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Identification of the initial population of a nonlinear predator-prey system backwards in time

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Abstract

We study for the first time the ill-posed backward problem for a contaminated nonlinear predator-prey system whose velocities of migration depend on the total average populations in the considered space domain. We propose a new regularized problem for which we are able to prove its unique solvability in Theorem 1. Moreover, under some mild assumptions on the true solution, we give useful and rigorous error estimates and convergence rates in both the L^2- and H^1-norms in Theorems 2 and 3, respectively. Furthermore, numerical simulations are performed to illustrate the accuracy and stability of the regularized solution.

1 Introduction

Studying the way predator-prey species interact and their mechanisms for survival represent a very important topic in the subject of population biology and ecology. In [20], the authors considered a degenerate predator-prey nonlinear system with homogeneous Dirichlet boundary conditions to describe the local interactions of prey and predator species where there is a direct movement of predators caused by a variation in prey. As far as numerical simulations are concerned, in [3], the authors studied numerical methods for obtaining spatio-temporal patterns described by a predator-prey model with time delay and diffusion.

More recently, the change in environment caused by pollution has affected the long term survival of species, human life style and biodiversity of habitat, and studying the effects of toxicant on populations attracted much attention. In the 80's, Hallam *et al.* [15, 16, 17] assessed the effects of a pollutant on an ecological system. In those works, the authors studied a single-species population by assuming that its growth rate density decreases linearly with the concentration of toxicant but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. Later on, Freedman and Shukla [14] studied a single species and on a predator-prey system by taking into account the introduction of toxicant from an external source, whilst Shukla and Dubey [21] studied the simultaneous effect of two toxicants, one being more toxic than the other, on a biological species. Dubey and Hussain [11] proposed a Lotka-Volterra diffusion model to study the interaction(namely, cooperation, competition

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and predator-prey) of two biological species in a polluted environment.

In this paper, although included, we shall not study the effect of toxicants from a contaminated/polluted environment but instead address a novel formulation and investigation of the backward contaminated problem for such a nonlinear predator-prey system with locally Lipschitz coupling interactions. Therefore, consider a two species predator-prey model with toxicant effect in a connected bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary $\partial\Omega$ over a time span $(0, T)$, where $T > 0$, with nonlocal diffusion terms. The forward model was introduced by [2], as follows:

$$\begin{cases} u_t - \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)\Delta u + \mathrm{div}(uK_1) = F(u, v, C_1), & (x,t) \in Q_T, \\ v_t - \mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right)\Delta v + \mathrm{div}(vK_2) = G(u, v, C_2), & (x,t) \in Q_T, \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t C_1 = k_1 C_3 - \chi_1 C_1 - m_1 C_1, & (x,t) \in Q_T, \\ \partial_t C_2 = k_2 C_3 - \chi_2 C_2 - m_2 C_2, & (x,t) \in Q_T, \\ \partial_t C_3 = -hC_3, \quad (x,t) \in Q_T, \end{cases} \tag{2}$$

where $Q_T = \Omega \times (0, T)$, with the initial value data

$$\begin{cases} u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \\ C_1(x,0) = C_{1,0}(x), \quad C_2(x,0) = C_{2,0}(x), \quad C_3(x,0) = C_{3,0}(x), \quad x \in \Omega, \end{cases} \tag{3}$$

and the boundary conditions

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega \times (0,T). \tag{4}$$

The homogeneous Dirichlet boundary conditions (4) express that the physical system is self-contained and that no populations at the boundary $\partial\Omega$ exist for any time $t \in (0, T)$, [20]. Other types of boundary conditions can also be considered.

In (1), u and v represent the density of the prey and the predator populations, respectively, depending on space $x \in \Omega$ and time $t \in (0, T)$, $\mathcal{D}_i$, $i = 1, 2$, represent the nonlocal diffusivities (velocities) of migration and the nonlinear sources

$$F(u, v, C_1) = ru\left(1 - \frac{u}{K}\right) - \beta_1 C_1 u - \frac{\mathrm{p}u}{1 + \mathrm{q}u}v, \quad G(u, v, C_2) = -av - \beta_2 C_2 v + e\frac{\mathrm{p}u}{1 + \mathrm{q}u}v, \tag{5}$$

where $a > 0$ is the rate of decay of the predator population, e is the conversion rate from prey to predator, $r > 0$ is the rate of logistical growth of the prey population, $K > 0$ is the carrying capacity, p is the reciprocal of the time spent by predator to catch prey and q/p is the manipulation time. The non-negative constants β_1 and β_2 represent the rates of the intrinsic growth rate associated with the uptake and toxicant, respectively. The functions $C_1(x, t)$, $C_2(x, t)$ and $C_3(x, t)$ represent the concentrations of toxicant/pollutant present in the prey species, predator species and environment, respectively. In (2), the terms $k_i C_3$, $-\chi_i C_i$ and $-m_i C_i$, $i = 1, 2$, represent the absorbing rate of the toxicant from the environment, excretion and depuration rates of the toxicant in the species, respectively, and $-hC_3$ is the loss rate of the toxicant due to self-volatilization. In the third equation in (2), it was assumed that the capacity of environment is large enough for the toxicant/pollutant in the environment, caused by excretion and uptake by both predator-prey polutations, to be neglected [24]. The convection terms $\mathrm{div}(uK_1)$ and $\mathrm{div}(vK_2)$ describe the change of the concentration at a given location due to the flows of velocities K_i, $i = 1, 2$.

A simpler version of (1), when $K_1 = K_2 = 0$ and $F(u,v,C_1)$, $G(u,v,C_2)$ are independent of C_1, C_2, is given by

$$\begin{cases} u_t - \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)\Delta u = F(u,v), & (x,t) \in Q_T, \\ v_t - \mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right)\Delta v = G(u,v), & (x,t) \in Q_T. \end{cases} \tag{6}$$

For existence, uniqueness and long time behavior of some classes of nonlocal nonlinear parabolic equations and systems, we refer to [1, 6, 7, 8, 13, 25].

One practical issue with the above direct model arises when the initial condition (3) at $t = 0$ is not available and the population dynamics is already evolving at $t > 0$. In such a situation, instead of (3), we prescribe/measure the population densities and toxicant concentrations at the later final time,

$$u(x,T) = g_1(x), \quad v(x,T) = g_2(x), \quad x \in \Omega, \tag{7}$$

$$C_1(x,T) = C_{1,T}(x), \quad C_2(x,T) = C_{2,T}(x), \quad C_3(x,T) = C_{3,T}(x), \quad x \in \Omega. \tag{8}$$

Notice that the backward problem (1), (2), (4)-(8) is not well-posed in the sense that the solution for u and v does not depend continuously on data (7), i.e. from the small noise made in measurement data (7), the corresponding solution may generate itself large and undesired errors, and standard computational procedures are not stable. One more important thing to remark is that for final data (8) in $L^2(\Omega)$ the problem given by the linear system (2) of ODEs with "frozen" x subject to (8) has a unique solution C_1, C_2, $C_3 \in C([0,T]; L^2(\Omega))$ which depends continuously on the input data (8). For example, the third equation in (2) can be solved independently of the other two to result in the unique solution $C_3(x,t) = e^{-h(t-T)}C_{3,T}(x)$. Therefore, in the system (1) we only need to consider u and v as unknowns and recast (1) as

$$\begin{cases} u_t - \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)\Delta u + \mathrm{div}(uK_1) = F(x,t,u,v), & (x,t) \in Q_T, \\ v_t - \mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right)\Delta v + \mathrm{div}(vK_2) = G(x,t,u,v), & (x,t) \in Q_T, \end{cases} \tag{9}$$

where the dependence on $C_i(x,t)$ have been embedded in the right-hand side terms of (9) by rewriting $F(u,v,C_1(x,t)) = F(x,t,u,v)$ and $G(u,v,C_2(x,t)) = G(x,t,u,v)$. Furthermore, where no confusion arises, we shall write $F(u,v)$ and $G(u,v)$ instead of $F(x,t,u,v)$ and $G(x,t,u,v)$, respectively.

Although there are many works on the backward heat parabolic equation, results on backward parabolic systems are rather scarce. For the backward problem given by the system (6) subject to (4) and (7), in principle, we can express the solution (u,v) by Fourier series with given data (g_1,g_2) and apply some spectral methods such as the Fourier truncation method for solving this problem. This method has recently been adopted in [23], where the backward problem for the system (6) with $\mathcal{D}_1$ and $\mathcal{D}_2$ positive constants and homogeneous Neumann boundary conditions has been solved. However, the use of spectral methods is not feasible when the convection terms $\mathrm{div}(uK_1)$ and $\mathrm{div}(vK_2)$ are present in (9). So, our problem is much more challenging. To the best of our knowledge, our problem given by (4), (7) and (9) has never been formulated or investigated before, though it is worth mentioning that in [22], the authors considered the backward in time nonlocal nonlinear problem for the population density u of a single migration species whose velocity of migration $\mathcal{D}$ depends on the total average population $\int_\Omega u dx$, given by [7, 9],

$$\begin{cases} u_t - \mathcal{D}\left(\int_\Omega u dx\right)\Delta u = F(u), & (x,t) \in Q_T, \\ \dfrac{\partial u}{\partial \eta}(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{10}$$

where η denotes the outward unit normal to the boundary $\partial\Omega$, and a regularization method was applied to construct a stable solution. The main idea of the method is stabilizing the ill-posed problem by using a small regularization parameter in the governing partial differential equation (PDE), and we follow this idea in equations (18) and (19) of subsection 3.2 below.

The outline of this paper is as follows. Section 2 gives some notations and assumptions used throughout the paper. Section 3 focuses on the construction of approximate problem by a regularization approach. We give an approximation for the locally Lipschitz reaction functions (5) and prove the existence of the unique regularized solution by using the Faedo-Galerkin method and Aubin-Lions lemma. In section 4, we give error estimates in both the L^2- and H^1-norms. In section 5, a numerical example is illustrated to corroborate our theoretical results. Finally, the conclusions are presented in Section 6.

2 Preliminaries

For a Banach space X, we denote by $L^p(0,T;X), L^\infty(0,T;X)$ and $C([0,T];X)$ the usual Banach spaces with the norms

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(\cdot,t)\|_X^p \mathrm{d}t\right)^{1/p} < \infty, \quad 1 \le p < \infty, \quad \|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_X < \infty,$$
$$\|u\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_X < \infty.$$

Let $\{\lambda_p\}_{p=1}^\infty$ be the eigenvalues of the Laplacian operator $-\Delta$ on the connected bounded domain Ω with homogeneous Dirichlet boundary condition (4), which satisfy $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_p \le ...$, with $\lambda_p \to \infty$ when $p \to \infty$. Let $\{\phi_p\}_{p=1}^\infty \subset H_0^1(\Omega)$ be the corresponding eigenfunctions, which form an orthonormal basis of $L^2(\Omega)$. Throughout this paper, we denote the inner product in $L^2(\Omega)$ by $\langle \cdot,\cdot \rangle$, and $\langle u(\cdot,t), \phi_p \rangle$ by $u_p(t)$. We also denote by $\|\cdot\|$ the norm $\|\cdot\|_{L^2(\Omega)}$. For $\sigma \ge 0$, we introduce the Gevrey space

$$G_\sigma(\Omega) = \left\{ U \in L^2(\Omega) : \|U\|_{G_\sigma(\Omega)} = \sqrt{\sum_{p=1}^\infty e^{\sigma\lambda_p} U_p^2} < +\infty \right\}, \text{where } U_p := \langle U, \phi_p \rangle. \tag{11}$$

Given $u(x,t)$ and $v(x,t) : \Omega \times [0,T] \to \mathbb{R}$, we denote $(u,v) : \Omega \times [0,T] \to \mathbb{R}^2$ defined by $(u,v)(x,t) := (u(x,t), v(x,t))$. Here, the norm of $(u,v) \in \mathbb{X} \times \mathbb{X}$ (for any Banach space $\mathbb{X}$) is defined as $\|(u,v)\|_{(\mathbb{X})^2} := \|u\|_{\mathbb{X}} + \|v\|_{\mathbb{X}}$. For $\mathbb{X} = L^2(\Omega)$, we also denote by $\|\cdot\|$ the norm $\|\cdot\|_{(L^2(\Omega))^2}$.

Next, we introduce the following assumptions:

(A_1) There exist positive constants m and M such that

$$m \le \mathcal{D}_i(\xi) \le M, \quad \forall\, \xi \in \mathbb{R};\ i = 1,2.$$

(A_2) There exists positive constants L_i, $i = 1,2$, such that for any ξ_1, $\xi_2 \in \mathbb{R}$, we have

$$|\mathcal{D}_i(\xi_1) - \mathcal{D}_i(\xi_2)| \le \mathrm{L}_i |\xi_1 - \xi_2|.$$

At this point, we remark that for fixed $t \in [0,T]$, $i = 1,2$ and for any $u_1(\cdot,t)$, $u_2(\cdot,t) \in L^2(\Omega)$, we have

$$\left|\mathcal{D}_i\left(\int_\Omega u_1(x,t)\mathrm{d}x\right) - \mathcal{D}_i\left(\int_\Omega u_2(x,t)\mathrm{d}x\right)\right| \le \mathrm{L}_i \left|\int_\Omega (u_1(x,t) - u_2(x,t))\mathrm{d}x\right| \le \mathrm{L}_i |\Omega| \left\| (u_1 - u_2)(\cdot,t)\right\|,$$

$i = 1,2$. Denote $\mathrm{L} := \max\{\mathrm{L}_1, \mathrm{L}_2\} |\Omega|$.

(A$_3$) Let $g_i^\varepsilon \in L^2(\Omega), i = 1, 2$, be noisy data which satisfy $\|g_1^\varepsilon - g_1\| + \|g_2^\varepsilon - g_2\| \le \varepsilon$, where $\varepsilon \ge 0$ represents the noise level.

(A$_4$) For sufficiently small ε, $g_i^\varepsilon(x) \ge 0$ a.e. $x \in \Omega, i = 1, 2$.

(A$_5$) $K_1, K_2 : \Omega \to \mathbb{R}^d$ belong to $L^\infty(\Omega)$.

Definition 1. A pair $(u, v) \in \left[L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))\right]^2$ is called a weak solution to the problem (4), (7) and (9) if (u, v) satisfies (7) and

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle u(\cdot,t),\varphi\Big\rangle + \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)\Big\langle \nabla u(\cdot,t),\nabla\varphi\Big\rangle - \Big\langle u(\cdot,t)K_1,\nabla\varphi\Big\rangle = \Big\langle F(\cdot,t,u,v),\varphi\Big\rangle, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\langle v(\cdot,t),\psi\Big\rangle + \mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right)\Big\langle \nabla v(\cdot,t),\nabla\psi\Big\rangle - \Big\langle v(\cdot,t)K_2,\nabla\psi\Big\rangle = \Big\langle G(\cdot,t,u,v),\psi\Big\rangle, \tag{13}$$

for all φ, $\psi \in H_0^1(\Omega)$ and for all $t \in [0, T]$.

