ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
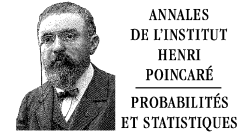
www.imstat.org/aihp

The renormalization transformation for two-type branching models

D. A. Dawson[a], A. Greven[b], F. den Hollander[c,d] Rongfeng Sun[d,e], and J. M. Swart[f]

[a]*School of Mathematics and Statistics, Carleton University, Ottawa K1S 5B6, Canada. E-mail: ddawson@math.carleton.ca*
[b]*Mathematisches Institut, Universität Erlangen–Nürnberg, Bismarckstraße 1 1/2, D-91054 Erlangen, Germany.*
E-mail: greven@mi.uni-erlangen.de
[c]*Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, the Netherlands. E-mail: denholla@math.leidenuniv.nl*
[d]*MA 7-5, Fakultät II – Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, 10623 Berlin. E-mail: sun@math.tu-berlin.de*
[e]*EURANDOM, P.O. Box 513, 5600 MB Eindhoven, the Netherlands*
[f]*ÚTIA, Pod vodárenskou věží 4, 18208 Praha 8, Czech Republic. E-mail: swart@utia.cas.cz*

Abstract. This paper studies countable systems of linearly and hierarchically interacting diffusions taking values in the positive quadrant. These systems arise in population dynamics for two types of individuals migrating between and interacting within colonies. Their large-scale space–time behavior can be studied by means of a renormalization program. This program, which has been carried out successfully in a number of other cases (mostly one-dimensional), is based on the construction and the analysis of a nonlinear renormalization transformation, acting on the diffusion function for the components of the system and connecting the evolution of successive block averages on successive time scales. We identify a general class of diffusion functions on the positive quadrant for which this renormalization transformation is well defined and, subject to a conjecture on its boundary behavior, can be iterated. Within certain subclasses, we identify the fixed points for the transformation and investigate their domains of attraction. These domains of attraction constitute the universality classes of the system under space–time scaling.

Résumé. Cet article étudie des systèmes dénombrables de diffusions en interaction hiérarchiques et linéaires vivant dans le quadrant positif. De tels systèmes apparaissent dans la dynamique d'individus de deux types qui migrent tout en interagissant dans des colonies. Le comportement à grande échelle et temps long peut être étudié en utilisant le programme de renormalisation. Ce programme, qui a permis de résoudre d'autres cas (principalement uni-dimensionnels) est basé sur la construction et l'analyse d'une transformation de renormalisation non linéaire, agissant sur la fonction de diffusion des composants du système et connectant l'évolution de blocs moyennés sur le temps à différentes échelles. Nous identifions une classe générale de fonctions de diffusion dans le quadrant positif pour lequel la transformation de renormalisation est bien définie et qui, sous une conjecture de comportement aux bords, peut-être itérée. À l'intérieur de certaines sous-classes, nous identifiens les points fixes de la transformation et étudions leurs domaines d'attraction. Ces domaines d'attraction constituent les classes d'universalité du système après changement d'échelle dans le temps et l'espace.

1. Introduction

1.1. *Model and background*

We are interested in the following system of coupled stochastic differential equations (SDE):

$$dX_{\eta,i}(t) = \sum_{\xi \in \Omega_N} a_N(\xi, \eta)\big[X_{\xi,i}(t) - X_{\eta,i}(t)\big]\,dt + \sqrt{2g_i\big(\vec{X}_\eta(t)\big)}\,dB_{\eta,i}(t), \quad \eta \in \Omega_N, i = 1, 2. \tag{1.1}$$

Here $a_N(\cdot, \cdot)$ is the transition rate kernel of a random walk on Ω_N, the hierarchical group (or lattice) of order N (see (1.3)), $\{\vec{X}_\eta\}_{\eta \in \Omega_N}$ with $\vec{X}_\eta = (X_{\eta,1}, X_{\eta,2})$ is a family of diffusions taking values in $[0, \infty)^2$, $g = (g_1, g_2)$ is a pair of diffusion functions on $[0, \infty)^2$, and $\{\vec{B}_\eta\}_{\eta \in \Omega_N}$ with $\vec{B}_\eta = (B_{\eta,1}, B_{\eta,2})$ is a family of independent standard Brownian motions on $\mathbb{R}^2$. As the initial condition, we take

$$\vec{X}_\eta(0) = \vec{\theta} = (\theta_1, \theta_2) \in [0, \infty)^2 \quad \forall \eta \in \Omega_N. \tag{1.2}$$

Equation (1.1) arises as the continuum limit of discrete models in *population dynamics*. In these models, individuals live in colonies labeled by the hierarchical group Ω_N. Each colony $\eta \in \Omega_N$ consists of two types of individuals, whose total masses are represented by the vector $\vec{X}_\eta$. Individuals *migrate* between colonies according to the migration kernel $a_N(\cdot, \cdot)$. At each colony, each individual undergoes *branching* at a rate that depends on the total masses of the two types of individuals present at that colony. The system in (1.1) arises in the so-called "small-mass–fast-branching" limit, where the number of individuals in each colony tends to infinity, the mass of each individual tends to zero, and the effective branching rate grows proportionally to the number of individuals in each colony. The drift term in (1.1) arises from the migration, which is the only source of interaction *between* colonies. The diffusion term in (1.1) arises from the branching, where $g_i(x)/x_i$ is the state-dependent branching rate of the ith type, which incorporates the interaction between individuals *within* a colony. For more background, see, e.g. [7,9,16,27], Chapters 9 and 10 in [19].

The goal of the present paper is to study the universality classes of the large-scale space–time behavior of (1.1). It turns out that, for the specific form of the migration kernel $a_N(\cdot, \cdot)$ given by (1.5) and in the limit as $N \to \infty$, (1.1) is susceptible to a renormalization analysis. The *renormalization program* for hierarchically interacting diffusions was introduced by Dawson and Greven [10,11] for diffusions taking values in $[0, 1]$. It has since been extended to several other state spaces (see [22,23] for an overview). We will give more detailed references in Section 1.3. First we outline the main ingredients of the renormalization program.

1.2. *Renormalization program*

The lattice in (1.1) is the *hierarchical group* of order N, which is defined as

$$\Omega_N = \left\{\eta = (\eta_i)_{i \in \mathbb{N}} \in \{0, 1, \ldots, N-1\}^{\mathbb{N}}: \sum_{i \in \mathbb{N}} \eta_i < \infty\right\}, \tag{1.3}$$

with coordinatewise addition modulo N. Define a shift $\phi: \Omega_N \to \Omega_N$ by $(\phi\eta)_i := \eta_{i+1}$ ($i \in \mathbb{N}$). On Ω_N, the *hierarchical distance* is defined as

$$d(\eta, \xi) = \min\big\{k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}: \phi^k\eta = \phi^k\xi\big\}, \tag{1.4}$$

which is an *ultrametric*, i.e., $d(\eta, \xi) \le d(\eta, \zeta) \vee d(\xi, \zeta)$ for all $\eta, \xi, \zeta \in \Omega_N$. We choose the random walk transition rate kernel in such a way that $a_N(\xi, \eta)$ depends only on the hierarchical distance between ξ and η. In view of what follows, we write a_N in the form

$$a_N(\xi, \eta) = \sum_{k \ge d(\xi, \eta)} c_{k-1} N^{1-2k}, \quad \xi, \eta \in \Omega_N, \xi \ne \eta, \tag{1.5}$$

where $(c_n)_{n\in\mathbb{N}_0}$ is a sequence of positive constants. Formula (1.5) says that the random walk associated with $a_N(\cdot,\cdot)$ jumps with rate c_{k-1}/N^{k-1} from η to an arbitrary site in the k-block $\{\xi \in \Omega_n: \phi^k\xi = \phi^k\eta\}$ around η.

The key objects in the renormalization analysis are the *k-block averages*:

$$Y_{\eta,i}^{[k]}(t) = \frac{1}{N^k} \sum_{\substack{\xi\in\Omega_N \\ \phi^k\xi=\eta}} X_{\xi,i}(t), \quad \eta \in \Omega_N, i = 1,2, k \in \mathbb{N}_0. \tag{1.6}$$

Using (1.5), we may rewrite (1.1) as

$$dX_{\eta,i}(t) = \sum_{k\geq 1} \frac{c_{k-1}}{N^{k-1}} \big[Y_{\phi^k\eta,i}^{[k]}(t) - X_{\eta,i}(t)\big]dt + \sqrt{2g_i(\vec{X}_\eta(t))}\,dB_{\eta,i}(t), \quad \eta \in \Omega_N, i = 1,2, \tag{1.7}$$

where each component $\vec{X}_\eta$ feels a drift towards the successive averages of k-blocks containing η. It can be seen that the evolution of the 1-block averages is described in law by the SDE

$$dY_{\eta,i}^{[1]}(tN) = \sum_{k\geq 1} \frac{c_k}{N^{k-1}} \big[Y_{\phi^k\eta,i}^{[k+1]}(tN) - Y_{\eta,i}^{[1]}(tN)\big]dt$$

$$+ \sqrt{\frac{2}{N} \sum_{\substack{\xi\in\Omega_N \\ \phi\xi=\eta}} g_i(\vec{X}_\xi(tN))}\,dB_{\eta,i}(t), \quad \eta \in \Omega_N, i = 1,2, \tag{1.8}$$

where $\vec{B}_\eta = (B_{\eta,1}, B_{\eta,2})$ is a family of independent standard two-dimensional Brownian motions. Note that in the limit $N \to \infty$, we expect both the drift and the diffusion term in (1.8) to be of order one, which means that $\vec{Y}_\eta^{[1]}$ evolves on the time scale tN.

Let us next see *heuristically* what happens if we let $N \to \infty$, the so-called *hierarchical mean-field limit*. If we let $N \to \infty$ in (1.7), then the only drift term that survives is

$$c_0\big[Y_{\phi\eta,i}^{[1]}(t) - X_{\eta,i}(t)\big]dt.$$

Furthermore, $\vec{Y}_{\phi\eta}^{[1]}(t) \to \vec{X}_{(\cdot)}(0) \equiv \vec{\theta}$ for all $t \geq 0$, because $\vec{Y}_{\phi\eta}^{[1]}$ evolves on the time scale tN. Therefore the system $\{\vec{X}_\eta(t)\}_{\eta\in\Omega_N}$ converges in law to an *independent* system of diffusions, each satisfying the autonomous SDE

$$dZ_i(t) = c_0(\theta_i - Z_i)\,dt + \sqrt{2g_i(\vec{Z}(t))}\,dB_i(t), \quad i = 1,2. \tag{1.9}$$

This kind of behavior is frequently referred to as the "McKean–Vlasov limit" and "propagation of chaos."

With the above fact in mind, we move one step up in the hierarchy. Since $\vec{X}_\xi(t)$ evolves on the time scale t, for each fixed t the family

$$\big\{\vec{X}_\xi(tN)\big\}_{\substack{\xi\in\Omega_N \\ \phi\xi=\eta}} \tag{1.10}$$

decouples and converges almost instantly to the equilibrium distribution of (1.9) with the drift towards $\vec{\theta}$ replaced by a drift towards the first block average $\vec{Y}_\eta^{[1]}(tN)$. Thus, we expect that

$$\frac{1}{N} \sum_{\substack{\xi\in\Omega_N \\ \phi\xi=\eta}} g_i(\vec{X}_\xi(tN)) \sim \int_{[0,\infty)^2} \Gamma_{\vec{Y}_\eta^{[1]}(tN)}^{c_0,g}(d\vec{x})g_i(\vec{x}) \quad \text{as } N \to \infty \text{ for fixed } t, \eta \in \Omega_N, i = 1,2, \tag{1.11}$$

where $\Gamma_{\vec{\theta}}^{c_0,g}$ denotes the *equilibrium distribution* of (1.9). Thus, if we set

$$(F_{c_0}g)_i(\vec{\theta}) = \int_{[0,\infty)^2} \Gamma_{\vec{\theta}}^{c_0,g}(d\vec{x})g_i(\vec{x}), \quad i = 1,2, \vec{\theta} \in [0,\infty)^2, \tag{1.12}$$

then by (1.11), for large N, the SDE (1.8) for the 1-block averages $\vec{Y}_\eta^{[1]}$ takes exactly the same form as the SDE (1.7) for the single components, provided that we rescale time by a factor N and replace the single component diffusion functions g_i by $(F_{c_0}g)_i$ $(i = 1, 2)$. Here, F_{c_0} plays the role of a *renormalization transformation* acting on the pair of diffusion functions $g = (g_1, g_2)$.

We can *iterate* the above procedure. The upshot of this is that, as $N \to \infty$, the k-block averages $\vec{Y}_\eta^{[k]}$ evolve on the time scale tN^k according to the SDE

$$dZ_i^{[k]}(t) = c_k\big(\theta_i - Z_i^{[k]}(t)\big)\,dt + \sqrt{2\big(F^{[k]}g\big)_i\big(\vec{Z}^{[k]}(t)\big)}\,dB_i(t), \quad i = 1, 2, \tag{1.13}$$

with diffusion functions $F^{[k]}g = (F^{[k]}g_1, F^{[k]}g_2)$ given by

$$F^{[k]}g = F_{c_{k-1}} \circ \cdots \circ F_{c_0}g, \quad k \in \mathbb{N}_0. \tag{1.14}$$

In fact, putting the successive iterates together and observing the sequence of block averages

$$\big(\vec{Y}_{\phi^k\eta}^{[k]}(sN^k), \vec{Y}_{\phi^{k-1}\eta}^{[k-1]}(sN^k), \dots, \vec{Y}_\eta^{[0]}(sN^k)\big) \tag{1.15}$$

on the time scale sN^k, as $N \to \infty$, we expect this sequence to converge in distribution to a *backward Markov chain*

$$\big(\vec{M}(-k), \vec{M}(-k+1), \dots, \vec{M}(0)\big), \tag{1.16}$$

the so-called *interaction chain*, where

(1) The starting position $\vec{M}(-k)$ is distributed as the weak solution of (1.13) at time s with initial condition $\vec{Z}^{[k]}(0) = \vec{\theta}$;
(2) for $0 \le j \le k - 1$, the transition probability kernel from $\vec{M}(-j-1)$ to $\vec{M}(-j)$ is given by

$$\mathbb{P}\big[\vec{M}(-j) \in d\vec{y} | \vec{M}(-j-1) = \vec{x}\big] = \Gamma_{\vec{x}}^{c_j, F^{[j]}g}(d\vec{y}), \tag{1.17}$$

where $\Gamma_{\vec{x}}^{c_j, F^{[j]}g}(\cdot)$ denotes the equilibrium distribution of (1.13) with k replaced by j.

The distribution of $\vec{M}(-k)$ depends on s because $\vec{Y}_{\phi^k\eta}^{[k]}(sN^k)$ evolves on the time scale sN^k, while the transition probability kernel from $\vec{M}(-j-1)$ to $\vec{M}(-j)$ for $0 \le j \le k-1$ is independent of t because, conditioned on $\vec{Y}_{\phi^{j+1}\eta}^{[j+1]}$, $\vec{Y}_{\phi^j\eta}^{[j]}$ equilibrates almost instantly on the time scale sN^k. Note that $(F^{[k]}g)_i(\vec{\theta}) = \mathbb{E}[g_i(\vec{M}(0))|\vec{M}(-k) = \vec{\theta}]$, where $\mathbb{E}$ denotes expectation with respect to the interaction chain.

With these heuristics in mind, the *renormalization program* consists of the following two steps:

(I) *Stochastic part:* Show that for all scales $k \in \mathbb{N}$, in the hierarchical mean-field limit $N \to \infty$, the block average in (1.6) converges in law to the solution of the SDE in (1.13), and the sequence of block averages in (1.15) converges in law to the interaction chain in (1.16).
(II) *Analytic part*: Analyze the renormalization transformation F_c and the iterates $F^{[n]}$, $n \in \mathbb{N}_0$.

Assuming that the stochastic part of the renormalization program can be completed, the large-scale space–time behavior of (1.1) in the limit $N \to \infty$ is characterized by the behavior of $F^{[n]}$ as $n \to \infty$, in particular, by its *fixed shapes* and their *universality classes*.

Here, by a fixed shape we mean a pair of diffusion functions $g = (g_1, g_2)$ such that $F_c g = \lambda g$ for some $c, \lambda > 0$. We speak of a *downgoing fixed shape, fixed point* or *upgoing fixed shape* depending on whether $\lambda < 1, = 1$, or > 1. Note that since the factor λ can always be absorbed in time-scaling, such fixed shapes correspond to models that are mapped into themselves after a suitable rescaling of space and time. Indeed, if we set $c_k = c\lambda^k$ $(k \ge 0)$, then such a fixed shape satisfies $F^{[k]}g = \lambda^k g$ because the SDE associated with $(c_k, F^{[k]}g)$ is simply a time change of the SDE associated with (c, g), which induces the same renormalization transformation. For the interacting model in (1.7), this means that the k-block averages evolve on the time scale $tN^k\lambda^k$ according to the diffusion function g. We note that our definition of a fixed shape deviates from the definition used in some earlier work, e.g. [21]. What is called a

fixed shape there is, in our terminology, a *joint fixed shape* for all $c > 0$, i.e., a g such that for all $c > 0$ there exists a $\lambda = \lambda(c)$ with $F_c g = \lambda g$.

By a universality class, we mean a set $\mathcal{G}$ of diffusion functions with the property that, given $(c_k)_{k \in \mathbb{N}_0}$, for each $g \in \mathcal{G}$ there exist scaling constants $(s_n)_{n \in \mathbb{N}}$ such that $s_n F^{[n]} g$ converges to the same limit (possibly up to a multiplicative constant). Typically, the limit will be a fixed shape or an *asymptotic fixed shape* (for the latter, see [21]). Note that each joint fixed shape gives rise to a universality class, namely all models within a given universality class exhibit the same large-scale space–time behavior.

Apart from being relevant in the study of large-scale space–time behavior, fixed shapes also give rise to continuum models, by taking the so-called *hierarchical mean-field continuum limit*, which is a spatial continuum limit of the hierarchical lattice Ω_N with $N \to \infty$. These continuum models also exhibit universality on small space–time scales, which is governed by the same renormalization transformation F_c and its iterates $F^{[n]}$, $n \in \mathbb{N}_0$. For more details, see [7] and [14].

The large-scale space–time behavior of (1.1) depends both on the diffusion function g and on the potential-theoretic properties of the random walk with transition rate kernel (1.5). Based on earlier work, we expect nontrivial universality classes to arise only when $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$, which is the "necessary and sufficient" condition for the random walk with transition rate kernel $a_N(\cdot, \cdot)$ on Ω_N to be recurrent (except for a side condition that becomes irrelevant in the limit $N \to \infty$; see [27]). For linear systems such as (1.1), the recurrence of the random walk is usually associated with *clustering*; see e.g. [8,11,30]. In our context, clustering means that the solution of (1.1) converges in law to a mixture of distributions, each of which is concentrated on the configuration $\vec{X}_\eta = \vec{x}$, $\eta \in \Omega_N$, for some $\vec{x} \in [0, \infty)^2$ with $g_1(\vec{x}) = g_2(\vec{x}) = 0$. The choice of $(c_n)_{n \in \mathbb{N}_0}$ determines the pattern of cluster formation, such as whether only small clusters appear, or only large clusters appear, or clusters of all scales appear. The latter is known as *diffusive clustering* (see e.g. [11,20]).

With the above facts in mind, the *analytic part* of the renormalization program can be more precisely formulated as follows.

1. Find classes of diffusion functions on which the renormalization transformations F_c and their iterates $F^{[n]}$, $n \in \mathbb{N}_0$, are well defined.
2. Determine all the (asymptotic) fixed shapes.
3. Determine the universality classes of diffusion functions that, for given $(c_n)_{n \in \mathbb{N}_0}$ and after appropriate rescaling, converge to these (asymptotic) fixed shapes, and determine the associated scaling constants.

1.3. *Literature*

The *full* renormalization program has been successfully carried out for hierarchically interacting diffusions taking values in:

(1) the compact interval $[0, 1]$ [2,10,11], where the Wright–Fisher diffusion is the unique fixed shape and is globally attracting with a scaling that is independent of the diffusion function;
(2) the halfline $[0, \infty)$ [3,12], where the Feller branching diffusion is the unique fixed point and is globally attracting with a scaling that depends on the asymptotic behavior of the diffusion function at infinity.

For higher-dimensional diffusions, the *analytic part* has been carried out for:

(3) isotropic diffusions taking values in a compact convex subset of $\mathbb{R}^d$ [24,31], where the diffusion function with constant curvature is the unique fixed shape and is globally attracting with a scaling that is independent of the diffusion function;
(4) a class of probability-measure-valued diffusions [13,15], where the Fleming–Viot process is the unique fixed shape and is globally attracting with a scaling that is independent of the diffusion function;
(5) a class of catalytic Wright–Fisher diffusions taking values in $[0, 1]^2$ [21], where the diffusion function for the first component is an autonomous Wright–Fisher diffusion and the diffusion function for the second component is an autonomous Wright–Fisher diffusion function multiplied by a *catalyzing function* depending only on the first component. The renormalization transformation effectively acts on the catalyzing function. There are four attracting shapes for the catalyzing function, depending on whether the initial catalyzing function is zero or strictly positive at the boundary points of $[0, 1]$, and these attracting shapes are globally attracting with a scaling that is independent of the catalyzing function.

The *stochastic part* for higher-dimensional diffusions has only been completed for interacting Fleming–Viot processes [13] and for mutually catalytic branching diffusions taking values in $[0, \infty)^2$ [7].

All previous studies deal with diffusions that have certain simplifying properties. In the one-dimensional cases (1) and (2), as well as in the two-dimensional case (5), the equilibrium of (1.9) is reversible. As a result, many explicit calculations can be performed that are crucial for the analysis. For certain diffusions with compact state space, which includes the cases (1), (3) and (4), there is a common underlying structure (called "invariant harmonics," see [30]) that allows the determination of the unique fixed shape and its domain of attraction. In all cases where the state space is compact, the scaling needed for convergence to an attracting shape depends only on $(c_n)_{n \in \mathbb{N}_0}$, not on the diffusion function g. This is different in case (2), where the state space is not compact. In all cases except case (5), the fixed shapes turn out to be joint fixed shapes for all $c > 0$.

The goal of the present paper is to carry out the *analytic part* of the renormalization program for a *general* class of branching diffusions taking values in $[0, \infty)^2$. The multi-dimensionality and the noncompactness of the state space pose significant challenges. Due to the multidimensionality, the well-definedness of the renormalization transformation is nontrivial. The structure of the fixed points/shapes turns out to be rather rich. In fact, we will prove that, under certain restrictions, the class of fixed points is a *4-parameter family of diffusions with independent branching, catalytic branching and mutually catalytic branching as the extremal fixed points*, and they are joint fixed points of F_c for all $c > 0$. Moreover, we will prove that all diffusion functions that are comparable to these fixed points in an appropriate sense fall in their domains of attraction.

1.4. *Outline*

The rest of the paper is organized as follows. In Section 2 we formulate our main results, which come with varying degrees of restrictions on the diffusion functions. Section 3 contains the proof of the ergodicity of the SDE (1.9), and basic properties of the renormalization transformation. Section 4 proves the identification of fixed points/shapes. Sections 5 and 6 identify the domains of attraction for the fixed points. In Appendices A and B we collect some technical results needed for the proofs.

2. Main results

In Section 2.1, we formulate a key class of diffusion functions $\mathcal{C}$, for which the SDE (1.9) has a unique weak solution. Section 2.2 contains a theorem on the ergodicity of the SDE (1.9), defines the renormalization transformation, formulates a subclass $\mathcal{H}_{0^+} \subset \mathcal{C}$ on which the renormalization transformation is well defined and, subject to a conjecture on the preservation of certain boundary properties, can be iterated. Section 2.3 gives the definition of certain generalized fixed points/shapes, and identifies some special fixed points/shapes. Section 2.4 contains results on the identification of fixed points/shapes in $\mathcal{H}_{0^+}$ under additional regularity assumptions. Section 2.5 contains our main result on the domains of attraction to the fixed points under further assumptions. Lastly, Section 2.6 provides a brief discussion of these results and lists some future challenges.

2.1. *Key class and uniqueness for the autonomous SDE*

The renormalization transformation F_c is based on (1.9), which is the SDE for the vector $\vec{X}(t) = (X_1(t), X_2(t)) \in [0, \infty)^2$ written out as

$$
\mathrm{d}X_1(t) = c\big[\theta_1 - X_1(t)\big]\,\mathrm{d}t + \sqrt{2g_1\big(X_1(t), X_2(t)\big)}\,\mathrm{d}B_1(t),
$$

$$
\mathrm{d}X_2(t) = c\big[\theta_2 - X_2(t)\big]\,\mathrm{d}t + \sqrt{2g_2\big(X_1(t), X_2(t)\big)}\,\mathrm{d}B_2(t), \tag{2.1}
$$

where $c > 0$, $\vec{\theta} = (\theta_1, \theta_2) \in [0, \infty)^2$, and $\vec{B}(t) = (B_1(t), B_2(t))$ are independent standard Brownian motions on $\mathbb{R}^2$. The corresponding generator is

$$
\big(L_{\vec{\theta}}^{c,g} f\big)(\vec{x}) = c \sum_{i=1}^{2}(\theta_i - x_i)\frac{\partial}{\partial x_i} f(\vec{x}) + \sum_{i=1}^{2} g_i(\vec{x})\frac{\partial^2}{\partial x_i^2} f(\vec{x}), \quad f \in C_c^2\big([0, \infty)^2\big). \tag{2.2}
$$

Note that, due to the absence of mixed partial derivatives, $L_{\vec{\theta}}^{c,g}$ can be interpreted as the generator of a two-type *branching diffusion* with state-dependent branching rates $g_i(\vec{x})/x_i$ $(i = 1, 2)$.

Abbreviate

$$A_1 = [0, \infty) \times \{0\}, \qquad A_2 = \{0\} \times [0, \infty). \tag{2.3}$$

We will say that a function $f: [0, \infty)^2 \to [0, \infty)$ has *boundary property*

(∂_1) if $\lim_{\vec{x} \to \vec{y}} \dfrac{f(\vec{x})}{x_1} = \gamma(\vec{y}) \; \forall \vec{y} \in A_1 \cup A_2$ with γ continuous and > 0 on $A_1 \cup A_2$,

(∂_2) if $\lim_{\vec{x} \to \vec{y}} \dfrac{f(\vec{x})}{x_2} = \gamma(\vec{y}) \; \forall \vec{y} \in A_1 \cup A_2$ with γ continuous and > 0 on $A_1 \cup A_2$,

(∂_{12}) if $\lim_{\vec{x} \to \vec{y}} \dfrac{f(\vec{x})}{x_1 x_2} = \gamma(\vec{y}) \; \forall \vec{y} \in A_1 \cup A_2$ with γ continuous and > 0 on $A_1 \cup A_2$. $\hspace{1cm}$ (2.4)

Throughout the paper, the pair $g = (g_1, g_2)$ will be assumed to be in the following class.

