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An Oil-Driven Endogenous Growth Model

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An Oil-Driven Endogenous Growth Model

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Abstract

In this paper we show that the abundance of a natural resource such as oil need not present a curse for the domestic economy, dooming the non-oil sector to secondary status and a long period of stagnation and decline. Rather oil revenues can themselves be source of economy wide growth. What is required is the judicious use of oil revenues, in our case the channelling oil revenues into government capital/infrastructure that will complement private capital. We show that in such cases economy wide growth need not arise at the expense of other government services. In the steady state government consumption can grow in line with private consumption, in our case at the rate dictated by household preferences.

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I.   Introduction

In this paper we are concerned with the effects of oil revenues on the non-oil domestic economy when the resource is owned by the government and affects the economy through the government budget constraint. There are at present a large number of papers that consider the Dutch Disease aspects of a rise in oil revenues through the contractionary effect this has on the non-oil sector (see, for example, Corden, 1984; van Wijnbergen, 1984a, 1984b). In addition there are papers that focus specifically on the role of oil revenue in relation to government expenditure and its interaction with monetary policy through the budget and/or balance of payments constraint (see, for example, Tekin-Koru and Ozmen, 2003; Kavand and Ferris, 2012). In this paper we investigate another dimension of having oil revenues appear in the government budget constraint by considering the role it can play in financing public capital. Turnovsky (1997), Cassou et. al. (1998) and Gomez (2004) have already established models where public capital is an important aspect of growth. We supplement this analysis by considering the optimal choice for allocating oil revenues (which are owned entirely by the government) to either consumption or investment spending and the implications of that choice for growth.

II.   Model

Consider an economy with the following aggregate production function for non-oil output:

$$y_t = A\left(\epsilon_t k_{gt}\right)^{\alpha} k_t^{1-\alpha} = A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha} k_t \quad or \quad \frac{y_t}{k_t} = A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha} \quad 0 < \alpha < 1 \tag{1}$$

Here y_t represents non-oil output, and k_t and k_{gt} represent, respectively, private and government capital. The production function is assumed to be linear homogeneous and thus can be written as an augmented AK model. With this production function, the constancy of $\frac{\epsilon_t k_g}{k}$ in the steady state means that the ratio $\frac{y}{k}$ will also be constant. Hence in the steady state $\epsilon k_g, k$ and y can all grow at a balanced rate and an endogenous growth model can be developed on that basis.

In this economy government capital is used in both the oil and the non-oil sectors. Hence ϵ in equation (1) above represents the share of government capital used outside the oil sector to enhance non-oil output, y_t. The non-oil production function exhibits constant return to scale in private, k_t, and government capital, $\epsilon_t k_{gt}$ while increases in public capital relative to private increase non-oil output at a diminishing rate. To focus on capital accumulation we assume no population growth and normalize its size to 1. This eliminates scale effects from the model and allows us to talk interchangeably of total or per capita values. Implicitly all labour is used in the non-oil sector and oil can only be exploited by using government capital.

Next we define the change in the government capital stock as:

$$\dot{k_{gt}} = h_t \tau_t y_t + \theta_t or_t, \tag{2}$$

where h_t is the share of government tax revenue and θ_t is the share of oil revenues, or_t, going into government investment. For convenience, we assume the depreciation rate on capital is zero. In (2)

government investment is an increasing function of both non-oil output, y_t, and oil revenues, or_t. In turn government investment is used either to supplement private production (through public infrastructure, for example) or to enhance oil output. Note that with $\theta_t = 0$ we get back to Turnovsky (1997) where public investment is a proportion only of non-oil output, i.e., $i_t = \dot{k}_{gt} = h_t \tau_t y_t$. At the other extreme $h_t = 0$ and public investment depends only on oil revenues, i.e.,

$$i_{gt} = \dot{k}_{gt} = \theta_t or_t.$$

Next we define the representative individual's utility function as:

$$U(t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{g_t^{1-\varphi}}{1-\varphi} \qquad \text{for } \sigma, \varphi \neq 1,$$

$$= \ln c_t + \ln g_t \qquad \text{for } \sigma, \varphi = 1 \tag{3}$$

Given that government consumption arrives as a non-tradeable good, the representative household's budget constraint can be formulated as:

$$\dot{a}_t = (1 - \tau_t)w_t + (1 - \tau_t)r_t a_t - c_t \tag{4}$$

where a_t represents the level of financial assets held by the representative household; w_t, the wage rate; r_t, the rate of interest; and τ_t, the tax rate on income. In a closed economy k_t equals a_t so that (4) can be rewritten as the differential equation:

$$s_t = \dot{k}_t = (1 - \tau_t)y_t - c_t \rightarrow \qquad \frac{\dot{k}_t}{k_t} = (1 - \tau_t)A\left(\frac{\epsilon k_{gt}}{k_t}\right)^\alpha - \frac{c_t}{k_t}. \tag{5}$$

The Government sector faces the following constraint:

$$g_t + \dot{k}_{gt} = \tau_t y_t + or_t \tag{6a}$$

which, given government investment, implies that government consumption and transfers become

$$g_t = (1 - h_t)\tau_t y_t + (1 - \theta_t)or_t. \tag{6b}$$

This also implies that the government budget constraint is satisfied each period.

