Collocation Boundary Element Method for the pricing of Geometric Asian Options

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Abstract

The Semi-Analytical method for pricing of Barrier Options (SABO) already applied in the context of European options is here extended to the evaluation of geometric Asian options with barriers. The validity of this approximation method, based on the use of collocation Boundary Element Method, is illustrated by numerical examples, where accuracy and stability of the presented approach are analyzed.

Keywords: Boundary Element Method, Fokker-Planck equation, Geometric Asian Options, Barrier Options, Greeks.

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1. Introduction

The availability of advanced numerical techniques and faster computer systems are often exploited for a more scientific approach to the problem of pricing financial products.
A new algorithm, the so-called SABO (Semi-Analytical method for pricing of Barrier Options), for the computation of European-style barrier options in the Black-Scholes and Heston models has been recently introduced in [1], [2], [3] and anticipated in [4] and [5] .
SABO has resulted to be stable and efficient in the special case of "barrier options" as it is based on Boundary Element Method that perfectly suits differential problems defined in unbounded domains whose data are assigned on a limited boundary. Computations are performed with high accuracy because of the implicit satisfaction of the solution far-field behavior and because of the low discretization costs. Moreover, the method provides a straight hedging computation. The essential requisite, that makes it not as general as other numerical methods, is that, for its application, we need the knowledge, at least in an approximated form, of the transition probability density related to the vanilla option problem.
This paper is aimed at implementing and testing the validity of SABO in the evaluation of continuously sampled geometric Asian options with barrier [6].

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Asian options are derivative contracts giving the holder the right to buy an asset for its average price over some prescribed period. Accordingly, their payoff at maturity depends on the average value of an underlying asset over some time interval; therefore we must keep track of more information about the asset price path than simply its present position. The average used in the calculation of the option's payoff can be defined in different ways: it can be an arithmetic average or a geometric average and the data could be discretely sampled or continuously sampled so that every realized asset price over the given period is used. Almost all Asian options are traded among practitioners with arithmetic average, but this work can be conceived as an intermediate and preparatory step, because the study of geometric case can give some information also about the evaluation of Asian barrier options with arithmetic mean (for which it is a lower bound and that can be used as control variate in Monte Carlo simulations) and because the mathematical foundations in the geometric case are well established and numerically easier to treat.

In presence of a "barrier", Asian option contracts get into existence or extinguish when the underlying asset reaches a certain barrier value.

With this additional condition w.r.t. plain vanilla contracts, the buyer get a reasonable protection against inconvenient fluctuations of the underlying price and the issuer can attain a better forecasting of the terminal position. In general Asian options, and in particular Asian barrier options, are less expensive than corresponding vanilla options and therefore they are more attractive.

For standard Asian options with geometric mean equipped with floating or fixed strike price, closed formula solutions are available [7], but if the contract involves non standard payoffs or arithmetic mean or barriers, numerical techniques are unavoidable. The pricing is then traditionally based on Monte Carlo methods [7], binomial/trinomial methods [8] or on domain methods, such as Finite Volume Methods [9] and Finite Difference methods [10]. Monte Carlo methods are affected by high computational costs and inaccuracy due to their slow convergence; domain methods have some troubles concerning stability: for path-dependent options, but also in the simpler Black-Scholes European option framework, there is the problem of degeneracy of the involved differential operator, pointed out for example in [11] and [12], in fact, for small volatility, the pricing PDE is convection dominated, leading to numerical problems in the form of spurious oscillations. For a quite complete survey and careful analysis of numerical methods available for arithmetic and geometric Asian options without barriers, the interested reader is referred to [13].

Anyway, barrier options are largely exchanged, as they are good products for hedging and investment and they are cheaper than vanilla options, but for Asian options we found in literature only the analysis of [14] which provides rigorous bounds in the arithmetic mean case. In this paper we illustrate how efficient, reliable and quite plain the application of SABO to continuously sampled geometric Asian option with barriers is. For clarity, the description is carried out in the case of call options with an up-and-out barrier and numerical examples concern only the case of fixed strike payoff but the method is very general w.r.t. these features. Unfortunately, the same can not be said referring to the extension to continuously sampled arithmetic Asian option, that, from a theoretical point of view, needs only some slight modifications but, practically, it collides with some numerical difficulties that will be the object of our next investigation.

The paper is structured as follows: in Sec. 2 there is an overview of the model problem, SABO method is described in Sec. 3, while in Sec. 4 there are some hints about performing hedging by SABO. At last in Sec. 5 two numerical examples related to a geometric Asian call option with fixed strike payoff and up-and-out barrier are presented and discussed.

2. The model problem

A geometric Asian option V is an option depending on the evolution of the stock price S_t (through the duration of the contract, assumed to be $[0,T]$) and on the geometric average of the stock price over some time interval : $\exp(A_t/t)$, having defined

$$A_t := \int_0^t \log(S_t)dt\,. \tag{1}$$

If the stochastic process S_t is modeled by the usual geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2}$$

where r denotes the risk free interest rate, σ the volatility and W_t a standard Wiener process, then, A_t is a lognormal stochastic process too.

With the classical hedging arguments applied in the Black-Scholes framework [12], it is possibile to conclude that the Asian option value $V(S,A,t)$ solves the following partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \log(S)\frac{\partial V}{\partial A} - rV = 0 \qquad S \in \mathbb{R}^+,\, A \in \mathbb{R},\, t \in [0,T) \tag{3}$$

Different final boundary conditions (payoffs) define different types of contract, such as:

$$\text{floating strike call} \qquad V(S,A,T) = \max\left(S - \exp\left(\frac{A}{T}\right), 0\right) \tag{4}$$

$$\text{floating strike put} \qquad V(S,A,T) = \max\left(\exp\left(\frac{A}{T}\right) - S, 0\right) \tag{5}$$

$$\text{fixed strike call} \qquad V(S,A,T) = \max\left(\exp\left(\frac{A}{T}\right) - E, 0\right) \tag{6}$$

$$\text{fixed strike put} \qquad V(S,A,T) = \max\left(E - \exp\left(\frac{A}{T}\right), 0\right) \tag{7}$$

for $S \in \mathbb{R}^+$, $A \in \mathbb{R}$ and E the strike price. Fixed strike Asian options are less expensive than vanilla options and guarantee that the average exchange rate realized during the year is above some level. Floating strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. However SABO can treat also other more unusual payoffs.

Explicit boundary conditions are not available in literature. Some boundary conditions are implicitly satisfied by V through its payoff behavior and they are such to assure existence and uniqueness of the Cauchy partial differential problem solution (issue that is discussed in Appendix A.1). Anyway, by stochastic considerations, it is possible to define the exact solution in an integral form as payoff expected value that can be therefore employed also with payoff contracts more general than (4)-(7):

$$V(S,A,t) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)d\widetilde{S}\,d\widetilde{A}\,. \tag{8}$$

The function $G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})$ is the *transition probability density function* (PDF), also known as *Green's function* or *fundamental solution* of the partial differential problem: as a function of $(S,A,t) \in \mathbb{R}^+ \times \mathbb{R} \times [0,T)$ the PDF solves (3) and, as a function of $(\widetilde{S},\widetilde{A},\widetilde{t})$, it solves the *backward Kolmogorov equation* adjoint of (3): *for each* $(S,A,t) \in \mathbb{R}^+ \times \mathbb{R} \times [0,T)$

$$\begin{cases} -\dfrac{\partial G}{\partial \widetilde{t}} + \dfrac{\sigma^2}{2}\widetilde{S}^2\dfrac{\partial^2 G}{\partial \widetilde{S}^2} + (2\sigma^2 - r)\widetilde{S}\dfrac{\partial G}{\partial \widetilde{S}} - \log(\widetilde{S})\dfrac{\partial G}{\partial \widetilde{A}} + (\sigma^2 - 2r)G = 0 & \widetilde{S} \in \mathbb{R}^+,\ \widetilde{A} \in \mathbb{R},\ \widetilde{t} > t \\ G(S,A,t;\widetilde{S},\widetilde{A},t) = \delta(S-\widetilde{S})\delta(A-\widetilde{A}) & \widetilde{S} \in \mathbb{R}^+,\ \widetilde{A} \in \mathbb{R} \end{cases} \tag{9}$$

where $\delta(\cdot,\cdot)$ represents the Dirac distribution[1]. The solution of problem (9) must satisfy suitable boundary conditions assuring that the Green identity[2] is verified. Look at [15] for the Differential Analysis on the matter.

