UNIVERSITÀ DI PARMA
ARCHIVIO DELLA RICERCA

University of Parma Research Repository

Multi-population Mean Field Games systems with Neumann boundary conditions

This is the peer reviewd version of the followng article:

Original
Multi-population Mean Field Games systems with Neumann boundary conditions / Cirant, Marco Alessandro. - In: JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. - ISSN 0021-7824. - 103:5(2015), pp. 1294-1315. [10.1016/j.matpur.2014.10.013]

Availability:
This version is available at: 11381/2857622 since: 2019-04-02T14:11:13Z

Publisher:
Elsevier Masson SAS

Published
DOI:10.1016/j.matpur.2014.10.013

Terms of use:
openAccess

Anyone can freely access the full text of works made available as "Open Access". Works made available

Publisher copyright

(Article begins on next page)

19 November 2022

Multi-population Mean Field Games systems with Neumann boundary conditions

Marco Cirant

Dipartimento di Matematica "F. Enriques"
Università di Milano
Via C. Saldini, 50
20133–Milano, Italy

Abstract

In this paper we discuss some existence and uniqueness results for multi-population stationary Mean Field Games systems with Neumann conditions at the boundary. We prove the existence of solutions through fixed-point and approximation arguments, assuming that the Hamiltonian functions are super-linear with respect to the gradient entry and the costs are regularizing functionals or local functions of the distributions. In the latter case we require uniform boundedness or some growth conditions on the costs, which assure that suitable a-priori estimates hold. We propose a sufficient hypothesis for uniqueness of solutions and some examples where multiplicity of solutions arises.

Keywords: Mean Field Games, Multi-population models, Ergodic stochastic control
2010 MSC: 35J47, 49N70, 35B45

1. Introduction

The objective of this paper is to study existence and uniqueness of solutions of the multi-population ergodic Mean Field Games system with Neumann boundary conditions

$$\begin{cases} -\nu_i \Delta u_i(x) + H^i(x, Du_i(x)) + \lambda_i = V^i[m](x), & \forall x \in \Omega \quad hj), \\ -\nu_i \Delta m_i(x) - \operatorname{div}(D_p H^i(x, Du_i(x)) m_i(x)) = 0 & \forall x \in \Omega \quad k), \\ \partial_n u_i(x) = 0, & \forall x \in \partial\Omega \quad hjn), \\ \nu_i \partial_n m_i(x) + m_i D_p H^i(x, Du_i(x)) \cdot n(x) = 0 & \forall x \in \partial\Omega \quad kn), \end{cases} \tag{MFG}$$

Email address: marco.cirant@unimi.it (Marco Cirant)

$i = 1, \dots, M$. Such a non-linear elliptic system appears within the theory of Mean Field Games (briefly MFG), a new branch of Dynamic Games proposed independently by Lasry, Lions [18, 19, 20] and Caines, Huang, Malhamé [16] which aims at modeling and analyzing complex decision processes involving a very large number of indistinguishable rational agents.

The system (MFG) captures equilibria of M populations of agents aiming at minimizing some long-time average criterion. Indeed, solutions of (MFG) provide optimal strategies for players of every population $i = 1, \dots, M$, who pay a cost that is proportional to their velocity and some function V^i that depends on their positions and the distributions $m_1, \dots, m_M$ of the populations. The state space of players is a bounded domain $\Omega \subset \mathbb{R}^d$ and their trajectories solve stochastic differential equations involving a Brownian diffusion $\sqrt{2\nu_i} B^i_t$ and reflection at the boundary $\partial\Omega$. The standard Dynamic Programming Principle associates the system of Hamilton-Jacobi-Bellman equations *hj)* with Neumann conditions *hjn)* to the infinite-horizon minimization problem. Kolmogorov equations *k)*, *kn)* characterize the (invariant) distributions of the populations, where every player chooses to implement the equilibrium strategy $-D_p H^i(\cdot, Du_i(\cdot))$. We point out that system (MFG) can be derived by taking the limit of Nash equilibria of games with NM players with suitable symmetry assumptions. This result has been obtained in [20] in the one population case and in [11] for $M \geq 2$, considering the torus as the state space.

In this paper we drop the assumption of periodicity, which is usually adopted in the MFG literature to avoid technical issues, and move to the setting of reflecting boundary, for which, to the best of our knowledge, no existence results are available; Achdou and Capuzzo-Dolcetta carried out numerical analysis on stationary models in [1, 2] with such conditions at the boundary.

Our aim is also to provide a general existence framework for the multi-population case. For $M \geq 2$, Feleqi [11] treated the periodic case with regularizing costs; Lachapelle and Wolfram [17] studied some two-population non-stationary models describing congestion in pedestrian crowds. In between the one-population and the multi-population setting, we mention that in [15, 24] it is considered a MFG where a population of "minor" agents interacts with a single "major" agent. For general theory of MFG see also [7, 8] and [21], where many techniques for MFG systems are developed.

We will make use of some well known result on Hamilton-Jacobi-Bellman and Kolmogorov equations, that will be presented in Section 2. The minimal assumptions (H) on the Hamiltonians

H^i are satisfied by functions which are superlinear with respect to the gradient entry, i.e.

$$H^i(x,p) = -b(x) \cdot p + R|p|^\gamma - H_0(x) \tag{1}$$

for some $R > 0$, $b, H_0 \in C^2(\overline{\Omega})$. In Section 3 we prove existence for (MFG) assuming the general (H) and that the costs V^i are regularizing, i.e. they map the set of probability measures with density in $W^{1,p}(\Omega)$ into a bounded set of $W^{1,\infty}(\Omega)$, and they are continuous with respect to the uniform convergence on $\overline{\Omega}$. The arguments in this case are quite standard and exploit the fixed point structure of the system: a solution u of *hj)*, *hjn)* for a given m is plugged into *k)*, *kn)* to produce a new vector of distributions μ. Once continuity of the map $m \mapsto \mu$ is verified, a fixed point is found by means of Schauder theorem.

In Section 4 we do not require anymore the costs to be regularizing with respect to the vector of distributions, and assume on the other hand V^i to be local functions of m, i.e. $V^i[m](x) = V^i(m(x))$. This case is much more tricky, as standard elliptic estimates are not sufficient in general for carrying over a fixed-point argument. In [9] existence is proved in the case $M = 1$ and with quadratic Hamiltonian by exploiting the Hopf-Cole change of variables, without any growth assumption on the cost; for more general Hamiltonians, a-priori estimates on solutions are presented in [14]. A full proof of existence for general MFG systems in the one population case and periodic space is provided in [13], where more general estimates are supported by a continuation argument. However, some growth conditions of V with respect to m are required.

Our approach relies in passing to the limit in approximating problems with smoothing costs, for which existence is proved in Section 3. Crucial a-priori estimates are obtained by exploiting the adjoint structure of (MFG) as in [14], Bernstein methods and fine bounds on the solutions of Kolmogorov equations, with the requirement that H^i are precisely of the form (1) (with $b \equiv 0$). We point out that the multi-population case differs from the single-population one as growth with respect to every m_i plays a role. In our hypotheses a precise behavior from above and from below of V^i is prescribed. We are also able to treat the case of local continuous costs that are uniformly bounded with respect to m. We finally present some examples of costs for which the existence results apply.

Section 5 is devoted to uniqueness of solutions of (MFG). The argument by Lasry and Lions presented in [20] is adapted to the multi-population case, where non-uniqueness of solutions has to be expected in general; we present indeed some examples of MFG systems that admit multiple solutions. In particular, we construct through a variational argument some "segregated" solutions of a two-population MFG system with aversion.

We finally observe that the natural sufficient condition for uniqueness (see (24)) is related to the hypothesis for existence with local unbounded costs ($\mathrm{V_{LU}}$) that we propose in Section 4, suggesting that such assumptions guarantee a "convex" structure of the problem (see Remark 15).

Acknowledgements

The author is very grateful to Prof. Martino Bardi for his precious advices during the preparation of this work and to the anonymous referee for his careful reading and suggestions that led to improve the manuscript.

2. Preliminaries

Throughout the paper Ω will be a C^2 bounded domain of $\mathbb{R}^d$. We will denote by $n(x)$ the outer normal vector at $x \in \partial\Omega$ and by $Df, Jf, D^2 f$ the gradient, Jacobian matrix and Hessian matrix of a function f, respectively. Moreover,

$$C_{\rm av}(\Omega) := \left\{ u \in C(\Omega) : \int_\Omega u = 0 \right\}, \quad \mathcal{P} = \left\{ m \in L^1(\Omega) : \int_\Omega m = 1 \right\}.$$

If one considers $V^i[m]$ as a function of x, i.e. m is frozen and $V^i(x) = V^i[m](x)$, equation (MFG), *hj)* is a stationary Hamilton-Jacobi-Bellman equation, for which existence and uniqueness of solutions (u_i, λ_i) are well understood. Suppose that the following set of hypotheses on the Hamiltonian function H holds:

(H) 1. $H \in C^1(\overline{\Omega} \times \mathbb{R}^d)$. There exist $B \in W^{1,\infty}(\Omega, \mathbb{R}^d), C > 0$, such that

$$H(x,p) \geq B(x) \cdot p - C \quad \forall x \in \Omega, p \in \mathbb{R}^d.$$

2. For all $x \in \Omega, p \in \mathbb{R}^d$

$$D_p H(x,p) \cdot p - H(x,p) \geq -C$$

for some $C > 0$.