The model assumes that oil resources and their revenues are owned by the government and that its real value can be increased by government investment:

$$or_t = \chi[(1 - \epsilon_t)k_g]^\beta \qquad \text{where } 0 < \beta < 1. \tag{7}$$

In (7) the flow of oil revenue increases with the amount of government capital but encounters diminishing returns in the amount of government capital used in the oil sector $(1 - \epsilon_t)k_{gt}$. The size of the parameter β reflects the efficiency by which public capital is used in the oil sector. With β close to zero, government's capital investment adds little to oil revenues but as β approaches one, oil revenues become proportional to government investment. Changes in parameter χ can be used to reflect the effects of other exogenous shocks that can affect the steady state of oil revenue growth such as: war,

sanctions against the economy, external changes to the foreign exchange rate etc. Taking the logarithms of both sides of (7) and the derivate with respect to time:

$$\left(\frac{\dot{or}}{or}\right) = \beta\left(\frac{(1\dot{-}\epsilon)}{(1-\epsilon)} + \frac{\dot{k_g}}{k_g}\right) \tag{7}'$$

The growth rate of oil revenue depends on the growth rate of the share of government capital devoted to oil sector, as well as the overall growth rate of public capital. With both constant in the steady state, the growth rate of oil revenues also becomes a constant.

The economy wide resource constraint combines the household and government budget constraints. This is found by combining (5) with (6a), i.e.,

$$\dot{k_t} = (1 - \tau_t)y_t - c_t \rightarrow i_t + c_t = (1 - \tau_t)y_t$$

plus

$$g_t + \dot{k_{gt}} = \tau_t y_t + or_t \rightarrow i_{gt} + g_t = \tau_t y_t + or_t$$

gives

$$c_t + i_t + i_{gt} + g_t = y_t + or_t. \tag{8}$$

When the actual levels are replaced by their behavioral equivalents, (8) can represent the equilibrium condition, $y_t^d = y_t^s$, where the left hand side represents household and government demands and the left hand side represents the aggregate supply available to the economy as a whole:

$$y_t^d = (c_t + i_t) + (i_{gt} + g_t) = y_t + or_t = y_t^s. \tag{9}$$

III. The Social Planner's Problem [following Turnovsky (1997)]:

A social planner determines the time paths of c_t, g_t, k_t, k_{gt} and τ_t and the values of h_t, θ_t, and v_t that maximizes the present value of household utility. This is represented as a solution to the Hamiltonian

$$H(c_t, g_t, h_t, \theta_t, \tau_t; k_t, k_{gt}) = U(c_t, g_t)e^{-\rho t} + e^{-\rho t}\mu_t\{(1-\tau_t)y_t - c_t\} + e^{-\rho t}\mu_{gt}\{\tau_t y_t + or_t - g_t\} \tag{10}$$

where μ_t and μ_{gt} are the co-state variables (current valued lagrangian multipliers) associated with the two capital stocks. Substituting in the functional forms of our problem, the problem can be restated as maximizing

$$H = \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{g_t^{1-\varphi}}{1-\varphi}\right]e^{-\rho t} + e^{-\rho t}\mu_t\left\{(1-\tau_t)Ak_t^{1-\alpha}(\epsilon_t k_{gt})^\alpha - c_t\right\} + e^{-\rho t}\mu_{gt}\left\{h_t\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^\alpha + \theta_t or_t\right\}.$$

Substituting (6b) for g_t and then (7) for or_t reduces the scale of the problem to

$$\max H = \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\left((1-h_t)\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha} + (1-\theta_t)\chi\left((1-\epsilon_t)k_{gt}\right)^{\beta}\right)^{1-\varphi}}{1-\varphi}\right]e^{-\rho t}$$

$$+ e^{-\rho t}\mu_t\left\{(1-\tau_t)Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha} - c_t\right\} + e^{-\rho t}\mu_{gt}\left\{h_t\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha} + \theta_t\chi((1-\epsilon_t)k_{gt})^{\beta}\right\}.\text{[1]} \quad (11)$$

The first order conditions for this internal maximum are:

$$\frac{\partial H}{\partial c_t} = 0 \rightarrow c_t^{-\sigma} = \mu_t \tag{12}$$

$$\frac{\partial H}{\partial h_t} = 0 \rightarrow g_t^{-\varphi}(\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha}) = \mu_{gt}\left(\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha}\right) \rightarrow g_t^{-\varphi} = \mu_{gt} \tag{13a}$$

$$\frac{\partial H}{\partial \theta_t} = 0 \rightarrow g_t^{-\varphi}\chi[(1-\epsilon_t)k_{gt}]^{\beta} = \mu_{gt}\chi[(1-\epsilon_t)k_{gt}]^{\beta} \rightarrow \quad g_t^{-\varphi} = \mu_{gt} \tag{13b}$$

$$\frac{\partial H}{\partial \epsilon_t} = 0 \rightarrow g_t^{-\varphi}[\alpha(1-h_t)\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha-1}k_{gt} - (1-\theta_t)\beta\chi[(1-\epsilon_t)k_{gt}]^{\beta-1}]$$

$$+ \mu_t(1-\tau_t)\alpha Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha-1}k_{gt}$$

$$+ \mu_{gt}\left\{\alpha h_t\tau_t Ak_t^{1-\alpha}(\epsilon_t k_{gt})^{\alpha-1}k_{gt} - \theta_t\beta\chi[(1-\epsilon_t)k_{gt}]^{\beta-1}k_{gt}\right\} = 0 \tag{14}$$

$$\frac{\partial H}{\partial \tau_t} = 0 \rightarrow g_t^{-\varphi}(1-h_t) = \mu_t - \mu_{gt}h_t \tag{15}$$