Denoting by $H[\cdot]$ the Heaviside step function, the closed form solution of problem (9) is

$$\begin{aligned} G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \;=\; & \frac{\sqrt{3}H[\widetilde{t}-t]}{\pi\sigma^2(\widetilde{t}-t)^2}\exp\left\{-\frac{2}{\sigma^2(\widetilde{t}-t)}\log^2\left(\frac{S}{\widetilde{S}}\right)\right. \\ & + \frac{6}{\sigma^2(\widetilde{t}-t)^2}\log\left(\frac{S}{\widetilde{S}}\right)\big(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\big) \\ & - \frac{6}{\sigma^2(\widetilde{t}-t)^3}\big(A-\widetilde{A}+(\widetilde{t}-t)\log(S)\big)^2 \\ & \left. - \left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-t)\right\}\left(\frac{\widetilde{S}}{S}\right)^{\frac{2r-\sigma^2}{2\sigma^2}}\frac{1}{\widetilde{S}} \end{aligned} \tag{10}$$

that satisfies

$$\int_{-\infty}^{+\infty}\int_{0}^{+\infty} G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})d\widetilde{S}\,d\widetilde{A} = \exp\big(-r(\widetilde{t}-t)\big)\,. \tag{11}$$

The attainment of expression (10) is related to theoretical results in Appendix A.1.

When considering fixed strike option, the exact solution can be evaluated also by another more efficient closed-formula of Black-Scholes type [16]:

[1]The *Dirac's delta distribution* satisfies the property that $\int_{-\infty}^{+\infty}\delta(y,x)f(x)dx = f(y)\,,\ \forall f \in \mathcal{C}_0^\infty(\mathbb{R})$.

[2]When considering the PDE $\mathcal{P}[u] = 0$ defined by the partial differential operator $\mathcal{P}$ applied to the unknown solution u then, a function G, satisfies the Green identity if

$$\langle \mathcal{P}[u], G\rangle - \langle u, \mathcal{P}^*[G]\rangle = 0$$

where $\mathcal{P}^*$ is the adjoint of operator $\mathcal{P}$ and $\langle\cdot\rangle$ is the L^2 scalar product.

- for the call option

$$C_{\text{fix}}(S,A,t) = S^*\mathcal{N}\left(\frac{\log\frac{S^*}{E} + (r+\frac{\sigma^{*2}}{2})(T-t)}{\sigma^*\sqrt{T-t}}\right) - Ee^{-r(T-t)}\mathcal{N}\left(\frac{\log\frac{S^*}{E} + (r-\frac{\sigma^{*2}}{2})(T-t)}{\sigma^*\sqrt{T-t}}\right)$$

$$S^* = S^{\frac{T-t}{T}}\exp\left(\frac{A}{T} + (\mu^* - r)(T-t)\right)$$

$$\mu^* = \left(r - q - \frac{\sigma^2}{2}\right)\frac{T-t}{2T} + \frac{\sigma^2}{6}\frac{(T-t)^2}{T^2}, \quad \sigma^* = \frac{\sigma}{\sqrt{3}}\frac{T-t}{T} \tag{12}$$

with $\mathcal{N}(\cdot)$ normal cumulative distribution;

- for the put option

$$P_{\text{fix}}(S,A,t) = Ee^{-r(T-t)}\mathcal{N}\left(-\frac{\log\frac{S^*}{E} + (r-\frac{\sigma^{*2}}{2})(T-t)}{\sigma^*\sqrt{T-t}}\right) - S^*\mathcal{N}\left(-\frac{\log\frac{S^*}{E} + (r+\frac{\sigma^{*2}}{2})(T-t)}{\sigma^*\sqrt{T-t}}\right). \tag{13}$$

These formulas satisfy the following put-call parity relation[3]:

$$\begin{aligned} C_{\text{fix}}(S,A,t) - P_{\text{fix}}(S,A,t) &= e^{-r(T-t)}\left(\mathbb{E}[e^{A/T}] - E\right) \\ &= S^* - Ee^{-r(T-t)}. \end{aligned} \tag{14}$$

In the special case $t = 0, A = 0$, the formula was first given in [7] and now it is implemented by Matlab® function `asianbykv`.

To Asian options we can apply some barriers, as often done with European options, in order to reduce their price and to ward against excessive fluctuations of stock price.

For this kind of geometric Asian options neither closed form solutions are available nor we have found some analysis in literature.

As example: a geometric Asian up-and-out barrier option is an option that is extinguished when the price of the underlying asset grows up enough to breach an assigned upper barrier B before the expiry date T. So let us consider the modeling partial differential problem:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \log(S)\frac{\partial V}{\partial A} - rV = 0 \qquad S \in (0,B),\ A \in \mathbb{R},\ t \in [0,T) \tag{15}$$

$$V(S,A,T) \quad \text{assigned} \qquad S \in (0,B),\ A \in \mathbb{R} \tag{16}$$

$$V(B,A,t) = 0 \qquad A \in \mathbb{R},\ t \in [0,T) \tag{17}$$

$$\text{asymptotic conditions of vanilla option} \qquad \{(S,A) : S = 0 \vee A \to -\infty \vee A \to +\infty\}. \tag{18}$$

The method, that we will illustrate in the following section for call options solving (15)-(18), is rather flexible; therefore it can be easily extended also to Asian call options with other types of barrier, that widen or contract (moving barriers), and to put options, too.

[3]Thanks to the property: $\mathcal{N}(d) + \mathcal{N}(-d) = 1$.

3. The SABO approach

SABO is the acronym of **S**emi-**A**nalytical method for the pricing of **B**arrier **O**ptions and substantially it is the application of Boundary Element Method (BEM) to the barrier option problems. The method is based on the below listed steps.

3.1. The integral representation formula for the solution of the partial differential problem

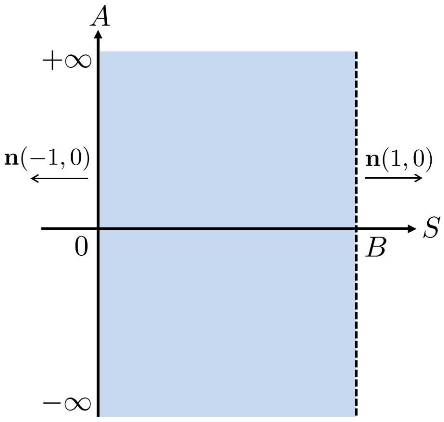

Figure 1: $(S, A) \in \Omega := (0, B) \times \mathbb{R}$, spatial domain of the differential problem (15)-(18) modeling an Asian option with up-and-out barrier.

Considering the differential problem (15) for an up-and-out barrier call option, the domain of investigation for $V(S, A, t)$ is $\Omega \times [0, T)$ having defined $\Omega := (0, B) \times \mathbb{R}$ as represented in Fig. 1. For such a problem the integral formulation (8) in the new domain has to be modified inserting one more term as established in the following Proposition.

Proposition: *Let* $V(S, A, T) \in \mathcal{C}(\mathbb{R}^2)$ *such that the following integrals are well defined in the sense of condition* (57). *Then*

$$\begin{aligned} V(S,A,t) &= \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(S,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} \\ &+ \int_t^T \int_{-\infty}^{+\infty} \frac{\sigma^2}{2} B^2 \frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t}) G(S,A,t;B,\widetilde{A},\widetilde{t}) d\widetilde{A}\, d\widetilde{t}\,. \end{aligned} \tag{19}$$

is solution of (15)-(18) *in* $\Omega \times [0, T)$.

Proof:

We will use some theory in [15] and [17].