Definition 2.1 (Class $\mathcal{C}$). *Let $\mathcal{C}$ be the class of functions $g(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}))$ satisfying:*

(i) *For $i = 1, 2$, g_i is continuous on $[0, \infty)^2$ and > 0 on $(0, \infty)^2$.*
(ii) *For $i = 1, 2$, g_i satisfies boundary property (∂_i) or (∂_{12}).*

Note that for $(g_1, g_2) \in \mathcal{C}$ we can write $g_i(\vec{x}) = x_i \gamma_i(\vec{x})$ or $g_i(\vec{x}) = x_1 x_2 \gamma_i(\vec{x})$ for some positive continuous function γ_i on $[0, \infty)^2$, depending on whether g_i satisfies boundary property (∂_i) or (∂_{12}). Note also that g_1 and g_2 vanish on A_2, respectively, A_1, which is necessary to guarantee that the diffusion stays within $[0, \infty)^2$. Thus, if we denote the *effective boundary* of g by

$$\partial g = \left\{ \vec{x} \in [0, \infty)^2 : g_1(\vec{x}) = g_2(\vec{x}) = 0 \right\}, \tag{2.5}$$

then ∂g can be either of the following:

$$A_1 \cap A_2, \qquad A_1, \qquad A_2, \qquad A_1 \cup A_2. \tag{2.6}$$

These boundary constraints allow for the system (2.1) to be treated as a perturbation of either of the following diffusions:

(1) *Independent branching*: $(g_1, g_2) = (b_1 x_1, b_2 x_2)$, $b_1, b_2 > 0$, $\partial g = A_1 \cap A_2$.
(2) *Catalytic branching*: either $(g_1, g_2) = (b_1 x_1, c_2 x_1 x_2)$, $b_1, c_2 > 0$, $\partial g = A_2$; or $(g_1, g_2) = (c_1 x_1 x_2, b_2 x_2)$, $c_1, b_2 > 0$, $\partial g = A_1$.
(3) *Mutually catalytic branching*: $(g_1, g_2) = (c_1 x_1 x_2, c_2 x_1 x_2)$, $c_1, c_2 > 0$, $\partial g = A_1 \cup A_2$.

Such a perturbation is behind the following result of [1] and [6], which provides the starting point of our analysis. The latter paper improves results in [17], where Hölder continuity is assumed rather than continuity.

Theorem 2.2 (Well-posedness of martingale problem [1,6]). *For all $c > 0$, $g \in \mathcal{C}$, $\vec{\theta} \in [0, \infty)^2$ and $\vec{x} \in [0, \infty)^2$, with the possible exception of the case when $\vec{x} = (0, 0)$, $\vec{\theta} \in (0, \infty)^2$, and either g_1 or g_2 satisfies boundary property (∂_{12}), the martingale problem associated with the generator in (2.2) has a unique solution with starting position $\vec{x}$.*

As a consequence of Theorem 2.2, the SDE (2.1) has a unique weak solution for all $\vec{\theta} \in [0, \infty)^2$ and $\vec{x} \in [0, \infty)^2$, with the possible exception of the case when $\vec{x} = (0, 0)$, $\vec{\theta} \in (0, \infty)^2$, and either g_1 or g_2 satisfies boundary property (∂_{12}). For each fixed $\vec{\theta} \in [0, \infty)^2$, the SDE (2.1) defines a Feller process satisfying the strong Markov property (see e.g. Theorem 4.4.2 in [19] and Corollary 11.1.5 in [29]).

Remark 1. *When $\vec{\theta} \in (0, \infty)^2$, $g \in \mathcal{C}$, g_1 and g_2 satisfy (∂_1), resp. (∂_2), the well-posedness of the martingale problem was established in [1] for all initial conditions $\vec{x} \in [0, \infty)^2$. When $\vec{\theta} \in (0, \infty)^2$, $g \in \mathcal{C}$, and g_1, g_2 both satisfy (∂_{12}), the well-posedness is established in [6] for all initial condition $\vec{x} \in [0, \infty)^2 \backslash \{(0, 0)\}$. Both [1] and [6] use local perturbation arguments and the results are not restricted to linear drift as considered here. Since the perturbation arguments are local, this implies that well-posedness also holds for mixed boundaries, i.e., g_1 satisfies (∂_1) and g_2 satisfies (∂_{12}), or vice versa. When either g_1 or g_2 satisfies (∂_{12}), Lemma 35 of [17] shows that, for all $\vec{x} \in [0, \infty)^2 \backslash (0, 0)$, with probability 1 the unique weak solution of (2.1) with initial condition $\vec{x}$ never hits $(0, 0)$, and hence we can restrict the state space to $[0, \infty)^2 \backslash \{(0, 0)\}$. When $\vec{\theta} \in \partial [0, \infty)^2$, the local analysis of [1] and [6] still applies until the diffusion first hits the absorbing boundary, at which time the diffusion becomes one-dimensional, a situation for which the well-posedness of the martingale problem is standard.*

Remark 2. *The proof given in [1] requires the drift to be strictly positive in each component on $\partial [0, \infty)^2$. However, as pointed out in [5], it is sufficient that the inward normal component of the drift is strictly positive on $\partial [0, \infty)^2$, which holds in our setting when $\vec{\theta} \in (0, \infty)^2$.*

Remark 3. *It would be considerably more difficult to deduce from Theorem 2.2 the well-posedness of the martingale problem for the system (1.1), for which one would need to restrict the state space. To deduce the Feller property, one would need to restrict the state space even further and impose growth conditions on the diffusion function g, typically $g_1(\vec{x}) + g_2(\vec{x}) = O(x_1^2 + x_2^2)$ (see, e.g., [7,28]). We will not resolve these issues here, since they belong to the stochastic part of the renormalization program, which remains open.*

2.2. *Equilibrium distribution and renormalization transformation*

Our first result shows that (2.1) has a unique equilibrium for the class $\mathcal{C}$. The proof will be given in Section 3.1. Henceforth $\mathcal{L}$ denotes law.

Theorem 2.3 (Equilibrium distribution). *For all $g \in \mathcal{C}$, $\vec{\theta} \in [0, \infty)^2$ and $c > 0$, (2.1) has a unique equilibrium distribution $\Gamma_{\vec{\theta}}^{c,g}$, which is continuous in $\vec{\theta}$ with respect to weak convergence of probability measures, and*

$$\mathcal{L}(\vec{X}(t)) \underset{t \to \infty}{\Longrightarrow} \Gamma_{\vec{\theta}}^{c,g} \quad \forall \vec{X}(0) \in [0, \infty)^2. \tag{2.7}$$

The convergence in (2.7) is crucial for the *stochastic part* of the renormalization program (not considered here), while the uniqueness of the equilibrium is crucial for the definition of the *renormalization transformation*, which we now define.

Definition 2.4 (Renormalization transformation). *The renormalization transformation F_c, acting on $g \in \mathcal{C}$, is defined as*

$$(F_c g)_i(\vec{\theta}) = \int_{[0, \infty)^2} g_i(\vec{x}) \Gamma_{\vec{\theta}}^{c,g}(d\vec{x}), \quad \vec{\theta} \in [0, \infty)^2, c > 0, i = 1, 2. \tag{2.8}$$

Henceforth we will denote expectation with respect to $\Gamma_{\vec{\theta}}^{c,g}$ by $\mathbb{E}_{\vec{\theta}}^{c,g}$.

Without restrictions on the growth of g at infinity, it is possible that $F_c g$ is infinite. We therefore need to consider a tempered subclass of $\mathcal{C}$.

Definition 2.5 (Class $\mathcal{H}_{0+}$).

(i) *For $a \geq 0$, let $\mathcal{H}_a \subset \mathcal{C}$ be the class of all $g \in \mathcal{C}$ satisfying*

$$g_1(x_1, x_2) + g_2(x_1, x_2) \leq C(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2), \quad (x_1, x_2) \in [0, \infty)^2, \tag{2.9}$$

for some $0 < C = C(g) < \infty$.

(ii) *Let*

$$\mathcal{H}_{0^+} = \bigcap_{a>0} \mathcal{H}_a. \tag{2.10}$$

Note that $\mathcal{H}_{0^+}$ is much larger than $\mathcal{H}_0$. In particular, $\mathcal{H}_{0^+}$ includes diffusion functions that along the axes grow faster than linear but slower than quadratic.

Our second result shows that F_c is well defined on the class $\mathcal{H}_a$ when $0 \le a < c$, preserves the effective boundary, and preserves the growth bound in (2.9) though with a different coefficient. The proof will be given in Section 3.2.

Theorem 2.6 (Finiteness, continuity, preservation of ∂g and growth bound). *For $c > 0$ and $0 \le a < c$, if $g \in \mathcal{H}_a$, then $F_c g$ is finite and continuous on $[0, \infty)^2$, $\partial F_c g = \partial g$, and $F_c g$ satisfies (2.9) with a replaced by $\frac{c}{c-a}a$.*

To proceed with our analysis, we need the following:

Conjecture 2.7 (Preservation of boundary properties). *Let $g \in \mathcal{H}_{0^+}$.*

(i) *For $i = 1, 2$, if g_i satisfies (∂_i), then so does $(F_c g)_i$ for all $c > 0$.*
(ii) *For $i = 1, 2$, if g_i satisfies (∂_{12}), then so does $(F_c g)_i$ for all $c > 0$.*

In Section 3.3 we will explain why this conjecture is plausible. Combining Theorem 2.6 with Conjecture 2.7, we get:

Corollary 2.8 (Preservation of class $\mathcal{H}_{0^+}$). *For all $c > 0$, the class $\mathcal{H}_{0^+}$ is preserved under F_c, i.e., $F_c g \in \mathcal{H}_{0^+}$ for all $g \in \mathcal{H}_{0^+}$.*

The latter is a key property, because it allows us to *iterate* F_c on $\mathcal{H}_{0^+}$ and investigate the orbit $F^{[n]} g = F_{c_{n-1}} \circ \cdots \circ F_{c_0} g$, $n \in \mathbb{N}_0$. We will not need Conjecture 2.7 or Corollary 2.8 until we study the iterates $F^{[n]}$ in Section 2.5.

The subquadratic growth bound imposed by $\mathcal{H}_{0^+}$ cannot be relaxed: we will see in Corollary 2.11 that F_c cannot be iterated indefinitely on $\mathcal{H}_a$ for any $a > 0$.

2.3. *Definition and examples of fixed points and fixed shapes*

We next give the definition of fixed points and fixed shapes of F_c. Generalizing our definition given in the Introduction, we allow for the case where $F_c g = \lambda g$ with λ not a constant but a diagonal matrix. These *generalized fixed shapes* do not give rise to universality classes as defined in Section 1.2, but they may be relevant for studying finer properties of the orbit $(F^{[n]} g)_{n \in \mathbb{N}_0}$.

Definition 2.9 (Generalized fixed shapes and points). *The pair $g = (g_1, g_2) \in \mathcal{H}_a$ with $a \in [0, c)$ is called a generalized fixed shape of F_c if*

$$F_c(g_1, g_2) = (\lambda_1 g_1, \lambda_2 g_2) \quad \text{for some } \lambda_1, \lambda_2 > 0. \tag{2.11}$$

If $\lambda_1 = \lambda_2$, then g is called a fixed shape, and if $\lambda_1 = \lambda_2 = 1$, then g is called a fixed point of F_c.

Our third result identifies a family of fixed points and (generalized) fixed shapes of F_c. The proof is nontrivial because of integrability issues, and will be given in Section 3.2.

Theorem 2.10 (Examples of fixed points and fixed shapes).

(i) *The pair*

$$(g_1, g_2) = (b_1 x_1 + c_1 x_1 x_2, b_2 x_2 + c_2 x_1 x_2) \tag{2.12}$$

is a fixed point of F_c in $\mathcal{H}_{0^+}$ for all $c > 0$ and all $b_1, b_2, c_1, c_2 \ge 0$ with $(b_1 + c_1)(b_2 + c_2) > 0$.

(ii) *The pair*

$$(g_1, g_2) = \left(a_1 x_1^2 + b_1 x_1 + c_1 x_1 x_2, a_2 x_2^2 + b_2 x_2 + c_2 x_1 x_2\right) \tag{2.13}$$

is a generalized fixed shape of F_c in $\mathcal{H}_{a_1 \vee a_2}$ for all $c > 0$, $0 < a_1, a_2 < c$ and $b_1, b_2, c_1, c_2 \geq 0$. The corresponding scaling constants are

$$\lambda_1 = \frac{c}{c - a_1}, \qquad \lambda_2 = \frac{c}{c - a_2}. \tag{2.14}$$

Diffusion functions of the form in (2.12) are *mixtures* of independent branching, catalytic branching and mutually catalytic branching (recall Section 2.1), all of which are in the class $\mathcal{H}_{0+}$. We will see in Theorem 2.15 that, under additional regularity conditions, such mixtures are the *only* fixed points of F_c. Diffusion functions of the form in (2.13) are mixtures of these fixed points and the Anderson branching diffusion $(g_1, g_2) = (a_1 x_1^2, a_2 x_2^2)$. The latter do not fall in the class $\mathcal{H}_{0+}$.

The following corollary of Theorem 2.10 shows that $F_c g$ cannot be defined for all $g \in \mathcal{H}_a$ with $a \geq c$, and F_c cannot be iterated indefinitely on $\mathcal{H}_a$ for any $a > 0$. The proof will be given in Section 3.2.

Corollary 2.11 (Divergence of iterated fixed shapes). *Let $g_i(\vec{x}) = \alpha_i x_i^2 + \beta_i x_i + \gamma_i x_1 x_2$ with $\alpha_i > 0$ and β_i, $\gamma_i \geq 0$, $i = 1, 2$. Let $(c_n)_{n \in \mathbb{N}_0}$ be the positive sequence that defines $F^{[n]}$ (see (1.14)). Let $n_0 = \min\{n \in \mathbb{N}: (\alpha_1 \vee \alpha_2) \sum_{i=0}^{n-1} c_i^{-1} \geq 1\}$. Then*

$$\left(\left(F^{[n]}g\right)_1, \left(F^{[n]}g\right)_2\right) = \left(\frac{1}{1 - \alpha_1 \sum_{i=0}^{n-1} c_i^{-1}} g_1, \frac{1}{1 - \alpha_2 \sum_{i=0}^{n-1} c_i^{-1}} g_2\right), \quad 0 \leq n < n_0, \tag{2.15}$$

while $(F^{[n_0]}g)_1 + (F^{[n_0]}g)_2 \equiv \infty$ on $(0, \infty)^2$.

2.4. *Identification of fixed points and fixed shapes*

Our fourth result rules out generalized fixed shapes in $\mathcal{H}_{0+}$ with an upgoing component. The proof will be given in Section 4.3.

Theorem 2.12 (No fixed shapes in $\mathcal{H}_{0+}$ with an upgoing component). *For $c > 0$, there is no $g \in \mathcal{H}_{0+}$ such that either $(F_c g)_1 = \lambda_1 g_1$ with $\lambda_1 > 1$ or $(F_c g)_2 = \lambda_2 g_2$ with $\lambda_2 > 1$.*

Our fifth result does the same for generalized fixed shapes with a downgoing component, but only under mild additional regularity conditions. The proof will be given in Section 4.3. Below, in line with general topological notation, $\liminf_{\vec{x} \to (\infty, \infty)}$ denotes the infimum of all limits along sequences tending to (∞, ∞).

Theorem 2.13 (Sufficient conditions for no downgoing fixed shapes in $\mathcal{H}_{0+}$). *Let $c > 0$.*

(i) *There is no $g \in \mathcal{H}_{0+}$ such that $F_c(g_1, g_2) = (\lambda_1 g_1, \lambda_2 g_2)$ with $0 < \lambda_1, \lambda_2 < 1$ and*

$$\liminf_{\vec{x} \to (\infty, \infty)} \left(\frac{g_1(\vec{x})}{x_1^2} + \frac{g_2(\vec{x})}{x_2^2}\right) = 0. \tag{2.16}$$

(ii) *There is no $g \in \mathcal{H}_{0+}$ such that $(F_c g)_1 = \lambda_1 g_1$ for some $0 < \lambda_1 < 1$ and g satisfies any of the following conditions*:

- $g_1 > 0 \quad on\ A_1 \setminus \{(0,0)\}$, $\tag{2.17}$

- $\displaystyle \liminf_{\vec{x} \to (\infty, \infty)} \frac{g_1(\vec{x})}{x_1 x_2} > 0.$ $\tag{2.18}$

A similar result holds with the indices 1 *and* 2 *interchanged.*

Remark. *Conditions* (2.16) *and* (2.18) *are complementary. Note that one particular case not covered by conditions* (2.16)–(2.18) *is when g_1 vanishes on both axes, $g_1(\vec{x}) = o(x_1 x_2)$ as $\vec{x} \to (\infty, \infty)$, and $g_2(\vec{x}) = x_1 x_2$. In that case we cannot rule out the possibility of g_1 being a downgoing fixed shape.*

In Theorem 2.10 we identified a 4-parameter family of fixed points. To show that these are the only fixed points, we need to impose strong additional regularity conditions.

Abbreviate

$$R_\infty = \big\{(0, \infty), (\infty, 0), (\infty, \infty)\big\} \tag{2.19}$$

and

$$h_{(\infty,0)}(\vec{x}) = x_1, \qquad h_{(0,\infty)}(\vec{x}) = x_2, \qquad h_{(\infty,\infty)}(\vec{x}) = x_1 x_2. \tag{2.20}$$

Definition 2.14 (Class $\mathcal{H}_0^r$). *Let $\mathcal{H}_0^r$ be the set of $g \in \mathcal{H}_0$ satisfying*

(i) $\quad \displaystyle\inf_{\vec{x} \in [s,\infty)^2} g_i(\vec{x}) > 0 \qquad\qquad \forall s > 0, i = 1, 2,$ $\hfill (2.21)$

(ii) $\quad \displaystyle\lim_{\vec{x} \to \vec{z}} \frac{g_i(\vec{x})}{h_{\vec{z}}(\vec{x})} = \lambda_{i,\vec{z}} \in [0, \infty) \quad \forall \vec{z} \in R_\infty, i = 1, 2.$ $\hfill (2.22)$

Note that $\mathcal{H}_0^r \subset \mathcal{H}_0 \subset \mathcal{H}_{0^+}$. Also note that, because g_1 vanishes on A_2 and g_2 on A_1, necessarily $\lambda_{1,(0,\infty)} = \lambda_{2,(\infty,0)} = 0$.

Our sixth result is the following. The proof will be given in Section 4.1.

Theorem 2.15 (Identification of fixed points in $\mathcal{H}_0^r$). *Let $c > 0$ and $g = (g_1, g_2) \in \mathcal{H}_0^r$. If $F_c(g_1, g_2) = (g_1, g_2)$, then*

$$\begin{aligned} g_1(\vec{x}) &= \lambda_{1,(\infty,0)} x_1 + \lambda_{1,(\infty,\infty)} x_1 x_2, \\ g_2(\vec{x}) &= \lambda_{2,(0,\infty)} x_2 + \lambda_{2,(\infty,\infty)} x_1 x_2, \end{aligned} \tag{2.23}$$

where $\lambda_{i,\vec{z}}, \vec{z} \in R_\infty$, are defined in (2.22).

2.5. *Domain of attraction of fixed points*

Our seventh and final result is on the domain of attraction of the iterated maps $F^{[n]} = F_{c_{n-1}} \circ \cdots \circ F_{c_0}$, $n \in \mathbb{N}_0$, for a fixed positive sequence $(c_n)_{n \in \mathbb{N}_0}$. We show that, provided $\inf_{n \in \mathbb{N}_0} c_n > 0$ and $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$, all diffusion functions that are comparable to a mixture of the fixed points fall into its domain of attraction. In Section 5, we will give the proof for the special case $c_n \equiv c$, while in Section 6, we prove the result for varying c_n.

Theorem 2.16 (Domain of attraction of fixed points). *Let $(c_n)_{n \in \mathbb{N}_0}$ be a sequence such that $\inf_{n \in \mathbb{N}_0} c_n > 0$ and $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$. Let $g \in \mathcal{H}_0^r$ be such that*

$$g_i(\vec{x}) \geq \alpha_i x_i + \beta_i x_1 x_2, \quad \alpha_i, \beta_i \geq 0, \alpha_i + \beta_i > 0, i = 1, 2. \tag{2.24}$$

Then

$$\lim_{n \to \infty} \big(F^{[n]} g\big)_i(\vec{\theta}) = \sum_{\vec{z} \in R_\infty} \lambda_{i,\vec{z}} h_{\vec{z}}(\vec{\theta}) \quad \forall \vec{\theta} \in [0, \infty)^2, i = 1, 2, \tag{2.25}$$

where $h_{\vec{z}}, \lambda_{i,\vec{z}}, \vec{z} \in R_\infty$, are defined in (2.20) *and* (2.22).

What this says is that under the iterates $F^{[n]}$, any g that is properly *minorized* and has the *same behavior at infinity* as a mixture of the fixed points, converges to that mixture pointwise as $n \to \infty$.

Remark 1. *Note that Theorem* 2.16 *implicitly assumes Conjecture* 2.7. *To be formally correct, in Theorem* 2.16 *we should replace $\mathcal{H}_0^r$ by the largest subclass of $\mathcal{H}_0^r$ that is preserved by F_c for all $c > 0$.*

Remark 2. *The condition $\inf_{n \in \mathbb{N}_0} c_n > 0$ means that we partially exclude the regime of large clusters (see e.g.* [11]). *We do not believe this assumption to be essential. As long as $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$, i.e., the associated random walk on Ω_N with transition rate kernel $a_N(\cdot, \cdot)$ is recurrent, we expect there to be universality and the convergence in* (2.25) *to hold.*

2.6. *Discussion and future challenges*

The results in Sections 2.2–2.5 constitute a *partial completion* of the analytic part of the renormalization program outlined in Section 1.2. We have formulated $\mathcal{H}_{0+}$ as the class on which the renormalization transformation is properly defined and, apart from Conjecture 2.7, it can be iterated. We have proved absence of upgoing fixed shapes in this class, and absence of downgoing fixed shapes under mild regularity conditions, given by (2.16)–(2.18). Furthermore, we have identified our 4-parameter family of fixed points in (2.12) as the only fixed points in a subclass $\mathcal{H}_0^r$ of the smaller class $\mathcal{H}_0$, given by the strong regularity conditions (2.21) and (2.22). Finally, we have found the domain of attraction of these fixed points in $\mathcal{H}_0^r$ supplemented with the lower bound (2.24), i.e., diffusion functions that are comparable to a mixture of the fixed shapes. There are several open problems remaining, the chief among which are:

(1) Verify Conjecture 2.7, i.e., establish that the renormalization transformation can be iterated on $\mathcal{H}_{0+}$.
(2) Remove assumptions (2.16)–(2.18) in the proof of the absence of downgoing fixed shapes in $\mathcal{H}_{0+}$.
(3) Show that the fixed points in (2.12) are the only fixed points in $\mathcal{H}_{0+}$. In particular, remove assumption (2.22) and the bound $g_1(\vec{x}) + g_2(\vec{x}) \leq C(1 + x_1)(1 + x_2)$ in $\mathcal{H}_0^r \subset \mathcal{H}_0$.
(4) Strengthen (2) and (3) by determining whether it is actually true that the fixed shapes in (2.13) are the only fixed shapes in $\mathcal{C}$.
(5) Study the orbit of $(F^{[n]}g)_{n \in \mathbb{N}_0}$ when the behavior of g at infinity is *different* from that of the fixed points. In that case we still expect convergence, but only after $F^{[n]}g$ is scaled with n in some appropriate manner. For diffusions on the halfline $[0, \infty)$, this study was successfully completed in [3], which raises some hope that it can be carried through on the quadrant as well.

The questions we treated in this paper and the open problems we just mentioned have close connections to probabilistic potential theory of diffusions and Markov chains taking values in the quadrant. Our proofs strongly lean on the observation that the fixed points we build are mixtures of *extremal universal harmonic functions* of the interaction chains described in Section 1.2. The problem of finding all fixed points then requires identifying the *universal Martin boundary* of these Markov chains. The reader interested in this point of view can find the necessary concepts in [26]. Harmonic functions have played an important role in earlier studies of the analytic part of the renormalization program. In particular, the convergence proofs in the cases (1), (3) and (4) mentioned in Section 1.3 all depend on a special property of these models, called "invariant harmonics" (see [30]). Case (2) uses moment equations combined with comparison arguments, while case (5) uses a representation in terms of a superprocess. Due to multi-dimensionality and noncompactness, these tools either do not apply or are insufficient for our model. However, our present methods have their limitations as well. In particular, in their present state they can only be used to prove convergence to *joint fixed points of F_c for all $c > 0$*, as opposed to fixed shapes, or cases where there might be different fixed points of F_c for different values of c. Moreover, we can treat only functions that are perturbations of these fixed points, albeit in a rather large class.

Another interesting question is to study multi-type branching models with more than two types. The class of random catalytic networks introduced in [17] and generalized in [25] provide a rich class of fixed points of the renormalization transformation. However our results here do not extend trivially to higher dimensions, because we need the well-posedness of the martingale problem (Theorem 2.2), which is more delicate in higher dimensions. Also, our proof of the formula (A.3) for the mixed moment $X_1 X_2$ does not extend to mixed moments of higher order.

3. Proofs of Theorems 2.3, 2.6, 2.10 and Corollary 2.11

In Section 3.1 we give the proof of Theorem 2.3, in Section 3.2 that of Theorems 2.6, 2.10 and Corollary 2.11. In Section 3.3 we discuss Conjecture 2.7. Along the way we need a proposition on moment equations for the equilibrium distribution $\Gamma_{\vec{\theta}}^{c,g}$, which will turn out to be fundamental in our analysis. This proposition is formulated and proved in Appendix A.

3.1. *Proof of Theorem* 2.3

We break down the proof of Theorem 2.3 into four parts: *existence*, *uniqueness*, *weak continuity* and *convergence*. For uniqueness and convergence, we need to distinguish between $\vec{\theta} \in (0, \infty)^2$ and $\vec{\theta} \in \partial[0, \infty)^2$.

