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Research Article

The High Contact Principle with Reward Functions Involving Initial Points

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This paper examines a class of general optimal stopping problems in which reward functions depend on initial points. Two points of view on the initial point are introduced: one is to view it as a constant, and the other is to view it as a constant process starting from the point. Based on the two different views, two versions of the generalized high contact principle are derived. Finally, we apply the generalized high contact principle to one example.

1. Introduction

The "high contact principle," first introduced by Samuelson [1] and McKean [2] and developed in greater depth by Øksendal [3] and Brekke and Øksendal [4], is a very useful tool to verify if a given function $h(x)$ is a solution to the following optimal stopping problem:

$$u(x) = \sup_{\tau} E\left[g\left(X_\tau^x\right)\right], \tag{1}$$

where g is a bounded continuously differentiable function in R^k, X_t^x is an Ito process satisfying the following stochastic differential equation:

$$dX_t = b\left(X_t\right)dt + \sigma\left(X_t\right)dB_t, \qquad X_0 = x, \tag{2}$$

$b : R^k \to R^k$ and $\sigma : R^k \to R^{k\times m}$ are Lipschitz continuous functions with at most linear growth, and B_t is an m-dimensional standard Brownian motion.

Generally, it is very hard to solve optimal stopping problem (1) directly. The usual way used in practice to finding a solution is to use the following two-step procedure. First, guess a solution based on some economic principles and common senses and second verify if it is indeed the solution using the high contact principle.

It is noted that the reward function g in optimal stopping problem (1) is assumed to be independent of the initial point x. That is, the existing high contact principle can only be applied to such a restricted case. However, there are many optimal stopping problems in economics and finance in which the reward functions do depend on the initial points.

More generally, a typical general optimal stopping problem has the following form:

$$u(x) = \sup_{\tau\ge 0} E\left[g\left(X_\tau^x, x\right)\right]. \tag{3}$$

The aim of this paper is to derive a generalized high contact principle which can be used to check if a given function h is a solution to (3). We use two approaches for deriving two versions of the generalized high contact principle: one is more practical from applied perspective and the other is more intuitive from theoretical perspective.

This paper is organized as follows. In Section 2, we review the three basic questions of the classical optimal stopping problems (1). Unfortunately, the answers to the same three basic questions for the general optimal stopping problems (3) are negative. Two counterexamples are provided in Section 3. Our main results, the two versions of the generalized high contact principle, are presented in Section 4. The application of the generalized high contact principle to one example is examined in Section 5. In the final section, Section 6, we discuss some remaining questions related to the general optimal stopping problems.

2. The Reward Functions Not Involving Initial Points

Following the Brekke and Øksendal approach [4], we begin this section with the three basic questions of the classical optimal stopping problem (1).

Question 1. If the optimal value u is known, can we find the continuation region D?

The answer is "yes." In this case, the continuation region D can be defined as follows:

$$D = \{x : u(x) > g(x)\}. \tag{4}$$

Thus, for any $x \in D$,

$$u(x) = E\left[g\left(X^x_{\tau_D}\right)\right], \tag{5}$$

where $\tau_D = \inf\{s > 0 : X^x_s \notin D\}$.

Question 2. If the continuation region D is known, can we find the optimal value u?

Again the answer is "yes." In this case, $u(x)$ is the solution to the following Dirichlet problem:

$$\begin{aligned} \mathscr{A}u(x) &= 0 \quad \text{for } x \in D \\ \lim_{x\to y} u(x) &= g(y) \quad \forall \text{regular } y \in \partial D, \end{aligned} \tag{6}$$

where $y \in \partial D$ is called regular if $\tau_D = 0$ a.s. P^x, $\mathscr{A}$ is the characteristic operator of X defined as follows:

$$\mathscr{A} = \sum_i b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad a = \sigma\sigma^T, \tag{7}$$

and b and σ are given in (2).

Question 3. If both the optimal value u and the continuation region D are not known, can we find them and how?

In this case, the optimal stopping is a problem of free boundary. Therefore, an additional boundary condition is needed to identify the boundary. The usual approach to identifying the extra boundary condition is the "high contact principle."

3. The Reward Functions Involving Initial Points

In this section, we consider a reward function g which depends not only on X_t but also on the initial point x. That is, $g = g(X^x_\tau, x)$.

Similarly, we ask the same three questions as in Section 2.