Remark 1. To illustrate the ill-posedness of the backward problem given by equations (4), (7) and (9), we consider a simpler version without the convection terms, $K_1 = K_2 = 0$, and with globally Lipschitz reactions (F, G) satisfying $F(0, 0) = G(0, 0) = 0$ and $\max\{\|F(u, v), \|G(u, v)\|\} \le \mathcal{K}(\|u\| + \|v\|)$ for some $\mathcal{K} \ge 0$. First, we consider the forward problem given by equations (1)-(4). Let $u_0 = v_0 = 0$ and $u_0^n = v_0^n = n\phi_n$, where ϕ_n is the $n-$th eigenfunction of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$. According to Theorem 1.2 ([2]), we have that the forward system (1)-(4) has unique weak solutions $(u_{\text{ex}}, v_{\text{ex}}) : u_{\text{ex}} = v_{\text{ex}} = 0$ and (u^n, v^n) with the initial data (u_0, v_0) and (u_0^n, v_0^n), respectively. In (12), by choosing $u = u^n$, $v = v^n$ and $\varphi = \phi_p$, multiplying both sides by $e^{\lambda_p \int_0^t \mathcal{D}_1(\int_\Omega u^n dx)ds}$, integrating the resulting equations from 0 to t, and summing up from $p = 1$ to ∞, we obtain

$$u^n(x,t) = \underbrace{n\phi_n(x)\exp\left(-\lambda_n\int_0^t \mathcal{D}_1\left(\int_\Omega u^n dx\right)ds\right)}_{H_1} + \underbrace{\sum_{p=1}^\infty \phi_p(x)\int_0^t \exp\left(-\lambda_p\int_s^t \mathcal{D}_1\left(\int_\Omega u^n dx\right)d\omega\right)\langle F(u^n,v^n),\phi_p\rangle ds}_{H_2}$$

With the assumption (A_1), we have that

$$\|H_1\|^2 = n^2\exp\left(-2\lambda_n\int_0^t \mathcal{D}_1\left(\int_\Omega u^n dx\right)ds\right) \le n^2e^{-2mt\lambda_n}.$$

Using Parseval's relation and Hölder's inequality, we obtain:

$$\|H_2\|^2 \le \sum_{p=1}^\infty T\int_0^t \langle F(u^n,v^n),\phi_p\rangle^2 ds = T\int_0^t \|F(u^n,v^n)\|^2 ds \le T\mathcal{K}^2\int_0^t (\|u^n\| + \|v^n\|)^2 ds.$$

With the above estimates, in the same manner for v^n, we have:

$$\|u^n(\cdot,t)\|^2 + \|v^n(\cdot,t)\|^2 \le 4n^2e^{-2mt\lambda_n} + 8T\mathcal{K}^2\int_0^t \left(\|u^n\|^2 + \|v^n\|^2\right)ds.$$

Using Grönwall's inequality we obtain

$$\|u^n(\cdot,t)\|^2 + \|v^n(\cdot,t)\|^2 \le 4n^2e^{-2mt\lambda_n}e^{8T\mathcal{K}^2t},$$

and choosing $t = T$, we have

$$\|u_T^n\|^2 + \|v_T^n\|^2 \le 4n^2 e^{-2mT\lambda_n} e^{8T^2\mathcal{K}^2} \to 0,$$

as $n \to \infty$. Now, returning to the backward problem, let $u_T = 0$ and $v_T = 0$ be the exact final data, which are perturbed by u_T^n and v_T^n, satisfying $u_T^n \to u_T$ and $v_T^n \to v_T$, as n tends to ∞. Suppose that the backward problem is well-posed, then with the above exact and noisy final data, there exist unique weak solutions with exact initial values $u_0 = 0$, $v_0 = 0$ and perturbed initial values u_0^n, v_0^n. But since $\|u_0^n - u_0\| = \|v_0^n - v_0\| = n \to \infty$, this shows the instability of the backward problem given by equations (4), (7) and (9).

3 The regularized problem

3.1 Setting the regularized problem

In this subsection, we present a regularized problem using a new quasi-reversibility type method. First, for technical reasons, we need to extend the source functions F and G in (5) so that they become defined for all u, $v \in \mathbb{R}$, by setting

$$\overline{F}(u,v,C_1) = \begin{cases} ru(1-\frac{u}{K}) - \beta_1 C_1 u - \frac{pu}{1+qu}v, & \text{if } u \ge 0, v \ge 0, \\ ru(1-\frac{u}{K}) - \beta_1 C_1 u, & \text{if } u \ge 0, v < 0, \\ 0, & \text{if } u < 0, \end{cases} \tag{14}$$

$$\overline{G}(u,v,C_2) = \begin{cases} -av - \beta_2 C_2 v + e\frac{pu}{1+qu}v, & \text{if } u \ge 0, v \ge 0, \\ -av - \beta_2 C_2 v, & \text{if } u < 0, v \ge 0, \\ 0, & \text{if } v < 0. \end{cases} \tag{15}$$

It is easy to show that $\overline{F}$ and $\overline{G}$ are locally Lipschitz functions with respect to u, v, i.e. for any $R > 0$, there exist Lipschitz non-negative constants $K_F(R)$, $K_G(R)$ such that

$$\left|\overline{F}\big(u_1, v_1, C_1(x,t)\big) - \overline{F}\big(u_2, v_2, C_1(x,t)\big)\right| \le K_F(R)\Big(|u_1 - u_2| + |v_1 - v_2|\Big),$$

$$\left|\overline{G}\big(u_1, v_1, C_1(x,t)\big) - \overline{G}\big(u_2, v_2, C_1(x,t)\big)\right| \le K_G(R)\Big(|u_1 - u_2| + |v_1 - v_2|\Big),$$

for all $\left\{(u_i, v_i) \in \mathbb{R}^2 \;\middle|\; |u_i| + |v_i| \le R\right\}, i = 1, 2$, and $(x,t) \in \overline{Q}_T = \overline{\Omega} \times [0,T]$. Notice that these Lipschitz constants tend to ∞ when $R \to \infty$, we cannot give the error estimate when the data (7) are noised, and standard regularization techniques are thus not applicable. To overcome these issues, we employ two sequences of globally Lipschitz functions F_ε and G_ε (with $\varepsilon > 0$) to approximate F and G, as follows:

$$F_\varepsilon(u,v) = \begin{cases} \overline{F}(u,v,C_1), & \text{if } |u| + |v| \le R^\varepsilon, \\ \overline{F}\left(\dfrac{R^\varepsilon u}{|u|+|v|}, \dfrac{R^\varepsilon v}{|u|+|v|}, C_1\right), & \text{if } |u| + |v| > R^\varepsilon, \end{cases} \tag{16}$$

$$G_\varepsilon(u,v) = \begin{cases} \overline{G}(u,v,C_2), & \text{if } |u| + |v| \le R^\varepsilon, \\ \overline{G}\left(\dfrac{R^\varepsilon u}{|u|+|v|}, \dfrac{R^\varepsilon v}{|u|+|v|}, C_2\right), & \text{if } |u| + |v| > R^\varepsilon. \end{cases} \tag{17}$$

For the sake of brevity, we write $F_\varepsilon(u,v)$, $G_\varepsilon(u,v)$ instead of $F_\varepsilon(u,v,C_1)$, $G_\varepsilon(u,v,C_2)$ or $F_\varepsilon(x,t,u,v)$, $G_\varepsilon(x,t,u,v)$. Here R^ε in (16) and (17) satisfies that $R^\varepsilon(\varepsilon) \to \infty$ when $\varepsilon \to 0$, and will be chosen later to obtain the convergence of the regularized solution to the true one.

The next lemma shows the globally Lipschitz property of F_ε and G_ε.

Lemma 1. The functions F_ε and G_ε given in (16) and (17), respectively, are globally Lipschitz functions with respect to u, v, i.e. there exist non-negative constants $K_F(R^\varepsilon)$ and $K_G(R^\varepsilon)$ such that for all $(x,t) \in \overline{Q}_T$ and $(u_i, v_i) \in \mathbb{R}^2, i = 1, 2$, we have

$$\begin{aligned}
\left|F_\varepsilon(u_1,v_1) - F_\varepsilon(u_2,v_2)\right| &\leq 2K_F(R^\varepsilon)\left(\left|u_1-u_2\right| + \left|v_1-v_2\right|\right),\\
\left|G_\varepsilon(u_1,v_1) - G_\varepsilon(u_2,v_2)\right| &\leq 2K_G(R^\varepsilon)(\left|u_1-u_2\right| + \left|v_1-v_2\right|).
\end{aligned}$$

Moreover, F_ε and G_ε satisfy

$$\begin{aligned}
\left\|F_\varepsilon(u_1,v_1)(\cdot,t) - F_\varepsilon(u_2,v_2)(\cdot,t)\right\| &\leq \sqrt{8}K_F(R^\varepsilon)\left\|((u_1,v_1)-(u_2,v_2))(\cdot,t)\right\|,\\
\left\|G_\varepsilon(u_1,v_1)(\cdot,t) - G_\varepsilon(u_2,v_2)(\cdot,t)\right\| &\leq \sqrt{8}K_G(R^\varepsilon)\left\|((u_1,v_1)-(u_2,v_2))(\cdot,t)\right\|,
\end{aligned}$$

for any fixed $t \in [0,T]$ and for any $u_i(\cdot,t)$, $v_i(\cdot,t) \in L^2(\Omega)$, $i = 1, 2$.

Proof. Because of the similarity between F_ε and G_ε, we need to consider only F_ε.
Case 1. $|u_1| + |v_1| \leq R^\varepsilon$, $|u_2| + |v_2| \leq R^\varepsilon$. Using (16) gives

$$\left|F_\varepsilon(u_1,v_1) - F_\varepsilon(u_2,v_2)\right| = \left|\overline{F}(u_1,v_1) - \overline{F}(u_2,v_2)\right| \leq K_F(R^\varepsilon)\left(|u_1-u_2| + |v_1-v_2|\right).$$

Case 2. $|u_1| + |v_1| \leq R^\varepsilon$, $|u_2| + |v_2| > R^\varepsilon$ (the same proof is used for the case $|u_1| + |v_1| > R^\varepsilon$, $|u_2| + |v_2| \leq R^\varepsilon$). Using (16), we obtain

$$\begin{aligned}
\left|F_\varepsilon(u_1,v_1) - F_\varepsilon(u_2,v_2)\right| &= \left|\overline{F}(u_1,v_1) - \overline{F}\left(\frac{R^\varepsilon u_2}{|u_2|+|v_2|}, \frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right)\right|\\
&\leq K_F(R^\varepsilon)\left(\left|u_1 - \frac{R^\varepsilon u_2}{|u_2|+|v_2|}\right| + \left|v_1 - \frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right|\right)\\
&\leq K_F(R^\varepsilon)\left[|u_1-u_2| + |v_1-v_2| + \left(|u_2|+|v_2|\right)\left(1 - \frac{R^\varepsilon}{|u_2|+|v_2|}\right)\right]\\
&\leq K_F(R^\varepsilon)\left(|u_1-u_2| + |v_1-v_2| + |u_2| + |v_2| - |u_1| - |v_1|\right)\\
&\leq 2K_F(R^\varepsilon)\left(|u_1-u_2| + |v_1-v_2|\right).
\end{aligned}$$

Case 3. $|u_1| + |v_1| > R^\varepsilon$, $|u_2| + |v_2| > R^\varepsilon$. Using (16), we have

$$\begin{aligned}
&\left|F_\varepsilon(u_1,v_1) - F_\varepsilon(u_2,v_2)\right|\\
&\quad= \left|\overline{F}\left(\frac{R^\varepsilon u_1}{|u_1|+|v_1|}, \frac{R^\varepsilon v_1}{|u_1|+|v_1|}\right) - \overline{F}\left(\frac{R^\varepsilon u_2}{|u_2|+|v_2|}, \frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right)\right|\\
&\quad\leq K_F(R^\varepsilon)\left(\left|\frac{R^\varepsilon u_1}{|u_1|+|v_1|} - \frac{R^\varepsilon u_2}{|u_2|+|v_2|}\right| + \left|\frac{R^\varepsilon v_1}{|u_1|+|v_1|} - \frac{R^\varepsilon v_2}{|u_2|+|v_2|}\right|\right)\\
&\quad\leq K_F(R^\varepsilon)\left[\frac{R^\varepsilon(|u_1-u_2| + |v_1-v_2|)}{|u_1|+|v_1|} + \frac{R^\varepsilon(|u_2|+|v_2|)}{|u_1|+|v_1|} - R^\varepsilon\right]\\
&\quad\leq 2K_F(R^\varepsilon)\left(|u_1-u_2| + |v_1-v_2|\right).
\end{aligned}$$

Next, we will prove the second statement of the lemma. From the previous inequality, we have

$$\begin{aligned}
\int_\Omega \left|F_\varepsilon(u_1,v_1) - F_\varepsilon(u_2,v_2)\right|^2 \mathrm{d}x &\leq 4K_F^2(R^\varepsilon)\int_\Omega \left(\left|u_1-u_2\right| + \left|v_1-v_2\right|\right)^2 \mathrm{d}x\\
&\leq 8K_F^2(R^\varepsilon)\left(\left\|u_1-u_2\right\|^2 + \left\|v_1-v_2\right\|^2\right) \leq 8K_F^2(R^\varepsilon)\left\|(u_1,v_1)-(u_2,v_2)\right\|^2.
\end{aligned}$$

The proof of Lemma 1 is completed. □

Throughout this paper, denote

$$K_{R^\varepsilon} := \max\left\{\sqrt{8}K_F(R^\varepsilon), \sqrt{8}K_G(R^\varepsilon)\right\}.$$

3.2 The well-posedness of the regularized problem

Next, we introduce a well-posed approximate problem to the ill-posed backward problem (4), (7) and (9) given by the following *perturbed regularized problem:*

$$\begin{cases} u^\varepsilon_t - \mathcal{D}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)(t)\Delta u^\varepsilon + \operatorname{div}(u^\varepsilon K_1) - Q^\alpha(u^\varepsilon) = F_\varepsilon(u^\varepsilon, v^\varepsilon), \quad (x,t) \in Q_T, \\ v^\varepsilon_t - \mathcal{D}_2\left(\int_\Omega v^\varepsilon \mathrm{d}x\right)(t)\Delta v^\varepsilon + \operatorname{div}(v^\varepsilon K_2) - Q^\alpha(v^\varepsilon) = G_\varepsilon(u^\varepsilon, v^\varepsilon), \quad (x,t) \in Q_T, \\ u^\varepsilon(x,T) = g^\varepsilon_1(x), \quad v^\varepsilon(x,T) = g^\varepsilon_2(x), \quad x \in \Omega, \\ u^\varepsilon(x,t) = v^\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{18}$$

Here $\alpha = \alpha(\varepsilon) > 0$ is the regularization parameter which satisfying $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$, and will be chosen later. The operator $Q^\alpha : L^2(\Omega) \to H^1_0(\Omega)$ is defined by

$$Q^\alpha(w) = \frac{1}{T}\sum_{p=1}^{\infty} \ln(1 + \alpha e^{TM_0\lambda_p})\langle w, \phi_p\rangle \phi_p(x), \quad \forall w \in L^2(\Omega), \tag{19}$$

where M_0 is a positive constant such that $M_0 > M$ with M given in (A_1). This operator was also introduced in [22]. We prove the existence of a weak solution to the problem (18) in the following theorem. The main tools used here are the Faedo-Galerkin method and the Aubin-Lions lemma [19].

