Balanced metrics on Cartan and Cartan–Hartogs domains

BALANCED METRICS ON CARTAN AND CARTAN–HARTOGS DOMAINS

ANDREA LOI, MICHELA ZEDDA

Abstract. This paper consists of two results dealing with balanced metrics (in S. Donaldson terminology) on noncompact complex manifolds. In the first one we describe all balanced metrics on Cartan domains. In the second one we show that the only Cartan–Hartogs domain which admits a balanced metric is the complex hyperbolic space. By combining these results with those obtained in [13] we also provide the first example of complete, Kähler-Einstein and projectively induced metric g such that αg is not balanced for all $\alpha > 0$.

1. Introduction

Let $\Omega \subset \mathbb{C}^d$ be a Cartan domain, i.e. an irreducible bounded symmetric domain, of complex dimension d and genus γ. For all positive real numbers μ consider the family of *Cartan-Hartogs* domains

$$M_\Omega(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C},\ |w|^2 < N_\Omega^\mu(z,z)\right\}, \tag{1}$$

where $N_\Omega(z,z)$ is the *generic norm* of Ω, i.e.

$$N_\Omega(z,z) = (V(\Omega)K(z,z))^{-\frac{1}{\gamma}}, \tag{2}$$

where $V(\Omega)$ is the total volume of Ω with respect to the Euclidean measure of the ambient complex Euclidean space and $K(z,z)$ is its Bergman kernel.

The domain Ω is called the *base* of the Cartan–Hartogs domain $M_\Omega(\mu)$ (one also says that $M_\Omega(\mu)$ is based on Ω). Consider on $M_\Omega(\mu)$ the metric $g(\mu)$ whose associated Kähler form $\omega(\mu)$ can be described by the (globally defined) Kähler potential centered at the origin

$$\Phi(z,w) = -\log(N_\Omega^\mu(z,z) - |w|^2). \tag{3}$$

These domains have been considered by several authors (see e.g. [15] and references therein). In [13] the authors of the present paper study when

2000 *Mathematics Subject Classification.* 53C55; 58C25.

Key words and phrases. Kähler metrics; balanced metrics; Hartogs domains.

The first author was supported by the M.I.U.R. Project "Geometric Properties of Real and Complex Manifolds"; the second author was supported by RAS through a grant financed with the "Sardinia PO FSE 2007-2013" funds and provided according to the L.R. 7/2007.

$(M_\Omega(\mu), \alpha\, g(\mu))$, for a positive constant α, admits a holomorphic and isometric (from now on *Kähler*) immersion f into the infinite dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$, i.e. $f^* g_{FS} = \alpha\, g(\mu)$, where g_{FS} denotes the Fubini–Study metric on $\mathbb{C}\mathrm{P}^\infty$ (when such a Kähler immersion exists, we say also that the metric is *projectively induced*). Recall that given homogeneous coordinates $[Z_0, \dots, Z_j, \dots]$ on $\mathbb{C}\mathrm{P}^\infty$, g_{FS} is the Kähler metric whose associated Kähler form ω_{FS} can be described in the open set $U_0 = \{Z_0 \neq 0\}$ by $\omega_{FS} = \frac{i}{2}\partial\bar\partial\Phi_{FS}$, where $\Phi_{FS} = \log(1 + \sum_{j=1}^\infty |z_j|^2)$ for $z_j = \frac{Z_j}{Z_0}$, $j = 1, \dots,$ affine coordinates on U_0. The main results obtained in [13] can be summarized in the following theorem (see also next section for a more detailed description of the Wallach set $W(\Omega)$ and for the definition of the integer a appearing in (c)).

Theorem LZ *Let $\Omega \subset \mathbb{C}^d$ be a Cartan domain of rank r, genus γ and dimension d and let g_B be its Bergman metric. Then the following results hold true:*

- (a) *$(\Omega, \beta g_B)$, $\beta > 0$, admits a equivariant Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ if and only if $\beta\gamma$ belongs to $W(\Omega) \setminus \{0\}$;*
- (b) *the metric $\alpha g(\mu)$, $\alpha > 0$, on the Cartan–Hartogs domain $M_\Omega(\mu)$ is projectively induced if and only if $(\alpha + m)\frac{\mu}{\gamma} g_B$ is projectively induced for every integer $m \geq 0$;*
- (c) *Let $\mu_0 = \gamma/(d+1)$ and $\Omega \neq \mathbb{C}\mathrm{H}^d$. Then the metric $\alpha g(\mu_0)$ on $M_\Omega(\mu_0)$ is Kähler-Einstein, complete, nonhomogeneous and projectively induced for all positive real number $\alpha \geq \frac{(r-1)(d+1)a}{2\gamma}$.*

In this paper we study balanced metrics (in S. Donaldson's terminology) on Cartan and Cartan–Hartogs domains. The main results are the following two theorems. In the first one we describe all balanced metrics on Cartan's domains, while the second one can be viewed as a characterization of the complex hyperbolic space among Cartan–Hartogs domains, in terms of balanced metrics (cfr. Example 1 below).