3. There exist $\mu > 0$, $\theta \in (0,1)$, $R_0 > 0$ such that

$$D_x H \cdot p + \frac{\theta}{d} H^2 + \mu |p|^2 + \mu [D_p H \cdot p - H] > 0$$

for all $|p| \geq R_0$, $x \in \Omega$, and

$$D_x H \cdot p + \frac{\theta}{d} H^2 + \mu |p|^2 + \mu [D_p H \cdot p - H] + t|p|^2 [D_p H \cdot n(x)] > 0$$

for all $|p| \geq R_0$, x in a neighborhood of $\partial\Omega$ and $t \in [0, C_0]$, where C_0 is the maximum among the negative part of the principal curvatures of $\partial\Omega$.

P.-L. Lions proved in [22] the following result.

Theorem 1. *Suppose that Ω is a bounded C^2 domain, $f \in W^{1,\infty}(\Omega)$, $\|f\|_{W^{1,\infty}(\Omega)} \leq \alpha$ for some $\alpha > 0$ and that (H) holds. Then, there exist unique $u \in C^2(\overline{\Omega}) \cap C_{\text{av}}(\Omega)$ and $\lambda \in \mathbb{R}$ that solve*

$$-\nu\Delta u(x) + H(x, Du(x)) + \lambda = f(x) \ \text{ in } \Omega, \quad \partial_n u(x) = 0 \ \text{ on } \partial\Omega.$$

Moreover,

$$|\lambda| \leq \|f\|_{L^\infty(\Omega)} \tag{2}$$

and

$$\|u\|_{W^{1,\infty}(\Omega)} \leq C \tag{3}$$

where C is a positive constant that depends on Ω, H and α.

Proof. See [22], Theorem II.1. □

A solution belonging to $\mathcal{P}$ of equation (MFG), *k)* with boundary conditions *kn)* is the density of the invariant measure of the optimal stochastic process driven by drift $b(x) := -D_p H^i(x, Du_i(x))$ with reflection at the boundary. For such an equation existence and uniqueness is established as well, at least under some regularity assumptions on b. We recall a result which will be used in the sequel.

Theorem 2. *Suppose that Ω is a C^2 bounded domain and $b \in L^\infty(\Omega, \mathbb{R}^d)$. Then, there exists a unique (weak) solution $m \in W^{1,2}(\Omega)$ of*

$$\nu \int_\Omega Dm \cdot D\phi = \int_\Omega m \, b \cdot D\phi \quad \forall \phi \in W^{1,2}(\Omega). \tag{4}$$

Moreover, $m \in W^{1,p}(\Omega)$ for all $p \geq 1$, $m \in C(\overline{\Omega})$ and it satisfies

$$\delta^{-1} \leq m(x) \leq \delta \quad \forall x \in \overline{\Omega}$$

for some $\delta > 0$ depending only on $\|b\|_{L^\infty(\Omega)}$.

Proof. See [6], Theorems II.4.4, II.4.5, II.4.7. □

Throughout the paper, V^i will be functionals that map $C^0(\Omega)$ functions into $C^0(\Omega)$ functions. In this situation, we may consider the following definition of solution of (MFG).

Definition 3. A solution of (MFG) will be a $3M$-uple (u, λ, m) such that

$$u_i \in C_{\rm av}(\Omega) \cap C^1(\overline{\Omega}), \quad \lambda_i \in \mathbb{R}, \quad m_i \in \mathcal{P} \cap W^{1,2}(\Omega) \quad \forall i = 1, \dots, M,$$

u_i, λ_i solve *hj)* in the standard viscosity sense, *hjn)* is satisfied pointwise and m_i solves *k)*, *kn)* in the weak (distributional) sense, namely

$$\nu_i \int_\Omega Dm_i \cdot D\phi + \int_\Omega m_i \, D_p H^i(x, Du_i(x)) \cdot D\phi = 0 \quad \forall \phi \in W^{1,2}(\Omega),\ i = 1, \dots, M.$$

Since $Du_i \in L^\infty(\Omega, \mathbb{R}^d)$ and $D_p H$ is continuous, $D_p H^i(\cdot, Du_i(\cdot)) \in L^\infty(\Omega, \mathbb{R}^d)$ and Theorem 2 let us conclude that m_i are continuous functions on Ω. As $V^i[m]$ are also continuous, the standard notion of viscosity solution can be applied to the Hamilton-Jacobi-Bellman equations *hj)*.

3. Existence in the non-local case.

In this section we prove a result on the existence of solutions to (MFG) under the assumption that the costs V^i are regularizing on the set of $W^{1,p}$ probability measures. We provide the full details of the proof, which exploits the fixed-point structure of the system and standard elliptic estimates. The delicate part of the argument is to verify the continuity of the operator defined on the set of probability measures, and manage the presence of Neumann boundary conditions.

Theorem 4. *Suppose that every H^i satisfies (H) and V^i is such that*

($\mathbf{V_{NL}}$)
1. *For some $\alpha > 0, p > d$, $\|V^i[m]\|_{W^{1,\infty}(\Omega)} \le \alpha$ for all $m \in [W^{1,p}(\Omega)]^M \cap \mathcal{P}^M$,*
2. *$m^{(n)} \to m$ uniformly on $\overline{\Omega} \Rightarrow V^i[m^{(n)}] \to V^i[m]$ uniformly on $\overline{\Omega}$.*

Then, there exists a solution (u, λ, m) of (MFG)*. Moreover $u_i \in C^2(\overline{\Omega})$ and $m_i \in W^{1,p}(\Omega)$ for all $p \ge 1$.*

Proof. Let $m_i \in W^{1,p}(\Omega) \cap \mathcal{P}$, $i = 1, \dots, M$ be fixed, where p is the constant that appears in ($\mathrm{V_{NL}}$) , and $F_i(x) := V^i[m_1, \dots, m_M](x)$; by Proposition 1 there exist solutions $(v_i, \lambda_i) \in (C^2(\overline{\Omega}) \cap C_{\rm av}(\Omega)) \times \mathbb{R}$, of

$$-\nu_i \Delta v_i + H^i(x, Dv_i) + \lambda_i = F_i(x) \ \text{ in } \Omega, \quad \partial_n v_i(x) = 0 \ \text{ on } \partial\Omega, \tag{5}$$

for all $i = 1, \dots, M$, together with estimates $\|v_i\|_{W^{1,\infty}(\Omega)} \leq C_i$, where the constants do not depend on m. If we let $b^i(x) := -D_p H^i(x, Dv_i)$, Proposition 2 guarantees existence (and uniqueness) of (weak) solutions $\mu_i \in W^{1,p}(\Omega) \cap \mathcal{P}$ of

$$-\nu_i \Delta \mu_i - \operatorname{div}(D_p H^i(x, Dv_i)\mu_i) = 0$$

satisfying Neumann boundary conditions, with estimates $\|\mu_i\|_{W^{1,p}(\Omega)} \leq \hat{C}_i$. Let now

$$K_i := \mathcal{P} \cap \{m \in W^{1,p}(\Omega)) : ||m||_{W^{1,p}(\Omega)} \leq \hat{C}_i\},$$

one has $\mu_i \in K_i$ independently on the initial m chosen, as V^i is uniformly bounded by hypothesis. It is consequently well defined the map

$$\Gamma : (m_i, \dots, m_N) \in \mathcal{K} := K_1 \times \cdots \times K_M \mapsto (v_1, \dots, v_M) \mapsto (\mu_1, \dots, \mu_M) \in \mathcal{K}. \tag{6}$$

Being every K_i compactly imbedded in $C(\overline{\Omega})$ ([12], Theorem 7.26), by showing that Γ is continuous (with respect to the standard $C(\overline{\Omega}) \times \cdots \times C(\overline{\Omega})$ topology), it is possible to apply the Schauder theorem ([12], Theorem 11.1) and obtain a fixed point of Γ. A fixed point $(m_1, \dots, m_M)$ will be a solution of (MFG), together with $(u_1, \dots, u_M)$ obtained by solving the Hamilton-Jacobi-Bellman equations.

Let $\{m^{(n)}\}$ be a sequence in $\mathcal{K}$ converging uniformly to some $m \in \mathcal{K}$; we first want to show that $v^{(n)} \to v$. Each $v_i^{(n)}$ solve

$$-\nu_i \Delta v_i^{(n)} + H^i(x, Dv_i^{(n)}) + \lambda_i^{(n)} = V^i[m^{(n)}](x) \quad \text{on } \Omega, \tag{7}$$

while each v_i is a solution of

$$-\nu_i \Delta v_i + H^i(x, Dv_i) + \lambda_i = V^i[m](x) \quad \text{on } \Omega; \tag{8}$$

we also know, thanks to (2) and (3), that the constants λ_i and $\lambda_i^{(n)}$ are bounded in absolute value by α, and $\|v_i^{(n)}\|_{W^{1,\infty}(\Omega)}, \|v_i\|_{W^{1,\infty}(\Omega)} \leq C_i$. We now consider any uniformly convergent subsequence $(v_1^{(n)}, \dots, v_M^{(n)}) \to (\bar{v}_1, \dots, \bar{v}_M)$ (by Ascoli-Arzelà there exists at least one). We begin by proving that $\lambda_i^{(n)} \to \lambda_i$ (reasoning as in [3]): fix $i = 1$ and consider some further converging subsequence $\lambda_1^{(n)} \to \bar{\lambda}_1$. Suppose by contradiction that $\bar{\lambda}_1 \neq \lambda_1$. Since by hypothesis $V^i[m^{(n)}] \to V^i[m]$ uniformly, we deduce that $\bar{v}_1$ is a solution in the viscosity sense of the limit equation

$$-\nu_1 \Delta \bar{v}_1 + H^1(x, D\bar{v}_1) + \bar{\lambda}_1 = V^1[m](x) \quad \text{on } \Omega,$$

with Neumann boundary conditions satisfied in generalized sense. Without loss of generality $\bar{\lambda}_1 > \lambda_1$ and $\bar{v}_1(y) > v_1(y)$ at some $y \in \Omega$, possibly adding a positive constant to $\bar{v}_1$; hence, there exists $\delta > 0$ such that

$$\begin{aligned} -\nu_1 \Delta \bar{v}_1 + H^1(x, D\bar{v}_1) - V^1 + \delta \bar{v}_1 = \delta \bar{v}_1 - \bar{\lambda}_1 \\ \leq \delta v_1 - \lambda_1 = -\nu_1 \Delta v_1 + H^1(x, Dv_1) - V^1 + \delta v_1. \end{aligned}$$

By comparison principle ([10]) it follows that $\bar{v}_1 \leq v_1$ in Ω, that is a contradiction. So, $\bar{\lambda}_1 = \lambda_1$ and $\lambda_1^{(n)} \to \lambda_1$.