Note that (13a) and (13b) are identical so that $h_t = \theta_t$. Also (13a) or (13b) in (15) yields $\mu_t = \mu_{gt}$. This in turn implies that $h_t = \theta_t = \tau_t$ and that

$$c_t^{-\sigma} = g_t^{-\varphi} \text{ or } g_t = c_t^{\sigma/\varphi}. \tag{16}$$

Using the equality of the multipliers also allows (14) to be simplified as

$$\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} = \beta\chi[(1-\epsilon_t)k_{gt}]^{\beta-1} \tag{14'}$$

This implies that ϵ_t is adjusted such that the marginal products of government capital in the private and oil sectors are equalised. Next,

$$\frac{\partial H}{\partial k_t} = -(e^{-\dot{\rho}t}\mu_t) \rightarrow$$

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[1] The maximization problem also requires the maintenance of the transversality conditions for private and government capital: $\lim_{t\to\infty}\mu_t k_t = 0$ and $\lim_{t\to\infty}\mu_{gt}k_{gt} = 0$; together with the initial conditions: $k_t = k_0$ and $k_{gt} = k_{g0}$ at t = 0.

$$e^{-\rho t}(g_t)^{-\varphi}(1-\alpha)(1-h_t)\tau_t A \frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} + e^{-\rho t}\mu_t(1-\alpha)(1-\tau_t)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha}$$

$$+ e^{-\rho t}\mu_{gt}\left\{(1-\alpha)h_t\tau_t A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha}\right\} = -e^{-\rho t}\dot{\mu_t} + \rho e^{-\rho t}\mu_t$$

Using $\mu_t = \mu_{gt} = g_t^{-\varphi}$ and dividing by $e^{-\rho t}\mu_t$, we find

$$(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} = -\frac{\dot{\mu_t}}{\mu_t} + \rho. \tag{17}$$

Taking the time derivative of (12) we find

$\dot{\mu_t} = -\sigma c_t^{-\sigma-1}\dot{c_t}$ which implies that $\frac{\dot{\mu_t}}{\mu_t} = \frac{-\sigma c_t^{-\sigma-1}}{c_t^{-\sigma}}\dot{c_t} = -\sigma\frac{\dot{c_t}}{c_t}$. Substituting this back into (17) we find the household's Euler equation as

$$\frac{\dot{c_t}}{c_t} = \frac{1}{\sigma}\left\{(1-\alpha)A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^\alpha - \rho\right\}. \tag{18}$$

The growth rate of consumption is a function of the gap between the marginal product of private capital and its opportunity cost, the marginal rate of time preference.

The corresponding condition for government capital is:

$$\frac{\partial H}{\partial k_{gt}} = -(e^{-\rho t}\dot{\mu_{gt}}) \to e^{-\rho t}(g_t)^{-\varphi}\left[\alpha(1-h_t)\tau_t A\frac{k_t^{1-\alpha}}{\epsilon_t k_{gt}^{1-\alpha}}\epsilon_t + \beta(1-\theta_t)\chi[(1-\epsilon_t)k_{gt}]^{\beta-1}(1-\epsilon_t)\right]$$

$$+ e^{-\rho t}\mu_t\alpha(1-\tau_t)A\frac{k_t^{1-\alpha}}{\epsilon_t k_{gt}^{1-\alpha}}\epsilon_t + e^{-\rho t}\mu_{gt}\left(\alpha h_t\tau_t A\frac{k_t^{1-\alpha}}{\epsilon_t k_{gt}^{1-\alpha}}\epsilon_t + \theta_t\beta\chi[(1-\epsilon_t)k_{gt}]^{\beta-1}(1-\epsilon_t)\right)$$

$$= -e^{-\rho t}\dot{\mu_{gt}} + \rho e^{-\rho t}\mu_{gt}.$$

Again using (13a)or (13b) and combining terms, this reduces to

$$-\frac{\dot{\mu_{gt}}}{\mu_{gt}} = \alpha A\frac{k_t^{1-\alpha}}{\epsilon_t k_{gt}^{1-\alpha}}\epsilon_t + (1-\epsilon_t)\beta\chi[(1-\epsilon_t)k_{gt}]^{\beta-1} - \rho, \tag{19}$$

where the first two terms on the right hand side of (19) represent the marginal product of government capital (optimally allocated between the oil and non-oil sectors). Using (14)' the first two right hand side terms can be combined to represent (19) as[2]

$$-\frac{\dot{\mu_{gt}}}{\mu_{gt}} = \alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \rho. \tag{20}$$

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[2]Note that the substitution in (19) could have been done the other way to derive $-\frac{\dot{\mu_{gt}}}{\mu_{gt}} = \beta\chi[(1-\epsilon)k_{gt}]^{\beta-1} - \rho$. This would have led to (20) being written in terms of the marginal product of government capital in the oil sector.