Question 4. If the optimal value u is known, can we define the continuation region the same as before? In other words, is

$$D = \{x : u(x) > g(x, x)\}? \tag{8}$$

Unfortunately, the answer is "no." Next is such a counterexample.

Example 1. Suppose that

$$u(x) = \sup_{\tau\ge 0} Eg\left(B^x_\tau, x\right), \tag{9}$$

where the reward function is

$$g\left(B^x_t, x\right) = e^{-(xB_t - a)^2} \tag{10}$$

and B_t denotes 1-dimensional Brownian motion starting at x; a is a constant $(x \ne a)$.

Obviously, $u(x) = 1$. If we follow the notion used in the classical case, then the "continuation region" D could be defined as follows:

$$D = \{x : u(x) > g(x, x)\}. \tag{11}$$

That is,

$$\begin{gathered} D = \left\{x : 1 > e^{-(x^2-a)^2}\right\} = \left\{x : x^2 \ne a\right\}, \\ \tau_D = \inf\left\{t \ge 0 : B_t = \pm\sqrt{a}\right\}. \end{gathered} \tag{12}$$

However,

$$u(x) \ne E\left[e^{-(xB_{\tau_D} - a)^2}\right]. \tag{13}$$

Therefore, the classical notion of the continuation region cannot be used here. In fact, the continuation region for this example is

$$D(x) = \left\{y : y \ne \frac{a}{x}\right\} \tag{14}$$

and the optimal stopping time, accordingly, is

$$\tau^* = \tau_D = \inf\left\{t \ge 0, B_t = \frac{a}{x}\right\}. \tag{15}$$

It is noted that this example shows that both the continuation region and the optimal stopping time may depend on initial point x. Thus, there does not exist a universal continuation region for every initial point x in general.

Question 5. If the continuation region D is known, is the optimal value $u(x)$ a solution to the following Dirichlet problem for $x \in D$? We have

$$\begin{aligned} \mathscr{A}u(x) &= 0 \quad \text{for } x \in D \\ \lim_{x\to y} u(x) &= g(y, y) \quad \forall \text{regular } y \in \partial D, \end{aligned} \tag{16}$$

where $\mathscr{A}$ is the characteristic operator of X defined in (7).

Unfortunately, it is not true either. Next is such a counterexample.

Example 2. Suppose that

$$u(x) = \sup_{\tau\ge 0} Eg\left(X^x_\tau, x\right), \tag{17}$$

where the reward function is

$$g\left(X_t^x, x\right) = X_t + x^2, \tag{18}$$

X_t satisfies

$$dX_t = X_t dB_t, \quad X_0 = x > 0, \tag{19}$$

and B_t denotes 1-dimensional standard Brownian motion.

Because $\{X_t\}_{t\geq 0}$ is a nonnegative martingale, then $u(x) = x + x^2$. The characteristic operator of process X_t is

$$\mathscr{A}(\cdot) = \frac{1}{2}x^2\frac{\partial^2(\cdot)}{\partial x^2}, \tag{20}$$

and $\mathscr{A}u(x) = x^2 > 0$. Therefore, $u(x)$ is not a solution of the above Dirichlet problem.

Question 6. If both the optimal value u and the continuation region D are not known, can the classical high contact principle be used to solve for u?

Clearly, the classical high contact principle stated in [4] cannot directly be used when the reward function depends on initial point due to the negative answers to the first two questions (Questions 4 and 5).

To solve the general optimal stopping problem (3), a new high contact principle is needed. We call it the generalized high contact principle which will be stated and proved in next section.

4. The Main Results

We will state and prove our main results in this section. To this end, we fix a domain G (the "solvency" set) in R^k. Let

$$\tau_G = \inf\left\{t > 0 : X_t \notin G\right\} \quad \text{(the bankruptcy time)}, \tag{21}$$

where the process X_t satisfies (2).

Let χ denote the set of all stopping times $\tau \leq \tau_G$. Consider the following problem: find $u(x)$ and $\tau^* \in \chi$ such that

$$u(x) = \sup_{\tau\in\chi} E\left[g\left(X_\tau^x, x\right)\right] = E\left[g\left(X_{\tau^*}^x, x\right)\right], \tag{22}$$

where $g(\cdot, x)$ is a bounded continuously differentiable function in R^k. For simplicity we also assume that the diffusion process X_t in (2) satisfies $\sum a_{i,j}\zeta_i\zeta_j \geq \delta\sum\zeta_i^2 (\delta > 0)$ with $a = [a_{i,j}] = (1/2)\sigma\sigma^T$.