We now show that $\bar{v}_1 = v_1$. By subtracting (8) to (7), we obtain

$$\begin{aligned} V^1[m^{(n)}](x) - V^1[m](x) - (\lambda_1^{(n)} - \lambda_1) \\ = -\nu_1 \Delta(v_1^{(n)} - v_1) + H^1(x, Dv_1^{(n)}) - H^1(x, Dv_1) \\ \geq -\nu_1 \Delta(v_1^{(n)} - v_1) + \frac{\partial H^1}{\partial p}(x, \xi)(Dv_1^{(n)} - Dv_1) \geq \\ -\nu_1 \Delta(v_1^{(n)} - v_1) - \left| \frac{\partial H^1}{\partial p}(x, \xi) \right| |Dv_1^{(n)} - Dv_1| \geq \\ -\nu_1 \Delta(v_1^{(n)} - v_1) - C|Dv_1^{(n)} - Dv_1|. \end{aligned}$$

where $\xi = \xi(x) \in [Dv_1^{(n)}(x), Dv_1(x)]$ and $C = \sup_{x \in \Omega, |\xi| \leq C_1} |\frac{\partial H^1}{\partial p}(x, \xi)|$. Set $w_1^{(n)} = v_1^{(n)} - v_1$, taking the limit as $n \to +\infty$, $w_1^{(n)} \to w_1 = \bar{v}_1 - v_1$ uniformly,

$$\begin{cases} -\nu_1 \Delta w_1 - C|Dw_1| \leq 0 & \text{on } \Omega \\ \frac{\partial w_1^{(n)}}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

again with Neumann boundary conditions that have to be intended in generalized viscosity sense. We want w_1 to be everywhere constant, and we suppose that w_1 reaches its maximum (that we may assume to be positive, by eventually adding a positive constant to w_1) on $\overline{\Omega}$ at a point *inside* the domain: in this case, the strong maximum principle in [4] implies that w_1 is constant. Furthermore, if that maximum was reached at some $x \in \partial\Omega$ we would have a contradiction (as in the proof of Theorem 2.1 in [5]); indeed, letting $M = u(x)$, we would have $u(y) < M$ for every $y \in \Omega$. We know that there exist $r > 0$ and a smooth function ϕ such that [1]

$$\begin{cases} -\nu_1 \Delta\phi - C|D\phi| > 0 & \text{su } B_r(x) \\ \frac{\partial\phi}{\partial n} > 0 & \text{su } \partial\Omega \cap B_r(x), \end{cases} \tag{10}$$

where $\phi(x) = 0$ and $\phi(y) > 0$ for all $y \in B_r(x) \cap \overline{\Omega} \setminus \{x\}$. The point x would be a local maximum of $w_1 - \phi$, that is impossible by (10) and the definition of viscosity subsolution, so $\bar{v}_1 - v_1$ is constant on Ω and $\bar{v}_1, v_1 \in C_{\rm av}(\Omega)$, hence $\bar{v}_1 = v_1$; by the same argument $\bar{v}_i = v_i$ for $i = 2, \ldots, M$.

Since the limit $\bar{v}$ is unique, we deduce that the entire sequence $(v_1^{(n)}, v_2^{(n)})$ converges to v. Let now $\mu_i^{(n)} \in W^{1,\infty}(\Omega) \cap \mathcal{P}$ be solutions of

$$-\nu_i \Delta\mu_i^{(n)} - \mathrm{div}(D_p H^i(x, Dv_i)\mu_i^{(n)}) = 0.$$

We prove (to obtain the continuity of Γ) that $\mu_i^{(n)} \to \mu_i$ uniformly, where $\mu = \Gamma(m)$. Notice that

$$\nu_i \Delta v_i^{(n)} = G_i(x) := H^i(x, Dv_i^{(n)}) + \lambda_i^{(n)} - V^i[m^{(n)}](x),$$

and the estimate $\|G_i\|_{L^\infty(\Omega)} \leq \hat{C}$ holds for some $\hat{C} > 0$ independent on n, hence, by standard $C^{1,\alpha}$ interior elliptic estimates (see for example [12], theorem 8.32) one has $\|v_i^{(n)}\|_{C^{1,\alpha}(\Omega')} < \infty$ on every $\Omega' \subset\subset \Omega$. Fix $i = 1$, then $Dv_1^{(n)} \to Dv_1$ uniformly on compacts in Ω and that easily implies that every converging subsequence $\mu_1^{(n)} \to \bar{\mu}_1$ is a (weak) solution of

$$-\nu_1 \Delta\mu - \mathrm{div}(D_p H^1(x, Dv_1)\mu) = 0,$$

that has μ_1 as a unique solution in $\mathcal{P}$. Similarly, $\mu_i^{(n)} \to \mu_i$ for every $i = 1, \ldots, M$.

□

Example 5. An example of costs satisfying ($\mathrm{V_{NL}}$) is given by

$$V^i[m](x) := W^i(m_1 \star \varphi(x), \ldots, m_M \star \varphi(x)) \star \varphi(x) \quad \forall x \in \overline{\Omega},$$

where $W^i \in C^0(\mathbb{R}^M)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a regularizing kernel. We recall that

$$m_i \star \varphi(x) := \int_\Omega \varphi(x-y) m_i(y) dy \quad \forall x \in \overline{\Omega},$$

[1] Take, for example, $\phi(x) = e^{-\rho s^2} - e^{-\rho|x-x_0|^2}$, with $\rho > 0$ large enough and $B_s(x_0)$ the external sphere $\overline{\Omega}$ at x.

so

$$|m_i \star \varphi(x)| \le (\sup_{\mathbb{R}^d} \varphi) \|m_i\|_{L^1(\Omega)} = (\sup_{\mathbb{R}^d} \varphi) \quad \forall x \in \Omega, m \in \mathcal{P}.$$

Moreover, $D(m_i \star \varphi) = m_i \star (D\varphi)$, hence

$$|D(m_i \star \varphi)(x)| \le (\sup_{\mathbb{R}^d} D\varphi) \|m_i\|_{L^1(\Omega)} = (\sup_{\mathbb{R}^d} D\varphi) \quad \forall x \in \Omega, m \in \mathcal{P}.$$

The two inequalities show that $(\mathrm{V_{NL}})$, 1. holds.

Given $m^{(n)} \to m$ uniformly on $\overline{\Omega}$, we have that

$$|(m_i^{(n)} - m_i) \star \varphi(x)| \le \int_\Omega |\varphi(x-y)(m_i^{(n)}(y) - m_i(y))| dy$$
$$\le \|\varphi\|_{L^1(\mathbb{R}^d)} \sup_{\overline{\Omega}} |(m_i^{(n)} - m_i)(y)| \to 0,$$

as $n \to \infty$ uniformly w.r.t $x \in \overline{\Omega}$, so also $(\mathrm{V_{NL}})$, 2. holds.

We observe that $V^i[m] \in C^\infty(\Omega)$, so a solution (u, m) of (MFG) produced by Theorem 4 belongs a-posteriori to $C^\infty(\Omega) \times C^\infty(\Omega)$ by standard elliptic regularity.

4. Existence in the local case.

In this section we focus on existence of solutions of (MFG) when the smoothing assumption on the costs V^i is dropped. Regularity of $V^i[m]$ uniform with respect to the vector of distributions m plays a substantial role in the proof of Theorem (4), providing strong a-priori estimates for solutions. We establish suitable estimates in the local case, exploiting the structure of the system and assuming boundedness *or* monotonicity of V^i.

Throughout this section the hamiltonians H^i will have the form

$$\begin{aligned} &H^i(x,p) = R_i|p|^\gamma - H_0^i(x), \quad R_i > 0, \gamma > 1, \\ &\partial_n H_0^i(x) \ge 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{11}$$

for some potential function $H_0^i \in C^2(\overline{\Omega})$. Although we believe that perturbations of such Hamiltonians could be considered in the subsequent proofs, we assume (11) to simplify the computations and focus on the main features of the problem.

In the first result we assume that V^i are continuous and bounded functions.

