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A MINIMUM-WAGE MODEL OF UNEMPLOYMENT AND GROWTH:

THE CASE OF A BACKWARD-BENDING DEMAND CURVE FOR LABOR

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We add a minimum wage and hence involuntary unemployment to a conventional two-sector model of a perfectly competitive economy with optimal saving and endogenous growth. Our resulting model highlights the possible case of a backward-bending demand curve for labor, along which a hike in the minimum wage might increase total employment. This possibility provides theoretical support for some controversial empirical studies, which challenge the textbook prediction of an inverse relationship between employment and the minimum wage. Our model also implies that a minimum-wage hike has negative implications for both the growth rate and lifetime utility.

JEL Classification Codes: E24, O41

Key Words: Optimal growth, Minimum wage, Learning by doing, Involuntary unemployment

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1. Introduction

This paper introduces a minimum wage and hence involuntary unemployment into the Ramsey (1928)-Cass (1965)-Koopmans (1965) two-factor model of optimal growth over an infinite horizon, as extended by Srinivasan (1964) to the two-good case.[1] To avoid an inherent problem of overdetermination, our minimum-wage model incorporates an endogenous rate of growth,[2] fuelled by technological improvements due to learning by doing. Within this framework, we investigate how a hike in the minimum wage affects employment, growth and welfare.

Contrary to conventional academic wisdom, our analysis shows that a minimum-wage hike can increase total employment, because of what may be called a backward-bending demand curve for labor. Notably, this outcome is possible even though we consider a perfectly competitive economy, in which firms are wage takers (not setters).[3] This result provides theoretical support for the controversial findings of Card and Krueger (1995), whose empirical

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[1] For an alternative approach to modelling growth with unemployment, see Cahuc and Michel (1996), who add a minimum wage to an overlapping-generations model with only one good (produced in two technologically different sectors) and three factors of production.

[2] As discussed below, the minimum wage fixes (via the conditions for profit maximization) the interest rate, which then determines the balanced-growth rate (from the household's Euler equation). For an alternative solution to the overdetermination problem in the absence of long-run growth, see Brecher, Chen and Yu (2013).

[3] It is well known that with wage-setting firms, minimum wages may increase employment, as first established by Stigler (1946) in the case of a monopsonist. Alternatively, if firms set wages optimally for efficiency-wage reasons, Manning (1995) shows that a minimum-wage hike may lead to a rise in employment. Flinn (2006) obtains this same result when the wage is determined instead by bargaining in the presence of search and matching frictions.

evidence challenges the simple textbook prediction of a negative relationship between employment and the minimum wage.[4]

We further demonstrate that the long-run rate of labor-augmenting technical progress is always negatively related to the minimum wage. This result is in consonance with Acemoglu (2010), who shows (among other things) how exogenous increases in the wage discourage innovation that raises the marginal product of labor.[5] However, his model does not allow him to analyze our case in which a higher wage might be associated with more employment.

One might reasonably conjecture that lifetime utility could rise if a hike in the minimum wage causes employment to increase. Our analysis, however, rejects this conjecture. Thus, any possible gain in employment must be outweighed by the definite contraction in the rate of growth, within the present representative-agent framework.[6]

Since the possibility of an employment-expanding hike in the minimum wage is our most surprising result, here is a brief preview of the underlying intuition. The wage hike lowers both the rate of return on capital and the growth rate, and hence may reduce the rate of interest net of growth. In this case, demand shifts from investment to consumption, thereby creating an excess demand for the consumption good and excess supply of the capital good. If the former good is relatively labor intensive, employment must increase to restore equilibrium.

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[4] Empirical surveys by Schmitt (2013) and Neumark, Salas and Wascher (2014) are respectively favorable and unfavorable to the Card-Krueger position.

[5] Whereas our assumptions include learning by doing and optimal saving/investment, he assumes that firms choose technology optimally, and that the stock of capital (or supply curve for this factor) is exogenously fixed.

[6] Of course, this framework allows us to consider only efficiency, not equity. As shown by Boadway and Cuff (2001) and Lee and Saez (2012), for example, a minimum wage might increase welfare for reasons related to interpersonal distribution.

Section 2 sets up the basic model, whose implications are explored in Sections 3 and 4 under alternative assumptions about the source of learning by doing. In section 5, we provide a numerical example to illustrate the possibility of a backward-bending demand curve for labor. Section 6 concludes with a summary of our main contributions.

2. Basic Model

Firms use capital and labor to produce capital itself and consumable output, which are called goods 1 and 2, respectively. The production function for each good is strictly quasi-concave, and exhibits constant returns to scale, with positive but diminishing marginal products. Assuming that firms maximize profits under perfect competition, we obtain the usual first-order conditions equating the marginal product of labor (capital) to the real wage (rental) rate. The two goods can be uniquely ranked according to their capital intensities per unit of labor, and there are no factor-intensity reversals. Although all variables (such as consumption, outputs, inputs, prices, assets, etc.) are functions of time, the time argument t is supressed for notational simplicity.