Theorem 1. *Suppose that the assumptions* $(A_1) - (A_5)$ *hold. Then, the problem* (18) *has a unique solution*

$$(u^\varepsilon, v^\varepsilon) \in \left[C([0,T]; L^2_+(\Omega)) \cap L^2(0,T; H^1_0(\Omega))\right]^2$$

in the weak sense, i.e.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle u^\varepsilon(\cdot,t), \varphi\Big\rangle + \mathcal{D}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\Big\langle \nabla u^\varepsilon(\cdot,t), \nabla\varphi\Big\rangle - \Big\langle u^\varepsilon(\cdot,t)K_1, \nabla\varphi\Big\rangle \\ = \Big\langle Q^\alpha(u^\varepsilon)(\cdot,t), \varphi\Big\rangle + \Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon), \varphi\Big\rangle, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle v^\varepsilon(\cdot,t), \psi\Big\rangle + \mathcal{D}_2\left(\int_\Omega v^\varepsilon \mathrm{d}x\right)\Big\langle \nabla v^\varepsilon(\cdot,t), \nabla\psi\Big\rangle - \Big\langle v^\varepsilon(\cdot,t)K_2, \nabla\varphi\Big\rangle \\ = \Big\langle Q^\alpha(v^\varepsilon)(\cdot,t), \psi\Big\rangle + \Big\langle G_\varepsilon(u^\varepsilon, v^\varepsilon), \psi\Big\rangle, \end{aligned} \tag{21}$$

for all φ, $\psi \in H^1_0(\Omega)$ *and for all* $t \in [0,T]$.

Proof. Define the following operators:

$$P^\alpha w = Q^\alpha w + M_0\Delta w,$$

$$\mathcal{B}_1\left(\int_\Omega u\mathrm{d}x\right) = M_0 - \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right), \quad \mathcal{B}_2\left(\int_\Omega v\mathrm{d}x\right) = M_0 - \mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right).$$

Notice that, from (A_1) and (A_2), we have:

$$0 < M_0 - M \le \mathcal{B}_i\left(\int_\Omega u\mathrm{d}x\right) \le M_0 - m,$$

$$\left|\mathcal{B}_1\left(\int_\Omega u_1\mathrm{d}x\right) - \mathcal{B}_1\left(\int_\Omega u_2\mathrm{d}x\right)\right| = \left|\mathcal{D}_1\left(\int_\Omega u_1\mathrm{d}x\right) - \mathcal{D}_1\left(\int_\Omega u_2\mathrm{d}x\right)\right| \le \mathrm{L}\big\|(u_1 - u_2)(\cdot,t)\big\|,$$

and

$$\left|\mathcal{B}_2\left(\int_\Omega u_1 \mathrm{d}x\right) - \mathcal{B}_2\left(\int_\Omega u_2 \mathrm{d}x\right)\right| = \left|\mathcal{D}_2\left(\int_\Omega u_1 \mathrm{d}x\right) - \mathcal{D}_2\left(\int_\Omega u_2 \mathrm{d}x\right)\right| \leq \mathrm{L}\big\|(u_1 - u_2)(\cdot,t)\big\|,$$

for any fixed $t \in [0,T]$ and $u(\cdot,t)$, $u_1(\cdot,t), u_2(\cdot,t) \in L^2(\Omega)$. The system (20) and (21) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle u^\varepsilon(\cdot,t),\varphi\Big\rangle - &\mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\Big\langle \nabla u^\varepsilon(\cdot,t), \nabla\varphi\Big\rangle - \Big\langle u^\varepsilon(\cdot,t)K_1, \nabla\varphi\Big\rangle \\ &= \Big\langle P^\alpha(u^\varepsilon)(\cdot,t),\varphi\Big\rangle + \Big\langle F_\varepsilon(u^\varepsilon,v^\varepsilon),\varphi\Big\rangle, \qquad (22)\\ \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle v^\varepsilon(\cdot,t),\psi\Big\rangle - &\mathcal{B}_2\left(\int_\Omega v^\varepsilon \mathrm{d}x\right)\Big\langle \nabla v^\varepsilon(\cdot,t), \nabla\psi\Big\rangle - \Big\langle v^\varepsilon(\cdot,t)K_2, \nabla\varphi\Big\rangle \\ &= \Big\langle P^\alpha(v^\varepsilon)(\cdot,t),\psi\Big\rangle + \Big\langle G_\varepsilon(u^\varepsilon,v^\varepsilon),\psi\Big\rangle. \qquad (23)\end{aligned}$$

We first state and prove the following technical lemma.

Lemma 2. (i) For any $w \in G_\sigma(\Omega)$ with $\sigma \geq 2M_0T$, we have

$$\big\|Q^\alpha(w)\big\| \leq \frac{\alpha}{T}\|w\|_{G_\sigma(\Omega)}.$$

(ii) For any $w \in L^2(\Omega)$ and $0 < \alpha$ sufficiently small, we have

$$\big\|P^\alpha(w)\big\| \leq \frac{1}{T}\ln\left(\frac{1}{\alpha}\right)\|w\|.$$

Proof. Using Parseval's equality and the inequality $\ln(1+a) \leq a, \forall a > 0$, we get

$$\big\|Q^\alpha(w)\big\|^2 = \frac{1}{T^2}\sum_{p=1}^\infty \ln^2\left(1+\alpha e^{TM_0\lambda_p}\right)\langle w,\phi_p\rangle^2 \leq \frac{\alpha^2}{T^2}\sum_{p=1}^\infty e^{2TM_0\lambda_p}w_p^2 \leq \frac{\alpha^2}{T^2}\|w\|^2_{G_\sigma(\Omega)}.$$

For the second statement, using Parseval's equality, it follows

$$\begin{aligned}\big\|P^\alpha(w)\big\|^2 &= \frac{1}{T^2}\sum_{p=1}^\infty \left(\ln\left(1+\alpha e^{TM_0\lambda_p}\right) - \ln\left(e^{TM_0\lambda_p}\right)\right)^2\langle w,\phi_p\rangle^2 \\ &= \frac{1}{T^2}\sum_{p=1}^\infty \ln^2\left(\alpha + e^{-TM_0\lambda_p}\right)\langle w,\phi_p\rangle^2 \leq \frac{1}{T^2}\ln^2\left(\frac{1}{\alpha}\right)\|w\|^2.\end{aligned}$$

□

We now divide the remaining proof of Theorem 1 into four parts.

Part 1. *Existence of the Galerkin approximate solution*

Let us consider the correspondingly $n-$dimensional space $\mathbb{U}_n = \mathrm{span}\langle\phi_1,...,\phi_n\rangle \subset H_0^1(\Omega)$.

For each $n \in \mathbb{N}^*$, we find the approximate solution $(u^{\varepsilon,n}, v^{\varepsilon,n})$ of the problem (22) and (23) in the following form:

$$u^{\varepsilon,n}(x,t) = \sum_{p=1}^n w_{np}^\varepsilon(t)\phi_p(x), \quad v^{\varepsilon,n}(x,t) = \sum_{p=1}^n \omega_{np}^\varepsilon(t)\phi_p(x), \qquad (24)$$

where $(u^{\varepsilon,n}, v^{\varepsilon,n})$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle u^{\varepsilon,n}(\cdot,t),\varphi\Big\rangle - \mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\Big\langle \nabla u^{\varepsilon,n}(\cdot,t),\nabla\varphi\Big\rangle - \Big\langle u^{\varepsilon,n}(\cdot,t)K_1,\nabla\varphi\Big\rangle$$
$$= \Big\langle P^\alpha(u^{\varepsilon,n})(\cdot,t),\varphi\Big\rangle + \Big\langle F_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),\varphi\Big\rangle, \quad (25)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle v^{\varepsilon,n}(\cdot,t),\psi\Big\rangle - \mathcal{B}_2\left(\int_\Omega v^{\varepsilon,n}\mathrm{d}x\right)\Big\langle \nabla v^{\varepsilon,n}(\cdot,t),\nabla\psi\Big\rangle - \Big\langle v^{\varepsilon,n}(\cdot,t)K_2,\nabla\varphi\Big\rangle$$
$$= \Big\langle P^\alpha(v^{\varepsilon,n})(\cdot,t),\psi\Big\rangle + \Big\langle G_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),\psi\Big\rangle, \quad (26)$$

for all φ, $\psi \in \mathbb{U}_n$, and the final conditions

$$u^{\varepsilon,n}(x,T) = \sum_{p=1}^{n}\langle g_1^\varepsilon,\phi_p\rangle\phi_p(x) =: g_1^{\varepsilon,n}(x), \quad v^{\varepsilon,n}(x,T) = \sum_{p=1}^{n}\langle g_2^\varepsilon,\phi_p\rangle\phi_p(x) =: g_2^{\varepsilon,n}(x). \quad (27)$$

Here $g_1^{\varepsilon,n} \to g_1^\varepsilon$, $g_2^{\varepsilon,n} \to g_2^\varepsilon$ strongly in $L^2(\Omega)$. Introducing (24) into (25) and(26), we obtain that the coefficients $w_{np}^\varepsilon(t)$ and $\omega_{np}^\varepsilon(t)$ are the solution of the system of $2n$ nonlinear ODEs:

$$\frac{\mathrm{d}w_{np}^\varepsilon}{\mathrm{d}t} - \lambda_p\mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)w_{np}^\varepsilon(t) - \Big\langle u^{\varepsilon,n}(\cdot,t)K_1,\nabla\phi_p\Big\rangle = \Big\langle P^\alpha(u^{\varepsilon,n})(\cdot,t),\phi_p\Big\rangle + \Big\langle F_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),\phi_p\Big\rangle,$$
$$\frac{\mathrm{d}\omega_{np}^\varepsilon}{\mathrm{d}t} - \lambda_p\mathcal{B}_2\left(\int_\Omega v^{\varepsilon,n}\mathrm{d}x\right)\omega_{np}^\varepsilon(t) - \Big\langle v^{\varepsilon,n}(\cdot,t)K_2,\nabla\phi_p\Big\rangle = \Big\langle P^\alpha(v^{\varepsilon,n})(\cdot,t),\phi_p\Big\rangle + \Big\langle G_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),\phi_p\Big\rangle,$$

for $p = \overline{1,n}$. Due to the continuity of $\mathcal{D}_1$, $\mathcal{D}_2$ and of F, G, by Peano's theorem, the system (25)-(27) has a local solution $(u^{\varepsilon,n}, v^{\varepsilon,n})$ in some interval $[T_m, T]$ for $0 \le T_m < T$. We now give an a priori estimate for $(u^{\varepsilon,n}, v^{\varepsilon,n})$, which is needed to extend $[T_m, T]$ to the whole interval $[0, T]$. In (25), for fixed t, taking $\varphi = u^{\varepsilon,n}(\cdot,t)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|u^{\varepsilon,n}(\cdot,t)\big\|^2 - \mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\big\|\nabla u^{\varepsilon,n}(\cdot,t)\big\|^2$$
$$= \Big\langle u^{\varepsilon,n}(\cdot,t)K_1,\nabla u^{\varepsilon,n}(\cdot,t)\Big\rangle + \Big\langle P^\alpha(u^{\varepsilon,n})(\cdot,t),u^{\varepsilon,n}(\cdot,t)\Big\rangle + \Big\langle F_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),u^{\varepsilon,n}(\cdot,t)\Big\rangle.$$

Integrating both sides of the above equation from t to T and using (27), we obtain

$$\|u^{\varepsilon,n}(\cdot,t)\|^2 + 2\int_t^T \mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\|\nabla u^{\varepsilon,n}(\cdot,s)\|^2\mathrm{d}s$$
$$= \|g_1^{\varepsilon,n}\|^2 - 2\underbrace{\int_t^T \Big\langle u^{\varepsilon,n}(\cdot,s)K_1,\nabla u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s}_{J_1} - 2\underbrace{\int_t^T \Big\langle P^\alpha(u^{\varepsilon,n})(\cdot,s),u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s}_{J_2}$$
$$- 2\underbrace{\int_t^T \Big\langle F_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n}),u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s}_{J_3}.$$

Consequently,

$$\|u^{\varepsilon,n}(\cdot,t)\|^2 + 2(M_0 - M)\int_t^T \|\nabla u^{\varepsilon,n}(\cdot,s)\|^2\mathrm{d}s \le \|g_1^{\varepsilon,n}\|^2 + 2|J_1| + 2|J_2| + 2|J_3|. \quad (28)$$

We first estimate $|J_1|$ by using Hölder's inequality, Cauchy's inequality, and (A_5) to yield

$$2|J_1| \le \frac{1}{M_0 - M}\int_t^T \|u^{\varepsilon,n}(\cdot,s)K_1\|^2 \mathrm{d}s + (M_0 - M)\int_t^T \|\nabla u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s$$
$$\le \frac{K_0^2}{M_0 - M}\int_t^T \|u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s + (M_0 - M)\int_t^T \|\nabla u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s, \tag{29}$$

where $K_0 := \max_{i=1,2} \|K_i\|_{[L^\infty(\Omega)]^d}$. The term $|J_2|$ is estimated by applying Hölder's inequality and Lemma 2 as follows:

$$|J_2| \le \int_t^T \|P^\alpha(u^{\varepsilon,n})(\cdot,s)\| \|u^{\varepsilon,n}(\cdot,s)\| \mathrm{d}s \le \frac{1}{T}\ln\left(\frac{1}{\alpha}\right)\int_t^T \|u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s. \tag{30}$$

For $|J_3|$, using Hölder's inequality and the globally Lipschitz property of F_ε from Lemma 1, it yields

$$|J_3| \le \int_t^T \|F_\varepsilon(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\| \|u^{\varepsilon,n}(\cdot,s)\| \mathrm{d}s$$
$$\le \int_t^T \left(\|F_\varepsilon(0,0)\| + K_{R^\varepsilon}\|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\|\right) \|u^{\varepsilon,n}(\cdot,s)\| \mathrm{d}s$$
$$= K_{R^\varepsilon}\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\| \|u^{\varepsilon,n}(\cdot,s)\| \mathrm{d}s. \tag{31}$$