Rewrite model equation (15) in the compact form

$$\frac{\partial V}{\partial \widetilde{t}}(\widetilde{S},\widetilde{A},\widetilde{t}) + \mathcal{L}[V](\widetilde{S},\widetilde{A},\widetilde{t}) = 0 \qquad \forall (\widetilde{S},\widetilde{A},\widetilde{t}) \in (0,B) \times \mathbb{R} \times [0,T) \tag{20}$$

having defined the operator

$$\mathcal{L}[V](\widetilde{S},\widetilde{A},\widetilde{t}) := \frac{\sigma^2}{2}\widetilde{S}^2 \frac{\partial^2 V}{\partial \widetilde{S}^2}(\widetilde{S},\widetilde{A},\widetilde{t}) + r\widetilde{S}\frac{\partial V}{\partial \widetilde{S}}(\widetilde{S},\widetilde{A},\widetilde{t}) + \log(\widetilde{S})\frac{\partial V}{\partial \widetilde{A}}(\widetilde{S},\widetilde{A},\widetilde{t}) - rV(\widetilde{S},\widetilde{A},\widetilde{t}) \tag{21}$$

and its adjoint operator

$$\mathcal{L}^*[V](\widetilde{S},\widetilde{A},\widetilde{t}) := \frac{\sigma^2}{2}\widetilde{S}^2\frac{\partial^2 V}{\partial \widetilde{S}^2}(\widetilde{S},\widetilde{A},\widetilde{t})+(2\sigma^2-r)\widetilde{S}\frac{\partial V}{\partial \widetilde{S}}(\widetilde{S},\widetilde{A},\widetilde{t})-\log(\widetilde{S})\frac{\partial V}{\partial \widetilde{A}}(\widetilde{S},\widetilde{A},\widetilde{t})+(\sigma^2-2r)V(\widetilde{S},\widetilde{A},\widetilde{t}) \tag{22}$$

Multiply (20) by the fundamental solution $G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})$; then subtract the PDE in (9) multiplied by $V(\widetilde{S},\widetilde{A},\widetilde{t})$ and integrate in time and space obtaining

$$\begin{aligned} 0 &= \int_t^T\int_{-\infty}^{+\infty}\int_0^B\left\{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\frac{\partial V}{\partial \widetilde{t}}(\widetilde{S},\widetilde{A},\widetilde{t})+V(\widetilde{S},\widetilde{A},\widetilde{t})\frac{\partial G}{\partial \widetilde{t}}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\right\}d\widetilde{S}\,d\widetilde{A}\,d\widetilde{t}\\ &+ \int_t^T\int_{-\infty}^{+\infty}\int_0^B\left\{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}[V](\widetilde{S},\widetilde{A},\widetilde{t})-V(\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}^*[G](S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\right\}d\widetilde{S}\,d\widetilde{A}\,d\widetilde{t}\,. \end{aligned} \tag{23}$$

The kernel in the second integral of (23) can be rewritten in a differential form

$$\begin{aligned} &G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}[V](\widetilde{S},\widetilde{A},\widetilde{t})-V(\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}^*[G](S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\\ &=\frac{\partial p_1}{\partial \widetilde{S}}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})+\frac{\partial p_2}{\partial \widetilde{A}}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \end{aligned} \tag{24}$$

with

$$\begin{aligned} p_1(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) &= V(\widetilde{S},\widetilde{A},\widetilde{t})\left(\widetilde{S}(r-\sigma^2)G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})-\frac{\sigma^2}{2}\widetilde{S}^2\frac{\partial G}{\partial \widetilde{S}}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\right)\\ &+\frac{\sigma^2}{2}\widetilde{S}^2\frac{\partial V}{\partial \widetilde{S}}(\widetilde{S},\widetilde{A},\widetilde{t})G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\\ p_2(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) &= \log(\widetilde{S})G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})V(\widetilde{S},\widetilde{A},\widetilde{t}) \end{aligned} \tag{25}$$

thus, by the divergence theorem,

$$\begin{aligned} &\int_t^T\int_{-\infty}^{+\infty}\int_0^B\left\{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}[V](\widetilde{S},\widetilde{A},\widetilde{t})-V(\widetilde{S},\widetilde{A},\widetilde{t})\mathcal{L}^*[G](S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\right\}d\widetilde{S}\,d\widetilde{A}\,d\widetilde{t}\\ &=\int_t^T\int_{\partial\Omega}(p_1,p_2)\cdot\mathbf{n}\,d\widetilde{S}\,d\widetilde{A}\,d\widetilde{t} \end{aligned} \tag{26}$$

where $\partial\Omega := \{(S,A) : S = 0 \vee S = B\}$.

Taking into account the boundary conditions (18) means that the integral at $\widetilde{S} = 0$ vanishes in fact the representation formula for vanilla option is (8). The truncation of vanilla option domain at the barrier with zero boundary condition (17) implies the integral representation (19) of the PDE problem (15)-(18):

$$\begin{aligned} &\int_t^T\int_{\partial\Omega}(p_1(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}),p_2(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}))\cdot\mathbf{n}\,d\widetilde{S}\,d\widetilde{A}\,d\widetilde{t}=\int_t^T\int_{-\infty}^{+\infty}p_1(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}\\ &=\int_t^T\int_{-\infty}^{+\infty}\frac{\sigma^2}{2}B^2\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})G(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}\,. \end{aligned}$$

The second term in the first integral in (23) can be integrated by parts in time

$$\int_t^T \int_{-\infty}^{+\infty} \int_0^B \left\{ G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \frac{\partial V}{\partial \widetilde{t}}(\widetilde{S},\widetilde{A},\widetilde{t}) + V(\widetilde{S},\widetilde{A},\widetilde{t}) \frac{\partial G}{\partial \widetilde{t}}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \right\} d\widetilde{S}\, d\widetilde{A}\, d\widetilde{t}$$

$$= \int_{-\infty}^{+\infty} \int_0^B \left\{ \int_t^T G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \frac{\partial V}{\partial \widetilde{t}}(\widetilde{S},\widetilde{A},\widetilde{t})\, d\widetilde{t} \right.$$

$$\left. + V(\widetilde{S},\widetilde{A},\widetilde{t}) G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \Big|_{\widetilde{t}=t}^{\widetilde{t}=T} - \int_t^T G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) \frac{\partial V}{\partial \widetilde{t}}(\widetilde{S},\widetilde{A},\widetilde{t})\, d\widetilde{t} \right\} d\widetilde{S}\, d\widetilde{A}$$

$$= \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(S,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} - \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},t) G(S,A,t;\widetilde{S},\widetilde{A},t) d\widetilde{S}\, d\widetilde{A} =$$

and taking into account the final condition in (9), it remains

$$= \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(S,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} - \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},t) \delta(S-\widetilde{S}) \delta(A-\widetilde{A}) d\widetilde{S}\, d\widetilde{A}$$

$$= \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(S,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} - V(S,A,t)\,.$$

The representation formula (19) follows immediately. $\square$

3.2. The boundary integral equation (BIE)

In the integral formula (19) $\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})$ is unknown. If we succeed in computing it, formula (19) gives us the solution of problem (15)-(18) over the whole domain $\Omega \times [0,T)$.

With this purpose, we take the limit for $S \to B$ in (19) and, using boundary condition (17), we obtain

$$\begin{aligned} 0 = V(B,A,t) &= \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(B,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} \\ &+ \int_t^T \int_{-\infty}^{+\infty} \frac{\sigma^2}{2} B^2 \frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t}) G(B,A,t;B,\widetilde{A},\widetilde{t}) d\widetilde{A}\, d\widetilde{t} \end{aligned} \tag{27}$$

i.e.

$$\int_t^T \int_{-\infty}^{+\infty} \frac{\sigma^2}{2} B^2 \frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t}) G(B,A,t;B,\widetilde{A},\widetilde{t}) d\widetilde{A}\, d\widetilde{t} = -\int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S},\widetilde{A},T) G(B,A,t;\widetilde{S},\widetilde{A},T) d\widetilde{S}\, d\widetilde{A} \tag{28}$$

in the sole unknown $\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})$.

The idea implemented by SABO is to approximate $\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})$ numerically solving (27) and then, inserting it in the representation formula (19), to recover the solution V at every desired point of the domain $\Omega \times [0,T)$.

3.3. The numerical approximation of the BIE solution

The approximation of the BIE unknown $\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})$ is found by collocation method as in [3] and it is structured as follows:

I) introduction of a uniform decomposition of the time interval $[0,T]$

$$\Delta t := \frac{T}{N_t}, \quad N_t \in \mathbb{N}^+, \qquad t_k := k\Delta t, \quad k = 0,\ldots,N_t \tag{29}$$

and time representation of the BIE unknown by piecewise constant basis functions

$$\varphi_k(\widetilde{t}) := H[\widetilde{t} - t_{k-1}] - H[\widetilde{t} - t_k], \quad k = 1, \ldots, N_t\,; \tag{30}$$

II) introduction of a quasi-uniform decomposition in the unbounded A-domain $\equiv \mathbb{R}$.

We know that the A-domain could be suitably truncated by $[A_{\min}, A_{\max}]$ setting $A_{\min} = A$ where A is the value at which $V(S, A, t)$ needs to be evaluated (in literature generally at $A = 0$) because the definition (1) implies that A_t grows with the passage of time if $S_t > 1$ (even if variables A and S have to be considered as independent). If S may assume values in the interval $[0, 1]$ it will be necessary to set a condition on $A_{\min}$ like that one just below proposed to determine $A_{\max}$.