Theorem 6. *Let Ω be a C^2 convex domain. If H^i has the form* (11) *and the costs V^i satisfy*

$$(\mathbf{V_{LB}}) \qquad V^i \in C^0(\mathbb{R}^M), \quad |V^i(m)| \le L \quad \forall m \in \mathbb{R}^M, i = 1, \dots, M$$

for some $L > 0$, then there exists a solution (u, λ, m) of (MFG). *Moreover $u_i \in C^{1,\delta}(\overline{\Omega})$ and $m_i \in W^{1,p}(\Omega)$ for some $0 < \delta < 1$ and for all $p \ge 1$.*

The second existence assertion of this section requires a growth assumption on V^i with respect to m from above and from below.

Theorem 7. *Let Ω be a C^2 convex domain. If H^i has the form* (11) *and $V^i \in C^1(\mathbb{R}^M)$ satisfy for all $m_i \ge 0$*

$$(\mathbf{V_{LU}}) \quad \begin{array}{l} i)\ \alpha \sum_{i=1}^M m_i^\gamma |v^i|^2 \le \sum_{i,j=1}^M \partial_{m_j} V^i(m) v^i \cdot v^j \text{ for all } v \in [\mathbb{R}^d]^M, \\ ii)\ -\overline{V} \le V^i(m) \le D(1 + \sum_{i=1}^M m_i^\eta), \end{array}$$

for some $\gamma \ge -1$ and $\alpha, \eta, D, \overline{V} > 0$ such that

$$\eta < \begin{cases} (\gamma + 2)/(d-2) & \text{if } d \ge 3, \\ +\infty & \text{else.} \end{cases}$$

Then there exists a solution (u, λ, m) of (MFG). *Moreover, $u_i \in C^{2,\delta}(\Omega) \cap C^1(\overline{\Omega})$ and $m_i \in W^{1,p}(\Omega)$ for some $0 < \delta < 1$ and for all $p \ge 1$.*

Before going into the details of proofs of Theorems 6 and 7 we provide some examples and state the a-priori estimates that will be needed.

Remark 8. Condition $(\mathrm{V_{LU}})$, *i)* holds when

$$\Phi + \Phi^T \text{ is positive semi-definite,} \tag{12}$$

where

$$\Phi := JV - \begin{bmatrix} \alpha m_1^\gamma & & 0 \\ & \ddots & \\ 0 & & \alpha m_M^\gamma \end{bmatrix} \tag{13}$$

JV denoting the Jacobian matrix of V. Indeed,

$$\sum_{i,j=1}^M \partial_{m_j} V^i(m) v^i \cdot v^j - \alpha \sum_{i=1}^M m_i^\gamma |v^i|^2 = \mathrm{tr}(\Phi W) \ge 0$$

where $W_{ij} = v^i \cdot v^j$, since (12) holds and V is symmetric and positive semi-definite for all $v^i, v^j \in \mathbb{R}^d$.

Moreover, suppose that for some $K > 0$ condition ($\mathrm{V_{LU}}$) is true for all $v \in [\mathbb{R}^d]^M$, $m_i \geq K$ for some i (not for all m in general) and in addition we have that

$$0 \leq \sum_{i,j=1}^{M} \partial_{m_j} V^i(m) v^i \cdot v^j$$

for all $v \in [\mathbb{R}^d]^M$, $m_1, \dots, m_M < K$. Then, the existence assertion of Theorem 7 still holds. Assumption ($\mathrm{V_{LU}}$) is indeed crucial when m becomes large.

Example 9. In the single population framework ($M = 1$), a typical model of unbounded cost is $V \in C^1(\mathbb{R})$, $V'(m) \geq 0$ for all $m \geq 0$ and

$$V(m_1) = m_1^\beta, \quad \beta > 0, m_1 \geq 1.$$

In this case, setting $\gamma = \beta - 1$ and $\eta = \beta$, if $d \geq 3$ the coefficients have to satisfy $\beta < \frac{\beta+1}{d-2}$. A sufficient condition for ($\mathrm{V_{LU}}$) turns out to be

$$\beta < 1/(d-3) \quad \text{if } d \geq 4,$$

and $\beta < \infty$ otherwise.

Example 10. If we now let

$$\begin{aligned} V^1(m_1, m_2) &= am_1 + bm_2, \\ V^2(m_1, m_2) &= cm_1 + am_2, \end{aligned} \tag{14}$$

with $a, b, c \in \mathbb{R}$ the mean field system can be interpreted as two populations interacting that behave in the same way with respect to themselves; ($\mathrm{V_{LU}}$) is then satisfied if

$$a > 0, \quad b, c \geq 0, \quad b + c < 2a.$$

if the space dimension is $d \leq 2$. In higher space dimension a sublinear growth with respect to m has to be required, but still monotonicity of V^i with respect to m_i should be leading.

Existence for (MFG) under the assumptions of Theorems 6, 7 is carried out by taking the limit as $\epsilon \to 0$ of the approximating problems

$$\begin{cases} -\nu_i \Delta u_{\epsilon,i}(x) + H^i(x, Du_{\epsilon,i}(x)) + \lambda_{\epsilon,i} = V^i_\epsilon[m_\epsilon](x), & \forall x \in \Omega \qquad i), \\ -\nu_i \Delta m_{\epsilon,i}(x) - \mathrm{div}(D_p H^i(x, Du_{\epsilon,i}(x)) m_{\epsilon,i}(x)) = 0 & \forall x \in \Omega \qquad ii), \\ \partial_n u_{\epsilon,i}(x) = 0 & \forall x \in \partial\Omega, \\ \partial_n m_{\epsilon,i}(x) = 0 & \forall x \in \partial\Omega. \end{cases} \tag{15}$$

where

$$V^i_\epsilon[m](x) := V^i(m_1 \star \varphi_\epsilon(x), \dots, m_M \star \varphi_\epsilon(x)) \star \varphi_\epsilon(x) \quad \forall x \in \overline{\Omega},$$

$\varphi_\epsilon(x) := \epsilon^{-d}\varphi(x/\epsilon)$, $\epsilon > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ is mollifier, i.e. $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$. We also require φ to be radial. We know that a solution of (15) exists for all $\epsilon > 0$ by virtue of Theorem 4 (see also Example 5). We present now some a-priori estimates for solutions of (15), independent on ϵ.

Lemma 11. *Let Ω be convex, $(u, \lambda, m) \in [C^3(\Omega)]^M \times \mathbb{R}^M \times [C^2(\overline{\Omega})]^M$ be a solution of* (15) *and let H^i be of the form* (11)*. Then,*

$$\sum_{i,j=1}^M \int_\Omega \partial_{m_j} V^i(m \star \varphi_\epsilon) D(m_i \star \varphi_\epsilon) \cdot D(m_j \star \varphi_\epsilon) dx \leq C, \tag{16}$$

for some $C = C(\nu, H^1, \dots, H^M, \Omega) > 0$.

Proof. We consider first the equations for population $i = 1$, apply the gradient operator D to equation *i)*, multiply it by Dm_1 and integrate over the domain to get

$$-\nu_1 \int_\Omega D(\Delta u_1) \cdot Dm_1 + \int_\Omega D(H^1(x, Du_1)) \cdot Dm_1 = \int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon) \star \varphi_\epsilon) \cdot Dm_1.$$

By integrating by parts the second term of the left hand side and using the Kolmogorov equation for m_1, together with boundary conditions, we obtain

$$-\int_\Omega \mathrm{tr}(D^2_{pp} H^1(x, Du_1)(D^2 u_1)^2) m_1 + R_1 \int_{\partial\Omega} m_1 D(|Du_1|^\gamma) \cdot n = \int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon) \star \varphi_\epsilon) \cdot Dm_1 + \int_\Omega DH^1_0 \cdot Dm_1 \tag{17}$$

Convexity of Ω implies that $D(|Du_1(x)|^\gamma) \cdot n(x) \leq 0$ for all $x \in \partial\Omega$. Since by standard properties of the convolution (and the fact that φ_ϵ is radial)

$$\int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon) \star \varphi_\epsilon) \cdot Dm_1 = \int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon)) \cdot D(m_1 \star \varphi_\epsilon),$$

so we integrate by parts the last term in (17), use the assumption (11) and exploit convexity of H^i with respect to p to obtain

$$\int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon)) \cdot D(m_1 \star \varphi_\epsilon) \leq \int_{\partial\Omega} m_1 \partial_n H^1_0 + \int_\Omega D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon)) \cdot D(m_1 \star \varphi_\epsilon) \leq \int_\Omega m_1 \Delta H^1_0 \leq \|\Delta H^1_0\|_{L^\infty(\Omega)}.$$

Hence, by the chain rule applied to the term $D(V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon))$,

$$\sum_j \int_\Omega \partial_{m_j} V^1(m_1 \star \varphi_\epsilon, \dots, m_M \star \varphi_\epsilon) D(m_j \star \varphi_\epsilon) \cdot D(m_1 \star \varphi_\epsilon) \leq \|\Delta H_0^1\|_{L^\infty(\Omega)}.$$

It suffices now to carry out the same computations for equations corresponding to populations $i = 2, \dots, M$ and sum over i to get (16). □

Regarding Kolmogorov equations of type (4), it is known that an estimate on the L^∞ norm of solutions $m \in \mathcal{P}$ follows from an L^∞ bound on the drift b. The next proposition states that for such an estimate to hold, a bound on the L^r norm of b and the L^q norm of m for some $q > 1$ and $r > d$ is sufficient.

Proposition 12. *Let $r > d$, $q > \frac{r}{r-1}$ and suppose that $b \in C(\overline{\Omega})$ satisfies $b \cdot n = 0$ on $\partial\Omega$. Moreover, $\|m\|_{L^q(\Omega)} \leq K$, $\|b\|_{L^r(\Omega)} \leq K$ for some $K > 0$. If m is a solution of (4), then*

$$\|m\|_{L^\infty(\Omega)} \leq C \tag{18}$$

for some $C = C(K, \nu, d, \Omega)$.

