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INVARIANT CONVEX SETS IN POLAR REPRESENTATIONS

LEONARDO BILIOTTI, ALESSANDRO GHIGI, AND PETER HEINZNER

ABSTRACT. We study a compact invariant convex set E in a polar representation of a compact Lie group. Polar rapresentations are given by the adjoint action of K on $\mathfrak{p}$, where K is a maximal compact subgroup of a real semisimple Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra, then $P = E \cap \mathfrak{a}$ is a convex set in $\mathfrak{a}$. We prove that up to conjugacy the face structure of E is completely determined by that of P and that a face of E is exposed if and only if the corresponding face of P is exposed. We apply these results to the convex hull of the image of a restricted momentum map.

The boundary of a compact convex set is the union of its faces. Among the faces, the simplest ones are the exposed ones. They are given by the intersection of the convex set with a supporting hyperplane. In [3, 4] we studied the convex hull $\widehat{\mathcal{O}}$ of a K-orbit $\mathcal{O}$ in $\mathfrak{p}$, where $\mathfrak{p}$ is given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a reductive Lie algebra $\mathfrak{g}$ and K acts on $\mathfrak{p}$ by the adjoint representation. In this paper we use the results of [4] and show that a substantial part of them holds for any K–invariant compact convex set E of $\mathfrak{p}$. More precisely we study the faces of E. We show in Proposition 1.2 that for a face F of E there exists a subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that F is a subset of $\mathfrak{p}^{\mathfrak{s}} = \{x \in \mathfrak{p} : [x, \mathfrak{s}] = 0\}$ and F is invariant with respect to the action of $K^{\mathfrak{s}} = \{h \in K : \mathrm{Ad}(h)(\mathfrak{s}) = \mathfrak{s}\}$, where Ad denotes the adjoint representation.

If we fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, then the set $P = E \cap \mathfrak{a}$ is convex and invariant with respect to the action of the normalizer $\mathcal{N}_K(\mathfrak{a}) = \{h \in K : \mathrm{Ad}(h)(\mathfrak{a}) = \mathfrak{a}\}$ of $\mathfrak{a}$ in K. The $\mathcal{N}_K(\mathfrak{a})$–action on P induces an action on the set of faces of P. Similarly K acts on the set of faces of E. Denote these sets by $\mathscr{F}(P)$ respectively by $\mathscr{F}(E)$. If σ is a face of P, let $\sigma^\perp$ denote the orthogonal complement in $\mathfrak{a}$ of the affine hull of σ (see Section 1). Our main result is

Theorem 0.1. *The map $\mathscr{F}(P) \to \mathscr{F}(E)$, $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ is well–defined and induces a bijection between $\mathscr{F}(P)/\mathcal{N}_K(\mathfrak{a})$ and $\mathscr{F}(E)/K$.*

2000 *Mathematics Subject Classification.* 22E46; 53D20 .

The first two authors were partially supported by FIRB 2012 "Geometria differenziale e teoria geometrica delle funzioni", by a grant of the Max-Planck Institut für Mathematik, Bonn and by GNSAGA of INdAM. The second author was also supported by PRIN 2009 MIUR "Moduli, strutture geometriche e loro applicazioni". The third author was partially supported by DFG-priority program SPP 1388 (Darstellungstheorie).

An application of Theorem 0.1 is the following result.

Theorem 0.2. *The faces of E are exposed if and only if the faces of P are exposed.*

Interesting K–invariant compact subsets of $\mathfrak{p}$ often arise as images of restricted momentum or gradient mappings. More precisely, let U be a compact connected Lie group which acts by biholomorphism and in a Hamiltonian fashion on a compact Kähler manifold Z with momentum map $\mu : Z \longrightarrow \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a connected Lie subgroup of $U^{\mathbb{C}}$ which is *compatible* with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that G is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. Let $X \subset Z$ be a G-invariant compact subset of Z. We have the restricted momentum map or the gradient map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ in the sense of [13] (see also Section 3) and we denote by $E = \widehat{\mu_{\mathfrak{p}}(X)}$ the convex hull of the K-invariant set $\mu_{\mathfrak{p}}(X)$. If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$ and π is the orthogonal projection onto $\mathfrak{a}$, then $\mu_{\mathfrak{a}} = \pi \circ \mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{a}$ is the gradient map with respect to $A = \exp(\mathfrak{a})$. Since $P = E \cap \mathfrak{a} = \widehat{\mu_{\mathfrak{a}}(X)}$ is a convex polytope (Proposition 3.1), we deduce the following.

Theorem 0.3. *All faces of $\widehat{\mu_{\mathfrak{p}}(X)}$ are exposed.*

A reformulation of Theorem 3.1 is that the faces of E correspond to maxima of components of the gradient map. This observation will be used to realize a close connection between the faces of E and parabolic subgroups of G. More precisely, for any face $F \subset E$ let $X_F := \mu_{\mathfrak{p}}^{-1}(F)$ and let $Q^F = \{g \in G : g \cdot X_F = X_F\}$. Then X_F is the set of maximum points of an appropriately chosen component of the gradient map and Q^F is a parabolic subgroup of G.

If X is a G-stable compact submanifold of Z, then for any face F, one can construct an open neighbourhood X_F^- of X_F in X, which is an analogue of an open Bruhat cell. Moreover there is a smooth deformation retraction of X_F^- onto X_F. See Theorem 3.1 for more details.

Acknowledgements. The first two authors are grateful to the Fakultät für Mathematik of Ruhr-Universität Bochum for the wonderful hospitality during several visits. They also wish to thank the Max-Planck Institut für Mathematik, Bonn for excellent conditions provided during their visit at this institution, where part of this paper was written.

1. Group theoretical description of the faces

We start by recalling the basic definitions and results regarding convex bodies. For more details see e.g. [18]. Let V be a real vector space with scalar product $\langle \cdot, \cdot \rangle$. A *convex body* $E \subset V$ is a convex compact subset of V. Let $\operatorname{Aff}(E)$ denote the affine span of E. The interior of E in $\operatorname{Aff}(E)$ is called the *relative interior* of E and is denoted by $\operatorname{relint} E$. By definition a *face* of E is a convex subset $F \subset E$ such that $x, y \in E$ and $\operatorname{relint}[x, y] \cap F \neq \emptyset$

implies $[x,y] \subset F$. A face distinct from E and $\emptyset$ is called a *proper face*. The *extreme points* of E are the points $x \in E$ such that $\{x\}$ is a face. We will denote by $\operatorname{ext} E$ the set of the extreme points of E. The set $\operatorname{ext} E$ completely determines the convex body E since the convex hull of $\operatorname{ext} E$ coincides with E and it is the smallest subset of E with this property. If F is a face of E, we denote by $\operatorname{Dir}(F)$ the vector subspace of V defined by $\operatorname{Aff}(F)$, i.e. $\operatorname{Aff}(F) = p + \operatorname{Dir}(F)$. We call $\operatorname{Dir}(F)$ the *direction* of F. Every vector $\beta \in V$ defines an *exposed face* $F = F_\beta(E) = \{x \in E : \langle x, \beta \rangle = \max_{y \in E} \langle y, \beta \rangle\}$ with $\operatorname{Dir}(F_\beta(E)) \subset \{\beta\}^\perp$. In general not all faces of a convex set are exposed, see Fig. 1 for an example. For any exposed face F the set

$$C_F = \{\beta \in V : F = F_\beta(E)\}, \tag{1}$$

is a convex cone. The faces of E are closed. If F_1 and F_2 are faces of E and they are distinct, then $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. Moreover the convex body E is the disjoint union of the relative interiors of its faces (see [18, p. 62]).