In [18], the authors suggest to choose $A_{\max} = T \log(E)$ but we are able to appropriately set $A_{\max}$ by (11) given that

$$\begin{aligned}
&\exp\big(-r(\widetilde{t}-t)\big) \; = \\
= \;& \int_{-\infty}^{+\infty}\int_0^{+\infty} G(S, A, t; \widetilde{S}, \widetilde{A}, \widetilde{t}) d\widetilde{S}\, d\widetilde{A} \\
= \;& \int_{-\infty}^{+\infty} \frac{1}{(\widetilde{t}-t)^{3/2}\sigma}\sqrt{\frac{3}{2\pi}} S^{\frac{3}{4}-\frac{3(2A-2\widetilde{A}+r(\widetilde{t}-t)^2)}{2(\widetilde{t}-t)^2\sigma^2}} \\
&\exp\left(-\frac{12\Big(2A-2\widetilde{A}+r(\widetilde{t}-t)^2\Big)^2+4(\widetilde{t}-t)^2\sigma^2\Big(-6A+6\widetilde{A}+5r(\widetilde{t}-t)^2\Big)+3(\widetilde{t}-t)^4\sigma^4+48(\widetilde{t}-t)^2\log^2(S)}{32(\widetilde{t}-t)^3\sigma^2}\right) d\widetilde{A} \\
= \;& \frac{1}{2}\exp\big(-r(\widetilde{t}-t)\big)\mathrm{Erf}\left[\sqrt{\frac{3}{2}}\frac{-4A+4\widetilde{A}-2r(\widetilde{t}-t)^2+(\widetilde{t}-t)^2\sigma^2-4(\widetilde{t}-t)\log(S)}{4(\widetilde{t}-t)^{3/2}\sigma}\right]\Bigg|_{\widetilde{A}\to-\infty}^{\widetilde{A}\to+\infty} \\
\approx \;& \frac{1}{2}\exp\big(-r(\widetilde{t}-t)\big)\mathrm{Erf}\left[\sqrt{\frac{3}{2}}\frac{-4A+4\widetilde{A}-2r(\widetilde{t}-t)^2+(\widetilde{t}-t)^2\sigma^2-4(\widetilde{t}-t)\log(S)}{4(\widetilde{t}-t)^{3/2}\sigma}\right]\Bigg|_{\widetilde{A}=0}^{\widetilde{A}=A_{\max}} .
\end{aligned}$$

therefore looking for the root of the following non linear equation in the unknown $A_{\max}$ (by Matlab `fzero` function with a tolerance equal to 10^{-10})

$$\frac{1}{\sqrt{\pi}}\int_{\sqrt{\frac{3}{2}}\frac{-4A-2r(\widetilde{t}-t)^2+(\widetilde{t}-t)^2\sigma^2-4(\widetilde{t}-t)\log(S)}{4(\widetilde{t}-t)^{3/2}\sigma}}^{\sqrt{\frac{3}{2}}\frac{-4A+4A_{\max}-2r(\widetilde{t}-t)^2+(\widetilde{t}-t)^2\sigma^2-4(\widetilde{t}-t)\log(S)}{4(\widetilde{t}-t)^{3/2}\sigma}} e^{-s^2}\, ds = 1\,, \tag{31}$$

setting $\widetilde{t} - t = T$ and S equal to the maximum value in the range for which $V(S, A, t)$ is requested. In the numerical examples, the two strategies produces about the same upper bound $A_{\max}$.

Actually we will consider two infinite elements in order to avoid such truncation but the choice of $A_{\min}$ and $A_{\max}$ suggests us the right choice for the discretization parameter ΔA and for the definition of the infinite elements:

$$\Delta A := \frac{A_{\max} - A_{\min}}{N_A}, \qquad A_h := A_{\min} + h\Delta A, \quad h = 0, \ldots, N_A \tag{32}$$

the BIE unknown is represented in the independent variable $\widetilde{A}$ by piecewise constant basis functions

$$\psi_h(\widetilde{A}) := H[\widetilde{A} - A_{h-1}] - H[\widetilde{A} - A_h], \quad h = 2, \ldots, N_A - 1 \tag{33}$$

with two infinite elements over $(-\infty, A_1)$ and $(A_{N_A-1}, +\infty)$ with related basis functions

$$\psi_1(\widetilde{A}) := H[A_1 - \widetilde{A}] \qquad \psi_{N_A}(\widetilde{A}) := H[\widetilde{A} - A_{N_A-1}]. \tag{34}$$

Anyway, note that considering piecewise constant basis functions on the interval $[A_{\min}, A_{\max}]$, as illustrated in [19], gives us a little computational saving (because the matrix is of Toeplitz type also w.r.t. A) and about the same good approximation results.

III) approximation of the BIE unknown

$$\frac{\partial V}{\partial \widetilde{S}}(B, \widetilde{A}, \widetilde{t}) \approx \sum_{k=1}^{N_t} \sum_{h=1}^{N_A} \alpha_h^{(k)} \psi_h(\widetilde{A}) \varphi_k(\widetilde{t}) \tag{35}$$

in the boundary integral equation (27);

IV) definition of the collocation points: as usual when considering piecewise constant trial functions, they are the centers of intervals $[A_{i-1}, A_i] \times [t_{j-1}, t_j]$

$$\overline{A}_i = \frac{A_i + A_{i-1}}{2}, \quad i = 1, \ldots, N_A \qquad \bar{t}_j = \frac{t_j + t_{j-1}}{2}, \quad j = 1, \ldots, N_t\,; \tag{36}$$

V) evaluation of (27) at the collocation points $(\overline{A}_i, \bar{t}_j)$ building a linear system of $N_A \times N_t$ equations:

for $i = 1, \ldots, N_A$, $j = 1, \ldots, N_t$

$$\begin{aligned} 0 \;&=\; \int_{-\infty}^{+\infty} \int_0^B V(\widetilde{S}, \widetilde{A}, T) G(B, \overline{A}_i, \bar{t}_j; \widetilde{S}, \widetilde{A}, T)\, d\widetilde{S}\, d\widetilde{A} \\ &+\; \int_{\bar{t}_j}^T \int_{-\infty}^{+\infty} \frac{\sigma^2}{2} B^2 \sum_{k=1}^{N_t} \sum_{h=1}^{N_A} \alpha_h^{(k)} \psi_h(\widetilde{A}) \varphi_k(\widetilde{t}) G(B, \overline{A}_i, \bar{t}_j; B, \widetilde{A}, \widetilde{t})\, d\widetilde{A}\, d\widetilde{t}\,. \end{aligned} \tag{37}$$

VI) resolution of the linear system

$$\mathcal{A}\alpha = \mathcal{F} \tag{38}$$

whose unknowns are the coefficients of linear representation in (35)

$$\alpha = (\alpha^{(k)}\big|_{k=1,\ldots,N_t}) = \big((\alpha_h^{(k)}\big|_{h=1,\ldots,N_A})\big|_{k=1,\ldots,N_t}\big). \tag{39}$$

The matrix entries are:

for $i = 1, \ldots, N_A,\; h = 1,\; j, k = 1, \ldots, N_{\Delta t}$

$$\begin{aligned} \mathcal{A}_{i1}^{(jk)} \;&=\; \frac{\sigma^2}{2} B^2 \int_{\bar{t}_j}^T \int_{-\infty}^{+\infty} \psi_1(\widetilde{A}) \varphi_k(\widetilde{t}) G(B, \overline{A}_i, \bar{t}_j; B, \widetilde{A}, \widetilde{t})\, d\widetilde{A}\, d\widetilde{t} \\ &=\; \frac{\sigma^2}{2} B^2 H[t_k - \bar{t}_j] \int_{\max(t_{k-1}, \bar{t}_j)}^{t_k} \int_{-\infty}^{A_1} G(B, \overline{A}_i, \bar{t}_j; B, \widetilde{A}, \widetilde{t})\, d\widetilde{A}\, d\widetilde{t} \\ &=\; \frac{\sigma^2}{2} B^2 H[t_k - \bar{t}_j] \int_{\max(t_{k-1}, \bar{t}_j)}^{t_k} \int_{-\infty}^{A_1} \frac{\sqrt{3}}{\pi \sigma^2 (\widetilde{t} - \bar{t}_j)^2 B} \\ &\quad\; \exp\left\{ -\frac{6\big(\overline{A}_i - \widetilde{A} + (\widetilde{t} - \bar{t}_j)\log(B)\big)^2}{\sigma^2 (\widetilde{t} - \bar{t}_j)^3} - \left(\frac{2r + \sigma^2}{2\sqrt{2}\sigma}\right)^2 (\widetilde{t} - \bar{t}_j) \right\} d\widetilde{A}\, d\widetilde{t} \end{aligned} \tag{40}$$