Theorem 1. *Let Ω be a Cartan domain of genus γ equipped with its Bergman metric g_B. The metric βg_B, $\beta > 0$, is balanced if and only if $\beta > \frac{\gamma - 1}{\gamma}$.*

Theorem 2. *Let $M_\Omega(\mu)$ be a Cartan-Hartogs domain based on the Cartan domain $\Omega \subset \mathbb{C}^d$. The metric $\alpha g(\mu)$ on $M_\Omega(\mu)$ is balanced if and only if $\alpha > d+1$ and $M_\Omega(\mu)$ is holomorphically isometric to the complex hyperbolic space $\mathbb{C}\mathrm{H}^{d+1}$, namely $\Omega = \mathbb{C}\mathrm{H}^d$ and $\mu = 1$.*

By combining these results with (c) in Theorem LZ we also obtain the first example of *complete, Kähler-Einstein and projectively induced metric g such that αg is not balanced for α varying in a continuous subset of the real numbers*. This is expressed by the following corollary.

Corollary 3. *Let $\Omega \subset \mathbb{C}^d$ be a Cartan domain of genus γ equipped with its Bergman metric g_B. Let $\mu_0 = \gamma/(d+1)$ and $\Omega \neq \mathbb{C}\mathrm{H}^d$. Then the metric $\alpha g(\mu_0)$ on $M_\Omega(\mu_0)$ is complete, Kähler-Einstein projectively induced and not balanced for all $\alpha \geq \frac{(r-1)(d+1)a}{2\gamma}$.*

The paper consists in three other sections. In Section 2 we recall the definition of balanced metrics. In Section 3 we describe all balanced metrics on Cartan domains and prove Theorem 1. Finally Section 4 is dedicated to the proof of Theorem 2.

The authors woud like to thank the anonymous referee for several useful remarks and comments which helped them to improve the exposition.

2. Balanced metrics

Let M be a n-dimensional complex manifold endowed with a Kähler metric g and let ω be the Kähler form associated to g, i.e. $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. Assume that the metric g can be described by a strictly plurisubharmonic real valued function $\Phi : M \to \mathbb{R}$, called a *Kähler potential* for g, i.e. $\omega = \frac{i}{2}\partial\bar\partial\Phi$.

Let $\mathcal{H}_\Phi$ be the weighted Hilbert space of square integrable holomorphic functions on (M, g), with weight $e^{-\Phi}$, namely

$$\mathcal{H}_\Phi = \left\{ f \in \mathrm{Hol}(M) \mid \int_M e^{-\Phi}|f|^2 \frac{\omega^n}{n!} < \infty \right\}, \tag{4}$$

where $\frac{\omega^n}{n!} = \det(\frac{\partial^2\Phi}{\partial z_\alpha \partial \bar z_\beta})\frac{\omega_0^n}{n!}$ is the volume form associated to ω and $\omega_0 = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar z_j$ is the standard Kähler form on $\mathbb{C}^n$. If $\mathcal{H}_\Phi \neq \{0\}$ we can pick an orthonormal basis $\{f_j\}$ and define its reproducing kernel by

$$K_\Phi(z,z) = \sum_{j=0}^N |f_j(z)|^2,$$

where $N+1$ denotes the complex dimension of $\mathcal{H}_\Phi \neq \{0\}$. Consider the function

$$\varepsilon_g(z) = e^{-\Phi(z)} K_\Phi(z,z). \tag{5}$$

As suggested by the notation it is not difficult to verify that ε_g depends only on the metric g and not on the choice of the Kähler potential Φ (which is defined up to an addition with the real part of a holomorphic function on M) or on the orthonormal basis chosen.

Definition. The metric g is *balanced* if the function ε_g is a positive constant.

A balanced metric g on M can be viewed as a particular projectively induced Kähler metric for which the Kähler immersion $f\colon M \to \mathbb{C}\mathrm{P}^N, N \leq \infty$, $x \mapsto [s_0(x), \dots, s_j(x), \dots]$, is given by the orthonormal basis $\{f_j\}$ of the Hilbert space $\mathcal{H}_\Phi$. Indeed the map f is well-defined since ε_g is a positive constant and hence for all $x \in M$ there exists $\varphi \in \mathcal{H}_\Phi$ such that $\varphi(x) \neq 0$. Moreover,

$$\begin{aligned} f^*\omega_{FS} =& \frac{i}{2}\partial\bar\partial \log \sum_{j=0}^{\infty} |f_j(z)|^2 \\ =& \frac{i}{2}\partial\bar\partial \log K_\Phi(z,z) \\ =& \frac{i}{2}\partial\bar\partial \log \varepsilon_g + \frac{i}{2}\partial\bar\partial \log e^{\Phi} \\ =& \frac{i}{2}\partial\bar\partial \log \varepsilon_g + \omega. \end{aligned}$$

Hence if g is balanced the map f is isometric.

