Stability of measures on Kähler manifolds

STABILITY OF MEASURES ON KÄHLER MANIFOLDS

LEONARDO BILIOTTI AND ALESSANDRO GHIGI

Abstract. Let (M, ω) be a Kähler manifold and let K be a compact group that acts on M in a Hamiltonian fashion. We study the action of $K^{\mathbb{C}}$ on probability measures on M. First of all we identify an abstract setting for the momentum mapping and give numerical criteria for stability, semi-stability and polystability. Next we apply this setting to the action of $K^{\mathbb{C}}$ on measures. We get various stability criteria for measures on Kähler manifolds. The same circle of ideas gives a very general surjectivity result for a map originally studied by Hersch and Bourguignon-Li-Yau.

Contents

1. Introduction

Let (M, ω) be a Kähler manifold and let K a compact connected Lie group. Assume that K acts on M in a Hamiltonian way with momentum mapping $\mu : M \longrightarrow \mathfrak{k}^*$. For ν a positive measure on M set

$$\mathfrak{F}(\nu) := \int_M \mu(x) d\nu(x). \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 53D20; Secondary 32M05, 14L24.

Key words and phrases. Kähler manifolds; moment maps; geometric invariant theory; probability measures.

The authors were partially supported by FIRB 2012 "Geometria differenziale e teoria geometrica delle funzioni", by a grant of the Max-Planck Institut für Mathematik, Bonn and by GNSAGA of INdAM. The first author was also supported by MIUR PRIN 2010-2011 "Varietà reali e complesse: geometria, topologia e analisi armonica". The second author was also supported by MIUR PRIN 2012 "Moduli, strutture geometriche e loro applicazioni".

This defines a map from the set of measures to $\mathfrak{k}^*$. This map has been studied at different levels of generality by Hersch [29], Millson and Zombro [36], Bourguignon, Peter Li and Yau [11] and ourselves [8]. Recall that $G := K^{\mathbb{C}}$ acts on M and hence on the set of measures on M. As in the quoted papers, we are interested in the following problem:

Problem 1.2. *Assume that* 0 *belongs to the interior of the convex envelope of* $\mu(M)$ *and let* ν *be a measure on* M*. Is there* $g \in G$ *such that* $\mathfrak{F}(g \cdot \nu) = 0$*?*

This question is motivated by an application to upper bounds for the first eigenvalue of the Laplacian on functions. For more details see 7.17 or the introduction to [8]. In this paper we concentrate on Problem 1.2 leaving aside the applications to eigenvalue estimates.

For a sufficiently regular measure a positive answer to Problem 1 is known in a few cases, namely $M = \mathbb{P}^1$ (Hersch in 1970 [29]), $M = \mathbb{P}^n$ (Bourguignon, Li and Yau in 1994 [11]) and M a flag manifold (ourselves in 2013 [8]). In all these cases ω is the symmetric metric and K is the connected component of the isometry group.

Our main theorem is a rather vast generalization of these results.

Theorem 1.3. *The answer to Problem 1.2 is positive for an arbitrary Kähler manifold with a Hamiltonian action of* K *and for any probability measure* ν *that is absolutely continuous with respect to smooth strictly positive measures.*

(See 5.26 and Definition 5.27 for the definition of this class of measures. See Theorem 6.14 for the actual result which is in fact stronger than stated here).

To study Problem 1.2 we cast it in a momentum mapping picture. The natural action to consider is the action of G on the set $\mathscr{P}(M)$ of Borel probability measures on M. This set does not seem to admit a reasonable symplectic structure. (But see [17] for something similar in the case of Euclidean space.) One could find some space admitting a symplectic structure and fibering over $\mathscr{P}(M)$ and one could lift the problem to this space. But this seems rather artificial. Instead it turns out that the part of the momentum mapping/G.I.T. picture which is needed can be developed on a topological space with no symplectic structure. Thus our first goal is to build up a theory for the momentum mapping that can be applied to the action of G on $\mathscr{P}(M)$. This is the content of Sections 2-4.

More precisely, given a Hausdorff topological space with a continuous G-action and a set of functions formally similar to the classical Kempf-Ness functions (see 2.6) we define an analogue of the momentum mapping and the usual concepts of stability (see Definition 3.1). The point of this construction is that one can characterize stability, semi-stability and polystability of a point x by numerical criteria, that is in terms of a function called *maximal weight* and denoted λ_x, which is defined on Tits boundary of G/K. See 2.23 for the definition of λ_x and Theorems 3.4, 4.17, 4.14 for the precise statements. We also establish a version of the Hilbert-Mumford criterion (Corollary 3.8) and the openness of the set of stable points (Corollary 3.10). In the classical case of a group action on a Kähler manifold these characterizations are due to Mundet i Riera [38, 39], Teleman [43], Kapovich, Leeb and Millson [31] and probably many others. In fact many of these ideas go back as far as Mumford [37, §2.2].

A different approach to stability on Kähler manifolds is based on the properties of invariant plurisubharmonic functions and the Mostow fibration instead of Kempf-Ness functions. This has been developed over the years by Azad, Heinzner, Huckleberry, Loeb, Loose, Schwarz and others [5], [21], [22], [23], [24], [25], [26]. It seems hard to apply this approach in our setting since it relies heavily on the fact that the space where the group acts is a complex space. A similar remark applies to the techniques used in [18]. The main tool there is the gradient flow of the momentum mapping squared. This is not available without some kind of differentiable structure.

The setting we have chosen to develop the theory is not necessarily the most natural, nor the most general. However it is well suited for the study of our problem, namely the stability of probability measures on a compact Kähler manifold. It should be of interest in other situations. Even in the classical case of a Hamiltonian action on a compact Kähler manifold, it offers a rather streamlined proof of the numerical criteria for stability, semi-stability and polystability.

In Section 5 we apply the abstract theory to the action of G on $\mathscr{P}(M)$ endowed with the weak topology. The map (1.1) turns out be the analogue of the momentum mapping in this setting. Using the Morse-Bott theory of the momentum mapping on M we are able to compute rather explicitly the maximal weight λ_ν of $\nu \in \mathscr{P}(M)$. Indeed fix a non-zero $v \in \mathfrak{k}$ and let $c_0 < \cdots < c_r$ be the critical values of $\mu^v := \langle \mu, v\rangle$. Let $C_i := (\mu^v)^{-1}(c_i)$ be the critical components. Set

$$W_i^u := \{x \in M : \lim_{t\to+\infty} \exp(itv)\cdot x \in C_i\}.$$

This is the *unstable manifold* of C_i for the gradient flow of μ^v. Denote by $\mathrm{e}(-v)$ the point of $\partial_\infty X$ corresponding to the geodesic $t \mapsto \exp(-itv)\cdot K$ in G/K. The maximal weight can be computed in terms of these Morse data and the result is rather clean:

$$\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=0}^{r} c_i \cdot \nu(W_i^u).$$

Based on this formula we get various stability criteria for measures. For example every measure that is absolutely continuous with respect to a smooth strictly positive measure is stable up to shifting the momentum mapping (Theorem 5.31).

In Section 6 we go back to Problem 1.2. The first step is analogous to something known in the finite-dimensional case: if the stabilizer of a measure ν is compact, then the restriction of $\mathfrak{F}$ to the orbit $G\cdot\nu$ is a submersion

(Theorem 6.4). Under a mild regularity assumption on ν the restriction of $\mathfrak{F}$ to $G \cdot \nu$ is in fact a smooth fibration onto the interior of the convex envelope of $\mu(M)$ (Theorem 6.14). Theorem 1.3 follows immediately.

Section 7 contains some applications. Using the previous results we get a precise characterization of stable, semi-stable and polystable measures on $\mathbb{P}^n$ (Theorems 7.6 and 7.8), extending previous work by Millson-Zombro and Donaldson.

Next we turn to the application to upper bounds for λ_1. We explain that the results in the paper give shorter and more conceptual proofs of some known statements. We also explain what is missing to get a very general estimate for λ_1 using these ideas.

Acknowledgements. The authors wish to thank Luigi Fontana, Daniel Greb, Peter Heinzner and Ignasi Mundet i Riera for interesting discussions/emails related to the subject of this paper. They also wish to thank the Max-Planck Institut für Mathematik, Bonn for excellent conditions provided during their visit at this institution, where they started working on the subject of this paper. Finally they wish to thank the referees for reading very carefully the manuscript.

2. Kempf-Ness functions

In this section we introduce an abstract setting for actions of complex reductive groups on topological spaces. More precisely, we define functions similar to the classical Kempf-Ness functions, from which we derive the whole momentum mapping picture. We start with some remarks on convex functions of symmetric spaces.

2.1. Let X be a symmetric space of the noncompact type. (Some of the following statements hold more generally if X is a Hadamard manifold or even a $CAT(0)$–space. We restrict to the case of a symmetric space which is the only one needed for the theory of the momentum mapping.) Two unit speed geodesics $\gamma, \gamma' : \mathbb{R} \to X$ are equivalent, denoted $\gamma \sim \gamma'$, if $\sup_{t>0} d(\gamma(t), \gamma'(t)) < +\infty$. The *Tits boundary* of X, denoted by $\partial_\infty X$, is the set of equivalence classes of unit speed geodesics in X. Assume that $X = G/K$ with G is a connected Lie group and $K \subset G$ a maximal compact subgroup. Put $o := K \in X$. For $v \in \mathfrak{k}$, let γ^v denote the geodesic $\gamma^v(t) := \exp(itv)K$. Mapping v to the tangent vector $\dot{\gamma}^v(0)$ yields an isomorphism $\mathfrak{k} \cong T_o X$. Since any geodesic ray in X is equivalent to a unique ray starting from o, the map

$$\mathrm{e} : S(\mathfrak{k}) \to \partial_\infty X, \ \mathrm{e}(v) := [\gamma^v], \tag{2.2}$$

where $S(\mathfrak{k})$ is the unit sphere in $\mathfrak{k}$, is a bijection. The *sphere topology* is the topology on $\partial_\infty X$ such that e is a homeomorphism. (For more details on the Tits boundary see for example [10, §I.2].)

2.3. Let X be a topological space. A continuous function $u : X \to \mathbb{R}$ is an *exhaustion* if for any $c \in \mathbb{R}$ the set $f^{-1}((-\infty, c])$ is compact. A continuous function is an exhaustion if and only if it is bounded below and proper. The following lemma is proven in greater generality in the paper [31, §3.1] by Kapovich, Leeb and Millson. Since it is basic to everything that follows, we recall its proof in detail. For more information on the geometry of convex functions on symmetric spaces and the relation with their Tits boundary see [31] especially pp. 313-316.

Lemma 2.4. *Let $u : X \to \mathbb{R}$ be a smooth convex function on X. If u is globally Lipschitz, the function*

$$u_\infty : \partial_\infty X \to \mathbb{R}, \quad u_\infty([\gamma]) := \lim_{t\to+\infty} (u \circ \gamma)'(t), \tag{2.5}$$

is well–defined. Moreover u is an exhaustion if and only if $u_\infty > 0$ on $\partial_\infty X$.

Proof. Since $f := u \circ \gamma$ is convex,

$$\frac{f(s)}{s} \leq f'(s) \leq \frac{f(t) - f(s)}{t - s} \quad \text{for } 0 < s < t.$$

Moreover the first two quantities are increasing in s, the third in t. Thus

$$\lim_{s\to+\infty} \frac{f(s)}{s} \leq \lim_{s\to+\infty} f'(s) \leq \lim_{t\to+\infty} \frac{f(t) - f(s)}{t - s}.$$

Since the last limit equals the first we get $\lim_{t\to+\infty} f'(t) = \lim_{t\to+\infty} f(t)/t$. If $\tilde{\gamma}$ is another geodesic and $d(\gamma(t), \tilde{\gamma}(t)) \leq C$, then $|u\circ\gamma(t)/t - u\circ\tilde{\gamma}(t)/t| \leq LC/t$, where L is a Lipschitz constant for u. Therefore

$$\lim_{t\to+\infty} (u \circ \gamma)'(t) = \lim_{t\to+\infty} \frac{u \circ \gamma(t)}{t} = \lim_{t\to+\infty} \frac{u \circ \tilde{\gamma}(t)}{t} = \lim_{t\to+\infty} (u \circ \tilde{\gamma})'(t).$$

This shows that u_∞ is well-defined. It is finite since $|f'(s)| \leq L$. Assume now that u is an exhaustion. Given γ there is $t_0 > 0$ such that for $t > t_0$, $u \circ \gamma(t) > u \circ \gamma(0)$. Thus

$$u_\infty[\gamma] = \lim_{t\to+\infty} \frac{u \circ \gamma(t) - u \circ \gamma(0)}{t} > 0.$$

Conversely assume that $u_\infty([\gamma]) > 0$ for any $[\gamma] \in \partial_\infty X$. Fix $x \in X$ and let S be the unit sphere in T_xX. For any $v \in S$ let $\gamma^v(t) = \exp_x(tv)$. Since $u_\infty([\gamma^v]) > 0$ there are $\varepsilon(v) > 0$ and T_v such that $u \circ \gamma^v(t) \geq 2\varepsilon(v)t$ for any $t \geq T_v$. By continuity there is a neighbourhood $U_v \subset S$ of v such that $u(\exp_x(T_v w)) \geq \varepsilon(v)T_v$ for any $w \in U_v$. Since $u \circ \gamma_w$ is convex, the function $u \circ \gamma_w(t)/t$ is increasing. Thus $u(\exp_x(tw)) \geq \varepsilon(v)t$ for any $t \geq T_v$ and any $w \in U_v$. If $S = \bigcup_{i=1}^k U_{v_i}$ set $T := \max\{T_{v_i}\}$ and $\varepsilon := \min\{\varepsilon(v_i)\}$. Then $u(\exp_x v) \geq \varepsilon|v|$ for any $v \in T_xX$ with $|v| \geq T$. This shows that u is an exhaustion function. □

2.6. Let $\mathscr{M}$ be a Hausdorff topological space and let K be a compact connected Lie group. Denote by $G = K^{\mathbb{C}}$ the complexification of K and assume that G acts from the left on $\mathscr{M}$ and that the action is continuous, that is the map $G\times\mathscr{M}\to\mathscr{M}$ is continuous. Starting with these data we are going to consider a function $\Psi:\mathscr{M}\times G\to\mathbb{R}$, subject to six conditions. The first four conditions are the following ones:

(P1) For any $x\in\mathscr{M}$ the function $\Psi(x,\cdot)$ is smooth on G.

(P2) The function $\Psi(x,\cdot)$ is left–invariant with respect to K: $\Psi(x,kg)=\Psi(x,g)$.

(P3) For any $x\in\mathscr{M}$, and any $v\in\mathfrak{k}$ and $t\in\mathbb{R}$
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x,\exp(itv))\geq 0.$$
Moreover
$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\bigg|_{t=0}\Psi(x,\exp(itv))=0$$
if and only if $\exp(\mathbb{C}v)\subset G_x$.

(P4) For any $x\in\mathscr{M}$, and any $g,h\in G$
$$\Psi(x,g)+\Psi(gx,h)=\Psi(x,hg).$$
(This equation is called the *cocycle condition*.)

2.7. Set
$$X:=G/K.$$
Fix an Ad–invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}$ and consider the corresponding Riemannian metric on X. If Ψ is a function satisfying (P1)–(P4), then by (P2) the function $g\mapsto\Psi(x,g^{-1})$ descends to a function on X:
$$\psi_x:X\to\mathbb{R},\quad \psi_x(gK):=\Psi(x,g^{-1}). \tag{2.8}$$
Using ψ_x instead of Ψ the cocycle condition reads
$$\psi_x(ghK)=\psi_x(gK)+\psi_{g^{-1}x}(hK). \tag{P4$'$}$$
We can now state our fifth assumption:

(P5) For all $x\in\mathscr{M}$, the function ψ_x is globally Lipschitz on X.

2.9. Let $\langle\cdot,\cdot\rangle:\mathfrak{k}^*\times\mathfrak{k}\to\mathbb{R}$ be the duality pairing. For $x\in\mathscr{M}$ define $\mathfrak{F}(x)\in\mathfrak{k}^*$ by requiring that
$$\begin{aligned}\langle\mathfrak{F}(x),v\rangle &= -d\psi_x(o)(\dot\gamma^v(0))=\\ &=\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0}\psi_x(\exp(-itv)K)=\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0}\Psi(x,\exp(itv)).\end{aligned} \tag{2.10}$$
(Notation as in 2.1.) The following is the last condition imposed on the function Ψ:

(P6) The map $\mathfrak{F}:\mathscr{M}\to\mathfrak{k}^*$ is continuous.

The following definition summarizes the above discussion.

Definition 2.11. *If K is a compact connected Lie group, $G = K^{\mathbb{C}}$ and $\mathscr{M}$ is a topological space with a continuous G–action, a* Kempf-Ness function *for $(\mathscr{M}, G, K)$ is a function*

$$\Psi : \mathscr{M} \times G \to \mathbb{R},$$

that satisfies conditions (P1)–(P6). We call $\mathfrak{F}$ the momentum mapping *of $(\mathscr{M}, G, K, \Psi)$.*

2.12. The basic example of a Kempf-Ness function is the following. Let (M, J) be a compact Kähler manifold of complex dimension n and let K be a compact connected Lie group. Assume that K acts almost effectively and holomorphically on M and that g is a K-invariant Kähler metric with Kähler form ω. If $v \in \mathfrak{k}$, let $v_M \in \mathfrak{X}(M)$ denote the fundamental vector field induced on M. Assume that the action of K on (M, ω) is Hamiltonian and fix a momentum mapping

$$\mu : M \to \mathfrak{k}^*.$$

If $v \in \mathfrak{k}$, set $\mu^v := \langle \mu, v\rangle$. That μ is a momentum mapping means that it is K-equivariant and that $d\mu^v = i_{v_M}\omega$. It is well-known that the action of K extends to a holomorphic action of the complexification $G := K^{\mathbb{C}}$.