Existence

Proof. If we denote the distribution of $\vec{X}(t)$ by μ_t, with $\mu_0 = \delta_{\vec{x}}$ for some arbitrary $\vec{x} \in [0, \infty)^2$, then it suffices to show that $\{\nu_t \colon \nu_t = \frac{1}{t} \int_0^t \mu_s \, ds\}_{t \geq 0}$ forms a tight family of distributions on $[0, \infty)^2$. Indeed, we can then find a sequence (t_n) tending to infinity such that ν_{t_n} converges weakly to a limiting distribution ν. Consequently, for any $f \in C_c^2([0, \infty)^2)$,

$$
\int \left(L_{\vec{\theta}}^{c,g} f \right)(\vec{x}) \nu(d\vec{x})
$$

$$
= \lim_{n \to \infty} \int \left(L_{\vec{\theta}}^{c,g} f \right)(\vec{x}) \nu_{t_n}(d\vec{x}) = \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \int \left(L_{\vec{\theta}}^{c,g} f \right)(\vec{x}) \mu_s(d\vec{x}) \, ds
$$

$$
= \lim_{n \to \infty} \frac{1}{t_n} \mathbb{E}_{\mu_0} \left[\int_0^{t_n} \left(L_{\vec{\theta}}^{c,g} f \right)(\vec{X}(s)) \, ds \right] = \lim_{n \to \infty} \frac{1}{t_n} \mathbb{E}_{\mu_0} \left[f(\vec{X}(t_n)) - f(\vec{X}(0)) \right] = 0, \tag{3.1}
$$

where the first line uses that ν_{t_n} converges weakly to ν, the second line uses the definition of ν_{t_n}, the third lines uses the definition of μ_s and Fubini, and the fourth line uses that $f(\vec{X}(t)) - f(\vec{X}(0)) - \int_0^t (L_{\vec{\theta}}^{c,g} f)(\vec{X}(s)) \, ds$ is a martingale and f is bounded. Since $\int (L_{\vec{\theta}}^{c,g} f)(\vec{x}) \nu(d\vec{x}) = 0$ for all $f \in C_c^2([0, \infty)^2)$, which form an algebra of functions that is dense in the space of continuous functions on $[0, \infty)^2$ vanishing at ∞, it follows from Theorem 4.9.17 in [19] that ν is an equilibrium distribution for (2.1).

Tightness of the family $\{\nu_t\}_{t \geq 0}$ follows from the following lemma.

Lemma 3.1 (Tightness estimate). *Let $(\vec{X}(t))_{t \geq 0}$ be the unique solution of the martingale problem for $L_{\vec{\theta}}^{c,g}$ with initial condition $\vec{X}(0) = \vec{x}$. Then*

$$
\mathbb{E}\left[X_i(t) - \theta_i \right] \leq (x_i - \theta_i) e^{-ct}, \quad i = 1, 2, t \geq 0. \tag{3.2}
$$

Proof. For any $\rho_1, \rho_2 > 0$, the function $f(t, \vec{x}) = \sum_{i=1}^2 \rho_i (x_i - \theta_i) e^{ct}$ satisfies

$$
\left(L_{\vec{\theta}}^{c,g} + \frac{\partial}{\partial t} \right) f(t, \vec{x}) = c \sum_{i=1}^2 \rho_i (\theta_i - x_i) e^{ct} + c \sum_{i=1}^2 \rho_i (x_i - \theta_i) e^{ct} = 0, \tag{3.3}
$$

and therefore the process $\sum_{i=1}^2 \rho_i (X_i(t) - \theta_i) e^{ct}$ is a local martingale. Introduce stopping times

$$
\tau_n = \inf \left\{ t \geq 0 \colon \sum_{i=1}^2 \rho_i X_i(t) \geq n \right\}, \quad n \in \mathbb{N}. \tag{3.4}
$$

Then

$$
\sum_{i=1}^{2} \rho_i (x_i - \theta_i) = \sum_{i=1}^{2} \rho_i \mathbb{E}\big[\big(X_i(t \wedge \tau_n) - \theta_i \big) e^{c(t \wedge \tau_n)} \big]
$$

$$
= \sum_{i=1}^{2} \rho_i \mathbb{E}\big[\big(X_i(t) - \theta_i \big) e^{ct} 1_{\{\tau_n > t\}} \big] + \sum_{i=1}^{2} \rho_i \mathbb{E}\big[\big(X_i(\tau_n) - \theta_i \big) e^{c\tau_n} 1_{\{\tau_n \leq t\}} \big]. \tag{3.5}
$$

For $n \geq \sum_{i=1}^{2} \rho_i \theta_i$, the second term in the right-hand side is nonnegative, so letting $n \to \infty$ we find that

$$
\sum_{i=1}^{2} \rho_i \mathbb{E}\big[X_i(t) - \theta_i \big] e^{ct} \leq \sum_{i=1}^{2} \rho_i (x_i - \theta_i). \tag{3.6}
$$

Since $\rho_1, \rho_2 > 0$ are arbitrary, we arrive at (3.2). $\qquad \square$

This completes the proof of the existence. $\qquad \square$

Uniqueness

Proof. We distinguish between $\vec{\theta}$ in the interior resp. on the boundary of $[0, \infty)^2$.

$\vec{\theta} \in (0, \infty)^2$: By Theorem 2.2, the unique weak solution $(\vec{X}(t))_{t \geq 0}$ of (2.1) is a strong Markov process. By Remark 1 following Theorem 2.2, we restrict the state space to be $[0, \infty)^2 \setminus \{(0, 0)\}$ for the cases where weak uniqueness is not known when $\vec{X}(0) = (0, 0)$. If $(\vec{X}(t))_{t \geq 0}$ has two distinct equilibrium distributions, then we can find two extremal equilibrium distributions μ and ν that are singular with respect to each other (see e.g. Theorem 6.9 in [32]). This implies that there exist $\vec{x}, \vec{y} \in [0, \infty)^2$ such that the transition kernels $p_t(\vec{x}, \cdot)$ and $p_t(\vec{y}, \cdot)$ are mutually singular for all $t > 0$. However, if $\vec{x}, \vec{y} \in (0, \infty)^2$, then we can first apply Theorem B.4 to transport the diffusions started at $\vec{x}$, resp. $\vec{y}$, to a common small neighborhood with positive probability, and subsequently apply Corollary B.3 to see that $p_t(\vec{x}, \cdot)$ and $p_t(\vec{y}, \cdot)$ cannot be singular for all $t > 0$. On the other hand, when either $\vec{x}$ or $\vec{y} \in \partial[0, \infty)^2$, it suffices to note that the drift in (2.1) forces the diffusion to enter $(0, \infty)^2$ instantly, which we justify shortly. Then, again by Theorem B.4, the diffusion can be kept in $(0, \infty)^2$ up to any fixed time with positive probability, which reduces it to the case $\vec{x}, \vec{y} \in (0, \infty)^2$.

We now show that, for $\vec{X}(0) = \vec{x} \in \partial[0, \infty)^2$, $(\vec{X}(t))_{t \geq 0}$ enters $(0, \infty)^2$ instantly. Consider first the case $\vec{X}(0) \in \{0\} \times (0, \infty)$. Let $\vec{X}(0) = (0, y)$ with $y > 0$, and let $\tau_\varepsilon = \inf\{t > 0 : |X_2(t) - X_2(0)| \geq y/2 \text{ or } X_1(t) \geq \varepsilon\}$. Then $X_1(t \wedge \tau_\varepsilon) - \int_0^{t \wedge \tau_\varepsilon} c(\theta_1 - X_1(s)) \, ds$ is a martingale, and

$$
\mathbb{E}\big[X_1(t \wedge \tau_\varepsilon) \big] = \mathbb{E}\bigg[\int_0^{t \wedge \tau_\varepsilon} c\big(\theta_1 - X_1(s)\big) \, ds \bigg]. \tag{3.7}
$$

Letting $t \to \infty$, we find that for ε small,

$$
\varepsilon \geq \mathbb{E}\big[X_1(\tau_\varepsilon) \big] = \mathbb{E}\bigg[\int_0^{\tau_\varepsilon} c\big(\theta_1 - X_1(s)\big) \, ds \bigg] \geq \frac{c\theta_1}{2} \mathbb{E}[\tau_\varepsilon]. \tag{3.8}
$$

Therefore $\mathbb{E}[\tau_\varepsilon] \to 0$ as $\varepsilon \downarrow 0$, which is possible only if $(\vec{X}(t))_{t \geq 0}$ enters $(0, \infty)^2$ instantly. The case $\vec{X}(0) \in (0, \infty) \times \{0\}$ is analogous. For $\vec{X}(0) = (0, 0)$, a similar argument shows that $(\vec{X}(t))_{t \geq 0}$ enters $[0, \infty)^2 \setminus \{(0, 0)\}$ instantly, which reduces it to the previous cases.

$\vec{\theta} \in \partial[0, \infty)^2$: If $\theta_1 = 0$, then $\mathbb{E}_{\vec{\theta}}^{c, g}[X_1] = \theta_1 = 0$ for any equilibrium distribution $\Gamma_{\vec{\theta}}^{c, g}$ by Proposition A.1. In particular, $\Gamma_{\vec{\theta}}^{c, g}$ is concentrated on $\{0\} \times [0, \infty)$. Furthermore, $(X_1(t))_{t \geq 0}$ is a local supermartingale, and hence $\{0\} \times [0, \infty)$ is an absorbing set. The equilibria for $(\vec{X}(t))_{t \geq 0}$ are therefore exactly the equilibria for $(\vec{X}(t))_{t \geq 0}$ restricted to the axis $\{0\} \times [0, \infty)$, which is a one-dimensional diffusion. The proof of the existence and the uniqueness of the equilibrium distribution for this one-dimensional diffusion can be deduced either from explicit calculations as in [3],

or from the same argument as above for the two-dimensional diffusion with $\vec{\theta} \in (0, \infty)^2$. The situation is similar if $\theta_2 = 0$. $\qquad\square$

Weak continuity

Proof. We will show that $\Gamma_{\vec{\theta}}^{c,g}$ is weakly continuous in $\vec{\theta}$. Let $(\vec{\theta}_n)$ be a sequence such that $\vec{\theta}_n \to \vec{\theta}$ in $[0, \infty)^2$. It suffices to show that $\{\Gamma_{\vec{\theta}_n}^{c,g}\}_{n \in \mathbb{N}}$ is tight, and that any weak limit point of $\Gamma_{\vec{\theta}_n}^{c,g}$ is an equilibrium distribution for the SDE (2.1), which must be the unique $\Gamma_{\vec{\theta}}^{c,g}$. Tightness of $\{\Gamma_{\vec{\theta}_n}^{c,g}\}_{n \in \mathbb{N}}$ follows from (A.2). Suppose that $\Gamma_{\vec{\theta}_n}^{c,g}$ converges weakly to a distribution ν. Then for any $f \in C_c^2([0, \infty)^2)$,

$$\int_{[0,\infty)^2} \left(L_{\vec{\theta}}^{c,g} f\right)(x)\nu(dx) = \int_{[0,\infty)^2} \left(L_{\vec{\theta}_n}^{c,g} f\right)(x)\Gamma_{\vec{\theta}_n}^{c,g}(dx) + \int_{[0,\infty)^2} \left[\left(L_{\vec{\theta}}^{c,g} - L_{\vec{\theta}_n}^{c,g}\right)f\right](x)\Gamma_{\vec{\theta}_n}^{c,g}(dx)$$

$$+ \int_{[0,\infty)^2} \left(L_{\vec{\theta}}^{c,g} f\right)(x)\left[\nu(dx) - \Gamma_{\vec{\theta}_n}^{c,g}(dx)\right], \tag{3.9}$$

where the first term is zero because $\Gamma_{\vec{\theta}_n}^{c,g}$ is an equilibrium distribution for the SDE in (2.1) with parameter $\vec{\theta}_n$, the second term tends to 0 as $n \to \infty$ because $f \in C_c^2([0, \infty)^2)$ and $\|L_{\vec{\theta}}^{c,g} f(x) - L_{\vec{\theta}_n}^{c,g} f(x)\|_\infty \to 0$ as $\theta_n \to \theta$, and the third term tends to 0 as $n \to \infty$ by the weak convergence of $\Gamma_{\vec{\theta}_n}^{c,g}$ to ν. Therefore $\int_{[0,\infty)^2} (L_{\vec{\theta}}^{c,g} f)(x)\nu(dx) = 0$ for all $f \in C_c^2([0, \infty)^2)$. By Theorem 4.9.17 in [19], it follows that ν must be an equilibrium distribution for (2.1), and hence $\nu = \Gamma_{\vec{\theta}}^{c,g}$. $\qquad\square$

Convergence

Proof. We again distinguish between $\vec{\theta}$ in the interior resp. on the boundary of $[0, \infty)^2$.

$\vec{\theta} \in (0, \infty)^2$: Firstly, note that by Theorem B.4 and the fact that $(X(t))_{t \geq 0}$ started from $\partial[0, \infty)^2$ enters $(0, \infty)^2$ instantly (see the paragraph containing (3.7) and (3.8)), the equilibrium distribution $\Gamma_{\vec{\theta}}^{c,g}$ must assign positive measure to every open subset of $(0, \infty)^2$.

Secondly, we show that for almost all $\vec{x} \in [0, \infty)^2$ with respect to $\Gamma_{\vec{\theta}}^{c,g}$, $\mathcal{L}(\vec{X}(t)|\vec{X}(0) = \vec{x})$ converges weakly to $\Gamma_{\vec{\theta}}^{c,g}$ as $t \to \infty$. We achieve this by showing that, for almost all $(\vec{x}, \vec{y}) \in [0, \infty)^2 \times [0, \infty)^2$ with respect to the product measure $\Gamma_{\vec{\theta}}^{c,g} \times \Gamma_{\vec{\theta}}^{c,g}$, we can couple two solutions $(\vec{X}(t))_{t \geq 0}$ and $(\vec{Y}(t))_{t \geq 0}$ of (2.1) starting from $\vec{x}$, resp. $\vec{y}$, such that $\lim_{t \to \infty} \mathbb{P}(\vec{X}(t) \neq \vec{Y}(t)) = 0$. This goes as follows.

Let $\varepsilon, \delta > 0$ be chosen as in Corollary B.3, where $b(\vec{x}) = c(\vec{\theta} - \vec{x})$ and $a(\vec{x}) = \begin{pmatrix} g_1(\vec{x}) & 0 \\ 0 & g_2(\vec{x}) \end{pmatrix}$ on $[0, \infty)^2$ (the definition of (a, b) in the rest of the plane $\mathbb{R}^2$ is irrelevant, for instance one may define it by reflection), $D = \{\vec{x} \in [0, \infty)^2 : \|\vec{x} - (1, 1)\| < \frac{1}{2}\}$ and $\vec{x}^* = (1, 1)$. Note that $a(\cdot)$ is nondegenerate on D for $g \in \mathcal{C}$. If $(\vec{X}(t))_{t \geq 0}$, $(\vec{Y}(t))_{t \geq 0}$ are two independent copies of the strong Markov process defined by (2.1), then the joint process $(\vec{X}(t), \vec{Y}(t))_{t \geq 0}$ is strong Markov and, by the same argument as for a single diffusion $(\vec{X}(t))_{t \geq 0}$, the joint process has a unique equilibrium given by the product measure $\Gamma_{\vec{\theta}}^{c,g} \times \Gamma_{\vec{\theta}}^{c,g}$, which implies that the stationary process $(\vec{X}(t), \vec{Y}(t))_{t \geq 0}$ with $\mathcal{L}(\vec{X}(0), \vec{Y}(0)) = \Gamma_{\vec{\theta}}^{c,g} \times \Gamma_{\vec{\theta}}^{c,g}$ is ergodic (see e.g. Theorem 6.9 in [32] and the remarks thereafter). Since $\Gamma_{\vec{\theta}}^{c,g} \times \Gamma_{\vec{\theta}}^{c,g}$ assigns positive measure to $B_\varepsilon(\vec{x}^*) \times B_\varepsilon(\vec{x}^*)$, by the ergodic theorem almost surely $(\vec{X}(t), \vec{Y}(t))_{t \geq 0}$ visits the set $B_\varepsilon(\vec{x}^*) \times B_\varepsilon(\vec{x}^*)$ after any finite time T. In particular, for almost all $(\vec{x}, \vec{y})$ with respect to $\Gamma_{\vec{\theta}}^{c,g} \times \Gamma_{\vec{\theta}}^{c,g}$, almost surely the Markov process $(\vec{X}(t), \vec{Y}(t))_{t \geq 0}$ starting from $(\vec{x}, \vec{y})$ visits $B_\varepsilon(\vec{x}^*) \times B_\varepsilon(\vec{x}^*)$ after any finite time T. For such a pair $(\vec{x}, \vec{y})$, we construct the coupled process as follows. Start the independent processes $(\vec{X}(t))_{t \geq 0}$ and $(\vec{Y}(t))_{t \geq 0}$ with initial conditions $\vec{x}$, resp. $\vec{y}$. Then $\tau = \inf\{t \geq 0 : (\vec{X}(t), \vec{Y}(t)) \in B_\varepsilon(\vec{x}^*) \times B_\varepsilon(\vec{x}^*)\} < \infty$ almost surely. By Corollary B.3, the conditional transition probability kernels $\mu_{\vec{X}} = \mathbb{P}(\vec{X}(\tau + \delta) \in \cdot | (\vec{X}(\tau), \vec{Y}(\tau)))$ and $\mu_{\vec{Y}} = \mathbb{P}(\vec{Y}(\tau + \delta) \in \cdot | (\vec{X}(\tau), \vec{Y}(\tau)))$ have a common part $\mu_{\vec{X}, \vec{Y}}$ with measure at least $\frac{1}{2}$. From $\mu_{\vec{X}} \times \mu_{\vec{Y}}$, we can take out $\mu_{\vec{X}, \vec{Y}} \times \mu_{\vec{X}, \vec{Y}}$, which has measure at least $\frac{1}{4}$, and couple $(\vec{X}(\tau + \delta + t))_{t \geq 0}$ and $(\vec{Y}(\tau + \delta + t))_{t \geq 0}$ so that they coincide for all $t \geq 0$ and evolve as the strong

Markov process defined by (2.1) with initial measure $\mu_{\vec{X},\vec{Y}}$. With respect to the remaining measure $\mu_{\vec{X}} \times \mu_{\vec{Y}} - \mu_{\vec{X},\vec{Y}} \times \mu_{\vec{X},\vec{Y}}$, we let $(\vec{X}(\tau + \delta + t))_{t \geq 0}$ and $(\vec{Y}(\tau + \delta + t))_{t \geq 0}$ continue to evolve independently. Since $\mu_{\vec{X}} \times \mu_{\vec{Y}} - \mu_{\vec{X},\vec{Y}} \times \mu_{\vec{X},\vec{Y}}$ is absolutely continuous with respect to $\mu_{\vec{X}} \times \mu_{\vec{Y}}$, a.s. $(\vec{X}(\tau + \delta + t), (\vec{Y}(\tau + \delta + t))_{t \geq 0}$ will visit $B_\varepsilon(\vec{x}^*) \times B_\varepsilon(\vec{x}^*)$ again. We can therefore iterate the above coupling procedure. Each iteration reduces the probability that $\vec{X}$ and $\vec{Y}$ have not been successfully coupled by a factor $\frac{1}{4}$. Continue the iteration indefinitely to get the desired coupling between $\vec{X}$ and $\vec{Y}$. We comment that, unlike in the context of Harris chains (see e.g. Section 5.6 of [18]) where one would need $\mathbb{P}(\vec{X}(\delta) \in \cdot | \vec{X}(0) = \vec{x})$ to be dominated from below by a positive measure uniformly for $\vec{x} \in B_\varepsilon(\vec{x}^*)$, to get a successful coupling it suffices that $\mathbb{P}(\vec{X}(\delta) \in \cdot | \vec{X}(0) = \vec{x})$ and $\mathbb{P}(\vec{X}(\delta) \in \cdot | \vec{X}(0) = \vec{y})$ overlap with probability at least α for some $\alpha > 0$ uniformly for all $\vec{x}, \vec{y} \in B_\varepsilon(\vec{x}^*)$.

Next we show that, for Lebesgue almost every $\vec{x} \in [0, \infty)^2$, $\mathcal{L}(\vec{X}(t) | \vec{X}(0) = \vec{x}) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$ as $t \to \infty$. Let $A = \{\vec{x} \in [0, \infty)^2: \mathcal{L}(\vec{X}(t) | \vec{X}(0) = \vec{x}) \not\Rightarrow \Gamma_{\vec{\theta}}^{c,g}\}$. By Theorem 2.2 and the remark following it, the process defined by (2.1) is Feller continuous, and therefore A is Borel-measurable. If A has positive Lebesgue measure, then we can find a simply connected bounded open domain $D \subset (0, \infty)^2$ with smooth boundary such that $A \cap D$ has positive Lebesgue measure. We have shown above that $\Gamma_{\vec{\theta}}^{c,g}(A) = 0$, and hence $\Gamma_{\vec{\theta}}^{c,g}(A \cap D) = 0$. If $(\vec{X}(t))_{t \geq 0}$ is the stationary solution of (2.1) with marginal distribution $\Gamma_{\vec{\theta}}^{c,g}$, then $\mathbb{E}[\int_0^T 1_{\vec{X}(t) \in A \cap D} \, dt] = 0$ for all $T > 0$. On the other hand, by Theorem B.5, we have for every $\vec{x} \in D$ that $\mathbb{E}[\int_0^{\tau_D} 1_{\vec{X}(t) \in A \cap D} \, dt | \vec{X}(0) = \vec{x}] > 0$. Since $\Gamma_{\vec{\theta}}^{c,g}$ assigns positive probability to D, we have

$$\int_D \mathbb{E}\left[\int_0^{\tau_D} 1_{\vec{X}(t) \in A \cap D} \, dt \,\middle|\, \vec{X}(0) = \vec{x}\right] \Gamma_{\vec{\theta}}^{c,g}(d\vec{x}) > 0.$$

By the monotone convergence theorem, we can choose T sufficiently large such that

$$\int_D \mathbb{E}\left[\int_0^{\tau_D \wedge T} 1_{\vec{X}(t) \in A \cap D} \, dt \,\middle|\, \vec{X}(0) = \vec{x}\right] \Gamma_{\vec{\theta}}^{c,g}(d\vec{x}) > 0,$$

the left-hand side of which is in turn dominated by $\mathbb{E}[\int_0^T 1_{\vec{X}(t) \in A \cap D} \, dt] = 0$, which is a contradiction. Therefore A has Lebesgue measure 0.

Lastly, we show that $\mathcal{L}(\vec{X}(t) | \vec{X}(0) = \vec{x}) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$ for all $\vec{x} \in [0, \infty)^2$. Indeed, for $\vec{x} \in (0, \infty)^2$, let $\varepsilon > 0$ be such that $B_\varepsilon(\vec{x}) \subset (0, \infty)^2$. By Corollary B.3 applied to $D = B_\varepsilon(\vec{x})$, the transition kernel $\mu_t^{B_\varepsilon(\vec{x})}(\vec{x}, \cdot)$ with killing at the boundary of $B_\varepsilon(\vec{x})$ is absolutely continuous with respect to Lesbesgue measure. Since, for Lebesgue almost every $\vec{y} \in B_\varepsilon(\vec{x})$, $\mathcal{L}(\vec{X}(t + s) | \vec{X}(t) = \vec{y}) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$ as $s \to \infty$ and $\mu_t^{B_\varepsilon(\vec{x})}(\vec{x}, B_\varepsilon(\vec{x})) \uparrow 1$ as $t \downarrow 0$ (see (B.3)), we have $\mathcal{L}(\vec{X}(t) | \vec{X}(0) = \vec{x}) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$. The case $\vec{x} \in \partial[0, \infty)^2$ follows from our previous observation that $\vec{X}(t)$ starting from $\vec{x}$ enters $(0, \infty)^2$ instantly (see (3.7) and (3.8)).

$\vec{\theta} \in \partial[0, \infty)^2$: Without loss of generality we may assume that $\theta_1 = 0$. If $X_1(0) = 0$, then $X_1(t) = 0$ for all $t \geq 0$ and $(X_1(t), X_2(t)) = (0, X_2(t))$ is effectively a one-dimensional diffusion with diffusion function $g_2(0, x_2)$. By the same argument as before, albeit much simpler, this one-dimensional diffusion is ergodic, and the convergence in (2.7) holds. If $X_1(0) \neq 0$, then it suffices to show that $X_1(t) \to 0$ a.s. and $\mathcal{L}(X_2(t)) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$ as $t \to \infty$, where $\Gamma_{\vec{\theta}}^{c,g}$ is taken as a measure on $[0, \infty)$.

Note that $X_1(t)$ is a local supermartingale and $X_1(t) \wedge 1$ is a bounded supermartingale, so that $X_1(t) \wedge 1 \to Y$ a.s. as $t \to \infty$ for some nonnegative random variable Y. By the bounded convergence theorem and (3.2),

$$\mathbb{E}[Y] = \lim_{t \to \infty} \mathbb{E}[X_1(t) \wedge 1] \leq \lim_{t \to \infty} X_1(0) e^{-ct} = 0. \tag{3.10}$$

Therefore $Y \equiv 0$ and $X_1(t) \to 0$ a.s. as $t \to \infty$.

To show that $\mathcal{L}(X_2(t)) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$ as $t \to \infty$, it suffices to show that $\mathbb{E}[\phi(X_2(t))] \to \mathbb{E}_{\vec{\theta}}^{c,g}[\phi(X_2)]$ as $t \to \infty$ for any $\phi \in C_c^2[0, \infty)$. Abbreviate

$$\alpha = \mathbb{E}_{\vec{\theta}}^{c,g}[\phi(X_2)] \quad \text{and} \quad u(t, \vec{x}) = \mathbb{E}[\phi(X_2(t)) \mid \vec{X}(0) = \vec{x}]. \tag{3.11}$$

For $\vec{X}(0) \in [0, \infty)^2$ with $X_1(0) = 0$, $(\vec{X}(t))_{t \geq 0}$ is effectively a one-dimensional diffusion that is ergodic, and hence $u(t, \vec{x}) \to \alpha$ as $t \to \infty$ for each $\vec{x} \in \{0\} \times [0, \infty)$. We claim that in fact $u(t, \vec{x}) \to \alpha$ uniformly on compact intervals of the form $\{0\} \times [0, K]$. To see why, note that if $Y(t)$ and $Z(t)$ are solutions of the one-dimensional SDE

$$\mathrm{d}X(t) = c\big(\theta_2 - X(t)\big)\mathrm{d}t + \sqrt{2g_2\big(0, X(t)\big)}\,\mathrm{d}B_t \tag{3.12}$$

with initial condition $Y(0) = y < Z(0) = z$, then $Z(t)$ stochastically dominates $Y(t)$ for all $t \geq 0$, i.e., if $F_{t,y}(v) = \mathbb{P}(Y(t) < v | Y(0) = y)$, then $F_{t,y}(v) \geq F_{t,z}(v)$ for all $t, v \geq 0$. Let $F_\infty(v) = \Gamma_{\vec{\theta}}^{c,g}(-\infty, v)$. Then, for any $x_2 \geq 0$, $F_{t,x_2}(v) \to F_\infty(v)$ as $t \to \infty$ for all but countably many $v \in [0, \infty)$. For any $x_2 \in [0, K]$, $K > 0$, we can write

$$u\big(t, (0, x_2)\big) = \int_0^\infty \phi(v)\,\mathrm{d}F_{t,x_2}(v) = -\int_0^\infty \phi'(v)F_{t,x_2}(v)\,\mathrm{d}v = \int_0^\infty \big(\phi'_-(v) - \phi'_+(v)\big)F_{t,x_2}(v)\,\mathrm{d}v, \tag{3.13}$$

where $\phi'_+(v) = \phi'(v) \vee 0$ and $\phi'_-(v) = -(\phi'(v) \wedge 0)$. Since

$$\int_0^\infty \phi'_-(v)F_{t,K}(v)\,\mathrm{d}v \leq \int_0^\infty \phi'_-(v)F_{t,x_2}(v)\,\mathrm{d}v \leq \int_0^\infty \phi'_-(v)F_{t,0}(v)\,\mathrm{d}v, \tag{3.14}$$

where both ends of the inequality tend to $\int_0^\infty \phi'_-(v)F_\infty(v)\,\mathrm{d}v$ by the bounded convergence theorem, $\int_0^\infty \phi'_-(v) \times F_{t,x_2}(v)\,\mathrm{d}v$ converges uniformly to $\int_0^\infty \phi'_-(v)F_\infty(v)\,\mathrm{d}v$ for $x_2 \in [0, K]$ as $t \to \infty$. A similar statement holds for $\int_0^\infty \phi'_+(v)F_{t,x_2}(v)\,\mathrm{d}v$. Therefore $u(t, \vec{x})$ converges uniformly to α on $\{0\} \times [0, K]$.