In the literature the function ε_g was first introduced under the name of *η-function* by J. Rawnsley in [16], later renamed as *θ-function* in [3]. The map f is called in [3] the *coherent states map*. It plays a fundamental role in the geometric quantization and quantization by deformation of a Kähler manifold. It also related to the Tian-Yau-Zelditch asymptotic expansion (see [9], [11], [12] and references therein).

Example 1. Notice that a projectively induced metric is not always balanced. For example, in [4] E. Calabi shows that the complex hyperbolic space $(\mathbb{C}\mathrm{H}^d, \alpha\, g_{hyp})$, endowed with a positive multiple of the hyperbolic metric g_{hyp}, is projectively induced for all $\alpha > 0$. (Here $\mathbb{C}\mathrm{H}^d = \{z \in \mathbb{C}^d \mid |z|^2 < 1\}$ and the Kähler form ω_{hyp} associated to g_{hyp} is given by $\omega_{hyp} = -\frac{i}{2}\partial\bar\partial \log(1 - |z|^2)$). Although, it is well-known that the weighted Hilbert space of square integrable holomorphic functions on $(\mathbb{C}\mathrm{H}^d, \alpha\, g_{hyp})$, i.e.

$$\mathcal{H}_{\alpha\Phi_{hyp}} = \left\{ \varphi \in \mathrm{Hol}(\mathbb{C}\mathrm{H}^d), \int_{\mathbb{C}\mathrm{H}^d} \left(1 - |z|^2\right)^{\alpha-(d+1)} |\varphi|^2 \frac{\omega_0^d}{d!} < \infty \right\},$$

is equal to $\{0\}$ for all $\alpha \leq d$. Similar considerations can be done for all Cartan domains (see Remark 6 below).

Remark 4. The definition of balanced metrics was originally given by S. Donaldson [6] in the case of a compact polarized Kähler manifold (M, g) and generalized in [2] (see also [5], [7], [10]) to the noncompact case. Here we give only the definition for those Kähler metrics which admit a globally defined potential such as the Cartan and Cartan–Hartogs domains treated in this paper.

3. Balanced metrics on Cartan domains

Let $(\Omega, \beta g_B)$, $\beta > 0$, denote a Cartan domain, i.e. an irreducible bounded symmetric domain of $\mathbb{C}^d$ endowed with a positive multiple of its Bergman metric g_B. Recall that g_B is the Kähler metric on Ω whose associated Kähler form ω_B is given by $\omega_B = \frac{i}{2}\partial\bar\partial \log K(z,z)$, where $K(z,z)$ is the reproducing kernel for the Hilbert space

$$\mathcal{H} = \left\{ \varphi \in \mathrm{Hol}(\Omega),\ \int_\Omega |\varphi|^2 \, \frac{\omega_0^d}{d!} < \infty \right\},$$

where $\omega_0 = \frac{i}{2}\sum_{j=1}^d dz_j \wedge d\bar z_j$ is the standard Kähler form of $\mathbb{C}^d$. A bounded symmetric domain $(\Omega, \alpha g_B)$ is uniquely determined by a triple of integers (r,a,b), where r represents the rank of Ω and a and b are positive integers. The genus γ of Ω is defined by $\gamma = (r-1)a + b + 2$ and the dimension d can be written as

$$d = r + \frac{r(r-1)}{2}a + rb. \qquad (6)$$

The table below summarizes the numerical invariants and the dimension of Ω according to its type (for a more detailed description of this invariants, which is not necessary in our approach, see e.g. [1]).

Table 1. Bounded symmetric domains, invariants and dimension.

Type	r	a	b	γ	dimension	
$\Omega_1[m,n]$	n	$0\ (n=1)$ $2\ (n>1)$	$m-n$	$n+m$	nm	$(n \le m)$
$\Omega_2[n]$	$[n/2]$	4	$0\ (n \text{ even})$ $2\ (n \text{ odd})$	$2n-2$	$n(n-1)/2$	$(n \ge 5)$
$\Omega_3[n]$	n	1	0	$n+1$	$n(n+1)/2$	$(n \ge 2)$
$\Omega_4[n]$	2	$n-2$	0	n	n	$(n \ge 5)$
$\Omega_V[16]$	2	6	4	12	16	
$\Omega_{VI}[27]$	3	8	0	18	27	

We give now the definition of the *Wallach set* of a Cartan domain Ω, referring the reader to [1], [8] and [17] for more details and results. The Wallach set, denoted by $W(\Omega)$, consists of all $\eta \in \mathbb{C}$ such that there exists a Hilbert space $\mathcal{H}_{\frac{\eta}{\gamma}}$ whose reproducing kernel is $K^{\frac{\eta}{\gamma}}$. This is equivalent to the requirement that $K^{\frac{\eta}{\gamma}}$ is positive definite, i.e. for all n-tuples of points $x_1, \dots, x_n$ belonging to Ω the $n \times n$ matrix $(K(x_\alpha, x_\beta)^{\frac{\eta}{\gamma}})$, is positive *semi-definite*. It turns out (cfr. [1, Cor. 4.4, p. 27] and references therein) that $W(\Omega)$ consists only of real numbers and depends on two of the domain's invariants, a and r. More precisely we have

$$W(\Omega) = \left\{0,\ \frac{a}{2},\ 2\frac{a}{2},\ \dots,\ (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2},\ \infty\right). \qquad (7)$$

The set $W_{dis} = \left\{0,\ \frac{a}{2},\ 2\frac{a}{2},\ \dots,\ (r-1)\frac{a}{2}\right\}$ and the interval $W_c = \left((r-1)\frac{a}{2},\ \infty\right)$ are called respectively the *discrete* and *continuous* part of the Wallach set of the domain Ω.