Along the production-possibility frontier, output of good i is given by $Q^i(p,K,L) \equiv Q^i(p,K,\lambda\ell)$ for $i = 1, 2$; where p stands for the relative price of good 1 in terms of 2; K and ℓ represent the economy's inputs of capital and labor, respectively; λ is the number of efficiency units per natural unit of labor; and $L \equiv \lambda\ell$. Given constant returns to scale, function Q^i is first-degree homogeneous in K and L. Thus, we can write,

$$Q^i(p,K,L) \equiv \lambda q^i(p,k,\ell) \equiv \lambda\ell\tilde{q}^i(p,\tilde{k}), \quad i = 1, 2, \tag{1}$$

where $k \equiv K/\lambda$ and $\tilde{k} \equiv k/\ell$.

To focus on the interesting situation in our two-good model, assume that the economy remains diversified in production (with both $Q^1 > 0$ and $Q^2 > 0$) throughout the analysis. Then, because a good's output responds positively to a rise in its relative price, $\partial Q^1 / \partial p > 0 > \partial Q^2 / \partial p$. Consequently, (1) implies that

$$q_p^1 > 0 > q_p^2, \; \tilde{q}_p^1 > 0 > \tilde{q}_p^2, \tag{2}$$

where subscripts of functions indicate partial derivatives (e.g., $q_p^1 \equiv \partial q^1 / \partial p$ and $\tilde{q}_k^1 \equiv \partial \tilde{q}^1 / \partial \tilde{k}$).

By the Rybczynski (1955) Theorem, $Q_K^1 > 0 > Q_K^2$ and $Q_L^1 < 0 < Q_L^2$ if good 1 is more capital intensive (per unit of labor) than good 2, whereas the signs of these derivatives are reversed under the opposite factor-intensity ranking. Thus, in light of (1),

$$q_k^i > 0 > q_k^j, \; \tilde{q}_k^i > 0 > \tilde{q}_k^j, \; q_\ell^i < 0 < q_\ell^j \text{ iff } K_i / L_i > K_j / L_j, \quad i, j = 1, 2, \tag{3}$$

where K_1 and K_2 are the inputs of capital used by industries 1and 2, respectively, while L_1 and L_2 are the corresponding inputs of labor in efficiency units.

We also have the following three well-known facts:

$$pq_p^1 + q_p^2 = p\tilde{q}_p^1 + \tilde{q}_p^2 = 0, \; q_k^1 + q_k^2 / p = \tilde{q}_k^1 + \tilde{q}_k^2 / p = r, \; pq_\ell^1 + q_\ell^2 = w, \tag{4}$$

where w is the real wage rate in terms of good 2 per efficiency unit of labor, and r represents the interest rate, which equals the marginal product of capital in sector 1. The first fact in (4) holds because the economy operates on the production-possibility frontier at the point where the marginal rate of transformation equals the product-price ratio. The remaining two facts stem from intersectoral equalization of each input's marginal value product.

According to the Stolper-Samuelson (1941) Theorem, a rise in the relative price of a good is associated with a rise in the real return to the factor used intensively in this good, and a fall in the other factor's real return. Thus,

$$dp / dw \lesseqgtr 0, \; dr / dp \gtreqless 0 \text{ iff } K_1 / L_1 \gtreqless K_2 / L_2. \tag{5}$$

Consumer behavior is consistent with that of a representative household, which competitively maximizes the present discounted value of lifetime utility, subject to a budget constraint. Specifically, this household maximizes

$$V \equiv \int_0^\infty e^{-\rho t} \ln C dt , \tag{6}$$

subject to

$$\dot{X} = rX + w\lambda\ell / p - C / p , \tag{7}$$

where ρ is the constant rate of time preference; C denotes total consumption (of good 2); the instantaneous utility function is $\ln C$, for simplicity of exposition;[7] X stands for total wealth in terms of good 1; and dots over variables indicate time derivatives (e.g., $\dot{X} \equiv \mathrm{d}X / \mathrm{d}t$).

The only control variable for this maximization problem is C at each point in time. Although the supply of labor is perfectly inelastic (with no disutility of effort), the household takes ℓ as given, because of involuntary unemployment due to a binding minimum-wage constraint that fixes the value of w. This value then determines p and hence r, via Samuelson's (1949) one-to-one correspondence between product and factor prices. Since the endowment of labor is normalized to equal 1 by choice of units, the rate of unemployment is $1-\ell$.