It has been proven by Mundet [38, §3] that one can always choose a function $\Psi^M : M \times G \to \mathbb{R}$ that has the properties (P1)–(P4). Set $\psi_x^M(gK) := \Psi^M(x, g^{-1})$. Then

$$-d\psi_x^M(o)(\dot\gamma^v(0)) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi^M(x, \exp(itv)) = \mu^v(x). \tag{2.13}$$

Hence $\mathfrak{F} = \mu$ and (P6) is satisfied. Moreover it follows from (2.22) that $d\psi_x^M(gK)(g\dot\gamma^v(0)) = d\psi^M_{g^{-1}x}(o)(\dot\gamma^v(0))$. Since M is compact, $||\mu||$ is a bounded function, so ψ_x^M is Lipschitz. Thus also (P5) holds.

In the case of a projective manifold with the restriction of the Fubini-Study metric ψ_x^M is the function originally defined by Kempf and Ness in [32].

Next we deduce some immediate consequences of Definition 2.11.

Proposition 2.14. *The map $\mathfrak{F} : \mathscr{M} \to \mathfrak{k}^*$ is K-equivariant.*

Proof. By the cocycle condition $\Psi(kx, \exp(itv)) = \Psi(x, \exp(itv)k) - \Psi(x, k)$. So using the left-invariance of $\Psi(x, \cdot)$ with respect to K we get

$$\begin{aligned}\langle \mathfrak{F}(kx), v\rangle &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(itv)k) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, k^{-1}\exp(itv)k) \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi\left(x, \exp(it\mathrm{Ad}(k^{-1})(v)\right) = \mathrm{Ad}^*(k)(\mathfrak{F}(x))(v).\end{aligned}$$

∎

2.15. Taking $g = h = e$ in the cocycle condition (P4) we obtain $\Psi(x, e) = 0$ and so $\Psi(x, k) = 0$ for every $k \in K$. Moreover, for any $x \in \mathscr{M}$ and for any $g, h \in G_x$ we have

$$\Psi(x, hg) = \Psi(x, g) + \Psi(x, h). \tag{2.16}$$

Lemma 2.17. *If $v \in \mathfrak{k}$ and $iv \in \mathfrak{g}_x$, then $\Psi(x, \exp(itv)) = \psi_x(\exp(-itv)K)$ is a linear function of t.*

Proof. By (2.16)

$$\begin{aligned}\Psi(x, \exp(i(t+s)v) &= \Psi(x, \exp(itv) \cdot \exp(isv)) = \\ &= \Psi(x, \exp(itv)) + \Psi(x, \exp(isv)).\end{aligned}$$

This shows that $t \mapsto \Psi(x, \exp(itv))$ is a morphism from $(\mathbb{R}, +)$ to itself. Since it is continuous, it is a linear map. □

2.18. The linear function considered in the previous lemma is an analogue of Futaki invariant in the geometry of Kähler-Einstein metrics, see e.g. [45, Chapter 3] and [44, p. 253].

Lemma 2.19. *The function ψ_x is geodesically convex on X. More precisely, if $v \in \mathfrak{k}$ and $\alpha(t) = g\exp(itv)K$ is a geodesic in X, then $\psi_x \circ \alpha$ is either strictly convex or affine. The latter case occurs if and only if $g\exp(\mathbb{C}v)g^{-1} \subset G_x$. In the case $g = e$, the function $\psi_x \circ \alpha$ is linear if $\exp(\mathbb{C}v) \subset G_x$ and strictly convex otherwise.*

Proof. Fix $t_0 \in \mathbb{R}$. Set $h := g\exp(it_0v)$. By (P4′)

$$\psi_x(\alpha(t_0 + s)) = \psi_x(h\exp(isv)K) = \psi_x(hK) + \psi_{h^{-1}x}(\exp(isv)K). \tag{2.20}$$

Hence

$$\begin{aligned}&\left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=t_0} \psi_x(\alpha(t)) = \\ &\qquad = \left.\frac{\mathrm{d}^2}{\mathrm{ds}^2}\right|_{s=0} \psi_{h^{-1}x}(\exp(isv)K) = \left.\frac{\mathrm{d}^2}{\mathrm{ds}^2}\right|_{s=0} \Psi(h^{-1}x, \exp(-isv)).\end{aligned}$$

Therefore (P3) yields convexity of $\psi_x \circ \alpha$. If $\psi_x \circ \alpha$ is not strictly convex at t_0, then by (P3) we conclude that $\exp(\mathbb{C}v) \subset G_{h^{-1}x}$. By the previous lemma $\psi_x(\exp(itv)K) = \Psi(h^{-1}x, \exp(-itv))$ is a linear function of t. By (2.20) we have $\psi_x(\alpha(t)) = \psi_x(hK) + \psi_x(\exp(i(t-t_0)v)K)$. This proves $\psi_x \circ \alpha$ is affine. Moreover from $\exp(\mathbb{C}v) \subset G_{h^{-1}x}$ it follows that $g\exp(\mathbb{C}v)g^{-1} \subset G_x$. The same computation shows that conversely if $g\exp(\mathbb{C}v)g^{-1} \subset G_x$ then $\psi_x \circ \alpha$ is affine. In case $g = e$ we know that $\psi_x(K) = 0$ by 2.16, so if the function is affine, it is in fact linear. □

2.21. The group G acts isometrically on X from the left: for $g \in G$ the map

$$L_g : X \to X, \qquad L_g(hK) := ghK,$$

is an isometry. We will sometimes write simply gx for $L_g(x)$. The cocycle condition (P4) is equivalent to the following identity between two functions and a constant:

$$L_g^* \psi_x = \psi_{g^{-1}x} + \psi_x(gK). \tag{2.22}$$

2.23. Since ψ_x is Lipschitz, we can apply the machinery of 2.1-2.3. For $x \in \mathscr{M}$ denote by λ_x the function $(\psi_x)_\infty$:

$$\lambda_x : \partial_\infty X \to \mathbb{R}, \qquad \lambda_x([\gamma]) := \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(\gamma(t)).$$

Set

$$\lambda : \mathscr{M} \times \partial_\infty X \longrightarrow \mathbb{R}, \quad \lambda(x,p) := \lambda_x(p). \tag{2.24}$$

By Lemma 2.4 λ_x and hence λ are well-defined and finite. We call λ_x the *maximal weight* of x. Using the notation defined in (2.2) for $v \in S(\mathfrak{k})$ we have

$$\lambda_x(\mathrm{e}(v)) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(\exp(itv)K) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(-itv)). \tag{2.25}$$

2.26. If $g \in G$ and γ is a unit speed geodesic, then also $g \circ \gamma$ is a unit speed geodesic and clearly $\gamma \sim \gamma' \Leftrightarrow g\circ\gamma \sim g \circ \gamma'$. Thus setting

$$g \cdot [\gamma] := [g \circ \gamma]$$

defines an action of G on $\partial_\infty X$. Using the bijection $\mathrm{e} : S(\mathfrak{k}) \to \partial_\infty X$, introduced in (2.2), we get a action of G on $S(\mathfrak{k})$:

$$g \cdot v := \mathrm{e}^{-1}(g \cdot e(v)). \tag{2.27}$$

This action is continuous with respect to the sphere topology on $\partial_\infty X$ (see e.g [10, p. 41]), but it is not smooth.

Lemma 2.28. *For any $x \in \mathscr{M}$, any $g \in G$ and any $p \in \partial_\infty X$*

$$\lambda_{g^{-1}x}(p) = \lambda_x(g \cdot p). \tag{2.29}$$

Proof. Assume that $p = [\gamma]$ for some geodesic γ in X. Then $g \cdot p = [g \circ \gamma]$. By (2.22) we have $\psi_{g^{-1}x} = L_g^* \psi_x - \psi_x(gK)$. Since $\psi_x(gK)$ does not depend on t,

$$\frac{\mathrm{d}}{\mathrm{dt}} \psi_{g^{-1}x}(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{dt}} \psi_x(g \cdot \gamma(t)).$$

The result follows immediately. □

Lemma 2.30. *Let $x \in \mathscr{M}$. The following conditions are equivalent:*

(1) $g \in G$ is a critical point of $\Psi(x, \cdot)$;
(2) $\mathfrak{F}(gx) = 0$;
(3) $g^{-1}K$ is a critical point of ψ_x.

Proof. Let $v \in \mathfrak{k}$. Using the cocycle condition (P4), one gets

$$\Psi(x, \exp(itv)g) = \Psi(x, g) + \Psi(gx, \exp(itv)).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(itv)g) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(gx, \exp(itv)) = \langle \mathfrak{F}(gx), v\rangle. \tag{2.31}$$

Since for any $k \in K$, $\Psi(x, kg) = \Psi(x, g)$, then $\mathfrak{F}(gx) = 0$ if and only if g is a critical point of $\Psi(x, \cdot)$ if and only if $g^{-1}K$ is a critical point of ψ_x. $\square$

2.32. Set

$$\mathfrak{F}^v(x) := \langle \mathfrak{F}(x), v\rangle.$$

If $t, t_0 \in \mathbb{R}$, then $\exp(i(t+t_0)v) = \exp(itv) \cdot \exp(it_0 v)$. Setting $g = \exp(it_0 v)$ in (2.31) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=t_0} \Psi(x, \exp(itv)) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(itv) \cdot \exp(it_0 v)) = \\ = \mathfrak{F}^v(\exp(it_0 v) \cdot x). \end{aligned} \tag{2.33}$$

2.34. Let U be compact Lie group and let $U^{\mathbb{C}}$ be its complexification which is a reductive complex algebraic group. The map $f : U \times i\mathfrak{u} \to U^{\mathbb{C}}$, $f(g, v) = g \cdot \exp v$ is a diffeomorphism. If $H \subset U^{\mathbb{C}}$ is a closed subgroup, set $L := H \cap U$ and $\mathfrak{p} := \mathfrak{h} \cap i\mathfrak{u}$. We say that H is *compatible* ([27, 28]) if $f(L \times \mathfrak{p}) = H$. The restriction of f to $L \times \mathfrak{p}$ is then a diffeomorphism onto H. It follows that L is a maximal compact subgroup of H and that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{p}$. Note that H has finitely many connected components.

Proposition 2.35. *If $\mathfrak{F}(x) = 0$, then G_x is compatible.*

Proof. Let $g \in G_x$. Then $g = k \exp(iv)$ for some $k \in K$ and $v \in \mathfrak{k}$. By Proposition 2.14, we have $\mathfrak{F}(\exp(iv)x) = 0$. Let $f(t) := \mathfrak{F}^v(\exp(itv)x)$. Then $f(0) = f(1) = 0$ and using (2.33)

$$\frac{\mathrm{d}}{\mathrm{dt}} f(t) = \frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}^v(\exp(itv)x) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(itv)) \geq 0.$$

Therefore $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(itv)) = 0$ for $0 \leq t \leq 1$. It follows from (P3) that $\exp(\mathbb{C}v) \subset G_x$, so $v \in \mathfrak{g}_x \cap i\mathfrak{k}$ and G_x is compatible. $\square$

3. Stability

Let $(\mathscr{M}, G, K)$ be as in the previous section, let Ψ be a Kempf-Ness function and let $x \in \mathscr{M}$.

Definition 3.1. *(1) x is* polystable *if $G \cdot x \cap \mathfrak{F}^{-1}(0) \neq \emptyset$.*
(2) x is stable *if it is polystable and $\mathfrak{g}_x$ is conjugate to a subalgebra of $\mathfrak{k}$.*
(3) x is semi–stable *if $\overline{G \cdot x} \cap \mathfrak{F}^{-1}(0) \neq \emptyset$.*
(4) x is unstable *if it is not semi–stable.*

3.2. The four conditions above are G-invariant in the sense that if a point x satisfies one of them, then every point in the orbit of x satisfies the same condition. This is clear from the definition for polystability, semi–stability and unstability. To check that stability is also G-invariant it is enough to recall that $\mathfrak{g}_{gx} = \mathrm{Ad}(g)(\mathfrak{g}_x)$.

Lemma 3.3. *If $\mathfrak{a} \subset \mathfrak{g}$ is a subalgebra which is conjugate to a subalgebra of $\mathfrak{k}$, then $\mathfrak{a} \cap i\mathfrak{k} = \{0\}$.*

Proof. It is enough to show that $\mathrm{Ad}(g)(\mathfrak{k}) \cap i\mathfrak{k} = \{0\}$ for any $g \in G$. Choose an embedding $j : G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ such that $j(K) \subset \mathrm{U}(n)$. If $v \in i\mathfrak{k}$ and $v = \mathrm{Ad}(g)w$ with $w \in \mathfrak{k}$, then $j(v)$ is a Hermitian matrix, hence with real eigenvalues, while $j(w)$ is skew-Hermitan with imaginary eigenvalues. Since these matrices are similar the only eigenvalue is 0. Thus $j(v) = 0$ and $v = 0$. □

Theorem 3.4. *The following conditions are equivalent: (1) $x \in \mathscr{M}$ is stable, (2) $\lambda_x > 0$ on $\partial_\infty X$, (3) ψ_x is an exhaustion function.*

Proof. (1) ⇒ (2). Assume that $\mathfrak{F}(gx) = 0$ for some $g \in G$. By Lemma 2.30, g is a critical point of $\Psi(x, \cdot)$. Set $y = gx$. We start by proving that $\lambda_y > 0$ on $\partial_\infty X$. Using (2.25) and the fact that $\Psi(y, \cdot)$ is a convex function we get

$$\lambda_y(\mathrm{e}(-v)) \geq \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(y, \exp(itv)) = \langle \mathfrak{F}(y), v \rangle = 0.$$

If $\lambda_y(\mathrm{e}(-v)) = 0$ for some $v \in S(\mathfrak{k})$, then $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(itv)) = 0$ for any $t \geq 0$. Using (P3) it follows that $\exp(\mathbb{C}v) \subset G_y$, so $iv \in \mathfrak{g}_y \cap i\mathfrak{k}$. Since x is stable, $\mathfrak{g}_y = \mathrm{Ad}(g)(\mathfrak{g}_x)$ is conjugate to a subalgebra of $\mathfrak{k}$, so Lemma 3.3 implies that $v = 0$. Since we are assuming $v \in S(\mathfrak{k})$ this is absurd. Therefore $\lambda_y > 0$ as desired. By Lemma 2.28 $\lambda_x(p) = \lambda_{g^{-1}y}(p) = \lambda_y(g \cdot p)$. Thus $\lambda_x > 0$ on $\partial_\infty X$. By (P5) ψ_x is Lipschitz continuous. So (2) ⇔ (3) is the content of Lemma 2.4. Finally we prove that (3) ⇒ (1). Since $\psi_x : X = G/K \to \mathbb{R}$ is an exhaustion, there is a minimum point $gK \in X$. Set $y := gx$. Thus $y \in G \cdot x \cap \mathfrak{F}^{-1}(0)$ and x is polystable. To complete the proof we need to show that $\mathfrak{g}_y \subset \mathfrak{k}$. Let $iv \in \mathfrak{g}_y \cap i\mathfrak{k}$. By Lemma 2.17 $\Psi(y, \exp(itv))$ is linear function of t. On the other hand (3) implies that this function is an exhaustion, if $v \neq 0$. Since a linear function cannot be an exhaustion, we conclude that $v = 0$. This proves that $\mathfrak{g}_y \cap i\mathfrak{k} = \{0\}$. By Proposition 2.35 the stabilizer G_y is compatible. In particular $\mathfrak{g}_y = (\mathfrak{g}_y \cap \mathfrak{k}) \oplus (\mathfrak{g}_y \cap i\mathfrak{k})$. So $\mathfrak{g}_y \subset \mathfrak{k}$. This proves that x is stable. □

Corollary 3.5. *If $x \in \mathscr{M}$ is stable, then G_x is compact.*

Proof. Let $g \in G$ be such that $\mathfrak{F}(gx) = 0$ and set $y = gx$. By Proposition 2.35 G_y is compatible, so has only finitely many connected components. Moreover G_y^0 is compact since $\mathfrak{g}_y \subset \mathfrak{k}$ as shown in the previous proof. It follows that G_y and $G_x = g^{-1}G_y g$ are both compact. □

3.6. If $\mathscr{M}'$ is a G-invariant subspace of $\mathscr{M}$, the restriction of Ψ to $G\times\mathscr{M}'$ is a Kempf-Ness function for $(\mathscr{M}', G, K)$. The functions λ and $\mathfrak{F}$ for $(\mathscr{M}', G, K)$ are simply the restrictions of those for $\mathscr{M}$.

3.7. If $K' \subset K$ is a closed subgroup and $G' := (K')^{\mathbb{C}}$, there are totally geodesic inclusions $X' := G'/K' \hookrightarrow X$ and $\partial_\infty X' \subset \partial_\infty X$. If Ψ is a Kempf-Ness function for $(G, K, \mathscr{M})$, then $\Psi^{K'} := \Psi|_{\mathscr{M}\times G'}$ is a Kempf-Ness function for $(G', K', \mathscr{M})$. The related functions are

$$\mathfrak{F}^{K'} : \mathscr{M} \to \mathfrak{k}'^*, \qquad \mathfrak{F}^{K'}(x) := \mathfrak{F}(x)|_{\mathfrak{k}'},$$
$$\psi_x^{K'} := \psi_x|_{X'}, \qquad \lambda^{K'} = \lambda|_{\mathscr{M}\times\partial_\infty X'}.$$

The following Corollary is analogous to the stability part in the Hilbert-Mumford criterion.