We are interested in invariant convex bodies in polar representations. A theorem of Dadok [6] asserts that we can restrict ourselves to the following setting.

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan involution θ and let B be the Killing form of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is the eigenspace decomposition of $\mathfrak{g}$ in 1 and -1 eigenspaces of θ and they are orthogonal under B. Moreover, B restricted to $\mathfrak{k}$, respectively $\mathfrak{p}$, is negative definite, respectively positive definite. In the sequel we denote $\langle \cdot, \cdot \rangle = B_{|\mathfrak{p} \times \mathfrak{p}}$ which is a K-invariant scalar product. Out object of study will be a K-stable convex body $E \subset \mathfrak{p}$. For for any $A, B \subset \mathfrak{p}$ we set

$$\begin{aligned} A^B &:= \{\eta \in A : [\eta, \xi] = 0, \text{for all } \xi \in B\} \\ G^B &:= \{g \in G : \operatorname{Ad} g(\xi) = \xi, \text{for all } \xi \in B\}, \\ K^B &:= K \cap G^B. \end{aligned}$$

where Ad denotes the adjoint representation. In the sequel we denote by $k \cdot x = \operatorname{Ad}(k)(x)$ the action of K on $\mathfrak{p}$ by linear isometries.

Faces of K–invariant convex bodies in $\mathfrak{p}$ are closely connected to orbits of subgroups of K which are given as centralizers. More precisely for any nonzero β in $\mathfrak{p}$ we have the Cartan decomposition $\mathfrak{g}^\beta = \mathfrak{k}^\beta \oplus \mathfrak{p}^\beta$ of the Lie algebra of the centralizer G^β of β in G.

Proposition 1.1. *Let $F = F_\beta(E)$ be an exposed face of E. Then*

a) *$F \subset \mathfrak{p}^\beta$ and F is K^β–stable;*
b) *$\operatorname{Dir}(F) \subset \beta^\perp$, where $\perp$ is in $\mathfrak{p}$.*

Proof. If $x \in F_\beta(E)$, then $\widehat{K \cdot x} \subset E$ since E is K-invariant. Moreover, we have

$$\max_{y \in E} \langle y, \beta \rangle = \max_{y \in \widehat{K \cdot x}} \langle y, \beta \rangle = \langle x, \beta \rangle.$$

Corollary 3.1 in [4] implies $F_\beta(\widehat{K\cdot x}) \subset \mathfrak{p}^\beta$. Therefore $x \in \mathfrak{p}^\beta$. This proves a). Part b) follows since F is contained in an affine hyperplane orthogonal to β. □

For an arbitrary face of E we have the following.

Proposition 1.2. *Let $F \subset E$ be a face. Then there exists an abelian subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that*

a) *$F \subset \mathfrak{p}^{\mathfrak{s}}$ and F is $K^{\mathfrak{s}}$–stable;*
b) *$\mathrm{Dir}(F) \subset \mathfrak{s}^\perp$;*

Proof. We may fix a maximal chain of faces $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ (see [3, Lemma 2]). If $k = 0$, then $F = E$ and $\mathfrak{s} = \{0\}$. Assume the theorem is true for a face contained in a maximal chain of length k. Then the claim is true for F_1 and consequently there exists $\mathfrak{s}_1 \subset \mathfrak{p}$ such that $F_1 \subset \mathfrak{p}^{\mathfrak{s}_1}$, F_1 is $K^{\mathfrak{s}_1}$-stable and $\mathrm{Dir}(F_1) \subset \mathfrak{s}_1^\perp$. F is an exposed face of F_1. Let $\beta' \in \mathfrak{p}^{\mathfrak{s}_1}$ such that $F = F_{\beta'}(F_1)$ and set $\mathfrak{s} := \mathbb{R}\beta' \oplus \mathfrak{s}_1$. Then $F \subset \mathfrak{p}^{\mathfrak{s}}$, F is $(K^{\mathfrak{s}_1})^{\beta'} = K^{\mathfrak{s}}$–stable and $\mathrm{Dir}(F) \subset \mathfrak{s}^\perp$. □

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $P = E \cap \mathfrak{a}$ is a convex subset of $\mathfrak{a}$ which is $\mathcal{N}_K(\mathfrak{a})$–stable. The proof of the following Lemma is given in [7].

Lemma 1.1. *(i) If $E \subset \mathfrak{p}$ is a K–invariant convex subset, then $E \cap \mathfrak{a} = \pi(E)$ and $K \cdot \pi(E) = E$. (ii) If $C \subset \mathfrak{a}$ is a $\mathcal{N}_K(\mathfrak{a})$-invariant convex subset, then $K \cdot C$ is convex and $\pi(K \cdot C) = C$.*

Lemma 1.2. *Let U be a compact Lie group and let $\mathfrak{g} \subset \mathfrak{u}^{\mathbb{C}}$ be a semisimple θ-invariant subalgebra. Then any Lie subgroup with finitely many connected components and with Lie algebra $\mathfrak{g}$ is closed and compatible.*

Proof. We fix an embedding $U \hookrightarrow \mathrm{U}(n)$ such that the Cartan involution $X \mapsto (X^{-1})^*$ of $\mathrm{GL}(n, \mathbb{C})$ restricts to θ. Then G is closed in $\mathrm{GL}(n, \mathbb{C})$ (see [16, p. 440] for a proof) and hence also in $U^{\mathbb{C}}$. Since $\mathfrak{g}$ is θ-invariant, also G is, and θ restricts to the Cartan involution of G. This shows that G is compatible. □

If $G \subset U^{\mathbb{C}}$ is compatible with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{g}$ is real reductive and there is a nondegenerate K–invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ which is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$ and such that $B(\mathfrak{k}, \mathfrak{p}) = 0$. Indeed, fix a U-invariant inner product $\langle\, , \rangle$ on $\mathfrak{u}$. Let $\langle\, , \rangle$ denote also the inner product on $i\mathfrak{u}$ such that multiplication by i be an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. Define B on $\mathfrak{u}^{\mathbb{C}}$ imposing $B(\mathfrak{u}, i\mathfrak{u}) = 0$, $B = -\langle\, , \rangle$ on $\mathfrak{u}$ and $B = \langle\, , \rangle$ on $i\mathfrak{u}$. Therefore B is $\mathrm{Ad}\, U^{\mathbb{C}}$–invariant and non-degenerate and its restriction to $\mathfrak{g}$ satisfies the above conditions.