Before stating the generalized high contact principle, let us give a heuristic argument for the optimal stopping problem. The argument is fairly intuitive, informal, and quite constructive.

Given x, let us define a new function

$$v_x(y) = \sup_{\tau\in\chi} E\left[g\left(X_\tau^y, x\right)\right], \tag{23}$$

where E denotes the expectation with respect to the law P^y of X_t given that $X_0 = y$. Obviously, $u(x) = v_x(x)$. For function $v_x(\cdot)$, first of all, we can see that $v_x(x) \geq g(x, x)$. Next we consider a small number $\delta > 0$:

$$v_x(x) \geq E\left[g\left(X_\tau^x, x\right)\right], \quad \text{for } \delta \leq \tau \in \chi. \tag{24}$$

Accordingly,

$$\begin{aligned} v_x(x) &\geq \sup_\tau\left\{E\left[g\left(X_\tau^x, x\right)\right] : \delta \leq \tau \in \chi\right\} \\ &= \sup_{\tau'}\left\{E\left[g\left(X_{\tau'}^{X_\delta^x}, x\right)\right] : 0 \leq \tau' \in \chi\right\} \\ &= Ev_x\left(X_\delta^x\right), \end{aligned} \tag{25}$$

where the first equality comes from the strong Markov property of the Ito diffusion process. That is,

$$\begin{aligned} E\left[g\left(X_{\tau'}^{X_\delta^x}, x\right)\right] &= E\left[g\left(X_{\tau-\delta}^{X_\delta^x,0}, x\right)\right] \\ &= E\left\{E\left[g\left(X_{\tau-\delta}^{y,0}, x\right)\right]_{y=X_\delta^x}\right\} \\ &= E\left\{E\left[g\left(X_\tau^{y,\delta}, x\right)\right]_{y=X_\delta^x}\right\} \\ &= E\left[g\left(X_\tau^x, x\right)\right]. \end{aligned} \tag{26}$$

From the stochastic analysis

$$X_\delta^x \cong x + \delta b(x) + \sigma(x) B_\delta. \tag{27}$$

Therefore, we have (approximately)

$$v_x(x) \geq Ev_x\left[x + \delta b(x) + \sigma(x) B_\delta\right]. \tag{28}$$

Assuming that $v_x(\cdot)$ is smooth enough, by Taylor's formula and the properties of Brownian motion process,

$$\begin{aligned} v_x(x) &\geq v_x(x) + \delta\sum_i b_i(x)\partial_i v_x(x) \\ &\quad + \frac{1}{2}\sum_{i,j} E[\sigma(x)B_\delta]_i[\sigma(x)B_\delta]_j\partial_{ij}^2 v_x(x) + o(\delta) \\ &= v_x(x) + \delta\sum_i b_i(x)\partial_i v_x(x) \\ &\quad + \frac{\delta}{2}\sum_{i,j} a_{ij}(x)\partial_{ij}^2 v_x(x) + o(\delta). \end{aligned} \tag{29}$$

Consequently after cancellation and letting $\delta \to 0$, we get

$$\sum b_i(x)\partial_i v_x(x) + \frac{1}{2}a_{ij}(x)\partial_{ij}^2 v_x(x) \leq 0. \tag{30}$$

In this case, on the one hand, it is similar to the variational inequality in the classical case; this suggests to us that this problem can be solved by the "high contact principle"; on the other hand, it is different from the classical case, because x in the reward function is fixed.

Next is our key result which rigorously justifies the above idea.

Theorem 3. *Let X_t, g be described as at the beginning of this section. For any given initial point x, if there exist an open set $D(x) \subset R^k$ with C^1-boundary and a function $q_x(y)$ on $\overline{D}(x)$*

such that (i) $q_x(\cdot) \in C^1(\overline{D}(x)) \cap C^2(D(x)), g(\cdot, x) \in C^2(G \setminus \overline{D}(x))$,

$$q_x(y) \geq g(y, x), \quad \forall y \in D(x),$$
$$\mathscr{A} g(y, x) \leq 0, \quad \forall y \in (G \setminus \overline{D}(x)); \tag{31}$$

and (ii) (the second order condition) $(D(x), q)$ *solves the following free boundary problem:*