Let $\vec{X}(0) \in [0, \infty)^2$ be arbitrary. By (3.2), $(X_2(t))_{t \geq 0}$ is tight, and hence for any $\varepsilon > 0$ we can choose K large enough so that $\mathbb{P}(X_2(t) > K) \leq \varepsilon$ for all $t \geq 0$. Since $u(t, \vec{x}) \to \alpha$ uniformly on $\{0\} \times [0, K]$, we can choose t_1 large enough so that $\sup_{x_2 \in [0, K]} |u(t_1, (0, x_2)) - \alpha| \leq \varepsilon/2$. Since $\{(\vec{X}(t))_{t \geq 0}\}_{\vec{X}(0) \in [0, \infty)^2}$ defines a Feller process (see the remark below Theorem 2.2), $u(t_1, \vec{x})$ is continuous in $\vec{x} \in [0, \infty)^2$. We can therefore choose $\delta > 0$ sufficiently small so that $\sup_{\vec{x} \in [0, \delta] \times [0, K]} |u(t_1, \vec{x}) - \alpha| \leq \varepsilon$. Since $X_1(t) \to 0$ a.s., we can choose t_2 large enough so that $\mathbb{P}(X_1(t) > \delta) \leq \varepsilon$ for all $t \geq t_2$. Then, by the Markov property, for any $t \geq t_1 + t_2$ we have

$$\begin{aligned} u\big(t, \vec{X}(0)\big) &= \mathbb{E}\big[u\big(t_1, \vec{X}(t - t_1)\big)\big] \\ &= \mathbb{E}\big[u\big(t_1, \vec{X}(t - t_1)\big)1_{\vec{X}(t-t_1) \in [0,\delta] \times [0,K]}\big] + \mathbb{E}\big[u\big(t_1, \vec{X}(t - t_1)\big)1_{\vec{X}(t-t_1) \notin [0,\delta] \times [0,K]}\big]. \end{aligned} \tag{3.15}$$

Since $\mathbb{P}(\vec{X}(t - t_1) \notin [0, \delta] \times [0, K]) \leq 2\varepsilon$ and $\|u\|_\infty \leq \|\phi\|_\infty$, $\alpha \leq \|\phi\|_\infty$, we easily verify from (3.15) that

$$\big|u\big(t, \vec{X}(0)\big) - \alpha\big| \leq \varepsilon + 4\varepsilon\|\phi\|_\infty \quad \text{for all } t \geq t_1 + t_2. \tag{3.16}$$

Since $\varepsilon > 0$ is arbitrary, $u(t, \vec{X}(0)) \to \alpha$ as $t \to \infty$, and hence $\mathcal{L}(\vec{X}(t)) \Rightarrow \Gamma_{\vec{\theta}}^{c,g}$. $\qquad\square$

3.2. *Proofs of Theorems* 2.6, 2.10 *and Corollary* 2.11

Proof of Theorem 2.6. Let $g = (g_1, g_2) \in \mathcal{H}_a$ for some $0 \leq a < c$. Then, by (2.9), there exists a $0 < C = C(g) < \infty$ such that

$$g_1(\vec{x}) + g_2(\vec{x}) \leq C(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2), \quad (x_1, x_2) \in [0, \infty)^2. \tag{3.17}$$

The finiteness of $F_c g$ follows from Proposition A.1(ii). If $\vec{\theta}_n \to \vec{\theta}$ for some $\vec{\theta} \in [0, \infty)^2$, then, by Proposition A.1(iii), g_1, g_2 are uniformly integrable with respect to $\{\Gamma_{\vec{\theta}_n}^{c,g}\}_{n \in \mathbb{N}}$. Combining this with the fact, shown in Theorem 2.3 and proved in Section 3.1, that $\Gamma_{\vec{\theta}_n}^{c,g}$ converges weakly to $\Gamma_{\vec{\theta}}^{c,g}$ as $\vec{\theta}_n \to \vec{\theta}$, we have $\mathbb{E}_{\vec{\theta}_n}^{c,g}[g_i(\vec{X})] \to \mathbb{E}_{\vec{\theta}}^{c,g}[g_i(\vec{X})]$, i.e., $(F_c g)_i(\vec{\theta}_n) \to (F_c g)_i(\vec{\theta})$ for $i = 1, 2$ (recall (2.8)).

By the moment equations (A.2) and (A.3), we have

$$(F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta}) = \mathbb{E}_{\vec{\theta}}^{c,g}\big[g_1(\vec{X}) + g_2(\vec{X})\big]$$

$$\leq \mathbb{E}_{\vec{\theta}}^{c,g}\big[C(1+X_1)(1+X_2) + a\big(X_1^2 + X_2^2\big)\big]$$

$$= C(1+\theta_1)(1+\theta_2) + a\big(\theta_1^2 + \theta_2^2\big) + \frac{a}{c}\big((F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta})\big). \tag{3.18}$$

Therefore

$$(F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta}) \leq \frac{c}{c-a}\big(C(1+\theta_1)(1+\theta_2) + a\big(\theta_1^2 + \theta_2^2\big)\big). \tag{3.19}$$

Consequently, if $g \in \mathcal{H}_{0^+}$, then $F_c g$ satisfies (3.19) for all $a > 0$, and so it satisfies the subquadratic growth bound imposed by the class $\mathcal{H}_{0^+}$.

To show $\partial F_c g = \partial g$, note that $F_c g \geq 0$ is obvious. If $\vec{\theta} \in (0,\infty)^2$, then the equilibrium distribution $\Gamma_{\vec{\theta}}^{c,g}$ has positive mass in $(0,\infty)^2$, and so $(F_c g)(\vec{\theta}) > 0$ follows from the fact that $g > 0$ on $(0,\infty)^2$. If $\theta_1 = 0$, then, by (A.2), $\Gamma_{\vec{\theta}}^{c,g}$ is concentrated on the vertical axis A_2. Since g_1 vanishes on A_2, it follows that $(F_c g)_1(\vec{\theta}) = 0$. Moreover, $(F_c g)_2(\vec{\theta}) = 0$ if and only if g_2 vanishes on A_2 (recall (2.5) and (2.6)). A similar result holds for $\theta_2 = 0$. $\square$

Proof of Theorem 2.10. Theorem 2.10(i) follows immediately from (A.2) and (A.3). To prove Theorem 2.10(ii), note that, by (A.2)–(A.4),

$$(F_c g)_1(\vec{\theta}) = \mathbb{E}_{\vec{\theta}}^{c,g}\big[a_1 X_1^2 + b_1 X_1 + c_1 X_1 X_2\big] = a_1 \mathbb{E}_{\vec{\theta}}^{c,g}\big[X_1^2\big] + b_1 \theta_1 + c_1 \theta_1 \theta_2$$

$$= a_1 \theta_1^2 + \frac{a_1}{c}(F_c g)_1 + b_1 \theta_1 + c_1 \theta_1 \theta_2 = g_1(\vec{\theta}) + \frac{a_1}{c}(F_c g)_1(\vec{\theta}). \tag{3.20}$$

Solving for $(F_c g)_1(\vec{\theta})$, we get $(F_c g)_1(\vec{\theta}) = \frac{c}{c-a_1} g_1(\vec{\theta})$. Similarly, we have $(F_c g)_2 = \frac{c}{c-a_2} g_2$ for $g_2 = a_2 x_2^2 + b_2 x_2 + c_2 x_1 x_2$. The assumption $(b_1 + c_1)(b_2 + c_2) > 0$ is meant to rule out the uninteresting case $g_1 = 0$ or $g_2 = 0$. $\square$

Proof of Corollary 2.11. Equation (2.15) follows from Theorem 2.10(ii) by induction. Note that if $\alpha_i \sum_{k=0}^{n_0-1} c_k^{-1} \geq 1$ for either $i = 1$ or 2, then the coefficient of x_i^2 in $(F^{[n_0-1]}g)_i(\vec{x})$ is $\alpha_i/[1 - \alpha_i \sum_{k=0}^{n_0-2} c_k^{-1}] \geq c_{n_0-1}$. To show $(F^{[n_0]}g)_1 + (F^{[n_0]}g)_2 = \infty$ on $(0,\infty)^2$, it therefore suffices to show $(F_c g)_1 + (F_c g)_2 \equiv \infty$ on $(0,\infty)^2$ for g of the form $g_i(\vec{x}) = \alpha_i x_i^2 + \beta_i x_i + \gamma_i x_1 x_2$ with $\alpha_1 \vee \alpha_2 \geq c$. Without loss of generality, assume $\alpha_1 \geq c$. The proof of Proposition A.1(ii) shows that the moment equations (A.2)–(A.4) are valid as long as $(F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta}) = \mathbb{E}_{\vec{\theta}}^{c,g}[g_1 + g_2] < \infty$. Assume $(F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta}) < \infty$ for some $\vec{\theta} \in (0,\infty)^2$. Then

$$\mathbb{E}_{\vec{\theta}}^{c,g}\big[X_1^2\big] = \theta_1^2 + \frac{1}{c}(F_c g)_1(\vec{\theta}) = \theta_1^2 + \frac{\alpha_1}{c}\mathbb{E}_{\vec{\theta}}^{c,g}\big[X_1^2\big] + \frac{\beta_1}{c}\theta_1 + \frac{\gamma_1}{c}\theta_1 \theta_2, \tag{3.21}$$

which is not possible for $\alpha_1 \geq c$. Therefore we must have $(F_c g)_1(\vec{\theta}) + (F_c g)_2(\vec{\theta}) = \infty$ for all $\vec{\theta} \in (0,\infty)^2$. $\square$

3.3. *Discussion of Conjecture* 2.7

In this section we explain why Conjecture 2.7 is plausible. We focus on the case where g_1, g_2 both satisfy boundary property (∂_{12}) in (2.4), i.e., $g_1(\vec{x}) = x_1 x_2 \gamma_1(\vec{x})$ and $g_2(\vec{x}) = x_1 x_2 \gamma_2(\vec{x})$ with $\gamma_1, \gamma_2 > 0$ continuous on $[0,\infty)^2$.

Consider the tilted equilibrium

$$\hat{\Gamma}_{\vec{\theta}}^{c,g}(d\vec{x}) = \frac{x_1 x_2}{\theta_1 \theta_2} \Gamma_{\vec{\theta}}^{c,g}(d\vec{x}), \quad \vec{\theta} \in (0,\infty)^2, \tag{3.22}$$

where (A.3) implies the proper normalization. The conjecture amounts to showing that, as $\vec{\theta} \to \vec{\theta}^* \in \partial[0,\infty)^2$, this tilted equilibrium converges weakly to some probability distribution on $[0,\infty)^2$, say $\hat{\Gamma}_{\vec{\theta}^*}^{c,g}(d\vec{x})$, that is weakly con-

tinuous in $\vec{\theta}^*$ and, in addition, $\gamma_i(\vec{x})$ is uniformly integrable with respect to $\hat{\Gamma}_{\vec{\theta}}^{c,g}(d\vec{x})$ for $\vec{\theta}$ in a small neighborhood of $\vec{\theta}^*$. Indeed, this observation is immediate from the identity

$$\int_{[0,\infty)^2} \gamma_i(\vec{x}) \hat{\Gamma}_{\vec{\theta}}^{c,g}(d\vec{x}) = \frac{1}{\theta_1 \theta_2} (F_c g)_i(\vec{\theta}), \quad i = 1, 2. \tag{3.23}$$

Now, recalling the generator in (2.2), we note that $\hat{\Gamma}_{\vec{\theta}}^{c,g}(d\vec{x})$ is the equilibrium associated with the time-changed diffusion given by the generator

$$\left(\hat{L}_{\vec{\theta}}^{c,g} f\right)(\vec{x}) = \frac{c(\theta_1 - x_1)}{x_1 x_2} \frac{\partial}{\partial x_1} f(\vec{x}) + \frac{c(\theta_2 - x_2)}{x_1 x_2} \frac{\partial}{\partial x_2} f(\vec{x}) + \gamma_1(\vec{x}) \frac{\partial^2}{\partial x_1^2} f(\vec{x}) + \gamma_2(\vec{x}) \frac{\partial^2}{\partial x_2^2} f(\vec{x}),$$

$$f \in C_c^2([0,\infty)^2), (\vec{\theta} - \vec{x}) \cdot \nabla f(\vec{x}) = 0 \text{ on } \partial[0,\infty)^2. \tag{3.24}$$

Let $\vec{\theta} \to \vec{\theta}^* = (\alpha, 0)$ for some $\alpha > 0$. Then, at least heuristically, we get a limiting generator

$$\left(\hat{L}_{(\alpha,0)}^{c,g} f\right)(\vec{x}) = \frac{c(\alpha - x_1)}{x_1 x_2} \frac{\partial}{\partial x_1} f(\vec{x}) - \frac{c}{x_1} \frac{\partial}{\partial x_2} f(\vec{x}) + \gamma_1(\vec{x}) \frac{\partial^2}{\partial x_1^2} f(\vec{x}) + \gamma_2(\vec{x}) \frac{\partial^2}{\partial x_2^2} f(\vec{x}),$$

$$f \in C_c^2([0,\infty)^2), (\vec{\theta}^* - \vec{x}) \cdot \nabla f(\vec{x}) = 0 \text{ on } \partial[0,\infty)^2 \setminus \{\vec{\theta}^*\}, \frac{\partial}{\partial x_1} f(\vec{\theta}^*) = \frac{\partial}{\partial x_2} f(\vec{\theta}^*) = 0. \tag{3.25}$$

Here, the diffusion part has no singularity at the boundary, but the drift part does. As the process approaches the vertical axis A_2 it feels a growing drift downwards and to the right, while as it approaches the horizontal axis A_1 it feels a growing drift horizontally towards $(\alpha, 0)$ and a constant drift downwards. Therefore, again heuristically, this generator describes a process that is *obliquely reflected in the direction of $(\alpha, 0)$ upon hitting A_2, and upon hitting A_1 jumps to $(\alpha, 0)$ instantly and then moves back into the interior by reflection*. Like the original diffusion with generator (2.2), this process ought to exist, be weakly unique, and have an ergodic equilibrium $\hat{\Gamma}_{\vec{\theta}^*}^{c,g}$ that is weakly continuous in $\vec{\theta}^* \in [0,\infty)^2$.

4. Proofs of Theorems 2.12, 2.13 and 2.15

Section 4.1 contains the proof of Theorem 2.15, which is an immediate consequence of Proposition 4.1. Section 4.2 contains some preliminary lemmas needed for the proof of Proposition 4.1. Section 4.3 provides the proof of Proposition 4.1 and of Theorems 2.12 and 2.13.

4.1. *Proof of Theorem* 2.15

The proof of Theorem 2.15 is based on an asymptotic analysis of the *homogeneous* Markov chain $\vec{M}^{c,g} = (\vec{M}^{c,g}(n))_{n \in \mathbb{N}_0}$ with transition probability kernel given by $p(\vec{\theta}, d\vec{y}) = \Gamma_{\vec{\theta}}^{c,g}(d\vec{y})$, the unique equilibrium distribution of (2.1). For $F_c g = g$, $\vec{M}^{c,g}$ is in fact the *interaction chain* in (1.16). Throughout the rest of the section, unless specified otherwise, we will denote the Markov chain $\vec{M}^{c,g}$ by $\vec{X}$. For $F_c g = g$ and $g \in \mathcal{H}_{0+}^r$, both g_1 and g_2 are harmonic functions of $\vec{X}$, i.e., both $(g_1(\vec{X}(n)))_{n \in \mathbb{N}_0}$ and $(g_2(\vec{X}(n)))_{n \in \mathbb{N}_0}$ are martingales. Theorem 2.15 then follows immediately from the following proposition.

Proposition 4.1 (Harmonic functions of $\vec{X} = \vec{M}^{c,g}$). *If $g \in \mathcal{H}_{0+}$ and satisfies* (2.21) *in the definition of $\mathcal{H}_0^r$, then every nonnegative harmonic function f of $\vec{X} = \vec{M}^{c,g}$, i.e., every f such that*

$$\mathbb{E}\left[f(\vec{X}(n)) | \vec{X}(0) = \vec{\theta}\right] = f(\vec{\theta}) \quad \forall \vec{\theta} \in [0,\infty)^2, n \in \mathbb{N}_0, \tag{4.1}$$

which furthermore satisfies the constraints

(i) $f(\vec{x}) \leq C(1 + x_1)(1 + x_2)$ *for some* $0 < C = C(f) < \infty$, $\qquad$ (4.2)

(ii) $\lim_{\vec{x} \to \vec{z}} f(\vec{x}) = 0$ $\qquad\qquad\qquad \forall \vec{z} \in \partial g$, $\qquad$ (4.3)

(iii) $\lim_{\vec{x} \to \vec{z}} \dfrac{f(\vec{x})}{h_{\vec{z}}(\vec{x})} = \lambda_{f,\vec{z}} \in [0, \infty)$ $\quad \forall \vec{z} \in R_\infty$, $\qquad$ (4.4)

is of the form

$$f(\vec{x}) = \sum_{\vec{z} \in R_\infty} \lambda_{f,\vec{z}} h_{\vec{z}}(\vec{x}) = \lambda_{f,(\infty,0)} x_1 + \lambda_{f,(0,\infty)} x_2 + \lambda_{f,(\infty,\infty)} x_1 x_2, \qquad (4.5)$$

with $h_{\vec{z}}, \vec{z} \in R_\infty$, *given by* (2.19) *and* (2.20).

The proof of Proposition 4.1 will be given in Section 4.3. The strategy is to first h-transform $\vec{X}$ (see Definition 4.3) to a new process $\vec{X}^h = (\vec{X}^h(n))_{n \in \mathbb{N}_0}$ using

$$h(\vec{x}) = (1 + x_1)(1 + x_2), \qquad (4.6)$$

i.e., $\vec{X}^h$ is defined as the homogeneous Markov chain with transition probability kernel

$$p(\vec{\theta}, d\vec{y}) = h(\vec{y}) \frac{\Gamma_{\vec{\theta}}^{c,g}(d\vec{y})}{h(\vec{\theta})},$$

which is well defined since $h(\vec{x})$ is a harmonic function of $\vec{M}^{c,g}$. The function f is harmonic for $\vec{M}^{c,g}$ if and only if f/h is harmonic for $\vec{X}^h$. The constraint in (4.2) guarantees that f/h is bounded, the constraints in (4.3) and (4.4) guarantee that f/h is continuous up to the boundary

$$R = \partial g \cup R_\infty, \qquad (4.7)$$

while the constraint in (2.21) guarantees that $\lim_{n \to \infty} \vec{X}^h(n) \in R$ a.s. It is then standard to show that f/h is uniquely determined by its values at R, which will imply (4.5).

The proofs of Theorems 2.12 and 2.13 are also based on an asymptotic analysis of the Markov chain $\vec{M}^{c,g}$, even though when g is not a fixed point of F_c, it no longer corresponds to the interaction chain in (1.16).

4.2. *Preliminary lemmas*

The key results in this section are Proposition 4.6 and Corollary 4.7.

Let $\vec{X} = \vec{M}^{c,g}$ be as stated before Proposition 4.1. First we list some moment equations for $\vec{X}(n)$, $n \in \mathbb{N}_0$, which follow immediately from Proposition A.1.

Lemma 4.2 (Moment equations for $\vec{X} = \vec{M}^{c,g}$). *Let $c > 0$, and $g \in \mathcal{H}_a$ for some $0 \leq a < c$. Fix $\vec{X}(0) = \vec{\theta} \in [0, \infty)^2$. Then for all $n \in \mathbb{N}_0$,*

$$\mathbb{E}[X_i(n)] = \theta_i, \quad i = 1, 2, \qquad (4.8)$$

$$\mathbb{E}[X_1(n) X_2(n)] = \theta_1 \theta_2. \qquad (4.9)$$

If $((F_c g)_1, (F_c g)_2) = (\lambda_1 g_1, \lambda_2 g_2)$ for some $\lambda_1, \lambda_2 > 0$, then

$$\mathbb{E}[g_i(\vec{X}(n))] = \lambda_i^n g_i(\vec{\theta}), \qquad\qquad i = 1, 2, \qquad (4.10)$$

$$\mathbb{E}[X_i^2(n)] = \theta_i^2 + \frac{1}{c} \sum_{j=1}^{n} \lambda_i^j g_i(\vec{\theta}), \quad i = 1, 2. \qquad (4.11)$$

In the proof of Theorem 2.15, we will need Doob's *h*-transform of a Markov chain, which we recall here. For more information on the *h*-transform, see e.g. Section 4.1 of [26].

Definition 4.3 (*h*-transform). *Let $X = (X(n))_{n \in \mathbb{N}_0}$ be a Markov chain with state space E and n-step transition probability kernel $p_n(x, dy)$. If h is a nonnegative (not identically zero) harmonic function of X, i.e., $(h(X(n)))_{n \in \mathbb{N}_0}$ is a nonnegative martingale, then the h-transform of X, denoted by X^h, is defined as the Markov chain on the space $\{x \in E : h(x) > 0\}$ with n-step transition probability kernel $p_n^h(x, dy) = p_n(x, dy)h(y)/h(x)$.*

The next two lemmas are immediate consequences of Definition 4.3.

Lemma 4.4 (Harmonic functions of X^h). *Let X, h and X^h be as in Definition 4.3. If f is a harmonic function of X, then f/h restricted to $\{x \in E : h(x) > 0\}$ is a harmonic function of X^h. The converse is true if $h(x) > 0$ for all $x \in E$.*

Lemma 4.5 (Absolute continuity of X^h w.r.t. X at bounded stopping times). *Let X, h and X^h be as in Definition 4.3. If $X(0) = X^h(0) = x \in E$ where $h(x) > 0$, and τ is a bounded stopping time, then the law of $X^h(\tau)$ is absolutely continuous with respect to the law of $X(\tau)$ with density $\frac{h(\cdot)}{h(x)}$.*

The next proposition is the key to establishing Proposition 4.1. Such a result is referred to as almost sure *extinction versus unbounded growth*, see e.g. [21].

Proposition 4.6 (Almost sure limit of *h*-transform of $\vec{X} = \vec{M}^{c,g}$). *Let $c > 0$, and let $g \in \mathcal{H}_{0^+}$ satisfy condition (2.21). Let $h(\vec{x}) = (1 + x_1)(1 + x_2)$ and let $\vec{X}^h$ be the h-transform of $\vec{X}$. Then, for any $\vec{X}^h(0) \in [0, \infty)^2$, almost surely, $\lim_{n \to \infty} \vec{X}^h(n) = \vec{X}^h(\infty)$ exists and $\vec{X}^h(\infty) \in R$ (see (4.7)).*

Before giving the proof of Proposition 4.6, which we defer to the end of this subsection, we first state and prove a corollary and another prerequisite lemma.

Corollary 4.7 (Trapping probabilities). *Let c, g, h, $\vec{X}^h$ and $\vec{X}^h(\infty)$ be as in Proposition 4.6.*

(i)

$$\mathbb{P}\big[\vec{X}^h(\infty) = (\infty, \infty)\big] = \frac{X_1^h(0) X_2^h(0)}{(1 + X_1^h(0))(1 + X_2^h(0))}. \tag{4.12}$$

(ii) *If $(0, \infty) \times \{0\} \notin \partial g$, then*

$$\mathbb{P}\big[\vec{X}^h(\infty) = (\infty, 0)\big] = \frac{X_1^h(0)}{(1 + X_1^h(0))(1 + X_2^h(0))}. \tag{4.13}$$

(iii) *If $\{0\} \times (0, \infty) \notin \partial g$, then*

$$\mathbb{P}\big[\vec{X}^h(\infty) = (0, \infty)\big] = \frac{X_2^h(0)}{(1 + X_1^h(0))(1 + X_2^h(0))}. \tag{4.14}$$

Proof. By Lemmas 4.2 and 4.4,

$$f_1(\vec{x}) = \frac{x_1 x_2}{(1 + x_1)(1 + x_2)}, \qquad f_2(\vec{x}) = \frac{x_1}{(1 + x_1)(1 + x_2)}, \qquad f_3(\vec{x}) = \frac{x_2}{(1 + x_1)(1 + x_2)}, \tag{4.15}$$

are bounded harmonic functions of $\vec{X}^h$, and therefore $(f_i(\vec{X}^h(n)))_{n \in \mathbb{N}_0}$, $i = 1, 2, 3$, are bounded martingales. Since, by Proposition 4.6, $\vec{X}^h(n) \to \vec{X}^h(\infty) \in R$ a.s. as $n \to \infty$, we have

$$f_i(\vec{X}^h(0)) = \mathbb{E}_{\vec{\theta}}^{c,g}\big[f_i(\vec{X}^h(\infty))\big], \quad i = 1, 2, 3. \tag{4.16}$$

Now (4.12)–(4.14) follow from the following observations: (1) $f_1((\infty, \infty)) = 1$ and $f_1 = 0$ on $R\setminus\{(\infty, \infty)\}$; (2) if $(0, \infty) \times \{0\} \notin \partial g$, then $f_2((\infty, 0)) = 1$ and $f_2 = 0$ on $R\setminus\{(\infty, 0)\}$; (3) if $\{0\} \times (0, \infty) \notin \partial g$, then $f_3((0, \infty)) = 1$ and $f_3 = 0$ on $R\setminus\{(0, \infty)\}$. $\qquad\square$

The proof of Proposition 4.6 in turn relies on the next lemma, which gives a lower bound for $\hat{\Gamma}^{c,g}_{\vec{\theta},h}(d\vec{x}) = \Gamma^{c,g}_{\vec{\theta}}(d\vec{x})h(\vec{x})/h(\vec{\theta})$, the transition kernel of $\vec{X}^h$ with $h(\vec{x}) = (1 + x_1)(1 + x_2)$, that is uniform in both g and $\vec{\theta}$. The uniformity in g is not needed for the proof of Proposition 4.6, but will be crucial for the proof of Theorem 2.16 in Section 5.