Let $\mathfrak{q}$ be a K–invariant subspace of $\mathfrak{p}$. Then $[\mathfrak{q}, \mathfrak{q}]$ is a K–invariant linear subspace of $\mathfrak{k}$ and therefore an ideal of $\mathfrak{k}$. Since K is compact, we have the following K–invariant splitting $\mathfrak{k} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{k}'$. In particular $\mathfrak{k}'$ is an ideal of $\mathfrak{k}$ commuting with $[\mathfrak{q}, \mathfrak{q}]$. Let $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ be a K–invariant splitting of $\mathfrak{p}$. Since

$$B([\mathfrak{q}, \mathfrak{q}'], \mathfrak{k}) = B(\mathfrak{q}, [\mathfrak{k}, \mathfrak{q}']) \subset B(\mathfrak{q}, \mathfrak{q}') = 0,$$

this shows that $[\mathfrak{q}, \mathfrak{q}'] = 0$ and so $[\mathfrak{q}', [\mathfrak{q}, \mathfrak{q}]] = [\mathfrak{q}, [\mathfrak{q}, \mathfrak{q}']] = 0$. Moreover $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ implies that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{q}'$ are compatible K–invariant commuting ideal of $\mathfrak{g}$.

If a K–invariant linear subspace $\mathfrak{q} \subset \mathfrak{p}$ is fixed, one gets decomposition of $\mathfrak{g}$, and so of G. This is decomposition is the content of the next Proposition. We will need it in the case where $F \subset \mathfrak{p}$ is a K–invariant convex body and $\mathfrak{q}$ is such that $\operatorname{Aff}(F) = x_0 + \mathfrak{q}$.

Proposition 1.3. *Let $G \subset U^{\mathbb{C}}$ be a compatible subgroup with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\mathfrak{q} \subset \mathfrak{p}$ be a linear K–invariant subspace. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ where $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{h}^{\perp_B}$. Then the following hold.*

a) *$\mathfrak{h}$ and $\mathfrak{h}'$ are compatible K–invariant commuting ideal of $\mathfrak{g}$;*
b) *Let K_1 be the connected Lie subgroup of G with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}, \mathfrak{h}]$. Then $K_1 \exp(\mathfrak{q})$ is a connected compatible subgroup of G and any two maximal subalgebras of $\mathfrak{q}$ are congiugate by an element of K_1.*
c) *Let K_2 be the connected Lie subgroup of G with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}', \mathfrak{h}']$. Then any two maximal subalgebras of $\mathfrak{q}'$ are congiugate by an element of K_2.*

Proof. We have proved (a) in the above discussion. Let $\mathfrak{b} := [\mathfrak{h}, \mathfrak{h}]$. Then $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{b}$ and $\mathfrak{b}$ is semisimple. Denote by B the connected subgroup of $U^{\mathbb{C}}$ with Lie algebra $\mathfrak{b}$. By Lemma 1.2 B is a closed subgroup of $U^{\mathbb{C}}$. Set $\mathfrak{z}_{\mathfrak{p}} := \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{p}$ and $\mathfrak{d} := \mathfrak{b} \oplus \mathfrak{a}$. Then $\mathfrak{d}$ is a reductive Lie algebra and $\exp \mathfrak{a}$ is a compatible abelian subgroup commuting with B. Thus $D := B \cdot \exp \mathfrak{a}$ is a connected closed subgroup with Lie algebra $\mathfrak{d}$. Moreover $D \cap U = B \cap U$ and $\exp(\mathfrak{b} \cap \mathfrak{p}) \cdot \exp \mathfrak{a} = \exp(\mathfrak{b} \cap \mathfrak{p} \oplus \mathfrak{a}) = \exp(\mathfrak{d} \cap \mathfrak{p})$. This shows that D is compatible. Since $D \cap U$ coincides with K_1 and D is connected the last statement in (b) follows from standard properties of compatible subgroups (see e.g. Prop. 7.29 in [16]; note that a connected compatible subgroup is a reductive group in the sense of [16, p. 446]). This proves (b). For (c) the same argument applies more directly. It is enough to observe that the connected Lie subgroup $H'' \subset G$ with Lie algebra $[\mathfrak{h}', \mathfrak{h}']$ is semisimple, compatible and connected and that $K_2 = H'' \cap U$. □

Remark 1.1. *The compatible subgroup G in the previous Proposition is not assumed to be connected. Nevertheless the constructions in (b) and (c) depend only on G^0. Thus considering G^0 in place of G makes no difference.*

Lemma 1.3. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a reductive Lie algebra and $\mathfrak{g}_i$ ideals. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal subalgebra, then $\mathfrak{a}_i := \mathfrak{a} \cap \mathfrak{p}_i$ is a maximal subalgebra of $\mathfrak{p}_i$ and $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$.*

If σ is a face of P, let $\sigma^{\perp}$ denote the orthogonal (inside $\mathfrak{a}$) to the direction of the affine hull of σ.

Lemma 1.4. *Let F be a face and let $\mathfrak{s}$ be as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing $\mathfrak{s}$. Set $\sigma := \pi(F)$. Then σ is a face of P, $\sigma = F \cap \mathfrak{a}$ and $F = K^{\sigma^\perp} \cdot \sigma$. Moreover F is a proper face if and only if $F \cap \mathfrak{a}$ is.*

Proof. By Proposition 1.2 $F \subset \mathfrak{p}^{\mathfrak{s}}$ is a $K^{\mathfrak{s}}$–stable convex set. By Lemma 1.1 we get $\sigma = \pi(F) = F \cap \mathfrak{a}$ and this is a face P by [3, Lemma 11]. Since $\mathrm{Dir}(F)$ is contained in the orthogonal complement of $\mathfrak{s}$, and $\mathrm{Dir}(\sigma) \subset \mathrm{Dir}(F)$, we have $\mathrm{Dir}(\sigma) \subset \mathfrak{a} \cap \mathfrak{s}^\perp$. Then $\sigma^\perp \subset \mathfrak{s}$. Hence $K^{\sigma^\perp} \cdot \sigma \subset K^{\mathfrak{s}} \cdot \sigma \subset F$. We prove the reverse inclusion. If $y \in F$, then $F \cap \widehat{K \cdot y}$ is a face of $\widehat{K \cdot y}$. Set $\tilde{\sigma} = \pi(F \cap \widehat{K \cdot y})$. We have $\tilde{\sigma} \subset \sigma$ and by Proposition 3.6 in [4] we also have that $F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma}$. On the other hand, $\sigma^\perp \subset \tilde{\sigma}^\perp$, so $K^{\tilde{\sigma}^\perp} \subset K^{\sigma^\perp}$ and

$$F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma} \subset K^{\sigma^\perp} \cdot \sigma.$$

This implies $F = K^{\sigma^\perp} \cdot \sigma$. Note that F is proper if σ is. It remains to prove that σ is proper, when F is proper.