$$\mathscr{A} q_x(y) = 0, \quad \forall y \in D(x),$$
$$q_x(y) = g(y, x), \quad \forall y \in \partial D(x), \tag{32}$$
$$\nabla_y q_x(y) = \nabla_y g(y, x), \quad \forall y \in \partial D(x) \cap G,$$

then extending $q_x(y)$ *to all of* G *by putting* $q_x(y) = g(y, x)$ *for* $y \in G \setminus D(x)$*, one has*

$$q(x) = u(x) = \sup_{\tau \in \chi} E\left[g(X_\tau^x, x)\right] = E\left[g(X_{\tau^*}^x, x)\right], \tag{33}$$

where $q(x) = q_x(x)$,

$$\tau^*(x) = \begin{cases} \tau_{D(x)} = \inf\{s \geq 0 : X_s^x \notin D(x)\}, & \textit{if } x \in D(x), \\ 0, & \textit{otherwise}, \end{cases}$$
$$D(x) = \{y : q_x(y) > g(y, x)\}. \tag{34}$$

Proof. Using the generalized Dynkin formula in [4] for the diffusion process X_t^y, we have

$$E\left[q_x\left(X_{\tau_V}^y\right)\right] = q_x(y) + E\int_0^{\tau_V} \mathscr{A} q_x\left(X_u^y\right) du \tag{35}$$

for all $y \in G$, where τ_V is the first exit time of $V \subset G$.

According to conditions (i) and (ii) of $\mathscr{A} q_x(\cdot)$, we get

$$E\left[q_x\left(X_{\tau_V}^y\right)\right] \leq q_x(y), \quad \forall y \in G, \tag{36}$$

and combine it with Theorem 12.16 in [5]; then, $q_x(\cdot)$ is a superharmonic function. Owing to the condition that $q_x(y) \geq g(y, x)$, so $q_x(\cdot)$ is a superharmonic majorant of g. We know that $v_x(y)$ in (23) is indeed the least superharmonic majorant of g. So

$$q_x(y) \geq v_x(y), \quad \forall y \in G. \tag{37}$$

When $y = x, q_x(x) \geq u(x)$.

To the opposite inequality, we note by condition (ii) that

$$q_x(y) = E\left[g\left(X_{\tau_{D(x)}}^y, x\right)\right], \tag{38}$$

where $\tau_{D(x)} = \inf\{s > 0 : X_s^y \notin D(x)\}$ and $D(x) = \{y : q_x(y) > g(y, x)\}$. From the definition of the $v_x(y)$, then

$$q_x(y) \leq v_x(y), \quad \forall y \in G. \tag{39}$$

When $y = x, q_x(x) \leq u(x)$. Thus, we get (33) for all $x \in G$ as well as τ^* defined in (34). □

It is noted that Theorem 3 tells us how to solve the general optimal stopping problem. First, fix x in the reward function and change the initial point of Ito diffusion process from x to y; then, the optimal value function becomes a new function $v_x(y)$ as (23) and get the form of $v_x(y)$ in terms of the conditions (i) and (ii); second, find $u(x)$ by letting $y = x$ in $v_x(y)$ and the optimal stopping time defined in (34).

Similar to [6], the case in Theorem 3 can be generalized to the time-inhomogeneous case involving an integral.

So far we have introduced our first point of view on the original initial point x, that is, to view it as a parameter or a constant. Next, we will introduce our second point of view on it. That is, instead of viewing it as a constant, we will view it as a process starting from x.

To this end, we change the initial point of the Ito diffusion process X_t from x to y:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad X_0 = y, \tag{40}$$

and define a constant Ito diffusion process Z_t as follows:

$$dZ_t = 0, \qquad Z_0 = x. \tag{41}$$

Put them together and we have a new Ito diffusion process $Y_t = Y_t^{(y,x)}$ in R^{2k} by

$$dY_t = \begin{pmatrix} b(X_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} \sigma(X_t) \\ 0 \end{pmatrix} dB_t = \widehat{b}(Y_t)\,dt + \widehat{\sigma}(Y_t)\,dB_t, \tag{42}$$

where

$$\widehat{b}(\eta) = \widehat{b}(\xi, x) = \begin{pmatrix} b(\xi) \\ 0 \end{pmatrix} \in R^{2k\times 1},$$
$$\widehat{\sigma}(\eta) = \widehat{\sigma}(\xi, x) = \begin{pmatrix} \sigma(\xi) \\ 0 \end{pmatrix} \in R^{2k\times m} \tag{43}$$

with $\eta = (\xi, x) \in R^k \times R^k$.