Defining $x \equiv X / \lambda$ and $c \equiv C / \lambda$, we can restate the household's problem as maximizing

$$V \equiv \int_0^\infty e^{-\rho t} \ln c dt + \int_0^\infty e^{-\rho t} \ln \lambda dt , \tag{8}$$

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[7] Our main results hold qualitatively for any utility function of the isoelastic form $(C^{1-\theta} - 1)/(1-\theta)$; where the constant θ is greater than zero, and equals the elasticity of the marginal utility of consumption, as well as the intertemporal elasticity of substitution. As $\theta \to 1$, this function approaches $\ln C$, which is the case that we adopt to simplify the exposition. For reasons explained by Barro and Sala-i-Martin (1995, p. 64), an isoelastic type of utility function is commonly assumed for consistency with a balanced-growth path.

subject to

$$\dot{x} = (r - g)x + w\ell / p - c / p\,; \tag{9}$$

where $g \equiv \dot{\lambda} / \lambda$, which is the economy's rate of growth due to technical progress of the labor-augmenting (Harrod-neutral) variety. The current-value Hamilton for this maximization problem is given by

$$H = \ln c + \ln\lambda + \mu[(r - g)x + w\ell / p - c / p]\,, \tag{10}$$

where the co-state variable μ can be interpreted as the shadow price of assets. The necessary conditions for a maximum include the following equations:

$$\partial H / \partial c = 1 / c - \mu / p = 0\,, \tag{11}$$

$$\dot{\mu} = \rho\mu - \partial H / \partial x = \mu(\rho + g - r)\,, \tag{12}$$

in addition to the $\dot{x}$ constraint (9), as well as the usual initial and transversality conditions.

Since output of good 2 is fully consumed,

$$c = q^2(p,k,\ell) \equiv \ell\tilde{q}^2(p,\tilde{k})\,. \tag{13}$$

From this equation and (11),

$$\ell = Z(p,k,\mu)\,,\ Z_p = (1/\mu - q_p^2)/q_\ell^2\,,\ Z_k = -q_k^2 / q_\ell^2\,,\ Z_\mu = -p/\mu^2 q_\ell^2\,. \tag{14}$$

We also have

$$\dot{K} = \lambda q^1(p,k,\ell)\,, \tag{15}$$

because all output of good 1 adds to the stock of capital. Then, differentiating $k\ (\equiv K/\lambda)$ with respect to time (while recalling that $g \equiv \dot{\lambda}/\lambda$), use (15) to obtain

$$\dot{k} = q^1(p,k,\ell) - gk\,. \tag{16}$$

Assume that output-based learning by doing occurs in one industry, and spreads automatically to the other industry, thereby causing (the economy-wide) λ to increase over time. Specifically, $\dot{\lambda}$ equals either $\lambda q^1(p,k,\ell)$ or $\lambda q^2(p,k,\ell)$, as learning occurs in the capital- or consumption-good sector, respectively. The first case is equivalent to learning by investing, an idea expounded originally by Arrow (1962). The second case could be called learning by consuming, in the spirit of Leibenstein's (1957) hypothesis that a worker's productivity depends on consumption for nutritional reasons. We now consider each of these two possibilities in turn.

3. Learning by Producing the Capital Good

If learning by doing occurs in the capital-good industry, $\dot{\lambda} = \lambda q^1(p,k,\ell)$ and thus

$$g = q^1(p,k,\ell) = \ell \tilde{q}^1(p,\tilde{k}). \tag{17}$$

This assumption about the growth rate allows us to rewrite (12) as

$$\dot{\mu} = \mu[\rho + q^1(p,k,\ell) - r], \tag{18}$$

and (16) as

$$\dot{k} = q^1(p,k,\ell)(1-k). \tag{19}$$

In steady-state equilibrium, $\dot{\mu} = \dot{k} = 0$. Then, (18) and (19) imply the following two equations, respectively:

$$r = \rho + q^1(p,k,\ell), \tag{20}$$

and (given $q^1 > 0$ under our above assumption about diversified production)

$$k = 1. \tag{21}$$

To determine the relationship between the minimum wage and national employment, substitute (21) into (20), and differentiate the resulting equation totally with respect to *p*, thereby yielding

$$d\ell / dp = [dr / dp - q_p^1(p,1,\ell)] / q_\ell^1(p,1,\ell)\,. \tag{22}$$

Multiply both sides of (22) by dp / dw, and use (3) with (5) to derive

$$d\ell / dw > 0 \text{ iff } dr / dp > q_p^1(p,1,\ell)\,. \tag{23}$$

In other words, we have the following result.

PROPOSITION 1: *A hike in the minimum wage increases the steady-state level of employment if and only if a rise in the relative price of the capital good (ceteris paribus) has a smaller impact on this good's output than on the rental rate*.