Combining (28)-(31), it follows

$$\|u^{\varepsilon,n}(\cdot,t)\|^2 + (M_0 - M)\int_t^T \|\nabla u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s$$
$$\le \|g_1^{\varepsilon,n}\|^2 + \left(\frac{2}{T}\ln\left(\frac{1}{\alpha}\right) + \frac{K_0^2}{M_0 - M}\right)\int_t^T \|u^{\varepsilon,n}(\cdot,s)\|^2 \mathrm{d}s + 2K_{R^\varepsilon}\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\| \|u^{\varepsilon,n}(\cdot,s)\| \mathrm{d}s. \tag{32}$$

By a similar argument with v, adding the resulting inequality to (32), we have

$$\|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,t)\|^2 + (M_0 - M)\int_t^T \|(\nabla u^{\varepsilon,n},\nabla v^{\varepsilon,n})(\cdot,s)\|^2 \mathrm{d}s$$
$$\le 2\|g_1^\varepsilon\|^2 + 2\|g_2^\varepsilon\|^2 + \left(\frac{4}{T}\ln\left(\frac{1}{\alpha}\right) + \frac{2K_0}{M_0 - M} + 4K_{R^\varepsilon}\right)\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\|^2 \mathrm{d}s$$
$$= C_1 + C_2\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\|^2 \mathrm{d}s. \tag{33}$$

Therefore,

$$\|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,t)\|^2 \le C_1 + C_2\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\|^2 \mathrm{d}s.$$

Applying Gronwall's inequality, we arrive at

$$\|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,t)\|^2 \le C_1 e^{(T-t)C_2} \le C_1 e^{TC_2}. \tag{34}$$

On the other hand, from (33), we have

$$\int_t^T \|(\nabla u^{\varepsilon,n},\nabla v^{\varepsilon,n})(\cdot,s)\|^2 \mathrm{d}s \le \frac{1}{M_0 - M}\left(C_1 + C_2\int_t^T \|(u^{\varepsilon,n},v^{\varepsilon,n})(\cdot,s)\|^2\right) \mathrm{d}s \le C_3. \tag{35}$$

From (34) and (35), we deduce that

$$(u^{\varepsilon,n}, v^{\varepsilon,n}) \text{ is bounded in } \left[L^\infty(0,T;L^2(\Omega))\right]^2, \tag{36}$$

$$(u^{\varepsilon,n}, v^{\varepsilon,n}) \text{ is bounded in } \left[L^2(0,T;H_0^1(\Omega))\right]^2. \tag{37}$$

Thus, from the theory of ODEs, we can extend the solution to the whole interval $[0,T]$.

Part 2. *Convergence of the Galerkin approximate solutions to the regularized solution*
From (25) we have that

$$u_t^{\varepsilon,n} = -\mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\Delta u^{\varepsilon,n} - \mathrm{div}(u^{\varepsilon,n}K_1) + P^\alpha(u^{\varepsilon,n}) + F_\varepsilon(u^{\varepsilon,n}, v^{\varepsilon,n}),$$

where we have used Green's formulae

$$\Big\langle -\mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\Delta u^{\varepsilon,n}(\cdot,t),\varphi\Big\rangle = \mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\Big\langle \nabla u^{\varepsilon,n}(\cdot,t),\nabla\varphi\Big\rangle,$$

$$\Big\langle -\mathrm{div}(u^{\varepsilon,n}(\cdot,t)K_1),\varphi\Big\rangle = \Big\langle u^{\varepsilon,n}(\cdot,t)K_1,\nabla\varphi\Big\rangle,$$

for all $\varphi \in H_0^1(\Omega)$. Thanks to (36), (37), Lemma 2, the globally Lipschitz property of F_ε, G_ε, and $\mathcal{B}_i$, $(i = 1,2)$, assumption (A_5) and the similarity between u, v, we obtain that

$$(u_t^{\varepsilon,n}, v_t^{\varepsilon,n}) \text{ is bounded in } \left[L^2(0,T;H^{-1}(\Omega))\right]^2. \tag{38}$$

From (36)-(38), by Banach-Alaoglu theorem, we can extract subsequences $u_k^{\varepsilon,n} = u^{\varepsilon,n}$ and $v_k^{\varepsilon,n} = v^{\varepsilon,n}$ (which we denote by the same symbols) such that

$$u^{\varepsilon,n} \rightharpoonup u^\varepsilon,\quad v^{\varepsilon,n} \rightharpoonup v^\varepsilon \text{ *-weakly in } L^\infty(0,T;L^2(\Omega)), \tag{39}$$

$$u^{\varepsilon,n} \rightharpoonup u^\varepsilon,\quad v^{\varepsilon,n} \rightharpoonup v^\varepsilon \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{40}$$

$$u_t^{\varepsilon,n} \rightharpoonup u_t^\varepsilon,\quad v_t^{\varepsilon,n} \rightharpoonup v_t^\varepsilon \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{41}$$

On the other hand, $H_0^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. From (40) and (41), using the Aubin-Lions compactness lemma, we have

$$u^{\varepsilon,n} \to u^\varepsilon,\quad v^{\varepsilon,n} \to v^\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{42}$$

Hence, by Riesz-Fischer theorem, we can extract subsequences $u_k^{\varepsilon,n} = u^{\varepsilon,n}$ and $v_k^{\varepsilon,n} = v^{\varepsilon,n}$ (which we denote by the same symbols) such that

$$u^{\varepsilon,n} \to u^\varepsilon,\quad v^{\varepsilon,n} \to v^\varepsilon \text{ a.e. in } Q_T. \tag{43}$$

Due to the continuity of $\mathcal{B}_i, i = 1,2$, we have

$$\mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right) \to \mathcal{B}_1\left(\int_\Omega u^\varepsilon\mathrm{d}x\right) \text{ strongly in } L^2(0,T),$$

$$\mathcal{B}_2\left(\int_\Omega v^{\varepsilon,n}\mathrm{d}x\right) \to \mathcal{B}_2\left(\int_\Omega v^\varepsilon\mathrm{d}x\right) \text{ strongly in } L^2(0,T).$$

Using Riesz-Fischer theorem, we have, up to some subsequences,

$$\mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right) \to \mathcal{B}_1\left(\int_\Omega u^\varepsilon\mathrm{d}x\right) \text{ a.e. in } Q_T, \tag{44}$$

$$\mathcal{B}_2\left(\int_\Omega v^{\varepsilon,n}\mathrm{d}x\right) \to \mathcal{B}_2\left(\int_\Omega v^\varepsilon\mathrm{d}x\right) \text{ a.e. in } Q_T. \tag{45}$$

From the linearity and boundedness of P^α, we have that

$$P^\alpha(u^{\varepsilon,n}) \to P^\alpha(u^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{46}$$

$$P^\alpha(v^{\varepsilon,n}) \to P^\alpha(v^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{47}$$

From the global Lipschitz property of F_ε, G_ε, we obtain

$$F_\varepsilon(u^{\varepsilon,n}, v^{\varepsilon,n}) \to F_\varepsilon(u^\varepsilon, v^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{48}$$

$$G_\varepsilon(u^{\varepsilon,n}, v^{\varepsilon,n}) \to G_\varepsilon(u^\varepsilon, v^\varepsilon) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{49}$$

Combining (37), (38), (44)-(49), we can pass in (25), (26) to the limit $n \to \infty$ to prove that (22), (23) hold for all φ, $\psi \in H_0^1(\Omega)$. By (40), we have that $u^\varepsilon(\cdot,t)$, $v^\varepsilon(\cdot,t) \in H_0^1(\Omega)$ a.e. $t \in [0,T]$. For fixed t, taking $\varphi = u^\varepsilon(\cdot,t)$ in (22), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^\varepsilon(\cdot,t)\right\|^2 - \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\left\|\nabla u^\varepsilon(\cdot,t)\right\|^2 - \left\langle u^\varepsilon(\cdot,t)K_1, \nabla u^\varepsilon\right\rangle \\ = \left\langle P^\alpha(u^\varepsilon)(\cdot,t), u^\varepsilon(\cdot,t)\right\rangle + \left\langle F_\varepsilon(u^\varepsilon,v^\varepsilon), u^\varepsilon(\cdot,t)\right\rangle. \end{aligned} \tag{50}$$

Then, by analogous arguments as for $(u^{\varepsilon,n}, v^{\varepsilon,n})$, but taking the supremum, it yields that

$$u^\varepsilon, v^\varepsilon \text{ are bounded in } C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{51}$$

Therefore,

$$(u^\varepsilon, v^\varepsilon) \in \left[C\left([0,T];L^2(\Omega)\right) \cap L^2\left([0,T];H_0^1(\Omega)\right)\right]^2.$$

On the other hand, we have

$$\left\langle g_1^{\varepsilon,n}, \varphi\right\rangle - \left\langle u^{\varepsilon,n}(\cdot,t), \varphi\right\rangle = \int_t^T \left\langle u_s^{\varepsilon,n}(\cdot,s), \varphi\right\rangle \mathrm{d}s, \quad \text{a.e. } t \in [0,T]. \tag{52}$$

From (41), (42), and since $g_1^{\varepsilon,n} \to g_1^\varepsilon$ strongly in $L^2(\Omega)$, we can pass in (52) to the limit $n \to \infty$, to obtain

$$\left\langle g_1^\varepsilon, \varphi\right\rangle - \left\langle u^\varepsilon(\cdot,t), \varphi\right\rangle = \int_t^T \left\langle u_s^\varepsilon(\cdot,s), \varphi\right\rangle \mathrm{d}s = \left\langle u^\varepsilon(\cdot,T), \varphi\right\rangle - \left\langle u^\varepsilon(\cdot,t), \varphi\right\rangle, \quad \text{a.e. } t \in [0,T]. \tag{53}$$

Thus, $u^\varepsilon(x,T) = g_1^\varepsilon(x)$. In a similar way, we have that $v^\varepsilon(x,T) = g_2^\varepsilon(x)$. This completes the proof of Part 2.

Part 3. *Non-negativity of the regularized solution*
Define $u_-^\varepsilon = \max\{-u^\varepsilon, 0\}$. Taking for fixed t, $\varphi = -u_-^\varepsilon(\cdot,t)$ in (22), we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle u^\varepsilon(\cdot,t) - u_-^\varepsilon(\cdot,t)\rangle - \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\langle \nabla u^\varepsilon(\cdot,t) - \nabla u_-^\varepsilon(\cdot,t)\rangle \\ = \left\langle u^\varepsilon(\cdot,t)K_1, -\nabla u_-^\varepsilon(\cdot,t)\right\rangle + \left\langle P^\alpha(u^\varepsilon)(\cdot,t), -u_-^\varepsilon(\cdot,t)\right\rangle + \left\langle F_\varepsilon(u^\varepsilon,v^\varepsilon), -u_-^\varepsilon(\cdot,t)\right\rangle. \end{aligned}$$

Because of the fact that $u_-^\varepsilon(x,t) \neq 0$ only if $u^\varepsilon(x,t) < 0$ (then ${}^\varepsilon(x,t) = -u_-^\varepsilon(x,t)$), using the linearity of P^α, we deduce that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_-^\varepsilon(\cdot,t)\right\|^2 - \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\left\|\nabla u_-^\varepsilon(\cdot,t)\right\|^2 \\ = \left\langle u_-^\varepsilon(\cdot,t)K_1, \nabla u_-^\varepsilon(\cdot,t)\right\rangle + \left\langle P^\alpha(u_-^\varepsilon)(\cdot,t), u_-^\varepsilon(\cdot,t)\right\rangle - \left\langle F_\varepsilon(u^\varepsilon,v^\varepsilon), u_-^\varepsilon(\cdot,t)\right\rangle. \end{aligned}$$

Integrating from t to T, it yields

$$\|u_-^\varepsilon(\cdot,t)\|^2 - \|u_-^\varepsilon(\cdot,T)\|^2 + 2\mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\int_t^T \|\nabla u_-^\varepsilon(\cdot,s)\|^2\mathrm{d}s$$

$$= \underbrace{-2\int_t^T \Big\langle u_-^\varepsilon(\cdot,s)K_1, \nabla u_-^\varepsilon(\cdot,s)\Big\rangle \mathrm{d}s}_{B_1} \underbrace{- 2\int_t^T \Big\langle P^\alpha(u_-^\varepsilon)(\cdot,s), u_-^\varepsilon(\cdot,s)\Big\rangle \mathrm{d}s}_{B_2}$$

$$\underbrace{+ 2\int_t^T \Big\langle F_\varepsilon(u^\varepsilon,v^\varepsilon), u_-^\varepsilon(\cdot,s)\Big\rangle \mathrm{d}s}_{B_3}.$$

Consequently,

$$\big\|u_-^\varepsilon(\cdot,t)\big\|^2 + 2(M_0-M)\int_t^T \big\|\nabla u_-^\varepsilon(\cdot,s)\big\|^2\mathrm{d}s \le \big|B_1\big| + \big|B_2\big| + \big|B_3\big|,$$

where, from (A_4) for ε sufficiently small we have that $\|u_-^\varepsilon(\cdot,T)\|^2 = 0$.

Applying some similar estimations as in (29) and (30) of Part 1, we have that

$$\big|B_1\big| \le \frac{K_0^2}{M_0-M}\int_t^T \big\|u_-^\varepsilon(\cdot,s)\big\|^2\mathrm{d}s + (M_0-M)\int_t^T \big\|\nabla u_-^\varepsilon(\cdot,s)\big\|^2\mathrm{d}s,$$

$$\big|B_2\big| \le \frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\int_t^T \big\|u^\varepsilon(\cdot,s)\big\|^2\mathrm{d}s.$$

For B_3, we have $B_3 = 0$, due to the facts that

- If $u^\varepsilon(x,t) > 0$, then $u_-^\varepsilon(x,t) = 0 \Rightarrow F_\varepsilon(u^\varepsilon,v^\varepsilon)u_-^\varepsilon(x,t) = 0$;
- If $u^\varepsilon(x,t) < 0$, from (14) and (16) then $F_\varepsilon(u^\varepsilon,v^\varepsilon) = 0 \Rightarrow F_\varepsilon(u^\varepsilon,v^\varepsilon)u_-^\varepsilon(x,t) = 0$.

Thus, we obtain

$$\|u_-^\varepsilon(\cdot,t)\big\|^2 \le C\int_t^T \|u_-^\varepsilon(\cdot,s)\|^2\mathrm{d}s.$$

Applying Gronwall's inequality, we have $u_-^\varepsilon = 0$ a.e. $(x,t) \in Q_T$. Following the same lines of the previous proof, we also have $v_-^\varepsilon = 0$ a.e. $(x,t) \in Q_T$.