Let $\mathrm{Aff}(E) = x_o + \mathfrak{q}_E$. Note that $\mathfrak{q}_E = \{x - y : x, y \in \mathrm{Aff}(E)\}$ implies that $\mathfrak{q}_E$ is K–invariant. Since K acts on $\mathfrak{p}$ by isometries, we may assume that x_o is orthogonal to $\mathfrak{q}$. Note that x_o is uniquely defined by this condition. It follows that x_o is a K fixed point and $E = x_0 + E_1$, where E_1 is a K–invariant convex body of $\mathfrak{q}_E$. Proposition 1.3 applied to $\mathfrak{q}_E$ yields K_1, K_2 such that $G_1 = K_1 \exp(\mathfrak{q}_E)$ is a connected compatible semisimple real Lie group, $K = K_1 \cdot K_2$ and for any $x \in E$ we have

$$K \cdot x = K \cdot (x_o + x_1) = x_o + K \cdot x_1 = x_o + K_1 \cdot x_1 = K_1 \cdot x.$$

since $\mathfrak{q}_E$ is fixed pointwise by K_2. By Lemma 1.3, $\mathfrak{a} = \mathfrak{a}_E \oplus \mathfrak{a}'_E$, where $\mathfrak{a}_E$ is a maximal abelian subalgebra of $\mathfrak{q}_E$ and $\mathfrak{a}'_E$ is a maximal abelian subalgebra of $\mathfrak{q}'_E$. Since $\pi(E) = \pi(x_o) + \pi(E_1)$ and $\mathrm{Dir}(E_1) = \mathfrak{q}_E$, it follows that the direction of $\pi(E)$ is $\mathfrak{a}_E$. If $\sigma = \pi(F) = \pi(E) = E \cap \mathfrak{a}$, then $\sigma^\perp = \mathfrak{a}'_E$ and so $K_1 \subset K^{\mathfrak{a}'_E}$. It follows that

$$F = K^{\mathfrak{a}'_E} \cdot (E \cap \mathfrak{a}) = K_1 \cdot (E \cap \mathfrak{a}) = K \cdot (E \cap \mathfrak{a}) = E.$$

where the last equality follows by Lemma 1.1. Hence, if F is proper, then $\sigma = \pi(F) \subsetneq \pi(E) = E \cap \mathfrak{a}$. ◻

Proposition 1.4. *Let F be a proper face and let $\mathfrak{s}$ as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing $\mathfrak{s}$. Then F is exposed if and only if $F \cap \mathfrak{a}$ is.*

Proof. Assume that there exists $\beta \in \mathfrak{p}$ such that $F = F_\beta(E)$. Since $F \cap \mathfrak{a} = \sigma$ is a proper face of P, the point β is not orthogonal to $\mathfrak{a}$. We have $\beta = \beta_1 \oplus \beta_2$, with $\beta_1 \in \mathfrak{a}$ different from zero and β_2 orthogonal to $\mathfrak{a}$. Therefore $F_\beta(E) \cap \mathfrak{a} = F_{\beta_1}(E) \cap \mathfrak{a} = F_{\beta_1}(P) = \sigma$. Now, assume that there exists $\beta \in \mathfrak{a}$ such that $\sigma = F_\beta(P)$. Let $F' := F_\beta(E)$. By Proposition 1.1 $F' \subset \mathfrak{p}^\beta$. Moreover $\mathfrak{a} \subset \mathfrak{p}^\beta$ since $\beta \in \mathfrak{a}$. By Lemma 1.4 the intersection of a face with

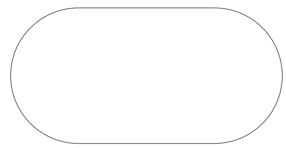

Figure 1.

$\mathfrak{a}$ determines the face. Since $F' \cap \mathfrak{a} = F_\beta(P) = \sigma = F \cap \mathfrak{a}$ we conclude that $F = F'$. Thus F is exposed. □

Remark 1.2. *Given a Weyl–invariant convex body $P \subset \mathfrak{a}$, set $E := K \cdot P$. By Lemma 1.1 E is a K-invariant convex body in $\mathfrak{p}$ and $P = E \cap \mathfrak{a}$. Thus a general P can be realized as $E \cap \mathfrak{a}$. A general Weyl–invariant convex body P can have non–exposed faces. For example take $G = U^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ and $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then $\mathfrak{a} = \mathbb{R}^2$ and the Weyl group is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ where the generators are given by the reflections on the axes. The picture in Fig. 1 is a Weyl–invariant P with exactly 4 non–exposed faces. By the Proposition also the corresponding body $E \subset i\mathfrak{su}(2) \oplus i\mathfrak{su}(2)$ has non–exposed faces.*

2. Proof of the main results

Let $\mathfrak{a} \subset \mathfrak{p}$ and define the following map

$$\Upsilon : \mathscr{F}(P) \longrightarrow \mathscr{F}(E), \quad \sigma \mapsto K^{\sigma^\perp} \cdot \sigma$$

Since σ is $\mathcal{N}_{K^{\sigma^\perp}}(\mathfrak{a})$-invariant, it follows from Lemma 1.1 that $\Upsilon(\sigma)$ is a face of E.

Theorem 0.1. *The map Υ induces a bijection between $\mathscr{F}(P)/\mathcal{N}_K(a)$ and $\mathscr{F}(E)/K$.*

Proof. Set $\mathcal{N} := \mathcal{N}_K(\mathfrak{a})$. We first show that Υ is $\mathcal{N}$-equivariant. Let $w \in \mathcal{N}$. Then $\sigma' = w\sigma$ implies $K^{\sigma'^\perp} = wK^{\sigma^\perp}w^{-1}$ and therefore $\Upsilon(\sigma') = w\Upsilon(\sigma)$. This means that the map

$$\tilde{\Upsilon} : \mathscr{F}(P)/\mathcal{N} \longrightarrow \mathscr{F}(E)/K, \quad [\sigma] \mapsto K^{\sigma^\perp} \cdot \sigma$$