Lemma 4.8 (Uniform lower bound on $\hat{\Gamma}^{c,g}_{\vec{\theta},h}(d\vec{x})$). *Let $\mathcal{A} \subset \mathcal{H}_{0^+}$.*

(i) *For any $\vec{\theta} \in [0, \infty)^2$, if*

$$\exists \varepsilon' > 0 \text{ such that } \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in B_{\varepsilon'}(\vec{\theta})}} g_i(\vec{x}) > 0 \quad \text{for } i = 1 \text{ or } i = 2 \tag{4.17}$$

with $B_{\varepsilon'}(\vec{\theta}) = \{\vec{x} \in [0, \infty)^2 : \|\vec{x} - \vec{\theta}\| \leq \varepsilon'\}$, then

$$\exists \varepsilon > 0 \text{ such that } \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in B_{\varepsilon}(\vec{\theta})}} \hat{\Gamma}^{c,g}_{\vec{x},h}\big([0, \infty)^2 \setminus B_{\varepsilon}(\vec{\theta})\big) > 0. \tag{4.18}$$

(ii) *For any $\alpha > 0$, if*

$$\exists \varepsilon', N' > 0 \text{ such that } \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N', \infty) \times [\alpha - \varepsilon', \alpha + \varepsilon']}} g_2(\vec{x}) > 0 \tag{4.19}$$

and

$$\forall a > 0, \exists C_a \in [0, \infty) \text{ such that, uniformly for all } \vec{x} \in [0, \infty)^2 \text{ and } g \in \mathcal{A}, \\ g_1(\vec{x}) + g_2(\vec{x}) \leq C_a(1 + x_1)(1 + x_2) + a\big(x_1^2 + x_2^2\big), \tag{4.20}$$

then

$$\exists \varepsilon, N > 0 \text{ such that } \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \hat{\Gamma}^{c,g}_{\vec{x},h}\big([0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]\big) > 0. \tag{4.21}$$

A statement similar to (4.21) holds for vertical strips of the form $[\alpha - \varepsilon, \alpha + \varepsilon] \times [N, \infty)$ if, in (4.19), g_2 is replaced by g_1 and $[N', \infty) \times [\alpha - \varepsilon', \alpha + \varepsilon']$ is replaced by $[\alpha - \varepsilon', \alpha + \varepsilon'] \times [N', \infty)$.

Proof. We first prove (4.18) and (4.21) with $\hat{\Gamma}^{c,g}_{\vec{x},h}$ replaced by $\Gamma^{c,g}_{\vec{x}}$. The main tool is the following moment equation valid for $g \in \mathcal{H}_{0^+}$, $\vec{\theta} \in [0, \infty)^2$ and $i = 1, 2$:

$$\mathbb{E}^{c,g}_{\vec{\theta}}\left[\frac{1}{(1 + X_i)^2}\right] = \frac{1}{1 + \theta_i}\mathbb{E}^{c,g}_{\vec{\theta}}\left[\frac{1}{1 + X_i}\right] + \frac{2}{c(1 + \theta_i)}\mathbb{E}^{c,g}_{\vec{x}}\left[\frac{g_i(\vec{X})}{(1 + X_i)^3}\right], \tag{4.22}$$

where $\vec{X} = (\vec{X}(t))_{t \geq 0}$ in this proof denotes the stationary solution of the SDE (2.1). By stationarity, $\mathcal{L}(\vec{X}(s)) = \Gamma^{c,g}_{\vec{\theta}}$ for all $s \geq 0$. Hence

$$M_i(t) = \frac{1}{1 + X_i(t)} - \frac{1}{1 + X_i(0)} - \int_0^t L^{c,g}_{\vec{\theta}}\left(\frac{1}{1 + x_i}\right)\bigg|_{\vec{x} = \vec{X}(s)} ds, \quad i = 1, 2, \tag{4.23}$$

are local martingales, where

$$L_{\vec{\theta}}^{c,g} = c(\theta_1 - x_1)\frac{\partial}{\partial x_1} + c(\theta_2 - x_2)\frac{\partial}{\partial x_2} + g_1(\vec{x})\frac{\partial^2}{\partial x_1^2} + g_2(\vec{x})\frac{\partial}{\partial x_2^2}. \tag{4.24}$$

Since $\mathbb{E}_{\vec{\theta}}^{c,g}[X_i(s)] = \theta_i$ and $\mathbb{E}_{\vec{\theta}}^{c,g}[g_i(\vec{X}(s))] = (F_c g)_i(\vec{\theta}) < \infty$ by Proposition A.1, we have

$$\mathbb{E}_{\vec{\theta}}^{c,g}\left[\sup_{0 \le s \le t} |M_i(s)|\right] \le 2 + \mathbb{E}_{\vec{\theta}}^{c,g}\left[\int_0^t \left(c|\theta_i - X_i(s)| + 2g_i(\vec{X}(s))\right) ds\right]$$

$$\le 2 + 2t\left(c\theta_i + (F_c g)_i(\vec{\theta})\right) < \infty. \tag{4.25}$$

Therefore $M_i = (M_i(t))_{t \ge 0}$, $i = 1, 2$, are in fact martingales, and $\mathbb{E}_{\vec{\theta}}^{c,g}[M_i(t)] = 0$. By the stationarity of $\vec{X}$, we have

$$\mathbb{E}_{\vec{\theta}}^{c,g}\left[L_{\vec{\theta}}^{c,g}\left(\frac{1}{1+x_i}\right)\Big|_{\vec{x}=\vec{X}(s)}\right] = \mathbb{E}_{\vec{\theta}}^{c,g}\left[-c \cdot \frac{1+\theta_i-1-X_i}{(1+X_i)^2} + \frac{2g_i(\vec{X})}{(1+X_i)^3}\right] = 0, \quad i = 1, 2. \tag{4.26}$$

Rearranging terms, we obtain (4.22).

(4.18): Suppose that (4.18) with $\hat{\Gamma}_{\vec{x},h}^{c,g}$ replaced by $\Gamma_{\vec{x}}^{c,g}$ is false. Then

$$\inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in B_\varepsilon(\vec{\theta})}} \Gamma_{\vec{x}}^{c,g}\left([0,\infty)^2 \setminus B_\varepsilon(\vec{\theta})\right) = 0 \quad \forall \varepsilon > 0. \tag{4.27}$$

By (4.17), we may assume without loss of generality that $\inf_{g \in \mathcal{A}, \vec{x} \in B_{\varepsilon_0}(\vec{\theta})} g_1(\vec{x}) = \delta > 0$ for some $\varepsilon_0 > 0$. In particular, $\inf_{g \in \mathcal{A}, \vec{x} \in B_\varepsilon(\vec{\theta})} g_1(\vec{x}) \ge \delta$ for all $\varepsilon \in [0, \varepsilon_0]$. Fix $\varepsilon \in [0, \varepsilon_0]$. Let $\vec{x}^{(n)} \in B_\varepsilon(\vec{\theta})$ and $g^{(n)} \in \mathcal{A}$ be chosen such that $\Gamma_{\vec{x}^{(n)}}^{c,g^{(n)}}([0,\infty)^2 \setminus B_\varepsilon(\vec{\theta})) = o(1)$ as $n \to \infty$. In (4.22) with $i = 1$, substitute $\vec{x}^{(n)}$ and $g^{(n)}$ for $\vec{\theta}$ and g. Then

$$\text{l.h.s.} \le \frac{1}{(1+\theta_1-\varepsilon)^2} + o(1),$$

$$\text{r.h.s.} \ge \frac{1}{(1+\theta_1)(1+\theta_1+\varepsilon)} + \frac{2}{c(1+\theta_1)} \times (1 - o(1)) \times \frac{\delta}{(1+\theta_1+\varepsilon)^3}, \tag{4.28}$$

where we applied Jensen's inequality to obtain $\frac{1}{(1+\theta_1)^2}$ in the estimate for the r.h.s. For $\varepsilon > 0$ sufficiently small and n sufficiently large, the above two equations are incompatible, and therefore (4.18) with $\hat{\Gamma}_{\vec{x},h}^{c,g}$ replaced by $\Gamma_{\vec{x}}^{c,g}$ holds. Since $h(\vec{x}) = (1+x_1)(1+x_2) \ge 1$ on $[0,\infty)^2$ and is bounded on $B_\varepsilon(\vec{\theta})$, it is easy to see by the definition of $\hat{\Gamma}_{\vec{x},h}^{c,g}$ that (4.18) also holds.

(4.21): The proof that (4.21) holds with $\hat{\Gamma}_{\vec{x},h}^{c,g}$ replaced by $\Gamma_{\vec{x}}^{c,g}$ is the same as above and we leave the details to the reader. To get (4.21), we argue as follows.

Choose $\varepsilon \in (0, \alpha)$ and $N_0 > 0$ such that

$$\beta_{\alpha,\varepsilon,N_0} = \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N_0,\infty) \times [\alpha-\varepsilon, \alpha+\varepsilon]}} \Gamma_{\vec{x}}^{c,g}\left([0,\infty)^2 \setminus [N_0,\infty) \times [\alpha-\varepsilon, \alpha+\varepsilon]\right) > 0. \tag{4.29}$$

By Proposition A.1, we have

$$\mathbb{E}_{\vec{x}}^{c,g}\left[(X_1 - x_1)^2\right] = \frac{1}{c}(F_c g)_1(\vec{x}), \quad g \in \mathcal{H}_{0^+}, \ \vec{x} \in [0,\infty)^2. \tag{4.30}$$

Therefore

$$\Gamma_{\vec{x}}^{c,g}\left\{\vec{y} \in [0,\infty)^2 : y_1 < \frac{x_1}{2}\right\} \le \Gamma_{\vec{x}}^{c,g}\left\{\vec{y} \in [0,\infty)^2 : |y_1 - x_1| \ge \frac{x_1}{2}\right\} \le \frac{4(F_c g)_1(\vec{x})}{cx_1^2}. \tag{4.31}$$

We claim that

$$\lim_{x_1 \to \infty} \sup_{\substack{g \in \mathcal{A} \\ x_2 \in [\alpha - \varepsilon, \alpha + \varepsilon]}} \frac{4(F_c g)_1(\vec{x})}{cx_1^2} = 0. \tag{4.32}$$

Assume (4.32) for the moment. Since $\beta_{\alpha, \varepsilon, N}$ is nondecreasing in N, we can choose $N > N_0$ sufficiently large such that

$$\inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \Gamma_{\vec{x}}^{c,g} \left\{ \vec{y} \in [0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon] : y_1 \geq \frac{x_1}{2} \right\} \geq \frac{\beta_{\alpha, \varepsilon, N_0}}{2}. \tag{4.33}$$

Then

$$\inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \hat{\Gamma}_{\vec{x}, h}^{c,g} \left([0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon] \right)$$

$$= \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \int_{[0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]} \frac{h(\vec{y})}{h(\vec{x})} \Gamma_{\vec{x}}^{c,g}(d\vec{y})$$

$$\geq \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \int_{\substack{\{y_1 \geq x_1/2, \\ [0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]\}}} \frac{(1 + y_1)(1 + y_2)}{(1 + x_1)(1 + x_2)} \Gamma_{\vec{x}}^{c,g}(d\vec{y})$$

$$\geq \inf_{\substack{g \in \mathcal{A} \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \frac{1 + x_1/2}{(1 + x_1)(1 + \alpha + \varepsilon)} \int_{\substack{\{y_1 \geq x_1/2, \\ [0, \infty)^2 \setminus [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]\}}} \Gamma_{\vec{x}}^{c,g}(d\vec{y})$$

$$\geq \frac{\beta_{\alpha, \varepsilon, N_0}}{4(1 + \alpha + \varepsilon)} > 0, \tag{4.34}$$

which establishes (4.21).

To verify (4.32), note that, by condition (4.20) and Proposition A.1,

$$\mathbb{E}_{\vec{x}}^{c,g}[g_1 + g_2] \leq \mathbb{E}_{\vec{x}}^{c,g} \left[C_a(1 + X_1)(1 + X_2) + a \left(X_1^2 + X_2^2 \right) \right]$$

$$= C_a(1 + x_1)(1 + x_2) + a \left(x_1^2 + x_2^2 \right) + \frac{a}{c} \mathbb{E}_{\vec{x}}^{c,g}[g_1 + g_2] \quad \forall g \in \mathcal{A}. \tag{4.35}$$

Solving for $\mathbb{E}_{\vec{x}}^{c,g}[g_1 + g_2]$, we get

$$\mathbb{E}_{\vec{x}}^{c,g}[g_1 + g_2] = (F_c g)_1(\vec{x}) + (F_c g)_2(\vec{x}) \leq \frac{c}{c - a} \left(C_a(1 + x_1)(1 + x_2) + a \left(x_1^2 + x_2^2 \right) \right) \quad \forall g \in \mathcal{A}.$$

Therefore

$$\limsup_{x_1 \to \infty} \sup_{\substack{g \in \mathcal{A} \\ x_2 \in [\alpha - \varepsilon, \alpha + \varepsilon]}} \frac{4(F_c g)_1(\vec{x})}{cx_1^2} \leq \frac{4ca}{c - a}. \tag{4.36}$$

Since $a > 0$ can be made arbitrarily small, (4.32) follows. $\qquad \square$

Proof of Proposition 4.6. By Lemma 4.2, $h_1(\vec{x}) = 1 + x_1$, $h_2(\vec{x}) = 1 + x_2$ and $h(\vec{x}) = (1 + x_1)(1 + x_2)$ are harmonic for $\vec{X}$. Hence, by Lemma 4.4, $h_1(\vec{x})/h(\vec{x}) = 1/(1 + x_2)$ and $h_2(\vec{x})/h(\vec{x}) = 1/(1 + x_1)$ are harmonic for $\vec{X}^h$. Therefore $(1/(1 + X_1^h(n)))_{n \in \mathbb{N}_0}$ and $(1/(1 + X_2^h(n)))_{n \in \mathbb{N}_0}$ are nonnegative martingales and, by the martingale convergence

theorem, $\vec{X}^h(n) \to \vec{X}^h(\infty) \in [0, \infty]^2$ a.s. as $n \to \infty$. We need to show that

(i) $\mathbb{P}[\vec{X}^h(\infty) \in [0, \infty)^2, \vec{X}^h(\infty) \notin \partial g] = 0.$

(ii) $\mathbb{P}[X_1^h(\infty) = \infty, X_2^h(\infty) \in (0, \infty)] = \mathbb{P}[X_2^h(\infty) = \infty, X_1^h(\infty) \in (0, \infty)] = 0.$

If (i) is false, then there exists a $\vec{\theta} \in [0, \infty)^2 \backslash \partial g$ such that, for all $B \subset [0, \infty)^2$ with $\vec{\theta} \in \text{int}(B)$, $\mathbb{P}[\vec{X}^h(n) \in B$ for all n large enough$] > 0$. In particular, we must have

$$\inf_{\vec{x} \in B} \frac{1}{h(\vec{x})} \int_{[0,\infty)^2 \backslash B} h(\vec{y}) \Gamma_{\vec{x}}^{c,g}(\mathrm{d}\vec{y}) = 0 \quad \forall B \subset [0, \infty)^2 \text{ with } \vec{\theta} \in \text{int}(B). \tag{4.37}$$

Otherwise, there is a uniform probability of escaping from B at each step, and $\vec{X}^h$ cannot be confined in B forever with positive probability.

If (ii) is false, then (considering without loss of generality the first part of (ii)) there exists an $\alpha \in (0, \infty)$ such that

$$\mathbb{P}[\vec{X}^h(n) \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon] \text{ for all } n \text{ large enough}] > 0 \quad \forall \varepsilon \in (0, \alpha), N > 0. \tag{4.38}$$

In particular, we must have

$$\inf_{\vec{x} \in [N,\infty) \times [\alpha-\varepsilon,\alpha+\varepsilon]} \frac{1}{h(\vec{x})} \int_{[0,\infty)^2 \backslash [N,\infty) \times [\alpha-\varepsilon,\alpha+\varepsilon]} h(\vec{y}) \Gamma_{\vec{x}}^{c,g}(\mathrm{d}\vec{y}) = 0 \quad \forall \varepsilon > 0, N > 0. \tag{4.39}$$

But both (4.37) and (4.39) contradict Lemma 4.8 applied to $\mathcal{A} = \{g\}$, where conditions (4.19) and (4.20) in Lemma 4.8 are easily verified by our assumption that $g \in \mathcal{H}_{0^+}$ and that g satisfies (2.21). Therefore we must have $\lim_{n\to\infty} \vec{X}^h(n) = \vec{X}^h(\infty) \in R$ a.s. $\qquad\square$

4.3. *Proofs of Proposition* 4.1 *and Theorems* 2.12 *and* 2.13

Proof of Proposition 4.1. Let f be a nonnegative harmonic function of $\vec{X} = \vec{M}^{c,g}$ satisfying the constraints in (4.2)–(4.4). Since x_1, x_2 and $x_1 x_2$ are harmonic for $\vec{X}$, so is $f_0(\vec{x}) = f(\vec{x}) - \lambda_{f,(0,\infty)} x_2 - \lambda_{f,(\infty,0)} x_1 - \lambda_{f,(\infty,\infty)} x_1 x_2$. Let $\vec{X}^h$ denote the h-transform of $\vec{X}$ with $h(\vec{x}) = (1 + x_1)(1 + x_2)$. Then, by Lemma 4.4, f_0/h is harmonic for $\vec{X}^h$, and so

$$\frac{f_0(\vec{\theta})}{h(\vec{\theta})} = \mathbb{E}\left[\frac{f_0(\vec{X}^h(n))}{h(\vec{X}^h(n))} \Big| \vec{X}^h(0) = \vec{\theta} \right] \quad \forall n \in \mathbb{N}, \vec{\theta} \in [0, \infty)^2. \tag{4.40}$$

Constraint (4.2) implies that f_0/h is bounded, constraint (4.4) implies that $\lim_{\vec{x} \to \vec{z}} f_0(\vec{x})/h(\vec{x}) = 0$ for all $\vec{z} \in R_\infty$, while constraints (4.3) and (4.4) imply that $\lim_{\vec{x} \to \vec{z}} f_0(\vec{x})/h(\vec{x}) = 0$ for all $\vec{z} \in \partial g$. Since, by Proposition 4.6, $\lim_{n\to\infty} \vec{X}^h(n) = \vec{X}^h(\infty) \in R(= \partial g \cup R_\infty)$ a.s., letting $n \to \infty$ in (4.40) and applying the bounded convergence theorem, we obtain $f_0/h \equiv 0$ and $f_0 \equiv 0$. Therefore $f(\vec{x}) = \lambda_{f,(0,\infty)} x_2 + \lambda_{f,(\infty,0)} x_1 + \lambda_{f,(\infty,\infty)} x_1 x_2$. $\qquad\square$

Proof of Theorem 2.12. Suppose the claim is false. Then, without loss of generality, we may assume that $((F_c g)_1, (F_c g)_2) = (\lambda_1 g_1, \lambda_2 g_2)$ for some $g \in \mathcal{H}_{0^+}$, $\lambda_1 > 1$, $\lambda_1 \geq \lambda_2 > 0$. By Definition 2.5, for any $a > 0$ there exists a $0 < C_a < \infty$ such that $g_1(\vec{x}) + g_2(\vec{x}) \leq C_a(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2)$. Fix $\vec{X}(0) = \vec{\theta} \in [0, \infty)^2$, then by Lemma 4.2, we have

$$\begin{aligned}
\lambda_1^n g_1(\vec{\theta}) &= \mathbb{E}[g_1(\vec{X}(n))] \\
&\leq \mathbb{E}[C_a(1 + X_1(n))(1 + X_2(n)) + a(X_1^2(n) + X_2^2(n))] \\
&\leq C_a(1 + \theta_1)(1 + \theta_2) + a(\theta_1^2 + \theta_2^2) + \frac{a}{c} \sum_{j=1}^n (\lambda_1^j g_1(\vec{\theta}) + \lambda_2^j g_2(\vec{\theta})).
\end{aligned} \tag{4.41}$$

Since $\lambda_1 > 1$ and $\lambda_1 \geq \lambda_2 > 0$, dividing both sides of the above inequality by λ_1^n and letting $n \to \infty$, we get

$$g_1(\vec{\theta}) \leq \frac{a\lambda_1}{c(\lambda_1 - 1)}\big[g_1(\vec{\theta}) + 1_{\lambda_1 = \lambda_2} g_2(\vec{\theta})\big]. \tag{4.42}$$

Since $a > 0$ can be made arbitrarily small, (4.42) implies that $g_1(\vec{\theta}) \leq 0$, which is a contradiction. $\square$

Proof of Theorem 2.13. (i) Assume that, for some $g \in \mathcal{H}_{0+}$ with $\liminf_{\vec{x} \to (\infty, \infty)}[g_1(\vec{x})/x_1^2 + g_2(\vec{x}))/x_2^2] = 0$, $F_c(g_1, g_2) = (\lambda_1 g_1, \lambda_2 g_2)$ for some $0 < \lambda_1, \lambda_2 < 1$. Fix $\vec{X}(0) = \vec{\theta} \in [0, \infty)^2$. By Lemma 4.2, we have

$$\mathbb{E}\big[\big(X_i(n) - \theta_i\big)^2\big] = \frac{1}{c} \sum_{k=1}^{n} \lambda_i^k g_i(\vec{\theta}) < \frac{\lambda_i}{c(1 - \lambda_i)} g_i(\vec{\theta}) \quad \forall n \in \mathbb{N}. \tag{4.43}$$

Next, choose $\vec{\theta}$ such that $\frac{\lambda_i g_i(\vec{\theta})}{c(1-\lambda_i)} \leq \frac{\theta_i^2}{16}$ for $i = 1, 2$, which is possible by the above assumptions. Then, by the Chebychev inequality,

$$\mathbb{P}\bigg(\vec{X}(n) \in \bigg[\frac{\theta_1}{2}, \frac{3\theta_1}{2}\bigg] \times \bigg[\frac{\theta_2}{2}, \frac{3\theta_2}{2}\bigg]\bigg|\vec{X}(0) = \vec{\theta}\bigg) \geq \frac{1}{2} \quad \forall n \in \mathbb{N}, \tag{4.44}$$

and hence

$$\mathbb{E}\big[g_i\big(\vec{X}(n)\big)\big] \geq \frac{1}{2} \inf_{\vec{x} \in [\theta_1/2, 3\theta_1/2] \times [\theta_2/2, 3\theta_2/2]} g_i(\vec{x}) > 0 \quad \forall n \in \mathbb{N}, \tag{4.45}$$

which contradicts the assumption that $\mathbb{E}[g_i(\vec{X}(n))] = \lambda_i^n g_i(\vec{\theta}) \to 0$ as $n \to \infty$.

(ii) We consider the conditions (2.17) and (2.18) separately.

(2.17): Assume that $(F_c g)_1 = \lambda_1 g_1$ with $\lambda_1 < 1$ and $g_1(x_1, 0) > 0$ for all $x_1 > 0$ for some $g \in \mathcal{H}_{0+}$. For $\vec{\theta} = (\theta_1, 0)$ with $\theta_1 \geq 0$, $\vec{\Gamma}_{\vec{\theta}}^{c,g}(d\vec{x})$ is supported on the horizontal axis A_1 and is in fact the equilibrium distribution of the one-dimensional diffusion

$$dX_1(t) = c(\theta_1 - X_1)\, dt + \sqrt{2g_1(X_1, 0)}\, dB_1(t). \tag{4.46}$$

Therefore the mapping $g_1(x_1, 0) \mapsto (F_c g)_1(x_1, 0)$ is the renormalization transformation for diffusions on the halfline which, by Lemma 2 and Theorem 2 in [3], cannot have a fixed shape with scaling constant $\lambda_1 \neq 1$.