Corollary 3.8. *A point $x \in \mathscr{M}$ is G-stable if and only if it is $T^{\mathbb{C}}$-stable for any compact torus $T \subset K$.*

Proof. Assume that x is G-stable. Then $\lambda_x > 0$. So for any torus $T \subset G$, we have $\lambda_x^T > 0$. By Theorem 3.4 x is $T^{\mathbb{C}}$–stable. Conversely assume that x is $T^{\mathbb{C}}$–stable for any compact torus $T \subset K$. If $v \in S(\mathfrak{k})$ choose a torus T such that $v \in \mathfrak{t}$. Then $\lambda_x(\mathrm{e}(v)) = \lambda_x^T(\mathrm{e}(v))$ and $\lambda_x^T(\mathrm{e}(v)) > 0$ by Theorem 3.4, since x is $T^{\mathbb{C}}$–stable. Hence $\lambda_x > 0$ on $\partial_\infty X$. Using again Theorem 3.4 we conclude that x is G-stable.

□

Lemma 3.9 ([43, Prop. 3.11 (5)])**.** *The function $\lambda : \mathscr{M} \times \partial_\infty X \longrightarrow \mathbb{R}$ is lower semicontinuous if $\partial_\infty X$ is endowed with the sphere topology (see 2.1).*

Proof. Fix $(x, p) \in \mathscr{M} \times \partial_\infty X$ with $p = \mathrm{e}(v)$ and $v \in S(\mathfrak{k})$. We need to show that for any $\varepsilon > 0$, there is a neighbourhood A of (x, v) in $\mathscr{M} \times S(\mathfrak{k})$, such that for $(x', v') \in A$ we have $\lambda(x', \mathrm{e}(v')) > \lambda(x, p) - \varepsilon$. By (2.25) and (2.33)

$$\lambda_x(\mathrm{e}(v)) = \lim_{t\to+\infty} \mathfrak{F}^v(\exp(-itv)\cdot x).$$

So given $\varepsilon > 0$, there is a $t_0 \in \mathbb{R}$ such that

$$\mathfrak{F}^v(\exp(-it_0v)\cdot x) > \lambda_x(p) - \varepsilon.$$

By the continuity of the G–action on $\mathscr{M}$ and (P6) the function

$$f : \mathscr{M} \times S(\mathfrak{k}) \longrightarrow \mathbb{R}, \quad f(x', v') := \mathfrak{F}^{v'}(\exp(-it_0v')\cdot x')$$

is continuous. Since $f(x, v) > \lambda_x(p) - \varepsilon$ there is a neighbourhood A of (x, v) in $\mathscr{M} \times S(\mathfrak{k})$ such that $f(x', v') > \lambda_x(p) - \varepsilon$ for any $(x', v') \in A$. Since the function $t \mapsto \mathfrak{F}^{v'}(\exp(-itv')\cdot x)$ is increasing, we get $\lambda(x', \mathrm{e}(v')) > \lambda_x(p) - \varepsilon$ for any $(x', v') \in A$, as desired. □

Corollary 3.10. *The set of stable points is open in $\mathscr{M}$.*

Proof. Let $\pi : \mathscr{M} \times \partial_\infty X \to \mathscr{M}$ be the projection on the first factor. Since $\partial_\infty X$ is compact in the sphere topology and $\mathscr{M}$ is Hausdorff, π is a closed map [13, Thm. 2.5, p. 227]. The set $A := \{(x,p) \in \mathscr{M} \times \partial_\infty X : \lambda(x,p) > 0\}$ is open in $\mathscr{M} \times \partial_\infty X$ since λ is lower semicontinuous. So its complement E is closed and $\pi(E)$ is closed in $\mathscr{M}$. By Theorem 3.4 the set of stable points is the complement of $\pi(E)$. Therefore it is open. □

4. Polystability and semi-stability

4.1. The aim of this section is to characterize polystability and semi-stability of $x \in \mathscr{M}$ in terms of the maximal weight λ_x. In the classical case of a Hamiltonian action on a Kähler manifold these characterizations are due to Mundet i Riera [39] and Teleman [43] respectively. Our results are more general, since we deal with a Kempf-Ness function on a topological space. Nevertheless our hypothesis are stronger from another point of view: (P5) is stronger than Assumption 1.2 in [39], while in the treatment of semi-stability we assume $\mathscr{M}$ compact, which is stronger than energy completeness (see [43, Def. 3.8]).

We start with some technical lemmata.

Lemma 4.2. *If $v, v' \in \mathfrak{k}$ commute and $iv' \in \mathfrak{g}_x$ then*

$$\lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(it(v+v'))) = \\ = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(itv)) + \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(itv')).$$

Proof. Using the commutativity and the cocycle condition (P4)

$$\begin{aligned} \Psi(x, \exp(it(v+v'))) &= \Psi(x, \exp(itv) \cdot \exp(itv')) \\ &= \Psi(\exp(itv') \cdot x, \exp(itv)) + \Psi(x, \exp(itv')) \\ &= \Psi(x, \exp(itv)) + \Psi(x, \exp(itv')). \end{aligned}$$

The result follows immediately. □

The previous lemma corresponds to [39, Lemma 3.8]. Note that there one should assume $iv' \in \mathfrak{g}_x$, which is the only case needed later in that paper.

Corollary 4.3. *If $\lambda_x \geq 0$, then $\Psi(x, \exp(iv)) = 0$ for any $v \in \mathfrak{k}$ such that $iv \in \mathfrak{g}_x$.*

Proof. The result is obvious if $v = 0$, see 2.15. If $v \neq 0$, we can assume by rescaling that $v \in S(\mathfrak{k})$. Set $f(t) := \Psi(x, \exp(itv))$. By Lemma 2.17 $f(t) = at$ for some $a \in \mathbb{R}$. Using (2.25) we get $\lambda(\mathrm{e}(-v)) = a$ and $\lambda_x(\mathrm{e}(v)) = -a$. Since $\lambda_x \geq 0$, $a = 0$ and $f \equiv 0$. □

Lemma 4.4. *Let x be a point in $\mathscr{M}$. If $\langle \mathfrak{F}(x), v\rangle = 0$ for any $v \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{k}_x)$, then $\mathfrak{F}(x) = 0$.*

Proof. Fix an Ad–invariant scalar product on $\mathfrak{k}$ and identify $\mathfrak{F}(x)$ with a vector in $\mathfrak{k}$. It is to enough to show that $\mathfrak{F}(x)$ belongs to $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{k}_x)$. See [43, Rem. 2.13] for the rest of the proof. □

4.5. Given a geodesic α in the symmetric space X, denote by $\alpha(+\infty)$ the equivalence class of α and by $\alpha(-\infty)$ the equivalence class of the geodesic $t \mapsto \alpha(-t)$.

Definition 4.6. *We say that p and $p' \in \partial_\infty X$ are* connected *if there exists a geodesic α in X such that $p = \alpha(\infty)$ and $p' = \alpha(-\infty)$.*

For $x \in \mathscr{M}$ set

$$Z(x) := \{p \in \partial_\infty X : \lambda_x(p) = 0\}. \tag{4.7}$$

Lemma 4.8. *If $\mathfrak{F}(x) = 0$, then $\mathfrak{g}_x = \mathfrak{k}_x \oplus i\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{k}_x$ and $Z(x) = \mathrm{e}(S(\mathfrak{q}))$.*

Proof. By Proposition 2.35 G_x is a compatible subgroup of G, so $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{p}$ with $\mathfrak{p} \subset i\mathfrak{k}$. Set $\mathfrak{q} := i\mathfrak{p}$. From Lemma 2.17 and (P3) it follows that for $v \in \mathfrak{k}$ the condition $iv \in \mathfrak{g}_x$ is equivalent to $\mathbb{C}v \subset \mathfrak{g}_x$. So $\mathfrak{q} \subset \mathfrak{k}_x$. To prove the last assertion fix $v \in S(\mathfrak{k})$. If $\mathrm{e}(v) \in Z(x)$, then $f(t) := \psi_x(\exp(itv)K)$ is convex and satisfies $f'(+\infty) = \lambda_x(\mathrm{e}(v)) = 0$ and $f'(0) = \mathfrak{F}^v(x) = 0$. Hence f is constant for $t > 0$, so $iv \in \mathfrak{g}_x$ by (P3) and $v \in \mathfrak{q}$. This proves that $\mathrm{e}^{-1}(Z(x)) \subset S(\mathfrak{q})$. Conversely, if $v \in S(\mathfrak{q})$, then $iv \in \mathfrak{g}_x$, so f is linear by Lemma 2.17. Moreover $f'(0) = \mathfrak{F}^v(x) = 0$. So $f \equiv 0$ and $\mathrm{e}(v) \in Z(x)$. □

4.9. For $u \in \mathfrak{k}$ denote by T_u the closure of $\exp(\mathbb{R}u)$ in K and denote by K^u the centralizer of u in K, i.e.

$$K^u := \{a \in K : \mathrm{Ad}\, a(u) = u\}.$$

Similarly G^u is the centralizer in G. Note that $G^u = (K^u)^{\mathbb{C}}$.

Lemma 4.10. *If $g \in G$ and $u \in S(\mathfrak{k})$, we have $\dim T_u = \dim T_{g\cdot u}$.*

The action of G on $S(\mathfrak{k})$ is defined in (2.27). The proof can be found in [39, Lemma 2.1].

Lemma 4.11. *Let $x \in \mathscr{M}$ and assume that $\lambda_x \geq 0$. Let $u \in \mathrm{e}^{-1}(Z(x))$ be such that*

$$\dim T_u = \max_{w \in \mathrm{e}^{-1}(Z(x))} \dim T_w. \tag{4.12}$$

Let $K' \subset K^u$ be a compact connected subgroup such that the morphism

$$T_u \times K' \to K^u, \quad (a, b) \mapsto ab, \tag{4.13}$$

is surjective and with finite kernel. Set $G' := (K')^{\mathbb{C}}$. If $\mathfrak{F}^u(x) = 0$, then x is G'-stable.

Proof. First of all we claim that $iu \in \mathfrak{g}_x$. In fact $\lambda_x(\mathrm{e}(u)) = 0$ and $\mathfrak{F}^u(x) = 0$ by hypothesis. So the convex function $f(t) = \psi_x(\exp(itu)K)$ satisfies $f'(0) = \lim_{t\to+\infty} f'(t) = 0$. It follows that f is constant on $[0, +\infty)$. Using

(P3) we conclude that $\exp(\mathbb{C}u) \subset G_x$. This proves the claim. Next set $X' := G'/K'$. Then $X' \hookrightarrow X = G/K$ since $G' \cap K = K'$. As noted in 3.7, $\Psi|_{\mathscr{M}\times G'}$ is a Kempf-Ness function for $(\mathscr{M}, G', K')$. We claim that $Z(x) \cap \partial_\infty X' = \emptyset$ and we argue by contradiction. Assume that there is $u' \in S(\mathfrak{k}')$, such that $\mathrm{e}(u') \in Z(x) \cap \partial_\infty X'$. Let $a > 0$. Since $[u', u] = 0$ and $iu \in \mathfrak{g}_x$, Lemma 4.2 yields

$$\lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(-it(au' + u)) =$$
$$= a \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(-itu')) + \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(-itu)) =$$
$$= a\lambda_x(\mathrm{e}(u')) + \lambda_x(\mathrm{e}(u)).$$

By assumption $\lambda_x(\mathrm{e}(u)) = \lambda_x(\mathrm{e}(u')) = 0$. It follows that for any $a \in \mathbb{R}$ the vector $(u + au')/|u + au'|$ belongs to $\mathrm{e}^{-1}(Z(x))$ We claim that for some a $\dim T_{u+au'} > \dim T_u$. Let $T' = \overline{\exp(\mathbb{R}u + \mathbb{R}u')}$ and $T_{u'} = \overline{\exp(\mathbb{R}u')}$. Since $T_{u'} \subset K'$ and $K' \cap T_u$ is finite, the morphism

$$f : T_u \times T_{u'} \longrightarrow T', \qquad f(a, b) = ab,$$

is a finite covering. Let $\{e_1, \dots, e_n\}$ (respectively $\{e'_1, \dots, e'_m\}$) be a basis of the lattice $\ker \exp \subset \mathfrak{t}_u$ (respectively $\ker \exp \subset \mathfrak{t}_{u'}$). If $u = X_1e_1 + \cdots + X_ne_n$ and $u' = Y_1e'_1 + \cdots + Y_me'_m$, then $u + au' = X_1e_1 + \cdots + X_ne_n + aY_1e'_1 + \cdots + aY_me'_m$. Denote by $T'_{u+au'}$ the closure of $\exp(\mathbb{R}(u + au'))$ in T'. Since f is a covering $\dim T_{u+au'} = \dim T'_{u+au'}$. Therefore

$$\dim T_{u+au'} = \dim_{\mathbb{Q}} \left(\mathbb{Q}X_1 + \cdots + \mathbb{Q}X_n + \mathbb{Q}\, aY_1 + \cdots + \mathbb{Q}\, aY_m\right).$$

(See e.g. [14, p. 61].) Since $u' \neq 0$, $Y_j \neq 0$ for some j. Choose a such that $aY_j \notin \mathbb{Q}X_1 + \ldots + \mathbb{Q}X_n$ and set $w := (u + au')/|u + au'|$. Then $\mathrm{e}(w) \in Z(x)$ and $\dim T_w = \dim T_{u+au'} > \dim T_u$. This contradicts (4.12). We have proved that $Z(x) \cap \partial_\infty X' = \emptyset$. Since $\lambda_x \geq 0$ on $\partial_\infty X$, we conclude that $\lambda_x > 0$ on $\partial_\infty X'$. By Theorem 3.4 x is G'-stable. □

Theorem 4.14 (Mundet i Riera)**.** *A point $x \in \mathscr{M}$ is polystable if and only if $\lambda_x \geq 0$ and for any $p \in Z(x)$ there exists $p' \in Z(x)$ such that p and p' are connected.*

Proof. Let $x \in \mathscr{M}$ be a point satisfying the condition in the theorem. If $Z(x) = \emptyset$, then $\lambda_x > 0$, so by Theorem 3.4 x is stable and a fortiori polystable. If $Z(x) \neq \emptyset$, choose $p \in Z(x)$ such that the dimension of the torus T_v, where $v := \mathrm{e}^{-1}(p)$ is the largest possible. In other words

$$\dim T_v = \max_{w \in \mathrm{e}^{-1}(Z(x))} \dim T_w.$$

By assumption there is a geodesic α in X such that $p = \alpha(+\infty)$ and $p' = \alpha(-\infty) \in Z(x)$, using the notation of 4.5. Assume that $\alpha(t) = g \exp(itu)K$.

Then $p = g \cdot \mathrm{e}(u)$ and $p' = g \cdot \mathrm{e}(-u)$. By (2.29)

$$\lambda_{g^{-1}\cdot x}(\mathrm{e}(u)) = \lambda_x(g \cdot \mathrm{e}(u)) = \lambda_x(p) = 0,$$
$$\lambda_{g^{-1}\cdot x}(\mathrm{e}(-u)) = \lambda_x(g \cdot \mathrm{e}(-u)) = \lambda_x(p') = 0.$$

Set $y := g^{-1} \cdot x$. We have just proved that the convex function $t \mapsto \psi_y(\exp(itu)K)$ has zero derivative at both $+\infty$ and $-\infty$. So it is constant and by (P3) $\exp(\mathbb{C}u) \subset G_y$. Moreover $\mathfrak{F}^u(y) = 0$ by (2.10). Since $\mathrm{e}(v) = p = g \cdot \mathrm{e}(u)$, Lemma 4.10 implies that $\dim T_u = \dim T_v$. From (2.29) we deduce that $Z(x) = g(Z(y))$ and that $\lambda_y \geq 0$. . Using again Lemma 4.10 we get

$$\dim T_u = \max_{w \in \mathrm{e}^{-1}(Z(y))} \dim T_w.$$

Fix a compact connected subgroup $K' \subset K^u$ such that the map (4.13) is a finite covering. Set $G' := (K')^{\mathbb{C}}$. By Lemma 4.11 $y \in \mathscr{M}$ is G'-stable. Denote by $\mathfrak{F}' : \mathscr{M} \to \mathfrak{k}'^*$ the momentum mapping of $(\mathscr{M}, G', K', \Psi|_{\mathscr{M}\times G'})$. Thus there is $h \in G' \subset G$ such that $\mathfrak{F}'(h \cdot y) = 0$. Set $z := h \cdot y$. The fact that $\mathfrak{F}'(z) = 0$, means that $\langle \mathfrak{F}(z), v\rangle = 0$, for any $v \in \mathfrak{k}'$, see 3.7. We claim that $\langle \mathfrak{F}(z), v\rangle = 0$ also for $v \in \mathfrak{t}_u$. First we prove that $\mathfrak{t}_u \subset \mathfrak{k}_z$. In fact let $\mathscr{M}'$ denote the set of points of $\mathscr{M}$ that are fixed by $\overline{\exp(\mathbb{C}u)}$. Note that $y \in \mathscr{M}'$. Since G^u preserves $\mathscr{M}'$, also $z \in \mathscr{M}'$, i.e. $\overline{\exp(\mathbb{C}u)} \subset G_z$, as desired. Now if $v \in \mathfrak{t}_u$, then $v \in \mathfrak{k}_z$, so by Lemma 2.19 the function $f(t) = \Psi(z, \exp(itv))$ is linear. Since $f'(-\infty) = \lambda_z(\mathrm{e}(v)) \geq 0$ and $f'(+\infty) = \lambda_z(\mathrm{e}(-v)) \geq 0$, we conclude that $f'(0) = \langle \mathfrak{F}(z), v\rangle = 0$ for any $v \in \mathfrak{t}_u$, as claimed. Since $\mathfrak{k}' \oplus \mathfrak{t}_u = \mathfrak{k}^u$, we have proved that $\langle \mathfrak{F}(z), v\rangle = 0$ for any $v \in \mathfrak{k}^u$. But $\mathfrak{t}_u \subset \mathfrak{k}_z$, so $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{k}_z) \subset \mathfrak{k}^u$. Therefore Lemma 4.4 implies that $\mathfrak{F}(z) = 0$. This finally proves that x is polystable. We have shown that the condition in the theorem implies polystability. To prove the opposite implication, assume that x is polystable. Then there exists $g \in G$ such that $\mathfrak{F}(g \cdot x) = 0$. Set $y = g \cdot x$ and fix $v \in \mathfrak{k}$. The function $f(t) = \Psi(y, \exp(-itv))$ is convex and

$$\lambda_y(\mathrm{e}(v)) = \lim_{t\to\infty} f'(t) \geq f'(0) = \langle \mathfrak{F}(y), v\rangle = 0.$$

This proves that $\lambda_y \geq 0$ on $\partial_\infty X$. By Lemma 2.28 also $\lambda_x \geq 0$. Next we check the condition on $Z(y)$. By Proposition 2.35 and Lemma 4.8, G_y is compatible with $\mathfrak{g}_y = \mathfrak{k}_y \oplus \mathfrak{q}$ and $Z(y) = e(S(\mathfrak{q}))$. If $\mathrm{e}(v) \in Z(y)$, then also $\mathrm{e}(-v) \in Z(y)$. Since $\mathrm{e}(v)$ and $\mathrm{e}(-v)$ are obviously connected, the condition in the theorem holds for $Z(y)$. Moreover $Z(x) = g^{-1}(Z(y))$, and so $Z(x)$ also satisfies the condition in the theorem. Indeed, if $p \in Z(x)$, then $g \cdot p \in Z(y)$, so there is $q \in Z(y)$ connected to $g \cdot p$, by a geodesic α, i.e. $\alpha(+\infty) = g \cdot p$ and $\alpha(-\infty) = q$. Then the geodesic $g^{-1} \circ \alpha$ connects p to $g^{-1} \cdot q \in Z(x)$. □

Example 4.15. If X is the unit disc, any two distinct points of $\partial_\infty X$ are connected. Thus in this case the condition in the theorem means that either $Z(x)$ is empty, or it contains at least two points. If $X = \mathbb{R}^n$, and $v \in S(\mathfrak{k}) = S^{n-1}$, then $\mathrm{e}(v)$ is connected only to $\mathrm{e}(-v)$. In this case the condition in the theorem amounts to saying that $\mathrm{e}^{-1}(Z(x)) \subset S^{n-1}$ is invariant by the antipodal map.

