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Asymptotic expansion for some local volatility models arising in finance

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Abstract
In this paper, we study the small noise asymptotic expansions for certain classes of local volatility models arising in finance. We provide explicit expressions for the involved coefficients as well as accurate estimates on the remainders. Moreover, we perform a detailed numerical analysis, with accuracy comparisons, of the obtained results by means of the standard Monte Carlo technique as well as exploiting the Polynomial Chaos Expansion approach.

Keywords Local volatility models · Small noise asymptotic expansions · Corrections to the Black–Scholes type models · Jump-diffusion models · Polynomial drift · Exponential drift · Polynomial Chaos Expansion method · Monte Carlo techniques

JEL Classification C02 · C33 · C63 · C65 · E43 · E47 · G11 · G12

1 Introduction

In the present paper, we shall provide small noise asymptotic expansions for some local volatility models (LVMs) arising in finance. Our approach is based on the rigorous

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results on asymptotic expansions for solutions of finite-dimensional SDE's obtained in Albeverio and Smii (2013) [following the approach proposed in Gardiner (2004, Sect. 6.2)]; some extensions to a class of SPDE's and infinite-dimensional SDE's have been presented in Albeverio et al. (2011, 2016a, b). In particular, we consider underlyings whose behavior is characterized by a stochastic volatility term of *small* amplitude ϵ with respect to which we perform a formal, based on Gardiner (2004, Sect. 6.2), resp. asymptotic, based on Albeverio and Smii (2013), expansion. The latter implies that the equation characterizing the particular LVM of interest is approximated by a finite recursive system of a number N of linear equations with random coefficients. We then exploit the solutions of the latter system to provide a formal, resp. an asymptotic, approximation of smooth functions of the original solution for the particular LVM of interest. In a similar way, we derive the corresponding approximation for the expected value of the related option price in a *risk- neutral setting*. Errors estimates and explicit expressions for the involved approximations are also provided for some specific cases, together with a detailed numerical analysis.

We would like to recall that LVMs are commonly used to analyze options markets where the underlying volatility strongly depends on the level of the underlying itself. Let us mention that although time homogeneous local volatilities are supposedly inconsistent with the dynamics of the equity index implied volatility surface, see, e.g., Mandelbrot et al. (2004), some authors, see, e.g., Crepey (2004), claim that such models provide the best average hedge for equity index options.

Let us also note that, particularly during recent years, different asymptotic expansions approaches to other particular problems in mathematical finance have been developed, see, e.g., Andersen and Lipton (2012), Bayer and Laurence (2014), Benarous and Laurence (2013), Benhamou et al. (2009), Breitung (1994), Cordoni and Di Persio (2015), Fouque et al. (2000), Friz et al. (2015), Fuji and Akihiko (2012), Gatheral et al. (2012), Gulisashvili (2012), Kusuoka and Yoshida (2000), Lütkebohmert (2004), Shiraya and Takahashi (2017), Takahashi and Tsuzuki (2014), Uchida and Yosida (2004) and Yoshida (2003), see also Albeverio et al. (2012), Imkeller et al. (2009), Peszat and Russo (2005) for applications of similar expansion to other areas.

The present paper is organized as follows: in Sect. 2, the basic general asymptotic expansion approach based on Albeverio and Smii (2013), is presented. Then, in Sect. 3 we apply the aforementioned results to important examples in financial mathematics. In particular, in Sect. 3 we study a perturbation up to the first order around the Black–Scholes model as well as a correction with jumps for the case of a generic smooth volatility function f. We then give more detailed results for the case of an exponential volatility function f, in Sect. 3.1 with Brownian motion driving, in Sect. 3.2 with an additional jump term. In Sect. 3.3, we shall present detailed corresponding results for the case of a polynomial volatility function f; in Sect. 3.4, we treat the case of corrections for f being a polynomial and the noise containing jumps. To validate our expansions, we present their numerical implementations obtained by exploiting the *Polynomial Chaos Expansion* approach as well as the standard *Monte Carlo* technique, also providing a detailed comparison between the two implementations in terms of accuracy.