Remark 5. If Ω has rank $r = 1$, namely Ω is the complex hyperbolic space $\mathbb{C}\mathrm{H}^d$, then $g_B = (d+1)g_{hyp}$. In this case (and only in this case) $W_{dis} = \{0\}$ and $W_c = (0, \infty)$ (cfr. Example 1).

We can now prove Theorem 1.

Proof of Theorem 1. Let d denote the complex dimension of Ω. It follows by standard results on bounded symmetric domains (see e.g. [8]) that the Hilbert space

$$\mathcal{H}_\beta = \left\{ \varphi \in \mathrm{Hol}(\Omega) \mid \int_\Omega \frac{1}{K^\beta} |\varphi|^2 \frac{\omega_B^d}{d!} < \infty \right\},$$

does not reduce to the zero dimensional space iff $\beta > \frac{\gamma-1}{\gamma}$. Observe that the notation of $\mathcal{H}_\beta$ is consistent with that of $\mathcal{H}_{\frac{\eta}{\gamma}}$ given above.

Hence, in order to prove that $\beta\, g_B$ is balanced for $\beta > \frac{\gamma-1}{\gamma}$, it remains to show that for $\beta > \frac{\gamma-1}{\gamma}$ the map $h_\beta : \Omega \to \mathbb{C}P^\infty,\ x \mapsto [\dots, h_\beta^j(x), \dots]$, where $\{h_\beta^j\}$ is an orthonormal basis of $\mathcal{H}_\beta$, is a well-defined map of Ω into $\mathbb{C}\mathrm{P}^\infty$ and it is Kähler i.e.

$$h_\beta^* g_{FS} = \beta g_B.$$

To prove that h_β is well-defined one needs to verify that for all $x \in \Omega$ there exists $\varphi \in \mathcal{H}_\beta$ such that $\varphi(x) \neq 0$. Assume, by contradiction, that there exists $x_0 \in \Omega$ such that $\varphi(x_0) = 0$ for all $\varphi \in \mathcal{H}_\beta$. Write $\Omega = G/K$, where G is a subgroup of $\mathrm{Aut}(\Omega) \cap \mathrm{Isom}(\Omega)$ which acts transitively on Ω. Then for all $g \in G$, $\varphi \circ g$ is an element of $\mathcal{H}_\beta$ which, by assumption, vanishes on x_0. Thus $0 = \varphi \circ g(x_0) = \varphi(gx_0)$ and since this holds true for all $g \in G$, h_β is the zero function. Hence $\mathcal{H}_\beta = \{0\}$, which is in contrast with the fact that $\mathcal{H}_\beta \neq \{0\}$ for $\beta > \frac{\gamma-1}{\gamma}$. In order to prove that h_β is Kähler notice that the function $\frac{\sum_{j=0}^\infty |h_\beta^j|^2}{K^\beta}$ is invariant by the group G and hence constant. Therefore

$$h_\beta^* \omega_{FS} = \frac{i}{2} \partial\bar\partial \log \sum_{j=0}^\infty |h_\beta^j|^2 = \beta\omega_B + \frac{i}{2}\partial\bar\partial \log \frac{\sum_{j=0}^\infty |h_\beta^j|^2}{K^\beta} = \beta\omega_B,$$

and we are done. □

Remark 6. By Theorem 1 the subset of the positive real numbers β for which $\beta\, g_B$ is balanced, i.e. $\left(\frac{\gamma-1}{\gamma}, \infty\right)$, is a proper subset of the continuous part of the set of β for which $\beta\, g_B$ is projectively induced, namely $\left((r-1)\frac{a}{2\gamma}, \infty\right)$ (see the definition (7) of the Wallach set $W(\Omega)$ and (a) of Theorem LZ). Thus for every Cartan domain there exists an infinite interval of positive real numbers β such that βg_B is projectively induced but not balanced.

Remark 7. Observe that it follows by Theorem 1 that, for all $\beta > \frac{\gamma-1}{\gamma}$, we have for some constant ξ

$$\int_\Omega N_\Omega^{\gamma(\beta-1)}\, h_\beta^j\, \bar{h}_\beta^k\, \omega_0^d = \xi\, \delta_{j,k}, \tag{8}$$

where N_Ω is the generic norm of Ω defined in (2) and h_β is the Kähler map defined in the proof of Theorem 1. In particular, the integral (8) is convergent and does not depend on j, k.