for $i=1,\dots,N_A,\ h=N_A,\ j,k=1,\dots,N_{\Delta t}$

$$
\begin{aligned}
\mathcal{A}^{(jk)}_{iN_A} &= \frac{\sigma^2}{2}B^2\int_{\bar{t}_j}^{T}\int_{-\infty}^{+\infty}\psi_{N_A}(\widetilde{A})\varphi_k(\widetilde{t})G(B,\overline{A}_i,\bar{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \\
&= \frac{\sigma^2}{2}B^2H[t_k-\bar{t}_j]\int_{\max(t_{k-1},\bar{t}_j)}^{t_k}\int_{A_{N_A-1}}^{+\infty}G(B,\overline{A}_i,\bar{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \\
&= \frac{\sigma^2}{2}B^2H[t_k-\bar{t}_j]\int_{\max(t_{k-1},\bar{t}_j)}^{t_k}\int_{A_{N_A-1}}^{+\infty}\frac{\sqrt{3}}{\pi\sigma^2(\widetilde{t}-\bar{t}_j)^2B} \\
&\quad \exp\left\{-\frac{6\big(\overline{A}_i-\widetilde{A}+(\widetilde{t}-\bar{t}_j)\log(B)\big)^2}{\sigma^2(\widetilde{t}-\bar{t}_j)^3}-\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-\bar{t}_j)\right\}d\widetilde{A}\,d\widetilde{t}
\end{aligned}
\tag{41}
$$

for $i=1,\dots,N_A,\ h=2,\dots,N_A-1,\ j,k=1,\dots,N_{\Delta t}$

$$
\begin{aligned}
\mathcal{A}^{(jk)}_{ih} &= \frac{\sigma^2}{2}B^2\int_{\bar{t}_j}^{T}\int_{-\infty}^{+\infty}\psi_{h}(\widetilde{A})\varphi_k(\widetilde{t})G(B,\overline{A}_i,\bar{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \\
&= \frac{\sigma^2}{2}B^2H[t_k-\bar{t}_j]\int_{\max(t_{k-1},\bar{t}_j)}^{t_k}\int_{A_{h-1}}^{A_h}G(B,\overline{A}_i,\bar{t}_j;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \\
&= \frac{\sigma^2}{2}B^2H[t_k-\bar{t}_j]\int_{\max(t_{k-1},\bar{t}_j)}^{t_k}\int_{A_{h-1}}^{A_h}\frac{\sqrt{3}}{\pi\sigma^2(\widetilde{t}-\bar{t}_j)^2B} \\
&\quad \exp\left\{-\frac{6\big(\overline{A}_i-\widetilde{A}+(\widetilde{t}-\bar{t}_j)\log(B)\big)^2}{\sigma^2(\widetilde{t}-\bar{t}_j)^3}-\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(\widetilde{t}-\bar{t}_j)\right\}d\widetilde{A}\,d\widetilde{t}
\end{aligned}
\tag{42}
$$

and, since equation (15) has constant parameters w.r.t. time and A variables, they depend only on the difference between time instants and between grid A-points. Further calculation can be found in Appendix A.2.

By consequence, $\mathcal{A}$ is a block upper triangular matrix with Toeplitz structure: the upper triangularity is due to the fact that the fundamental solution is defined by (9) only for $\widetilde{t}>t$ implying that the matrix entries are non trivial only for $k\geq j$; the Toeplitz structure is due to the dependence on time differences.

$$
\mathcal{A}=\begin{bmatrix}
A^{(0)} & A^{(1)} & A^{(2)} & \cdots & A^{(N_t-1)} \\
0 & A^{(0)} & A^{(1)} & \cdots & A^{(N_t-2)} \\
0 & 0 & A^{(0)} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & A^{(1)} \\
0 & 0 & \cdots & 0 & A^{(0)}
\end{bmatrix}.
\tag{43}
$$

This allows to solve the linear system by block backward substitution and to compute only the entries in the last column blocks with considerable computational saving.

The rhs entries are:

for $i = 1, \dots, N_A$, $j = 1, \dots, N_t$

$$\begin{aligned}
\mathcal{F}_i^{(j)} &= -\int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(B,\overline{A}_i,\overline{t}_j;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} \\
&= -\int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\sqrt{3}}{\pi\sigma^2(T-\overline{t}_j)^2}\exp\left\{-\frac{2}{\sigma^2(T-\overline{t}_j)}\log^2\left(\frac{B}{\widetilde{S}}\right)\right. \\
&+ \frac{6}{\sigma^2(T-\overline{t}_j)^2}\log\left(\frac{B}{\widetilde{S}}\right)\left(\overline{A}_i-\widetilde{A}+(T-\overline{t}_j)\log(B)\right) \qquad (44)\\
&- \frac{6}{\sigma^2(T-\overline{t}_j)^3}\left(\overline{A}_i-\widetilde{A}+(T-\overline{t}_j)\log(B)\right)^2 \\
&- \left.\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(T-\overline{t}_j)\right\}\left(\frac{\widetilde{S}}{B}\right)^{\frac{2r-\sigma^2}{2\sigma^2}}\frac{1}{\widetilde{S}}\,d\widetilde{S}\,d\widetilde{A}\,.
\end{aligned}$$

Look at Appendix A.3 for a useful analysis of rhs entries computation in the specific case of fixed strike payoff, to which numerical results presented afterwards are referred.

3.4. The numerical approximation of option price

Once system (38) is solved, the knowledge of α (that determines the approximation of BIE solution) implies the possibility of computing $V(S,A,t)$ at any point $(S,A,t) \in \Omega\times[0,T)$ introducing (35) in the integral representation formula (19) [4]

$$\begin{aligned}
V(S,A,t) &\approx \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)G(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} \\
&+ \frac{\sigma^2}{2}B^2\sum_{k=\text{floor}\frac{t}{\Delta t}+1}^{N_t}\sum_{h=1}^{N_A}\alpha_h^{(k)}\int_{\max(t,t_{k-1})}^{t_k}\int_{-\infty}^{+\infty}\psi_h(\widetilde{A})G(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}\,.
\end{aligned} \qquad (45)$$

The main advantage of this method is that we can avoid to evaluate the solution over a grid considering only the evaluation at the points of interest.

4. Hedging

This section highlights the easy and straightforward evaluation of the Greeks obtained by deriving the representation formula (19) and using SABO, for $t \in [0,T)$, $S \in (0,B)$

- $$\begin{aligned}
\mathbf{\Delta}(S,A,t) := \frac{\partial V}{\partial S}(S,A,t) &= \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\partial G}{\partial S}(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} \\
&+ \frac{\sigma^2}{2}B^2\int_t^T\int_{-\infty}^{+\infty}\frac{\partial V}{\partial \widetilde{S}}(B,\widetilde{A},\widetilde{t})\frac{\partial G}{\partial S}(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t}
\end{aligned} \qquad (46)$$

with

$$\frac{\partial G}{\partial S}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) = G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\frac{-12(A-\widetilde{A})-(\widetilde{t}-t)^2(2r-\sigma^2)-4(\widetilde{t}-t)\log(\widetilde{S}S^2)}{2\sigma^2 S(\widetilde{t}-t)^2}\,. \qquad (47)$$

[4] floor[·]:=function that rounds its argument to the nearest integers towards minus infinity.

- $$\mathbf{\Gamma}(S,A,t) := \frac{\partial^2 V}{\partial S^2}(S,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\partial^2 G}{\partial S^2}(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} + \frac{\sigma^2}{2}B^2\int_t^T\int_{-\infty}^{+\infty}\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})\frac{\partial^2 G}{\partial S^2}(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \tag{48}$$

with

$$\frac{\partial^2 G}{\partial S^2}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) = \frac{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})}{4\sigma^4S^2(\widetilde{t}-t)^4}\Bigg\{4\Big(6(A-\widetilde{A})+r(\widetilde{t}-t)^2\Big)^2-16\sigma^2(\widetilde{t}-t)^3-\sigma^4(\widetilde{t}-t)^4 + 16(\widetilde{t}-t)\log(S^2\widetilde{S})\Big(6(A-\widetilde{A})+r(\widetilde{t}-t)^2+(\widetilde{t}-t)\log(S^2\widetilde{S})\Big)\Bigg\}\,. \tag{49}$$