Thus, Y_t is an Ito diffusion process starting at (y, x). Let $P^{(y,x)}$ denote the probability law of Y_t and let $E^{(y,x)}$ denote the expectation with respect to $P^{(y,x)}$. In terms of Y_t, (22) can be rewritten as

$$u(x) = u(x, x) = \sup_{\tau \in \chi} E^{(x,x)}\left[g(X_\tau, x)\right]$$
$$= \sup_{\tau \in \chi} E^{(x,x)}\left[g(Y_\tau)\right] = E^{(x,x)}\left[g(Y_{\tau^*})\right] \tag{44}$$

which is a special case of the problem

$$u(y, x) = \sup_{\tau \in \chi} E^{(y,x)}\left[g(X_\tau, x)\right]$$
$$= \sup_{\tau \in \chi} E^{(y,x)}\left[g(Y_\tau)\right] = E^{(y,x)}\left[g(Y_{\tau^*})\right], \tag{45}$$

where the optimal stopping time associated with $u(y, x)$ is denoted by τ^*.

From the above, we have the second version of the generalized high contact principle.

Theorem 4. *Let g be described as the above, let X_t be described as in (40), and let $G \subset R^k$ be an open set and denote $W = G\times G$. For any given initial point (y,x), if there exist an open set $D \subset R^{2k}$ with C^1-boundary and a function $\phi(y,x)$ on $\overline{D}$ such that (i) $\phi(\cdot,\cdot) \in C^1(\overline{D}) \cap C^2(D)$, $g(\cdot,\cdot) \in C^2(W \setminus \overline{D})$,*

$$\begin{aligned} &\phi(y,x) \geq g(y,x), \quad \forall (y,x) \in D, \\ &\mathscr{A} g(y,x) \leq 0, \quad \forall (y,x) \in \left(G \setminus \overline{D}\right); \end{aligned} \tag{46}$$

and (ii) (the second order condition) (D,ϕ) solves the following free boundary problem:

$$\begin{aligned} &\mathscr{A}\phi(y,x) = 0, \quad \forall (y,x) \in D, \\ &\phi(y,x) = g(y,x), \quad \forall (y,x) \in \partial D, \\ &\nabla_y \phi(y,x) = \nabla_y g(y,x), \quad \forall (y,x) \in \partial D \cap W, \end{aligned} \tag{47}$$

then extending $\phi(y,x)$ to all of W by putting $\phi(y,x) = g(y,x)$ for $(y,x) \in W \setminus D$ and letting $y = x$, one has

$$\phi(x,x) = u(x) = \sup_{\tau\in\chi} E^{(x,x)}\left[g(X_\tau, x)\right] = E^{(x,x)}\left[g(X_{\tau^*}, x)\right], \tag{48}$$

where

$$\begin{aligned} &\tau^* = \begin{cases} \tau_D = \inf\{s \geq 0; (X_s^x, x) \notin D\}, & \textit{if } (x,x) \in D, \\ 0, & \textit{otherwise}, \end{cases} \\ &D = \{(y,x) : \phi(y,x) > g(y,x)\}. \end{aligned} \tag{49}$$

Proof. Because X_t is uniformly elliptic in G and by conditions (i), (ii), and $(iii)'$ of Theorem 3 in [4], we get

$$\begin{aligned} \phi(y,x) = u(y,x) &= \sup_{\tau\in\chi} E^{(y,x)}\left[g(X_\tau, x)\right] \\ &= E^{(y,x)}\left[g(Y_{\tau^*})\right], \quad \forall (y,x) \in W, \end{aligned} \tag{50}$$

where

$$\begin{aligned} &\tau^* = \begin{cases} \tau_D = \inf\{s \geq 0; (X_s^y, x) \notin D\}, & \text{if } (y,x) \in D, \\ 0, & \text{otherwise}, \end{cases} \\ &D = \{(y,x) : \phi(y,x) > g(y,x)\}. \end{aligned} \tag{51}$$

Let $y = x$, and we get the result. □

We will end this section with a comparison of the two versions of the generalized high contact principle. In Theorem 4 we view x in the reward function as a degenerate diffusion process and then the problem becomes the classical case with two diffusion processes. This method is much easier to understand theoretically, but it leads to double the number of the dimensions of the domain of the test function and accordingly the continuation region. Generally, the higher the dimension of the domain is, the more difficult it is to guess the solution. Thus, the first version of the generalized high contact principle, Theorem 3, is much easier to use than the second version of the generalized high contact principle, Theorem 4.