is well-defined. Next, we prove that $\tilde{\Upsilon}$ is injective. Assume for some $g \in K$ $g \cdot F_1 = F_2$ where $F_1 = \Upsilon(\sigma_1)$ and $F_2 = \Upsilon(\sigma_2)$. Since $F_2 = K^{\sigma_2^\perp} \cdot \sigma_2$, the face F_2 is a $K^{\sigma_2^\perp}$–invariant convex body. Moreover $\sigma_2 \subset \mathfrak{a} \subset \mathfrak{p}^{\sigma_2^\perp}$ and $\mathfrak{p}^{\sigma_2^\perp}$ is $K^{\sigma_2^\perp}$–invariant. Therefore F_2 is contained in $\mathfrak{p}^{\sigma_2^\perp}$. It follows that $\mathrm{Aff}(F_2) = x_o + \mathfrak{q}_{F_2}$, where $\mathfrak{q}_{F_2}$ is a $K^{\sigma_2^\perp}$ invariant subspace of $\mathfrak{p}^{\sigma_2^\perp}$, x_o is a fixed $K^{\sigma_2^\perp}$ point and it is orthogonal orthogonal to $\mathfrak{q}_{F_2}$. We apply Proposition 1.3 to the group $G^{\sigma_2^\perp}$ and $\mathfrak{q}_{F_2}$. Thus $\mathfrak{h}_{F_2} = [\mathfrak{q}_{F_2}, \mathfrak{q}_{F_2}] \oplus \mathfrak{q}_{F_2}$ and its orthogonal complement in $\mathfrak{g}^{\sigma_2^\perp}$, that we denote by $\mathfrak{h}'_{F_2}$, are commuting ideal. The Proposition 1.3 also yields subgroups $K_1, K_2 \subset K^{\sigma_2 \perp}$ such that any two maximal subalgebras in $\mathfrak{q}_{F_2}$, respectively $\mathfrak{q}'_{F_2}$, are interchanged by

K_1, respectively K_2. Since $\sigma_2 \subset \mathfrak{a}$, also $\mathrm{Dir}(\sigma_2) \subset \mathfrak{a}$ and we may decompose $\mathfrak{a} = \mathrm{Dir}(\sigma_2) \oplus \sigma_2^\perp$. But $\mathrm{Dir}(\sigma_2)$ is contained also in $\mathfrak{q}_{F_2}$ since $\sigma_2 \subset F_2$. So $\sigma_2^\perp \subset \mathfrak{q}_{F_2}^\perp \cap \mathfrak{p} = \mathfrak{q}'_{F_2}$. By dimension $\mathrm{Dir}(\sigma_2)$ is a maximal subalgebra in $\mathfrak{q}_{F_2}$ and $\sigma_2^\perp$ is a maximal subalgebra in $\mathfrak{q}'_{F_2}$. On other hand from $g \cdot F_1 = F_2$ it follows that $g \cdot \mathrm{Dir}(\sigma_1) \subset \mathfrak{q}_{F_2}$ and $g \cdot \sigma_1^\perp \subset \mathfrak{q}_{F_2}$, and they are also maximal subalgebras in these spaces. By the Proposition 1.3 (b) and (c) there exist $k_1 \in K_1, k_2 \in K_2$ such that

$$(k_1 g) \cdot \mathrm{Dir}(\sigma_1) = \mathrm{Dir}(\sigma_2)$$
$$(k_2 g) \cdot \sigma_1^\perp = \sigma_2^\perp.$$

Since x_0 is fixed by the larger group $K^{\sigma_2^\perp}$ it follows that $k_1 g \sigma_1 = \sigma_2$. Moreover $k_1 k_2 = k_2 k_1$ since $[\mathfrak{h}_{F_2}, \mathfrak{h}'_{F_2}] = 0$. For the same reason $\mathfrak{q}'_{F_2}$ is fixed pointwise by K_1 and $\mathfrak{q}_{F_2}$ is fixed pointwise by K_2. Set $k = k_1 k_2$ and $w = kg$. Then $k \in K^{\sigma_2^\perp}$ and $w \in K$. We get

$$w \cdot \mathrm{Dir}(\sigma_1) = \mathrm{Dir}(\sigma_2)$$
$$w \cdot \sigma_1^\perp = \sigma_2^\perp.$$

Thus $w \cdot \mathfrak{a} = \mathfrak{a}$, i.e. $w \in \mathcal{N}$. Since $k \in K^{\sigma_2^\perp}$, $w \cdot F_1 = (kg) \cdot F_1 = k \cdot F_2 = F_2$. Since $\sigma_1 = (x_0 + \mathrm{Dir}(\sigma_1)) \cap F_1$ and similarly for F_2, we conclude that $w\sigma_1 = \sigma_2$. Finally we prove that $\tilde{\Theta}$ is surjective. Let $F \subset \widehat{\mathcal{O}}$ be a face. Then $F \subset \mathfrak{p}^{\mathfrak{s}}$ for some abelian subalgebra $\mathfrak{s} \in \mathfrak{p}$. Then there exists $k \in K$ such that $k \cdot \mathfrak{a} \subset \mathfrak{p}^{\mathfrak{s}}$. Therefore $k^{-1} \cdot F \subset \mathfrak{p}^{(k^{-1}\cdot \mathfrak{s})}$ and $\mathfrak{a} \subset \mathfrak{p}^{(k^{-1}\cdot\mathfrak{s})}$. By Proposition 1.4, $k \cdot F = K^{\sigma^\perp} \cdot \sigma$ where $\sigma = (k \cdot F) \cap \mathfrak{a}$ and so $\tilde{\Upsilon}$ is surjective. □

As an application of the above theorem and Proposition 1.4, we get the following result.

Theorem 0.2. *The faces of E are exposed if and only if the faces of P are exposed.*

Proof. By the above Theorem, the map $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ induces a bijection between $\mathscr{F}(P)/\mathcal{N}$ and $\mathscr{F}(E)/K$. Hence, keeping in mind that if $F_1 = kF_2$, then F_1 is exposed if and only if F_2, the result follows from Proposition 1.4. □

Remark 2.1. *We have proven Theorems 0.1 and 0.2 under the assumption that G is a connected real semisimple Lie group. From this it follows that both theorems hold true for any connected compatible subgroup of $U^{\mathbb{C}}$, since such a subgroup is real reductive in the sense of* [16, p. 446] *and thus it is the product of a semisimple connected subgroup and an abelian subgroup, see e.g.* [16, p. 453].