- $$\mathbf{\Theta}(S,A,t) := \frac{\partial V}{\partial t}(S,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\partial G}{\partial t}(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} + \frac{\sigma^2}{2}B^2\int_t^T\int_{-\infty}^{+\infty}\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})\frac{\partial G}{\partial t}(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \tag{50}$$

with

$$\frac{\partial G}{\partial t}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) = \frac{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})}{8(\widetilde{t}-t)^4\sigma^2}\Bigg\{-144(A-\widetilde{A})^2-(\widetilde{t}-t)^3\big(-16\sigma^2-(\widetilde{t}-t)(2r+\sigma^2)^2\big) - 16(\widetilde{t}-t)\Big(6(A-\widetilde{A})\log(S\widetilde{S})+(\widetilde{t}-t)\big(\log(\widetilde{S})^2+\log(S)\log(S\widetilde{S})\big)\Big)\Bigg\}\,. \tag{51}$$

- $$\boldsymbol{\rho}(S,A,t) := \frac{\partial V}{\partial r}(S,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\partial G}{\partial r}(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} + \frac{\sigma^2}{2}B^2\int_t^T\int_{-\infty}^{+\infty}\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})\frac{\partial G}{\partial r}(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \tag{52}$$

with

$$\frac{\partial G}{\partial r}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) = G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})\frac{-(\widetilde{t}-t)(2r+\sigma^2)+2\log(\widetilde{S}/S)}{2\sigma^2} \tag{53}$$

- $$\mathbf{Vega}(S,A,t) := \frac{\partial V}{\partial \sigma}(S,t) = \int_{-\infty}^{+\infty}\int_0^B V(\widetilde{S},\widetilde{A},T)\frac{\partial G}{\partial \sigma}(S,A,t;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} + \frac{\sigma^2}{2}B^2\int_t^T\int_{-\infty}^{+\infty}\frac{\partial V}{\partial\widetilde{S}}(B,\widetilde{A},\widetilde{t})\frac{\partial G}{\partial \sigma}(S,A,t;B,\widetilde{A},\widetilde{t})\,d\widetilde{A}\,d\widetilde{t} \tag{54}$$

with

$$\frac{\partial G}{\partial \sigma}(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t}) = \frac{G(S,A,t;\widetilde{S},\widetilde{A},\widetilde{t})}{-4\sigma^3(\widetilde{t}-t)^3}\Bigg\{-48(A-\widetilde{A})^2-(\widetilde{t}-t)^3\Big(4r^2(\widetilde{t}-t)-8\sigma^2-(\widetilde{t}-t)\sigma^4\Big) - 8(\widetilde{t}-t)\Big(6\log(S)\big(2(A-\widetilde{A})+(\widetilde{t}-t)\log(S)\big) + \log(S/\widetilde{S})\big(-6(A-\widetilde{A})+r(\widetilde{t}-t)^2-6(\widetilde{t}-t)\log(S)\big)+2(\widetilde{t}-t)\log(S/\widetilde{S})^2\Big)\Bigg\}\,. \tag{55}$$

Substituting $\frac{\partial V}{\partial \widetilde{S}}(B, \widetilde{A}, \widetilde{t})$ by its approximation (known once the linear system (38) is solved) in (46), (48), (50), (52) and (54) we can straightforwardly compute the related Greeks taking exactly into account possible discontinuities in parameters too (see [1]).
Observe that the "secondary" unknowns can be evaluated even without computing the primary unknown V and only at the point (S, A, t) where its derivatives are required. This advantage is even more evident when considering (52) and (54) because a finite difference algorithm needs to be rerun at least twice to compute them.

5. Numerical results

Numerical examples concern the pricing problem of a call Asian option with an up-and-out barrier (15)-(18) and fixed strike payoff (6), but the application of SABO, in the general form illustrated in Sec. 3, to other payoff functions is straightforward.

- *1st example*

The only possible way to check our numerical results is to observe that, if the barrier is far away from the stock prices interval of interest, then the option price is not significantly influenced by the barrier and therefore formula (12) is a close approximation for a call Asian option with an up-and-out barrier.
In this example we use the same finance parameters found in the Release Notes of Matlab® R2017a in the section "Pricing Asian Options", referring to the use of function `asianbykv` that provides the analytical solution to geometric Asian option get by [7] and included in the formula (12).

B	T	E	r	σ
150	1	90	0.035	0.2

Table 1: Fixed strike up-and-out call option data.

The results obtained by the Matlab specialists are plotted in Fig. 2. As expected, geometric Asian option behaves as lower bound w.r.t. arithmetic one and they are both cheaper w.r.t. European option.

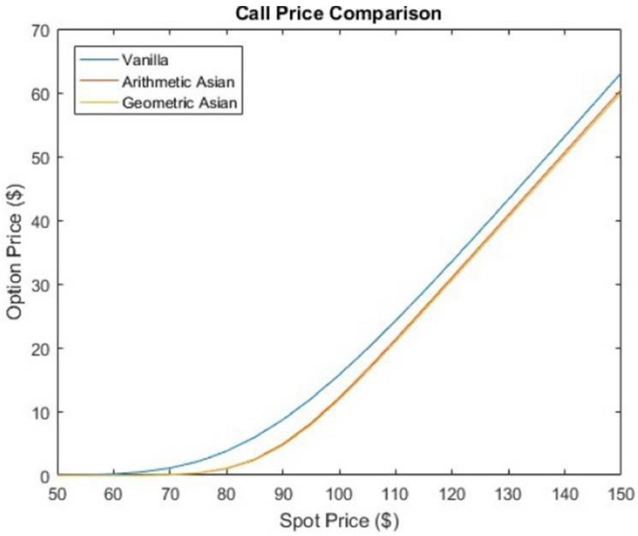

Figure 2: Image taken from the Release Notes of Matlab R2017a, comparing European, geometric Asian and arithmetic Asian options without barriers and with parameters in Table 1.

Actually, if B is far from the interest region $S \in [50, 150]$, for example $B = 200$, then the value of the barrier Asian call option computed by SABO algorithm overlaps the call price without barrier. After this check, we set the barrier at $B = 150$ in order to observe its effective influence on the option price. We computed a fixed strike up-and-out call option approximation at $t = 0$ and $A = 0$, by setting $A_{\min} = 0$, $A_{\max} = 6$ and $N_t = N_A = 20$ obtaining results in Fig. 3 on the left.

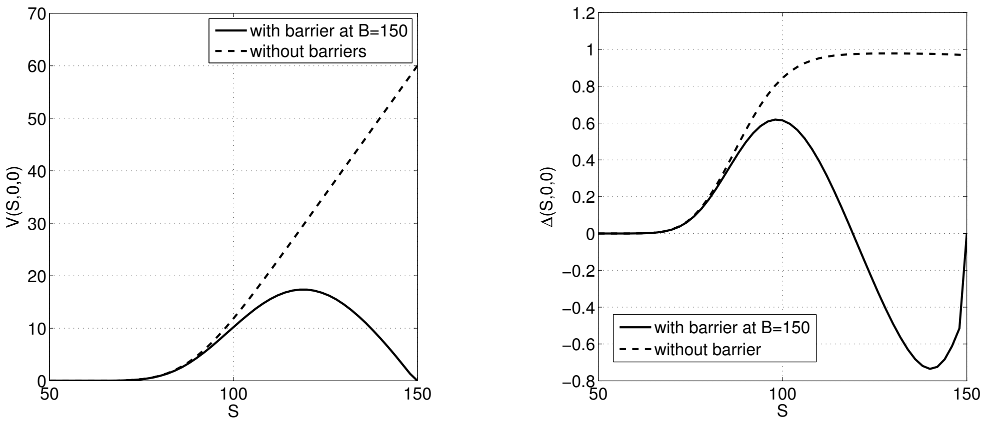

Figure 3: Call up-and-out Geometric Asian option values and the associated $\mathbf{\Delta}$-values obtained by SABO.

At the moment, we can only report results obtained by SABO without comparison with other numerical methods that are anyway in preparation [19] and currently unavailable in other literature: we can observe that the solution appears to be smooth and, in compliance with previsions, it stays under the option value without barriers with a behavior analogous to that of European barrier options (look at [2]).

Concerning convergence, we can observe in Tab. 2 the stabilization of digits at $S = 100, 120, 140$

and the increasing (quadrupling) of CPU computational time[5] in relation to the refinement (doubling) of mesh. As expected ([2]), the nearer the barrier the slower the convergence.

$N_t = N_A$	$S = 100$	$S = 120$	$S = 140$	CPU time(sec)
10	10.2170	17.3650	8.0877	$3.0 \cdot 10^0$
20	10.1480	17.2561	7.9929	$1.1 \cdot 10^1$
40	10.1419	17.2960	8.1400	$4.3 \cdot 10^1$
80	10.1432	17.3061	8.1507	$1.7 \cdot 10^2$
160	10.1438	17.3086	8.1551	$6.9 \cdot 10^2$
320	10.1439	17.3094	8.1566	$3.0 \cdot 10^3$

Table 2: $V(S, 0, 0)$ evaluated by SABO at $S = 100, 120, 140$.