The following important lemma is due to Kapovich, Leeb and Millson [31, Lemma 3.4].

Lemma 4.16. *Let X be a symmetric space and let let $u : X \to \mathbb{R}$ be a smooth convex function. If $u_\infty \geq 0$ on $\partial_\infty X$, then there is a sequence $\{p_n\}$ in X such that $||\nabla u_\infty(p_n)|| \to 0$.*

The next theorem uses ideas from the proof of Theorem 4.3 in [43], with simplifications due to the previous lemma and the compactness hypothesis.

Theorem 4.17. *If $\mathscr{M}$ is compact, then a point $x \in \mathscr{M}$ is semi-stable if and only if $\lambda_x \geq 0$.*

Proof. Assume that $x \in \mathscr{M}$ and that $\lambda_x \geq 0$. By Lemma 4.16 there is a sequence of points $\{p_n\} \subset X$ such that $d\psi_x(p_n) \to 0$. Write $p_n = g_n K$. Using (2.22) we get $d(L^*_{g_n}\psi_x) = d\psi_{g_n^{-1}x}$. So

$$d\psi_{g_n^{-1}x}(o) = d\psi_x(g_n K) \circ dL_{g_n}(o) = d\psi_x(p_n) \circ dL_{g_n}(o).$$

Since L_g is an isometry of X,

$$||\mathfrak{F}(g_n^{-1}x)|| = ||d\psi_{g_n^{-1}x}(o)|| = ||d\psi_x(p_n)|| \to 0.$$

Since $\mathscr{M}$ is compact, we can assume that $g_n^{-1}x$ converges to some point $y \in \overline{G \cdot x}$. By (P6) we get immediately that $\mathfrak{F}(y) = 0$. So x is semi–stable. Conversely, assume that x is semi-stable, i.e. there is $y \in \overline{G \cdot x} \cap \mathfrak{F}^{-1}(0)$. Pick a net of elements $g_\alpha \in G$ such that $g_\alpha x \to y$ (if $\mathscr{M}$ satisfies the first countability axiom one can just take a sequence). Assume by contradiction that there is some $p \in \partial_\infty X$ such that $\lambda_x(p) < 0$ and set $v := \mathrm{e}^{-1}(p) \in S(\mathfrak{k})$. Since $\mathfrak{F}(y) = 0$, y is polystable, so by Theorem 4.14 $\lambda_y \geq 0$. Write $g_\alpha \cdot p = \mathrm{e}(v_\alpha)$ for $v_\alpha \in S(\mathfrak{k})$. By passing to a subnet we can assume that $v_\alpha \to w$. Set $q = \mathrm{e}(w)$. Then $g_\alpha \cdot p \to q$ in the sphere topology. By Lemma 3.9 we have

$$\lambda_y(q) \leq \liminf_{\alpha} \lambda_{g_\alpha \cdot x}(g_\alpha \cdot p).$$

But $\lambda_{g_\alpha \cdot x}(g_\alpha \cdot p) = \lambda_x(p) < 0$ and $\lambda_y(q) \geq 0$. Thus we get a contradiction. This proves that $\lambda_x \geq 0$.

□

5. Measures

5.1. If M is a compact manifold, denote by $\mathscr{M}(M)$ the vector space of finite signed Borel measures on M. These measures are automatically Radon [16, Thm. 7.8, p. 217]. Denote by $C(M)$ the space of real continuous function on M. It is a Banach space with the sup–norm. By the Riesz Representation

Theorem [16, p.223] $\mathscr{M}(M)$ is the topological dual of $C(M)$. The induced norm on $\mathscr{M}(M)$ is the following one:

$$||\nu|| := \sup\left\{\int_M f d\nu : f \in C(M), \sup_M |f| \leq 1\right\}. \tag{5.2}$$

We endow $\mathscr{M}(M)$ with the weak-$*$ topology as dual of $C(M)$. Usually this is simply called the *weak topology* on measures. We use the symbol $\nu_\alpha \rightharpoonup \nu$ to denote the weak convergence of the net $\{\nu_\alpha\}$ to the measure ν. Denote by $\mathscr{P}(M) \subset \mathscr{M}(M)$ the set of Borel probability measures on M. We claim that $\mathscr{P}(M)$ is a compact convex subset of $\mathscr{M}(M)$. Indeed the cone of positive measures is closed and $\mathscr{P}(M)$ is the intersection of this cone with the closed affine hyperplane $\{\nu \in \mathscr{M}(M) : \nu(M) = 1\}$. Hence $\mathscr{P}(M)$ is closed. For a positive measure $|\nu| = \nu$, so $\mathscr{P}(M)$ is contained in the closed unit ball in $\mathscr{M}(M)$, which is compact in the weak topology by the Banach-Alaoglu Theorem [15, p. 425]. Since $C(M)$ is separable, the weak topology on $\mathscr{P}(M)$ is metrizable [15, p. 426].

5.3. If $f : X \to Y$ is a measurable map between measurable spaces and ν is a measure on X, the *image measure* $f_*\nu$ is defined by $f_*\nu(A) := \nu(f^{-1}A)$. It satisfies the *change of variables formula*

$$\int_Y u(y) d(f_*\nu)(y) = \int_X u(f(x)) d\nu(x). \tag{5.4}$$

Lemma 5.5. *Let M be a compact manifold. If G is a Lie group acting continuously on M, the map*

$$G \times \mathscr{P}(M) \to \mathscr{P}(M), \quad (g, \nu) \mapsto g_*\nu, \tag{5.6}$$

defines a continuous action of G on $\mathscr{P}(M)$ provided with the weak topology.

Proof. The map is obviously an action. To check the continuity let $g_\alpha \to g$ and $\nu_\alpha \rightharpoonup \nu$ be converging nets in G and $\mathscr{P}(M)$ respectively (since both are metrizable spaces considering sequences would be enough). Fix a distance d on M inducing the manifold topology. We claim that $g_\alpha \to g$ uniformly on (M, d). Given $\varepsilon > 0$ the set $A := \{(h, x) : d(h \cdot x, g \cdot x) < \varepsilon\}$ is open in $G \times M$. Since $\{g\} \times M \subset A$ and M is compact, there is a neighbourhood U of g such that $U \times M \subset A$. There is α_0 such that $g_\alpha \in U$ for $\alpha \geq \alpha_0$, so

$$\sup_{x \in M} d(g_\alpha \cdot x, g \cdot x) < \varepsilon, \quad \text{for } \alpha \geq \alpha_0.$$

This proves the claim. Since any $\varphi \in C(M)$ is uniformly continuous, it follows that $\varphi \circ g_\alpha \to \varphi \circ g$ uniformly on M. To prove the continuity of the action we need to show that $g_{\alpha *}\nu_\alpha \rightharpoonup g_*\nu$, i.e.

$$\int_M \varphi \, d(g_{\alpha *}\nu_\alpha) \to \int_M \varphi \, d(g_*\nu), \quad \text{for any } \varphi \in C(M).$$

In fact

$$\Big|\int_M \varphi\, d(g_{\alpha *}\nu_\alpha) - \int_M \varphi\, d(g_*\nu)\Big| = \Big|\int_M \varphi \circ g_\alpha\, d\nu_\alpha - \int_M \varphi \circ g\, d\nu\Big| \leq$$
$$\leq \Big|\int_M \varphi \circ g_\alpha\, d\nu_\alpha - \int_M \varphi \circ g\, d\nu_\alpha\Big| + \Big|\int_M \varphi \circ g\, d\nu_\alpha - \int_M \varphi \circ g\, d\nu\Big|.$$

Since $\nu_\alpha(M) = 1$, the first term is bounded by $||\varphi \circ g_\alpha - \varphi \circ g||$, so tends to 0. The second term tends to 0, because $\nu_\alpha \rightharpoonup \nu$. □

5.7. The map (5.6) is not continuous if $\mathscr{P}(M)$ is endowed with the topology coming from the norm (5.2). For example if $g_n \to g$ in G, but $g_n \cdot x \neq g \cdot x$ for some point $x \in M$, then, denoting by δ_x the Dirac measure supported at x, we have $g_{n*}\delta_x = \delta_{g_n \cdot x}$ and $||g_{n*}\delta_x - g_*\delta_x|| = 2$.

Lemma 5.8. *Let X be a vector field on M with flow $\{\varphi_t\}$. If $\nu \in \mathscr{M}(M)$ and X vanishes ν–almost everywhere, then $\varphi_{t*}\nu = \nu$ for any t.*

Proof. Set $N := \{x \in M : X(x) \neq 0\}$. Then $\nu(N) = 0$ and for any $t \in \mathbb{R}$ and any $x \notin N$, $\varphi_t(x) = x$. In particular both N and $M - N$ are φ_t-invariant. If $A \subset M$ is measurable, then

$$\varphi_{-t}(A) = \varphi_{-t}((A - N) \sqcup (N \cap A)) = (A - N) \sqcup \varphi_{-t}(N \cap A).$$

Since $\varphi_{-t}(N \cap A) \subset N$, $\varphi_{t*}\nu(A) = \nu(\varphi_{-t}(A)) = \nu(A - N) = \nu(A)$. □

5.9. In the sequel for $g \in G$ and $\nu \in \mathscr{P}(M)$, we will use the notation

$$g \cdot \nu := g_*\nu.$$

5.10. From now on we assume that M is a compact Kähler manifold, that K is a compact connected Lie group acting on M in a Hamiltonian fashion with momentum mapping μ and complexification G. All the notation will be as in 2.12.

Corollary 5.11. *If $v \in \mathfrak{g}$ and $v_M(x) = 0$ for every x outside a set of ν–measure zero, then $\exp(\mathbb{C}v) \subset G_\nu$.*

Proof. Since $(iv)_M = J(v_M)$ the result follows immediately from the above Lemma. □

Proposition 5.12. *Let M, G, K and μ be as in 2.12. The function*

$$\Psi^{\mathscr{P}} : \mathscr{P}(M) \times G \to \mathbb{R}, \quad \Psi^{\mathscr{P}}(\nu, g) := \int_M \Psi^M(x, g) d\nu(x) \tag{5.13}$$

is a Kempf-Ness function for $(\mathscr{P}(M), G, K)$. The corresponding function on the symmetric space $X = G/K$ is

$$\psi_\nu^{\mathscr{P}} : X \to \mathbb{R}, \quad \psi_\nu^{\mathscr{P}}(gK) := \Psi^{\mathscr{P}}(\nu, g^{-1}) = \int_M \psi_x^M(gK) d\nu(x). \tag{5.14}$$

The momentum mapping $\mathfrak{F} : \mathscr{P}(M) \to \mathfrak{k}^*$ *is given by formula* (1.1)*, i.e.*

$$\mathfrak{F}(\nu) := \int_M \mu(x) d\nu(x). \tag{5.15}$$

Proof. Since Ψ^M is left-invariant with respect to K, also $\Psi^{\mathscr{P}}$ is left-invariant with respect to K. Fix $\nu \in \mathscr{P}(M)$. By differentiation under the integral sign $\Psi^{\mathscr{P}}(\nu, \cdot)$ is a smooth function on G and for $v \in \mathfrak{k}$ we have

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi^{\mathscr{P}}(\nu, \exp(itv)) = \int_M \left(\frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi^M(x, \exp(itv)) \right) d\nu(x) \geq 0,$$

since the integrand is non-negative by (P3). If

$$\left. \frac{\mathrm{d}^2}{\mathrm{dt}^2} \right|_{t=0} \Psi^{\mathscr{P}}(\nu, \exp(itv)) = 0,$$

then

$$\left. \frac{\mathrm{d}^2}{\mathrm{dt}^2} \right|_{t=0} \Psi^M(x, \exp(itv)) = 0 \qquad \nu\text{-almost everywhere.}$$

Again by (P3) this implies that $v_M = 0$ ν-almost everywhere. By Corollary 5.11 it follows that $\exp(\mathbb{C}v) \subset G_\nu$. We have proved (P1)–(P3). The cocycle condition for $\Psi^{\mathscr{P}}$ follows immediately from the cocycle condition for Ψ^M. Indeed,

$$\begin{aligned} \Psi^{\mathscr{P}}(\nu, gh) &= \int_M \Psi^M(x, gh) d\nu(x) = \int_M \Psi^M(x, g) d\nu(x) + \int_M \Psi^M(gx, h) d\nu \\ &= \int_M \Psi^M(x, g) d\nu(x) + \int_M \Psi^M(y, h) d(g \cdot \nu) \\ &= \Psi^{\mathscr{P}}(\nu, g) + \Psi^{\mathscr{P}}(g \cdot \nu, h). \end{aligned}$$

Fix $\nu \in \mathscr{P}(M)$. It is immediate to verify that the function $\psi^{\mathscr{P}}$ defined as in (2.8) is given by (5.14) Next we verify that $\psi^{\mathscr{P}}_\nu$ is Lipschitz. Next we compute the momentum mapping.

$$\begin{aligned} \langle \mathfrak{F}(\nu), v \rangle = \int_M \left. \frac{\mathrm{d}}{\mathrm{dt}} \right|_{t=0} \Psi^M(x, \exp(itv)) d\nu(x) = \int_M \langle \mu(x), v \rangle d\nu(x) = \\ = \langle \int_M \mu(x) d\nu(x), v \rangle. \end{aligned}$$

This proves that the map $\mathfrak{F}$ defined as in (2.10) is given by (5.15). Therefore it is clearly continuous on $\mathscr{P}(M)$, i.e. (P6) holds. Finally we verify that $\psi^{\mathscr{P}}_\nu$ is a Lipschitz function. Indeed denoting by $o = K$ the origin in X

$$\begin{aligned} (d\psi^{\mathscr{P}}_\nu)_{go}(dL_g(\dot\gamma^v(0))) = d\psi^{\mathscr{P}}_{g^{-1}\cdot\nu}(o)(\dot\gamma^v(0)) = -\langle \mathfrak{F}(g^{-1} \cdot \nu), v \rangle = \\ = -\int_M \langle \mu(x), v \rangle d(g^{-1} \cdot \nu)(x). \end{aligned}$$

Keeping in mind that M is compact and recalling that L_g is an isometry of X, we get $||(d\psi_\nu^{\mathscr{P}})_{go}|| \leq ||\mu||_{L^\infty}$. Thus $\psi_\nu^{\mathscr{P}}$ is a Lipschitz function, i.e. (P5) holds. □

Theorem 5.16 (Linearization Theorem)**.** *Let M, G, K and μ be as in 2.12. If x is a fixed point of G, then there exist an open subset $S \subset T_xM$, stable under the isotropy representation of G, an open G-stable neighbourhood Ω of x in M and a G-equivariant biholomorphism $h : S \to \Omega$. One can further require that $h(0) = x$ and $dh_0 = \mathrm{id}_{T_xM}$.*

For the proof see [26, §14], [22], [25] and [42].

5.17. Fix $v \in S(\mathfrak{k})$. The *gradient flow* of a function $f \in C^\infty(M)$ is usually defined as the flow of the vector field $-\operatorname{grad} f$. Let $\{\varphi_t\}$ denote the gradient flow of μ^v. Since $\operatorname{grad}\mu^v = Jv_M = (iv)_M$, we have $\varphi_t(x) = \exp(-itv)\cdot x$.

Proposition 5.18. *For any $x \in M$ the limit*

$$\alpha(x) := \lim_{t\to-\infty}\varphi_t(x) = \lim_{t\to+\infty}\exp(itv)\cdot x. \tag{5.19}$$

exists.