The necessary and sufficient condition in (23) may be satisfied if good 1 is capital intensive, since (2) and (5) imply that q_p^1 and dr / dp are both greater than zero in this case.[8] However, under the opposite factor-intensity ranking, dr / dp is less than zero, in which case the (necessary and sufficient) condition in (23) cannot be satisfied. Thus, Proposition 1 describes a scenario that is possible only if the capital good is capital intensive.

For an intuitive understanding of Proposition 1, suppose that good 1 is relatively intensive in capital. Then, a minimum-wage hike lowers *p*, leading to a fall in *r* and—at the initial level of employment—a drop in $g\,[= q^1(p,1,\ell)]$. If *r* falls more than *g*, there is a reduction in the net rate of return $r - g$ on capital per efficiency unit.[9] This reduction tends to discourage saving and hence encourage consumption. In fact, with employment held constant temporarily, *c* rises more

[8] As the elasticity of technical substitution (along an isoquant) approaches zero for both goods, so does q_p^1, but not dr / dp.

[9] Since a rise in λ tends to lower $k\ (\equiv K / \lambda)$, we can interpret $g\ (\equiv \dot{\lambda} / \lambda)$ as a depreciation rate, and hence $r - g$ as the net rate of interest.

than $q^2(p,1,\ell)$, creating an excess demand for good 2.[10] To clear this excess demand for the labor-intensive good, employment must rise.

To determine the relationship between *w* and *c*, begin by using (12) and (16) with (1), while setting $\dot{\mu} = \dot{k} = 0$ to obtain $\tilde{q}^1(p,\tilde{k}) = (r-\rho)\tilde{k}$. Then, differentiate this equation totally with respect to *p*, thereby yielding $d\tilde{k}/dp = (\tilde{q}^1_p - \tilde{k}dr/dp)/(g-\tilde{q}^1_k)$. Thus, differentiating $\tilde{q}^2(p,\tilde{k})$ totally with respect to *p*—while using (4), (17) and (20)—confirm that

$d\tilde{q}^2/dp = (\rho p\tilde{q}^1_p - \tilde{k}\tilde{q}^2_k dr/dp)/(g-\tilde{q}^1_k)$.

The numerator of this expression is positive because $\tilde{q}^1_p > 0$ by (2), while (3) and (5) imply that $\tilde{q}^2_k$ and dr/dp are always opposite in sign, no matter what the factor-intensity ranking of the two goods. The denominator is positive or negative if good 1 is intensive in labor or capital, respectively.[11] Thus, from the sign of dp/dw in (5), it is clear that

$$d\tilde{q}^2/dw > 0. \tag{24}$$

This condition and (13) imply the following result relating dc/dw and $d\ell/dw$.

PROPOSITION 2: *If (but not only if) a hike in the minimum wage raises the steady-state level of employment, there is a corresponding rise in consumption per efficiency unit.*

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[10] More formally, as $r-g-\rho$ falls below 0, we have $\dot{\mu} > 0$ by (12), hence $\dot{c} < 0$ by (11), and thus $\dot{x} < 0$ because (by well-known reasoning) consumption depends positively on wealth. With $\dot{x} < 0$, $\dot{X}/\lambda\,(=\dot{x}+xg) < q^1$ (since $g = q^1$ and initially $x = k = 1$), implying $\dot{X} < Q^1$. In light of this inequality and the instantaneous budget constraint ($p\dot{X} + C = pQ^1 + Q^2$), clearly $C > Q^2$, indicating an excess demand for good 2.

[11] In the former case, $\tilde{q}^1_k < 0$ by (3). In the latter case, set $\dot{\mu} = 0$ in (12) and use (4), to yield $\tilde{q}^1_k = r - \tilde{q}^2_k/p > r = \rho + g > g$; where the first inequality follows from the fact that now $\tilde{q}^2_k < 0$.

To see the minimum wage's impact on the rate of growth, set $\dot{\mu} = 0$ in (12) and use this equation to obtain $dg / dw < 0$, since $dr / dw = (dr / dp)dp / dw < 0$ by (5). In other words, we have the following result.

PROPOSITION 3: *A hike in the minimum wage unambiguously lowers the steady-state rate of growth.*

To show that steady-state equilibrium is saddle-path stable, consider Figure 1, which is the phase diagram for the dynamic system of (18) and (19). For the sake of concreteness, suppose that the capital good is capital intensive, although the stability analysis would be essentially the same under the opposite factor-intensity ranking.

The schedule for $\dot{k} = 0$ is a horizontal line at a height equal to 1, because of (21). The vertical arrows of motion point toward this line, to reflect the fact that $\dot{k} \overset{>}{\underset{<}{=}} 0$ as $k \overset{<}{\underset{>}{=}} 1$, in accordance with (19). By the following argument, the (generally non-linear) schedule for $\dot{\mu} = 0$ is negatively sloped, and is associated with horizontal arrows that point away from it.