3. Convex hull of the gradient map image

Let U be a compact connected Lie group and $U^{\mathbb{C}}$ its complexification. Let (Z, ω) be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that U acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. Let $G \subset U^{\mathbb{C}}$ be a closed connected subgroup of $U^{\mathbb{C}}$ which is compatible with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that G is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a K-equivariant map $\mu_{i\mathfrak{p}} : Z \longrightarrow (i\mathfrak{p})^*$. Using a fixed U-invariant scalar product $\langle\, ,\, \rangle$ on $\mathfrak{u}$, we identify $\mathfrak{u} \cong \mathfrak{u}^*$. We also denote by $\langle\, ,\, \rangle$ the scalar product on $i\mathfrak{u}$ such that multiplication by i be an isometry of $\mathfrak{u}$ onto $i\mathfrak{u}$. For $z \in Z$ let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p}$. In other words we require that $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$, for any $\beta \in \mathfrak{p}$. Then we view $\mu_{i\mathfrak{p}}$ as a map

$$\mu_{\mathfrak{p}} : Z \to \mathfrak{p},$$

which is called the *G-gradient map* or *restricted momentum map* associated to μ. For the rest of the paper we fix a G-stable *compact subset* $X \subset Z$ and we consider the gradient map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ restricted on X. We also set

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu^{-i\beta}.$$

We will now study the convex hull of $\mu_{\mathfrak{p}}(X)$, that we denote by E. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and let $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $\pi \circ \mu_{\mathfrak{p}} =: \mu_{\mathfrak{a}}$ is the gradient map associated to $A = \exp(\mathfrak{a})$. Let Z^A be the set of fixed points of A. We note that $\mu_{\mathfrak{a}}$ is locally constant on Z^A since $\operatorname{Ker} \mathrm{d}\mu_{\mathfrak{a}}(x) = (\mathfrak{a} \cdot x)^{\perp}$ (see [15]). Let $\mathfrak{b}$ a subspace of $\mathfrak{a}$ and let $X^{\mathfrak{b}} = \{p \in X : \xi_X(p) = 0 \text{ for all } \xi \in \mathfrak{b}\}$, where ξ_X is the vector field induced by the A action on X. Then the map $\mu_{\mathfrak{b}} : X^{\mathfrak{b}} \longrightarrow \mathfrak{b}$, that is the composition of $\mu_{\mathfrak{p}}$ with the orthogonal projection onto $\mathfrak{b}$, is locally constant ([11], Section 3). Since $X^{\mathfrak{b}}$ is compact, $\mu_{\mathfrak{b}}(X^{\mathfrak{b}})$ is a finite set. In [11] it also shown that for any $y \in X^{(\mathfrak{b})} := \{p \in X : \mathfrak{a}_p = \mathfrak{b}\}$, where $\mathfrak{a}_p := \{\xi \in \mathfrak{a} : \xi_X(p) = 0\}$, we have that $\mu_{\mathfrak{a}}(A \cdot y) \subset \mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ is an open subset of the affine space $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (the orthogonal complements are taken in $\mathfrak{a}$). Moreover $\mu_{\mathfrak{a}}(A \cdot y)$ is a convex subset of $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (see [10]) and therefore $\mu_{\mathfrak{a}}(\overline{A \cdot y}) = \overline{\mu_{\mathfrak{a}}(A \cdot y)}$ is a convex body.

Let $P := \widehat{\mu_{\mathfrak{a}}(X)}$. If $\beta \in \mu_{\mathfrak{a}}(X)$ is an extremal point of P, and $y \in \mu_{\mathfrak{a}}^{-1}(\beta)$, then $\mu_{\mathfrak{a}}(A \cdot y)$ is an open neighborhood of $\mu_{\mathfrak{a}}(y)$ in $\mu_{\mathfrak{a}}(y) + \mathfrak{a}_y^{\perp}$ and it is contained in $\mu_{\mathfrak{a}}(X) \subset P$. Since $\mu_{\mathfrak{a}}(y)$ is an extremal point, it follows that $\mathfrak{a}_y^{\perp} = \{0\}$ and so y is a fixed point of A. Since X is compact, the set X^A has finitely many connected components. Therefore P has finitely many extremal points, i.e. it is a polytope. We have shown the following.

Proposition 3.1. *Let $X \subset Z$ be a G-invariant compact subset of Z. Then the image $\mu_{\mathfrak{a}}(X^A)$ is a finite set $\{c_1, \dots, c_p\}$ and $P = \widehat{\mu_{\mathfrak{a}}(X)}$ is the convex hull of $c_1, \dots, c_p$.*

As a corollary we get the following result.

Theorem 0.3. *Let $X \subset Z$ be a G-invariant compact subset of Z. Then every face of $E = \widehat{\mu_\mathfrak{p}(X)}$ is exposed.*

Proof. Since

$$\pi(E) = \pi(\widehat{\mu_\mathfrak{p}(X)}) = \widehat{\mu_\mathfrak{a}(X)},$$

by Lemma 1.1 (i) we conclude that $E \cap \mathfrak{a} = \pi(E) = P$ and by Proposition 3.1, Remark 2.1 and Theorem 0.2 we get that every face of E is exposed. □

We call P the *momentum polytope*. If $G = U^\mathbb{C}$ and X is a complex connected submanifold of Z, then $P = \mu_\mathfrak{a}(X)$ by the Atiyah-Guillemin-Sternberg convexity theorem [1, 8]. The same holds for X an irreducible semi-algebraic subset of a Hodge manifold Z [17, 11, 5].

Since any proper face F of E is exposed, the set C_F defined in (1) is a non-empty convex cone in $\mathfrak{p}$. Set

$$K^F := \{g \in K : g \cdot F = F\}.$$

By Proposition 5 in [3] we have $C_F^{K^F} := \{\beta \in C_F : K^F \cdot \beta = \beta\} \neq \emptyset$. This means that for a proper face F one can find a K^F–invariant vector β such that $F_\beta(E) = F$. For $\beta \in \mathfrak{p}$, denote by X^β the set of points of X that are fixed by $\exp(\mathbb{R}\beta)$. If $\beta \in C_F$, let

$$X^\beta_{\max} := \{x \in X : \mu^\beta_\mathfrak{p}(x) = \max_X \mu^\beta_\mathfrak{p}\}.$$

Since the function $\mu^\beta_\mathfrak{p}$ is K^β–invariant the set $X^\beta_{\max}$ is K^β–invariant. Moreover $X^\beta_{\max}$ is a union of finitely many connected components of X^β and X^β is G^β-stable. Every connected component of G^β meets K^β. This implies that G^β leaves $X^\beta_{\max}$ invariant. Next we show that $X^\beta_{\max}$ does not depend on the choice of β in C_F.

Lemma 3.1. *If $\beta \in C_F$, then $X^\beta_{\max} = \mu_\mathfrak{p}^{-1}(F)$. Moreover F is the convex hull of $\mu_\mathfrak{p}(X^\beta_{\max})$.*

Proof. Fix $x \in X$. Then $\mu_\mathfrak{p}(x) \in F$ if and only if $\langle \mu_\mathfrak{p}(x), \beta\rangle = \max_{v\in E}\langle v, \beta\rangle$. Moreover $\max_{v\in E}\langle v, \beta\rangle = \max_{v\in\mu_\mathfrak{p}(X)}\langle v, \beta\rangle = \max_X \mu^\beta_\mathfrak{p}$. So $x \in \mu_\mathfrak{p}^{-1}(F)$ if and only if x is a maximum of $\mu^\beta_\mathfrak{p}(x)$ restricted to X. This shows that $X^\beta_F = \mu_\mathfrak{p}^{-1}(F)$. The inclusion $\mu_\mathfrak{p}(X^\beta_F) \subset F$ follows from the definition and therefore $\widehat{\mu_\mathfrak{p}(X^\beta_F)} \subset F$. By [3, Lemma 3] $\operatorname{ext} F = \operatorname{ext} E \cap F$, so $\operatorname{ext} F \subset \mu_\mathfrak{p}(X) \cap F = \mu_\mathfrak{p}(X^\beta_F)$. It follows that $F = \widehat{\mu_\mathfrak{p}(X^\beta_F)}$. □

Motivated by the above Lemma we set $X_F := X^\beta_{\max}$ where β is any vector in C_F. We also set

$$Q^F = \{g \in G : g \cdot X_F = X_F\}.$$

Q^F is a closed Lie subgroup of G.

