JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 4, Number 1, Winter 1992

STURM-LIOUVILLE PROBLEMS AND HAMMERSTEIN OPERATORS

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ABSTRACT. It is shown that a generally complex-valued function of a real variable is a solution of a classical Sturm-Liouville eigenvalue problem if and only if a related two-parameter eigenvalue problem for a pair of integral operators, one of which is of Hammerstein type, admits a real solution belonging to a cone in a Krein space.

1. Introduction. Let $q, w : [a, b] \equiv I \to R$; $q, w \in L[a, b]$ where a, b are finite real numbers. We define the sets E^0, E^+, E^-, respectively, by $\{x \in I : w(x) = 0\}$, $\{x \in I : w(x) > 0\}$, $\{x \in I : w(x) < 0\}$ and we assume that $\mu(E^0) = 0$, $\mu(E^+) > 0$, $\mu(E^-) > 0$, where μ is Lebesgue measure.

We now consider the Dirichlet problem associated with the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda w(x)y, \tag{1.1}$$

on $a < x < b$, where

$$y(a) = y(b) = 0. \tag{1.2}$$

The existence and asymptotic behavior of the real eigenvalues of this problem has been treated elsewhere and we refer the interested reader to [**1, 3**] for details. We emphasize here that there are no sign restrictions on the coefficients q, w above. Of specific interest here is the existence or nonexistence of *nonreal* eigenvalues and their related eigenfunctions. This question dates back to the pioneering studies of Otto Haupt and Roland Richardson, see the survey paper [**5**] for these and other

1980 *Mathematics subject classification.* Primary 47H12, 45C05, Secondary 34B20.
Key words and phrases. Sturm-Liouville, Hammerstein, Krein space.
This research is partially funded by an NSERC-Canada grant.

historical references. In these studies the authors each claimed the possible existence of nonreal eigenvalues, although neither one gave an example of such an occurrence. For such an example, see [**6**].

It is essentially clear that since q, w are real-valued, the eigenfunctions corresponding to nonreal eigenvalues must necessarily be complex-valued and cannot be assumed to be real-valued (as is the case when the eigenvalues are real).

A basic question here is also the magnitude of a nonreal eigenvalue of (1.1-2). More specifically, we seek *a priori* estimates on the real/imaginary parts of such eigenvalues along with a solution of the more fundamental problem of their existence; in this respect, see [**1,7**] where this last question is treated in specific cases.

We show in this paper that the question of the existence of a nonreal eigenvalue of (1.1-2) is intimately related, actually equivalent, to the existence of a fixed point in a cone of a Krein space associated with a two-parameter nonlinear integral operator.

2. Basic results and terminology. A *Krein space* is a Hilbert space $(H, (\ ,\ ))$ on which there is a generally indefinite inner-product, $[\ \ ,\ \ ]$, which allows for a decomposition of H as

$$H = H^+[+]H^-$$

where $(H^+, [\ \ ,\ \ ])$, $(H^-, -[\ \ ,\ \ ])$, are Hilbert spaces and the spaces H^+, H^- are orthogonal with respect to $[\ \ ,\ \ ]$. The indefinite inner-product $[\ \ ,\ \ ]$ is then related to the Hilbert space inner product $(\ \ ,\ \ )$ via the Gram operator, J, where for $f, g \in H$,

$$[f, g] = (Jf, g). \tag{2.1}$$

Actually, $J = P_+ - P_-$ with $P_\pm$ being orthoprojectors on $H^\pm$, respectively. The Gram operator is a self-adjoint involution on H whose inverse is bounded as an operator on H. The norm of an element in a Krein space is understood to be its norm as an element of the Hilbert space. We refer to [**4**] for further information on Krein spaces and their operators. In the case under consideration, the Krein space is the weighted Lebesgue space

$$H \equiv L^2_w[a, b] = \{f : I \to \mathbf{C} | \int_a^b |f|^2 |w|\, dx < \infty\}$$

with the usual norm induced by the standard inner product (,) where

$$(f,g) = \int_a^b f\bar{g}|w|\,dx$$

while the indefinite inner product on H is now defined by

$$[f,g] = \int_a^b f\bar{g}w\,dx.$$

The relation (2.1) holds with J defined on H by

$$(Jf)(x) = (\operatorname{sgn} w(x))f(x),$$

that is, the Gram operator is simply multiplication by the signum functions, $\operatorname{sgn} w$, given as usual by $\operatorname{sgn} w(x) = +1, -1$, depending upon whether $w(x) > 0$ or $w(x) < 0$, respectively.

Note that the set C of nonnegative *isotropic vectors*, i.e., those fs for which $[f,f] = 0$, is a cone in H although it is not convex.

Next, by a solution of (1.1) is meant a function $f : I \to \mathbf{C}$ which is absolutely continuous along with f' and such that f satisfies (1.1) a.e. on I. It is readily shown using a quadratic form argument that any nonreal eigenfunction of (1.1-2) corresponding to a nonreal eigenvalue is an isotropic vector in H, i.e.,

$$\int_a^b |y|^2 w\,dx = 0.$$

3. The main result. We assume for simplicity that $\lambda = 0$ is not an eigenvalue of (1.1-2) and denote by $G(x,t)$ the corresponding Green function. This is not a severe restriction and it can always be assumed that $\lambda = 0$ is not an eigenvalue of (1.1-2), e.g., [**1, 6, 8**], a result which can be shown using Prüfer arguments.