To determine the sign of this schedule's slope, differentiate (20) totally with respect to μ (holding p and hence r constant) while using (14) to obtain $dk / d\mu = (pq_\ell^1 / \mu^2 q_\ell^2) / (q_k^1 - q_\ell^1 q_k^2 / q_\ell^2)$. With this equation, use (4) to find that $dk / d\mu = (w - pq_\ell^1) pq_\ell^1 / (wq_k^1 - rpq_\ell^1)\mu^2 q_\ell^2 < 0$; where this inequality follows from the signs of the Rybczynski derivatives in (3). In other words the schedule for $\dot{\mu} = 0$ is negatively sloped.

Starting from any point on this schedule, an increase in μ (at constant k) would lower ℓ by (14) and (3), thus raising output of capital-intensive good 1. The resulting increase in q^1 would make $\dot{\mu} > 0$, in accordance with (18). Therefore, the horizontal arrows point away from the schedule for $\dot{\mu} = 0$.

Beginning at any arbitrary point in Figure 1, μ jumps instantaneously at time 0 to reach the saddle path, represented by the dashed curve (generally non-linear). Then, the economy moves continuously along this path toward the steady-state equilibrium, which corresponds to point S at which the schedules for $\dot{k} = 0$ and $\dot{\mu} = 0$ intersect each other.

Let the schedules in Figure 1 correspond to the situation after a hike in the minimum wage. Suppose also that the pre-hike economy is in steady-state equilibrium at point A, which must be on the $\dot{k} = 0$ schedule, whose position (at the constant height of $k = 1$) is independent of *w*. Then, the wage hike causes the economy to jump (via an instant change in μ) to the new equilibrium at point S. We therefore have the following result.

Proposition 4: *A hike in the minimum wage causes the economy to jump immediately from the initial to the new steady-state equilibrium.*

Thus, the corresponding changes in total employment, aggregate consumption and economic growth (as described by Propositions 1, 2 and 3, respectively) all occur simultaneously with the wage hike, without any transitional dynamics.

A possible rise in *c* (by Proposition 2) and definite fall in *g* (by Proposition 3) would affect lifetime utility positively and negatively, respectively, since (8) can be rewritten as

$$V \equiv \int_0^\infty e^{-\rho t} \ln c dt + \int_0^\infty e^{-\rho t} \ln(\lambda_0 e^{gt}) dt = \ln \lambda_0 / \rho + \ln c / \rho + g / \rho^2, \qquad (25)$$

where *c* and *g* remain constant at their steady-state levels, while λ_0 represents the value of λ at the instant when the wage hike occurs. However, regardless of whether the decrease in g is accompanied by a fall or rise in c, the Appendix shows that $dV/dw < 0$ unambiguously. We thus have the following result.

PROPOSITION 5: *A hike in the minimum wage definitely lowers the level of lifetime utility.*

4. Learning by Producing the Consumption Good.

For this case, replace (17) by

$$g = q^2(p,k,\ell) = \ell\tilde{q}^2(p,\tilde{k}). \tag{26}$$

Proposition 3 and condition (24) still hold, because they are derived without the use of (17). Thus, a minimum-wage hike lowers ℓ by (26) and reduces c by (13), contrary to Propositions 1 and 2, respectively.

Despite the replacement of (17) by (26), it is straightforward to show that steady-state equilibrium remains saddle-path stable, although the schedules for $\dot{k} = 0$ and $\dot{\mu} = 0$ become negatively sloped and vertically linear, respectively. These schedule modifications imply that a hike in the minimum wage changes the steady-state level of k. Thus, rather than switching instantly between steady-state equilibria, the economy first jumps from the initial equilibrium to the new saddle path, and then follows this path over time toward the new equilibrium.

In view of this dynamic process of adjustment, Proposition 4 no longer holds. Nevertheless, it is possible (but tedious) to verify that Proposition 5 remains valid if the wage hike is small. Whether this proposition similarly extends for a large hike is a technically challenging question for future research. The challenge arises from the facts that the transition between steady states is not instant in the present (unlike the previous) case, and the precise shape of the saddle path is difficult (or impossible) to characterize outside the neighborhood of steady-state equilibrium.

5. Numerical Analysis

This section provides a numerical example of the case in which a minimum-wage hike increases the level of employment, assuming (for reasons suggested above) that technological progress occurs through learning by doing in the capital-good sector, and that the consumption good is relatively labor intensive. An important by-product of this exercise is to demonstrate the existence of a unique steady-state equilibrium in our model, for each value of the wage within a specified range. Starting from a position of full employment in the present example, successive hikes in the minimum wage first decrease but then increase employment, illustrating what we call a backward-bending demand curve for labor.[12]

To construct our example, we adopt a CES type of production function for each industry. More specifically, suppose that

$$Y_i = [a_i K_i^{\sigma_i} + \left(1 - a_i\right)(\lambda \ell_i)^{\sigma_i}]^{1/\sigma_i} = \lambda \ell_i (a_i \tilde{k}_i^{\sigma_i} + 1 - a_i)^{1/\sigma_i}, \quad i = 1, 2, \tag{27}$$

where Y_i and $\ell_i (\equiv L_i / \lambda)$, respectively, denote output produced and (natural units of) labor employed by sector *i*; $\tilde{k}_i \equiv K_i / \lambda \ell_i$; a_i and σ_i are constants; and $1/(1 - \sigma_i)$ is the elasticity of substitution between capital and labor in production of good *i*.