4. Balanced metrics on Cartan–Hartogs domains

In order to prove Theorem 2 we need the following two lemmata. The first one gives an explicit description of the Kähler immersions of a $d+1$-dimensional Cartan–Hartogs domain $(M_\Omega(\mu), \alpha g(\mu))$ into $\mathbb{C}\mathrm{P}^\infty$ while the second one describes a necessary condition for the metric $\alpha g(\mu)$ to be balanced.

Lemma 8. *If $f\colon M_\Omega(\mu) \to \mathbb{C}\mathrm{P}^\infty$ is a holomorphic map such that $f^*\omega_{FS} = \alpha\,\omega(\mu)$ then up to unitary transformation of $\mathbb{C}\mathrm{P}^\infty$ it is given by*

$$f = \left[1, s, h_{\frac{\mu\,\alpha}{\gamma}}, \dots, \sqrt{\frac{(m+\alpha-1)!}{(\alpha-1)!m!}}\, h_{\frac{\mu(\alpha+m)}{\gamma}}\, w^m, \dots\right], \tag{9}$$

where $s = (s_1, \dots, s_m, \dots)$ with

$$s_m = \sqrt{\frac{(m+\alpha-1)!}{(\alpha-1)!m!}}\, w^m,$$

and $h_k = (h_k^1, \dots, h_k^j, \dots)$ denotes the sequence of holomorphic maps on Ω such that the immersion $\tilde{h}_k = (1, h_k^1, \dots, h_k^j, \dots)$, $\tilde{h}_k : \Omega \to \mathbb{C}\mathrm{P}^\infty$, satisfies $\tilde{h}_k^*\omega_{FS} = k\omega_B$, i.e.

$$1 + \sum_{j=1}^{\infty} |h_k^j|^2 = \frac{1}{N^{\gamma\, k}}. \tag{10}$$

Proof. Since the immersion is isometric, by (3) we have $f^*\Phi_{FS} = -\alpha \log(N_\Omega^\mu(z,z) - |w|^2)$, which is equivalent to

$$\frac{1}{(N^\mu - |w|^2)^\alpha} = \sum_{j=0}^{\infty} |f_j|^2,$$

for $f = [f_0, \dots, f_j, \dots]$. If we consider the power expansion around the origin of the left hand side with respect to w, $\bar{w}$, we get

$$\sum_{k=1}^{\infty} \left[\frac{\partial^{2k}}{\partial w^k \partial \bar{w}^k} \frac{1}{(N^\mu - |w|^2)^\alpha}\right]_0 \frac{|w|^{2k}}{k!^2} = \sum_{k=1}^{\infty} \left[\frac{\partial^{2k}}{\partial w^k \partial \bar{w}^k} \frac{1}{(1 - |w|^2)^\alpha}\right]_0 \frac{|w|^{2k}}{k!^2}$$
$$= \left(\sum_{k=0}^{\infty} |w|^2\right)^\alpha - 1.$$

The power expansion with respect to z and $\bar z$ reads

$$\sum_{j,k}\left[\frac{\partial^{|m_j|+|m_k|}}{\partial z^{m_j}\partial\bar z^{m_k}}\frac{1}{(N^\mu-|w|^2)^\alpha}\right]_0\frac{z^{m_j}\bar z^{m_k}}{m_j!m_k!}=\sum_{j,k}\left[\frac{\partial^{|m_j|+|m_k|}}{\partial z^{m_j}\partial\bar z^{m_k}}\frac{1}{N^{\mu\alpha}}\right]_0\frac{z^{m_j}\bar z^{m_k}}{m_j!m_k!}$$
$$=\sum_{j=1}^{\infty}|h^j_{\frac{\mu\alpha}{\gamma}}|^2,$$

where the last equality holds since by (10) $\sum_{j=1}^\infty h^j_{\frac{\mu\alpha}{\gamma}}$ is the power expansion of $\frac{1}{N^{\mu\alpha}}-1$. Here we are using Calabi's multi index notation, namely we arrange every d-tuple of nonnegative integers as the sequence $m_j=(m_{j,1},\dots,m_{j,d})$ with nondecreasing order, that is $m_0=(0,\dots,0)$, $|m_j|\leq|m_{j+1}|$, with $|m_j|=\sum_{\alpha=1}^d m_{j,\alpha}$. Further z^{m_j} denotes the monomial in d variables $\prod_{\alpha=1}^d z_\alpha^{m_{j,\alpha}}$ and $m_j!=m_{j,1}!\cdots m_{j,d}!$.