Proof. Consider the set $L_\alpha(x)$ formed by all points $y \in M$ such that there is a sequence $t_n \to -\infty$ with $y = \lim_{n\to\infty}\varphi_{t_n}(x)$. It follows from the compactness of M that $L_\alpha(x)$ is a non–empty subset of M. Moreover $L_\alpha(x)$ is invariant under the flow. (See [30, Ex. 1 p. 164] for more details.) Fix a point $y \in L_\alpha(x)$ and fix a sequence $\{t_n\}$ such that $y = \lim_{n\to\infty}\varphi_{t_n}(x)$ and $t_n \to -\infty$. We claim that y is a fixed point of the flow, i.e. $v_M(y) = 0$. In fact if $v_M(y) \neq 0$, one can linearize the vector field v_M on a neighbourhood U of y. Since φ_t is a gradient flow one can assume that for any point $z \in U$ the flow $\varphi_t(z) = \exp(-itv)\cdot z$ lies out of U for t sufficiently negative. But there is n_0 such that $\varphi_{t_n}(x) \in U$ for $n \geq n_0$. So $z = \varphi_{t_{n_0}}(x) \in U$, and $\varphi_{t_n - t_{n_0}}(z) = \varphi_{t_n}(x)$ belongs to U for any n, although $t_n - t_{n_0} \to -\infty$. This yields a contradiction and proves that necessarily $v_M(y) = 0$ as desired. Set $T := \overline{\exp(\mathbb{R}v)}$. Then T and its complexification $T^{\mathbb{C}}$ fix y. We get an isotropy action $T^{\mathbb{C}} \to \mathrm{GL}(T_yM)$, defined by $a \mapsto da_y$. The Linearization Theorem 5.16 tells us that one can find an open subset $S \subset T_yM$, invariant under the isotropy action, an open $T^{\mathbb{C}}$–invariant subset $\Omega \subset M$ and a $T^{\mathbb{C}}$–equivariant biholomorphism $h : S \to \Omega$ such that $h(0) = y$, $dh_0 = \mathrm{id}_{T_yM}$. Since $\exp(\mathbb{C}v) \subset T^{\mathbb{C}}$ and $\varphi_t = \exp(-itv)$, we also have an action of $\mathbb{R}$ on T_yM, given by $t \mapsto (d\varphi_t)_y$. The infinitesimal generator of this action is the symmetric operator H corresponding to the Hessian of $-\mu^v$ at y, see [28, p. 215]. In other words $(d\varphi_t)_y = \exp(tH)$. Denote by $T_yM = V_+ \oplus V_0 \oplus V_-$ the decomposition corresponding to the sign of the eigenvalues of H. Since S is invariant by $\exp(\mathbb{R}H)$, we have necessarily $V_+ \oplus V_- \subset S$. Fix a small ball $B(0,r) \subset S$. There is n_0 such that $h^{-1}(\varphi_{t_n}(x)) \in B(0,r)$ for any $n \geq n_0$. Set $w := h^{-1}(\varphi_{t_{n_0}}(x))$. Then $\varphi_{t_n}(x) = \varphi_{t_n - t_{n_0}}(\varphi_{t_{n_0}}(x))$ and $h^{-1}(\varphi_{t_n}(x)) = \exp((t_n - t_{n_0})H)\cdot w$. Let $w = w_+ + w_0 + w_-$ be the decomposition with

$w_\pm \in V_\pm$ and $w_0 \in V_0$. If the component w_+ were nonzero, we would have $\lim_{t\to-\infty} ||\exp(tH)w|| = +\infty$. Instead $\exp((t_n - t_{n_0})H)\cdot w = h^{-1}(\varphi_{t_n}(x)) \in B(0,r)$ for any $n \geq n_0$. Therefore $w_+ = 0$. It follows that

$$\lim_{t\to-\infty} \exp(tH)w = w_0,$$
$$\lim_{t\to-\infty} \varphi_t(x) = \lim_{t\to-\infty} \varphi_t(\varphi_{t_{n_0}}(x)) = h(w_0).$$

This proves that the limit exists. □

5.20. Fix again $v \in S(\mathfrak{k})$ and the notation of 5.17. By Frankel Theorem (see e.g. [4, Thm. 2.3, p. 109] or [35, p. 180]) the function μ^v is a Morse-Bott function with critical points of even index, therefore all its local maximum points are global maximum points and all its level sets are connected. Let $c_0 < \cdots < c_r$ be the critical values of μ^v and let $C_i := (\mu^v)^{-1}(c_i)$. Since the level sets of μ^v are connected, the C_i's are exactly the connected components of $\mathrm{Crit}(\mu^v)$. Set

$$W_i^u := \{x \in M : \alpha(x) \in C_i\}, \tag{5.21}$$

This is the *unstable manifold* of the critical component C_i for the gradient flow of μ^v. It follows from the previous Proposition that

$$M = \bigsqcup_{i=0}^{r} W_i^u. \tag{5.22}$$

For any i the map

$$\alpha|_{W_i^u} : W_i^u \to C_i$$

is a smooth fibration with fibres diffeomorphic to $\mathbb{R}^{l_i}$ where l_i is the index (of negativity) of the critical submanifold C_i. Since all local maximum points of μ^v are global maximum points, we have $\dim_{\mathbb{R}} C_i + l_i = \dim_{\mathbb{R}} M$ if and only if $i = r$. This means that W_i^u is open only for $i = r$. It follows from (5.22) that W_r^u is also dense.

Theorem 5.23. *With the notation above we have*

$$\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=0}^{r} c_i \cdot \nu(W_i^u).$$

Proof. Using the definition of λ_ν, (5.13) and differentiation under the integral we get

$$\lambda_\nu(\mathrm{e}(-v)) = \lim_{t\to+\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi^{\mathscr{P}}(\nu, \exp(itv)) =$$
$$= \lim_{t\to+\infty} \int_M \left(\frac{\mathrm{d}}{\mathrm{dt}} \Psi^M(x, \exp(itv)) \right) d\nu(x).$$

By (2.33)

$$\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=t_o} \Psi^M(x, \exp(itv)) = \mu^v(\exp(it_0 v)\cdot x).$$

Since μ^v is a bounded we can apply the dominated convergence theorem:

$$\begin{aligned}\lambda_\nu(\mathrm{e}(-v)) = \lim_{t\to+\infty}\int_M \mu^v(\exp(itv)\cdot x)d\nu(x) = \\ = \int_M \left(\lim_{t\to+\infty}\mu^v(\exp(itv)\cdot x)\right)d\nu(x) = \int_M \mu^v(\alpha(x))d\nu(x) = \\ = \sum_{i=0}^r \int_{W_i^u}\mu^v(\alpha(x))d\nu(x).\end{aligned}$$

For $x\in W_i^u$, $\alpha(x)\in C_i$, so $\mu^v(\alpha(x)) = c_i$. Thus

$$\int_{W_i^u}\mu^v(\alpha(x))d\nu(x) = c_i\cdot\nu(W_i^u).$$

This proves the theorem. □

5.24. Let $E(\mu)$ denote the convex hull of $\mu(M)\subset\mathfrak{k}^*$ and let $\Omega(\mu)$ denote the interior of $E(\mu)$ as a subset of $\mathfrak{k}^*$. Remark that the action of K on M is *almost effective* (i.e. if $v\in\mathfrak{k}$ and $v_M = 0$, then $v = 0$) if and only if $\mu(M)$ is full in $\mathfrak{k}^*$ (i.e. it is not contained in any affine hyperplane) if and only if $\Omega(\mu)$ is non-empty.

Lemma 5.25. *If ν is a probability measure on M, then $\mathfrak{F}(\nu)\in E(\mu)$.*

Proof. If $\nu\in\mathscr{P}(M)$, then $\mathfrak{F}(\nu)$ is the center of gravity of the measure $\mu_*\nu\in\mathscr{P}(\mu(M))$. Therefore $\mathfrak{F}(\nu)$ lies in the convex hull of $\mu(M)$. □

5.26. On a differentiable manifold there is no preferred measure, but there is a well defined class of measures: those that in any chart have a smooth strictly positive density with respect to the Lebesgue measure of the chart. These are called *smooth strictly positive measures* on M. Any two such measures are absolutely continuous with respect to one another.

Definition 5.27. *Let $\mathscr{AC}(M)$ denote the set of the probability measures on M that are absolutely continuous with respect to one smooth strictly positive measure (and hence with respect to any such measure).*

Definition 5.28. *Let $\mathscr{W}(M, K, \omega, \mu)$ (or $\mathscr{W}(M, K)$ for brevity) denote the set of probability measures on M that satisfy the following condition: for every $v\in S(\mathfrak{k})$, the open unstable manifold has full measure. In the notation of 5.20 this means that $\nu(W_r^u) = 1$.*

5.29. In words $\nu\in\mathscr{W}(M,K)$ if ν is concentrated on the open unstable manifold. Since the other unstable manifolds are submanifolds of positive codimension in M, it is clear that $\mathscr{AC}(M)\subset\mathscr{W}(M,K)$.

An easy consequence of Theorem 5.23 is the following result.

Corollary 5.30. *If $\nu \in \mathscr{W}(M,K)$, then for any $v \in S(\mathfrak{k})$*

$$\lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M} \mu^v(x).$$

Proof. By Theorem 5.23, $\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=0}^r c_i \nu(W_i^v)$. Since $\nu(W_r^v) = 1$, $\nu(W_i^v) = 0$ for $i = 0, \dots, r-1$, so $\lambda_\nu(\mathrm{e}(-v)) = c_r = \max_{x\in M} \mu^v(x)$. □

Theorem 5.31. *If $\nu \in \mathscr{W}(M,K)$ and $0 \in \Omega(\mu)$, then ν is stable. In particular it is polystable, so there is $g \in G$ such that $\mathfrak{F}(g_*\nu) = 0$.*

Proof. Since $0 \in \Omega(\mu)$, for each $v \neq 0$ the function μ^v attains both positive and negative values. So the maximum of μ^v is positive. By the previous corollary $\lambda_\nu(\mathrm{e}(-v)) > 0$. The result follows applying Theorem 3.4. □

Corollary 5.32. *If $0 \in \Omega(\mu)$, then the set $\mathscr{P}_s(M) := \{\nu \in \mathscr{P}(M) :$ ν is stable$\}$ is open and dense in $\mathscr{P}(M)$.*

Proof. By Corollary 3.10 the set $\mathscr{P}_s(M)$ is open. By Theorem 5.31 any smooth measure is stable. Hence it is enough to prove that smooth measures are dense. It is easy to check that for any Dirac measure δ_y there exists a sequence of smooth measures ν_n such that $\nu_n \rightharpoonup \delta_y$. Hence any convex combination of Dirac measures is the weak limit of a sequence of smooth measures. Since $\mathscr{P}(M)$ is a compact convex set in $\mathscr{M}(M)$ (endowed with the weak topology) and its extremal points are exactly the Dirac measures, the Krein-Milman Theorem [15, p. 440] implies that convex combinations of Dirac measures are dense. Therefore also smooth probability measures are dense and so $\mathscr{P}_s(M)$ is dense in $\mathscr{P}(M)$. (See Lemma 3.6 in [31, p. 316] for a similar argument.) □

5.33. We point out that for an almost effective action, up to shifting the momentum mapping μ, the condition $0 \in \Omega(\mu)$ is always satisfied. This is the content of the following lemma. It implies that $0 \in \Omega(\mu)$ when K is semisimple and the action is almost effective. In the following we fix an Ad–invariant scalar product on $\mathfrak{k}$ and we think of the momentum mapping as a $\mathfrak{k}$-valued map.

Lemma 5.34. *Let (M,ω), K and μ be as in 2.12. Assume that the action is almost effective. Then $\Omega(\mu) \cap \mathfrak{z}(\mathfrak{k}) \neq \emptyset$.*

Proof. Fix a maximal torus $T \subset K$ and let $\mathfrak{t}$ be its Lie algebra and $\pi : \mathfrak{k} \to \mathfrak{t}$ the orthogonal projection. Then $P := \pi(\mu(M))$ is the momentum polytope for the T-action. Let $\{v_1, \dots, v_q\}$ be the set of vertices of P. Then $b := (v_1 + \dots + v_q)/q$ is the centroid of P [6, p. 60]. We claim that $b \in \operatorname{int} P$. In fact, if b were a boundary point, there would be a a proper face containing b. Since any face of a polytope is exposed [41], there would exist a nonzero vector $u \in \mathfrak{k}$ such that $\langle b, u\rangle = c$, where $c := \max_{x\in P}\langle x, u\rangle$. So b would belong to the face $F_u(P) = \{y \in P : \langle x, u\rangle = c\}$. After reordering the vertices we can assume that $\langle v_j, u\rangle = c$ if $1 \leq j \leq p$ and $\langle v_j, u\rangle < c$ if $p < j \leq q$. This means that $v_1, \dots, v_p$ are the extremal points of $F_u(P)$. So there would exist $\lambda_i \in [0,1]$ such that $b = \lambda_1 v_1 + \cdots + \lambda_p v_p$. Therefore

$$\sum_{i=1}^{p} \left(\lambda_i - \frac{1}{q}\right) v_i = \sum_{i>p} \frac{1}{q} v_i.$$

But the left hand side lies in the hyperplane $\{x \in \mathfrak{t} : \langle x, u\rangle = c\}$, while the right hand side lies in the half-space $\{x \in \mathfrak{t} : \langle x, u\rangle < c\}$. Thus we have $p = q$ and $P = F_u(P)$. But this is absurd since $F_u(P)$ is a proper face. Thus we have proved that $b \in \operatorname{int} P$. By the main Theorem in [9] the faces of $E(\mu)$ correspond to the faces of P. Hence the interior of $E(\mu)$ corresponds to the interior of P, that is $K \cdot \operatorname{int} P = \Omega(\mu)$. So $b \in \Omega(\mu)$. Finally $\mu(M)$ is K-invariant, so $P = \pi(\mu(M))$ is invariant for the Weyl group $\mathcal{W} = \mathcal{W}(K,T)$. Therefore also b is fixed by $\mathcal{W}$. This proves that b lies in $\mathfrak{z}(\mathfrak{k})$. □

Corollary 5.35. *Let (M, ω) be a compact Kähler manifold and let K be a compact group acting on M almost effectively and in Hamiltonian fashion with momentum mapping $\mu : M \to \mathfrak{k}^*$. If $\nu \in \mathscr{AC}(M)$, then G_ν is compact. If $\nu \in \mathscr{AC}(M)$ is K–invariant, then $G_\nu = K$.*

Proof. By the above lemma, up to shifting μ by an element of the center we can assume that $0 \in \Omega(\mu)$. By Theorem 5.31 ν is stable. By Corollary 3.5 G_ν is compact. If ν is K–invariant, then $K \subset G_\nu$. Since K is a maximal compact subgroup of G, we get $G_\nu = K$. □

6. The construction of Hersch and Bourguignon-Li-Yau

6.1. In various situations it is interesting to know how the map $\mathfrak{F}$ behaves on the orbit $G \cdot \nu$, where $\nu \in \mathscr{P}(M)$. Set

$$F_\nu : G \longrightarrow \mathfrak{k}^*, \quad F_\nu(a) := \mathfrak{F}(a \cdot \nu). \tag{6.2}$$

This map was used for the first time by Hersch [29], in the case $M = S^2$, to get upper bounds for λ_1. For the same purpose it was generalized by Bourguignon, Li and Yau [11] to the case $M = \mathbb{P}^n(\mathbb{C})$. We further generalized it to arbitrary flag manifolds in [8]. In this section we prove rather general theorems that extend the results in these papers to actions on arbitrary Kähler manifolds.

Recall from 2.12 that μ is the momentum mapping with respect to the symplectic form ω and that g is the Kähler metric corresponding to ω. For $X, Y \in \mathfrak{X}(M)$ set

$$(X,Y)_{L^2(g,\nu)} := \int_M g(X(x), Y(x)) d\nu(x).$$

In all this section we assume that the action of K is almost effective.

Lemma 6.3. *If $v, w \in \mathfrak{k}$ and $\nu \in \mathscr{P}(M)$, then*

$$\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \mathfrak{F}(\exp(itv)\cdot\nu), w\rangle = (v_M, w_M)_{L^2(g,\nu)}.$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \mathfrak{F}(\exp(itv)\cdot\nu), w\rangle = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \int_M \mu(y)\, d(\exp(itv)\cdot\nu)(y), w\rangle =$$

$$= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \int_M \mu^w(y)\, d(\exp(itv)\cdot\nu)(y) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \int_M \mu^w(\exp(itv)\cdot x)\, d\nu(x) =$$

$$= \int_M i_{w_M}\omega(Jv_M)\, d\nu = \int_M g(w_M, v_M) d\nu = (v_M, w_M)_{L^2(g,\nu)}.$$

□

Theorem 6.4. *If $\nu \in \mathscr{P}(M)$ and G_ν is compact, then the map $F_\nu : G \to \mathfrak{k}^*$ defined in* (6.2) *is a smooth submersion and its image is contained in $\Omega(\mu)$.*

Proof. If $a \in G$ and $v \in \mathfrak{k}$, consider the curve $\alpha(t) := \exp(itv)a$. Set $\tilde\nu := a\cdot\nu$. Then for any $w \in \mathfrak{k}$

$$\langle dF_\nu(\dot\alpha(0)), w\rangle = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \mathfrak{F}(\exp(itv)\cdot\tilde\nu), w\rangle = (v_M, w_M)_{L^2(g,\tilde\nu)}.$$

If $dF_v(\dot\alpha(0)) = 0$, choose $w = v$. Then $||v_M||_{L^2(g,\tilde\nu)} = 0$, i.e. $v_M = 0$ $\tilde\nu$–a.e. By Corollary 5.11 $\exp(\mathbb{C}v) \subset G_{\tilde\nu}$. Since $G_{\tilde\nu}$ is compact, $v = 0$. This proves that dF_ν is injective on the subspace $dR_a(e)(i\mathfrak{k}) \subset T_aG$. By dimension reasons $T_aG = \ker dF_\nu \oplus dR_a(e)(i\mathfrak{k})$ and dF_ν is onto. Therefore $F_\nu(G)$ is an open subset of $\mathfrak{k}^*$. Since it is contained in $E(\mu)$ we have $F_\nu(G) \subset \Omega(\mu)$. □

6.5. The first assertion of the previous theorem is analogous to a fact that is well-known in the classical theory of Hamiltonian actions on a Kähler manifold: if the stabilizer of a point is compact, the restriction of the momentum mapping to its orbit is a submersion. See e.g. [26, Prop. 6.1].