Proof. From [25], Theorem 3.1, we know that the following a-priori estimate on m holds:

$$\|m\|_{W^{1,p}(\Omega)} \leq C(\|\nu \Delta m\|_{W^{-1,p}(\Omega)} + \|m\|_{W^{-1,p}(\Omega)}),$$

for all $p > 1$ and a constant C that depends on p, ν, d, Ω. Using equation (4) and Holder inequality, for all test functions $\phi \in C^\infty(\overline{\Omega})$,

$$\left| \int_\Omega \nu \Delta m \, \phi \right| \leq \int_\Omega |mb \cdot D\phi| \leq \|m\|_{L^q(\Omega)} \|b\|_{L^r(\Omega)} \|D\phi\|_{L^{p'}(\Omega)}$$

and similarly

$$\left| \int_\Omega m \, \phi \right| \leq \int_\Omega |m\phi| \leq \|m\|_{L^q(\Omega)} \left(\int_\Omega dx \right)^{1/r} \|\phi\|_{L^{p'}(\Omega)},$$

setting p, p' such that

$$\frac{1}{r} + \frac{1}{q} + \frac{1}{p'} = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

That leads to

$$\|m\|_{W^{1,p}(\Omega)} \leq C,$$

as $p > 1$ by the choice of r, q. Plus, C depends only on K and fixed data of the problem. If $p > d$ we are done by using Sobolev embeddings (by which m is continuous and bounded on the whole domain). Else,

$$\|m\|_{L^{p^*}(\Omega)} \leq C,$$

with $p^* \geq q + \epsilon$ for some $\epsilon > 0$ that does not depend on q by the hypothesis $r > d$. Iterating the last two estimates and setting $q = p^*$ at each time, a bootstrap argument let us conclude in a finite number of steps that

$$\|m\|_{L^\infty(\Omega)} \leq C.$$

□

We are now ready to provide the

Proofs of Theorems 6 and 7. Step 1. Since H^i satisfies *(H)* and $(\mathrm{V_{NL}})$ holds for $V^i_\epsilon[m]$, by Theorem 4 there exists $(u_\epsilon, \lambda_\epsilon, m_\epsilon) \in [C^2(\overline{\Omega})]^M \times \mathbb{R}^M \times [W^{1,p}(\Omega)]^M$ solution of (15). We will denote $(u, \lambda, m) := (u_\epsilon, \lambda_\epsilon, m_\epsilon)$ during the next steps of proof.

Step 2: $\|V^i_\epsilon[m]\|_{L^q(\Omega)} \leq C$ for some $q > d$. This is easily verified if $(\mathrm{V_{LB}})$ holds. If $(\mathrm{V_{LU}})$ holds, we know that if (u, λ, m) is a solution of the system, by hypothesis i) and Lemma 11

$$\alpha \int_\Omega \sum_{i=1}^M (m_i \star \varphi_\epsilon)^\gamma |D(m_i \star \varphi_\epsilon)|^2 \leq C$$

for some $C > 0$; hence

$$\int_\Omega (m_i \star \varphi_\epsilon)^{\frac{d(\gamma+2)}{d-2}} \leq C, \quad i = 1, \dots, M \tag{19}$$

by Sobolev inequality if $d \geq 3$, otherwise $\|m_i \star \varphi_\epsilon\|_{L^p(\Omega)}$ is bounded for every $p \geq 1$ (the positive constant C may vary throughout the proof, but it never depends on ϵ). Moreover,

$$\|V^i_\epsilon[m]\|_{L^q(\Omega)} \leq \|V^i(m \star \varphi_\epsilon)\|_{L^q(\Omega)} \leq D\|1 + \sum (m_i \star \varphi_\epsilon)^\eta\|_{L^q(\Omega)}, \quad i = 1, \dots, M$$

and the last term is bounded by a positive constant for some $q > d$ because of (19) adn $(\mathrm{V_{LU}})$ (if $d \leq 2$ this is true just by requiring V^i to have polynomial growth).

Step 3: $\|Du\|_{L^r(\Omega)} \leq C = C(r)$ for all $r \geq 1$, which follows from Theorem 19. Indeed, the ergodic constants λ_i are bounded from below by $-L + \min_{\overline{\Omega}} H^i_0$ or $-\overline{V} + \min_{\overline{\Omega}} H^i_0$ by maximum principle if ($\mathrm{V_{LB}}$) or ($\mathrm{V_{LU}}$) hold respectively, and the L^q norm of the left hand sides of the Hamilton-Jacobi-Bellman equations (15), *i)* are uniformly bounded due to Step 2. We also obtain that

$$\|H^i(x, Du_i(x))\|_{L^{\bar r}(\Omega)}, \|D_pH^i(x, Du_i(x))\|_{L^{\bar r}(\Omega)} \le C, \quad i = 1, \dots, M \tag{20}$$

with $\bar r$ as large as we need.

Step 4: $\|m\|_{L^\infty(\Omega)} \le C$. We first multiply (15), *ii)* by $\log m_i$ and integrate by parts to get

$$\nu_i \int_\Omega \frac{|Dm_i|^2}{m_i} + \int_\Omega D_pH^i(x, Du_i) \cdot Dm_i = 0,$$

hence

$$\nu_i \int_\Omega \frac{|Dm_i|^2}{m_i} \le \frac{\nu_i}{2} \int_\Omega \frac{|Dm_i|^2}{m_i} + \frac{1}{2\nu_i} \int_\Omega m_i |D_pH^i(x, Du_i)|^2,$$

so

$$\|D\sqrt{m_i}\|^2_{L^2(\Omega)} \le C\|m_i\|_{L^{\bar p/2}(\Omega)} \| \, |D_pH^i(Du_i)|^2\|_{L^{(\bar p/2)'}(\Omega)} \tag{21}$$

for all $\bar p \ge 2$.

We use now the Gagliardo-Nirenberg interpolation inequality (see [23])

$$\|\sqrt{m_i}\|_{L^{\bar p}(\Omega)} \le C(\|D\sqrt{m_i}\|^{1/2}_{L^2(\Omega)} \|\sqrt{m_i}\|^{1/2}_{L^2(\Omega)} + \|\sqrt{m_i}\|^{1/2}_{L^2(\Omega)}),$$

which holds for

$$\frac{1}{\bar p} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{4}$$

if $d \ge 2$ and for all $\bar p \ge 1$ if $d = 1$. Since $\int_\Omega m_i = 1$, it follows that

$$\|m_i\|^{1/2}_{L^{\bar p/2}(\Omega)} \le C(\|D\sqrt{m_i}\|^{1/2}_{L^2(\Omega)} + 1),$$

therefore, adjusting C

$$\|m_i\|^2_{L^{\bar p/2}(\Omega)} \le C(\|D\sqrt{m_i}\|^2_{L^2(\Omega)} + 1).$$

We plug now (21) into the last inequality to conclude that

$$\|m_i\|_{L^{\bar p/2}(\Omega)} \le C, \tag{22}$$

$\bar p$ as above, as $\| \, |D_pH^i(Du_i)|^2\|_{L^{(\bar p/2)'}(\Omega)}$ is bounded by (20). We observe that $\bar p/2 > 1$ for all d.

By choosing $\bar r$ large enough so that $\bar p/2 > \bar r/(\bar r - 1)$, we deduce using Proposition 12 that

$$\|m_i\|_{L^\infty(\Omega)} \le C, \quad i = 1, \dots, M,$$

which produces an uniform upper bound for $\|V^i_\epsilon[m]\|_{L^\infty(\Omega)}$ and consequently for the constants λ_i as well.

Step 5: $\|u\|_{C^{1,\delta'}(\overline{\Omega})}, \|m\|_{W^{1,p}(\Omega)} \leq C$ for some $\delta' > 1$ and for all $p \geq 1$. By Sobolev imbedding theorems $\|u_i\|_{C^0(\overline{\Omega})} \leq C$. Equations (15), *i)* can be rewritten as

$$\nu_i \Delta u_i(x) = H^i(x, Du_i(x)) + \lambda_i - V^i_\epsilon[m](x) \quad \forall x \in \Omega, \tag{23}$$

By standard elliptic regularity it is possible to conclude that

$$\|u\|_{W^{2,q}(\Omega)} \leq C(\|\nu \Delta u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}),$$

and the right hand side is uniformly bounded. Choosing q large enough let us conclude using again Sobolev imbedding theorems that $\|u_i\|_{C^{1,\delta}(\overline{\Omega})} \leq C$. The estimate for $\|m_i\|_{W^{1,p}(\Omega)}$ follows from the fact that $\|Du_i\|_{L^\infty(\Omega)} \leq C$ and Theorem 2.

