$. Since the projections from X to Z and from $\mathrm{Kum}^{2,3}(A)$ to $\mathrm{Kum}^2(A)$ are both generically finite of degree three, we get a birational morphism from X to $\mathrm{Kum}^{2,3}(A)$. But by Lemma 5.10 the exceptional locus of this birational morphism is at most one-dimensional. So it has to be an isomorphism. □

Remark 5.13. The positive dimensional fibers of the projection map from $\mathrm{Kum}^{2,3}(A)$ to $\mathrm{Kum}^2(A)$ live above the 81 three-torsion points of A. Indeed, if P is such a point, then the fat point defined by $I(P)^2$ is a length three subscheme that defines a point in $\mathrm{Kum}^2(A)$. Since this subscheme contains all the length two schemes supported at P, we get a fiber isomorphic to $\mathbf{P}(T_P A) \simeq \mathbf{P}^1$. All the other length three schemes supported at P are curvilinear, hence define a unique tangent. In particular we get an identification of D_{Y_8} with $A[3]$, provided we have fixed an origin (in the preimage of D_{Y_8} in A via the map $A \to D_{Y_6}$).

Theorem 5.14. *T is isomorphic to* $\mathrm{Hilb}^2(A)$.

Proof. In the proof of Theorem 5.11 we constructed a morphism $\eta_X : X \to \mathrm{Hilb}^2(A)$. In fact the construction shows that this is the composition of a morphism from $\eta_T : T \to \mathrm{Hilb}^2(A)$ with the projection $X \to T$. Since this morphism is birational, as well as the projection $\mathrm{Kum}^{2,3}(A) \to \mathrm{Hilb}^2(A)$, η_T is birational. Since T has trivial canonical bundle, such a birational morphism must be an isomorphism. □

Remark 5.15. The group structure of A allows us to define a surjective morphism $A \times \mathrm{Kum}^1(A)$ to $\mathrm{Hilb}^2(A)$ which is an étale cover of degree sixteen. This is the étale cover whose existence is predicted by the Beauville–Bogomolov decomposition.

6 On the group structure of A

In this section we geometrically describe the group structure of A. This is an analogue of the usual description of the group structure of a plane cubic curve from its intersection with lines. It is worth mentioning that, in a different context, Donagi [Don80] provided a geometric characterization of the group law for the n-dimensional abelian variety parametrizing the $(n-1)$-dimensional linear subspaces of the intersection of two general quadrics in $\mathbf{P}^{2n+1}$, which is known to be the jacobian of a hyperelliptic curve of genus n.

Recall what we have established so far. If we choose two distinct points P, Q of $A \subset \mathbf{P}(V_9^\vee)$, the corresponding point z in $\mathrm{Hilb}^2(A) \simeq T$ maps to a point $[V_1] \in D_{Y_4} \subset \mathbf{P}(V_9)$. If this point is not on D_{Y_6}, it defines a flag $V_1 \subset V_3 \subset V_6 \subset V_9$ such that ω belongs to $V_{199} + V_{369} + V_{666}$. Moreover, its three preimages z, z', z'' in T yield additional subspaces (V_5, V_7), (V_5', V_7'), (V_5'', V_7''), with

$$V_1 \subset V_3 \subset V_5, V_5', V_5'' \subset V_6 \subset V_7, V_7', V_7'' \subset V_9,$$

such that ω belongs to $V_{199} + V_{559} + V_{777}$, and to the corresponding spaces with (V_5, V_7) replaced by (V_5', V_7') and (V_5'', V_7''). Since $V_7 = P \cap Q$, this implies that if we contract ω by an equation of the hyperplane P and an equation of the hyperplane Q, we get a vector in V_1. With a slight abuse of notation, we write

$$[\omega(P, Q, \cdot)] = [V_1] \in D_{Y_4} \subset \mathbf{P}(V_9).$$

This yields a simple description of the map from $\mathrm{Hilb}^2(A) \simeq T$ to D_{Y_4}. Moreover, this is enough to characterize the point $R \in A$ such that (P, Q, R) belongs to Σ:

Proposition 6.1. *Let $P, Q \in \mathbf{P}(V_9^\vee)$ be general points of A. Then the unique point $R \in A$ such that (P, Q, R) belongs to Σ is characterized by the condition*

$$[\omega(P, R, \cdot)] = [\omega(Q, R, \cdot)] = [\omega(P, Q, \cdot)].$$

Proof. The previous remarks show that R verifies the required condition. There remains to prove that it is uniquely characterized by it.

Let $V_7 := P \cap Q$, and let us choose $v_P \in Q \setminus P$ and $v_Q \in P \setminus Q$. Let us decompose ω with respect to the direct sum $V_9 = \mathbb{C}v_P \oplus \mathbb{C}v_Q \oplus V_7$, as

$$\omega = v_P \wedge v_Q \wedge v_1 + v_P \wedge \alpha + v_Q \wedge \beta + \sigma,$$

with $\alpha, \beta \in \wedge^2 V_7$ and $\sigma \in \wedge^3 V_7$. In particular v_1 generates V_1. Since P belongs to A, $v_Q \wedge v_1 + \alpha$ has rank (at most) four (and since also Q belongs to A, $v_Q \wedge v_1 - \beta$ also has rank (at most) four). This means that α itself has rank at most four, and $v_Q \wedge v_1 \wedge \alpha \wedge \alpha = 0$, or equivalently $v_1 \wedge \alpha \wedge \alpha = 0$.