(2.18): Assume that $(F_c g)_1 = \lambda_1 g_1$ with $\lambda_1 \in (0, 1)$ for some $g \in \mathcal{H}_{0+}$ such that $\liminf_{\vec{x} \to (\infty, \infty)} g_1(\vec{x})/x_1 x_2 = \varepsilon > 0$. Then the h-transformed Markov chain $\vec{X}^h$ with $h(\vec{x}) = (1 + x_1)(1 + x_2)$ satisfies $\mathbb{E}[(g_1/h)(\vec{X}^h(n))] = \lambda_1^n (g_1/h)(\vec{X}^h(0))$. If $X_1^h(0), X_2^h(0) > 0$, then, by Corollary 4.7,

$$\mathbb{P}\big[\vec{X}^h(\infty) = (\infty, \infty)\big] = \frac{X_1^h(0) X_2^h(0)}{(1 + X_1^h(0))(1 + X_2^h(0))} > 0 \tag{4.47}$$

and

$$0 = \lim_{n \to \infty} \lambda_1^n \frac{g_1(\vec{X}^h(0))}{h(\vec{X}^h(0))} = \lim_{n \to \infty} \mathbb{E}\bigg[\frac{g_1(\vec{X}^h(n))}{h(\vec{X}^h(n))}\bigg] \tag{4.48}$$

$$\geq \frac{X_1^h(0) X_2^h(0)}{(1 + X_1^h(0))(1 + X_2^h(0))} \liminf_{\vec{x} \to (\infty, \infty)} \frac{g_1(\vec{x})}{h(\vec{x})} > 0, \tag{4.49}$$

which is a contradiction. $\square$

5. Proof of Theorem 2.16 with constant c_n

Proof. Assume $c_n \equiv c > 0$, in which case $F^{[n]} = F_c^n$. The proof is based on an analysis of the *interaction chain* introduced in Section 1.2. Let g satisfy the conditions in Theorem 2.16. Let $\vec{X} = (\vec{X}(-n))_{n \in \mathbb{N}_0}$ be the (inhomogeneous) *backward Markov chain* on $[0, \infty)^2$ with transition probability kernel

$$\mathbb{P}\big(\vec{X}(-n) \in d\vec{x} | \vec{X}(-n-1) = \vec{\theta}\big) = \Gamma_{\vec{\theta}}^{c, F_c^n g}(d\vec{x}). \tag{5.1}$$

Denote the transition probability kernel from time $-m$ to time $-n > -m$ by $K^{-m,-n}(\vec{x}, d\vec{u})$. By Proposition A.1, the functions 1, x_1, x_2 and $x_1 x_2$ are harmonic for $\vec{X}$. Let $\vec{X}^h = (\vec{X}^h(-n))_{n \in \mathbb{N}_0}$ denote the h-transform of $\vec{X}$ with $h(\vec{x}) = (1 + x_1)(1 + x_2)$. Then 1, $\frac{x_1}{1+x_1}$, $\frac{x_2}{1+x_2}$ and $\frac{x_1 x_2}{h(\vec{x})}$ are harmonic for $\vec{X}^h$. Now change variables and let

$$\vec{Y}(-n) = \phi\big(\vec{X}^h(-n)\big), \tag{5.2}$$

with $\phi : [0, \infty)^2 \to [0, 1)^2$ given by

$$\phi(x_1, x_2) = \left(\frac{x_1}{1 + x_1}, \frac{x_2}{1 + x_2} \right). \tag{5.3}$$

Then $\vec{Y} = (\vec{Y}(-n))_{n \in \mathbb{N}_0}$ is a backward Markov chain on $[0, 1)^2$ with 1, y_1, y_2 and $y_1 y_2$ harmonic. Denote its transition probability kernel from time $-m$ to time $-n > -m$ by $\hat{K}^{-m,-n}(\vec{y}, d\vec{v})$. Then $\hat{K}^{-m,-n}$ and $K^{-m,-n}$ are related via

$$\int_{[0,\infty)^2} f(\vec{x}) K^{-m,-n}(\vec{\theta}, d\vec{x}) = h(\vec{\theta}) \int_{[0,1)^2} \left(\frac{f}{h} \circ \phi^{-1} \right)(\vec{y}) \hat{K}^{-m,-n}\big(\phi(\vec{\theta}), d\vec{y}\big) \quad \forall f \text{ measurable.}$$

In particular,

$$\big(F_c^j g\big)_i(\vec{\theta}) = \int_{[0,\infty)^2} g_i(\vec{x}) K^{-j,0}(\vec{\theta}, d\vec{x})$$

$$= h(\vec{\theta}) \int_{[0,1)^2} \left(\frac{(F_c^N g)_i}{h} \circ \phi^{-1} \right)(\vec{y}) \hat{K}^{-j,-N}\big(\phi(\vec{\theta}), d\vec{y}\big), \quad 0 \le N \le j, i = 1, 2, \tag{5.4}$$

since $(F_c^j g)_i(\vec{\theta}) = \mathbb{E}[(F_c^N g)_i(\vec{X}(-N)) | \vec{X}(-j) = \vec{\theta}]$ for all $0 \le N \le j$. For $j \in \mathbb{N}$, if we let

$$\vec{Y}^{(j)} = \big(\vec{Y}^{(j)}(-n)\big)_{n \in \mathbb{N}_0} \tag{5.5}$$

denote the Markov chain $\vec{Y}$ started at time $-j$ with $\vec{Y}^{(j)}(-j) = \phi(\vec{\theta})$, and for all $-n < -j$ set $\vec{Y}^{(j)}(-n) = \phi(\vec{\theta})$, then we can rewrite (5.4) as

$$\big(F_c^j g\big)_i(\vec{\theta}) = h(\vec{\theta}) \mathbb{E}\left[\left(\frac{g_i}{h} \circ \phi^{-1} \right)\big(\vec{Y}^{(j)}(0)\big) \right] = h(\vec{\theta}) \mathbb{E}\left[\left(\frac{(F_c^N g)_i}{h} \circ \phi^{-1} \right)\big(\vec{Y}^{(j)}(-N)\big) \right]. \tag{5.6}$$

To establish (2.25), and hence Theorem 2.16 for $c_n \equiv c$, we need the following lemma, the proof of which is postponed.

Lemma 5.1. *For any fixed $N \in \mathbb{N}_0$, all weak limit points of $\{\vec{Y}^{(j)}(-N)\}_{j \in \mathbb{N}}$ as $j \to \infty$ are supported on $\phi(R_\infty) \cup ([0, 1) \times \{0\}) \cup (\{0\} \times [0, 1))$.*

We first complete the proof subject to Lemma 5.1. Without loss of generality, take $i = 1$. Note that, since $g \in \mathcal{H}_0^r$, we have $g_1(\vec{x}) + g_2(\vec{x}) \le C(1 + x_1)(1 + x_2)$ for some $C > 0$. Consequently, by the moment equations (A.2) and (A.3), the family of functions

$$\left\{ \left(\frac{(F_c^k g)_1}{h} \circ \phi^{-1} \right)(\vec{y}) \right\}_{k \in \mathbb{N}_0, \vec{y} \in [0,1)^2} \tag{5.7}$$

is uniformly bounded. Now fix $\vec{\theta} \in [0, \infty)^2$. If $\{j'_m\}_{m \in \mathbb{N}}$ is any subsequence along which $\lim_{m \to \infty}(F_c^{j'_m} g)_1(\vec{\theta})$ exists, then we can find a further subsequence $\{j_m\}_{m \in \mathbb{N}}$ such that $\vec{Y}^{(j_m)}$ converges weakly to a limit $\vec{Y}^\infty = (\vec{Y}^\infty(-n))_{n \in \mathbb{N}_0}$ as $([0, 1]^2)^{\mathbb{N}}$-valued random variables with the product topology. In particular, $\vec{Y}^{(j_m)}(-N)$ converges weakly to $\vec{Y}^\infty(-N)$ for each $N \in \mathbb{N}_0$.

By Theorem 2.6, the family

$$\left\{ \left(\frac{(F_c^k g)_1}{h} \circ \phi^{-1} \right)(\vec{y}) \right\}_{k \in \mathbb{N}_0} \tag{5.8}$$

is continuous on $[0, 1)^2$. In fact, it is also continuous at $\phi(R_\infty)$ with

$$\left(\frac{(F_c^k g)_1}{h} \circ \phi^{-1} \right)(\vec{z}) = \lambda_{1,\vec{z}} \quad \forall k \in \mathbb{N}_0, \vec{z} \in \phi(R_\infty). \tag{5.9}$$

Indeed, this follows from these observations: (1) $g \in \mathcal{H}_0^r$, and hence $((g_1/h) \circ \phi^{-1})(\vec{z}) = \lambda_{1,\vec{z}}$ for $\vec{z} \in \phi(R_\infty)$ and is continuous at $\vec{z}$; (2) by (5.6), $(F_c^k g)_1(\vec{\theta})/h(\vec{\theta}) = \mathbb{E}[(g_1/h) \circ \phi^{-1})(\vec{Y}^{(k)}(0))]$; (3) because $Y_i^{(k)}$, $i = 1, 2$, are martingales while $\phi(R_\infty) = \{(1, 0), (0, 1), (1, 1)\}$ are extremal in $[0, 1]^2$, it follows from the Markov inequality that $\hat{K}^{-k,0}(\phi(\vec{\theta}), d\vec{y})$ converges weakly to the point mass at $\vec{z}$ as $\phi(\vec{\theta}) \to \vec{z}$ for $\vec{z} \in \phi(R_\infty)$. By Lemma 5.1, we can now substitute j_m for j in (5.6) and take the limit $m \to \infty$, to obtain

$$\lim_{m \to \infty} \left(F_c^{j_m} g \right)_1(\vec{\theta}) = h(\vec{\theta}) \mathbb{E}\left[\left(\frac{(F_c^N g)_1}{h} \circ \phi^{-1} \right)(\vec{Y}^\infty(-N)) \right] \quad \forall N \in \mathbb{N}_0. \tag{5.10}$$

Denote the distribution of $\vec{Y}^\infty(-N)$ by μ_N. Again by Lemma 5.1, μ_N is concentrated on $\phi(R_\infty) \cup [0, 1) \times \{0\} \cup \{0\} \times [0, 1)$. Consequently, because $((F_c^N g)_1/h \circ \phi^{-1})(\vec{y})$ vanishes on $\{0\} \times [0, 1]$, we have

$$\lim_{m \to \infty} \left(F_c^{j_m} g \right)_1(\vec{\theta}) = h(\vec{\theta}) \left(\mu_N\{(1, 1)\} \lambda_{1,(\infty,\infty)} + \int_0^1 \left(\frac{(F_c^N g)_1}{h} \circ \phi^{-1} \right)(y_1, 0) \mu_N(dy_1 \times \{0\}) \right). \tag{5.11}$$

Since y_1, y_2, $y_1 y_2$ are bounded continuous functions on $[0, 1]^2$ and since $\mathbb{E}[Y_i^{(j_m)}(-N)] = \phi_i(\vec{\theta})$ and $\mathbb{E}[Y_1^{(j_m)}(-N)Y_2^{(j_m)}(-N)] = \phi_1(\vec{\theta})\phi_2(\vec{\theta})$ with $\phi = (\phi_1, \phi_2)$, we must also have $\int y_i \mu_N(d\vec{y}) = \phi_i(\vec{\theta})$ and $\int y_1 y_2 \times \mu_N(d\vec{y}) = \phi_1(\vec{\theta})\phi_2(\vec{\theta})$. By our property of the support of μ_N, we thus find

$$\mu_N\{(1, 1)\} = \phi_1(\vec{\theta})\phi_2(\vec{\theta}) = \frac{\theta_1 \theta_2}{h(\vec{\theta})}, \tag{5.12}$$

$$\int y_1 \mu_N(dy_1 \times \{0\}) = \int y_1(1 - y_2) \mu_N(d\vec{y}) = \phi_1(\vec{\theta})(1 - \phi_2(\vec{\theta})) = \frac{\theta_1}{h(\vec{\theta})}. \tag{5.13}$$

Therefore

$$\left| \lim_{m \to \infty} \left(F_c^{j_m} g \right)_1(\vec{\theta}) - \lambda_{1,(\infty,\infty)} \theta_1 \theta_2 - \lambda_{1,(\infty,0)} \theta_1 \right|$$

$$\leq h(\vec{\theta}) \sup_{y_1 \in [0,1]} \left| \left(\frac{(F_c^N g)_1}{h} \circ \phi^{-1} \right)(y_1, 0) - \lambda_{1,(\infty,0)} y_1 \right|$$

$$= h(\vec{\theta}) \sup_{x > 0} \left| \frac{(F_c^N g)_1(x, 0) - \lambda_{1,(\infty,0)} x}{1 + x} \right|. \tag{5.14}$$

Next, note that $((F_c^N g)_1)_{N \in \mathbb{N}_0}$ restricted to $(0, \infty) \times \{0\}$ are the iterates of the renormalization transformation acting on diffusion functions on the halfline with initial diffusion function $g_1(x, 0)$. Since $\lim_{x \to \infty} g_1(x, 0)/x = \lambda_{1,(\infty,0)} \in [0, \infty)$, Theorem 5 of [3] implies that $\sup_{x > 0} |(F_c^N g)_1(x, 0) - \lambda_{1,(\infty,0)} x|/(1 + x) \to 0$ as $N \to \infty$. (The case

$\lambda_{1,(\infty,0)} = 0$ is not included in Theorem 5 in [3], but an examination of the proof shows that the same result holds.) Since N can be taken arbitrarily large in (5.14), we have established the convergence in (2.25) along the subsequence $\{j_m\}_{m\in\mathbb{N}}$. Since $\{(F_c^j g)_1(\vec{\theta})\}_{j\in\mathbb{N}_0}$ is uniformly bounded, (2.25) now follows and the proof of Theorem 2.16 for $c_n \equiv c$ is complete. $\qquad\square$

We now prove Lemma 5.1.

Proof of Lemma 5.1. We must prove that the weak limit of $\{\vec{Y}^{(j_m)}\}_{m\in\mathbb{N}}$, written $\vec{Y}^\infty$, satisfies

$$\mathbb{P}\big(\vec{Y}^\infty(-N) \in \phi(R_\infty) \cup [0,1) \times \{0\} \cup \{0\} \times [0,1)\big) = 1 \quad \forall N \in \mathbb{N}_0. \tag{5.15}$$

The proof consists of the following three steps:

(A) Show that $(Y_i^\infty(-n))_{n\in\mathbb{N}_0}$, $i = 1, 2$, are backward martingales on $[0,1]$, i.e.,

$$\mathbb{E}\big[Y_i^\infty(-k)\big|\big(Y_i^\infty(-n)\big)_{n\geq k+1}\big] = Y_i^\infty(-k-1), \quad i = 1, 2, \tag{5.16}$$

implying that $\lim_{n\to\infty} \vec{Y}^\infty(-n) = \vec{Y}^\infty(-\infty)$ exists a.s. by the backward martingale convergence theorem (see e.g. Section 4.6 in [18]).

(B) Show that $\mathbb{P}\{\vec{Y}^\infty(-\infty) \in \phi(R_\infty) \cup [0,1) \times \{0\} \cup \{0\} \times [0,1)\} = 1$.

(C) Show that $\mathbb{P}\{\vec{Y}^\infty(-N) \in \phi(R_\infty) \cup [0,1) \times \{0\} \cup \{0\} \times [0,1)\} = 1$ for all $N \in \mathbb{N}_0$.

Since $(Y_i^{(j_m)}(-n))_{n\in\mathbb{N}_0}$, $m \in \mathbb{N}$, $i = 1, 2$, are bounded backward martingale sequences, (A) follows from a general result on weak limits of backward martingale sequences, which we state as Lemma 5.2. The proof of (B) given below uses Lemma 4.8, which relies on uniform lower and upper bounds on $\{F_c^n g\}_{n\in\mathbb{N}_0}$, where assumptions (2.24) and $g \in \mathcal{H}_0$ are crucial. The proof of (C) given below is achieved after approximating $\vec{Y}^\infty$ by the Markov chains $\vec{Y}^{(j_m)}$ and using the fact that $Y_i^{(j_m)}$, $i = 1, 2$, are martingales. Note that it is not clear if $\vec{Y}^\infty$ is a Markov chain, because $\vec{Y}^{(j_m)}$ take values in $[0,1)^2$ while $\vec{Y}^\infty$ takes values in $[0,1]^2$. Even though the transition kernels of $\vec{Y}^{(j_m)}$ are consistent for m sufficiently large, they may not be (weakly) continuously extendable to $[0,1]^2 \setminus [0,1)^2$.

Lemma 5.2 (Weak limits of backward martingales). *For $j \in \mathbb{N}$, let $Z^{(j)} = (Z^{(j)}(-n))_{n\in\mathbb{N}_0}$ be a backward martingale, i.e.,*

$$\mathbb{E}\big[Z^{(j)}(-k)\big|\big(Z^{(j)}(-n)\big)_{n\geq k+1}\big] = Z^{(j)}(-k-1). \tag{5.17}$$

If $\{Z^{(j)}(0)\}_{j\in\mathbb{N}}$ are uniformly integrable, and $Z^{(j)}$ converges weakly to a random variable $Z^\infty = (Z^\infty(-n))_{n\in\mathbb{N}_0}$ in the space $\mathbb{R}^\mathbb{N}$ with the product topology, then $(Z^\infty(-n))_{n\in\mathbb{N}_0}$ is also a backward martingale.

Proof. Since $\{Z^{(j)}(0)\}_{j\in\mathbb{N}_0}$ are uniformly integrable, we have

$$\forall \varepsilon > 0, \exists N > 0 \text{ such that } \mathbb{E}\big[\big|Z^{(j)}(0)\big|1_{|Z^{(j)}(0)|\geq N}\big] \leq \varepsilon \quad \forall j \in \mathbb{N}, \tag{5.18}$$

which is easily seen to be equivalent to

$$\forall \varepsilon > 0, \exists N > 0 \text{ such that } \mathbb{E}\big[\big(\big|Z^{(j)}(0)\big| - N\big)^+\big] \leq \varepsilon \quad \forall j \in \mathbb{N}. \tag{5.19}$$

Since $f(x) = (|x| - N)^+$ is a convex function, for all $j, k \in \mathbb{N}$ we have, by Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\big[\big(\big|Z^{(j)}(-k)\big| - N\big)^+\big] &= \mathbb{E}\big[f\big(Z^{(j)}(-k)\big)\big] \\
&= \mathbb{E}\big[f\big(\mathbb{E}\big[Z^{(j)}(0)\big|\big(Z^{(j)}(-n)\big)_{n\geq k}\big]\big)\big] \\
&\leq \mathbb{E}\big[\mathbb{E}\big[f\big(Z^{(j)}(0)\big)\big|\big(Z^{(j)}(-n)\big)_{n\geq k}\big]\big] \\
&= \mathbb{E}\big[f\big(Z^{(j)}(0)\big)\big] = \mathbb{E}\big[\big(\big|Z^{(j)}(0)\big| - N\big)^+\big].
\end{aligned} \tag{5.20}$$

Therefore $\{Z^{(j)}(-n)\}_{j \in \mathbb{N}, n \in \mathbb{N}_0}$ is a uniformly integrable family.

For each $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$, and any bounded continuous function $f: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, the martingale property of $Z^{(j)}$ implies that

$$\mathbb{E}\big[f\big((Z^{(j)}(-n))_{n \geq k+1}\big)\big(Z^{(j)}(-k) - Z^{(j)}(-k-1)\big)\big] = 0. \tag{5.21}$$

Since $Z^{(j)}$ converges weakly to Z^{∞}, and $\{Z^j(-k)\}_{j \in \mathbb{N}}$ and $\{Z^j(-k-1)\}_{j \in \mathbb{N}}$ are uniformly integrable, we may pass to the limit $j \to \infty$ and obtain

$$\mathbb{E}\big[f\big((Z^{\infty}(-n))_{n \geq k+1}\big)\big(Z^{\infty}(-k) - Z^{\infty}(-k-1)\big)\big] = 0. \tag{5.22}$$

Indeed, the latter is easily verified by applying Skorohod's representation theorem, which allows for a coupling between $\{Z^{(j)}\}_{j \in \mathbb{N}}$ and Z^{∞} such that the convergence is a.s. From (5.22) we have

$$\mathbb{E}\big[f\big((Z^{\infty}(-n))_{n \geq k+1}\big)\mathbb{E}\big[Z^{\infty}(-k) - Z^{\infty}(-k-1)\big|(Z^{\infty}(-n))_{n \geq k+1}\big]\big] = 0, \tag{5.23}$$

which implies that

$$\mathbb{E}\big[Z^{\infty}(-k) - Z^{\infty}(-k-1)\big|(Z^{\infty}(-n))_{n \geq k+1}\big] = 0 \quad \text{a.s.,} \tag{5.24}$$

and thus establishes the martingale property for Z^{∞}. $\qquad\square$

We are now ready to verify (B) and (C).

(B): Note that

$$\phi(R_{\infty}) \cup \big([0, 1) \times \{0\}\big) \cup \big(\{0\} \times [0, 1)\big) = \big([0, 1] \times \{0\}\big) \cup \big(\{0\} \times [0, 1]\big) \cup (1, 1). \tag{5.25}$$

Suppose that (B) fails. Then there exists a $\vec{u} \in (0, 1]^2 \setminus (1, 1)$ in the support of the distribution of $\vec{Y}^{\infty}(-\infty)$. In particular, for each $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $N(\varepsilon) > 0$ such that

$$\mathbb{P}\big\{\vec{Y}^{\infty}(-n) \in \hat{B}_{\varepsilon/2}(\vec{u}) \ \forall n \geq N(\varepsilon)\big\} > \delta(\varepsilon), \tag{5.26}$$

where $\hat{B}_{\varepsilon/2}(\vec{u}) = \{\vec{y} \in [0, 1]^2 : \|\vec{y} - \vec{u}\| \leq \varepsilon/2\}$. Since $\vec{Y}^{(j_m)}$ converges weakly to $\vec{Y}^{\infty}$ as $m \to \infty$, for each $M \in \mathbb{N}$ we can find an $m^* = m^*(M)$ sufficiently large such that

$$\mathbb{P}\big\{\vec{Y}^{(j_{m^*})}(-n) \in \hat{B}_{\varepsilon}(\vec{u}) \cap (0, 1)^2 \ \forall N(\varepsilon) \leq n \leq N(\varepsilon) + M\big\} \geq \frac{1}{2}\delta(\varepsilon). \tag{5.27}$$

We now derive a contradiction with Lemma 4.8 as follows. By assumption (2.24) and the fact that $g \in \mathcal{H}_0^r$, implying $g_1(\vec{x}) + g_2(\vec{x}) \leq C(1 + x_1)(1 + x_2)$ for some $0 < C = C(g) < \infty$, $F_c^n g$ satisfy the same upper and lower bounds for all $n \in \mathbb{N}$. It is then easy to check that in Lemma 4.8 with $\mathcal{A} = \{F_c^n g\}_{n \in \mathbb{N}_0}$ condition (4.17) is satisfied for all $\vec{\theta} \in (0, \infty)^2$, and conditions (4.19) and (4.20) are satisfied for all $\alpha > 0$ (and the analogue of (4.19) for vertical strips). Since the transition kernel $\hat{K}^{-n-1, -n}(\phi(\vec{\theta}), d\vec{y})$ is related to the biased equilibrium measure $\hat{\Gamma}_{\vec{\theta}, h}^{c, F_c^n g}(d\vec{x})$ through the coordinate change ϕ, Lemma 4.8(i) and (ii) imply that, for $\vec{u} \in (0, 1]^2 \setminus (1, 1)$ and $\varepsilon > 0$ sufficiently small,

$$\inf_{\substack{n \in \mathbb{N}_0 \\ \vec{v} \in \hat{B}_{\varepsilon}(\vec{u}) \cap [0, 1)^2}} \hat{K}^{-n-1, -n}\big(\vec{v}, [0, 1)^2 \setminus \hat{B}_{\varepsilon}(\vec{u})\big) > 0. \tag{5.28}$$

This uniform rate of escape from $\hat{B}_{\varepsilon}(\vec{u})$ contradicts (5.27), where M can be chosen to be arbitrarily large while $\delta(\varepsilon) > 0$ remains fixed.

(C): For $\varepsilon > 0$, let

$$U_{\varepsilon} = \Big\{\vec{y} \in [0, 1]^2 : \inf_{\vec{z} \in \phi(R_{\infty}) \cup [0, 1) \times \{0\} \cup \{0\} \times [0, 1)} \|\vec{y} - \vec{z}\| \leq \varepsilon\Big\}. \tag{5.29}$$

Since $\lim_{n\to\infty}\vec{Y}^\infty(-n) = \vec{Y}^\infty(-\infty)$ a.s., we can choose $M = M(\varepsilon)$ sufficiently large such that $\mathbb{P}(\vec{Y}^\infty(-M) \in U_\varepsilon) > 1 - \varepsilon$. Since $\vec{Y}^{(j_m)}(-M) \to \vec{Y}^\infty(-M)$ in distribution as $m \to \infty$, we can choose $m^*(M)$ sufficiently large such that $\mathbb{P}(\vec{Y}^{(j_m)}(-M) \in U_{2\varepsilon}) > 1 - 2\varepsilon$ for all $m \geq m^*$. By the geometry of $\phi(R_\infty) \cup ([0,1) \times \{0\}) \cup (\{0\} \times [0,1))$ and the fact that $Y_i^{(j_m)}(\cdot)$, $i = 1, 2$, are martingales for the Markov chain $(\vec{Y}^{(j_m)}(-n))_{n\in\mathbb{N}_0}$, an elementary application of the Chebychev inequality shows that, for all $m \geq m^*$ and $L > 2$,

$$\mathbb{P}\big(\vec{Y}^{(j_m)}(0) \in U_{2L\varepsilon}\big) \geq (1 - 2\varepsilon)\left(1 - \frac{2}{L}\right). \tag{5.30}$$

By the weak convergence of $\vec{Y}^{(j_m)}(0)$ to $\vec{Y}^\infty(0)$ as $m \to \infty$, the same holds for $\vec{Y}^\infty(0)$. Now let $\varepsilon \to 0$ and $L \to \infty$ such that $\varepsilon L \to 0$. Then we find that

$$\mathbb{P}\big(\vec{Y}^\infty(0) \in \phi(R_\infty) \cup ([0,1) \times \{0\}) \cup (\{0\} \times [0,1))\big) = 1.$$

The same argument works for $\vec{Y}^\infty(-N)$ for any $N \in \mathbb{N}_0$. $\square$

6. Proof of Theorem 2.16 with varying c_n

Proof. The proof of Theorem 2.16 with varying c_n follows the same line of argument as that for constant c_n, except for a few technical differences, which we now outline. For the rest of the section, let $(\vec{X}(-n))_{n\in\mathbb{N}_0}$ denote the backward time-inhomogeneous Markov chain with transition kernels

$$\mathbb{P}\big(\vec{X}(-n) \in d\vec{x} | \vec{X}(-n-1) = \vec{\theta}\big) = \Gamma_{\vec{\theta}}^{c_n, F^{[n]}g}(d\vec{x}), \tag{6.1}$$

and let $(\vec{X}^h(-n))_{n\in\mathbb{N}_0}$ denote $\vec{X}$ h-transformed by $h(\vec{x}) = (1 + x_1)(1 + x_2)$, which is still a harmonic function for $\vec{X}$. Both $\vec{X}$ and $\vec{X}^h$ generalize their counterparts in Section 5. We proceed by first establishing the analogue of Lemma 5.1, where $\{\vec{Y}^{(j)}\}_{j\in\mathbb{N}}$ are now defined in terms of our current $\vec{X}$ and $\vec{X}^h$.

The proof of Lemma 5.1 in Section 5 is based on Lemma 4.8, which no longer applies in our current context, because if c_n can be arbitrarily large, then we lose the uniformity of the escape probability with respect to $\{\Gamma_{\vec{x}}^{c_n, F^{[n]}g}\}_{n\in\mathbb{N}_0}$. So, the first task is to formulate a suitable analogue of Lemma 4.8 for our current $\vec{X}$ and $\vec{X}^h$, which would imply the analogue of Lemma 5.1 for the present context. In the derivation of Theorem 2.16 for constant c_n from Lemma 5.1, we used the following fact from [3]: for the renormalization transformation F_c acting on one-dimensional diffusion functions $f : [0, \infty) \to [0, \infty)$, where f is positive and continuous on $(0, \infty)$, locally Lipschitz at 0, $f(0) = 0$ and $\lim_{x\to\infty} f(x)/x = \lambda \in [0, \infty)$, we have $\sup_{x>0} |(F_c^n f)(x) - \lambda x|/(1 + x) \to 0$ as $n \to \infty$. Our second task is therefore to establish the analogous result for $F^{[n]}f$. The two technical points outlined above will be addressed in Lemma 6.1 and Proposition 6.2.

Observe that, by Proposition A.1, for all $-m \leq -n \leq 0$ and $\vec{\theta} \in [0, \infty)^2$, the backward Markov chain $\vec{X}$ satisfies the moment equations

$$\mathbb{E}\big[\vec{X}(-n)|\vec{X}(-m) = \vec{\theta}\big] = \vec{\theta}, \tag{6.2}$$

$$\mathbb{E}\big[X_1(-n)X_2(-n)|\vec{X}(-m) = \vec{\theta}\big] = \theta_1\theta_2, \tag{6.3}$$

$$\mathbb{E}\big[X_i(-n)^2|\vec{X}(-m) = \vec{\theta}\big] = \theta_i^2 + \left(\sum_{j=n}^{m-1} \frac{1}{c_j}\right)(F^{[m]}g)_i(\vec{\theta}), \quad i = 1, 2, \tag{6.4}$$

$$\mathbb{E}\big[g_i(\vec{X}(0))|\vec{X}(-m) = \vec{\theta}\big] = (F^{[m]}g)_i(\vec{\theta}), \qquad i = 1, 2. \tag{6.5}$$

From the point of view of variance increment, (6.4) indicates that the natural time associated with $(\vec{X}(-n))_{n\geq 0}$ is not n, but rather $\sum_{i=0}^{n-1} c_i^{-1}$. Therefore to obtain a uniform bound on escape probabilities for the Markov chain $\vec{X}^h$, we formulate the analogue of Lemma 4.8 as follows.