As example of Greek, we have straightforwardly computed $\boldsymbol{\Delta}$ by (46) (plotted in Fig. 3 on the right). Some reference values to check the reliability of formula (46) can be given by the approximation of $\boldsymbol{\Delta}$ with the fourth order formula

$$\boldsymbol{\Delta} = \frac{-V(S + 2\Delta S, 0, 0) + 8V(S + \Delta S, 0, 0) - 8V(S - \Delta S, 0, 0) + V(S - 2\Delta S, 0, 0)}{12\Delta S} \tag{56}$$

applied to SABO values V computed by $N_t = N_A = 320$ and reducing ΔS. These values gives us also an idea of the refinement level of S-grid to be used in finite difference methods.

Formula (56)

ΔS	$S = 100$	$S = 120$	$S = 140$
8	0.6103	-0.0375	-0.8209
4	0.6142	-0.0378	-0.7741
2	0.6144	-0.0378	-0.7742
1	0.6144	-0.0378	-0.7742

SABO

$N_t = N_A$	$S = 100$	$S = 120$	$S = 140$
10	0.6214	-0.0491	-0.6895
20	0.6149	-0.0441	-0.7356
40	0.6142	-0.0383	-0.7643
80	0.6144	-0.0380	-0.7733
160	0.6144	-0.0379	-0.7742
320	0.6144	-0.0378	-0.7742

Table 3: $\boldsymbol{\Delta}(S, 0, 0)$ evaluated by formula (56) on the left and by SABO on the right, at $S = 100, 120, 140$.

- *2nd example*

In this example we use some troubling finance parameters found in [20] as example in the numerical evaluation of arithmetic Asian options.

B	T	E	r	σ
110	1	100	0.15	0.05

[5]All the numerical simulations have been performed with a laptop computer: CPU Intel i5, 4Gb RAM.

Table 4: Fixed strike up-and-out call option data.

In this set of data $r >> \sigma^2$ implying that the diffusive term in Eq. (15) degenerates becoming negligible; this kind of situation is well know also in the simpler Black and Scholes context (and in general in advection-diffusion problems whose transport term dominates the diffusive term) and it causes instabilities in finite difference schemes that can be handled by mesh refinements or weakening accuracy by artificial diffusive terms.

In this case we have decided to set the barrier at $B = 110$ and to compute a fixed strike up-and-out call option approximation at $t = 0$ and $A = 0$, by setting $A_{\max} = 5$. SABO too needs a mesh of at least $N_t = N_A = 50$ basis functions to obtain a shape without spurious oscillation as in Fig. 4 but no other particular tricks.

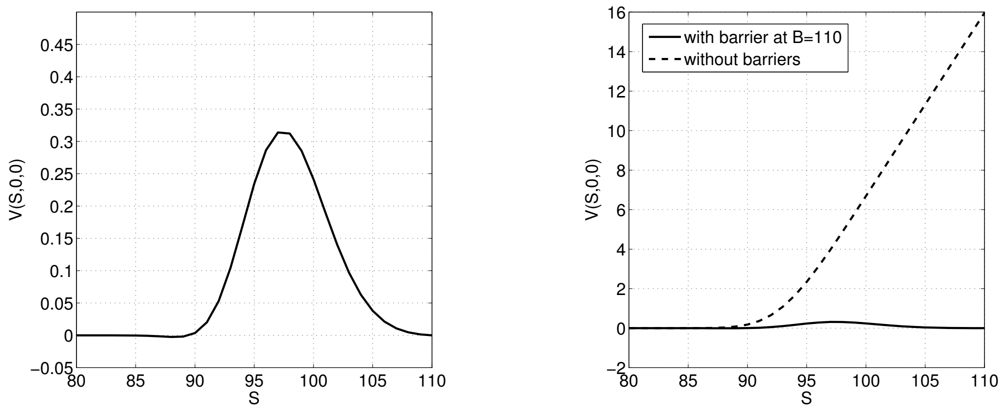

Figure 4: On the left: geometric Asian up-and-out call option obtained by SABO with data in Tab. 4. On the right: comparison between geometric Asian option values with up-and-out barrier and without barriers.

Concerning convergence, we can observe in Tab. 5 the stabilization of digits at $S = 100, 120, 140$ together with the CPU computational time in relation to the mesh refinement.

$N_t = N_A$	$S = 90$	$S = 97$	$S = 104$	CPU time
50	0.00363	0.31381	0.06284	$3.4 \cdot 10^1$
100	0.01103	0.32914	0.06219	$1.4 \cdot 10^2$
200	0.01499	0.32992	0.06232	$5.7 \cdot 10^2$
400	0.01603	0.33004	0.06233	$2.4 \cdot 10^3$

Table 5: $V(S, 0, 0)$ evaluated by SABO at $S = 90, 97, 104$.

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Appendix A.1: Feynman-Kac formula for vanilla Asian options

With the following changes of variables [10]:

$$V(S,A,t) = u(x,y,\tau)\exp\left(-x\frac{2r-\sigma^2}{2\sqrt{2}\sigma} - \tau\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2\right),$$

$$\tau = T - t\,, \qquad x = \frac{\sqrt{2}}{\sigma}\log(S)\,, \qquad y = \frac{A\sqrt{2}}{\sigma}\,,$$

Eq. 3 reduces to the forward equation

$$x\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial \tau} = 0 \qquad x \in (-\infty,+\infty),\; y \in (-\infty,+\infty),\; \tau \in (0,T]\,.$$

Provided that the initial datum $u(x,y,0)$ is assigned such that

$$\int_{\mathbb{R}^2} e^{-c(x^2+y^2)} u(x,y,0)dx\,dy < \infty \qquad \text{for some positive constant } c\,,$$

the existence and uniqueness of the solution are assured and u can be formulated by the Feynman-Kac formula ([21])

$$u(x,y,\tau) = \int_{\mathbb{R}^2} G_u(x,y,\tau;\xi,\eta,0)u(x,y,0)d\xi\,d\eta \tag{57}$$

in terms of the fundamental solution ([22])

$$G_u(x,y,\tau;\xi,\eta,s) = \frac{\sqrt{3}H[\tau-s]}{2\pi(\tau-s)^2}\exp\left(-\frac{(x-\xi)^2}{\tau-s} + \frac{3(x-\xi)\big(y-\eta+(\tau-s)x\big)}{(\tau-s)^2} - \frac{3\big(y-\eta+(\tau-s)x\big)^2}{(\tau-s)^3}\right)$$

which satisfies the adjoint problem

$$\begin{aligned} -\xi\frac{\partial G_u}{\partial \eta} + \frac{\partial^2 G_u}{\partial \xi^2} + \frac{\partial G_u}{\partial s} &= 0 \qquad \xi,\eta \in \mathbb{R}\,,\; s < \tau \\ G_u(x,y,\tau;\xi,\eta,\tau) &= \delta(x-\xi)\delta(y-\eta) \qquad \xi,\eta \in \mathbb{R}\,. \end{aligned}$$

Appendix A.2: computation of matrix entries

Starting from the definition of matrix entries in (40)-(42), let $\xi = i - h,\ \xi = -N_A + 1, \ldots, N_A - 1$ and $\ell = k - j,\ \ell = 0, \ldots, N_t - 1$, perform the change of variable $\widetilde{t} = \Delta t(\tau + k - 1)$ getting $\widetilde{t} - \overline{t}_j = \Delta t(\tau + k - j - 1/2) = \Delta t(\tau + \ell - 1/2)$ and analytically integrate w.r.t. $\widetilde{A}$, obtaining[6]

$$
\begin{aligned}
\mathcal{A}_{ih}^{(jk)} &= \frac{B\sqrt{3}}{2\pi} \int_{\frac{1}{2}-\frac{1}{2}H[\ell]}^{1} \int_{A_{h-1}}^{A_h} \frac{1}{\Delta t(\tau + \ell - 1/2)^2} \\
&\quad \exp\left\{ -\frac{6\big(\overline{A}_i - \widetilde{A} + \Delta t(\tau + \ell - 1/2)\log(B)\big)^2}{\sigma^2 \Delta t^3 (\tau + \ell - 1/2)^3} - \left(\frac{2r + \sigma^2}{2\sqrt{2}\sigma}\right)^2 \Delta t(\tau + \ell - 1/2) \right\} d\widetilde{A}\, d\tau \\
&= \frac{\sigma B \Delta t}{4\sqrt{2\pi}} \int_{\frac{1}{2}-\frac{1}{2}H[\ell]}^{1} \frac{\exp\left\{ -\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2 \Delta t(\tau + \ell - 1/2) \right\}}{\sqrt{\Delta t(\tau + \ell - 1/2)}} \\
&\quad \left\{ \mathrm{Erf}\left[\frac{\sqrt{6}\big(\Delta A(\xi + \frac{1}{2}) + \Delta t(\tau + \ell - 1/2)\log(B)\big)}{\sigma \Delta t^{3/2} (\tau + \ell - 1/2)^{3/2}} \right] \right. \\
&\quad \left. -\mathrm{Erf}\left[\frac{\sqrt{6}\big(\Delta A(\xi - \frac{1}{2}) + \Delta t(\tau + \ell - 1/2)\log(B)\big)}{\sigma \Delta t^{3/2} (\tau + \ell - 1/2)^{3/2}} \right] \right\} d\tau =: \mathcal{A}_{\xi}^{(\ell)}\,.
\end{aligned}
\tag{58}
$$