Convergence as $\epsilon \to 0$ under (V_{LB}) . By virtue of the estimates of Step 5, we can extract subsequences such that

$$\begin{aligned} \lambda_{\epsilon,i} &\to \lambda_i \in \mathbb{R} \\ u_{\epsilon,i} &\to u_i \in C^{1,\delta}(\overline{\Omega}) \\ m_{\epsilon,i} &\to m_i \in C^0(\overline{\Omega}) \\ m_{\epsilon,i} &\rightharpoonup m_i \in W^{1,p}(\Omega) \end{aligned}$$

as $\epsilon \to 0$ for $\delta < \delta'$ and for all $i = 1, \ldots, M$. By uniform convergence of $D(u_\epsilon)_i$ and weak convergence of $(m_\epsilon)_i$, passing to the limit in Kolmogorov equations (15), *ii)* let us conclude that u_i and m_i solve (MFG) *ii)*, *iv)* in weak sense. In order to pass to the limit in the Hamilton-Jacobi-Bellman equations, we need to prove that $V^i_\epsilon[m_\epsilon] \to V^i[m]$ locally uniformly. Indeed,

$$m_{\epsilon,i} \star \varphi_\epsilon - m_i = (m_{\epsilon,i} - m_i) \star \varphi_\epsilon + (m_i \star \varphi_\epsilon - m_i)$$

and both terms of the sum converge to zero locally uniformly. Therefore $V^i(m_\epsilon \star \varphi_\epsilon) \to V^i(m)$ locally uniformly and similarly $V^i_\epsilon[m_\epsilon] \to V^i(m)$. Hence (u_i, λ_i, m_i) solve equations (MFG) *i)*, *iii)* in viscosity sense.

Convergence as $\epsilon \to 0$ under (V_{LU}) . Arguments in this case are very similar to the previous one, but differentiability of V^i improves convergence. Indeed, the bound on $\|m_\epsilon\|_{W^{1,p}(\Omega)}$ implies by Sobolev imbedding a bound on $\|m_\epsilon\|_{C^{0,\delta}(\overline{\Omega})}$ if p is chosen large enough. Hence, $\|m_\epsilon \star \varphi_\epsilon\|_{C^{0,\delta}(\overline{K})} \leq C$

and similarly $\|V^i_\epsilon[m_\epsilon]\|_{C^{0,\delta}(\overline{K})} \leq C$, for all $K \subset\subset \Omega$. Combining (23) and standard Schauder estimates we conclude that

$$u_{\epsilon,i} \to u_i \in C^{2,\delta''}(\Omega) \quad \textit{locally}$$

for some $\delta'' > 1$. Hence, convergence for (15), *i)* is pointwise. □

5. Uniqueness.

In the one-population case, uniqueness of solutions for (MFG) has been proven to hold when the hamiltonian is convex and the cost V is increasing with respect to the density m; from the game point of view, players tend to *avoid* regions where the distribution m is high. This result has been obtained by Lasry and Lions [20] using deeply the structure of the mean field system and an argument that is not standard in classical theory of elliptic systems. A generalization of their argument that appears natural in the multi-population context is the following.

Theorem 13. *Suppose that the following L^2 monotonicity condition on the costs V^i holds*

$$\int_\Omega \sum_{i=1}^M \left(V^i[m](x) - V^i[\bar{m}](x)\right)(m_i(x) - \bar{m}_i(x))dx \geq 0 \quad \forall m, \bar{m} \tag{24}$$

and that the hamiltonians $H^i(x,\cdot)$ are strictly convex for every $x \in \Omega$. Then, uniqueness of (classical) solutions for (MFG) *holds.*

Proof. Let (u, λ, m) and $(\bar{u}, \bar{\lambda}, \bar{m})$ be two solutions of (MFG). We multiply *i)* by $(m_i - \bar{m}_i)$ and *ii)* by $(u_i - \bar{u}_i)$, subtract, integrate by parts, use the fact that $m_i, \bar{m}_i \in \mathcal{P}$ and sum for $i = 1, \ldots, M$ to get

$$-\sum_{i=1}^M \int_\Omega m_i[H^i(x, D\bar{u}_i) - H^i(x, Du_i) - D_pH^i(x, Du_i)\cdot(D\bar{u}_i - Du_i)]dx$$
$$-\sum_{i=1}^M \int_\Omega \bar{m}_i[H^i(x, Du_i) - H^i(x, D\bar{u}_i) - D_pH^i(x, D\bar{u}_i)\cdot(Du_i - D\bar{u}_i)]dx$$
$$= \int_\Omega \sum_{i=1}^M \left(V^i[m](x) - V^i[\bar{m}](x)\right)(m_i(x) - \bar{m}_i(x))dx$$

The left hand side of the equation is non-positive (the Hamiltonians H^i are convex with respect to p) and the right hand side is non-negative by assumption, so they both have to be zero. Moreover

$m_i > 0$ on Ω and H^i are strictly convex, i.e.

$$H^i(x, p+q) - H^i(x,p) - D_p H^i(x,p)\cdot q = 0 \Rightarrow q = 0 \quad \forall x \in \Omega, p, q \in \mathbb{R}^d,$$

so $Du_i = D\bar{u}_i$ on Ω for all i. Hence, u_i and $\bar{u}_i$ differ by a constant, but they have to be equal because they both belong to $C_{\rm av}(\Omega)$, and therefore $\lambda_i = \bar{\lambda}_i$ as well. Since uniqueness holds for Kolmogorov equations, we conclude that $m_i = \bar{m}_i$.

□

Remark 14. Suppose that $V^i \in C^1(\mathbb{R}^M)$ are local costs. Then, a sufficient condition for (24) is that

$$JV + JV^T \text{ is positive semi-definite,}$$

as an easy consequence of Lagrange's theorem:

$$\sum_i (V^i(m) - V^i(\bar{m}))(m_i - \bar{m}_i) = \sum_{i,j} \partial_{m_j} V^i(\xi)(m_i - \bar{m}_i)(m_j - \bar{m}_j)$$

for some $\xi = \xi(m, \bar{m})$. We observe that this condition is slightly weaker than ($\mathrm{V_{LU}}$) *i)* (which requires some sort of positive-definiteness of JV), that is the "bound from below" for existence for (MFG) with local costs (see Remark 8).

Remark 15. It has been pointed out that one-population Mean Field Games system are connected to optimal control of partial differential equations, see [20], Section 2.6 or [19]. In particular, assume that $M = 1$, H is convex, and consider the following minimization problem

$$\inf_\alpha \left\{ \int_\Omega L(x,\alpha)m + \int_\Omega \Phi(m) \right\} \tag{25}$$

where L is the Legendre transform of H, $\Phi \in C^1(\mathbb{R})$ and $m \in \mathcal{P}$ is the state corresponding to the control $\alpha \in L^\infty(\Omega, \mathbb{R}^d)$, which is the (unique) probability distribution that solves the Kolmogorov equation

$$-\nu\Delta m - \mathrm{div}(\alpha m) = 0 \quad \text{in } \Omega,$$

and satisfies the boundary condition $\nu\partial_n m + m\alpha\cdot n = 0$. Then, it can be verified that if $\bar{\alpha}$ is an optimal control, namely it minimizes the expression in (25) among some set of admissible controls, then the corresponding state $\bar{m}$, the dual state $\bar{u}$ which is determined by $\bar{\alpha} = -D_p H(x, D\bar{u})$ and the optimal value λ solve (MFG) with $V[m] = \Phi'(m)$.

The optimal control interpretation carries over naturally to the multi-population case under the following assumption on the costs V^i, which we suppose to be differentiable: there exists $\Psi \in C^2(\mathbb{R}^M)$ such that

$$\partial_{m_i}\Psi(m_1,\dots,m_M) = V^i(m_1,\dots,m_M) \quad \forall m_1,\dots,m_M \text{ and } i=1,\dots,M. \tag{26}$$

If the set of costs satisfies such kind of "gradient" condition, we are led to consider the minimization problem

$$\inf_{\alpha_1,\dots,\alpha_M}\left\{\int_\Omega L^1(x,\alpha_1)m_1 + \ldots + \int_\Omega L^M(x,\alpha_M)m_M + \int_\Omega \Psi(m_1,\dots,m_M)\right\}, \tag{27}$$

where L^i is the Legendre transform of H^i for all $i=1,\dots,M$, α_i is the control implemented by the i-th population and the states $m_1,\dots,m_M$ solve respectively

$$-\nu_i\Delta m_i - \operatorname{div}(\alpha_i m_i) = 0 \quad \text{in } \Omega.$$

As in the single population case, a minimizer $\bar\alpha_1,\dots,\bar\alpha_M$ of (27) provides a solution to (MFG). Note that assumption (26) implies a certain symmetry in the system, as $D^2\Psi$ turns out to be symmetric, and consequently $\partial_{m_j}V^i = \partial_{m_i}V^j$ for all $i,j=1,\dots,M$.

Moreover, it is worth to observe that (27) becomes a *convex* optimization problem as soon as Ψ is a convex function. This is equivalent to ask $D^2\Psi$ to be positive semidefinite, that is $\{\partial_{m_j}V^i\}$ positive semidefinite, which is equivalent to the uniqueness condition (24).

Example 16. Consider the two-populations Example 10. Then, (24) and therefore uniqueness of solutions is ensured if

$$a \geq 0, \quad |b+c| \leq 2a.$$

We mention that in [17], a non-stationary version of (MFG) with two populations is considered, from a theoretical and numerical point of view. A model with linear costs of the form (14) is studied, where $a=2$ and $b,c=\lambda\geq 0$. Existence and uniqueness of solutions is obtained in the "convex" case $\lambda \leq 2$, that is precisely the interval where (24) is satisfied.