Lemma 6.2. *$v_1 \wedge \alpha \wedge \alpha = 0$ if and only if there exist u, v, w such that $\alpha = v_1 \wedge u + v \wedge w$.*

Proof. If α has rank six or more, then $v_1 \wedge \alpha \wedge \alpha = 0$ would imply $v_1 = 0$, which is not the case. If α has rank two, $\alpha \wedge \alpha = 0$. So suppose that α has rank exactly four, which means that there exists a unique four-plane L such that α belongs to $\wedge^2 L$. Then $\alpha \wedge \alpha$ is a generator of $\wedge^4 L$, and $v_1 \wedge \alpha \wedge \alpha = 0$ means that v_1 belongs to L. The conclusion easily follows, since if we choose a generic vector u in L, the line generated by α and $v_1 \wedge u$ in $\mathbf{P}(\wedge^2 L)$ will meet the quadric of rank two tensors at another point. □

Applying this Lemma also to $v_Q \wedge v_1 - \beta$, we deduce that there exist $u, v, w, u', v', w' \in V_7$ such that

$$\omega = v_P \wedge v_Q \wedge v_1 + v_P \wedge (v_1 \wedge u + v \wedge w) + v_Q \wedge (v_1 \wedge u' + v' \wedge w') + \sigma.$$

Generically, $v_P, v_Q, v_1, u, v, w, u', v', w'$ is a basis of V_9. Note that $\omega(P, R, \cdot)$ is the contraction of $v_Q \wedge v_1 + v_1 \wedge u + v \wedge w$ by R (considered as a linear form). In particular it will be proportional to v_1 if and only if v_1, v, w belong to the hyperplane R. Similarly $\omega(Q, R, \cdot)$ is generated by v_1 if and only if v_1, v', w' belong to R. So $R \supset U_5 = \langle v, w, v', w', v_1 \rangle$. If we let $U_4 = \langle v_P, v_Q, u, u' \rangle$, we are thus looking for $R \in \mathbf{P}(U_4^\vee) \cong \mathbf{P}^3$ such that $\omega(R, \cdot, \cdot)$ has rank four.

Let us decompose σ further with respect to the decomposition $V_7 = \mathbb{C}u \oplus \mathbb{C}u' \oplus U_5$: there exist $\sigma_0 \in U_5$, $\sigma_u, \sigma_{u'} \in \wedge^2 U_5$ and $\tilde{\sigma} \in \wedge^3 U_5$ such that

$$\sigma = u \wedge u' \wedge \sigma_0 + u \wedge \sigma_u + u' \wedge \sigma_{u'} + \tilde{\sigma}.$$

As a consequence, we get

$$\omega(R, \cdot, \cdot) = a \wedge v_1 + b \wedge \sigma_0 + \tau,$$

where $a = R(v_P + u')v_Q + R(u - v_Q)v_P - R(v_P)u - R(v_Q)u'$ and $b = R(u)u' - R(u')u$ belong to U_4, while

$$\tau = R(v_P)v \wedge w + R(v_Q)v' \wedge w' + R(u)\sigma_u + R(u')\sigma_{u'}$$

belongs to $\wedge^2 U_5$ (here again we denoted by the same letter R a linear form whose kernel is the hyperplane R). If a and b are dependent and for example $b = 0$, we need that $R(u) = R(u') = 0$. Then $\omega(R, \cdot, \cdot) = a \wedge v_1 + R(v_P)v \wedge w + R(v_Q)v' \wedge w'$ never has rank four or less, unless it is zero. The case where $b \neq 0$ is similar.

If a and b are independent, since generically v_1 and σ_0 are independent, the only way for $\omega(R, \cdot, \cdot)$ to have rank at most four is that $v_1 \wedge \sigma_0 \wedge \tau = 0$. Since the map $v_1 \wedge \sigma_0 \wedge : \wedge^2 V_5 \longrightarrow \wedge^4 V_5$ has rank three, this yields three linear conditions that determine $(R(v_P), R(v_Q), R(u), R(u'))$ uniquely up to scalar. So the hyperplane R is uniquely determined. □

The point $[\omega(P, Q, \cdot)] = [V_1] \in D_{Y_4}$ should really be thought of as the line joining P and Q, in analogy to the line joining two points on a plane cubic, and that defines a unique third point. From this perspective, the space of "lines" is D_{Y_4}, or rather its desingularization $Z = \mathrm{Kum}^2(A)$.

Once we have chosen an origin O of A, exactly as for plane cubics we can then recover the group structure on A by applying Proposition 6.1 twice: starting from two general points $P, Q \in A$, we first find the point R such that (P, Q, R) belongs to Σ; then from the two points $O, R \in A$, we deduce the sum $P + Q \in A$.

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