Lemma 6.6. *If $\nu \in \mathscr{P}(M)$ and G_ν is compact, then the function ψ_ν^M, defined in* (5.14)*, is strictly convex.*

Proof. Convexity is proven in Lemma 2.19. We need to check strict convexity. Using the cocycle condition we can restrict to geodesics passing through $o = K \in X$. Let $\alpha(t) = \exp(itv)K$ be such a geodesic. If

$$0 = \frac{\mathrm{d}^2}{\mathrm{dt}^2}\bigg|_{t=t_0} \psi_\nu^{\mathscr{P}} \circ \alpha(t) = \int_M \frac{\mathrm{d}^2}{\mathrm{dt}^2}\psi_x^M(\alpha(t)) d\nu(x),$$

then

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\psi_x^M(\alpha(t)) = 0 \quad \nu-\text{a.e. in } M.$$

By (P3) this implies that $v_M = 0$ ν-a.e., so $\exp(\mathbb{C}v) \subset G_\nu$. Since G_ν is compact, $v = 0$. This proves that $\psi_\nu^{\mathscr{P}}$ is strictly convex along geodesics of

X that pass through the origin $o = K$. The usual argument with the cocycle condition yields strict convexity along any geodesic of X. $\square$

6.7. Assume now that $K = T$ is a compact torus, so $G = T^{\mathbb{C}}$. Set

$$\widehat{F}_\nu : \mathfrak{t} \to \mathfrak{t}^*, \qquad \widehat{F}_\nu(v) := F_\nu(\exp(iv)).$$

Denote by P the momentum polytope i.e. the image of $\mu : M \to \mathfrak{t}^*$. By the Atiyah-Guillemin-Sternberg convexity theorem [3, 20] P is a polytope.

Proposition 6.8. *If $\nu \in \mathscr{W}(M, T, \omega, \mu)$, then $\widehat{F}_\nu$ is a diffeomorphism of $\mathfrak{t}$ onto the interior of the momentum polytope.*

Proof. Let $\pi : \mathfrak{t} \to X := T^{\mathbb{C}}/T$ be the diffeomorphism $\pi(v) := \exp(-iv)T$. Since T is abelian, the geodesics of X are images under the map π of affine lines in $\mathfrak{t}$. Since $\psi_\nu^{\mathscr{P}}$ is strictly convex on X, the function $f := \psi_\nu^{\mathscr{P}} \circ \pi : \mathfrak{t} \to \mathbb{R}$, is strictly convex on $\mathfrak{t}$. Note that

$$f(v) = \int_M \psi_x^M(\exp(-iv)T) d\nu(x).$$

Using the commutativity of T, the cocycle condition (P4$'$) and (2.13) we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \psi_x^M(\exp(-i(w+tv))) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \psi_x^M(\exp(-iw)\exp(-itv)T)$$
$$= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Big(\psi^M_{\exp(iw)x}(\exp(-itv)T) + \psi_x^M(\exp(-iw)T)\Big) = \langle \mu(\exp(iw)\cdot x), v\rangle$$

$$df(w)v = \int_M \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \psi_x^M(\exp(-i(w+tv))T) d\mu(x) =$$
$$= \langle F_\nu(\exp(iw)), v\rangle = \langle \widehat{F}_\nu(w), v\rangle.$$

Hence $df(w) = \widehat{F}_\nu(w)$. Since f is strictly convex, a basic result in convex analysis [19, p. 122] ensures that $df : \mathfrak{t} \to \mathfrak{t}^*, x \mapsto df(x)$ is a diffeomorphism onto an open convex subset U of $\mathfrak{t}^*$. Therefore $\widehat{F}_\nu(\mathfrak{t}) = df(\mathfrak{t}) = U$ is an open convex subset of $\mathfrak{t}^*$ and $\widehat{F}_\nu : \mathfrak{t} \to U$ is a diffeomorphism. By Lemma 5.25 $U \subset P$. Since U is an open subset of $\mathfrak{t}$ we have $U \subset \operatorname{int} P$. We need to show that $U = \operatorname{int} P$. Assume by contradiction that $U \subsetneq \operatorname{int} P$. Since both U and $\operatorname{int} P$ are convex, we get $\overline{U} \subsetneq P$. Fix $x_0 \in P - \overline{U}$ and $x_1 \in U$. Set $x_t := (1-t)x_0 + tx_1$ and $\tau := \inf\{t \in [0,1] : x_t \in \overline{U}\}$. Since $\overline{U}$ is closed $x_\tau \in \overline{U}$ and $\tau \in (0,1)$. Moreover $x_\tau \in \partial U$. Since $x_1 \in U \subset \operatorname{int} P$ and $\tau > 0$, it follows that $x_\tau \in \operatorname{int} P$. So $y := x_\tau \in \partial U \cap \operatorname{int} P$. Any boundary point of a compact convex set lies in some exposed face, i.e. it admits a support hyperplane [41]. So there is $v \in \mathfrak{t}$, $v \neq 0$, such that

$$\langle y, v\rangle = \max_{\overline{U}} \langle \cdot, v\rangle = \sup_U \langle \cdot, v\rangle = \sup_{w \in \mathfrak{t}} \langle \widehat{F}_\nu(w), v\rangle.$$

Since $\nu \in \mathscr{W}(M, T, \omega, \mu)$, Corollary 5.30 yields that

$$\lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M} \mu^v(x).$$

Moreover

$$\lambda_\nu(\mathrm{e}(-v)) = \lim_{t\to+\infty} \int_M \mu^v(\exp(itv)\cdot x) d\nu(x) = \lim_{t\to+\infty} \langle \widehat{F}_\nu(\exp(itv), v\rangle.$$

(See the proof of Theorem 5.23.) Therefore

$$\sup_{w\in\mathfrak{t}} \langle \widehat{F}_\nu(w), v\rangle \geq \lambda_\nu(\mathrm{e}(-v)).$$

Summing up

$$\langle y, v\rangle = \sup_{w\in\mathfrak{t}} \langle \widehat{F}_\nu(w), v\rangle \geq \lambda_\nu(\mathrm{e}(-v)) = \max_{x\in M} \mu^v(x) = \max_P \langle \cdot, v\rangle.$$

This means that the linear function $\langle \cdot, v\rangle$ attains its maximum on P at the point $y \in \operatorname{int} P$. Since P is a convex set, this implies $v = 0$, a contradiction. Therefore we have indeed $\overline{U} = P$ and $U = \operatorname{int} P$. □

Theorem 6.9. *If $\nu \in \mathscr{W}(M, T, \omega, \mu)$ and the action of T on M is almost effective, then $F_\nu : T^{\mathbb{C}} \to \operatorname{int} P$ is a surjective submersion with compact fibres.*

Proof. Since $T^{\mathbb{C}}$ is abelian, the map F_ν is T-invariant: if $k \in T$, then $F_\nu(kg) = \mathrm{Ad}(k)F_\nu(g) = F_\nu(g)$. Let $\varphi : T \times \mathfrak{t} \to T^{\mathbb{C}}$ be the diffeomorfism $\varphi(k, v) := k \cdot \exp(iv)$ and let $\mathrm{pr}_2 : T \times \mathfrak{t} \to \mathfrak{t}$ be the projection on the second factor. Then $F_\nu = \widehat{F}_\nu \circ \mathrm{pr}_2 \circ \varphi^{-1}$. Therefore F_ν is a proper submersion onto $\operatorname{int} P$ and its the fibres are the T–cosets. □

6.10. Consider the following example: $M = S^2$ with $T = S^1$ acting by rotation around the z-axis; ν is the measure concentrated at the South pole. Then the group $T^{\mathbb{C}} = \mathbb{C}^*$ acts as complex dilations of the Riemann sphere leaving the South pole fixed. Thus the map F_ν is constant. In particular it is not a submersion from $\mathbb{C}^*$ to $\mathfrak{t} \cong \mathbb{R}$. Clearly ν does not belong to $\mathscr{W}(S^2, T)$. So in some sense the previous result is sharp. We are going to prove a similar result in the non-abelian case. This will be very general, though not as sharp as the abelian one. Indeed we will need a technical condition that is dealt with in the next lemma.

Lemma 6.11. *Let M be a compact manifold and let K be a compact Lie group acting continuously on M. Let $\nu_0 \in \mathscr{P}(M)$ be K-invariant and let $\nu \in \mathscr{P}(M)$ be absolutely continuous with respect to ν_0. Let k_n be a sequence in K converging to k. Then $k_n \cdot \nu \to k \cdot \nu$ in the norm* (5.2).

Proof. By the Radon-Nikodym theorem there is a non-negative function $\varphi \in L^1(M, \nu_0)$ such that $d\nu = \varphi \cdot d\nu_0$. We claim that

$$||k_n \cdot \nu - k \cdot \nu|| \leq ||\varphi \circ k_n^{-1} - \varphi \circ k||_{L^1(\nu_0)}.$$

If f is a bounded measurable function on M, then

$$\int_M f\, d(k\cdot\nu) = \int_M f(kx)\varphi(x)d\nu_0(x) = \int_M f(y)\varphi(k^{-1}\cdot y)d\nu_0(y),$$

and similarly for k_n. So

$$||k_n\cdot\nu - k\cdot\nu|| =$$
$$= \sup\left\{\int_M fd(k_n\cdot\nu) - \int_M fd(k\cdot\nu) : f\in C(M), \sup_M |f|\le 1\right\} =$$
$$= \sup\left\{\int_M f(\varphi\circ k_n^{-1} - \varphi\circ k^{-1})d\nu_0 : f\in C(M), \sup_M |f|\le 1\right\} \le$$
$$\le ||\varphi\circ k_n^{-1} - \varphi\circ k^{-1}||_{L^1(\nu_0)}.$$

This proves the claim. Given $\varepsilon > 0$ fix a continuous function φ_0 such that

$$||\varphi - \varphi_0||_{L^1(\nu_0)} < \varepsilon.$$

Using the K–invariance of ν_0 we get

$$||\varphi\circ k_n^{-1} - \varphi\circ k^{-1}||_{L^1(\nu_0)} \le$$
$$\le ||\varphi\circ k_n^{-1} - \varphi_0\circ k_n^{-1}||_{L^1(\nu_0)} + ||\varphi_0\circ k_n^{-1} - \varphi_0\circ k^{-1}||_{L^1(\nu_0)} +$$
$$+ ||\varphi_0\circ k^{-1} - \varphi\circ k^{-1}||_{L^1(\nu_0)} =$$
$$= 2||\varphi - \varphi_0||_{L^1(\nu_0)} + ||\varphi_0\circ k_n^{-1} - \varphi_0\circ k^{-1}||_{L^1(\nu_0)} <$$
$$< 2\varepsilon + ||\varphi_0\circ k_n^{-1} - \varphi_0\circ k^{-1}||_{L^1(\nu_0)}.$$

As φ_0 is continuous there is $\delta > 0$ such that $|\varphi_0(x)-\varphi_0(y)| < \varepsilon$ if $d(x,y) < \delta$. The action of K on M being continuous and M being compact imply that $k_n \to k$ uniformly on M (see the proof of Lemma 5.5). Thus there is n_0 such that for any $n \ge n_0$ and for any $x \in M$, $d(k_n\cdot x, k\cdot x) < \delta$. Therefore for $n \ge n_0$

$$||k_n\cdot\nu - k\cdot\nu|| \le ||\varphi\circ k_n^{-1} - \varphi\circ k^{-1}||_{L^1(\nu_0)} < (2+\nu_0(M))\varepsilon = 3\varepsilon.$$

This proves the lemma. □

Proposition 6.12. *Let (M,ω) be a Kähler manifold and let K be a compact group acting isometrically and almost effectively on M with momentum mapping $\mu : M \to \mathfrak{k}^*$. If $\nu_0 \in \mathscr{P}(M)$ is K-invariant and $\nu \in \mathscr{W}(M,K)$ is absolutely continuous with respect to ν_0, then $F_\nu(G) = \Omega(\mu)$ and $F_\nu : G \to \Omega(\mu)$ is a fibration with compact connected fibres.*

Proof. Since $\nu \in \mathscr{W}(M,K)$, we know that after appropriately shifting μ, the measure ν is stable (Theorem 5.31, hence G_ν is compact (Corollary 3.5). By Theorem 6.4 F_ν is of maximal rank. Therefore its image is an open subset of $\mathfrak{k}$. Since it is contained in $E(\mu)$, it is in fact contained in the interior of $E(\mu)$, i.e. in $\Omega(\mu)$. We claim that if F_ν is regarded as a map $F_\nu : G \to \Omega(\mu)$, then it is proper. Indeed let $\{g_n\} \subset G$ be a sequence such that $\{F_\nu(g_n)\}$ converges to a point of $\Omega(\mu)$. We have to show that some subsequence of $\{g_n\}$ converges. Let $T \subset G$ be a maximal torus. Since

$G = KT^{\mathbb{C}}K$, we write $g_n = q_n \exp(iv_n)k_n^{-1}$ with $k_n, q_n \in K$ and $v_n \in \mathfrak{t}$. Passing to subsequences we may assume that $k_n \to k$ and $q_n \to q$. From the fact that $F_\nu(kg) = \mathrm{Ad}(k)F_\nu(g)$, we immediately deduce that the sequence $\{F_\nu(\exp(iv_n)k_n^{-1})\}$ is also convergent in $\Omega(\mu)$. We claim that

$$F_\nu(\exp(iv_n)k_n^{-1}) - F_\nu(\exp(iv_n)k^{-1}) \to 0. \tag{6.13}$$

Fix $w \in \mathfrak{k}$. Then

$$\begin{aligned}
\langle F_\nu(\exp(iv_n)k_n^{-1}), w\rangle &= \int_M \mu^w(\exp(iv_n)k_n^{-1} \cdot x)d\nu(x) = \\
&= \int_M \mu^w(\exp(iv_n)y)d(k_n^{-1}\cdot \nu)(y).\\
\langle F_\nu(\exp(iv_n)k_n^{-1}) - F_\nu(\exp(iv_n)k^{-1}), w\rangle &= \\
&= \int_M \mu^w \circ \exp(iv_n)\, d(k_n^{-1}\cdot\nu) - \int_M \mu^w \circ \exp(iv_n)\, d(k^{-1}\cdot \nu).\\
\Big|\langle F_\nu(\exp(iv_n)k_n^{-1} - F_\nu(\exp(iv_n)k^{-1}, w\rangle\Big| &\leq \\
&\leq ||\mu^w \circ \exp(iv_n)||_{C(M)} \cdot ||k_n^{-1}\cdot\nu - k^{-1}\cdot\nu|| \leq \\
&\leq \sup_M |\mu| \cdot ||k_n^{-1}\cdot\nu - k^{-1}\cdot \nu||.
\end{aligned}$$

Therefore

$$\Big|F_\nu(\exp(iv_n)k_n^{-1} - F_\nu(\exp(iv_n)k^{-1}\Big| \leq \sup_M |\mu| \cdot ||k_n^{-1}\cdot\nu - k^{-1}\cdot\nu||.$$

Since ν_0 is K–invariant and ν is absolutely continuous with respect to ν_0, the previous lemma ensures that $||k_n^{-1}\cdot\nu - k^{-1}\cdot\nu|| \to 0$. Thus (6.13) is proved. Since $\{F_\nu(\exp(iv_n)k_n^{-1})\}$ is convergent in $\Omega(\mu)$, it follows that also $\{F_\nu(\exp(iv_n)k^{-1})\}$ converges to some point of $\Omega(\mu)$. And the same is true for $F_\nu(k\exp(iv_n)k^{-1}) = \mathrm{Ad}(k)F_\nu(\exp(iv_n)k)$. The points $k\exp(iv_n)k^{-1}$ belong to the maximal torus $S := kTk^{-1}$. Let $\pi : \mathfrak{k}^* \to \mathfrak{s}^*$ denote the restriction. Then $\mu_S := \pi\circ\mu : M \to \mathfrak{s}^*$ is a momentum mapping for the action of S on M. Let P denote the momentum polytope. Then $P = \pi(E(\mu))$. Note also that $\pi(\Omega(\mu)) \subset P^0$ since π is obviously an open map. By Theorem 6.9 the map

$$F_\nu^S : S^{\mathbb{C}} \longrightarrow \mathfrak{s}^*, \quad F_\nu^S(a) := \int_M \mu_S(a\cdot x)d\nu(x),$$

is a proper submersion onto $\mathrm{int}\, P$. (Note that $\mathscr{W}(M,K) \subset \mathscr{W}(M,T)$.) But $F_\nu^S = \pi\circ F_\nu|_{S^{\mathbb{C}}}$, that is the following diagram commutes:

$$\begin{array}{ccccc}
S^{\mathbb{C}} & \hookrightarrow & G & \xrightarrow{F_\nu} & \mathfrak{k}^* \\
 & \searrow^{F_\nu^S} & & & \downarrow \pi \\
 & & & & \mathfrak{s}^*.
\end{array}$$

Since $\{k\exp(iv_n)k^{-1}\} \subset S^{\mathbb{C}}$ and $F^S_\nu(k\exp(iv_n)k^{-1})$ converges to some point of P^0, we conclude that the sequences $\{k\exp(iv_n)k^{-1}\}$ and $\{\exp(iv_n)\}$ admit convergent subsequences. This proves that $F_\nu : G \to \Omega(\mu)$ is proper. Hence it is a closed map. As it is also open, it is onto. Moreover it is a locally trivial fibration by Ehresmann theorem. Since the base is contractible, the fibration is trivial, i.e. G is diffeomorphic to a product $\Omega \times F$ where F is the fibre. Since G is connected it follows that F is connected. □

Theorem 6.14. *Let (M, g, ω) be a Kähler manifold and let K be a compact group acting isometrically and almost effectively on M with momentum mapping $\mu : M \to \mathfrak{k}^*$. If $\nu \in \mathscr{AC}(M)$, then $F_\nu(G) = \Omega(\mu)$ and $F_\nu : G \to \Omega(\mu)$ is a fibration with compact connected fibres.*

Proof. Fix a K-invariant Riemannian metric on M and denote by ν_0 the the normalized Riemannian measure. Then ν is absolutely continuous with respect to ν_0 and $\nu \in \mathscr{W}(M, K)$. Thus the theorem follows from the previous proposition. □

7. Applications

7.1. In this section we describe some applications of the results obtained in the previous sections. In 7.2–7.15 we give an explicit characterization of stable, semi-stable and polystable measures on $\mathbb{P}^n$. These have attracted interest in various contexts, e.g. in the study of balanced metrics [12] and in the study of the loop space of S^2 [36].