2 The asymptotic expansion

2.1 The general setting

We shall consider the following stochastic differential equation (SDE), indexed by a parameter $\epsilon \geq 0$

$$\begin{cases} \mathrm{d}X_t^\epsilon = \mu^\epsilon\left(X_t^\epsilon\right)\mathrm{d}t + \sigma^\epsilon\left(X_t^\epsilon\right)\mathrm{d}L_t, \\ X_0^\epsilon = x_0^\epsilon \in \mathbb{R}, \quad t \in [0,\infty) \end{cases} \tag{1}$$

where $L_t, t \in [0,\infty)$, is a real-valued, d-dimensional, Lévy process of jump-diffusion type, subject to some restrictions which will be specified later on and $\mu^\epsilon : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma^\epsilon : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are Borel measurable functions for any $\epsilon \geq 0$ satisfying some additional technical conditions in order to have existence and uniqueness of strong solutions, e.g., locally Lipschitz and sublinear growth at infinity, see, e.g., Applebaum (2009), Arnold (1974), Mandrekar and Rüdiger (2015), Gihman and Skorokhod (1972), Imkeller et al. (2009) and Shreve (2004). If the Lévy process L_t has a jump component, then X_t^ϵ in Eq. (1) has to be understood as $X_{t-}^\epsilon := \lim_{s\uparrow t} X_s^\epsilon$, see, e.g., Mandrekar and Rüdiger (2015) for details.

Hypothesis 1 *Let us assume that:*

(i) $\mu^\epsilon, \sigma^\epsilon \in C^{k+1}(\mathbb{R}^d)$ *in the space variable, for any fixed value* $\epsilon \geq 0$ *and for all* $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$*;*

(ii) *the maps* $\epsilon \mapsto \alpha^\epsilon(x)$*, where* $\alpha = \mu, \sigma$*, are in* $C^M(I)$ *in* ϵ*, for some* $M \in \mathbb{N}$*, for every fixed* $x \in \mathbb{R}$ *and where* $I := [0, \epsilon_0]$*,* $\epsilon_0 > 0$.

Our goal is to show that under Hypothesis 1 and some further smoothness conditions on μ^ϵ and σ^ϵ (needed for the construction of the random coefficients X_t^i, $i = 0, 1, \ldots, N$ appearing in (2)), a solution X_t^ϵ of Eq. (1) can be represented as a power series with respect to the parameter ϵ, namely

$$X_t^\epsilon = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + \cdots + \epsilon^N X_t^N + R_N(t,\epsilon), \tag{2}$$

where $X^i : [0,\infty) \to \mathbb{R}$, $i = 0, \ldots, N$, are continuous functions, while $|R_N(t,\epsilon)| \leq C_N(t)\epsilon^{N+1}$, $\forall N \in \mathbb{N}$ and $\epsilon \geq 0$, for some $C_N(t)$ independent of ϵ, but in general dependent of randomness, through $X_t^0, X_t^1, \ldots, X_t^N$. For $n \in \mathbb{N}$, the functions X_t^i are determined recursively as solutions of random differential equations in terms of the X_t^j, $j \leq i-1$, $\forall i \in \{1, \ldots, N\}$.

Before giving the proof of the validity of the expression in Eq. (2), let us recall the following result, see, e.g., Giaquinta and Modica (2000).

Lemma 1 *Let* f *be a real (resp. complex)-valued function in* $C^{M+1}(B(x_0,r))$*,* $r > 0$*,* $x_0 \in \mathbb{R}$ *for some* $M \in \mathbb{N}_0$*, where* $(B(x_0,r)$ *denotes the ball of center* x_0 *and radius* r.