5.1. Some non-uniqueness examples.

In this final section we will present some two-populations examples ($M=2$) where (24) does not hold and (MFG) admits multiple solutions. We recall that if the Hamiltonian functions are quadratic, i.e.

$$H^i(x,p) = \frac{1}{2}|p|^2,$$

the so-called Hopf-Cole change of variables $v_i := e^{-\frac{u_i}{2\nu_i}}$, $m_i = v_i^2$ reduces (MFG) to (see for example [20])

$$\begin{aligned}
&-2\nu_1^2 \Delta v_1 + (V^1[v_1^2, v_2^2] - \lambda_1)v_1 = 0 \quad \text{in } \Omega \\
&-2\nu_2^2 \Delta v_2 + (V^2[v_1^2, v_2^2] - \lambda_2)v_2 = 0, \\
&v_i > 0, \textstyle\int v_i^2 = 1, \\
&\partial_n v_i = 0 \text{ on } \partial\Omega.
\end{aligned} \tag{28}$$

We will consider some examples of local linear costs of the form (14).

Proposition 17. *Let $d = 1$, $\nu_1 = \nu_2 = 2^{-1/2}$ and $\Omega = (0, 1)$. Then, there exist $a > 0$, $b < 0$ such that the system*

$$\begin{cases}
-\nu_1 u_1''(x) + \frac{1}{2}|u_1'(x)|^2 + \lambda_1 = am_1(x) + bm_2(x), & \text{in } \Omega, \\
-\nu_2 u_2''(x) + \frac{1}{2}|u_2'(x)|^2 + \lambda_2 = bm_1(x) + am_2(x), \\
-\nu_i m_i''(x) - (u_i'(x)m_i(x))' = 0, \\
u_i'(0) = u_i'(1) = m_i'(0) = m_i'(1) = 0 & \text{on } \partial\Omega
\end{cases} \tag{29}$$

has at least two different solutions.

Proof. By the Hopf-Cole transform we might consider (28). Particular solutions of (28) are given by $(v_1, v_2) = (\varphi, \varphi)$, $\lambda_i = \lambda$, where φ, λ solve

$$-\varphi'' = \lambda\varphi - (a + b)\varphi^3 \quad \varphi > 0, \quad \int \varphi^2 = 1,$$

with $\varphi' = 0$ on the boundary. As it is pointed out in [20] (p. 11) this equation has (at least) two solutions in dimension $d = 1$ if $-(a + b)$ is positive and large enough, that is true, for example, when $a > 0$ and $a + b << 0$. Indeed, $\varphi = 1, \lambda = a + b$ provide a solution. A non-constant solution is obtained by solving the minimization problem

$$\min_{\varphi \in H^1(\Omega), \int_\Omega \varphi^2 = 1} \frac{1}{2} \int_\Omega (\varphi')^2 + \frac{a + b}{4} \int_\Omega \varphi^4.$$

□

In the previous example V^i is strictly increasing with respect to m_i, but uniqueness fails due to the leading dependence of the costs with respect to m_{-i}.

Non-uniqueness issues arise also when considering different parameters a, b, c in (14).

Proposition 18. *Let $d = 1$ and $\Omega = (-1/2, 1/2)$. Then, there exists $\nu_0 > 0$ such that for all $\nu \in (0, \nu_0)$, the system*

$$\begin{cases} -\nu u_1''(x) + \frac{1}{2}|u_1'(x)|^2 + \lambda_1 = m_2(x), \quad \text{in } \Omega, \\ -\nu u_2''(x) + \frac{1}{2}|u_2'(x)|^2 + \lambda_2 = m_1(x), \\ -\nu m_i''(x) - (u_i'(x) m_i(x))' = 0, \\ u_i'(-1/2) = u_i'(1/2) = m_i'(-1/2) = m_i'(1/2) = 0 \quad \text{on } \partial\Omega \end{cases} \tag{30}$$

has at least two different solutions. Moreover, the non-constant solution (u, m, λ) satisfies

$$\lambda_{1,2} \leq 48\nu^2. \tag{31}$$

System (30) describes (long-time average) equilibria of two "xenophobic" populations, where the cost paid by every individual is increasing with respect to the distribution of the individuals of the other population at his position. One should expect some equilibrium configuration where the two distributions are concentrated in different parts of the domain Ω. Let (u, m, λ) be a solution of (30) satisfying (31). It holds that

$$\int_\Omega m_1 m_2 \leq \lambda_1.$$

This inequality can be easily obtained by multiplying the Hamilton-Jacobi-Bellman equation for u_1 by m_1 and the Kolmogorov equation for m_1 by u_1 and integrating by parts. Since

$$\int_\Omega m_1 m_2 \leq 48\nu^2,$$

we shall conclude that as $\nu \to 0$ the distributions m_1, m_2 become more and more *segregated*, in the sense that $\int_\Omega m_1 m_2 \to 0$.

Proof of Proposition 18. By virtue of the Hopf-Cole transform we consider (28). Particular solutions of (28) are given by $(v_1(x), v_2(x)) = (\varphi(x), \varphi(-x))$, $\lambda_i = \lambda$, where φ, λ solve

$$-2\nu^2 \varphi''(x) + \varphi^2(-x)\varphi(x) = \lambda\varphi(x), \quad \varphi(x) > 0 \quad \forall x \in \Omega, \quad \int \varphi^2 = 1, \tag{32}$$

with $\varphi' = 0$ on the boundary. The constant function $\varphi \equiv 1$ together with $\lambda = 1$ solves (32). Equation (32) is also linked to the variational problem

$$\min_{\varphi \in H^1(\Omega), \int_\Omega \varphi^2 = 1} J(\varphi), \quad J(\varphi) := \nu^2 \int_\Omega (\varphi'(x))^2 dx + \frac{1}{4} \int_\Omega \varphi^2(x)\varphi^2(-x) dx.$$

Indeed, for all $v \in H^1(\Omega)$

$$J(\varphi + \epsilon v) =$$

$$J(\varphi) + \epsilon 2\nu^2 \int_\Omega \varphi' v' + \frac{\epsilon}{2}\left(\int_\Omega \varphi^2(x)\varphi(-x)v(-x)dx + \int_\Omega \varphi^2(-x)\varphi(x)v(x)dx\right) + o(\epsilon) =$$

$$J(\varphi) + \epsilon\left(2\nu^2 \int_\Omega \varphi'(x)v'(x)dx + \int_\Omega \varphi^2(-x)\varphi(x)v(x)dx\right) + o(\epsilon),$$

so a minimum of J constrained to $\int_\Omega \varphi^2 = 1$ solves (weakly) (32). It is also positive by standard comparison principle arguments. Such a minimizer exists as minimizing sequences of J are bounded in H^1. We show now that if ν is sufficiently small, the minimum is not achieved at $\varphi \equiv 1$. Let

$$\bar{\varphi} := \max\{0, \sqrt{24}x\}.$$

Then, $\int_\Omega \bar{\varphi}^2 = 1$ and

$$J(\bar{\varphi}) = \nu^2 \int_\Omega (\varphi'(x))^2 dx = 12\nu^2 < \frac{1}{4} = J(1)$$

if $\nu < \nu_0 := \sqrt{1/48}$. Hence, for such values of ν the minimizer φ of J is not the constant function. To obtain (31), we multiply (32) by φ and integrate by parts to get

$$\lambda = 2\nu^2 \int_\Omega (\varphi')^2 + \int_\Omega \varphi^2(-x)\varphi^2(x)dx = 4J(\varphi) - 2\nu^2 \int_\Omega (\varphi')^2 \le 4J(\bar{\varphi}) = 48\nu^2.$$

□

Appendix A. An a-priori estimate for HJB equations.

We derive an a-priori estimate on the gradient of solutions of HJB equations with superlinear Hamiltonian by applying the integral Bernstein method, introduced in [22]. In this version of the method, integral estimates are obtained by employing the equation solved by $|Du|^2$. For the convenience of the reader, we present a detailed proof in the case of Neumann boundary conditions, adapting the arguments of [22].

Theorem 19. *Let Ω be convex, $f \in L^q(\Omega)$ with $q > d$, $\gamma > 1$, $R > 0$, and $u \in C^3(\overline{\Omega})$ be a solution of*

$$\begin{cases} -\nu\Delta u + R|Du|^\gamma + \lambda = f(x) & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

Let $r \geq 1$ be fixed. Suppose that $\lambda \geq \lambda_0$ for some $\lambda_0 \in \mathbb{R}$ and $\|f\|_{L^q(\Omega)} \leq f_0$ for some $f_0 > 0$, then

$$\|Du\|_{L^r(\Omega)} \leq C, \tag{A.2}$$

with $C = C(\nu, R, \gamma, r, d, \lambda_0, f_0)$

Proof. Set $w = |Du|^2$, so

$$D_j w = 2\sum_i D_i u\, D_{ij} u, \quad D_{jj} w = 2\sum_i ((D_{ij}u)^2 + D_i u\, D_{ijj} u).$$

It holds that

$$|Dw| \leq 2d^2 |Du||D^2 u|. \tag{A.3}$$

By differentiating the equation (A.1) in Ω with respect to D_j and taking the sums for $j = 1, \ldots, d$ one obtains

$$-\nu\Delta w + R\gamma|Du|^{\gamma-2} Du \cdot Dw + 2|D^2u|^2 = 2Df \cdot Du.$$

We multiply it by w^p, with $p \geq 1$ that will be fixed later; through all the proof we will denote by C a constant that depends upon ν, R, γ, r, d and emphasize with C_p the dependance upon p. One has

$$-\nu \int_\Omega \Delta w\, w^p + 2\int_\Omega |D^2u|^2 w^p = -R\gamma \int_\Omega |Du|^{\gamma-2} Du \cdot Dw\, w^p + 2\int_\Omega Df \cdot Du\, w^p$$