In the rest of the section we make some remarks on the relation of our results to the problem of upper bounds for λ_1 in the style of Bourguignon-Li-Yau [11].

7.2. We wish to characterize stable, semi-stable and polystable measures on $\mathbb{P}^n$ endowed with the Fubini-Study metric and the standard action of $\mathrm{SL}(n+1,\mathbb{C})$. Let $v \in \mathfrak{su}(n+1)$, $v \neq 0$, let $c_0 < \cdots < c_r$ be the eigenvalues of iv and let $V := \mathbb{C}^{n+1} = V_0 \oplus \cdots \oplus V_r$ be the eigenspace decomposition of v, so that v acts on V_i as the multiplication by $-ic_i$. Set

$$E_i := \bigoplus_{j \leq i} V_j, \quad L_i := \mathbb{P}(E_i), \quad L_{-1} := \emptyset. \tag{7.3}$$

Lemma 7.4. *The critical values of μ^v are $c_0 < \cdots < c_r$. The critical component corresponding to c_i is $C_i = \mathbb{P}(V_i)$ and the unstable manifold of C_i is $W^u_i = L_i - L_{i-1}$.*

Proof. Fix a point $x = [z] \in \mathbb{P}^n(\mathbb{C})$ and decompose z according to the eigenspace decomposition: $z = z_0 + \cdots + z_r$. Then

$$\mu^v(x) = \frac{i\langle v(z), z\rangle}{|z|^2} = \sum_{i=0}^{r} c_i \frac{|z_i|^2}{|z|^2} \tag{7.5}$$

Hence $C_i = (\mu^v)^{-1}(c_i) = \mathbb{P}(V_i)$. Next let i be such that $z_i \neq 0$ and $z_j = 0$ for $j > i$. Then

$$\begin{gathered}\exp(itv)\cdot x = [e^{c_0 t} z_0 + \cdots + e^{c_i t} z_i] = \\ = [e^{(c_0 - c_i)t} z_1 + \cdots + e^{(c_{i-1} - c_i)t} z_{i-1} + z_i], \\ \lim_{t\to+\infty} \exp(itv)\cdot x = [z_i].\end{gathered}$$

It follows that $\{[z] \in \mathbb{P}^n : z = z_0 + \cdots + z_i \text{ with } z_i \neq 0\} = L_i - L_{i-1} \subset W_i^u$. Since both $\{W_i^u\}$ and $\{L_i - L_{i-1}\}$ are partitions of $\mathbb{P}^n$ we deduce $W_i^u = L_i - L_{i-1}$. □

Theorem 7.6. *A measure $\nu \in \mathscr{P}(\mathbb{P}^n)$ is stable (respectively semi-stable) with respect to the* $\mathrm{SL}(n+1,\mathbb{C})$*-action if and only if for any proper linear subspace $L \subset \mathbb{P}^n$*

$$\nu(L) < \frac{\dim L + 1}{n+1} \quad \Big(\textit{respectively } \nu(L) \leq \frac{\dim L + 1}{n+1}\Big). \tag{7.7}$$

Proof. Fix $\nu \in \mathscr{P}(\mathbb{P}^n)$. Assume first that the strict inequality holds in (7.7). Fix $v \in S(\mathfrak{su}(n+1))$ and use the notation fixed above. Set $a_i := \nu(L_i)$, so $0 = a_{-1} \leq a_0 \leq \cdots \leq a_r = 1$ and $\nu(W_i^u) = a_i - a_{i-1}$ by the previous lemma. By Theorem 3.4 to prove that ν is stable it is enough to show that $\lambda_\nu > 0$ on $\partial_\infty X$, where $X = \mathrm{SL}(n+1,\mathbb{C})/\mathrm{SU}(n+1)$. We apply Theorem 5.23 to compute $\lambda_\nu(\mathrm{e}(-v))$:

$$\begin{aligned}\lambda_\nu(\mathrm{e}(-v)) = \sum_{i=0}^r c_i(a_i - a_{i-1}) = \sum_{i=0}^r c_i a_i - \sum_{i=1}^r c_i a_{i-1} = \\ = c_r + \sum_{i=0}^{r-1} a_i (c_i - c_{i+1}).\end{aligned}$$

Set $\varepsilon_i := \dim E_i$. By (7.7) $a_i < \varepsilon_i/(n+1)$. Since $c_i - c_{i+1} < 0$ we get

$$\lambda_\nu(\mathrm{e}(-v)) > c_r + \frac{1}{n+1}\sum_{i=0}^{r-1}(c_i - c_{i+1})\varepsilon_i.$$

$$\begin{aligned}\sum_{i=0}^{r-1}(c_i - c_{i+1})\varepsilon_i = c_0\varepsilon_0 + \sum_{i=1}^{r-1} c_i(\varepsilon_i - \varepsilon_{i-1}) - c_r\varepsilon_{r-1} = \\ = \sum_{i=0}^{r-1} c_i \dim V_i - c_r \varepsilon_{r-1}.\end{aligned}$$

Since $v \in \mathfrak{su}(n+1)$,

$$\sum_{i=0}^r c_i \dim V_i = \operatorname{tr} v = 0.$$

So

$$\sum_{i=0}^{r-1} c_i \dim V_i - c_r\varepsilon_{r-1} = -c_r \dim V_r - c_r\varepsilon_{r-1} = -(n+1)c_r.$$

Summing up

$$\lambda_\nu(\mathrm{e}(-v)) > c_r - \frac{(n+1)c_r}{n+1} = 0.$$

So (7.7) implies that $\lambda_\nu > 0$ on $\partial_\infty X$, hence that ν is stable. The same computation using Theorem 4.17 yields the result for semi-stability. Conversely let us prove that condition (7.6) is necessary for stability. Let $L \subset \mathbb{P}^n$ be a proper linear subspace of dimension d. Let $V_0 \subset \mathbb{C}^{n+1}$ be such that $L = \mathbb{P}(V_0)$ and let V_1 be the orthogonal complement to V_0. Thus $\dim V_0 = d+1$, $\dim V_1 = n-d$. Set $c_0 = (d-n)$ and $c_1 = d+1$. Let v be the operator that acts as multiplication by $-ic_j$ on V_j. Since $d < n$, $v \in \mathfrak{su}(n+1)$. If $\nu \in \mathscr{P}(\mathbb{P}^n)$ is stable, then

$$0 < \lambda_\nu(\mathrm{e}(-v)) = c_0\nu(L)) + c_1(1-\nu(L)) = c_1 - (c_1 - c_0)\nu(L),$$

$$\text{hence} \quad \nu(L) < \frac{c_1}{c_1 - c_0} = \frac{d+1}{n+1}.$$

Thus (7.7) is a necessary condition for stability. Also in this case the argument for semi-stability is identical. □

The next result gives a complete characterization of the measures on $\mathbb{P}^n$ that are polystable with respect to the standard action of $\mathrm{SL}(n+1,\mathbb{C})$.

Theorem 7.8. *Let $\nu \in \mathscr{P}(\mathbb{P}^n)$. Then ν is polystable if and only if there exists a splitting $\mathbb{C}^{n+1} = V_0 \oplus \cdots \oplus V_r$ and measures $\nu_j \in \mathscr{P}(\mathbb{P}(V_j))$ such that ν_j is stable with respect to $\mathrm{SL}(V_j,\mathbb{C}))$ for $j = 0,\ldots,r$ and*

$$\nu = \sum_{j=0}^{r} \frac{\dim V_j}{n+1}\nu_j.$$

We start with two technical lemmata.

Lemma 7.9. *Let $\nu \in \mathscr{P}(\mathbb{P}^n)$ be such that $\mathfrak{F}(\nu) = 0$. Then there exists an orthogonal splitting $\mathbb{C}^{n+1} = V_0 \oplus \cdots \oplus V_r$ such that ν is concentrated on $\mathbb{P}(V_0) \cup \cdots \cup \mathbb{P}(V_r)$ and it is stable with respect to $\mathrm{SL}(V_0,\mathbb{C}) \times \cdots \times \mathrm{SL}(V_r,\mathbb{C})$.*

Proof. Since ν is polystable $\lambda_\nu \geq 0$. If $Z(\nu) = \emptyset$, then ν is stable and the theorem is trivially proved. Assume $Z(\nu) \neq \emptyset$. Let $v \in Z(\nu)$ be such that

$$\dim T_v = \max_{w \in \mathrm{e}^{-1}(Z(\nu))} \dim T_w,$$

where $T_w = \overline{\exp(\mathbb{R}w)}$. By the same argument as in the proof of Lemma 4.11, $\exp(\mathbb{C}v) \subset G_\nu$. If K' is a compact subgroup of K^v such that $T_v \cdot K' = K^v$ and $K' \cap T_v$ is finite, then ν is stable with respect to $G' = (K')^{\mathbb{C}}$. Let $\alpha : M \longrightarrow \mathrm{Crit}(\mu^v)$ be the map defined in (5.19). By Proposition 5.18 it is well-defined. If $f \in C(M)$, then $\lim_{t\to+\infty} f(\exp(itv)\cdot x) = f(\alpha(x))$. It follows that $\exp(itv)\cdot\nu \rightharpoonup \alpha\cdot\nu$ for $t \to +\infty$. Since $\exp(\mathbb{C}v) \subset G_\nu$, we conclude that $\nu = \alpha\cdot\nu$. So ν is concentrated on the image of α, i.e. on the critical sets of μ^v. Using the notation of 7.2 and Lemma 7.4 this means that ν is concentrated on $\mathbb{P}(V_0)\cup\cdots\cup\mathbb{P}(V_r)$. Moreover, $\mathrm{SU}(V_0)\times\cdots\times\mathrm{SU}(V_j) \subset K^v$ and the intersection $(\mathrm{SU}(V_0)\times\cdots\times\mathrm{SU}(V_r))\cap T_v$ is finite due the fact that T_v is contained in the center of K^v. This proves that we may choose K' so that $\mathrm{SU}(V_0)\times\cdots\times\mathrm{SU}(V_r) \subset K'$. Since ν is stable for G' then it is stable for $\mathrm{SL}(V_0,\mathbb{C})\times\cdots\times\mathrm{SL}(V_r,\mathbb{C}) \subset G'$ concluding the proof. □

7.10. Let V be a complex vector space and let h be a Hermitian product on V. The group $\mathrm{SU}(V,h)$ acts on $\mathbb{P}(V)$ with momentum mapping μ given by the formula

$$\mu([v]) = -i\left(P_v - \frac{\mathrm{id}_V}{n+1}\right),$$

where P_v denotes the h-orthogonal projection $V \to \mathbb{C}v$. The construction in Section 5 yields a momentum mapping $\mathfrak{F} : \mathscr{P}(\mathbb{P}(V)) \to \mathfrak{su}(V,h)$. If h' is a second Hermitian product on V, and $h'(z,z') = h(Lz,Lz')$ with $L \in \mathrm{GL}(V)$ h-self-adjoint, then $L\circ P'_v = P_{Lv}\circ L$, so $\mathrm{Ad}\,L\circ\mu' = \mu\circ L$. Similarly $\mathrm{Ad}\,L\circ\mathfrak{F}' = \mathfrak{F}\circ L$, so $\mathfrak{F}^{-1}(0) = L\left((\mathfrak{F}')^{-1}(0)\right)$. It follows immediately that the stability of a measure $\nu \in \mathscr{P}(\mathbb{P}(V))$ does not depend on the choice of h. This problem is central in [43]. It would be interesting to develop the arguments in that paper in the setting of Section 2.

Lemma 7.11. *Let $\nu \in \mathscr{P}(\mathbb{P}^n)$. Let $\mathbb{C}^{n+1} = V_0\oplus\cdots\oplus V_r$ be an orthogonal splitting such that ν is concentrated on $\mathbb{P}^n(V_0)\cup\cdots\cup\mathbb{P}(V_r)$. Then $\mathfrak{F}(\nu) = 0$ if and only if $\nu(\mathbb{P}(V_j)) = \dim V_j/(n+1)$ and $\mathfrak{F}_j(\nu_j) = 0$, where $\mathfrak{F}_j : \mathscr{P}(\mathbb{P}(V_j)) \to \mathfrak{su}(V_j)$ is the momentum mapping with respect to the natural $\mathrm{SU}(V_j,\mathbb{C})$-action on $\mathbb{P}(V_j)$.*

Proof. By assumption $\nu = \sum_{j=0}^r a_j\nu_j$, where ν_j is a probability measure on $\mathbb{P}(V_j)$, $a_j := \nu(\mathbb{P}(V_j)) \geq 0$ and $\sum_{j=0} a_j = 1$. Let μ be the momentum mapping on $\mathbb{P}^n$ and μ_j the momentum mapping on $\mathbb{P}(V_j)$ for the natural $\mathrm{SU}(V_j)$-action. Set for simplicity $V := \mathbb{C}^{n+1}$ and $d_j := \dim V_j$. If $[z] \in \mathbb{P}(V_j)$, then

$$\mu([z]) = \mu_j([z]) + i\left(\frac{\mathrm{id}_V}{n+1} - \frac{\mathrm{id}_{V_j}}{d_j}\right).$$

Hence

$$\mathfrak{F}(\nu) = \sum_{j=0}^{r} a_j \int_{\mathbb{P}(V_j)} \mu([z]) d\nu_j([z]) =$$

$$= \sum_{j=0}^{r} a_j \mathfrak{F}_j(\nu_j) + i \sum_{j=0}^{r} a_j \left(\frac{\mathrm{id}_V}{n+1} - \frac{\mathrm{id}_{V_j}}{d_j} \right) =$$

$$= \sum_{j=0}^{r} a_j \mathfrak{F}_j(\nu_j) + i \left(\frac{\mathrm{id}_V}{n+1} - \sum_{j=0}^{r} \frac{a_j}{d_j} \mathrm{id}_{V_j} \right) =$$

$$= \sum_{j=0}^{r} a_j \mathfrak{F}_j(\nu_j) + i \sum_{j=0}^{r} \left(\frac{1}{n+1} - \frac{a_j}{d_j} \right) \mathrm{id}_{V_j} .$$

Since $\mathfrak{F}_j(\nu_j) \in \mathfrak{su}(V_j)$ the terms in the last sum are all orthogonal to each other. Thus $\mathfrak{F}(\nu) = 0$ if and only if every term vanishes. □

Proof of Theorem 7.8. If ν is stable, then the theorem holds with $r = 0$. Assume that ν is polystable but not stable. Then there exists $g \in \mathrm{SL}(n+1, \mathbb{C})$ such that $\mathfrak{F}(g \cdot \nu) = 0$. Set $\nu' = g \cdot \nu$. By Lemma 7.9 there exists an orthogonal splitting $\mathbb{C}^{n+1} = W_0 \oplus \cdots \oplus W_r$ such that ν' is concentred on $\mathbb{P}(W_0) \cup \cdots \cup \mathbb{P}(W_r)$ and ν' is stable with respect to $G := \mathrm{SL}(W_0, \mathbb{C}) \times \cdots \times \mathrm{SL}(W_r, \mathbb{C})$. Therefore $\nu' = \sum_{j=0}^{r} a_j \nu'_j$ where $\nu'_j \in \mathscr{P}(\mathbb{P}(W_j))$ and $\sum_{j=0}^{r} a_j = 1$. By Lemma 7.11, $a_j = d_j/(n+1)$ and $\mathfrak{F}_j(\nu'_j) = 0$. In particular ν'_j is polystable with respect to $\mathrm{SL}(W_j, \mathbb{C})$ for $j = 0, \ldots, r$. The stabilizer of ν'_j in $\mathrm{SL}(W_j)$ is contained G_ν. Since ν is G-stable, G_ν is compact, so the same holds for the stabilizer in $\mathrm{SL}(W_j)$ of ν'_j. Hence ν'_j is actually stable with respect to $\mathrm{SL}(W_j)$ for any j. Set $V_j := g^{-1}(W_j)$ for $j = 0, \ldots, r$. Then $\mathbb{C}^{n+1} = V_0 \oplus \cdots \oplus V_r$. The measures $\nu_j := g^{-1} \cdot \nu'_j \in \mathscr{P}(\mathbb{P}(V_j))$ are stable with respect to $\mathrm{SL}(V_j)$ since g is an isomorphism and as observed in 7.10 we do not need to care about the Hermitian product. Finally

$$\nu = \sum_{j=0}^{r} \frac{d_j}{n+1} \nu_j. \tag{7.12}$$

We have proved that the condition in the theorem is necessary for polystability. Vice versa assume that there exists a splitting $\mathbb{C}^{n+1} = V_0 \oplus \cdots \oplus V_r$ such that (7.12) holds, where $\nu_j \in \mathscr{P}(\mathbb{P}(V_j))$ is stable. Fix a Hermitian scalar product h on $V = \mathbb{C}^{n+1}$, such that the above splitting is orthogonal. By Lemma 7.11 ν is polystable when we consider on $\mathscr{P}(\mathbb{P}^n)$ the action of $\mathrm{SU}(V, h)$. As noted in 7.10 the choice of the Hermitian product does not matter. So ν is polystable also with respect to the standard action of $\mathrm{SU}(n+1)$. □

7.13. In [12, §2.2] Donaldson considers the ν-balanced metrics on $\mathcal{O}_{\mathbb{P}^n}(1)$ where $\nu \in \mathscr{P}(\mathbb{P}^n)$. He establishes two conditions on ν ensuring that there exists a ν-balanced metric. The existence of a ν-balanced metric is equivalent to the polystability of ν with respect to the action of $\mathrm{SL}(n+1,\mathbb{C})$ on $\mathbb{P}^n$. We now show how to recover Donaldson's conditions. The two conditions are the following:

(1) for any non-trivial linear function λ on $\mathbb{C}^{n+1}$ the function $\log(\frac{|\lambda(z)|}{|z|})$ is ν-integrable.
(2) $\nu = \sum_{i=1}^r \lambda_i \delta_{y_i}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^r \lambda_i = 1$ and such that
$$\frac{\nu(L)}{\dim L + 1} < \frac{1}{n+1},$$

for any proper linear subspace $L \subset \mathbb{P}^n$. We prove that if $\nu \in \mathscr{P}(\mathbb{P}^n)$ satisfies one of the above conditions, then it is stable. If the first condition holds, then $\nu(H) = 0$ for any hyperplane. Hence it satisfies the assumption of Theorem 7.6 and so it is stable. The second condition is also clearly covered by Theorem 7.6.