Part 4. *Uniqueness of the solution*
Suppose that

$$(u_1^\varepsilon, v_1^\varepsilon), (u_2^\varepsilon, v_2^\varepsilon) \in \big[C([0,T];L_+^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))\big]^2 \tag{54}$$

are two weak solutions of the problem (18). Define

$$U(x,t) = (u_1^\varepsilon - u_2^\varepsilon)(x,t), \quad V(x,t) = (v_1^\varepsilon - v_2^\varepsilon)(x,t).$$

Then,

$$U(x,T) = V(x,T) = 0.$$

From (22), we have

$$\Big\langle (\partial_t u_1^\varepsilon - \partial_t u_2^\varepsilon)(\cdot,t),\varphi\Big\rangle - \mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right)\Big\langle \nabla u_1^\varepsilon(\cdot,t),\nabla\varphi\Big\rangle + \mathcal{B}_1\left(\int_\Omega u_2^\varepsilon \mathrm{d}x\right)\Big\langle \nabla u_2^\varepsilon(\cdot,t),\nabla\varphi\Big\rangle$$
$$= \Big\langle (u_1^\varepsilon - u_2^\varepsilon)(\cdot,t)K_1,\nabla\varphi\Big\rangle + \Big\langle P^\alpha(u_1^\varepsilon)(\cdot,t) - P^\alpha(u_2^\varepsilon)(\cdot,t),\varphi\Big\rangle + \Big\langle F_\varepsilon(u_1^\varepsilon,v_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon,v_2^\varepsilon),\varphi\Big\rangle.$$

Consequently,

$$\Big\langle U_t(\cdot,t),\varphi\Big\rangle - \mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right)\Big\langle \nabla U(\cdot,t),\nabla\varphi\Big\rangle - \left(\mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right) - \mathcal{B}_1\left(\int_\Omega u_2^\varepsilon \mathrm{d}x\right)\right)\Big\langle \nabla u_2^\varepsilon(\cdot,t),\nabla\varphi\Big\rangle$$
$$= \Big\langle U(\cdot,t)K_1,\nabla\varphi\Big\rangle + \Big\langle P^\alpha(U)(\cdot,t),\varphi\Big\rangle + \Big\langle F_\varepsilon(u_1^\varepsilon,v_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon,v_2^\varepsilon),\varphi\Big\rangle. \tag{55}$$

For fixed t, taking $\varphi = U(\cdot,t)$, and integrating from t to T, we obtain

$$\big\|U(\cdot,T)\big\|^2 - 2\int_t^T \mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right)\big\|\nabla U(\cdot,s)\big\|^2 \mathrm{d}s - \underbrace{2\int_t^T \Big\langle U(\cdot,t)K_1,\nabla U(\cdot,t)\Big\rangle \mathrm{d}s}_{I_4}$$
$$= \big\|U(\cdot,t)\big\|^2 + \underbrace{2\int_t^T \left(\mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right) - \mathcal{B}_1\left(\int_\Omega u_2^\varepsilon \mathrm{d}x\right)\right)\Big\langle \nabla u_2^\varepsilon(\cdot,s),\nabla U(\cdot,s)\Big\rangle \mathrm{d}s}_{I_1}$$
$$+ \underbrace{2\int_t^T \Big\langle P^\alpha(U)(\cdot,s),U(\cdot,s)\Big\rangle \mathrm{d}s}_{I_2} + \underbrace{2\int_t^T \Big\langle F_\varepsilon(u_1^\varepsilon,v_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon,v_2^\varepsilon),U(\cdot,s)\Big\rangle \mathrm{d}s}_{I_3}. \tag{56}$$

We first estimate I_1 by using Hölder's inequality and (A$_2$) to yield

$$\begin{aligned}
|I_1| &\le 2\int_t^T \left|\mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right) - \mathcal{B}_1\left(\int_\Omega u_2^\varepsilon \mathrm{d}x\right)\right| \big\|\nabla u_2^\varepsilon(\cdot,s)\big\|\big\|\nabla U(\cdot,s)\big\| \mathrm{d}s \\
&\le \frac{1}{M_0 - M}\int_t^T \left|\mathcal{B}_1\left(\int_\Omega u_1^\varepsilon \mathrm{d}x\right) - \mathcal{B}_1\left(\int_\Omega u_2^\varepsilon \mathrm{d}x\right)\right|^2 \big\|\nabla u_2^\varepsilon(\cdot,s)\big\|^2 \mathrm{d}s \\
&\quad + (M_0 - M)\int_t^T \big\|\nabla U(\cdot,s)\big\|^2 \mathrm{d}s \\
&\le \frac{\mathrm{L}^2}{M_0 - M}\int_t^T \big\|U(\cdot,s)\big\|^2\big\|\nabla u_2^\varepsilon(\cdot,s)\big\|^2 \mathrm{d}s + (M_0 - M)\int_t^T \big\|\nabla U(\cdot,s)\big\|^2 \mathrm{d}s.
\end{aligned} \tag{57}$$

For I_2, using Lemma 2, we have

$$|I_2| \le \frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\int_t^T \big\|U(\cdot,s)\big\|^2 \mathrm{d}s. \tag{58}$$

Next, we estimate I_3 using Lemma 1 as

$$|I_3| \le 2\int_t^T \Big\|F_\varepsilon(u_1^\varepsilon,v_1^\varepsilon) - F_\varepsilon(u_2^\varepsilon,v_2^\varepsilon)\Big\|\big\|U(\cdot,s)\big\| \mathrm{d}s \le 2K_{R^\varepsilon}\int_t^T \big\|(U,V)(\cdot,s)\big\|\big\|U(\cdot,s)\big\| \mathrm{d}s. \tag{59}$$

For I_4, using Hölder's inequality and (A_5) gives

$$\begin{aligned}
|I_4| &\le \frac{1}{M_0 - M}\int_t^T \|U(\cdot,s)K_1\|^2 \mathrm{d}s + (M_0 - M)\int_t^T \|\nabla U(\cdot,s)\|^2 \mathrm{d}s \\
&\le \frac{K_0^2}{M_0 - M}\int_t^T \|U(\cdot,s)\|^2 \mathrm{d}s + (M_0 - M)\int_t^T \|\nabla U(\cdot,s)\|^2 \mathrm{d}s.
\end{aligned} \tag{60}$$

Combining (56) - (60), we deduce that

$$\begin{aligned}\|U(\cdot,t)\|^2 \le \frac{\mathrm{L}^2}{M_0-M}\int_t^T \|U(\cdot,s)\|^2\|\nabla u_2^\varepsilon(\cdot,s)\|^2\mathrm{d}s + \left(\frac{K_0^2}{M_0-M}+\frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\right)\int_t^T \|U(\cdot,s)\|^2\mathrm{d}s\\ + 2K_{R^\varepsilon}\int_t^T \|(U,V)(\cdot,s)\|\|U(\cdot,s)\|\mathrm{d}s. \qquad (61)\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|V(\cdot,t)\|^2 \le \frac{\mathrm{L}^2}{M_0-M}\int_t^T \|V(\cdot,s)\|^2\|\nabla v_2^\varepsilon(\cdot,s)\|^2\mathrm{d}s + \left(\frac{K_0^2}{M_0-M}+\frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\right)\int_t^T \|V(\cdot,s)\|^2\mathrm{d}s\\ + 2K_{R^\varepsilon}\int_t^T \|(U,V)(\cdot,s)\|\|V(\cdot,s)\|\mathrm{d}s. \qquad (62)\end{aligned}$$

By adding (61) to (62), we arrive at

$$\|(U,V)(\cdot,t)\|^2 \le 2\|U(\cdot,t)\|^2 + 2\|V(\cdot,t)\|^2 \le C_4\int_t^T \|(U,V)(\cdot,s)\|^2\mathrm{d}s,$$

where the positive constant C_4 depends on u_2^ε and v_2^ε. Using Gronwall's inequality, we obtain

$$\|(U,V)(\cdot,t)\|^2 \le 0,$$

which implies that $U = V = 0$, or $(u_1^\varepsilon, v_1^\varepsilon) = (u_2^\varepsilon, v_2^\varepsilon)$ a.e. $(x,t) \in Q_T$. This completes the proof of Theorem 1. □

4 Error estimates

Now, we shall prove some estimations for the error between the solution to problem (4), (7) and (9) and the solution $(u^\varepsilon, v^\varepsilon)$ to the regularized problem (18) in the L^2- and H^1-norms.

4.1 Estimation in L^2-norm

Theorem 2. *Suppose that the assumptions $(A_1)-(A_5)$ hold and that the problem (4), (7) and (9) has a unique weak solution (u,v) satisfying*

$$(u,v) \in \left[L^2(0,T;G_\sigma(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)\right]^2$$

with $\sigma \ge 2M_0T$. Choose $\alpha = \alpha(\varepsilon) > 0$ such that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\alpha} = l < \infty. \qquad (63)$$

Then, there exists $C_0 = C_0(u,v) \ge 0$ such that

$$\|\left((u^\varepsilon, v^\varepsilon) - (u,v)\right)(\cdot,t)\| \le C_0\alpha^{t/T}e^{2K_{R^\varepsilon}T}, \quad \forall t \in [0,T]. \qquad (64)$$

Proof. Let us define

$$\mathcal{X}(x,t) := e^{q(t-T)}(u^\varepsilon - u)(x,t), \quad \mathcal{Y}(x,t) := e^{q(t-T)}(v^\varepsilon - v)(x,t),$$

where $q = q(\alpha) > 0$ is a positive constant, which will be chosen later. From (12) and (20), we have

$$\begin{aligned}
&\Big\langle (u^\varepsilon_t - u_t)(\cdot,t), \varphi \Big\rangle + \mathcal{D}_1\left(\int_\Omega u^\varepsilon\right)\Big\langle \nabla u^\varepsilon(\cdot,t), \nabla\varphi\Big\rangle - \mathcal{D}_1\left(\int_\Omega \mathrm{d}x\right)\Big\langle \nabla u(\cdot,t), \nabla\varphi\Big\rangle \\
&= \Big\langle Q^\alpha(u^\varepsilon)(\cdot,t), \varphi\Big\rangle + \Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon) - \overline{F}(u,v), \varphi\Big\rangle + \Big\langle (u^\varepsilon - u)(\cdot,t)K_1, \nabla\varphi\Big\rangle,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\Big\langle \mathcal{X}_t(\cdot,t), \varphi\Big\rangle &- \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\Big\langle \nabla\mathcal{X}(\cdot,t), \nabla\varphi\Big\rangle \\
&+ e^{q(t-T)}\left(\mathcal{D}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right) - \mathcal{D}_1\left(\int_\Omega u \mathrm{d}x\right)\right)\Big\langle \nabla u(\cdot,t), \nabla\varphi\Big\rangle \\
&= q\Big\langle \mathcal{X}(\cdot,t), \varphi\Big\rangle + e^{q(t-T)}\Big\langle Q^\alpha(u)(\cdot,t), \varphi\Big\rangle + \Big\langle P^\alpha(\mathcal{X})(\cdot,t), \varphi\Big\rangle \\
&+ e^{q(t-T)}\Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon) - \overline{F}(u,v), \varphi\Big\rangle + \Big\langle \mathcal{X}(\cdot,t)K_1, \nabla\varphi\Big\rangle. \qquad (65)
\end{aligned}$$

For fixed t, taking $\varphi = \mathcal{X}(\cdot,t)$ and integrating from t to T, we obtain

$$\begin{aligned}
&\|\mathcal{X}(\cdot,T)\|^2 - \|\mathcal{X}(\cdot,t)\|^2 - 2\int_t^T \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right)\|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s - 2\int_t^T q\|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s \\
&= \underbrace{2\int_t^T e^{q(s-T)}\Big\langle Q^\alpha(u)(\cdot,s), \mathcal{X}(\cdot,s)\Big\rangle \mathrm{d}s}_{K_1} + \underbrace{2\int_t^T e^{q(s-T)}\Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon) - \overline{F}(u,v), \mathcal{X}(\cdot,s)\Big\rangle \mathrm{d}s}_{K_2} \\
&+ \underbrace{2\int_t^T \Big\langle P^\alpha(\mathcal{X})(\cdot,s), \mathcal{X}(\cdot,s)\Big\rangle \mathrm{d}s}_{K_3} + \underbrace{2\int_t^T \Big\langle \mathcal{X}(\cdot,s)K_1, \nabla\mathcal{X}(\cdot,s)\Big\rangle \mathrm{d}s}_{K_4} \\
&\underbrace{- 2\int_t^T e^{q(s-T)}\left(\mathcal{D}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right) - \mathcal{D}_1\left(\int_\Omega u \mathrm{d}x\right)\right)\Big\langle \nabla u(\cdot,s), \nabla\mathcal{X}(\cdot,s)\Big\rangle \mathrm{d}s}_{K_5}. \qquad (66)
\end{aligned}$$

Applying Hölder's inequality and Lemma 2, we have

$$\begin{aligned}
|K_1| &\le 2\int_t^T \|Q^\alpha(u)(\cdot,s)\|\|\mathcal{X}(\cdot,s)\|\mathrm{d}s \le \frac{2\alpha}{T}\int_t^T \|u(\cdot,s)\|_{G_\sigma(\Omega)}\|\mathcal{X}(\cdot,s)\|\mathrm{d}s \\
&\le \frac{\alpha^2}{T^2}\int_t^T \|u(\cdot,s)\|^2_{G_\sigma(\Omega)}\mathrm{d}s + \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s. \qquad (67)
\end{aligned}$$

Next, we estimate K_2. Notice that since $R^\varepsilon \to \infty$ when $\varepsilon \to 0$, with the assumption u, $v \in L^\infty(Q_T)$, we can choose a sufficiently small ε, such that $(u,v) \in \mathbb{B}_{R^\varepsilon}$, or $\overline{F}(u,v) = F_\varepsilon(u,v)$. Thus, we obtain

$$\begin{aligned}
|K_2| &\le 2\int_t^T e^{q(s-T)}\left|\Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon) - F_\varepsilon(u,v), \mathcal{X}(\cdot,s)\Big\rangle\right|\mathrm{d}s \\
&\le 2K_{R^\varepsilon}\int_t^T e^{q(s-T)}\|\left((u^\varepsilon, v^\varepsilon) - (u,v)\right)(\cdot,s)\|\|\mathcal{X}(\cdot,s)\|\mathrm{d}s = 2K_{R^\varepsilon}\int_t^T \|(\mathcal{X},\mathcal{Y})(\cdot,s)\|\|\mathcal{X}(\cdot,s)\|\mathrm{d}s. \qquad (68)
\end{aligned}$$

For K_3, using Lemma 2, it follows

$$|K_3| \le 2\int_t^T \|P^\alpha(\mathcal{X})(\cdot,t)\|\|\mathcal{X}(\cdot,s)\|\mathrm{d}s \le \frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s. \qquad (69)$$