Taking the time derivative of (13), represented as g, we find the Euler equation for government spending as

$$\frac{\dot{g}_t}{g_t} = \frac{1}{\varphi}\left\{\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \rho\right\}, \tag{21}$$

that is, the growth rate of government services is an increasing function of the difference between the productivity of government capital in producing government services and the rate of time preference. These two differential equations in combination with the ones defining changes in the capital stocks fully describe the movement of the economy. Repeating the latter for convenience as (22) and (23),

$$\dot{k}_t = (1-\tau_t)A k_t^{1-\alpha}\left(\epsilon_t k_{gt}\right)^{\alpha} - c_t, \qquad \rightarrow \qquad \frac{\dot{k}_t}{k_t} = (1-\tau_t)A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha} - \frac{c_t}{k_t}, \tag{22}$$

and

$$\dot{k}_{gt} = h_t \tau_t A k_t^{1-\alpha}\left(\epsilon_t k_{gt}\right)^{\alpha} + \theta_t \chi\left[(1-\epsilon_t)k_{gt}\right]^{\beta}$$

$$= h_t \tau_t \epsilon_t k_{gt} A\left(\frac{\epsilon k_{gt}}{k_t}\right)^{\alpha-1} + \frac{(1-\epsilon_t)\theta_t k_{gt}}{\beta}\beta\chi\left[(1-\epsilon_t)k_{gt}\right]^{\beta-1}$$

$$\rightarrow \frac{\dot{k}_{gt}}{k_{gt}} = \left[\epsilon_t h_t \tau_t + \frac{\theta_t(1-\epsilon_t)\alpha}{\beta}\right]A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} \tag{23}$$

after using (14)'.

IV.   Steady State

Because of the production and oil revenue externalities, output will grow through time. Nevertheless even though the equilibrium levels of variables will not be stationary, they could all grow at the same or a related rate through time. This characteristic is used to solve for a balanced growth equilibrium. We later ask whether such a steady state exists and whether the model will converge to that steady state. Note first that from (16), $g_t = c_t^{\sigma/\varphi}$, so that

$$\frac{\dot{g}_t}{g_t} = \left(\frac{\sigma}{\varphi}\right)\frac{\dot{c}_t}{c_t}, \tag{24}$$

that is, government consumption will grow faster or slower than consumption depending on its relative weight in utility. From (13a) and (13b) we have already found that $h_t = \theta_t = \tau_t$.

Next, from oil revenues function in (7), note that $\frac{\partial or}{\partial k_g} = \chi\beta(1-\epsilon_t)[(1-\epsilon_t)k_g]^{\beta-1} = \beta\frac{or_t}{k_g}$. Then using (14)' in the left hand side, we find

$$\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1}(1-\epsilon_t) = \beta\frac{or_t}{k_g}, \text{ that in the steady state becomes}$$

$$\beta \left(\frac{or}{k_g}\right)^{ss} = \frac{(1-\epsilon^{ss})\alpha A}{\left(\epsilon^{ss}\left(\frac{k_g}{k}\right)^{ss}\right)^{1-\alpha}}. \tag{25}$$

In the steady state, the marginal product of public capital in the oil sector varies inversely with $\left(\frac{k_g}{k}\right)^{ss}$.[3]

Next we define $x_t = \frac{k_{gt}}{k_t}$ and $z_t = \frac{c_t}{k_t}$.This implies,from (21) and (22), that

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{k}_{gt}}{k_{gt}} - \frac{\dot{k}_t}{k_t} = \left[\epsilon_t h_t \tau_t + \frac{\theta_t(1-\epsilon_t)\alpha}{\beta} - (1-\tau_t)\epsilon_t x_t\right] A(\epsilon_t x_t)^{\alpha-1} + z_t. \tag{26}$$

In the same manner, from (18) and (21)

$$\frac{\dot{z}_t}{z_t} = \frac{\dot{c}_t}{c_t} - \frac{\dot{k}_t}{k_t} = \frac{1}{\sigma}\{(1-\alpha)A(\epsilon_t x_t)^\alpha - \rho\} - (1-\tau_t)A(\epsilon_t x_t)^\alpha \epsilon_t + z_t$$

$$= \left[\frac{(1-\alpha)-\sigma\epsilon_t(1-\tau_t)}{\sigma}\right] A(\epsilon_t x_t)^\alpha - \frac{\rho}{\sigma} + z_t. \tag{27}$$

Then if we impose the condition for a steady state, $\frac{\dot{x}_t}{x_t} = 0$, we find from (26) that

$$z^{ss} = \left[\epsilon^{ss} h^{ss} \tau^{ss} + \frac{\alpha\theta^{ss}(1-\epsilon^{ss})}{\beta} - (1-\tau^{ss})\epsilon^{ss}x^{ss}\right] A(\epsilon^{ss}x^{ss})^{\alpha-1} \tag{28}$$

Imposing the steady state condition $\frac{\dot{z}_t}{z_t} = 0$ in (27) we find

$$z^{ss} = \frac{\rho}{\sigma} - \left[\frac{(1-\alpha)-\sigma\epsilon^{ss}(1-\tau_t)}{\sigma}\right] A(\epsilon^{ss}x^{ss})^\alpha \tag{29}$$

We now have two equations to solve for x^{ss} and z^{ss} given the steady state values of the parameters $\epsilon_t, h_t, \theta_t$ and τ_t, knowing that $h_t = \theta_t = \tau_t$. Hence we need two more equations for a steady state solution.