7.14. Millson and Zombro [36] studied measures on the sphere $S^2 \subset \mathbb{R}^3$. If $i : S^2 \hookrightarrow \mathbb{R}^3$ denotes the inclusion, they studied the following map

$$\mathscr{P}(S^2) \longrightarrow \mathbb{R}^3 \qquad \nu \mapsto \int_{S^2} i(x) d\nu(x),$$

that assigns to a measure its center of mass. Since the inclusion i is the momentum mapping μ for the $\mathrm{SO}(3)$-action on S^2, this is exactly the map (5.15). Their main result asserts that if ν has no atom of mass greater than or equal to $1/2$, then there exists $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $\mathfrak{F}(g\nu) = 0$. This follows directly from Theorem 7.6. In fact Theorem 7.6 says that this condition is equivalent to stability. The proof in [36] is based on the notion of conformal center of mass. This technique is rather complicated and seems to be limited to S^2. We also point out that Corollary 5.32 (relying on Corollary 3.10) generalizes Proposition 3.3 of [36]. Furthermore Millson and Zombro defined a measure $\nu \in \mathscr{P}(S^2)$ to be *semi-stable* if there is no atom of mass greater than $1/2$, and *nice semi-stable* if it has two atoms each of mass $1/2$. By Theorem 7.6 our notion of semi-stable coincides with theirs and by Theorem 7.8 a measure is nice semi-stable if and only if it is polystable but not stable.

7.15. Let $M \subset \mathbb{P}^n$ be a projective manifold endowed by the Fubini study metric. Let $K = \{g \in \mathrm{SU}(n+1) : g(M) = M\}$ and let $G = K^{\mathbb{C}}$. The K-action on M is Hamiltonian and a momentum mapping is given by the restriction to M of the momentum mapping of $\mathbb{P}^n$. Let $\nu \in \mathscr{P}(M)$ be such that for any linear subspace $L \subset \mathbb{P}^n$,

$$\nu(M \cap L) < \frac{\dim L + 1}{n+1} \left(\text{respectively } \nu(M \cap L) \leq \frac{\dim L + 1}{n+1} \right). \tag{7.16}$$

If $\xi \in \mathfrak{k}$, then the unstable manifolds are given by $W_j^\xi = (M \cap L_j) - (M \cap L_{j-1})$. Hence the computation of $\lambda_\nu(\mathrm{e}(-v))$ works as for the projective space $\mathbb{P}^n$, showing that ν is stable, respectively semi-stable.

7.17. We wish to recall briefly the original motivation for the map F_ν defined in (6.2). Let (M, g) be a compact Riemannian manifold. Given functions $f_1, \ldots, f_r \in C^\infty(M)$ such that $\int_M f_j \operatorname{vol}_g = 0$, the Rayleigh theorem yields the upper bound

$$\lambda_1(M, g) \leq \frac{\sum_{j=1}^r \int_N |\nabla f_j|_g^2 \operatorname{vol}_g}{\sum_{j=1}^r \int_N f_j^2 \operatorname{vol}_g}. \tag{7.18}$$

Assume that M admits a special metric $\hat{g}$ and that $\hat{f}_j, \ldots, \hat{f}_r$ are eigenfunctions of λ_1 for this metric. Assume moreover that there is a large group G acting on M. If g is another Riemannian metric on M, one might look for functions f_j of the form $f_j = a^* \hat{f}_j$ for some $a \in G$. If there is some $a \in G$ that makes the integral of all these functions vanish, one gets the bound (7.18). If moreover one is able to compute the right hand side of the bound, then one gets an interesting estimate.

In the paper [29] of Hersch $M = S^2$, $\hat{g}$ is the round metric and $G = \mathrm{PSL}(2, \mathbb{C})$ acting by Möbius transformations. The functions $\hat{f}_j$ are the three coordinate functions x, y, z. Hersch was able to show that if g is an arbitrary Riemannian metric on S^2 (normalized to have volume 4π), then there is $a \in G$ such that $\int_M a^* x \operatorname{vol}_g = \int_M a^* y \operatorname{vol}_g = \int_M a^* z \operatorname{vol}_g = 0$. Moreover he was able to show that the right hand side in (7.18) is equal to 2. Thus he showed that $\lambda_1(S^2, g) \leq 2$.

Bourguignon, Li and Yau [11] realized that this method applies also to estimate $\lambda_1(\mathbb{P}^n(\mathbb{C}), g)$ if g is a *Kähler* metric. In [8] we recast the method of Hersch-Bouguignon-Li-Yau in terms of momentum mapping and applied it when M is an arbitrary compact Hermitian symmetric space, $\hat{g}$ is the symmetric metric, $G = \mathrm{Aut}(M)$ and the functions $\hat{f}_j$ are the components of the momentum mapping $\mu : M \to \mathfrak{k}^*$ for $K := \mathrm{Isom}(M, \hat{g})$. (Related papers include [2], [7], [34] and [40]).

In [29], [11] and [8] the estimation of λ_1 proceeds in two steps: the first one is to find $a \in G$ such that $\int_M a^* f_j \operatorname{vol}_g = 0$. The second step is to actually compute the right hand side in (7.18). The map (6.2) is the tool to deal with the first step. Set $\nu := \operatorname{vol}_g / \operatorname{Vol}(M, g)$. Since $\hat{f}_j$ are the components of μ, $\int_M a^* \hat{f}_j \operatorname{vol}_g = 0$ for all j if and only if $\int_M \mu(ax) d\nu(x) = 0$. Thus the first step amounts to proving that the measure ν is G–polystable! Theorem 5.31 represents a very general solution to the first step. The estimate we get is the following one.

Theorem 7.19. *Let (M, g, ω) be a Kähler manifold and let K be a compact group acting isometrically and almost effectively on M with momentum mapping $\mu : M \to \mathfrak{k}^*$. Fix an* Ad*-invariant scalar product on $\mathfrak{k}$ and let $e_1, \ldots, e_r$ be an orthonormal basis of $\mathfrak{k}$. Set $\mu_j := \langle \mu, e_j \rangle$. If g is any Kähler* *metric on M then there is $a \in G$ such that*

$$\lambda_1(M,g) \leq \frac{\sum_{j=1}^r \int_M |\nabla(a^* f_j)|^2 \operatorname{vol}_g}{\int_M a^*(|\mu|^2) \operatorname{vol}_g} \tag{7.20}$$

7.21. The second step is more mysterious. At the moment we are not able to compute the right hand side in (7.20) in any reasonable geometric situation, except the known ones, i.e. Hermitian symmetric spaces. We believe that this computation can be carried out in much greater generality and that it would yield very interesting estimates. We leave this problem for future investigations. It is important to notice that in the case of symmetric spaces the Kähler-Einstein metric (i.e. the symmetric metric) maximizes λ_1 among Kähler metrics in $c_1(M)$. Now thanks to the work of Apostolov, Jakobson and Kokarev there are examples of Fano manifolds where this does not happen, see [1, Cor. 3.4].

7.22. We want to explain the relation between the results in [8] and those in the present paper. Let M be a flag manifold. This means that K acts transitively on the complex manifold M. By Bott-Borel-Weyl theorem, any flag manifold is the unique complex K-orbit in $\mathbb{P}(V)$ for some irreducible representation $\tau : G \longrightarrow \mathrm{GL}(V)$. In [8] we defined τ-admissible measure as those $\nu \in \mathscr{P}(M)$ such that $\nu(H \cap M) = 0$ for any hyperplane $H \subset \mathbb{P}(V)$. It is immediate that a τ–admissible measure satisfies condition (7.16), so it is stable. In [8, Thm. 3] we proved that for a τ-admissible ν the map $F_\nu : G \longrightarrow \Omega(\mu)$ is surjective. By (7.15) the complement of the open unstable manifold of any $v \in \mathfrak{k}$ is contained in a hyperplane section of M. So if ν is τ-admissible, it is concentrated on the open unstable manifold. Therefore $\nu \in \mathscr{W}(M, K, \omega, \mu)$. This means that the assumptions in Theorem 6.9 (i.e. when restricting to a torus) are weaker than those of [8, Thm. 3]. Moreover the conclusion is stronger since we prove that F_ν is a surjective submersion. In the nonabelian case, Theorem 6.14 treats a slightly smaller class of measures. The proofs are completely different. We expect that a more general surjectivity result holds in the nonabelian case, but we leave it for future investigations.

7.23. In [8] we used the map F_{ν_0}, where ν_0 is the K-invariant measure on a flag manifold, to get a diffeomorphism between X and $\Omega(\mu)$. The following proposition shows that this holds in greater generality.

Proposition 7.24. *Let (M, ω) be a Kähler manifold and K a compact connected Lie group. Assume that K acts almost effectively on M with momentum mapping μ. If $\nu_0 \in \mathscr{AC}(M)$ is K–invariant, then F_{ν_0} descends to a map*

$$\tilde{F}_{\nu_0} : X = G/K \to \mathfrak{k}$$

that is a diffeomorphism of the symmetric space X onto $\Omega(\mu)$.

Proof. Since $k \cdot \nu_0 = \nu_0$,

$$F_{\nu_0}(ak) = \int_M \mu(ak \cdot x) d\nu_0(x) = \int_M \mu(a \cdot y) d\nu_0(y) = F_{\nu_0}(a).$$

Hence F_{ν_0} descends to a map on G/K. By Theorem 6.14 it is a local diffeomorphism and a proper map. Hence a covering map. Since $\Omega(\mu)$ is contractible, it follows that F_{ν_0} is a diffemorphism onto $\Omega(\mu)$. □

7.25. In [8, Thm. 2, p. 239] we proved that if $M = K/K_o$ is the flag manifold given by the complex K-orbit in $\mathbb{P}(V)$ for an irreducible representation $\tau : G \to \mathrm{GL}(V)$, then the map F_{ν_0} extends to the Satake compactification $\overline{X}^S_\tau$ of $X = G/K$ yielding a homeomorphism between $\overline{X}^S_\tau$ and the convex hull of the momentum image of M, which is a coadjoint orbit. Such a homeomorphism exists for any symmetric space of non-compact type by a theorem of Korányi [33]. If M is an arbitrary compact Kähler manifold with a Hamiltonian action of K, by Proposition 7.24 the convex hull $E(\mu)$ is a K-equivariant compactification of G/K where $G = K^{\mathbb{C}}$. In [9] we completely described the faces of $E(\mu)$. More precisely we proved that the faces of $E(\mu)$ are exposed and correspond to maxima of components of the momentum mapping. We used this fact to realize a close connection between the faces of $E(\mu)$ and parabolic subgroups of G. Hence, as for the Satake compatifications, boundary components of G/K are related to parabolic subgroups of G. We think that it would be interesting to further analyze these connections.

References

[1] V. Apostolov, D. Jakobson, and G. Kokarev. An extremal eigenvalue problem in Kähler geometry. *J. Geom. Phys.*, 91:108–116, 2015.

[2] C. Arezzo, A. Ghigi, and A. Loi. Stable bundles and the first eigenvalue of the Laplacian. *J. Geom. Anal.*, 17(3):375–386, 2007.

[3] M. F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[4] M. Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[5] H. Azad and J.-J. Loeb. Plurisubharmonic functions and the Kempf-Ness theorem. *Bull. London Math. Soc.*, 25(2):162–168, 1993.

[6] M. Berger. *Geometry. I.* Universitext. Springer-Verlag, Berlin, 1987. Translated from the French by M. Cole and S. Levy.

[7] L. Biliotti and A. Ghigi. Homogeneous bundles and the first eigenvalue of symmetric spaces. *Ann. Inst. Fourier (Grenoble)*, 58(7):2315–2331, 2008.

[8] L. Biliotti and A. Ghigi. Satake-Furstenberg compactifications, the moment map and λ_1. *Amer. J. Math.*, 135(1):237–274, 2013.

[9] L. Biliotti, A. Ghigi, and P. Heinzner. Invariant convex sets in polar representations. *Preprint, arXiv:1411.6041*, 2014.

[10] A. Borel and L. Ji. *Compactifications of symmetric and locally symmetric spaces.* Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 2006.

[11] J.-P. Bourguignon, P. Li, and S.-T. Yau. Upper bound for the first eigenvalue of algebraic submanifolds. *Comment. Math. Helv.*, 69(2):199–207, 1994.

[12] S. K. Donaldson. Some numerical results in complex differential geometry. *Pure Appl. Math. Q.*, 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):571–618, 2009.
[13] J. Dugundji. *Topology.* Allyn and Bacon Inc., Boston, Mass., 1966.
[14] J. J. Duistermaat and J. A. C. Kolk. *Lie groups.* Universitext. Springer-Verlag, Berlin, 2000.
[15] N. Dunford and J. T. Schwartz. *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.
[16] G. B. Folland. *Real analysis.* Pure and Applied Mathematics. John Wiley & Sons, New York, second edition, 1999.
[17] W. Gangbo, H. K. Kim, and T. Pacini. Differential forms on Wasserstein space and infinite-dimensional Hamiltonian systems. *Mem. Amer. Math. Soc.*, 211(993):vi+77, 2011.
[18] V. Georgulas, J. W. Robbin, and D. A. Salamon. The moment-weight inequality and the Hilbert-Mumford criterion. *Preprint. arXiv:1311.0410*, 2013.
[19] V. Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1994.
[20] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.
[21] P. Heinzner. Geometric invariant theory on Stein spaces. *Math. Ann.*, 289(4):631–662, 1991.
[22] P. Heinzner and A. Huckleberry. Kählerian potentials and convexity properties of the moment map. *Invent. Math.*, 126(1):65–84, 1996.
[23] P. Heinzner and A. Huckleberry. Analytic Hilbert quotients. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 309–349. Cambridge Univ. Press, Cambridge, 1999.
[24] P. Heinzner, A. T. Huckleberry, and F. Loose. Kählerian extensions of the symplectic reduction. *J. Reine Angew. Math.*, 455:123–140, 1994.
[25] P. Heinzner and F. Loose. Reduction of complex Hamiltonian G-spaces. *Geom. Funct. Anal.*, 4(3):288–297, 1994.
[26] P. Heinzner and G. W. Schwarz. Cartan decomposition of the moment map. *Math. Ann.*, 337(1):197–232, 2007.
[27] P. Heinzner, G. W. Schwarz, and H. Stötzel. Stratifications with respect to actions of real reductive groups. *Compos. Math.*, 144(1):163–185, 2008.
[28] P. Heinzner and H. Stötzel. Critical points of the square of the momentum map. In *Global aspects of complex geometry*, pages 211–226. Springer, Berlin, 2006.
[29] J. Hersch. Quatre propriétés isopérimétriques de membranes sphériques homogènes. *C. R. Acad. Sci. Paris Sér. A-B*, 270:A1645–A1648, 1970.
[30] M. W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.
[31] M. Kapovich, B. Leeb, and J. Millson. Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity. *J. Differential Geom.*, 81(2):297–354, 2009.
[32] G. Kempf and L. Ness. The length of vectors in representation spaces. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 233–243. Springer, Berlin, 1979.
[33] A. Korányi. Remarks on the Satake compactifications. *Pure Appl. Math. Q.*, 1(4, part 3):851–866, 2005.
[34] E. Legendre and R. Sena-Dias. Toric aspects of the first eigenvalues. *arXiv:1505.01678*, 2015. Preprint.
[35] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[36] J. J. Millson and B. Zombro. A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space. *Invent. Math.*, 123(1):35–59, 1996.
[37] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, third edition, 1994.
[38] I. Mundet i Riera. A Hitchin-Kobayashi correspondence for Kähler fibrations. *J. Reine Angew. Math.*, 528:41–80, 2000.
[39] I. Mundet i Riera. A Hilbert-Mumford criterion for polystability in Kaehler geometry. *Trans. Amer. Math. Soc.*, 362(10):5169–5187, 2010.
[40] F. Panelli and F. Podestà. On the first eigenvalue of invariant Kähler metrics. *Math. Z.*, 281(1-2):471–482, 2015.
[41] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.
[42] R. Sjamaar. Holomorphic slices, symplectic reduction and multiplicities of representations. *Ann. of Math. (2)*, 141(1):87–129, 1995.
[43] A. Teleman. Symplectic stability, analytic stability in non-algebraic complex geometry. *Internat. J. Math.*, 15(2):183–209, 2004.
[44] R. P. Thomas. Notes on GIT and symplectic reduction for bundles and varieties. In *Surveys in differential geometry. Vol. X*, volume 10 of *Surv. Differ. Geom.*, pages 221–273. Int. Press, Somerville, MA, 2006.
[45] G. Tian. *Canonical metrics in Kähler geometry.* Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld.

Università degli Studi di Parma
E-mail address: leonardo.biliotti@unipr.it

Università degli Studi di Pavia
E-mail address: alessandro.ghigi@unipv.it