For K_4, using Hölder's inequality, Cauchy's inequality and (A_5), we deduce that

$$|K_4| \leq 2K_0 \int_t^T \|\mathcal{X}(\cdot,s)\| \|\nabla\mathcal{X}(\cdot,s)\| \mathrm{d}s \leq \frac{K_0^2}{M_0-M} \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s + (M_0-M) \int_t^T \|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s. \tag{70}$$

Let us denote

$$E := \max\left\{ \|(u,v)\|_{(L^2(0,T;G_\sigma(\Omega)))^2}, \|(u,v)\|_{(L^\infty(0,T;H_0^1(\Omega)))^2} \right\}.$$

Using (A$_2$), Hölder's inequality and Cauchy's inequality, we obtain

$$\begin{aligned}
|K_5| &\leq 2\mathrm{L} \int_t^T e^{q(s-T)} \| (u^\varepsilon - u)(\cdot,s) \| \|\nabla u(\cdot,s)\| \|\nabla\mathcal{X}(\cdot,s)\| \mathrm{d}s \\
&\leq 2\mathrm{L} \int_t^T \|\mathcal{X}(\cdot,s)\| \|\nabla u(\cdot,s)\| \|\nabla\mathcal{X}(\cdot,s)\| \mathrm{d}s \\
&\leq \frac{\mathrm{L}^2}{M_0-M} \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \|\nabla u(\cdot,s)\|^2 \mathrm{d}s + (M_0-M) \int_t^T \|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s \\
&\leq \frac{\mathrm{L}^2 E^2}{M_0-M} \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s + (M_0-M) \int_t^T \|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s. \qquad (71)
\end{aligned}$$

Combining (66)-(71), and choosing $q = \frac{1}{T}\ln(\frac{1}{\alpha})$, we have

$$\begin{aligned}
&\|\mathcal{X}(\cdot,t)\|^2 + 2(M_0-M) \int_t^T \|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s + \frac{2}{T}\ln\left(\frac{1}{\alpha}\right) \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s \\
&\leq \|\mathcal{X}(\cdot,t)\|^2 + 2\int_t^T \mathcal{B}_1\left(\int_\Omega u^\varepsilon \mathrm{d}x\right) \|\nabla\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s + \frac{2}{T}\ln\left(\frac{1}{\alpha}\right) \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s \\
&\leq \varepsilon^2 + |K_1| + |K_2| + |K_3| + |K_4| + |K_5|,
\end{aligned}$$

or

$$\begin{aligned}
\|\mathcal{X}(\cdot,t)\|^2 \leq \varepsilon^2 + \frac{\alpha^2}{T^2} \int_t^T \|u(\cdot,s)\|^2_{G_\sigma(\Omega)} \mathrm{d}s + \left(1 + \frac{K_0^2}{M_0-M} + \frac{\mathrm{L}^2E^2}{M_0-M}\right) \int_t^T \|\mathcal{X}(\cdot,s)\|^2 \mathrm{d}s \\
+ 2K_{R^\varepsilon} \int_t^T \|(\mathcal{X},\mathcal{Y})(\cdot,s)\| \|\mathcal{X}(\cdot,s)\| \mathrm{d}s. \qquad (72)
\end{aligned}$$

In a similar manner, we obtain the estimate for $\|\mathcal{Y}(\cdot,t)\|^2$, and summing with (72), it yields

$$\|(\mathcal{X},\mathcal{Y})(\cdot,t)\|^2 \leq 2\|\mathcal{X}(\cdot,t)\|^2 + 2\|\mathcal{Y}(\cdot,t)\|^2 \leq 4\varepsilon^2 + C_5\alpha^2 + (C_6 + 4K_{R^\varepsilon}) \int_t^T \|(\mathcal{X},\mathcal{Y})(\cdot,s)\|^2 \mathrm{d}s. \tag{73}$$

Applying Gronwall's inequality, we arrive at

$$\|(\mathcal{X},\mathcal{Y})(\cdot,t)\|^2 \leq \left(4\varepsilon^2 + C_5\alpha^2\right) \exp\left(C_6T + 4K_{R^\varepsilon}T\right),$$

which leads to

$$\| ((u^\varepsilon, v^\varepsilon) - (u,v))(\cdot,t)\|^2 = e^{2q(T-t)} \|(\mathcal{X},\mathcal{Y})(\cdot,t)\|^2 \leq \left(\frac{C_7\varepsilon^2}{\alpha^2} + C_8\right) \alpha^{\frac{2t}{T}} e^{4K_{R^\varepsilon}T}.$$

The proof of Theorem 2 is completed. □

Remark 2. We now give an example of parameter choice to obtain the convergence of the approximate solution. Let us choose $\alpha = \varepsilon$ and R^ε such that

$$K_{R^\varepsilon} \leq \frac{1}{4T} \ln\left(\ln^{2\beta}\left(\frac{1}{\varepsilon}\right)\right) \text{ for some } \beta > 0.$$

Then, $\left\|((u^\varepsilon, v^\varepsilon) - (u,v))(\cdot,t)\right\|$ is of order $\varepsilon^{t/T}\ln^\beta(\frac{1}{\varepsilon})$, which tends to 0, as $\varepsilon \searrow 0$, for all $t \in (0,T]$.

4.2 Estimation in $H^1(\Omega)$-norm

We first state and prove the following lemma.

Lemma 3. Assume (A_5) and (A_6) : $\mathrm{div}(K_i) \in L^\infty(\Omega)$, $i = 1, 2$. Then, there exist constants $0 \leq b(K_i)$, $i = 1, 2$, such that

$$\left\|\mathrm{div}(K_i w)\right\|^2 \leq b(K_i)\left\|w\right\|^2_{H^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{74}$$

Proof. We have

$$\begin{aligned}
\left\|\mathrm{div}(K_i w)\right\|^2 &= \left\|\mathrm{div}(K_i)w + \sum_{j=1}^d K_{ij}\partial_{x_j} w\right\|^2 \\
&= \left\|\mathrm{div}(K_i)w\right\|^2 + 2\left\langle \mathrm{div}(K_i)w, \sum_{j=1}^d K_{ij}\partial_{x_j} w\right\rangle + \left\|\sum_{j=1}^d K_{ij}\partial_{x_j} w\right\|^2 \\
&\leq b_1\left\|w\right\|^2 + b_2\left\langle w, \sum_{j=1}^d \partial_{x_j} w\right\rangle + b_3\left\|\sum_{j=1}^d \partial_{x_j} w\right\|^2 \leq b\left(\left\|w\right\|^2 + \left\|\nabla w\right\|^2\right) = b\left\|w\right\|^2_{H^1(\Omega)},
\end{aligned}$$

where we have used the Hölder inequality. □

Now the following theorem gives an estimation for the error in the H^1-norm.

Theorem 3. *Suppose that the assumptions $(A_1), (A_2), (A_4) - (A_6)$ hold. Furthermore, assume*

$$(A_7): \quad g_1, g_2, g_1^\varepsilon, g_2^\varepsilon \in H^1(\Omega) \text{ satisfying } \left\|g_1^\varepsilon - g_1\right\|_{H^1(\Omega)} + \left\|g_2^\varepsilon - g_2\right\|_{H^1(\Omega)} \leq \varepsilon.$$

Choose $\alpha = \alpha(\varepsilon) > 0$ satisfying (63). Assume that the problem (4), (7) and (9) has a unique weak solution

$$(u,v) \in \left[L^2(0,T; G_\sigma(\Omega)) \cap L^\infty(0,T; H^2(\Omega)) \cap L^\infty(Q_T)\right]^2$$

with $\sigma \geq 2M_0 T$. Then, there exists $C_0^* = C_0^*(u,v) \geq 0$ such that

$$\left\|\left((u^\varepsilon, v^\varepsilon) - (u,v)\right)(\cdot,t)\right\|_{[H^1(\Omega)]^2} \leq C_0^* \alpha^{t/T} \exp\left(4K_{R^\varepsilon}T + \frac{8K^2_{R^\varepsilon}}{M_0 - M}T\right), \quad \forall t \in [0,T]. \tag{75}$$

Proof. We first prove that the solution to the problem (22) and (23) satisfies Δu^ε, $\Delta v^\varepsilon \in L^2([0,T]; L^2(\Omega))$. Since the basis $\{\phi_p\}_{p=1}^\infty \subset H_0^1(\Omega)$, we have that

$$\Delta u^{\varepsilon,n} = -\sum_{p=1}^n \lambda_p w^\varepsilon_{np}(t)\phi_p(x), \quad \Delta v^{\varepsilon,n} = -\sum_{p=1}^n \lambda_p \omega^\varepsilon_{np}(t)\phi_p(x)$$

also lie in $H_0^1(\Omega)$ a.e. $t \in [0,T]$, where $(u^{\varepsilon,n}, v^{\varepsilon,n})$ is the Galerkin approximate solution satisfying (25) and (26).

For fixed t, taking $\varphi = \Delta u^{\varepsilon,n}(\cdot,t)$ in (25) and integrating from t to T, we obtain

$$
\begin{aligned}
&\left\|\nabla u^{\varepsilon,n}(\cdot,t)\right\|^2 + 2\int_t^T \mathcal{B}_1\left(\int_\Omega u^{\varepsilon,n}\mathrm{d}x\right)\left\|\Delta u^{\varepsilon,n}(\cdot,s)\right\|^2\mathrm{d}s \\
&= \left\|\nabla g_1^{\varepsilon,n}\right\|^2 + 2\int_t^T \Big\langle P^\alpha(u^{\varepsilon,n})(\cdot,s), \Delta u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s \\
&\quad + 2\int_t^T \Big\langle F_\varepsilon(u^{\varepsilon,n}, v^{\varepsilon,n}), \Delta u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s - 2\int_t^T \Big\langle \mathrm{div}(u^{\varepsilon,n}(\cdot,s)K_1), \Delta u^{\varepsilon,n}(\cdot,s)\Big\rangle\mathrm{d}s \\
&\le \left\|\nabla g_1^\varepsilon\right\|^2 + 2\int_t^T \Big(\left\|P^\alpha(u^{\varepsilon,n})(\cdot,s)\right\| + \left\|F_\varepsilon(u^{\varepsilon,n}, v^{\varepsilon,n})\right\| + \left\|\mathrm{div}\,(u^{\varepsilon,n}(\cdot,s)K_1)\right\|\Big)\left\|\Delta u^{\varepsilon,n}(\cdot,s)\right\|\mathrm{d}s \\
&\le C_9 + C_{10}\int_t^T \left(\left\|(u^{\varepsilon,n}, v^{\varepsilon,n})(\cdot,s)\right\|^2 + \left\|u^{\varepsilon,n}(\cdot,s)\right\|^2_{H^1(\Omega)}\right)\mathrm{d}s + (M_0 - M)\int_t^T \|\Delta u^{\varepsilon,n}(\cdot,s)\|^2\mathrm{d}s,
\end{aligned}
$$

where we have used (A_7), Hölder's inequality, Cauchy's inequality, Lemmas 2 and 3, and the Lipschitz property of F_ε. Hence, using (36) and (37) we arrive at $\|\Delta u^{\varepsilon,n}\|_{L^2(0,T;L^2(\Omega))} \le C_{12}$. Then, the limit function u^ε also satisfies this estimate. Using the same arguments for v^ε, we have Δu^ε, $\Delta v^\varepsilon \in L^2(0,T;L^2(\Omega))$. From the hypothesis u, $v \in L^\infty(0,T;H^2(\Omega))$, we obtain that $\Delta\mathcal{X}$, $\Delta\mathcal{Y} \in L^2(0,T;L^2(\Omega))$. For fixed t, taking $\varphi = \lambda_p\langle\mathcal{X}(\cdot,t),\phi_p\rangle\phi_p(x)$ in (65), summing from $p=1$ to ∞ and then integrating from t to T, we get

$$
\begin{aligned}
&\left\|\nabla\mathcal{X}(\cdot,T)\right\|^2 - \left\|\nabla\mathcal{X}(\cdot,t)\right\|^2 - 2q\int_t^T \left\|\nabla\mathcal{X}(\cdot,s)\right\|^2\mathrm{d}s \\
&\quad - 2\int_t^T \mathcal{B}_1\left(\int_\Omega u^\varepsilon\mathrm{d}x\right)(s)\left\|\Delta\mathcal{X}(\cdot,s)\right\|^2\mathrm{d}s \\
&= \underbrace{-2\int_t^T e^{q(s-T)}\left(\mathcal{D}_1\left(\int_\Omega u^\varepsilon\mathrm{d}x\right) - \mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)\right)(s)\Big\langle\Delta u(\cdot,s), \Delta\mathcal{X}(\cdot,s)\Big\rangle\mathrm{d}s}_{G_1} \\
&\quad \underbrace{-2\int_t^T e^{q(t-T)}\Big\langle Q^\alpha(u)(\cdot,s), \Delta\mathcal{X}(\cdot,s)\Big\rangle\mathrm{d}s}_{G_2} + \underbrace{2\int_t^T \Big\langle P^\alpha(\nabla\mathcal{X})(\cdot,s), \nabla\mathcal{X}(\cdot,s)\Big\rangle\mathrm{d}s}_{G_3} \\
&\quad \underbrace{-2\int_t^T e^{q(t-T)}\Big\langle F_\varepsilon(u^\varepsilon, v^\varepsilon) - \overline{F}(u,v), \Delta\mathcal{X}(\cdot,s)\Big\rangle\mathrm{d}s}_{G_4} + \underbrace{2\int_t^T \Big\langle \mathrm{div}(\mathcal{X}(\cdot,s)K_1), \Delta\mathcal{X}(\cdot,s)\Big\rangle\mathrm{d}s}_{G_5}. \qquad (76)
\end{aligned}
$$

The above terms make sense because of the linearity of P^α, Q^α, the Lipschitz property of F_ε, and the fact that $\Delta\mathcal{X}$, $\Delta\mathcal{Y}$, Δu, Δv, $\mathrm{div}(\mathcal{X}K_1) \in L^2(0,T;L^2(\Omega))$. Using Hölder's inequality, Cauchy's inequality and the assumption (A_2), it yields

$$
\begin{aligned}
|G_1| &\le 2\mathrm{L}\int_t^T \left\|\mathcal{X}(\cdot,s)\right\|\left\|\Delta u(\cdot,s)\right\|\left\|\Delta\mathcal{X}(\cdot,s)\right\|\mathrm{d}s \\
&\le \frac{2\mathrm{L}^2(E^*)^2}{M_0 - M}\int_t^T \left\|\mathcal{X}(\cdot,s)\right\|^2\mathrm{d}s + \frac{M_0 - M}{2}\int_t^T \left\|\Delta\mathcal{X}(\cdot,s)\right\|^2\mathrm{d}s, \qquad (77)
\end{aligned}
$$

where

$$
E^* := \max\left\{\left\|(u,v)\right\|_{[L^2(0,T;G_\sigma(\Omega))]^2}, \left\|(u,v)\right\|_{[L^\infty(0,T;H^2(\Omega))]^2}\right\}.
$$