5. One Application

In this section, we apply the first version of generalized high contact principle to one example: the perpetual American put.

Example 5 (the perpetual American put). Suppose a stock value process is as follows:

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad S_0 = x, \tag{52}$$

where B_t denotes 1-dimensional standard Brownian motion and r, σ are constant.

The perpetual American put option is a contract signed at time 0 which entitles the buyer to sell one unit of the stock at any time $t \geq 0$ at a price $K = f(x) > 0$ (the exercising price is a function of x). So his discounted reward is $e^{-rt}(f(x) - X_t)^+$ if exercising the option at time t. To determine such an optimal striking time τ^*, we must solve the following optimal stopping problem:

$$\begin{aligned} u(x) &= \sup_{\tau\in\mathscr{T}} E\left[e^{-r\tau}\left(f(x) - S_\tau^x\right)^+ I_{\{\tau<\infty\}}\right] \\ &= E\left[e^{-r\tau^*}\left(f(x) - S_{\tau^*}^x\right)^+ I_{\{\tau^*<\infty\}}\right], \end{aligned} \tag{53}$$

where $\mathscr{T} = \{$all stopping times that take value in $[0,\infty)\}$.

It is noted that the reward function

$$e^{-r\tau}\left(f(x) - S_\tau^x\right)^+ I_{\{\tau<\infty\}} \tag{54}$$

depends on the initial point x. Thus, the classical high contact principle cannot be used to solve optimal stopping problem (53).

Because no point can be exercised if $S_\tau^x > f(x)$,

$$u(x) = \sup_{\tau\in\mathscr{T}} E\left[e^{-r\tau}\left(f(x) - S_\tau^x\right) I_{\{\tau<\infty\}}\right]. \tag{55}$$

First we fix x in the reward function and change the initial value of the diffusion process to y; that is,

$$v_x(y) = \sup_{\tau\in\mathscr{T}} E\left[e^{-r\tau}\left(f(x) - S_\tau^y\right) I_{\{\tau<\infty\}}\right]. \tag{56}$$

From direct calculation, we get

$$v_x(y) = \begin{cases} \left(f(x) - y^*\right)\left(\dfrac{y}{y^*}\right)^{-(2r/\sigma^2)}, & \forall y \geq y^*, \\ f(x) - y, & \text{otherwise}, \end{cases} \tag{57}$$

where $y^* = (2r/(2r+\sigma^2))f(x)$. Let $y = x$, and we get the solution

$$\begin{aligned} &u(x) = \begin{cases} \left(f(x) - y^*\right)\left(\dfrac{x}{y^*}\right)^{-(2r/\sigma^2)}, & \forall x \geq y^*, \\ f(x) - x, & \text{otherwise}, \end{cases} \\ &\tau^*(x) = \begin{cases} \tau_D = \inf\{s \geq 0 : X_s^x \leq y^*\}, & \text{if all } x \geq y, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{58}$$

where $y^* = (2r/(2r+\sigma^2))f(x)$.

6. Some Remaining Problems

We will end the paper with some comments on some remaining questions.

It is well known that it is very hard to solve a typical partial differential equation in general. However, as we know, function $u(x) = E[g(X_t^x)]$ is the solution of the following partial differential equation:

$$\mathscr{A}u = 0. \tag{59}$$

That is, solving the classical optimal stopping problem (1) provides a method for solving a class of partial differential equations (59).

Moreover, (59) plays a key role in the "high contact principle" since it sets up the connection between optimal stopping problem and free boundary problem. Equation (59) provides a general method for guessing a solution to the classical optima stopping problem (1) since the solution to (1) must be a solution to (59).

Likewise, we also want to know what kind of partial differential equation $u(x) = E[g(X_t^x, x)]$ should satisfy. Unfortunately, we have not succeeded in finding such a PDE yet. Similar to the classical case, there are also two reasons why we are interested in the PDE. One is that it provides a method for guessing a solution candidate to the general optimal stopping problem (3) since the solution to (3) must be a solution to the PDE if it exists. The other one is that solving the general optimal stopping (3) would provide a method for solving a class of such PDEs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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