First defining $v_t = \frac{c_t}{g_t}$, we find that $\frac{\dot{v}_t}{v_t} = \frac{\dot{c}_t}{c_t} - \frac{\dot{g}_t}{g_t}$ and then using (18) and (20), we find

$$\frac{\dot{v}_t}{v_t} = \frac{1}{\sigma}\left\{(1-\alpha)A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^\alpha - \rho\right\} - \frac{1}{\varphi}\left\{\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \rho\right\}.$$

Rearranging to combine terms, we find

$$\frac{\dot{v}_t}{v_t} = A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1}\left[\frac{\varphi(1-\alpha)\left(\frac{\epsilon_t k_{gt}}{k_t}\right)-\alpha\sigma}{\sigma\varphi}\right] - \rho\frac{(\varphi-\sigma)}{\sigma\varphi} \tag{30}$$

In the steady state with $x^{ss} = \left(\frac{k_g}{k}\right)^{ss}$

$$\left(\frac{\dot{v}_t}{v_t}\right)^{SS} = A(\epsilon^{SS}x^{SS})^{\alpha-1}\left[\frac{\varphi(1-\alpha)\epsilon^{SS}x^{SS}-\alpha\sigma}{\sigma\varphi}\right] - \rho\frac{(\varphi-\sigma)}{\sigma\varphi}. \tag{31}$$

From (24) and the definition of v_t, we can also find $\frac{\dot{v}_t}{v_t} = \frac{\dot{c}_t}{c_t} - \left(\frac{\sigma}{\varphi}\right)\frac{\dot{c}_t}{c_t} = \left[\frac{\varphi-\sigma}{\varphi}\right]\frac{\dot{c}_t}{c_t}$. Inserting the steady state value of this as the left hand side of (31) and using (18), we find

$$\left[\frac{\varphi-\sigma}{\varphi}\right]\frac{1}{\sigma}\{(1-\alpha)A(\epsilon x^{SS})^\alpha - \rho\} = A(\epsilon^{SS}x^{SS})^{\alpha-1}\left[\frac{\varphi(1-\alpha)\epsilon^{SS}x^{SS}-\alpha\sigma}{\sigma\varphi}\right] - \rho\frac{(\varphi-\sigma)}{\sigma\varphi},$$

which simplifies to

$$\epsilon^{SS}x^{SS} = \frac{\alpha}{(1-\alpha)}. \tag{32}$$

Equation (32) implies that in the steady state, the more intensive is the role of public capital in non-oil production function (the close α is to 1) the larger should the share of public capital be devoted to this sector. Moreover, any external shock which causes a steady state increase in the relative size of government capital (ceteris paribus) will cause a decrease in the share of public capital used in the non-oil sector (ϵ^{SS}).[4]

Finally let $w_t = \frac{g_t}{k_{gt}}$. Then taking the time derivative and using (21) and (23), we find

$$\frac{\dot{w}_t}{w_t} = \frac{\dot{g}_t}{g_t} - \frac{\dot{k}_{gt}}{k_{gt}} = \frac{1}{\varphi}\left\{\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \rho\right\} - \left[\epsilon_t h_t \tau_t + \frac{\theta_t(1-\epsilon_t)\alpha}{\beta}\right]A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1}$$

$$= \frac{1}{\varphi}\left\{\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \rho\right\} - \left[\frac{\epsilon_t h_t \tau_t}{\alpha} + \frac{\theta_t(1-\epsilon_t)}{\beta}\right]\alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1}$$

$$= \alpha A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha-1} - \frac{\rho}{\varphi}. \tag{33}$$

In the steady state this becomes

$$\left(\frac{\dot{w}_t}{w_t}\right)^{SS} = \left[\frac{1}{\varphi} - \frac{\epsilon^{SS}h^{SS}\tau^{SS}}{\alpha} + \frac{\theta^{SS}(1-\epsilon^{SS})}{\beta}\right]\alpha A(\epsilon^{SS}x^{SS})^{\alpha-1} - \frac{\rho}{\varphi}. \tag{34}$$

In addition, note that we can write $w_t = \frac{g_t}{c_t}\cdot\frac{c_t}{k_t}\cdot\frac{k_t}{k_{gt}}$ where the latter two ratios are constant. This implies that $\frac{\dot{w}_t}{w_t} = \frac{\dot{v}_t}{v_t} = \left(\frac{\sigma}{\varphi}\right)\frac{\dot{c}_t}{c_t} - \frac{\dot{c}_t}{c_t} = \left(\frac{\sigma-\varphi}{\varphi}\right)\frac{\dot{c}_t}{c_t}$. Inserting this into (33) and using (18):

$$\left(\frac{\sigma-\varphi}{\varphi}\right)\frac{1}{\sigma}\{(1-\alpha)A(\epsilon^{SS}x^{SS})^\alpha - \rho\} = \left[\frac{1}{\varphi} - \frac{\epsilon^{SS}h^{SS}\tau^{SS}}{\alpha} + \frac{\theta^{SS}(1-\epsilon^{SS})}{\beta}\right]\alpha A(\epsilon^{SS}x^{SS})^{\alpha-1} - \frac{\rho}{\varphi}$$

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[4] When $\epsilon^{SS} = 1$ then $x^{SS} = \frac{k_g^{SS}}{k^{SS}} = \frac{\alpha}{(1-\alpha)}$ and the optimal proportion of government to private capital depends only on the characteristics of the non-oil production function. However whenever $\epsilon^{SS} < 1$ then $\frac{\alpha}{(1-\alpha)x^{SS}} < 1 \rightarrow \frac{\alpha}{(1-\alpha)} < x^{SS}$. This means that he optimal proportion of government to private capital will always be larger than that implied by the non-oil sector alone.