Given $\beta \in \mathfrak{p}$ define the following subgroups:

$$G^{\beta+} = \{g \in G : \lim_{t \mapsto -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists}\},$$
$$G^{\beta-} = \{g \in G : \lim_{t \mapsto +\infty} \exp(-t\beta) g \exp(t\beta) \text{ exists}\},$$
$$R^{\beta+} = \{g \in G : \lim_{t \mapsto -\infty} \exp(t\beta) g \exp(-t\beta) = e\},$$
$$R^{\beta-} = \{g \in G : \lim_{t \mapsto +\infty} \exp(-t\beta) g \exp(t\beta) = e\}.$$

$G^{\beta+}$ (respectively $G^{\beta-}$) is a parabolic subgroup, $R^{\beta+}$ (respectively $R^{\beta-}$) is its unipotent radical and G^β is a Levi factor. Therefore $G^{\beta+} = G^\beta \rtimes R^{\beta+}$ (respectively $G^{\beta-} = G^\beta \rtimes R^{\beta-}$).

Lemma 3.2. *$Q^F \cap K = K^F$.*

Proof. If $g \in Q^F \cap K$, then $g \cdot X_F = X_F$. Since $\mu_{\mathfrak{p}}$ is a K-invariant map, $g \cdot \mu_{\mathfrak{p}}(X_F) = \mu_{\mathfrak{p}}(X_F)$. Taking the convex hull of both sides and using Lemma 3.1 we get that $g \cdot F = F$, thus $g \in K^F$. Conversely, if $g \in K^F$, the equivariance of $\mu_{\mathfrak{p}}$ yields $X_F = \mu_{\mathfrak{p}}^{-1}(F) = \mu_{\mathfrak{p}}^{-1}(g \cdot F) = gX_F$, thus $g \in Q^F$. □

We are now ready to prove the connection between the set of the faces of E and parabolic subgroups of G.

Proposition 3.2. *Q^F is a parabolic subgroup of G. Moreover $Q^F = G^{\beta+}$ for every $\beta \in C_F^{K^F}$.*

Proof. Observe that by definition Q^F is a closed subgroup of G. Let $\beta \in C_F^{K^F}$. Then $F = F_\beta(E)$ and, by definition of K^F, we get $K^F = K^\beta$. The set $X_F = \{x \in X : \mu_{\mathfrak{p}}^\beta(x) = \max_X \mu_{\mathfrak{p}}^\beta\}$ is G^β-stable. Fix $p \in X_F$ and consider the orbit $G \cdot p$, which is a smooth submanifold contained in X. By Proposition 2.5 in [13] (see also Proposition 2.1 in [4]) we get that $\xi_X(x) = 0$ for any $\xi \in \mathfrak{r}^{\beta+}$ and for any $x \in X_F$. Therefore $G^{\beta+} \cdot p \subset X_F$. Hence $G^{\beta+} \subset Q^F$ and the Lie algebra $\mathfrak{q}^F$ of Q^F is parabolic. On the other hand by Lemma 3.2, we have $\mathfrak{q}^F \cap \mathfrak{k} = \mathfrak{g}^{\beta+} \cap \mathfrak{k} = \mathfrak{k}^\beta$ and so by Lemma 3.7 [4] we conclude that $\mathfrak{q}^F = \mathfrak{g}^{\beta+}$. Since $Q^F \subset N_G(\mathfrak{g}^{\beta+}) = G^{\beta+}$ we get $Q^F = G^{\beta+}$. □

Remark 3.1. *If $\beta' \in C_F^{K^F}$, then $Q_F = G^{\beta'+} = G^{\beta+}$. By Lemma 2.8 in* [4]*, we have $[\beta, \beta'] = 0$, $G^\beta = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$.*

Let $Q^{F-} = \Theta(Q^F)$, where $\Theta : G \longrightarrow G$ denotes the Cartan involution. The subgroup Q^{F-} is parabolic and depends only on F. The subgroup $L^F := Q^F \cap Q^{F-}$ is a Levi factor of both Q^F and Q^{F-}. Let $\beta \in C_F^{K^F}$. Then $Q^F = G^{\beta+}$, $L^F = G^\beta$ and we have the projection

$$\pi^{\beta+} : G^{\beta+} \longrightarrow G^\beta, \qquad \pi^{\beta+}(g) = \lim_{t \mapsto +\infty} \exp(t\beta) h \exp(-t\beta),$$

respectively

$$\pi^{\beta+} : G^{\beta-} \longrightarrow G^{\beta}, \qquad \pi^{\beta-}(g) = \lim_{t\mapsto -\infty} \exp(t\beta)h\exp(-t\beta).$$

Lemma 3.3. *If $\beta \in C_F^{K^F}$, then the projections $\pi^{\beta+}$ and $\pi^{\beta-}$ depend only on F.*

Proof. Let $g \in G^{\beta+}$. We know that $g = hr$, where $h \in G^{\beta}$ and $r \in R^{\beta+}$. Then

$$\pi^{\beta+}(g) = \lim_{t\mapsto +\infty} \exp(t\beta)g\exp(-t\beta) = h \lim_{t\mapsto +\infty} \exp(t\beta)r\exp(-t\beta) = h.$$

Since $G^{\beta} = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$ the decomposition $g = hr$ is the same for both groups and $\pi^{\beta+}(g) = \pi^{\beta'+}(g)$. The same argument works for $\pi^{\beta-}$. $\square$

Now assume that X is a G-stable *compact submanifold* of Z.

For $\beta \in C_F^{K_F}$ set $X_F^{\beta-} := \{p \in X : \lim_{t\mapsto +\infty} \exp(t\beta)\cdot p \in X_F\}$. Then the map

$$p^{\beta-} : X_F^{\beta-} \longrightarrow X_F, \qquad p^{\beta-}(x) = \lim_{t\mapsto +\infty} \exp(t\beta)\cdot x \tag{2}$$

is well-defined, G^{β}-equivariant, surjective and its fibers are $R^{\beta-}$-stable.