The integration w.r.t. τ in (58) is then simply performed by the adaptive quadrature Matlab `quad` function. Pay attention that inaccuracy may here appear, due to "jumps" of the integrand function: Erf function has two horizontal asymptotes and when the absolute value of the error function argument is greater than 2 the function gradient tends to 0; this implies that, if both error function arguments rapidly go beyond 2 or below -2, the integrand, while remaining a smooth function, has very steep descents to zero as shown in Fig. 5.

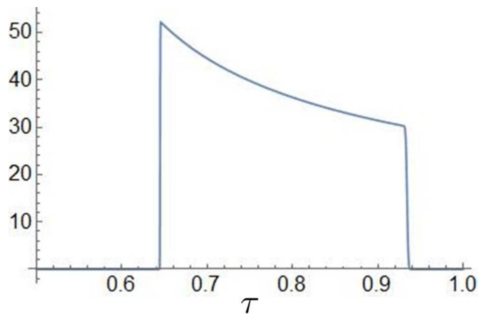

Figure 5: Behavior of the integrand function in (58) for example when $\sigma = 0.2$, $r = 0.035$, $B = 1000$, $i = 1$, $h = 2$, $\ell = 0$, $\Delta A = 0.5$, $\Delta t = 0.25$.

[6] Remembering the definition of the *error function*

$$\mathrm{Erf}[z] := \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

and of its complement $\mathrm{Erfc}[z] := 1 - \mathrm{Erf}[z]$.

In order to obtain a greater accuracy saving computational time, it is however sufficient to set the upper and lower interval bounds inputs at

$$\text{Upper} := \min\left\{\tau \,\middle|\, \frac{\sqrt{6}\big(\Delta A(\xi - \frac{1}{2}) + \Delta t(\tau + \ell - 1/2)\log(B)\big)}{\sigma \Delta t^{3/2}(\tau + \ell - 1/2)^{3/2}} \gtrsim 2.5\right\}$$

$$\text{Lower} := \max\left\{\tau \,\middle|\, \frac{\sqrt{6}\big(\Delta A(\xi + \frac{1}{2}) + \Delta t(\tau + \ell - 1/2)\log(B)\big)}{\sigma \Delta t^{3/2}(\tau + \ell - 1/2)^{3/2}} \lesssim -2.5\right\}$$

if Upper and/or Lower belong to $\left[\frac{1}{2} - \frac{1}{2}H[\ell], 1\right]$.

Appendix A.3: computation of rhs entries

The computation of rhs entries can be developed in the specific case of fixed strike payoff, starting from (44)

$$
\begin{aligned}
\mathcal{F}_i^{(j)} &= -\int_{-\infty}^{+\infty}\int_0^B \max\left(e^{\frac{\widetilde{A}}{T}}-E,0\right) G(B,\overline{A}_i,\bar{t}_j;\widetilde{S},\widetilde{A},T)\,d\widetilde{S}\,d\widetilde{A} \\
&= -\frac{\sqrt{3}B^{-\frac{2r-\sigma^2}{2\sigma^2}}}{\pi\sigma^2(T-\bar{t}_j)^2}\int_0^B \widetilde{S}^{\frac{2r-3\sigma^2}{2\sigma^2}}\int_{T\log(E)}^{+\infty}\left(e^{\frac{\widetilde{A}}{T}}-E\right)\exp\left\{-\frac{2}{\sigma^2(T-\bar{t}_j)}\log^2\left(\frac{B}{\widetilde{S}}\right)\right. \\
&+ \frac{6}{\sigma^2(T-\bar{t}_j)^2}\log\left(\frac{B}{\widetilde{S}}\right)\left(\overline{A}_i-\widetilde{A}+(T-\bar{t}_j)\log(B)\right) \\
&- \frac{6}{\sigma^2(T-\bar{t}_j)^3}\left(\overline{A}_i-\widetilde{A}+(T-\bar{t}_j)\log(B)\right)^2 \\
&- \left.\left(\frac{2r+\sigma^2}{2\sqrt{2}\sigma}\right)^2(T-\bar{t}_j)\right\}d\widetilde{A}\,d\widetilde{S} \\
&= \frac{B^{-\frac{2r-\sigma^2}{2\sigma^2}}}{2\sigma\sqrt{2\pi(T-\bar{t}_j)}}\int_0^B \widetilde{S}^{\frac{2r-3\sigma^2}{2\sigma^2}+\frac{T-\bar{t}_j}{2T}}\exp\left(-\frac{(T-\bar{t}_j)^2(2r+\sigma^2)^2+4\log^2\left(\frac{B}{\widetilde{S}}\right)}{8\sigma^2(T-\bar{t}_j)}\right) \\
&\quad \left\{E\widetilde{S}^{-\frac{T-\bar{t}_j}{2T}}\,\mathrm{Erf}\left[\sqrt{\frac{3}{2}}\,\frac{2(\widetilde{A}-\overline{A}_i)-2(T-\bar{t}_j)\log(B)+(T-\bar{t}_j)\log(\frac{B}{\widetilde{S}})}{\sigma(T-\bar{t}_j)^{\frac{3}{2}}}\right]\right. \\
&+ B^{\frac{T-\bar{t}_j}{2T}}\exp\left(\frac{\overline{A}_i}{T}+\frac{\sigma^2(T-\bar{t}_j)^3}{24T^2}\right) \\
&\quad \left.\mathrm{Erf}\left[\frac{12\,T(\overline{A}_i-\widetilde{A})+\sigma^2(T-\bar{t}_j)^3+12\,T(T-\bar{t}_j)\log(B)-6\,T(T-\bar{t}_j)\log(\frac{B}{\widetilde{S}})}{2\sqrt{6}\,T\sigma(T-\bar{t}_j)^{\frac{3}{2}}}\right]\right\}_{\widetilde{A}=T\log(E)}^{\widetilde{A}->+\infty} d\widetilde{S} \\
&= \frac{B^{-\frac{2r-\sigma^2}{2\sigma^2}}}{2\sigma\sqrt{2\pi(T-\bar{t}_j)}}\int_0^B \widetilde{S}^{\frac{2r-3\sigma^2}{2\sigma^2}+\frac{T-\bar{t}_j}{2T}}\exp\left(-\frac{(T-\bar{t}_j)^2(2r+\sigma^2)^2+4\log^2\left(\frac{B}{\widetilde{S}}\right)}{8\sigma^2(T-\bar{t}_j)}\right) \\
&\quad \left\{E\widetilde{S}^{-\frac{T-\bar{t}_j}{2T}}\,\mathrm{Erfc}\left[\sqrt{\frac{3}{2}}\,\frac{-2\overline{A}_i-2(T-\bar{t}_j)\log(B)+2T\log(E)+(T-\bar{t}_j)\log(\frac{B}{\widetilde{S}})}{\sigma(T-\bar{t}_j)^{\frac{3}{2}}}\right]\right. \\
&+ B^{\frac{T-\bar{t}_j}{2T}}\exp\left(\frac{\overline{A}_i}{T}+\frac{\sigma^2(T-\bar{t}_j)^3}{24T^2}\right)\Bigg(-2+ \\
&\quad \left.\mathrm{Erfc}\left[\frac{12\,T\overline{A}_i+\sigma^2(T-\bar{t}_j)^3-6\,T\big(-2(T-\bar{t}_j)\log(B)+2\,T\log(E)+(T-\bar{t}_j)\log(\frac{B}{\widetilde{S}})\big)}{2\sqrt{6}\,T\sigma(T-\bar{t}_j)^{\frac{3}{2}}}\right]\Bigg)\right\}d\widetilde{S}\,.
\end{aligned}
\tag{59}
$$