Let $\lambda = \alpha + i\beta$, $\beta \neq 0$ be an eigenvalue of (1.1-2) and $y(x) = r(x)e^{i\theta(x)}$ a corresponding nonreal eigenfunction. This substitution has also been used by my colleague S.G. Halvorsen to treat such quantities. Here $r(x) \geq 0$ and $\theta(x)$ is an angular variable. It follows that $r(a) = r(b) = 0$ on account of (1.2).

Lemma. *For a given nonreal eigenvalue* $\lambda = \alpha + i\beta$, $\beta \neq 0$, *a corresponding eigenfunction* $y = re^{i\theta}$ *has no zeros in the interval* (a, b), *i.e.,* $r(x) > 0$ *for* $x \in (a, b)$.

Proof. For, assume, on the contrary, that $r(x_0) = 0$ for $a < x_0 < b$. Since $r \in C^1(a, b)$ and $r(x) \geq 0$, it follows that $r'(x_0) = 0$. Thus, $y(x_0) = y'(x_0) = 0$ and so $y \equiv 0$ by uniqueness. This contradiction proves the result. □

This lemma sharpens former results [**6**] and elucidates the numerical evidence for this phenomenon as reported in [**2**].

The corresponding equations for r and θ are now

$$-r'' + q(x)r = \alpha w(x)r - r\theta'^2 \tag{3.1}$$

and

$$r\theta'' + 2r'\theta' + \beta w(x)r = 0. \tag{3.2}$$

Note that $r(x) \geq 0$ and $r(a) = r(b) = 0$. Use of the integrating factor r in (3.2) readily gives

$$r^2(x)\theta'(x) = -\beta \int_a^x r^2 w \, dt. \tag{3.3}$$

Inserting (3.3) into (3.1), we obtain the nonlinear integrodifferential equation

$$-r'' + q(x)r = \alpha w(x)r + \frac{\beta^2}{r^3}\left(\int_a^x r^2 w \, dt\right)^2 \tag{3.4}$$

$$r(a) = r(b) = 0, \tag{3.5}$$

for $r = |y|$. Using Green's function $G(x, t)$ above, the last equation reduces to

$$r(x) = \alpha \int_a^b G(x,t)r(t)w(t)\,dt + \beta^2 \int_a^b G(x,t)r^{-3}(t)\left(\int_a^t r^2 w \, ds\right)^2 dt$$

or

$$r \equiv \alpha K r + \beta^2 N r$$

where K is compact as an operator on the Krein space H defined above [**7**], and N is a nonlinear integral operator of Hammerstein type. It follows that if $\lambda = \alpha + i\beta$, $\beta \neq 0$, is an eigenvalue of (1.1-2) with eigenfunction $y = re^{i\theta}$, then $r = |y|$ is a fixed point of the operator $\alpha K + \beta^2 N$. Such a fixed point is necessarily in the cone C

$$C = \{f \in H | f(x) > 0 \text{ for } x \in \text{int}\,(I),\ f(a) = f(b) = 0,\ [f, f] = 0\}$$

of the Krein space H.

On the other hand, if for some real pair α, β the operator $\alpha K + \beta^2 N$ has a fixed point r in C, then r, r' are absolutely continuous on I, r satisfies (3.4-5) a.e. on I and θ defined by (3.3) is absolutely continuous along with θ', and the resulting function $y = re^{i\theta}$ satisfies (1.1-2). We have proved the following result.

Theorem. *Let H denote the Krein space $L^2_w[a, b]$ endowed with the indefinite inner product $[\quad,\quad]$ defined above. Let C denote the* (*real*) *cone*

$$C = \{f \in H | f(x) > 0 \textit{ for } x \in \textit{int}\,(I),\ f(a) = f(b) = 0,\ [f, f] = 0\}.$$

Then the Sturm-Liouville problem (1.1-2) *has a nonreal eigenvalue $\alpha + i\beta$, $\beta \neq 0$, if and only if $\alpha K + \beta^2 N$ has a* (*nontrivial*) *fixed point in C.*

4. Concluding remarks. The operator N, viewed as an operator on the Krein space H is not compact. This is most easily seen by choosing f to be the characteristic function of the set E^+ defined at the outset and noting that $||Nf|| = \infty$; thus, N is unbounded as an operator with domain H. This operator remains unbounded even if we restrict its domain to the space of nonnegative continuous functions on I which vanish at a and b.

The question is now to determine a dense subspace of the Krein space on which N is bounded, if possible. It would also be useful if one had a Krein space version of the Krein-Rutman theorem from which existence results for fixed points of our operator might follow.

We can transport results from the linear problem to the nonlinear problem above [**5**]. For example, it follows from the results in [**6**] that there are at most finitely many pairs of real numbers $\{\alpha, \beta^2\}$ with $\beta \neq 0$ such that the operator $\alpha K + \beta^2 N$ has a unique fixed point in C. Thus, so long as $\beta \neq 0$, there can only be finitely many fixed points in C for a given problem. On the other hand, it follows from the results in [**1**] that if there is such a fixed point in C, for a given pair $\{\alpha, \beta^2\}$ with $\beta \neq 0$, then the operator K itself has no (nontrivial) fixed points in C. These considerations make the formulation of a general existence result for the fixed points of $\alpha K + \beta^2 N$ a nontrivial undertaking.

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