More generally one can consider $p^{\beta-}$ as a map from $X^{\beta-} = \{y \in X : \lim_{t\mapsto +\infty} \exp(t\beta)\cdot x$ exists $\}$ to X^{β}. In general however this map is not even continuous [14, Example 4.2]. To ensure continuity and smoothness it is enough that the topological Hilbert quotient $X^{\beta-}//G^{\beta}$ exists. Using the notation of [14] and choosing $r = \max_X \mu_{\mathfrak{p}}^{\beta}$, we have $X_F = X_{\max}^{\beta} = X_r^{\beta}$ and $X_r^{\beta-} = X_F^{\beta-}$. Thus Prop. 4.4 of [14] applies and yields that $X_F^{\beta-}$ is an open $G^{\beta-}$–stable subset of X and that (2) is smooth deformation retraction onto X_F. Using $\pi^{\beta-}$ one defines an action of $Q^{F-} = G^{\beta-}$ on X_F by setting $g\cdot x = \pi^{\beta-}(q)\cdot x$. This just depends on F. With respect to this action the map $p^{\beta-}$ becomes Q^{F-}–equivariant.

Lemma 3.4. *The set $X_F^{\beta-}$ and the map $p^{\beta-}$ do not depend on the choice of $\beta \in C_F^{K^F}$.*

Proof. Set $\Gamma = \exp(\mathbb{R}\beta)$. If $p \in X_F$ by the Slice Theorem [13, Thm. 3.1] there are open neighborhoods $S_p \subset T_pX$ and $\Omega_p \subset X$ and a Γ-equivariant diffeomorphism $\Psi_p : S_p \longrightarrow \Omega_p$, such that $0 \in S_p$, $p \in \Omega_p$, $\Psi_p(0) = p$. Since p is a maximum of $\mu_{\mathfrak{p}}^{\beta}$ restricted to X, the following orthogonal splitting $T_pX = V_0 \oplus V_-$ with respect to the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ holds. Here V_0 denotes the kernel of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ and V_- denotes the sum of eigenspaces of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ corresponding to negative eigenvalues. We also point out that $V_0 = T_pX_F$ and $S_p = \{x_0 + x_- : x_0 \in S_p \cap V_0, x_- \in V_-\}$, see [15]. It follows that $\Omega_p \subset X_F^{\beta-}$. Set $\Omega := \bigcup_{p\in X_F} \Omega_p$. By what we just proved, $\Omega \subset X_F^{\beta-}$. On the other hand Ω is an open Γ-invariant neighbourhood of X_F, so $X_F^{\beta-} \subset \Omega$. So $X_F^{\beta-} = \Omega$. If β' is another vector of $C_F^{K^F}$, set

$B = \exp(\mathbb{R}\beta \oplus \mathbb{R}\beta')$. This is a compatible abelian subgroup and $X_F \subset X^B$. So we may choose the open subsets Ω_p above to be B-stable. Therefore we get $X^{\beta'-} = \Omega$ as well. This proves that $X_F^{\beta-} = X_F^{\beta'-}$.

Next we show that $p^{\beta-} = p^{\beta'-}$. First observe that $p^{\beta-}(y) = p^{\beta'-}(y)$ if $y \in \Omega$. Indeed if $y \in \Omega_p$ we can study the limit using the diffeomorphism $\Psi_p : S_p \to \Omega_p$. The decomposition $T_pX = V_0 \oplus V_-$ is the same for β and β' since they commute and attain their maxima on X_F. Therefore if $x = \Psi_p^{-1}(y) = x_0 + x_-$, then

$$p^{\beta-}(y) = \Psi_p(x_0) = p^{\beta'-}(y). \tag{3}$$

If $p \in X_F^{\beta-}$ and $q = \lim_{t\mapsto+\infty} \exp(t\beta)\cdot p \in X_F$, there is $t_1 \in \mathbb{R}$, such that $\exp(t\beta)\cdot p \in \Omega$. Therefore

$$\begin{aligned}\lim_{t\mapsto+\infty} \exp(t\beta')\cdot p &= \lim_{t\mapsto+\infty} \exp(t\beta')(\exp(t_1\beta')\cdot p)\\ &= \lim_{t\mapsto+\infty} \exp(t\beta)(\exp(t_1\beta')\cdot p) \text{ (by 3)}\\ &= \exp(t_1\beta')(\lim_{t\mapsto+\infty} \exp(t\beta)\cdot p)\\ &= \lim_{t\mapsto+\infty} \exp(t\beta)\cdot p.\end{aligned}$$

□

By the above Lemma if F is a face and $\beta \in C_F^{K^F}$, we can set $X_F^- := X_F^{\beta-}$ and $p^{F-} := p^{\beta-} : X_F^- \longrightarrow X_F$.

Theorem 3.1. *For any face $F \subset E$, the set X_F is closed and L^F-stable, X_F^- is an open Q^{F-}–stable neighborhood of X_F in X and the map p^{F-} is a smooth Q^{F-}–equivariant deformation retraction of X_F^- onto X_F.*

References

[1] M. F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14 (1):1–15, 1982.

[2] L. Biliotti and A. Ghigi. Satake-Furstenberg compactifications, the moment map and λ_1. *Amer. J. Math.* 135 (1): 237–274, (2013).

[3] L. Biliotti, A. Ghigi, and P. Heinzner. Coadjoint orbitopes. *Osaka Math. J.* 51 (4) 935–968 (2014).

[4] L. Biliotti, A. Ghigi, and P. Heinzner. Polar orbitopes. *Comm. Anal. Geom.* 21 (3): 579–606, (2013).

[5] L. Biliotti, A. Ghigi, and P. Heinzner. A remark on the gradient map. *Documenta Mathematica*, Vol. 19 (2014), 1017-1023.

[6] J. Dadok. Polar coordinates induced by actions of compact Lie groups. *Trans. Amer. Math. Soc.*, 288(1):125–137, 1985.

[7] V. M. Gichev. Polar representations of compact groups and convex hulls of their orbits. *Differential Geom. Appl.*, 28(5):608–614, 2010.

[8] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[9] G. Heckman. *Projection of orbits and asymptotic behaviour of multiplicities of compact Lie groups.* 1980. PhD thesis.

[10] P. Heinzner and H. Stötzel. Semistable points with respect to real forms *Math. Ann.* 338 (2007), 1–9.
[11] P. Heinzner and P. Schützdeller. Convexity properties of gradient maps. *Adv. Math.*, 225(3):1119–1133, 2010.
[12] P. Heinzner and G. W. Schwarz. Cartan decomposition of the moment map. *Math. Ann.*, 337(1):197–232, 2007.
[13] P. Heinzner, G. W. Schwarz, and H. Stötzel. Stratifications with respect to actions of real reductive groups. *Compos. Math.*, 144(1):163–185, 2008.
[14] P. Heinzner, G. W. Schwarz, and H. Stötzel. Stratifications with respect to actions of real reductive groups. arXiv:math/0611491v2.
[15] P. Heinzner and H. Stötzel. Critical points of the square of the momentum map. In *Global aspects of complex geometry*, pages 211–226. Springer, Berlin, 2006.
[16] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.
[17] B. Kostant. On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup. (4)*, 6:413–455 (1974), 1973.
[18] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

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