Lemma 6.1 (Uniform rate of escape of $(\vec{X}^h(-n))_{n \geq 0}$ from small balls and thin strips). *Let $(c_n)_{n \in \mathbb{N}_0}$ and g be as in Theorem 2.16. Let $(\vec{X}(-n))_{n \in \mathbb{N}_0}$ denote the inhomogeneous backward Markov chain with transition kernel (6.1), and let $(\vec{X}^h(-n))_{n \in \mathbb{N}_0}$ denote $(\vec{X}(-n))_{n \in \mathbb{N}_0}$ h-transformed by $h(\vec{x}) = (1 + x_1)(1 + x_2)$. There exists an increasing sequence $(n_k)_{k \in \mathbb{N}_0} \subset \mathbb{N}_0$ with $n_0 = 0$ such that $\sum_{i=n_k}^{n_{k+1}-1} c_i^{-1} \in [\Lambda^{-1}, \Lambda]$ for some $\Lambda > 1$ for all $k \in \mathbb{N}_0$. For $A \subset [0, \infty)^2$, denote $\tau_A^{-m} = \inf\{-j \geq -m : \vec{X}^h(-j) \notin A\}$. Then*

(i) *For each $\vec{\theta} \in (0, \infty)^2$, there exists $\varepsilon > 0$ such that*

$$\inf_{\substack{k \in \mathbb{N}_0 \\ \vec{x} \in B_\varepsilon(\vec{\theta})}} \mathbb{P}\big(\tau_{B_\varepsilon(\vec{\theta})}^{-n_{k+1}} \leq -n_k \big| \vec{X}^h(-n_{k+1}) = \vec{x}\big) > 0. \tag{6.6}$$

(ii) *For each $\alpha > 0$, there exist $\varepsilon, N > 0$ such that*

$$\inf_{\substack{k \in \mathbb{N}_0 \\ \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}} \mathbb{P}\big(\tau_{[N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]}^{-n_{k+1}} \leq -n_k \big| \vec{X}^h(-n_{k+1}) = \vec{x}\big) > 0, \tag{6.7}$$

$$\inf_{\substack{k \in \mathbb{N}_0 \\ \vec{x} \in [\alpha - \varepsilon, \alpha + \varepsilon] \times [N, \infty)}} \mathbb{P}\big(\tau_{[\alpha - \varepsilon, \alpha + \varepsilon] \times [N, \infty)}^{-n_{k+1}} \leq -n_k \big| \vec{X}^h(-n_{k+1}) = \vec{x}\big) > 0. \tag{6.8}$$

Proof. The existence of the increasing sequence $(n_k)_{k \in \mathbb{N}_0}$ with the prescribed property follows immediately from our assumptions that $\inf_{n \in \mathbb{N}_0} c_n > 0$ and $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$. The rest of the proof parallels that of Lemma 4.8. First we prove (6.6)–(6.8) with $\vec{X}^h$ replaced by $\vec{X}$. By (4.22), for each $m \in \mathbb{N}_0$ and $\vec{x} \in [0, \infty)^2$, $i = 1, 2$, conditioned on $\vec{X}(-m-1) = \vec{x}$, we have

$$\mathbb{E}\left[\frac{1}{(1 + X_i(-m))^2}\right] = \frac{1}{1 + x_i}\mathbb{E}\left[\frac{1}{1 + X_i(-m)}\right] + \frac{2}{c_m(1 + x_i)}\mathbb{E}\left[\frac{(F^{[m]}g)_i(\vec{X}(-m))}{(1 + X_i(-m))^3}\right]$$

$$\geq \frac{1}{(1 + x_i)^2} + \frac{2}{c_m(1 + x_i)}\mathbb{E}\left[\frac{(F^{[m]}g)_i(\vec{X}(-m))}{(1 + X_i(-m))^3}\right], \tag{6.9}$$

where we applied Jensen's inequality. Conditioned on $\vec{X}(-n_{k+1}) = \vec{x}$, we can apply (6.9) iteratively to obtain, for $i = 1, 2$,

$$\mathbb{E}\left[\frac{1}{(1 + X_i(-n_k))^2}\right] \geq \frac{1}{(1 + x_i)^2} + \sum_{m=n_k}^{n_{k+1}-1} \frac{2}{c_m}\mathbb{E}\left[\frac{1}{1 + X_i(-m - 1)}\frac{(F^{[m]}g)_i(\vec{X}(-m))}{(1 + X_i(-m))^3}\right]. \tag{6.10}$$

If (6.6) fails when $\vec{X}^h$ is replaced by $\vec{X}$, then there exists $\vec{\theta} \in (0, \infty)^2$ such that, for all $\varepsilon > 0$, there exist sequences $k^{(l)} \uparrow \infty$ and $\vec{x}^{(l)} \in B_\varepsilon(\vec{\theta})$ (depending on ε) such that

$$\lim_{l \to \infty} \mathbb{P}\big(\tau_{B_\varepsilon(\vec{\theta})}^{-n_{k^{(l)}+1}} \leq -n_{k^{(l)}} \big| \vec{X}(-n_{k^{(l)}+1}) = \vec{x}^{(l)}\big) = 0. \tag{6.11}$$

Now we apply (6.10) to $\vec{X}(-n)$ for $-n_{k^{(l)}+1} \leq -n \leq -n_{k^{(l)}}$ with $\vec{X}(-n_{k^{(l)}+1}) = \vec{x}^{(l)}$. By (6.11), as $l \to \infty$, the two sides of (6.10) satisfy

$$\text{l.h.s.} \leq \frac{1}{(1 + \theta_i - \varepsilon)^2} + o(1), \tag{6.12}$$

$$\text{r.h.s.} \geq \frac{1}{(1 + \theta_i + \varepsilon)^2} + \sum_{m=n_{k^{(l)}}}^{n_{k^{(l)}+1}-1} \frac{2}{c_m}\big(1 - o(1)\big)\frac{\delta}{(1 + \theta_i + \varepsilon)^4}, \tag{6.13}$$

where in (6.13) we have used the assumption that $g_i(\vec{x}) \geq \alpha_i x_i + \beta_i x_1 x_2$ for some $\alpha_i, \beta_i \geq 0$ and $\alpha_i + \beta_i > 0$, $i = 1, 2$, which implies that $\{F^{[m]}g\}_{m \in \mathbb{N}_0}$ satisfy the same lower bound and $(F^{[m]}g)_i(\vec{x}) \geq \delta > 0$ uniformly for $\vec{x} \in B_\varepsilon(\vec{\theta})$

and $m \in \mathbb{N}_0$. Since $\sum_{m=n_k}^{n_{k-1}} \frac{2}{c_m} \geq \Lambda^{-1} > 0$ uniformly for all $k \in \mathbb{N}_0$, (6.12) and (6.13) are incompatible for $\varepsilon > 0$ sufficiently small and $l \in \mathbb{N}$ sufficiently large. Therefore (6.6) must hold for $\vec{X}$ in place of $\vec{X}^h$. The proof of (6.7) and (6.8) for $\vec{X}$ in place of $\vec{X}^h$ is similar, and we leave the details to the reader.

To verify that (6.6) also holds for $\vec{X}^h$, we apply Lemma 4.5 and note that $h(\vec{x}) = (1 + x_1)(1 + x_2)$ is bounded uniformly from above for $\vec{x} \in B_\varepsilon(\vec{\theta})$, and bounded uniformly from below by 1 for $\vec{x} \in [0, \infty)^2$. The proof of (6.7) and (6.8) for $\vec{X}^h$ is essentially the same as its counterpart in the proof of Lemma 4.8. Note that by Lemma 4.5, the law of $\vec{X}^h(\tau_{[N,\infty)\times[\alpha-\varepsilon,\alpha+\varepsilon]}^{-n_{k+1}} \wedge (-n_k))$ conditioned on $\vec{X}^h(-n_{k+1}) = \vec{x} \in [N, \infty) \times [\alpha - \varepsilon, \alpha + \varepsilon]$ is absolutely continuous with respect to the law of $\vec{X}(\tau_{[N,\infty)\times[\alpha-\varepsilon,\alpha+\varepsilon]}^{-n_{k+1}} \wedge (-n_k))$ conditioned on $\vec{X}(-n_{k+1}) = \vec{x}$, where the density is $\frac{h(\cdot)}{h(\vec{x})}$. As in the proof of Lemma 4.8, it suffices to show that for any fixed $0 < \varepsilon < \alpha < \infty$,

$$\lim_{x_1 \to \infty} \sup_{\substack{k \in \mathbb{N}_0 \\ x_2 \in [\alpha-\varepsilon,\alpha+\varepsilon]}} \mathbb{P}\big(\tau_{[x_1/2,\infty)\times\mathbb{R}}^{-n_{k+1}} \leq -n_k | \vec{X}(-n_{k+1}) = \vec{x}\big) = 0. \tag{6.14}$$

Since $(X_1(-n))_{n \leq n_{k+1}}$ is a martingale, by Doob's inequality and (6.2)–(6.4), we have

$$\mathbb{P}\big(\tau_{[x_1/2,\infty)\times\mathbb{R}}^{-n_{k+1}} \leq -n_k | \vec{X}(-n_{k+1}) = \vec{x}\big)$$

$$\leq \mathbb{P}\Bigg(\sup_{-n_{k+1} \leq -n \leq -n_k} \big|X_1(-n) - x_1\big| \geq \frac{x_1}{2} \Big| \vec{X}(-n_{k+1}) = \vec{x} \Bigg)$$

$$\leq \frac{16}{x_1^2} \mathbb{E}\big[\big(X_1(-n_k) - x_1\big)^2 | \vec{X}(-n_{k+1}) = \vec{x}\big]$$

$$= \frac{16}{x_1^2} \Bigg(\sum_{n=n_k}^{n_{k+1}-1} \frac{1}{c_n} \Bigg) \big(F^{[n_{k+1}]}g\big)_1(\vec{x}). \tag{6.15}$$

Note that $\sum_{n=n_k}^{n_{k+1}-1} \frac{1}{c_n} \leq \Lambda$ uniformly in k. Since $g \in \mathcal{H}_0^r$, we have $g_1(\vec{x}) + g_2(\vec{x}) \leq K(1 + x_1)(1 + x_2)$ for some $K \in (0, \infty)$, and by Proposition A.1, $\{F^{[n]}g\}_{n \in \mathbb{N}_0}$ all share the same upper bound. Equation (6.14) then follows immediately. □

Remark. *Note that* (6.6)–(6.8) *with* $\vec{X}$ *in place of* $\vec{X}^h$ *are proved using only the assumptions that* $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$ *and,* $\{F^{[n]}g\}_{n \in \mathbb{N}_0}$ *have a uniform lower bound which is positive and uniformly bounded away from 0 on* $(a, \infty)^2$ *for each* $a > 0$. *Only in deriving* (6.7) *and* (6.8) *from their analogues for* $\vec{X}$, *did we use the assumptions that* $\inf_{n \in \mathbb{N}_0} c_n > 0$ *and,* $\{F^{[n]}g\}_{n \in \mathbb{N}_0}$ *have a uniform upper bound* $\phi = (\phi_1, \phi_2)$, *where* $\phi_1(x_1, x_2)$ *grows sub-quadratically in* x_1 *and* $\phi_2(x_1, x_2)$ *grows sub-quadratically in* x_2.

Using Lemma 6.1 and the fact that $1, x_1, x_2, x_1 x_2$ are still harmonic functions for the Markov chain $\{\vec{X}(-n)\}_{n \in \mathbb{N}_0}$, we deduce the analogue of Lemma 5.1 in our present context by the same arguments as in the original proof. To deduce Theorem 2.16 with varying c_n from the analogue of Lemma 5.1, we need to address the second technical point outlined at the beginning of this section.

Proposition 6.2 (Convergence to fixed points under $F^{[n]}$: the halfline case). *Let* $(c_n)_{n \in \mathbb{N}_0}$ *satisfy* $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$. *Let* $f(x) : [0, \infty) \to [0, \infty)$ *be positive and continuous on* $(0, \infty)$, *locally Lipschitz at* 0, $f(0) = 0$ *and* $\lim_{x \to \infty} x^{-1} f(x) = \lambda \in [0, \infty)$. *Then we have*

$$\lim_{n \to \infty} \sup_{x > 0} \left| \frac{(F^{[n]}f)(x) - \lambda x}{1 + x} \right| = 0, \tag{6.16}$$

where $F^{[n]}$ *are renormalization transformations acting on one-dimensional diffusion functions.*

Proof. Note that we do not require $\inf_{n \in \mathbb{N}_0} c_n > 0$ as in Lemma 6.1. The case $c_n \equiv c$ is covered by Theorem 5 of [3]. Here we give a proof along the same line of argument as we have been pursuing so far in this section for the proof of Theorem 2.16 with varying c_n, except that we do not need to appeal to the current proposition.

As in Section 2.3 of [3], we make use of the concave upper envelope f^+ and the convex lower envelope f^- of f. It is easy to see that f^+ and, f^- in the case $\lambda > 0$, satisfy the same constraints as specified for f in the proposition. Since, for any $c > 0$, F_c is convexity preserving and order preserving by Proposition 3 of [3], together with Jensen's inequality we have, for each $x \in [0, \infty)$, $(F^{[n]} f^+)(x) \downarrow f_\infty^+(x)$ and $(F^{[n]} f^-)(x) \uparrow f_\infty^-(x)$ for some f_∞^+ and f_∞^- as $n \to \infty$, and $(F^{[n]} f^-)(x) \le (F^{[n]} f)(x) \le (F^{[n]} f^+)(x)$ for all $n \in \mathbb{N}_0$. We claim that it suffices to show that $f_\infty^+(x) = f_\infty^-(x) = \lambda x$. Indeed,

$$\sup_{x > 0} \left| \frac{(F^{[n]} f)(x) - \lambda x}{1 + x} \right| = \sup_{y \in (0,1)} \left| (1 - y)(F^{[n]} f) \circ \phi_1^{-1}(y) - \lambda y \right|,$$

where $\phi_1(x) = \frac{x}{1+x}$. Since F_c preserves the slope at infinity, we have that

$$
\begin{aligned}
\psi_n^+(y) &= (1 - y)(F^{[n]} f^+) \circ \phi_1^{-1}(y), \\
\psi_n(y) &= (1 - y)(F^{[n]} f) \circ \phi_1^{-1}(y), \\
\psi_n^-(u) &= (1 - y)(F^{[n]} f^-) \circ \phi_1^{-1}(y),
\end{aligned}
\tag{6.17}
$$

are all continuous functions on $[0, 1]$. If $f_\infty^+(x) = f_\infty^-(x) = \lambda x$, then, on $[0, 1]$, $\psi_n^+(y)$ decreases monotonically to λy as $n \to \infty$, while $\psi_n^-(y)$ increases monotonically to λy as $n \to \infty$. Since the monotone convergence of a sequence of continuous functions to a continuous limit is necessarily uniform on compacts, the sup-norm convergence of $\psi_n(y)$ to λy on $[0, 1]$ follows since ψ_n is sandwiched between ψ_n^+ and ψ_n^-.

The proof that $f_\infty^+(x) = \lim_{n \to \infty}(F^{[n]} f^+)(x) = \lambda x$ and $f_\infty^-(x) = \lim_{n \to \infty}(F^{[n]} f^-)(x) = \lambda x$ now follows the same argument as that used for Theorem 2.16 with varying c_n. First consider the case $\{F^{[n]} f^+\}_{n \in \mathbb{N}_0}$ with $\lambda > 0$. In the proof of Theorem 2.16 with varying c_n, we replace $\vec{X}$ there by the $[0, \infty)$-valued Markov chain $(X(-n))_{n \in \mathbb{N}_0}$ with transition kernels

$$\mathbb{P}(X(-n) \in \cdot | X(-n-1) = x) = \Gamma_x^{c_n, F^{[n]} f^+}(\cdot);$$

$\vec{X}^h$ is replaced by X^h, which is the h-transform of X by the harmonic function $h(x) = 1 + x$; $\phi(\vec{x})$ is replaced by $\phi_1(x) = \frac{x}{1+x}$; in Lemma 5.1, the relevant boundary points now consist of only $\{0\} \cup \{1\}$. Lastly, because of the one-dimensional setting, we only need to establish the analogue of (6.6). By the remark following the proof of Lemma 6.1, the only assumptions we need here are $\sum_{n \in \mathbb{N}_0} c_n^{-1} = \infty$ and, a uniform lower bound on $\{F^{[n]} f^+\}_{n \in \mathbb{N}_0}$ which is positive and bounded away from 0 on $[a, \infty)$ for each $a > 0$. Note that f^- provides such a lower bound. The case $\{F^{[n]} f^-\}_{n \in \mathbb{N}_0}$ with $\lambda > 0$ is identical. For the case $\lambda = 0$, we only need to consider $\{F^{[n]} f^+\}_{n \in \mathbb{N}_0}$. Everything remains the same, except that the uniform lower bound on $\{F^{[n]} f^+\}_{n \in \mathbb{N}_0}$ is now provided by f_∞^+. Indeed, as a limit of concave functions, f_∞^+ is also concave, hence either $f_\infty^+ \equiv 0$, in which case we are done, or f_∞^+ is positive and nondecreasing on $(0, \infty)$, which is sufficient for the proof of the analogue of (6.6) to go through. □

With Lemma 6.1 and Proposition 6.2, we can now proceed as in the proof of Theorem 2.16 for constant c_n and extend it to varying c_n. We leave the details to the reader. □

Appendix A. Moment equations and estimates

Proposition A.1 (Moment equations and estimates). *Let $g \in \mathcal{C}$, $\vec{\theta} \in [0, \infty)^2$, $c > 0$ and let $\Gamma_{\vec{\theta}}^{c,g}$ be any equilibrium distribution of (2.1) with generator (2.2). Let $\vec{X} = (X_1, X_2)$ be a random variable with distribution $\Gamma_{\vec{\theta}}^{c,g}$. Then:*

(i) *For any $f(\vec{x}) \in C_b^2([0, \infty)^2)$ that differs from a function with compact support by only a constant,*

$$\mathbb{E}_{\vec{\theta}}^{c,g}\big[(L_{\vec{\theta}}^{c,g} f)(\vec{X})\big] = \mathbb{E}_{\vec{\theta}}^{c,g}\left[c \sum_{i=1}^{2}(\theta_i - X_i)\frac{\partial}{\partial x_i} f(\vec{X}) + \sum_{i=1}^{2} g_i(\vec{X})\frac{\partial^2}{\partial x_i^2} f(\vec{X})\right] = 0. \tag{A.1}$$

(ii) *For all $g \in \mathcal{H}_a$ with $0 \le a < c$, all $\vec{\theta} \in [0, \infty)^2$ and $i = 1, 2$,*

$$\mathbb{E}_{\vec{\theta}}^{c,g}[X_i] = \theta_i, \tag{A.2}$$

$$\mathbb{E}_{\vec{\theta}}^{c,g}[X_1 X_2] = \theta_1 \theta_2, \tag{A.3}$$

$$\mathbb{E}_{\vec{\theta}}^{c,g}\big[X_i^2\big] = \theta_i^2 + \frac{1}{c}\mathbb{E}_{\vec{\theta}}^{c,g}\big[g_i(\vec{X})\big] = \theta_i^2 + \frac{1}{c}(F_c g)_i(\vec{\theta}), \tag{A.4}$$

where all expectations are finite.

(iii) *Let $g \in \mathcal{H}_a$ with $0 \le a < c$, and let K be any compact subset of $[0, \infty)^2$. Then*

$$\sup_{c' \ge c, \vec{\theta} \in K} \mathbb{E}_{\vec{\theta}}^{c',g}\big[(X_1 + X_2 + 2)^2 \log(X_1 + X_2 + 2)\big] < C_{c,K,g} \tag{A.5}$$

for some $C_{c,K,g} < \infty$ depending only on c, K and g. Consequently, g_1 and g_2 are uniformly integrable with respect to $\{\Gamma_{\vec{\theta}}^{c',g}\}_{c' \ge c, \vec{\theta} \in K}$.

Proof. (i) This part follows from the observation that, with our choice of f,

$$f(\vec{X}(t)) - f(\vec{X}(0)) - \int_0^t (L_{\vec{\theta}}^{c,g} f)(\vec{X}(s)) \, ds \tag{A.6}$$

is a martingale. Taking expectation and noting the stationarity of the distribution of $\vec{X}(t)$, we obtain (A.1).

(ii) We first prove that the expectations in (A.2)–(A.4) are all finite. Once this is settled, the equalities will follow easily.

Finiteness: Let $h \in C_b^2([0, \infty))$ be such that $h(r) = r$ for $r \in [0, 1]$, h is constant on $[3, \infty)$, $h' \in [0, 1]$ and $h'' \in [-1, 0]$. Let $h_n(r) = nh(\frac{r}{n})$. Then $h_n' \in [0, 1]$, $h_n'' \in [-\frac{1}{n}, 0]$, and $h_n(r) \uparrow r$, $h_n'(r) \uparrow 1$, $h_n''(r) \to 0$ as $n \to \infty$.

(A.2): We apply (A.1) for $f(x_1, x_2) = h_n(\rho_1 x_1 + \rho_2 x_2)$ with fixed $\rho_1, \rho_2 > 0$. Since (in the formulas below we suppress the argument)

$$\begin{aligned}
\partial_{x_i} h_n(\rho_1 x_1 + \rho_2 x_2) &= \rho_i h_n', \\
\partial_{x_i}^2 h_n(\rho_1 x_1 + \rho_2 x_2) &= \rho_i^2 h_n'',
\end{aligned} \tag{A.7}$$

and $h_n(\rho_1 x_1 + \rho_2 x_2)$ differs from a function with compact support by a constant, by substituting the partials into (A.1), we get

$$\mathbb{E}_{\vec{\theta}}^{c,g}\left[c \sum_{i=1}^{2} \rho_i(\theta_i - X_i)h_n' + \sum_{i=1}^{2} \rho_i^2 g_i(\vec{X})h_n''\right] = 0, \tag{A.8}$$

which can be rewritten as

$$\begin{aligned}
c\mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1 X_1 + \rho_2 X_2)h_n'\big] &= c\mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1\theta_1 + \rho_2\theta_2)h_n'\big] + \mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1^2 g_1 + \rho_2^2 g_2)h_n''\big] \\
&\le c\mathbb{E}_{\vec{\theta}}^{c,g}[\rho_1\theta_1 + \rho_2\theta_2]
\end{aligned} \tag{A.9}$$

since $h_n'' \le 0$ and $g_1, g_2 \ge 0$. By monotone convergence as $n \to \infty$, we get

$$\rho_1 \mathbb{E}_{\vec{\theta}}^{c,g}[X_1] + \rho_2 \mathbb{E}_{\vec{\theta}}^{c,g}[X_2] \le \rho_1\theta_1 + \rho_2\theta_2. \tag{A.10}$$

Since ρ_1, ρ_2 are arbitrary, we obtain $\mathbb{E}_{\vec{\theta}}^{c,g}[X_i] \le \theta_i < \infty$, $i = 1, 2$.

(A.3): Here we apply (A.1) for $f(x_1, x_2) = h_n((1 + x_1)(1 + x_2))$. The calculations are similar to that for (A.2), which we skip.

(A.4): Here we apply (A.1) for $f(x_1, x_2) = h_n(\rho_1 x_1^2 + \rho_2 x_2^2)$ with fixed $\rho_1, \rho_2 > 0$. Since

$$\partial_{x_i} h_n\big(\rho_1 x_1^2 + \rho_2 x_2^2\big) = 2\rho_i x_i h_n',$$
$$\partial_{x_i}^2 h_n\big(\rho_1 x_1^2 + \rho_2 x_2^2\big) = 2\rho_i h_n' + 4\rho_i^2 x_i^2 h_n'', \tag{A.11}$$

by substituting the partials into (A.1), we get

$$\mathbb{E}_{\vec{\theta}}^{c,g}\big[2c\rho_1 X_1(\theta_1 - X_1)h_n' + 2c\rho_2 X_2(\theta_2 - X_2)h_n' + 2(\rho_1 g_1 + \rho_2 g_2)h_n' + 4\big(\rho_1^2 X_1^2 g_1 + \rho_2^2 X_2^2 g_2\big)h_n''\big] = 0.$$

Rearranging terms, we obtain

$$2c\mathbb{E}_{\vec{\theta}}^{c,g}\big[\big(\rho_1 X_1^2 + \rho_2 X_2^2\big)h_n'\big] = 2c\mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1\theta_1 X_1 + \rho_2\theta_2 X_2)h_n'\big] + 2\mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1 g_1 + \rho_2 g_2)h_n'\big]$$
$$+ \mathbb{E}_{\vec{\theta}}^{c,g}\big[4\big(\rho_1^2 X_1^2 g_1 + \rho_2^2 X_2^2 g_2\big)h_n''\big] \tag{A.12}$$
$$\le 2c\big(\rho_1\theta_1^2 + \rho_2\theta_2^2\big) + 2\mathbb{E}_{\vec{\theta}}^{c,g}\big[(\rho_1 g_1 + \rho_2 g_2)h_n'\big].$$

Since $g \in \mathcal{H}_a$ with $0 \le a < c$, we have $g_1(\vec{x}) + g_2(\vec{x}) \le C(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2)$. Substituting this bound into (A.12) and setting $\rho_1 = \rho_2 = 1$, using the fact that $\mathbb{E}_{\vec{\theta}}^{c,g}[X_i] \le \theta_i$ and $\mathbb{E}_{\vec{\theta}}^{c,g}[X_1 X_2] < \infty$, and rearranging terms, we get

$$2(c - a)\mathbb{E}_{\vec{\theta}}^{c,g}\big[\big(X_1^2 + X_2^2\big)h_n'\big] < C' < \infty. \tag{A.13}$$

By monotone convergence as $n \to \infty$, we obtain $\mathbb{E}_{\vec{\theta}}^{c,g}[X_i^2] < \infty$. This also implies $\mathbb{E}_{\vec{\theta}}^{c,g}[g_i] < \infty$.