Finally, the power expansion with respect to z, $\bar z$, w, $\bar w$ reads

$$\sum_{m=1}^{\infty}\sum_{j,k}\left[\frac{\partial^{|m_j|+|m_k|}}{\partial z^{m_j}\partial\bar z^{m_k}}\frac{\partial^{2m}}{\partial w^m\partial\bar w^m}\frac{1}{(N^\mu-|w|^2)^\alpha}\right]_0\frac{z^{m_j}\bar z^{m_k}w^m\bar w^m}{m_j!m_k!m!^2}$$
$$=\sum_{m=1}^{\infty}\sum_{j,k}\left[\frac{\partial^{|m_j|+|m_k|}}{\partial z^{m_j}\partial\bar z^{m_k}}\frac{(m+\alpha-1)!}{(\alpha-1)!m!N^{\mu(\alpha+m)}}\right]_0\frac{z^{m_j}\bar z^{m_k}}{m_j!m_k!}|w|^{2m}$$
$$=\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{(m+\alpha-1)!}{(\alpha-1)!m!}|w|^{2m}|h^j_{\frac{\mu(\alpha+m)}{\gamma}}|^2,$$

where we are using (10) again. It follows by the previous power series expansions, that the map f given by (9) is a Kähler immersion of $(M_\Omega(\mu),\alpha g(\mu))$ into $\mathbb{C}\mathrm{P}^\infty$. By Calabi's rigidity Theorem (cfr. [4]) all other Kähler immersions are given by $U\circ f$, where U is a unitary transformation of $\mathbb{C}P^\infty$. □

Lemma 9. *If $\alpha\, g(\mu)$ is balanced then $\alpha>d+1$ and $\alpha\mu>\gamma-1$.*

Proof. Assume that $\alpha\, g(\mu)$ is balanced. Then it is projectively induced and by Lemma 8, up to unitary transformation of $\mathbb{C}\mathrm{P}^\infty$, the Kähler immersion $f: M_\Omega(\mu)\to\mathbb{C}\mathrm{P}^\infty$, $f=[f_0,\dots,f_j,\dots]$, is given by (9). By Section 2 $\{f_j\}_{j=0,1,\dots}$ is an orthonormal basis for the weighted Hilbert space

$$\mathcal{H}_{\alpha\Phi}=\left\{\varphi\in\mathrm{Hol}(M_\Omega(\mu))\ \Big|\ \int_{M(\mu)}\left(N_\Omega^\mu-|w|^2\right)^\alpha|\varphi|^2\frac{\omega(\mu)^{d+1}}{(d+1)!}<\infty\right\},\quad(11)$$

where up to the multiplication with a positive constant

$$\frac{\omega(\mu)^{d+1}}{(d+1)!}=\frac{N_\Omega^{\mu(d+1)-\gamma}}{(N_\Omega^\mu-|w|^2)^{d+2}}\frac{\omega_0^{d+1}}{(d+1)!},$$

as it follows by a long but straightforward computation of the determinant of the metric $g(\mu)$. Thus, in particular we have

$$\int_{M_\Omega(\mu)}(N_\Omega^\mu-|w|^2)^\alpha f_j\bar f_k\frac{\omega(\mu)^{d+1}}{(d+1)!}=$$
$$\int_{M_\Omega(\mu)}(N_\Omega^\mu-|w|^2)^{\alpha-(d+2)}N_\Omega^{\mu(d+1)-\gamma}f_j\bar f_k\frac{\omega_0^{d+1}}{(d+1)!}=\lambda\,\delta_{jk},$$

for some constant λ independent from j and k. It follows by (9) that the integral

$$\int_{M_\Omega(\mu)}(N_\Omega^\mu-|w|^2)^{\alpha-(d+2)}N_\Omega^{\mu(d+1)-\gamma}|h^j_{\frac{\mu\alpha}{\gamma}}|^2\frac{\omega_0^{d+1}}{(d+1)!},$$

is convergent. Passing to polar coordinates one gets

$$\frac{\pi}{(d+1)!}\int_\Omega N_\Omega^{\mu(d+1)-\gamma}|h^j_{\frac{\mu\alpha}{\gamma}}|^2\int_0^{N_\Omega^\mu}(N_\Omega^\mu-\rho)^{\alpha-(d+2)}d\rho\,\omega_0^d.$$

The integral

$$\int_0^{N_\Omega^\mu}(N_\Omega^\mu-\rho)^{\alpha-(d+2)}d\rho,$$

is convergent iff $\alpha-(d+2)>-1$, i.e. iff $\alpha>d+1$. Further, when $\alpha>d+1$, going on with computations gives

$$\frac{\pi}{(d+1)!}\frac{1}{(\alpha-(d+2)+1)}\int_\Omega N_\Omega^{\mu\alpha-\gamma}|h^j_{\frac{\mu\alpha}{\gamma}}|^2\omega_0^d.$$

By Remark 7 this last integral converges and does not depends on j iff $\alpha\mu>\gamma-1$, and we are done. □

We are now in the position of proving Theorem 2.