We are going to estimate separately each of the four terms appearing in the equation. Since $Dw \cdot n \leq 0$ (p. 236 [22]),

$$\begin{aligned}
-\nu \int_\Omega \Delta w\, w^p &= \nu \int_\Omega Dw \cdot D(w^p) - \nu \int_{\partial\Omega} w^p Dw \cdot n \geq \frac{4p\nu}{(p+1)^2} \int_\Omega |D(w^{(p+1)/2})|^2 \\
&\geq C\frac{4p}{(p+1)^2} \left(\int_\Omega |w|^{\frac{(p+1)d}{d-2}}\right)^{\frac{d-2}{d}} - C\frac{4p}{(p+1)^2}\int_\Omega w^{p+1} \\
&\geq C\frac{4p}{(p+1)^2} \left(\int_\Omega |w|^{\frac{(p+1)d}{d-2}}\right)^{\frac{d-2}{d}} - C\frac{4p}{(p+1)^2}\left(\int_\Omega w^{p+\gamma}\right)^{\frac{p+1}{p+\gamma}}
\end{aligned}$$

by the Sobolev embedding theorem and Holder inequality also. Then,

$$\begin{aligned}
2\int_\Omega |D^2u|^2 w^p &\geq \\
&\int_\Omega |D^2u|^2 w^p + \int_\Omega \sum_i (D_{ii}u)^2 w^p \geq \int_\Omega |D^2u|^2 w^p + \frac{1}{d}\int_\Omega (\Delta u)^2 w^p \\
&\geq \int_\Omega |D^2u|^2 w^p + \frac{R^2}{4d\nu^2}\int_\Omega |Du|^{2\gamma} w^p - \frac{\lambda_0^2}{d\nu^2}\int_\Omega w^p - \frac{2}{d\nu^2}\int_\Omega f^2 w^p,
\end{aligned}$$

using the equation (A.1) and applying to the term $(R|Du|^\gamma+\lambda-f)^2$ the inequality $(a-b)^2 \geq \frac{a^2}{2}-2b^2$ for every $a, b \in \mathbb{R}$ twice. For the third term we have that

$$\begin{aligned} -R\gamma\int_\Omega |Du|^{\gamma-2}Du\cdot Dw\, w^p &= -R\gamma\int_\Omega |Du|^{\gamma-2}Du\cdot D\left(\frac{1}{p+1}w^{p+1}\right) \\ &= \frac{R\gamma}{p+1}\int_\Omega \mathrm{div}(|Du|^{\gamma-2}Du)w^{p+1} - \frac{R\gamma}{p+1}\int_{\partial\Omega}|Du|^{\gamma-2}w^{p+1}Du\cdot n \\ &= \frac{R\gamma}{p+1}\int_\Omega D(|Du|^{\gamma-2})\cdot Du\, w^{p+1} + \frac{R\gamma}{p+1}\int_\Omega |Du|^{\gamma-2}\Delta u\, w^{p+1} \\ &\leq \frac{2d^2R\gamma(\gamma-1)}{p+1}\int_\Omega |Du|^{\gamma-2}|D^2u|\, w^{p+1} \leq \frac{C}{p+1}\int_\Omega w^{p+\gamma} + \frac{C}{p+1}\int_\Omega |D^2u|^2w^p, \end{aligned}$$

by the estimate (A.3). Finally,

$$\begin{aligned} 2\int_\Omega Df\cdot Du\, w^p = -2\int_\Omega f\mathrm{div}(Du\, w^p) + 2\int_{\partial\Omega} f\, w^p Du\cdot n = \\ -2\int_\Omega f\Delta u\, w^p - 2\int_\Omega fDu\cdot D(w^p) \leq 2d\int_\Omega |f||D^2u|w^p + 4d^2(p-1)\int_\Omega |f||D^2u|w^p \\ \leq \frac{1}{2}\int_\Omega |D^2u|^2w^p + C_p\int_\Omega |f|^2w^p, \end{aligned}$$

and by putting all the estimates together

$$\begin{aligned} C\frac{4p}{(p+1)^2}\left(\int_\Omega |w|^{\frac{(p+1)d}{d-2}}\right)^{\frac{d-2}{d}} + \frac{1}{2}\int_\Omega |D^2u|^2w^p + \\ + \frac{R^2}{4d\nu^2}\int_\Omega w^{p+\gamma} - \frac{\lambda_0^2}{d\nu^2}\int_\Omega w^p - \frac{2}{d\nu^2}\int_\Omega f^2w^p \\ \leq \frac{C}{p+1}\int_\Omega w^{p+\gamma} + \frac{C}{p+1}\int_\Omega |D^2u|^2w^p + C_p\int_\Omega f^2w^p + C\frac{4p}{(p+1)^2}\left(\int_\Omega w^{p+\gamma}\right)^{\frac{p+1}{p+\gamma}}. \end{aligned}$$

One may choose p sufficiently large in order to have

$$\begin{aligned} C\frac{4p}{(p+1)^2}\left(\int_\Omega |w|^{\frac{(p+1)d}{d-2}}\right)^{\frac{d-2}{d}} + \frac{R^2}{8d\nu^2}\int_\Omega w^{p+\gamma} \\ \leq C_p\int_\Omega f^2w^p + C\frac{4p}{(p+1)^2}\left(\int_\Omega w^{p+\gamma}\right)^{\frac{p+1}{p+\gamma}} + \frac{\lambda_0^2}{d\nu^2}\int_\Omega w^p \\ \leq C_p\left(\int_\Omega |w|^{\frac{(p+1)d}{d-2}}\right)^{\frac{(d-2)p}{(p+1)d}}\left(\int_\Omega |f|^{2\beta}\right)^{\frac{1}{\beta}} + C\frac{4p}{(p+1)^2}\left(\int_\Omega w^{p+\gamma}\right)^{\frac{p+1}{p+\gamma}} + C_p\left(\int_\Omega w^{p+\gamma}\right)^{\frac{p}{p+\gamma}} \end{aligned} \quad (A.4)$$

using Holder inequality, with $\beta = \alpha'$ and $\alpha = (p+1)d/(d-2)p$. Since $2\beta \to d$, choosing p large enough we conclude that (A.2) holds.

□

References

[1] Y. Achdou, Finite difference methods for mean field games, in: Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2013, pp. 1–47.

[2] Y. Achdou, I. Capuzzo-Dolcetta, Mean field games: numerical methods, SIAM J. Numer. Anal. 48 (3) (2010) 1136–1162.

[3] M. Arisawa, P.-L. Lions, On ergodic stochastic control, Comm. Partial Differential Equations 23 (11-12) (1998) 2187–2217.

[4] M. Bardi, F. Da Lio, On the strong maximum principle for fully nonlinear degenerate elliptic equations, Arch. Math. (Basel) 73 (4) (1999) 276–285.

[5] G. Barles, F. Da Lio, On the boundary ergodic problem for fully nonlinear equations in bounded domains with general nonlinear Neumann boundary conditions, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (5) (2005) 521–541.

[6] A. Bensoussan, Perturbation methods in optimal control, Wiley/Gauthier-Villars Series in Modern Applied Mathematics, John Wiley & Sons Ltd., Chichester, 1988.

[7] A. Bensoussan, J. Frehse, P. Yam, Mean field games and mean field type control theory, Springer Briefs in Mathematics, Springer, New York, 2013.

[8] P. Cardaliaguet, Notes on mean field games, www.ceremade.dauphine.fr/cardalia/MFG100629.pdf.

[9] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, Long time average of mean field games, Netw. Heterog. Media 7 (2) (2012) 279–301.

[10] M. G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1) (1992) 1–67.

[11] E. Feleqi, The Derivation of Ergodic Mean Field Game Equations for Several Populations of Players, Dyn. Games Appl. 3 (4) (2013) 523–536.

[12] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, 3rd ed., Springer-Verlag, 2001.

[13] D. A. Gomes, S. Patrizi, V. Voskanyan, On the existence of classical solutions for stationary extended mean field games, Nonlinear Anal. 99 (2014) 49–79.

[14] D. A. Gomes, G. E. Pires, H. Sánchez-Morgado, A-priori estimates for stationary mean-field games, Netw. Heterog. Media 7 (2) (2012) 303–314.

[15] M. Huang, Large-population LQG games involving a major player: the Nash certainty equivalence principle, SIAM J. Control Optim. 48 (5) (2009/10) 3318–3353.

[16] M. Huang, R. P. Malhamé, P. E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, Commun. Inf. Syst. 6 (3) (2006) 221–251.

[17] A. Lachapelle, M.-T. Wolfram, On a mean field game approach modeling congestion and aversion in pedestrian crowds, Transportation Research Part B: Methodological 45 (10) (2011) 1572–1589.

[18] J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. I. Le cas stationnaire, C. R. Math. Acad. Sci. Paris 343 (9) (2006) 619–625.

[19] J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal, C. R. Math. Acad. Sci. Paris 343 (10) (2006) 679–684.

[20] J.-M. Lasry, P.-L. Lions, Mean field games, Jpn. J. Math. 2 (1) (2007) 229–260.

[21] P.-L. Lions, Cours au collège de france, http://www.college-de-france.fr.

[22] P.-L. Lions, Quelques remarques sur les problemes elliptiques quasilineaires du second ordre, Journal d'Analyse Mathmatique 45 (1985) 234–254.

[23] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa (3) 13 (1959) 115–162.

[24] M. Nourian, P. E. Caines, ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents, SIAM J. Control Optim. 51 (4) (2013) 3302–3331.

[25] M. Schechter, On L^p estimates and regularity. I, Amer. J. Math. 85 (1963) 1–13.