Equality: Having thus proved that the expectations in (A.2)–(A.4) are finite, we are now ready to prove that equality holds. To that end, return to (A.9). Since $\mathbb{E}_{\vec{\theta}}^{c,g}[\rho_1^2 g_1 + \rho_2^2 g_2] < \infty$, $h_n'' \in [-\frac{1}{n}, 0]$ and $h_n'' \to 0$ as $n \to \infty$, (A.2) follows by applying the dominated convergence theorem. By the same argument, (A.4) follows by applying the dominated convergence theorem to (A.12), provided that

$$\big(\rho_1^2 x_1^2 g_1 + \rho_2^2 x_2^2 g_2\big)\big|h_n''\big(\rho_1 x_1^2 + \rho_2 x_2^2\big)\big| \le C\big(x_1^2 + x_2^2\big) \tag{A.14}$$

for some $C < \infty$ independent of n. To see the latter, note that $h_n''(r) = \frac{1}{n}h''(\frac{r}{n}) = \frac{r}{n}h''(\frac{r}{n})\frac{1}{r} \le \frac{3}{r}$, since $h'' \in [-1, 0]$ and $h''(\frac{r}{n}) \ne 0$ only when $\frac{r}{n} \le 3$. The bound in (A.14) then follows readily.

To verify (A.3), we apply (A.1) for $f(x_1, x_2) = h_n((x_1 + x_2)^2)$ instead of $h_n((1 + x_1)(1 + x_2))$. This gives

$$\mathbb{E}_{\vec{\theta}}^{c,g}\big[2c(\theta_1 + \theta_2 - X_1 - X_2)(X_1 + X_2)h_n' + 2(g_1 + g_2)h_n' + 4(X_1 + X_2)^2(g_1 + g_2)h_n''\big] = 0. \tag{A.15}$$

Since $(x_1 + x_2)^2 h_n''((x_1 + x_2)^2) \le 3$ and $\mathbb{E}_{\vec{\theta}}^{c,g}[g_1 + g_2] < \infty$, we can apply the dominated convergence theorem in (A.15) as $n \to \infty$. Then, together with (A.2) and (A.4), we obtain (A.3).

(iii) This part follows from similar computations as in part (ii). Let $c' \ge c$ be arbitrary, and abbreviate $\bar{X}_i = 1 + X_i$, $\bar{\theta}_i = 1 + \theta_i$, $\bar{x}_i = 1 + x_i$ for $i = 1, 2$. We first show that

$$\mathbb{E}_{\vec{\theta}}^{c',g}\big[\bar{X}_1 \bar{X}_2 \log(\bar{X}_1 + \bar{X}_2)\big] < \infty \tag{A.16}$$

by applying (A.1) to $h_n(\bar{x}_1 \bar{x}_2 \log(\bar{x}_1 + \bar{x}_2))$. Then we apply (A.1) to $h_n((\bar{x}_1 + \bar{x}_2)^2 \log(\bar{x}_1 + \bar{x}_2))$ to prove (A.5).

(A.16): Let $f(x_1, x_2) = h_n(\bar{x}_1 \bar{x}_2 \log(\bar{x}_1 + \bar{x}_2))$, which differs from a function with compact support by a constant. Since

$$\partial_{x_1} h_n\big(\bar{x}_1 \bar{x}_2 \log(\bar{x}_1 + \bar{x}_2)\big) = \left(\bar{x}_2 \log(\bar{x}_1 + \bar{x}_2) + \frac{\bar{x}_1 \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\right)h_n',$$

$$\partial_{x_1}^2 h_n\big(\bar{x}_1 \bar{x}_2 \log(\bar{x}_1 + \bar{x}_2)\big) = \left(\frac{\bar{x}_2}{\bar{x}_1 + \bar{x}_2} + \frac{\bar{x}_2^2}{(\bar{x}_1 + \bar{x}_2)^2}\right)h_n' + \left(\bar{x}_2 \log(\bar{x}_1 + \bar{x}_2) + \frac{\bar{x}_1 \bar{x}_2}{\bar{x}_1 + \bar{x}_2}\right)^2 h_n'',$$

and since the same holds if we interchange the indices 1 and 2, by substituting the partials into (A.1) and noting that $\partial_{x_1}^2 h_n \le 2h_n'$, $\partial_{x_2}^2 h_n \le 2h_n'$, we get

$$\mathbb{E}_{\vec{\theta}}^{c',g}\left[c'(\bar{\theta}_1 - \bar{X}_1)\left(\bar{X}_2 \log(\bar{X}_1 + \bar{X}_2) + \frac{\bar{X}_1\bar{X}_2}{\bar{X}_1 + \bar{X}_2} \right) h_n' \right.$$
$$\left. + c'(\bar{\theta}_2 - \bar{X}_2)\left(\bar{X}_1 \log(\bar{X}_1 + \bar{X}_2) + \frac{\bar{X}_1\bar{X}_2}{\bar{X}_1 + \bar{X}_2} \right) h_n' + 2(g_1 + g_2)h_n' \right] \ge 0. \tag{A.17}$$

Rearranging terms and noting that $\frac{\bar{X}_1\bar{X}_2}{\bar{X}_1+\bar{X}_2} < \bar{X}_1 \wedge \bar{X}_2$, we find that

$$2c'\mathbb{E}_{\vec{\theta}}^{c',g}\left[\bar{X}_1\bar{X}_2 \log(\bar{X}_1 + \bar{X}_2)h_n' \right] \le \mathbb{E}_{\vec{\theta}}^{c',g}\left[c'(\bar{\theta}_1\bar{X}_2 + \bar{X}_1\bar{\theta}_2)\left(1 + \log(\bar{X}_1 + \bar{X}_2)\right) + 2(g_1 + g_2) \right]. \tag{A.18}$$

By assumption, $g_1(\vec{x}) + g_2(\vec{x}) \le C(1 + x_1)(1 + x_2) + a(x_1^2 + x_2^2)$. Substituting this bound into (A.18), applying monotone convergence as $n \to \infty$, and noting that (A.4) implies that $\mathbb{E}_{\vec{\theta}}^{c',g}[X_1^2 + X_2^2] \le \phi(\theta_1, \theta_2)$ for some quadratic polynomial ϕ depending only on c and g, we easily verify that

$$\mathbb{E}_{\vec{\theta}}^{c',g}\left[\bar{X}_1\bar{X}_2 \log(\bar{X}_1 + \bar{X}_2) \right] \le \tilde{\phi}(\theta_1, \theta_2) \tag{A.19}$$

for some cubic polynomial $\tilde{\phi}$ depending only on c and g.

By applying (A.1) to $h_n((\bar{x}_1 + \bar{x}_2)^2 \log(\bar{x}_1 + \bar{x}_2))$ and using (A.19), it can be shown that

$$\mathbb{E}_{\vec{\theta}}^{c',g}\left[(\bar{X}_1 + \bar{X}_2)^2 \log(\bar{X}_1 + \bar{X}_2) \right] \le \hat{\phi}(\theta_1, \theta_2) \tag{A.20}$$

for some cubic polynomial $\hat{\phi}$ depending only on c and g. The uniform bound in (A.5) then follows. The calculations, which we omit, are similar as before.

Since $g_1(\vec{x}) + g_2(\vec{x}) \le C(\bar{x}_1^2 + \bar{x}_2^2)$ for some $C < \infty$, which by (A.20) is uniformly integrable with respect to $\{\Gamma_{\vec{\theta}}^{c',g}\}_{c' \ge c, \vec{\theta} \in K}$ for any compact $K \subset [0, \infty)^2$, it follows that g_1 and g_2 are also uniformly integrable. $\square$

Remark. *By similar computations, it can be shown that* (A.5) *is still valid when the logarithm in the left-hand side of the inequality is raised to an arbitrary power.*

Appendix B. Properties of uniformly elliptic diffusions

In this Appendix, we list some facts about uniformly elliptic diffusions that are needed in the proof of Theorem 2.3. We thank S. R. S. Varadhan for pointing out some of the relevant results and references on uniformly elliptic diffusions.

Theorem B.1 (Uniformly elliptic diffusions in $\mathbb{R}^d$). *Let $b: \mathbb{R}^d \to \mathbb{R}^d$ be a bounded measurable map, and let $a: \mathbb{R}^d \to S_d$ be a continuous map, where S_d is the space of symmetric nonnegative definite $d \times d$ real matrices. Assume further that $a(\cdot)$ is uniformly elliptic, i.e., there exists $0 < \Lambda < \infty$ such that for all $\vec{x}, \vec{\theta} \in \mathbb{R}^d$, $\vec{\theta} \ne 0$,*

$$\Lambda^{-1} \le \frac{\langle \vec{\theta}, a(\vec{x})\vec{\theta} \rangle}{\langle \vec{\theta}, \vec{\theta} \rangle} \le \Lambda.$$

Then, for each $\vec{x} \in \mathbb{R}^d$, the martingale problem with generator

$$Lf = \sum_{i,j=1}^{d} a_{ij}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j} f(\vec{x}) + \sum_{i=1}^{d} b_i(\vec{x}) \frac{\partial}{\partial x_i} f(\vec{x}), \quad f \in C_c^2(\mathbb{R}^d), \tag{B.1}$$

has a unique solution $\mathbb{P}^{\vec{x}}$ in the space of probability measures on $\Omega = C([0, \infty), \mathbb{R}^d)$ with $\mathbb{P}^{\vec{x}}(\omega \in \Omega : \omega(0) = \vec{x}) = 1$. The family of solutions $\{\mathbb{P}^{\vec{x}}\}_{\vec{x} \in \mathbb{R}^d}$ defines a strong Feller and strong Markov process that admits a transition probability

density $p_t(\vec{x}, \vec{y})$ with respect to Lebesgue measure for each $t > 0$ and $\vec{x} \in \mathbb{R}^d$. Furthermore, for each $t > 0$ and $\vec{x}^* \in \mathbb{R}^d$,

$$\lim_{\vec{x} \to \vec{x}^*} \left\| p_t(\vec{x}, \cdot) - p_t(\vec{x}^*, \cdot) \right\|_1 = \lim_{\vec{x} \to \vec{x}^*} \int_{\mathbb{R}^d} \left| p_t(\vec{x}, \vec{y}) - p_t(\vec{x}^*, \vec{y}) \right| d\vec{y} = 0.$$

Proof. All facts follow from results in [29]. For the well-posedness of the martingale problem, see Theorem 7.2.1 therein. For the strong Markov property, see Theorem 6.2.2. For the strong Feller property, see Theorem 7.2.4. For the existence of the transition density, see Theorem 9.1.9 and Lemma 9.2.2. Lastly, for the L_1-continuity of the transition density, see Theorem 11.4.3. $\qquad\square$

Theorem B.2 (Diffusions restricted to bounded domains). *Let a and b satisfy the conditions in Theorem* B.1, *and let $\{\mathbb{P}^{\vec{x}}\}_{\vec{x} \in \mathbb{R}^d}$ denote the family of solutions to the martingale problem with coefficients (a, b) in* (B.1). *If $\bar{a} : \mathbb{R}^d \to S_d$ and $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$ are locally bounded measurable maps with $\bar{a} = a$ and $\bar{b} = b$ on a bounded open set D, then for any $\vec{x} \in D$ and any solution $\bar{\mathbb{P}}^{\vec{x}}$ to the martingale problem with coefficients $(\bar{a}, \bar{b})$, $\bar{\mathbb{P}}^{\vec{x}} = \mathbb{P}^{\vec{x}}$ on $\mathcal{F}_{\tau_D}$, the sigma-field on Ω generated by the family of projection maps $\{\pi_s : \Omega \to \mathbb{R}^d | \pi_s(\omega) = \omega(s \wedge \tau_D)\}_{s \geq 0}$, where $\tau_D(\omega) = \inf\{t \geq 0: \omega(t) \notin D\}$.*

Proof. See Theorem 10.1.1 in [29]. $\qquad\square$

Corollary B.3 (Transition density for diffusions restricted to bounded domains). *Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a locally bounded measurable map, and let $a : \mathbb{R}^d \to S_d$ be continuous such that the martingale problem with coefficients a and b in* (B.1) *is well-posed. Assume further that a is nondegenerate on $\overline{D}$ for a simply connected bounded open set $D \subset \mathbb{R}^d$ with smooth boundary. For any $\vec{x} \in D$, if $\mathbb{P}^{\vec{x}}$ is the solution of the martingale problem starting from $\vec{x}$, then, for each $t > 0$, the measure $\mu_t^D(\vec{x}, \cdot)$ on Borel-measurable sets defined by $\mu_t^D(\vec{x}, \cdot) = \mathbb{P}^{\vec{x}}(\omega: t < \tau_D(\omega), \omega(t) \in \cdot)$ admits a density $p_t^D(\vec{x}, \vec{y})$ with respect to Lebesgue measure. Furthermore, for each $\vec{x}^* \in D$, there exist $\varepsilon, \delta > 0$ sufficiently small such that, for all $\vec{x}, \vec{x}' \in B_\varepsilon(\vec{x}^*)$, the ball of radius ε centered at $\vec{x}^*$, the overlap between $\mu_\delta^D(\vec{x}, \cdot)$ and $\mu_\delta^D(\vec{x}', \cdot)$ satisfies*

$$\frac{\mu_\delta^D(\vec{x}, D) + \mu_\delta^D(\vec{x}', D) - \| p_\delta^D(\vec{x}, \cdot) - p_\delta^D(\vec{x}', \cdot) \|_1}{2} \geq \frac{1}{2}. \tag{B.2}$$

Proof. By our assumptions on a, b and D, we can find coefficients $(\bar{a}, \bar{b})$ on $\mathbb{R}^d$ such that $(\bar{a}, \bar{b}) = (a, b)$ on D, $(\bar{a}, \bar{b})$ are bounded, a is continuous and uniformly elliptic on $\mathbb{R}^d$. For instance, we can define $\bar{b} = b$ on D and $\bar{b} \equiv 0$ on $\mathbb{R}^d \backslash D$, define $\bar{a} = a$ on $\overline{D}$ and $\bar{a} \equiv I$ on $\mathbb{R}^d \backslash B$ where B is a large open ball containing $\overline{D}$, and on $B \backslash \overline{D}$ define $\bar{a}$ to be the harmonic interpolation between its values on ∂B and ∂D. By Theorem B.1, the martingale problem with coefficients $(\bar{a}, \bar{b})$ has a unique family of solutions $\{\bar{P}^{\vec{x}}\}_{\vec{x} \in \mathbb{R}^d}$, which is strong Markov and admits a transition density $\bar{p}_t(\vec{x}, \vec{y})$ for all $t > 0$ and $\vec{x} \in \mathbb{R}^d$. By Theorem B.2, for $\vec{x} \in D$, $\bar{\mathbb{P}}^{\vec{x}} = \mathbb{P}^{\vec{x}}$ on $\mathcal{F}_{\tau_D}$. In particular, $\mu_t^D(\vec{x}, \cdot) = \bar{\mu}_t^D(\vec{x}, \cdot) = \bar{\mathbb{P}}^{\vec{x}}\{\omega: t < \tau_D(\omega), \omega(t) \in \cdot\}$. Since $\bar{\mu}_t^D(\vec{x}, \cdot)$ is absolutely continuous with respect to $\bar{\mathbb{P}}^{\vec{x}}(\omega: \omega(t) \in \cdot)$ with density $\bar{p}_t(\vec{x}, \vec{y})$, $\mu_t^D(\vec{x}, \cdot) = \bar{\mu}_t^D(\vec{x}, \cdot)$ also admits a density $p_t^D(\vec{x}, \vec{y})$ with respect to Lebesgue measure for all $\vec{x} \in D$ and $t > 0$.

It is not difficult to see that the left-hand side of (B.2) is the mass of the maximal positive measure that is dominated by both $\mu_\delta^D(\vec{x}, \cdot)$ and $\mu_\delta^D(\vec{x}', \cdot)$. To verify (B.2), fix $\vec{x}^* \in D$ and choose $\varepsilon' > 0$ such that $B_{2\varepsilon'}(\vec{x}^*) \subset D$. Then we can choose $\delta > 0$ sufficiently small such that, for all $\vec{x} \in B_{\varepsilon'}(\vec{x}^*)$, $\mathbb{P}^{\vec{x}}(\tau_D \leq \delta) \leq \frac{1}{5}$. To verify this claim, note that, given $\vec{z} \in B_{\varepsilon'}(\vec{x}^*)$, if we define $f(\vec{x}) = \|\vec{x} - \vec{z}\|^2 = \sum_{i=1}^d (x_i - z_i)^2$, then

$$f\big(\vec{X}(t \wedge \tau_D)\big) - f\big(\vec{X}(0)\big) - \int_0^{t \wedge \tau_D} Lf\big(\vec{X}(s)\big) ds$$

is a martingale, where $(\vec{X}(s))_{s \geq 0}$ has law $\mathbb{P}^{\vec{z}}$. In particular,

$$\begin{aligned}
\big(\varepsilon'\big)^2 \mathbb{P}^{\vec{z}}(\tau_D \leq \delta) &\leq \mathbb{E}\big[\big\| \vec{X}(\delta \wedge \tau_D) - \vec{z} \big\|^2\big] \\
&= \mathbb{E}\left[\int_0^{\delta \wedge \tau_D} 2 \sum_{i=1}^d \big(b_i\big(\vec{X}(s)\big)\big(X_i(s) - z_i\big) + a_{ii}\big(\vec{X}(s)\big)\big) ds \right] \leq \delta C_{D,a,b}, \tag{B.3}
\end{aligned}$$

where $C_{D,a,b}$ depends only on D and (a,b) on D. Therefore $\mathbb{P}^{\vec{z}}(\tau_D \leq \delta) \leq \delta C_{D,a,b}(\varepsilon')^{-2}$ uniformly for all $\vec{z} \in B_{\varepsilon'}(\vec{x}^*)$. Choosing δ sufficiently small, we then verify the claim.

Applying Theorem B.1 to $\{\bar{P}^{\vec{x}}\}_{\vec{x} \in \mathbb{R}^d}$, we can choose $\varepsilon \in (0, \varepsilon')$ small such that, for all $\vec{x} \in B_{\varepsilon}(\vec{x}^*)$, $\|\bar{p}_\delta(\vec{x}, \cdot) - \bar{p}_\delta(\vec{x}^*, \cdot)\|_1 \leq \frac{1}{10}$, and hence, for all $\vec{x}, \vec{x}' \in B_{\varepsilon}(\vec{x}^*)$, $\|\bar{p}_\delta(\vec{x}, \cdot) - \bar{p}_\delta(\vec{x}', \cdot)\|_1 \leq \frac{1}{5}$. Since for $\vec{z} \in B_{\varepsilon}(\vec{x}^*)$, $\|\bar{p}_\delta(\vec{z}, \cdot) - \bar{p}_\delta^D(\vec{z}, \cdot)\|_1 = \mathbb{P}^{\vec{z}}(\tau_D \leq \delta) \leq \frac{1}{5}$, we have $\|\bar{p}_\delta^D(\vec{x}, \cdot) - \bar{p}_\delta^D(\vec{x}', \cdot)\|_1 \leq \frac{3}{5}$ for all $\vec{x}, \vec{x}' \in B_{\varepsilon}(\vec{x}^*)$. Finally, note that, for $\vec{x}, \vec{x}' \in B_{\varepsilon}(\vec{x}^*)$, $\mu_\delta^D(\vec{x}, D) = 1 - \bar{P}^{\vec{x}}(\tau_D \leq \delta) \geq 1 - \frac{1}{5}$ and the same holds for $\mu_\delta^D(\vec{x}', D)$, hence, substitution of all the estimates into the left-hand side of (B.2) yields the desired result. $\qquad\square$

Remark. *Note that the constant on the right-hand side of* (B.2) *can be made arbitrarily close to* 1 *by choosing* ε, δ *sufficiently small.*

Theorem B.4 (Support theorem for uniformly elliptic diffusions). *Let* a, b, D *and* $\{P^{\vec{x}}\}_{\vec{x} \in D}$ *be as in Corollary* B.3. *For any* $\vec{x} \in D$, $\varepsilon > 0$, *and any continuous function* $\psi : [0, t] \to D$ *with* $\psi(0) = \vec{x}$,

$$\mathbb{P}^{\vec{x}}\left(\omega: \sup_{0 \leq s \leq t} |\omega(s) - \psi(s)| \leq \varepsilon\right) > 0.$$

Proof. The support theorem is a classic result of Stroock and Varadhan. The statement above follows Theorem 2.5 in Chapter V of [4] and Theorem B.2. $\qquad\square$

Theorem B.5 (Occupation time measure for uniformly elliptic diffusions). *Let* a, b, D *and* $\{\mathbb{P}^{\vec{x}}\}_{\vec{x} \in D}$ *be as in Corollary* B.3. *If* $A \subset D$ *has positive Lesbegue measure, then, for all* $\vec{x} \in D$, $\mathbb{E}^{\vec{x}}[\int_0^{\tau_D} 1_{\omega(s) \in A} \, ds] > 0$, *where* $\mathbb{E}^{\vec{x}}$ *denotes expectation with respect to* $\mathbb{P}^{\vec{x}}$, *and* $\tau_D = \inf\{t \geq 0: \omega(t) \notin D\}$.

Proof. The statement above follows from Theorem 8.5 in Chapter V of [4] (which goes back to Krylov) in combination with the support theorem, Theorem B.4, and the Girsanov transformation (see Theorem 7.2.2 in [29]). $\qquad\square$

Acknowledgments

References

[1] S. R. Athreya, M. T. Barlow, R. F. Bass and E. A. Perkins. Degenerate stochastic differential equations and super-Markov chains. *Probab. Theory Related Fields* **123** (2002) 484–520. MR1921011

[2] J.-B. Baillon, P. Clément, A. Greven and F. den Hollander. On the attracting orbit of a non-linear transformation arising from renormalization of hierarchically interacting diffusions, Part I: The compact case. *Canad. J. Math.* **47** (1995) 3–27. MR1319687

[3] J.-B. Baillon, P. Clément, A. Greven and F. den Hollander. On the attracting orbit of a non-linear transformation arising from renormalization of hierarchically interacting diffusions, Part II: The non-compact case. *J. Funct. Anal.* **146** (1997) 236–298. MR1446381

[4] R. F. Bass. *Diffusions and Elliptic Operators*. Springer, New York, 1998. MR1483890

[5] R. F. Bass and E. A. Perkins. Countable systems of degenerate stochastic differential equations with applications to super-Markov chains. *Electron. J. Probab.* **9** (2004) 634–673. MR2110015

[6] R. F. Bass and E. A. Perkins. Degenerate stochastic differential equations arising from catalytic branching networks. Preprint.

[7] J. T. Cox, D. A. Dawson and A. Greven. *Mutually Catalytic Super Branching Random Walks: Large Finite Systems and Renormalization Analysis*. Amer. Math. Soc., Providence, RI, 2004. MR2074427

[8] J. T. Cox and A. Greven. Ergodic theorems for infinite systems of locally interacting diffusions. *Ann. Probab.* **22** (1994) 833–853. MR1288134

[9] D. A. Dawson, L. G. Gorostiza and A. Wakolbinger. Degrees of transience and recurrence and hierarchical random walks. *Potential Anal.* **22** (2005) 305–350. MR2135263

[10] D. A. Dawson and A. Greven. Multiple scale analysis of interacting diffusions. *Probab. Theory Related Fields* **95** (1993) 467–508. MR1217447

[11] D. A. Dawson and A. Greven. Hierarchical models of interacting diffusions: Multiple time scales, phase transitions and cluster formation. *Probab. Theory Related Fields* **96** (1993) 435–473. MR1234619

[12] D. A. Dawson and A. Greven. Multiple space–time analysis for interacting branching models. *Electron. J. Probab.* **1** (1996) 1–84. MR1423467

[13] D. A. Dawson, A. Greven and J. Vaillancourt. Equilibria and quasi-equilibria for infinite collections of interacting Fleming–Viot processes. *Trans. Amer. Math. Soc.* **347** (1995) 2277–2360. MR1297523

[14] D. A. Dawson, A. Greven and I. Zähle. Continuum limits of multitype population models on the hierarchical group. In preparation.

[15] D. A. Dawson and P. March. Resolvent estimates for Fleming–Viot operators and uniqueness of solutions to related martingale problems. *J. Funct. Anal.* **132** (1995) 417–472. MR1347357

[16] D. A. Dawson and E. A. Perkins. Long-time behavior and coexistence in a mutually catalytic branching model. *Ann. Probab.* **26** (1998) 1088–1138. MR1634416

[17] D. A. Dawson and E. A. Perkins. On the uniqueness problem for catalytic branching networks and other singular diffusions. *Illinois J. Math.* **50** (2006) 323–383. MR2247832

[18] R. Durrett. *Probability: Theory and Examples*, 2nd edition. Duxbury Press, Belmont, CA, 1996. MR1609153

[19] S. N. Ethier and T. G. Kurtz. *Markov Processes – Characterization and Convergence*. Wiley, New York, 1986. MR0838085

[20] K. Fleischmann and A. Greven. Diffusive clustering in an infinite system of hierarchically interacting Fisher–Wright diffusions. *Probab. Theory Related Fields* **98** (1994) 517–566. MR1271108

[21] K. Fleischmann and J. M. Swart. Renormalization analysis of catalytic Wright–Fisher diffusions. *Electron. J. Probab.* **11** (2006) 585–654. MR2242657

[22] A. Greven. Renormalization and universality for multitype population models. In *Interacting Stochastic Systems* 209–246. J.-D. Deuschel and A. Greven, Eds. Springer, Berlin, 2005. MR2118576

[23] F. den Hollander. Renormalization of interacting diffusions. In *Complex Stochastic Systems* 219–233. O. E. Barndorff–Nielsen, D. R. Cox and C. Klüppelberg, Eds. Chapman & Hall, Boca Raton, 2001. MR1893414

[24] F. den Hollander and J. M. Swart. Renormalization of hierarchically interacting isotropic diffusions. *J. Stat. Phys.* **93** (1998) 243–291. MR1656371

[25] S. Kliem. Degenerate stochastic differential equations for catalytic branching networks. Preprint. Available at arXiv:0802.0035v1.

[26] R. G. Pinsky. *Positive Harmonic Functions and Diffusion*. Cambridge Univ. Press, 1995. MR1326606

[27] S. Sawyer and J. Felsenstein. Isolation by distance in a hierarchically clustered population. *J. Appl. Probab.* **20** (1983) 1–10. MR0688075

[28] T. Shiga and A. Shimizu. Infinite-dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.* **20** (1980) 395–416. MR0591802

[29] D. W. Stroock and S. R. S. Varadhan. *Multidimensional Diffusion Processes*. Springer, New York, 1979. MR0532498

[30] J. M. Swart. Clustering of linearly interacting diffusions and universality of their long-time distribution. *Probab. Theory Related Fields* **118** (2000) 574–594. MR1808376

[31] J. M. Swart. Uniqueness for isotropic diffusions with a linear drift. *Probab. Theory Related Fields* **128** (2004) 517–524. MR2045951

[32] S. R. S. Varadhan. *Probability Theory*. Amer. Math. Soc., Providence, RI, 2001. MR1852999