$$\left\{\left(\frac{\sigma-\varphi}{\varphi}\right)\frac{1}{\sigma}(1-\alpha)A(\epsilon^{SS}x^{SS})^{\alpha}-\rho\left(\frac{\sigma-\varphi}{\varphi}\right)\frac{1}{\sigma}\right\}=\left[\frac{1}{\varphi}-\frac{\epsilon^{SS}h^{SS}\tau^{SS}}{\alpha}+\frac{\theta^{SS}(1-\epsilon^{SS})}{\beta}\right]\alpha A(\epsilon^{SS}x^{SS})^{\alpha-1}-\frac{\rho}{\varphi}$$

$$\rightarrow\left\{\left(\frac{\sigma-\varphi}{\varphi\sigma}\right)(1-\alpha)A(\epsilon^{SS}x^{SS})^{\alpha}\right\}=\left[\frac{1}{\varphi}-\frac{\epsilon^{SS}h^{SS}\tau^{SS}}{\alpha}+\frac{\theta^{SS}(1-\epsilon^{SS})}{\beta}\right]\alpha A(\epsilon^{SS}x^{SS})^{\alpha-1}-\frac{\rho}{\sigma}. \tag{34$'$}$$

Equations (27), (28), (32) and (34)' all holding simultaneously are necessary for a steady state, but the steady state solution implied is difficult to describe analytically. Hence we illustrate the simultaneous solutions for x^{SS} and z^{SS} given the steady state values of ε^{SS} and $h^{SS}=\tau^{SS}=\theta^{SS}$ in the section that follows.

IV.   Phase Diagrams:

a.   The $\dot{z}=0$ curve(s)

From (27), the $\dot{z}_t=0$ schedule is given (other than for $z_t=0$) by equation (29). This repeated below as

$\dot{z}_t=0 \rightarrow z_t=\frac{\rho}{\sigma}-\left[\frac{(1-\alpha)-\sigma\epsilon^{SS}(1-\tau^{SS})}{\sigma}\right]A(\epsilon^{SS}x_t)^{\alpha}$ which implies that when $x=0 \rightarrow z=\frac{\rho}{\sigma}$. Solving for its slope we find

$$\frac{dz}{dx}=-\alpha A\epsilon_t\left[\frac{(1-\alpha)}{\sigma}-\epsilon^{SS}(1-\tau^{SS})\right](\epsilon^{SS}x_t)^{\alpha-1} \tag{35}$$

which can be positive or negative depending upon whether $\left\{\frac{(1-\alpha)}{\sigma}-\epsilon^{SS}(1-\tau^{SS})\right\}> or <0$. There are then two cases:

        i. If $\left\{\frac{(1-\alpha)}{\sigma}-\epsilon^{SS}(1-\tau^{SS})\right\}>0 \Rightarrow \frac{dz}{dx}<0$. Hence it starts from the axis at $z=\frac{\rho}{\sigma}$ and decreases

until it crosses the axis at $x=\frac{1}{\epsilon_t}\left[\frac{\frac{\rho}{A}}{(1-\alpha)-\sigma\epsilon^{SS}(1-\tau^{SS})}\right]^{\frac{1}{\alpha}}>0$. From (27) it can be seen that values of z above (below) this line result in $\dot{z}_t>0 \ (<0)$. This is represented in Figure 1a below.
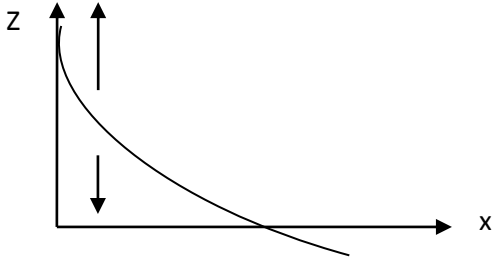

Figure 1a

ii. If, on the other hand, $\left\{\frac{(1-\alpha)}{\sigma} - \epsilon^{SS}(1 - \tau^{SS})\right\} < 0 \Rightarrow \frac{dz}{dx} > 0$, then the $\dot{z} = 0$ curve departs

from the vertical axis at $z = \frac{\rho}{\sigma}$ and increases, but with z increasing at a decreasing rate. Similarly, values

of z above (below) the $\dot{z} = 0$ curve result in further increases (decreases) in z. This is illustrated in Figure
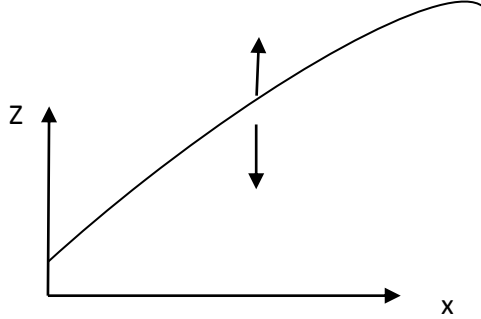
1b.

Figure 1b

b. The $\dot{x}_t = 0$ curve

From (26) the $\dot{x}_t = 0$ curve schedule can be determined (for other than $x_t = 0$) by equation (29). This is

repeated for convenience as $z^{SS} = \left[\epsilon^{SS}h^{SS}\tau^{SS} + \frac{\alpha\theta^{SS}(1-\epsilon^{SS})}{\beta} - (1 - \tau^{SS})\epsilon^{SS}x^{SS}\right]A(\epsilon^{SS}x^{SS})^{\alpha-1}$ which

implies that when $z_t = 0$, $x^0 = \frac{h^{SS}\tau^{SS}}{(1-\tau^{SS})} + \frac{\theta^{SS}(1-\epsilon^{SS})\alpha}{\beta\epsilon^{SS}(1-\tau^{SS})} > 0$.It follows that $\dot{x}_t = 0$ crosses horizontal axis at

x^0. Next,

$\frac{dz}{dx} =$

$(\epsilon^{SS})^2 h^{SS}\tau^{SS}A(1 - \alpha)(\epsilon^{SS}x_t)^{\alpha-2} + \frac{\epsilon^{SS}\theta^{SS}(1-\epsilon^{SS})\alpha}{\beta}A(1 - \alpha)(\epsilon^{SS}x_t)^{\alpha-2} + (1 - \tau^{SS})A\epsilon^{SS}\alpha(\epsilon^{SS}x_t)^{\alpha-1} >$