Given constant returns to scale and perfect competition, we can think of a representative firm in each industry. Subject to (27), this firm chooses K_i and ℓ_i to maximize profits, given by

$$\pi_i = p_i Y_i - \omega \lambda \ell_i - r K_i, \quad i = 1, 2, \tag{28}$$

[12] Of course, this curve is a general-equilibrium (rather than Marshallian) one.

where $\omega \equiv w/p$,which represents the real wage in terms of good 1;[13] p_i is the nominal price of good *i*; and $p_1 \equiv 1$ by choice of units (implying that ω and *r*, respectively, are also equal to the wage and rental rates in nominal terms). The first-order conditions for profit maximization are

$$p_i a_i \tilde{k}_i^{\sigma_i - 1}(a_i \tilde{k}_i^{\sigma_i} + 1 - a_i)^{1/\sigma_i - 1} = r\,, \quad i = 1,2\,, \tag{29}$$

$$p_i(1-a_i)\,(a_i \tilde{k}_i^{\sigma_i} + 1 - a_i)^{1/\sigma_i - 1} = \omega\,, \quad i = 1,2\,. \tag{30}$$

Using (29) and (30) for sector 1 (while recalling that $p_1 \equiv 1$ by normalization), obtain the following two equations, respectively:

$$r = a_1[a_1 + (1-a_1)/\tilde{k}_1^{\sigma_1}]^{1/\sigma_1 - 1}, \tag{31}$$

$$\tilde{k}_1^{\sigma_1} = \{[\omega/(1-a_1)]^{\sigma_1/(1-\sigma_1)} - (1-a_1)\}/a_1\,. \tag{32}$$

Combining (29) and (30) for sector 2, verify that

$$\tilde{k}_2 = [(\omega/r)a_2/(1-a_2)]^{1/(1-\sigma_2)}\,. \tag{33}$$

In light of (31) – (33), the rental rate and capital/labor ratios in both sectors are each a function of ω.

From growth definition (17), Euler equation (20) and production function (27) for sector 1, this sector's employment is

$$\ell_1 = (r-\rho)/(a_1 \tilde{k}_1^{\sigma_1} + 1 - a_1)^{1/\sigma_1}, \tag{34}$$

which is a function of ω (via *r* and $\tilde{k}_1$). Then, ω also determines each of the remaining

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[13] Although it is natural to specify the minimum wage in terms of the consumption good (as in previous sections), here both factor rewards (ω and *r*) are expressed in terms of the same (capital-good) units, for expositional convenience. Since the Stolper-Samuelson Theorem implies that $d\omega/dw > 0$, ω can be used as a proxy for *w*, without loss of generality.

variables: $k_1 \equiv K_1 / \lambda = \tilde{k}_1 \ell_1$; $k_2 \equiv K_2 / \lambda = 1 - k_1$, given (21); $\ell_2 = k_2 / \tilde{k}_2$; and $\ell = \ell_1 + \ell_2$. Thus, for any value of ω, steady-state equilibrium (if it exists) is unique.

Figure 2 illustrates the wage-employment relationship for the following parameter values: $\rho = 0.04$, $a_1 = 0.6$, $a_2 = 0.2$, $\sigma_1 = -1.5$ and $\sigma_2 = -25$.[14] As this illustration confirms, a backward-bending demand curve for labor is indeed possible.[15] Along the positively sloped portion of this curve, a hike in the minimum wage leads to an increase in employment for the economy as a whole.

6. Conclusion

Our main contribution is a new mechanism whereby a minimum-wage hike can stimulate total employment and hence reduce involuntary unemployment. This mechanism operates within a standard two-sector model of optimal saving/investment, with endogenous growth due to learning by doing. In this model, a hike in the minimum wage may reduce the net rate of interest adjusted for growth, thereby creating an excess demand for the consumption good. If this good is relatively labor intensive, total employment must rise to restore equilibrium, along a backward-bending demand curve for labor. Regardless of what actually happens to employment,

[14] These particular values are chosen for diagrammatic clarity only, without full-blown calibration, which is beyond the scope of the present paper.

