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# ON LIOUVILLE'S TWELVE SQUARES THEOREM

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## Abstract

A simple proof is given of a formula for the number of representations of a positive integer as the sum of twelve squares.

## 1. Introduction

Let $q$ be a complex variable with $|q| < 1$. Following [1, p. 6] we set

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{1.1}$$

Then, as in [1, p. 120], we set

$$x := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z := \varphi^2(q). \tag{1.2}$$

Let $\mathbb{N}$ denote the set of positive integers. For $k, n \in \mathbb{N}$ we define

$$\sigma_k(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k. \tag{1.3}$$

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If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. The Eisenstein series $E_{2k}(q)$ is defined by

$$E_{2k}(q) := 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{1.4}$$

where $\zeta$ denotes the Riemann zeta function. For brevity we set

$$R(q) := E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{1.5}$$

It is shown in [1, pp. 127, 128] that

$$R(q) = (1 - 33x - 33x^2 + x^3) z^6 \tag{1.6}$$

and

$$R(q^4) = \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right) z^6. \tag{1.7}$$

Ramanujan's discriminant function $\Delta(q)$ is defined by

$$\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}. \tag{1.8}$$

From [4, eq. (26), p. 392], we have

$$\Delta(q^2) := \frac{1}{256} x^2 (1-x)^2 z^{12}. \tag{1.9}$$

We define integers $b(n)$ $(n \in \mathbb{N})$ by

$$\sum_{n=1}^{\infty} b(n) q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^{12} \tag{1.10}$$

so that

$$\sum_{n=1}^{\infty} b(n) q^n = \Delta(q^2)^{1/2} = \frac{1}{16} x(1-x) z^6. \tag{1.11}$$

We make use of (1.1), (1.2), (1.6), (1.7), (1.10) and (1.11) to determine a formula for the number $r_{12}(n)$ of representations of $n$ $(n \in \mathbb{N})$ as a sum of twelve squares, that is, for the quantity

$$r_{12}(n) := \text{card}\{(x_1, \ldots, x_{12}) \in \mathbb{Z}^{12} \mid n = x_1^2 + \cdots + x_{12}^2\},$$

where $\mathbb{Z}$ denotes the set of all integers. We prove

**Theorem.** *Let* $n \in \mathbb{N}$. *Then*

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16b(n).$$

## 2. Proof of Theorem

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} r_{12}(n)q^n &= \varphi^{12}(q) \\
&= z^6 \\
&= -\frac{1}{63}(1 - 33x - 33x^2 + x^3)z^6 \\
&\quad + \frac{64}{63}\left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right)z^6 + x(1-x)z^6 \\
&= -\frac{1}{63}R(q) + \frac{64}{63}R(q^4) + 16\sum_{n=1}^{\infty} b(n)q^n \\
&= 1 + \sum_{n=1}^{\infty}(8\sigma_5(n) - 512\sigma_5(n/4) + 16b(n))q^n.
\end{aligned}$$

Equating coefficients of $q^n$ $(n \in \mathbb{N})$, we obtain the asserted formula for $r_{12}(n)$.

From (1.10) we see that

$$b(n) = 0, \quad \text{if } n \equiv 0 \pmod 2. \tag{2.1}$$

Hence

$$r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4), \quad \text{if } n \equiv 0 \pmod 2. \tag{2.2}$$

This result was stated by Liouville [3] in a slightly different form. For other formulae for $r_{12}(n)$, see [2].

## References

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