Proof of Theorem 2. Since by Theorem 1 the hyperbolic metric αg_{hyp} is balanced iff $\alpha>d+1$, the sufficient condition is verified (recall that for the hyperbolic metric we have $\mu=1$ and $\gamma=d+2$). For the necessary part, assume that $\alpha\,g(\mu)$ is balanced. By Lemma 9 we can assume $\alpha>d+1$ and $\alpha\mu>\gamma-1$. Following the same approach as in Lemma 9, this gives that the integral

$$\int_{M_\Omega(\mu)}(N_\Omega^\mu-|w|^2)^{\alpha-(d+2)}N^{\mu(d+1)-\gamma}f_j\bar f_k\frac{\omega_0^{d+1}}{(d+1)!}$$

is zero for $j \neq k$ and does not depend on j otherwise. By (9) this implies that the following integral

$$\int_{M_\Omega(\mu)} (N_\Omega^\mu - |w|^2)^{\alpha-(d+2)} N^{\mu(d+1)-\gamma} \frac{(m+\alpha-1)!}{(\alpha-1)!m!} |h^j_{\frac{\mu(\alpha+m)}{\gamma}}|^2 |w|^{2m} \frac{\omega_0^{d+1}}{(d+1)!} =$$
$$\frac{\pi}{(d+1)!} \frac{(m+\alpha-1)!}{(\alpha-1)!m!} \int_\Omega N_\Omega^{\mu(d+1)-\gamma} |h^j_{\frac{\mu(\alpha+m)}{\gamma}}|^2 \int_0^{N_\Omega^\mu} (N_\Omega^\mu - \rho)^{\alpha-(d+2)} \rho^m \omega_0^d =$$
$$\frac{\pi\, m!}{(d+1)!} \frac{(m+\alpha-1)!}{(\alpha-1)!\, m!} \frac{1}{(\alpha-(d+2)+1)\cdots(\alpha-(d+2)+m)} \cdot$$
$$\cdot \int_\Omega N_\Omega^{\mu(d+1)-\gamma} |h^j_{\frac{\mu(\alpha+m)}{\gamma}}|^2 \int_0^{N_\Omega^\mu} (N_\Omega^\mu - \rho)^{\alpha-(d+2)+m} \omega_0^d =$$
$$\frac{\pi}{(d+1)!} \frac{(m+\alpha-1)!}{(\alpha-1)!} \frac{1}{(\alpha-(d+2)+1)\cdots(\alpha-(d+2)+m+1)} \cdot$$
$$\cdot \int_\Omega N_\Omega^{\mu(\alpha+m)-\gamma} |h^j_{\frac{\mu(\alpha+m)}{\gamma}}|^2 \omega_0^d, \tag{12}$$

does not depend on the choice of m and j. Since $\alpha\mu > \gamma - 1$ implies $\frac{\mu(\alpha+m)}{\gamma} > \frac{\gamma-1}{\gamma}$, Remark 7 yields that $\int_\Omega N_\Omega^{k-\gamma} |h^j_{\frac{k}{\gamma}}|^2 \omega_0^d$ is constant for all j and thus (12) does not depend on j (observe also that for $j=0$ one obtains the term s_m of s in Lemma 8 and for $j = m = 0$ we recover the first term of f, $f_0 = 1$). Thus if $\alpha\, g(\mu)$ is balanced the quantity

$$\frac{\pi}{(d+1)!} \frac{(m+\alpha-1)!}{(\alpha-1)!} \frac{1}{(\alpha-(d+2)+1)\cdots(\alpha-(d+2)+m)} \int_\Omega N_\Omega^{\mu(\alpha+m)-\gamma} \omega_0^d$$

does not depend on m. By [14, Prop. 2.1, p. 358] this is equivalent to ask that the quantity

$$\frac{(m+\alpha-1)!}{(\alpha-d-1)\cdots(\alpha-d+m-2)} \frac{F(\mu(\alpha+m)-\gamma)}{F(0)}$$

does not depend on m, where

$$\frac{F(s)}{F(0)} = \prod_{j=1}^r \frac{\Gamma\left(s+1+\frac{(j-1)a}{2}\right)\Gamma\left(b+2+\frac{(r+j-2)a}{2}\right)}{\Gamma\left(1+\frac{(j-1)a}{2}\right)\Gamma\left(s+b+2+\frac{(r+j-2)a}{2}\right)},$$

for Γ the usual Gamma function and (a,b,r) the domain's invariants described in Table 1. Deleting the terms which do not depend on m and changing the orders of terms in the argument of the Gamma functions, we get

$$(\alpha+m-1)\cdots(\alpha+m-d) \prod_{j=1}^r \frac{\Gamma\left(\mu(\alpha+m)-\gamma+1+\frac{(j-1)a}{2}\right)}{\Gamma\left(\mu(\alpha+m)-\gamma+b+2+\frac{(r+j-2)a}{2}\right)} =$$

$$(\alpha+m-1)\cdots(\alpha+m-d)\prod_{j=1}^{r}\frac{\Gamma\left(\mu(\alpha+m)-\gamma+1+\frac{(j-1)a}{2}\right)}{\Gamma\left(\mu(\alpha+m)-\gamma+b+2+\frac{(2r-j-1)a}{2}\right)}=$$

$$(\alpha+m-1)\cdots(\alpha+m-d)\prod_{j=1}^{r}\frac{\Gamma\left(\mu(\alpha+m)-\gamma+1-\frac{a}{2}+\frac{ja}{2}\right)}{\Gamma\left(\left[\mu(\alpha+m)-\gamma+1-\frac{a}{2}+\frac{ja}{2}\right]+b+1+ra-ja\right)}.$$