[15] Below this curve's lower bound (where $\ell = 1$), the minimum wage is not a binding constraint. Above the curve's upper bound, the interest rate would be less than the rate of time preference, implying (absurdly) a negative output of good 1 in (20). Although Minhas (1962) shows that factor-intensity reversal must occur at some wage/rental ratio when constant elasticities of technical substitution differ between industries, the first good in the present example is always more capital intensive than the second between the above-mentioned bounds.

we also show that the minimum-wage hike has negative implications for both the growth rate and lifetime utility.

Appendix

From (25),

$$dV/dp = (dc/dp)/c\rho + (dg/dp)/\rho^2. \tag{A1}$$

After differentiating (13) totally with respect to *p*, use (4), (21) and (22) to obtain

$$dc/dp = (q_\ell^2 dr/dp - wq_p^1)/q_\ell^1. \tag{A2}$$

From (17) and (20),

$$dg/dp = dr/dp. \tag{A3}$$

Substituting (A2) and (A3) into (A1) yields

$$dV/dp = [(q_\ell^2/q_\ell^1 q^2 + 1/\rho)dr/dp - wq_p^1/q_\ell^1 q^2]/\rho, \tag{A4}$$

after using (13).

Note that

$$q_\ell^2/q_\ell^1 = (q^2/k_2)/(q^1/k_1), \tag{A5}$$

because the ratio of Rybczynski derivatives for labor equals the ratio of average products for capital.[16] Since capital is fully utilized, (21) can be rewritten as

$$k_1 + k_2 = 1. \tag{A6}$$

Substitute (A5) and (A6) into (A4), multiply both sides of the resulting equation by dp/dw, and use (20) to verify that

$$dV/dw = \{[1 - r/(q^1/k^1)](dr/dw)/\rho k_2 - wq_p^1(dp/dw)/q_\ell^1 q^2\}/\rho. \tag{A7}$$

Since the average product of each factor exceeds its marginal product,

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[16] See Brecher's (1974) discussion of the slope of the well-known Rybczynski line, introduced earlier by Mundell (1957).

$$q^1 / k^1 > r\,. \tag{A8}$$

From (5),

$$dr / dw < 0\,. \tag{A9}$$

It is also true that

$$q_\ell^1 dp / dw > 0\,, \tag{A10}$$

from (3) and (5).

Using (A7) - (A10) and (2), we see that

$$dV / dw < 0\,. \tag{A11}$$

This confirms Proposition 5.

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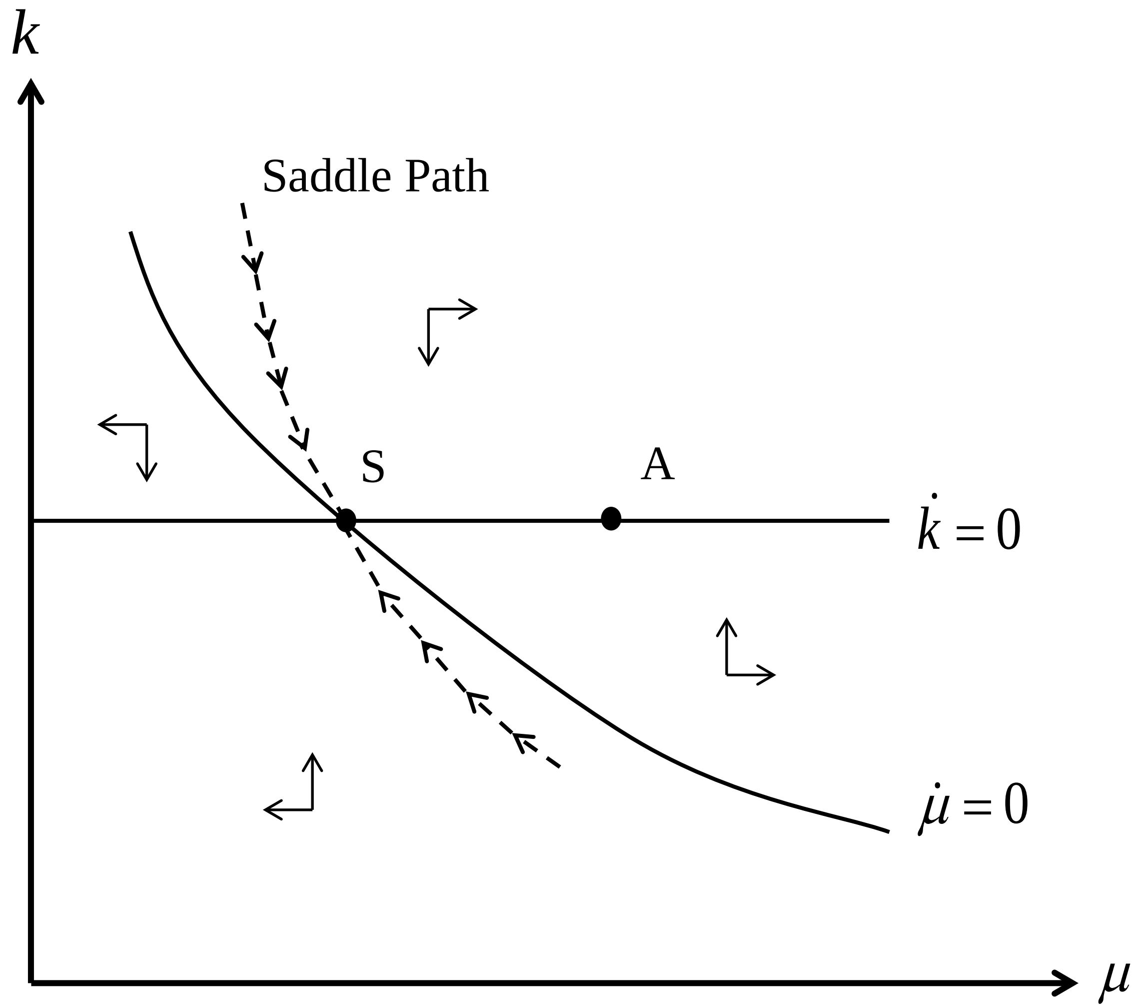