Using Hölder's inequality, Cauchy's inequality and Lemma 2, we obtain

$$
\begin{aligned}
|G_2| &\leq 2\int_t^T \big\|Q^\alpha(u)(\cdot,s)\big\|\big\|\Delta\mathcal{X}(\cdot,s)\big\|\mathrm{d}s \qquad (78)\\
&\leq \frac{2}{M_0-M}\int_t^T \big\|Q^\alpha(u)(\cdot,s)\big\|^2\mathrm{d}s + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s\\
&\leq \frac{2\alpha^2}{(M_0-M)T^2}\int_t^T \big\|u(\cdot,s)\big\|^2_{G_\sigma(\Omega)}\mathrm{d}s + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s\\
&\leq \frac{2\alpha^2(E^*)^2}{(M_0-M)T^2} + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s.
\end{aligned}
$$

Thanks to Lemma 2, we have

$$
|G_3| \leq 2\int_t^T \big\|P^\alpha(\nabla\mathcal{X})(\cdot,s)\big\|\big\|\nabla\mathcal{X}(\cdot,s)\big\|\mathrm{d}s \leq \frac{2}{T}\ln\left(\frac{1}{\alpha}\right)\int_t^T \big\|\nabla\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s. \qquad (79)
$$

With an analogous argument as in section 4.1, we can choose a sufficiently small ε such that $\overline{F}(u,v) = F_\varepsilon(u,v)$ a.e. $(x,t)\in Q_T$. Therefore,

$$
\begin{aligned}
|G_4| &= 2\int_t^T e^{q(s-T)}\Big|\Big\langle F_\varepsilon(u^\varepsilon,v^\varepsilon) - F_\varepsilon(u,v), \Delta\mathcal{X}(\cdot,s)\Big\rangle\Big|\mathrm{d}s\\
&\leq 2\int_t^T e^{q(s-T)}\big\|F_\varepsilon(u^\varepsilon,v^\varepsilon) - F_\varepsilon(u,v)\big\|\big\|\Delta\mathcal{X}(\cdot,s)\big\|\mathrm{d}s\\
&\leq 2K_{R_\varepsilon}\int_t^T \big\|(\mathcal{X},\mathcal{Y})(\cdot,s)\big\|\big\|\Delta\mathcal{X}(\cdot,s)\big\|\mathrm{d}s\\
&\leq \frac{2K^2_{R_\varepsilon}}{M_0-M}\int_t^T \big\|(\mathcal{X},\mathcal{Y})(\cdot,s)\big\|^2\mathrm{d}s + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s. \qquad (80)
\end{aligned}
$$

For G_5, using Hölder, Cauchy inequalities and Lemma 3, we obtain

$$
\begin{aligned}
|G_5| &\leq 2\int_t^T \big\|\mathrm{div}(\mathcal{X}(\cdot,s)K_1)\big\|\big\|\Delta\mathcal{X}(\cdot,s)\big\|\mathrm{d}s\\
&\leq \frac{2}{M_0-M}\int_t^T \big\|\mathrm{div}(\mathcal{X}(\cdot,s)K_1)\big\|^2\mathrm{d}s + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s\\
&\leq \frac{2b}{M_0-M}\int_t^T \big\|\mathcal{X}(\cdot,s)\big\|^2_{H^1(\Omega)}\mathrm{d}s + \frac{M_0-M}{2}\int_t^T \big\|\Delta\mathcal{X}(\cdot,s)\big\|^2\mathrm{d}s. \qquad (81)
\end{aligned}
$$

Choosing again $q = \frac{1}{T}\ln\left(\frac{1}{\alpha}\right)$, from (76)-(81), we deduce

$$
\big\|\nabla\mathcal{X}(\cdot,t)\big\|^2 \leq \varepsilon^2 + C_{11}\alpha^2 + C_{12}\int_t^T \big\|\mathcal{X}(\cdot,s)\big\|^2_{H^1(\Omega)}\mathrm{d}s + \frac{2K^2_{R^\varepsilon}}{M_0-M}\int_t^T \big\|(\mathcal{X},\mathcal{Y})(\cdot,s)\big\|^2\mathrm{d}s.
$$

In a similar manner, we obtain the estimate for $\mathcal{Y}$. Hence,

$$
\begin{aligned}
\big\|(\nabla\mathcal{X},\nabla\mathcal{Y})(\cdot,t)\big\|^2 &\leq 2\big\|\nabla\mathcal{X}(\cdot,t)\big\|^2 + 2\big\|\nabla\mathcal{Y}(\cdot,t)\big\|^2\\
&\leq 4\varepsilon^2 + 4C_{11}\alpha^2 + 2C_{12}\int_t^T \big\|(\mathcal{X},\mathcal{Y})(\cdot,s)\big\|^2_{H^1(\Omega)}\mathrm{d}s + \frac{8K^2_{R^\varepsilon}}{M_0-M}\int_t^T \big\|(\mathcal{X},\mathcal{Y})(\cdot,s)\big\|^2\mathrm{d}s. \qquad (82)
\end{aligned}
$$

Combining (82) and (73), it yields

$$\begin{aligned}\left\|(\mathcal{X},\mathcal{Y})(\cdot,t)\right\|^2_{[H^1(\Omega)]^2} &\le 2\left\|(\mathcal{X},\mathcal{Y})(\cdot,t)\right\|^2 + 2\left\|(\nabla\mathcal{X},\nabla\mathcal{Y})(\cdot,t)\right\|^2\\ &\le C_{13}\varepsilon^2 + C_{14}\alpha^2 + \left(C_{15} + 8K_{R^\varepsilon} + \frac{16K^2_{R^\varepsilon}}{M_0 - M}\right)\int_t^T \left\|(\mathcal{X},\mathcal{Y})(\cdot,s)\right\|^2_{[H^1(\Omega)]^2}\mathrm{d}s.\end{aligned}$$

Applying Gronwall's inequality, we arrive at

$$\left\|(\mathcal{X},\mathcal{Y})(\cdot,t)\right\|^2_{[H^1(\Omega)]^2} \le \left(C_{16}\varepsilon^2 + C_{17}\alpha^2\right)\exp\left(8K_{R^\varepsilon}T + \frac{16K^2_{R^\varepsilon}T}{M_0 - M}\right).$$

Thus, by taking $q = \frac{1}{T}\ln(\frac{1}{\alpha})$, or

$$\left\|\left((u^\varepsilon,v^\varepsilon) - (u,v)\right)(\cdot,t)\right\|^2_{[H^1(\Omega)]^2} = \frac{\alpha^{2t/T}}{\alpha^2}\left\|(\mathcal{X},\mathcal{Y})(\cdot,t)\right\|^2_{[H^1(\Omega)]^2},$$

we can deduce (75). This completes the proof of Theorem 3. □

Remark 3. We can now give another example of parameter choice different from that in Remark 2. If we choose $\alpha = \varepsilon$ and R^ε such that

$$8K_{R^\varepsilon} + \frac{16K^2_{R^\varepsilon}}{M_0 - M} \le \frac{1}{T}\ln\left(\ln^{2\beta}\left(\frac{1}{\varepsilon}\right)\right) \text{ for some } \beta > 0,$$

then, $\left\|((u^\varepsilon,v^\varepsilon) - (u,v))(\cdot,t)\right\|_{(H^1(\Omega))^2}$ is of order $\varepsilon^{t/T}\ln^\beta\left(\frac{1}{\varepsilon}\right)$, which again tends to 0 as $\varepsilon \searrow 0$, for all $t \in (0,T]$.

Remark 4. Expressions (64) and (75) in Theorems 2 and 3, respectively, yield stability estimates for any $t \in (0,T]$. At $t = 0$, we can follow Theorem 13 of [22] step by step to obtain a logarithmic error estimate, but then we need to add one more condition on the true solution (u,v), namely that $u,v \in C^1(0,T;L^2(\Omega))$.

Remark 5. In both Theorems 2 and 3 it is assumed that u and v belong to $L^2(0,T;G^\sigma(\Omega))$, where the Gevrey space of functions $G^\sigma(\Omega)$ has been defined in (11). At this stage, there are unknown to us sufficient conditions on the data entering the problem given by equations (4), (7) and (9) to ensure that the solution $(u,v) \in (L^2(0,T;G^\sigma(\Omega))^2$, but we point out to references [4, 5, 12] for some useful results on Gevrey regularity for parabolic equations.

5 Numerical results and discussion

In this section, we consider some examples to show the instability of the backward problem and illustrate the theoretical results of regularization. We implement the predator-prey model in the one-dimensional domain $\Omega = (0,\pi)$ and $T = 1$. Let $\mathcal{D}_1 = \mathcal{D}_2 = 1$ and $K_1 = K_2 = 0.02$ and take the following ecological parameters

$$r = 0.3, \quad K = 2, \quad a = 0.3, \quad \beta_1 = \beta_2 = 0, \quad e = 0.9, \quad \mathrm{p} = 0.9, \quad \mathrm{q} = 0.2. \tag{83}$$

which are characteristic to a predator-prey system in a polluted environment [2, 24]. As we have proved in section 3, the Galerkin approximate solution $(u^{\varepsilon,n},v^{\varepsilon,n})$ satisfying (25) and (26) converges to our regularized solution, so we choose $(u^{\varepsilon,n},v^{\varepsilon,n})$ as our numerical regularized solution. Taking $\alpha(\varepsilon) = \varepsilon$, we have to find $w^\varepsilon_{np}(t)$ and $\omega^\varepsilon_{np}(t)$ that are the solutions of the $2n$ system of ODEs

$$\begin{cases}\dfrac{\mathrm{d}w^\varepsilon_{np}}{\mathrm{d}t} + \mathcal{D}_1\lambda_p w^\varepsilon_{np} - K_1\left\langle u^{\varepsilon,n}(\cdot,t),\nabla\phi_p\right\rangle = \dfrac{\ln\left(1+\varepsilon e^{TM_0\lambda_p}\right)}{T}w^\varepsilon_{np} + \left\langle F(u^{\varepsilon,n}(\cdot,t),v^{\varepsilon,n}(\cdot,t)),\phi_p\right\rangle,\\ \dfrac{\mathrm{d}\omega^\varepsilon_{np}}{\mathrm{d}t} + \mathcal{D}_2\lambda_p\omega^\varepsilon_{np} - K_2\left\langle v^{\varepsilon,n}(\cdot,t),\nabla\phi_p\right\rangle = \dfrac{\ln\left(1+\varepsilon e^{TM_0\lambda_p}\right)}{T}\omega^\varepsilon_{np} + \left\langle G(u^{\varepsilon,n}(\cdot,t),v^{\varepsilon,n}(\cdot,t)),\phi_p\right\rangle,\end{cases} \tag{84}$$

for $p=\overline{1,n}$, where $u^{\varepsilon,n}$ and $v^{\varepsilon,n}$ are given by (24). The Laplacian operator has eigenfunctions, satisfying the homogeneous Dirichlet boundary condition: $\phi_p(x) = \sqrt{\frac{2}{\pi}}\sin(px)$, with corresponding eigenvalues, $\lambda_p = p^2, p \in \mathbb{N}^*$. This sequence $\{\phi_p(x)\}_{p=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$.

A uniform grid of mesh-points (x_i, t_k) is used to discretize the space and time intervals

$$\Delta x = \frac{\pi}{N_x}, \quad x_i = (i-1)\Delta x, \quad i = \overline{1, N_x+1},$$
$$\Delta t = \frac{T}{N_t}, \quad t_k = (k-1)\Delta t, \quad k = \overline{1, N_t+1}.$$

The inner product in $L^2(0,\pi)$ can be approximated by the composite Simpson's rule of numerical integration

$$\int_0^\pi f(x)\mathrm{d}x \approx \frac{\Delta x}{3}\sum_{i=1}^{N_x+1} c_i f(x_i),$$

where $c_i = \begin{cases} 1, & \text{if } i=1 \text{ or } i=N_x+1, \\ 2, & \text{if } i=2l+1, \\ 4, & \text{if } i=2l \end{cases}$. Consequently, the discrete norm in $L^2(\Omega)$, namely the ℓ^2-norm can be defined by

$$\left\|w\right\|_{\ell^2(\Omega)} = \sqrt{\frac{\Delta x}{3}\sum_{i=1}^{N_x+1} c_i w^2(x_i)}.$$

The absolute errors are evaluated by $\epsilon_u = \left\|u^{\varepsilon,n} - u\right\|_{\ell^2(\Omega)}$ and $\epsilon_v = \left\|v^{\varepsilon,n} - v\right\|_{\ell^2(\Omega)}$.
We find the coefficients w^ε_{np} and ω^ε_{np} satisfying (84) by the finite difference method (FDM). Denoting by w_p^k and ω_p^k the $p-$th coefficients at $t=t_k$, we have the following relations:

$$\begin{cases} \frac{w_p^{k+1}-w_p^k}{\Delta t} + \mathcal{D}_1\lambda_p w_p^{k+1} - K_1\left\langle u^{\varepsilon,n}(\cdot,t_{k+1}), \nabla\phi_p\right\rangle_{\ell^2(\Omega)} & = \frac{1}{T}\ln(1+\varepsilon e^{TM_0\lambda_p})w_p^{k+1} \\ & +\left\langle F\left(u^{\varepsilon,n}(\cdot,t_{k+1}), v^{\varepsilon,n}(\cdot,t_{k+1})\right), \nabla\phi_p\right\rangle_{\ell^2(\Omega)}, \\ \frac{\omega_p^{k+1}-\omega_p^k}{\Delta t} + \mathcal{D}_2\lambda_p \omega_p^{k+1} - K_2\left\langle v^{\varepsilon,n}(\cdot,t_{k+1}), \nabla\phi_p\right\rangle_{\ell^2(\Omega)} & = \frac{1}{T}\ln(1+\varepsilon e^{TM_0\lambda_p})\omega_p^{k+1} \\ & +\left\langle G\left(u^{\varepsilon,n}(\cdot,t_{k+1}), v^{\varepsilon,n}(\cdot,t_{k+1})\right), \nabla\phi_p\right\rangle_{\ell^2(\Omega)}. \end{cases}$$

Example 1. Let the final data (7) be given by

$$g_1(x) = g_2(x) = 0, \quad x \in (0,\pi), \tag{85}$$

which is perturbed as

$$g_i^\varepsilon(x) = \frac{\varepsilon\text{rand}(\text{size}(x))}{2\sqrt{\pi}}, \quad i=1,2,$$

where rand(size(x)) is a random vector, having values in $[0,1]$ (so that the assumptions (A_3) and (A_4) are satisfied), with the same size as the discretised vector x. Then, the problem (4), (7) and (9) admits the trivial solution $u_{\text{ex}}(x,t) = v_{\text{ex}}(x,t) = 0$. Choose $T=1$, $N_x = N_t = 100$ and $M_0 = 1.000002$. We have