Since the quantity $b+1+ra-ja$ is a positive integer, by the well-known property $\Gamma(z+1)=z\Gamma(z)$ we get

$$\frac{(\alpha+m-1)\cdot\ldots\cdot(\alpha+m-d)}{\prod_{j=1}^{r}\prod_{k=0}^{b+a(r-j)}\left(\mu(\alpha+m)+1-\gamma-\frac{a}{2}+\frac{ja}{2}+k\right)}. \tag{13}$$

It is easy to verify by using (6) that numerator and denominator regarded as polynomial in the variable m have the same degree d. If the above quantity does not depend on m, then it must be equal to its limit as m goes to infinity, i.e to $1/\mu^d$. Thus by (13) we get

$$\mu^d(\alpha+m-1)\cdot\ldots\cdot(\alpha+m-d)=\prod_{j=1}^{r}\prod_{k=0}^{b+a(r-j)}\left(\mu(\alpha+m)+1-\gamma-\frac{a}{2}+\frac{ja}{2}+k\right),$$

from which it follows $\mu=1$ (comparing the terms not depending on $\alpha+m$). Setting $\mu=1$ and comparing the terms of degree $d-1$ in m one gets

$$\sum_{k=1}^{d}(\alpha-k)=\sum_{j=1}^{r}\sum_{k=0}^{b+a(r-j)}\left(\alpha+1-\gamma-\frac{a}{2}+\frac{ja}{2}+k\right),$$

that is,

$$d\left(\alpha-\frac{d+1}{2}\right)=r\left(\alpha+1+\frac{a}{2}(r-1)-\gamma+\frac{b}{2}\right)\left(b+1+\frac{a}{2}(r-1)\right),$$

which by definition of γ and by (6) leads to the following second order equation in r:

$$ar^2+2\left(b+1-\frac{3}{2}a\right)r+2(a-b-1)=0.$$

This equation is satisfied by $r=1$, i.e. $\Omega=\mathbb{C}\mathrm{H}^d$. On the other hand, if $r\neq 1$ we can assume $a\neq 0$ and the second solution is given by $r=2(a-b-1)/a$, condition which is not satisfied by any Cartan domain (see Table 1), and this concludes the proof of the theorem.

□

References

[1] J. Arazy, *A Survey of Invariant Hilbert Spaces of Analytic Functions on Bounded Symmetric Domains*, Contemporary Mathematics 185 (1995).

[2] C. Arezzo, A. Loi, *Moment maps, scalar curvature and quantization of Kähler manifolds*, Comm. Math. Phys. 243 (2004), 543-559.

[3] M. Cahen, S. Gutt, J. Rawnsley, *Quantization of Kähler manifolds. I: Geometric interpretation of Berezin's quantization*, J. Geom. Physics 7 (1990), 45–62.

[4] E. Calabi, *Isometric Imbeddings of Complex Manifolds*, Ann. Math. 58 (1953), 1-23.

[5] F. Cuccu and A. Loi, *Global symplectic coordinates on complex domains*, J. Geom. and Phys. 56 (2006), 247-259.

[6] S. Donaldson, *Scalar Curvature and Projective Embeddings, I*, J. Diff. Geometry 59 (2001), 479-522.

[7] M. Engliš, *Weighted Bergman kernels and balanced metrics*, RIMS Kokyuroku 1487 (2006), 40–54.

[8] J. Faraut, A. Koranyi, *Function Spaces and Reproducing Kernels on Bounded Symmetric Domains*, Journal of Functional Analysis 88 (1990), 64–89.

[9] T. Gramchev, A. Loi, *TYZ expansion for the Kepler manifold*, Comm. Math. Phys. 289, (2009), 825-840.

[10] A. Greco, A. Loi, *Radial balanced metrics on the unit disk* J. Geom. Phys. 60 (2010), 53-59.

[11] A. Loi, *Regular quantizations of Kähler manifolds and constant scalar curvature metrics*, J. Geom. Phys. 53 (2005), 354-364.

[12] A. Loi, *Regular quantizations of Kähler manifolds and constant scalar curvature metrics*, J. Geom. Phys. 53 (2005), 354-364.

[13] A. Loi, M. Zedda, *Kähler–Einstein submanifolds of the infinite dimensional projective space*, to appear in Mathematische Annalen.

[14] G. Roos, Keiping Lu, Weiping Yin, *New classes of domains with explicit Bergman kernel*, Science in China 47, no. 3 (2004), 352–371.

[15] G. Roos, A. Wang, W. Yin, L. Zhang, *The Kähler-Einstein metric for some Hartogs domains over bounded symmetric domains*, Science in China 49 (September 2006).

[16] J. Rawnsley, *Coherent states and Kähler manifolds*, Quart. J. Math. Oxford (2), n. 28 (1977), 403–415.

[17] H. Upmeier, *Index theory for Toeplitz operators on bounded symmetric domains*, Représentations des groupes et analyse complexe (Luminy, 1986), 89–94, Journées SMF, 24, Univ. Poitiers, Poitiers (1986).

Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

E-mail address: `loi@unica.it; michela.zedda@gmail.com`