Figure 1: Phase Diagram

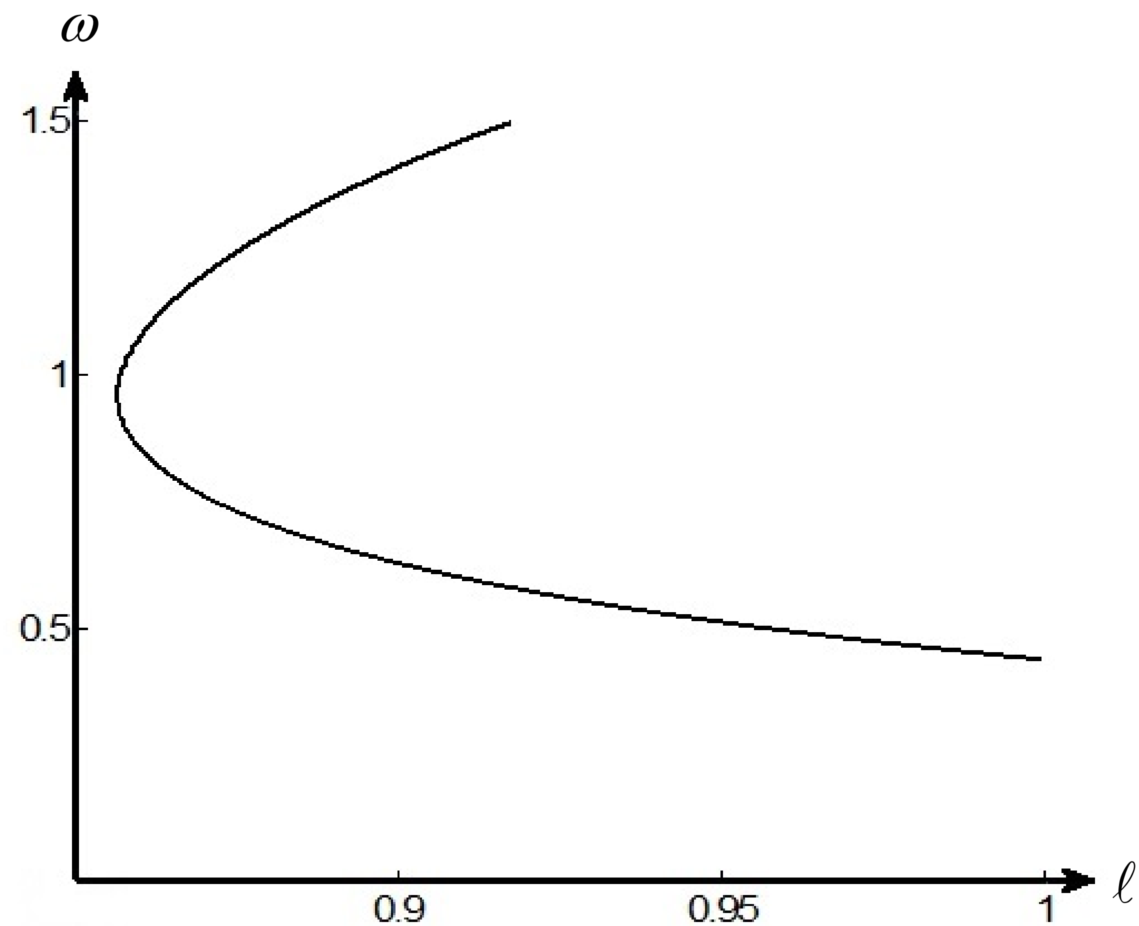

Figure 2: Backward-Bending Demand Curve for Labor