0 with $\frac{d^2z}{dx^2} < 0$. $\hspace{4cm}$ (36)

For values of x to the right (left) of the curve, $\dot{x}_t < 0$ (> 0).Hence the $\dot{x} = 0$ curve can be represented as
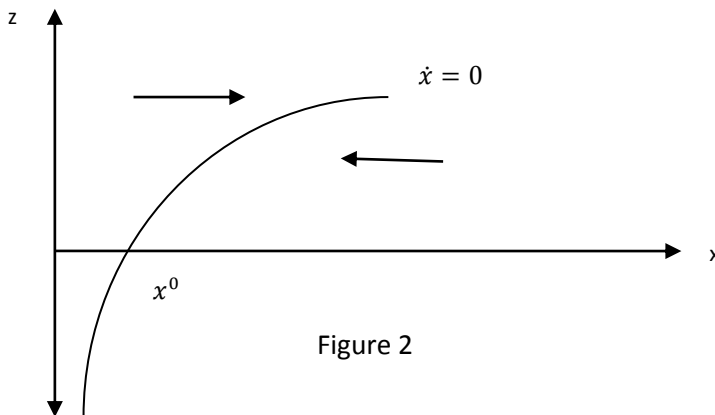
Figure 2.

Figure 2

c.   Steady State and Stability Condition

Combining the two sets of steady state conditions from sections a. and b. above, there are two possibilities for $x^{ss} > 0$ and $z^{ss} > 0$. The first combines Figures 1a and 2.
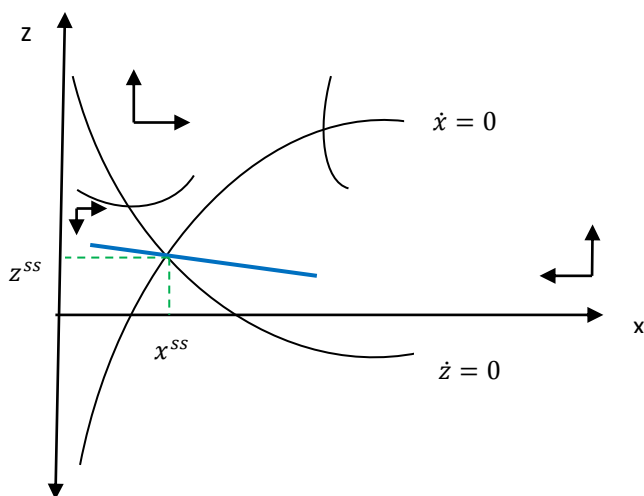

Figure 3a

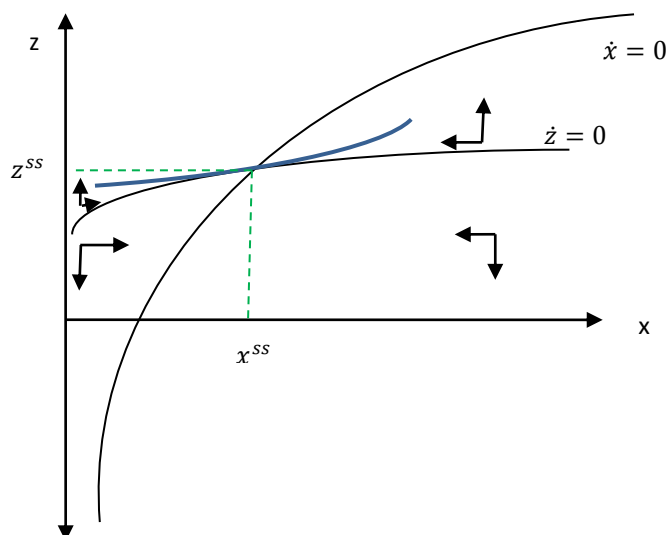The second represents the combination of Figures 2b and 3.

Figure 3b

Note that in both cases there is one stable and one unstable root so that the saddle-path property generates a unique adjustment path to the steady state. Then with x^{ss} and z^{ss} in hand, $h^{ss} = \tau^{ss} = \theta^{ss}$ and ϵ^{ss} can be determined from (32) and (34)'.

V. Private and Government Consumption in the Steady State

Once the steady state values of $x^{SS} = \left(\frac{k_g}{k}\right)^{SS}$, $z^{SS} = \left(\frac{c}{k}\right)^{SS}$, ε^{SS} and $h^{SS} = \tau^{SS} = \theta^{SS}$ are determined, the other steady state values fall readily into place. That is, from the non-oil production function, $\frac{y_t}{k_t} = A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha}$, the steady state output-capital ratio can be seen to be $\left(\frac{y}{k}\right)^{SS} = A(\epsilon^{SS} x^{SS})^{\alpha}$. Using this and (18) we find the growth rate of consumption in the steady state as

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left\{(1-\alpha)A\left(\frac{\epsilon_t k_{gt}}{k_t}\right)^{\alpha} - \rho\right\} \quad\rightarrow\quad \left(\frac{\dot{c}}{c}\right)^{SS} = \frac{1}{\sigma}\left\{(1-\alpha)\left(\frac{y}{k}\right)^{SS} - \rho\right\}. \tag{37}$$

Then because $x^{SS} = \left(\frac{k_g}{k}\right)^{SS}$, $z^{SS} = \left(\frac{c}{k}\right)^{SS}$ are both constants, private and government capital must also grow at the same rate as consumption. In the Appendix we provide a proof that this growth rate is indeed positive.