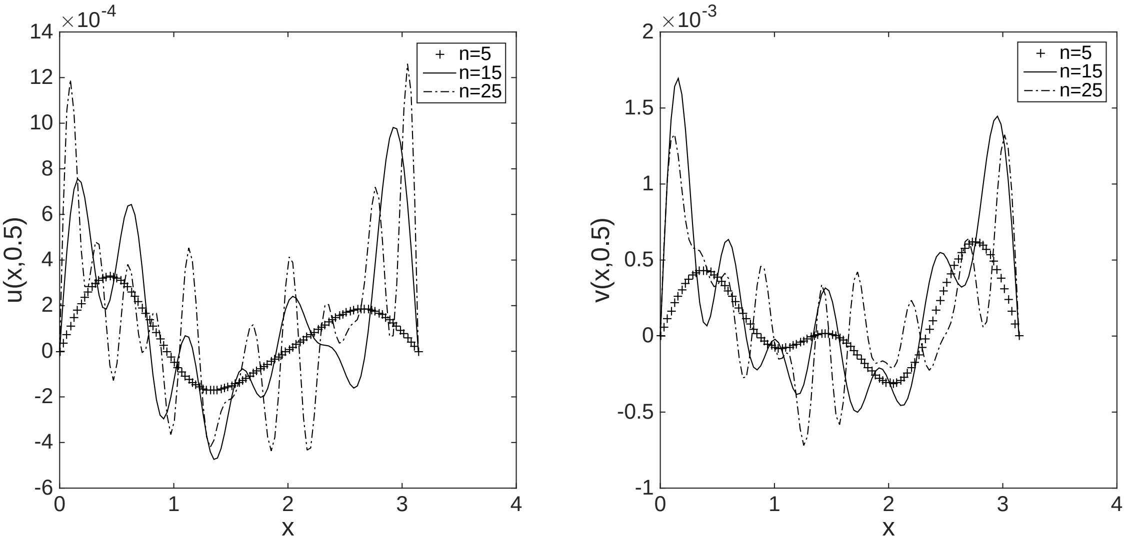

Figure 1: The regularized solutions at $t = 0.5$ for $\varepsilon = 10^{-4}$ with $n = 5$ (+++), $n = 15$ (——) and $n = 25$ ($- \cdot -$).

that $n = 26$ is the maximum number of Fourier coefficients that the term $\exp(TM_0\lambda_n)$ can be computed in Matlab program. The unregularized solution has been obtained to be very large and unbounded of $\mathcal{O}(10^{40})$ and therefore is not presented. In order to alleviate this instability, the regularized solution shown in Figure 1 illustrates the stabilizing effect with significantly reduced errors of the orders $\mathcal{O}(10^{-4})$ to $O(10^{-3})$ for $n \in \{5, 15, 25\}$ for $\varepsilon = 10^{-4}$ noise. The number of terms considered in the approximations (24) has also a regularization character which can be inferred from Figure 1, where the numerical solutions start to manifest instabilities for larger n.

Table 1 shows the errors ϵ_u and ϵ_v at various times $t \in \{0.25, 0.5, 0.75\}$ for various amounts of noise $\varepsilon \in \{10^{-5}, 10^{-4}, 10^{-3}\}$ with $n = 10$. From this table, we can observe that the errors at $t = 0.5$ are greater than those at $t = 0.75$ and smaller than those at $t = 0.25$, as expected since the instability increases as we proceed backwards in time. Furthermore, for smaller errors in input data, the results obtained are more accurate, which verifies the theoretical stability result in Theorem 2.

Table 1: The errors at $t \in \{0.25, 0.5, 0.75\}$ for various amounts of noise $\varepsilon \in \{10^{-5}, 10^{-4}, 10^{-3}\}$, with $n = 10$, for Example 1.

ε	$\epsilon_u(0.25)$	$\epsilon_v(0.25)$	$\epsilon_u(0.5)$	$\epsilon_v(0.5)$	$\epsilon_u(0.75)$	$\epsilon_v(0.75)$
10^{-5}	0.0046	0.0076	0.0004	0.0003	1.6E-5	1.9E-5
10^{-4}	0.0066	0.0120	0.0008	0.0014	1.1E-4	9.7E-5
10^{-3}	0.0148	0.0298	0.0038	0.0029	5.8E-4	8.9E-4

Example 2. In case that the analytical solution for the problem (4), (7) and (9) is not available, we first solve the forward problem given by the first equation in (3), (4) and (9) to numerically simulate the final

data (7) at $t = T =$, perturb them as

$$g_i^\varepsilon(x) = g_i(x)\left(1 + \frac{\varepsilon(2\text{rand}(\text{size}(x)) - 1)}{\|g_1\| + \|g_2\|}\right), \quad i = 1, 2,$$

and then use this noisy data to construct the regularized solution of the inverse problem. We use the fourth-order finite difference scheme to solve the forward problem as follows:

$$\begin{aligned}
\frac{u_i^{k+1} - u_i^k}{\Delta t} &- \mathcal{D}_1 \frac{-u_{i-2}^{k+1} + 16u_{i-1}^{k+1} - 30u_i^{k+1} + 16u_{i+1}^{k+1} - u_{i+2}^{k+1}}{12\Delta x^2} \\
&+ K_1 \frac{u_{i-2}^{k+1} - 8u_{i-1}^{k+1} + 8u_{i+1}^{k+1} - u_{i+2}^{k+1}}{12\Delta x} = F(u_i^k, v_i^k), \\
\frac{v_i^{k+1} - v_i^k}{\Delta t} &- \mathcal{D}_2 \frac{-v_{i-2}^{k+1} + 16v_{i-1}^{k+1} - 30v_i^{k+1} + 16v_{i+1}^{k+1} - v_{i+2}^{k+1}}{12\Delta x^2} \\
&+ K_2 \frac{v_{i-2}^{k+1} - 8v_{i-1}^{k+1} + 8v_{i+1}^{k+1} - v_{i+2}^{k+1}}{12\Delta x} = G(u_i^k, v_i^k).
\end{aligned}$$

We take $N_x = N_t = 100$, $n = 2$ and $M_0 = 5$, and consider two cases of input data, belonging or not to the Gevrey space (11).

Case 1. Let the initial data (first equation in (3)) be

$$u(x, 0) = \sin x, \quad v(x, 0) = \sin x, \tag{86}$$

which belong to the Gevrey space (11) for any $\sigma > 0$. The errors between the forward and backward solutions at $t \in \{0.25, 0.5, 0.75\}$ for $\epsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ are given in Table 2. The approximated forward and backward solutions are shown in Figures 2 and 3, and very good agreement can be observed.

Table 2: The errors at $t \in \{0.25, 0.5, 0.75\}$ for various amounts of noise $\varepsilon \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$, for Example 2, Case 1.

ε	$\epsilon_u(0.25)$	$\epsilon_v(0.25)$	$\epsilon_u(0.5)$	$\epsilon_v(0.5)$	$\epsilon_u(0.75)$	$\epsilon_v(0.75)$
10^{-4}	0.0218	0.0325	0.0122	0.0167	0.0063	0.0079
10^{-3}	0.0767	0.1087	0.0362	0.0599	0.0139	0.0244
10^{-2}	0.4683	0.4807	0.2439	0.2902	0.0976	0.1279
10^{-1}	0.7878	0.8779	0.4897	0.5962	0.2379	0.3107

Case 2. Let the initial data (first equation in (3)) be

$$u(x, 0) = \frac{1}{2}(\pi - x)x, \quad v(x, 0) = x \sin x, \tag{87}$$

which does not belong to a Gevrey space (11) for any $\sigma > 0$. The errors between the forward and backward solutions at $t \in \{0.25, 0.5, 0.75\}$ for $\epsilon \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ are given in Table 3. From this table it can be seen that the errors e_u and e_v decrease monotonically, as ϵ decreases.

6 Conclusion

In this paper, the coupled nonlinear system of parabolic PDEs (9) governing the predator-prey species interactions, with nonlinear sources, convection and diffusion depending nonlocally on the total average

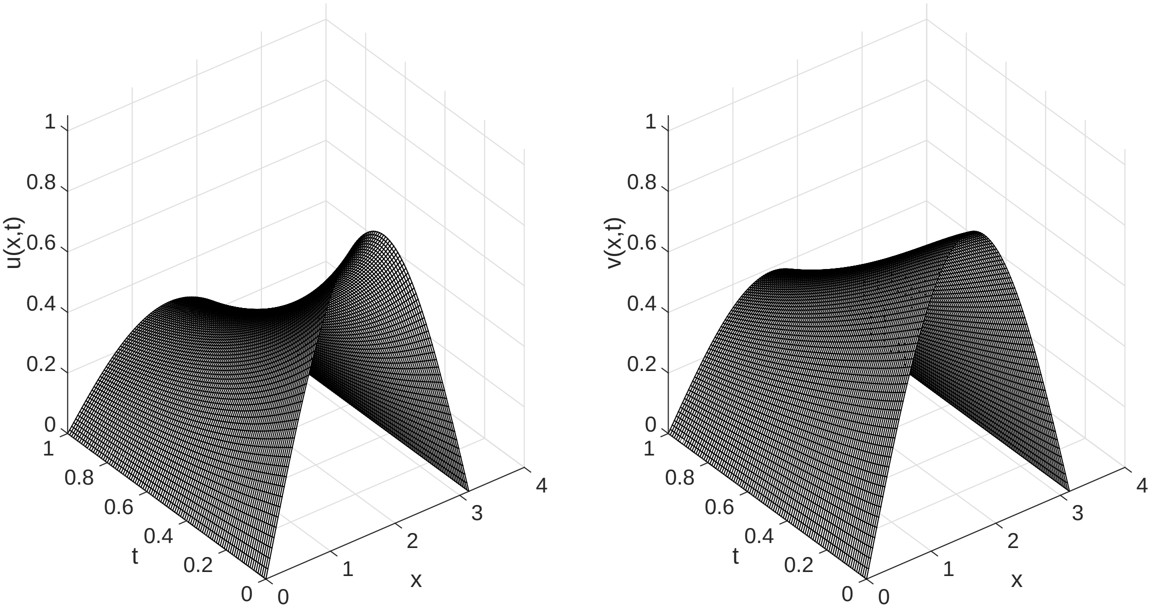

Figure 2: The solution to the forward problem corresponding to Example 2, Case 1.

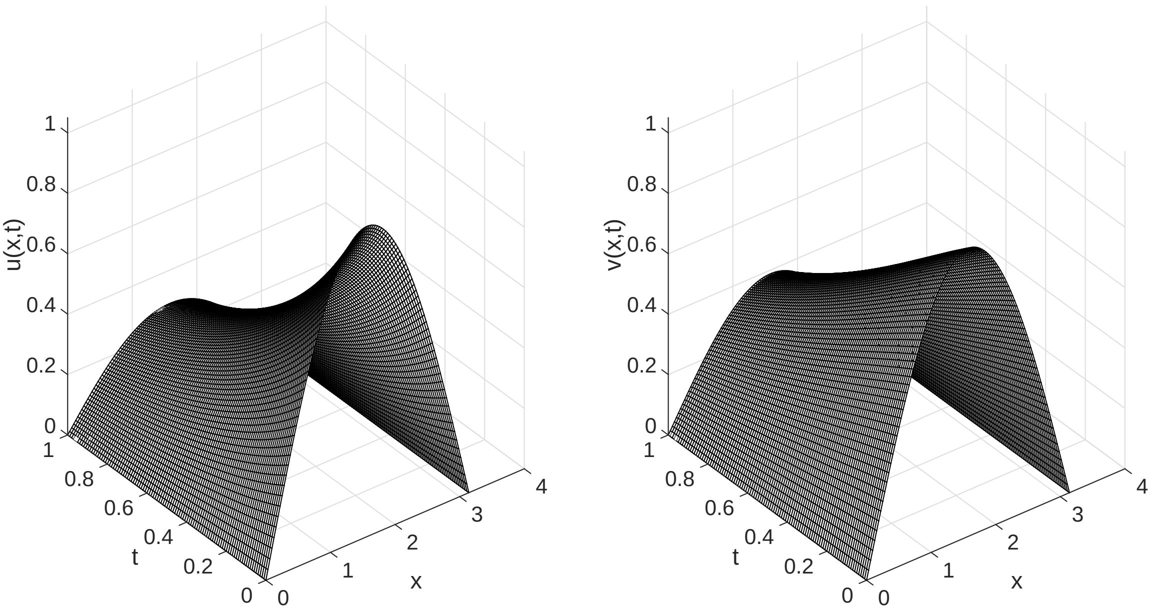

Figure 3: The regularized solution to the backward problem for $\varepsilon = 10^{-4}$ for Example 2, Case 1.

Table 3: The errors at $t \in \{0.25, 0.5, 0.75\}$ for various amounts of noise $\varepsilon \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$, for Example 2, Case 2.

ε	$\epsilon_u(0.25)$	$\epsilon_v(0.25)$	$\epsilon_u(0.5)$	$\epsilon_v(0.5)$	$\epsilon_u(0.75)$	$\epsilon_v(0.75)$
10^{-4}	0.0552	0.2899	0.0276	0.1133	0.0118	0.0420
10^{-3}	0.0907	0.3347	0.0405	0.1470	0.0154	0.0571
10^{-2}	0.5648	0.8172	0.2696	0.4820	0.0997	0.2128
10^{-1}	0.9312	1.4358	0.5382	0.9763	0.2447	0.5080

population densities has been investigated. In particular, the initial condition is not specified and has to be determined from the knowledge of the predator and prey populations at a later time. The resulting backward problem (4), (7) and (9) is ill-posed and we have proposed a new regularization method for solving it in a well-posed manner. Furthermore, we have established rigorously the error estimates (64) and (75) in the L^2- and H^1-norms, respectively.

The inclusion of stochasticity [10] in the backward model would make the model even more general and practical, but this investigation is deferred to a future work. Also, the inverse problem of determining the diffusivity/velocity $\mathcal{D}\left(\int_\Omega u\mathrm{d}x\right)$ (or their system counterpart $\mathcal{D}_1\left(\int_\Omega u\mathrm{d}x\right)$ and $\mathcal{D}_2\left(\int_\Omega v\mathrm{d}x\right)$ in (10) (or (6) or (9)) from the knowledge/measurement of the population density (or the flux) on a subportion $\Gamma \subset \partial\Omega$ for all time $t \in (0, T)$, would be of much interest, see [18] for a particular theoretical study.

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