From (21), the steady state growth rate of government capital can be written as

$$\left(\frac{\dot{g}}{g}\right)^{SS} = \frac{1}{\varphi}\left\{\alpha\frac{A(\epsilon^{SS} x^{SS})^{\alpha}}{\epsilon^{SS} x^{SS}} - \rho\right\} = \frac{1}{\varphi}\left\{(1-\alpha)\left(\frac{y}{k}\right)^{SS} - \rho\right\}, \tag{38}$$

where (32) was used in the last step. It follows that when $\sigma = \varphi$ private and government consumption will grow at the same rate. Otherwise $\left(\frac{\dot{c}}{c}\right)^{SS} > \left(\frac{\dot{g}}{g}\right)^{SS}$ if $\varphi > \sigma$ and $\left(\frac{\dot{c}}{c}\right)^{SS} < \left(\frac{\dot{g}}{g}\right)^{SS}$ when $\varphi < \sigma$.

VI. Conclusion

In this paper we have shown that the abundance of a natural resource such as oil need not present a curse for the domestic economy, dooming the non-oil sector to secondary status and a long period of stagnation and decline. Rather oil revenues can themselves be source of economy wide growth. What is required is simply the judicious use of oil revenues, in our case the channelling of oil revenues into privately productive government capital. As we have also shown, the resulting economy wide growth and development need not be at the expense of other government services. In the steady state government consumption can grow in line with private consumption, in our case at the rate dictated by household preferences.

What is perhaps even more encouraging is that the conditions under which endogenous growth can arise are not that stringent. First, the natural resource must be an important generator of revenue to the government and, second, government capital investments must be complementary with private capital. While not all countries are fortunate enough to find an exogenous source of revenue, a good many countries have their equivalent to oil. In addition, the economic literature provides many examples of government investment being complementary to private capital (Karras, 1996; Katrakilidis and Tabakis, 2001; Rashid, 2005). For less well developed economies many authors suggest ways of combining synergies to create an even greater potential for complementarity (Evans, 1996).

Appendix

A proof that the growth rate in the steady state is positive requires a demonstration that

$\left(\frac{\dot{c}}{c}\right)^{SS} = \frac{1}{\sigma}\left\{(1-\alpha)\left(\frac{y}{k}\right)^{SS} - \rho\right\} > 0$, that in turn requires $(1-\alpha)\left(\frac{y}{k}\right)^{SS} - \rho > 0$.

Proof:

From equation (17) in the text we have:

$$(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} = -\frac{\dot{\mu}_t}{\mu_t} + \rho. \tag{A1}$$

This can be rewritten as:
$$\frac{\dot{\mu}_t}{\mu_t} = -\left\{(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} - \rho\right\} \tag{A2}$$

Solving this first order differential equation, we find

$$\mu_t = \mu_0 e^{-\int_0^t\left\{(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} - \rho\right\}dt} \tag{A3}$$

Next the transversality condition for private capital in the model is $\lim_{t\to\infty}\mu_t k_t = 0$.

Inserting (A3) into the transversality condition we find,

$$\lim_{t\to\infty} k_t \mu_0 e^{-\int_0^t\left\{(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} - \rho\right\}dt} = 0 \tag{A4}$$

Dividing both sides of the equation by μ_0 and k_{gt} :

$$\lim_{t\to\infty}\frac{k_t}{k_{gt}} e^{-\int_0^t\left\{(1-\alpha)A\frac{(\epsilon_t k_{gt})^\alpha}{k_t^\alpha} - \rho\right\}dt} = 0 . \tag{A5}$$

We know that through the time $\frac{k_t}{k_{gt}}$ tends asymptotically to the constant steady state value $\left(\frac{k}{k_g}\right)^{SS}$ and

using $\left(\frac{y}{k}\right)^{SS} = A(\epsilon^{SS}x^{SS})^\alpha$ or $\left(\frac{y}{k}\right)^{SS} = A\left(\epsilon^{SS}\left(\frac{k}{k_g}\right)^{SS}\right)^\alpha$, the transversality condition can be written as :

$$\lim_{t\to\infty}\left(\frac{k}{k_g}\right)^{SS} e^{-\int_0^t\left\{(1-\alpha)\left(\frac{y}{k}\right)^{SS} - \rho\right\}dt} = 0. \tag{A6}$$

Because the transversality condition must be satisfied in the steady state, it follows that

$(1-\alpha)A\left(\frac{y}{k}\right)^{SS} - \rho > 0$ so that $\left(\frac{\dot{c}}{c}\right)^{SS} = \left(\frac{\dot{k}}{k}\right)^{SS} = \left(\frac{\dot{k}_g}{k_g}\right)^{SS} > 0.$

In the same way we can show that the transversality condition $\lim_{t\to\infty}\mu_{gt} k_{gt} = 0$ implies $\left(\frac{\dot{g}}{g}\right)^{SS} > 0.$

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