Then, for any $x \in B(x_0, r)$ the following Taylor expansion formula holds

$$f(x) = \sum_{p=0}^{M} \frac{D^p f(x_0)}{p!}(x - x_0)^p + R_M\left(D^{M+1} f(x_0, x)\right),$$

with $D^p f(x_0) := D^p f(x)|_{x=x_0}$ the pth derivative at x_0 and

$$R_M\left(f^{(M+1)}(x_0, x)\right) := (x - x_0)^{M+1} C_M(x_0, x),$$

with

$$C_M(x_0, x) := \frac{M+1}{(M+1)!}\int_0^1 (1-s)^M D^{M+1} f(x_0 + s(x - x_0))\mathrm{d}s.$$

We have

$$\begin{aligned} |C_M(x_0, x)| &\leq \frac{M+1}{(M+1)!}\int_0^1 (1-s)^M \sup_{x \in B(x_0,r)} |D^{M+1} f(x_0 + s(x - x_0))|\mathrm{d}s \\ &=: \tilde{C}_M(x_0) < +\infty \end{aligned} \tag{3}$$

and also

$$|R_M\left(f^{(M+1)}(x_0, x)\right)| \leq |C_M(x, x_0)||x - x_0|^{M+1} \leq \tilde{C}_M(x_0)|x - x_0|^{M+1}, \ M \in \mathbb{N}_0.$$

With this lemma in mind, let us then consider a function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, and $f_\epsilon(x) := f(\epsilon, x)$, $\epsilon \geq 0$, $x \in \mathbb{R}$. If we then suppose that for any fixed $x \in \mathbb{R}$, f is of class $C^{K+1}(I)$ in ϵ for some $K \in \mathbb{N}_0$, $I = [0, \epsilon_0]$, $\epsilon_0 > 0$, we can write the Taylor expansion of f around $\epsilon = 0$, w.r.t. $\epsilon \in I$ for any fixed $x \in \mathbb{R}$, as follows:

$$f_\epsilon(x) = \sum_{j=0}^{K} f_j(x)\epsilon^j + R_K^{f_\epsilon}(\epsilon, x), \tag{4}$$

where f_j is the jth coefficient in the expansion provided by Lemma 1, while $\sup_x |R_K^{f_\epsilon}(\epsilon, x)| \leq C_{K,f}\epsilon^{K+1}$ for some $C_{K,f} > 0$, independent of ϵ. Assume in addition that $x \mapsto f_j(x)$ are in C^{M+1}, $j = 0, \ldots, K$, for some $M \in \mathbb{N}_0$, then, applying Lemma 1 to the function f_j in $B(x_0, r)$, $r > 0$, we obtain

$$f_\epsilon(x) = \sum_{j=0}^{K} \epsilon^j \left[\sum_{\gamma=0}^{M} \frac{D^\gamma f_j(x_0)}{\gamma!}(x - x_0)^\gamma + R_M(f_j^{(M+1)}(x_0, x))\right] + R_K^{f_\epsilon}(\epsilon, x), \tag{5}$$

with $R_M(f_j^{(M+1)}(x_0, x))$ estimated as in Lemma 1 (with f_j replacing f) and $R_K^{f_\epsilon}(\epsilon, x)$ as in (4).

Let us now take $x = x(\epsilon)$ assuming $\epsilon \mapsto x(\epsilon)$ in C^{N+1}, with $0 \le \epsilon \le \epsilon_0$, $0 < \epsilon_0 < 1$ and $x(0) = x_0 \in \mathbb{R}$. Then, by Lemma 1

$$x(\epsilon) = \sum_{j=0}^{N} \epsilon^j x_j + R_N^x(\epsilon), \quad N \in \mathbb{N}_0, \quad x_j \in \mathbb{R}, \, j = 0, 1, \ldots, N, \tag{6}$$

with f replaced by x, M replaced by N, x by ϵ, x_0 by 0 and $R_M(f^{(M+1)}(x_0, x))$ by $R_N^x(\epsilon)$. In particular,

$$|R_N^x(\epsilon)| \le \tilde{C}_N(0)\epsilon^{N+1}, \tag{7}$$

with $\tilde{C}_N(0)$ independent of ϵ.

Plugging (6) into (5), we get

$$\begin{aligned} f_\epsilon(x(\epsilon)) &= \sum_{j=0}^{K} \epsilon^j \left[\sum_{\gamma=0}^{M} \frac{D^\gamma f_j(x_0)}{\gamma!} (x(\epsilon) - x_0)^\gamma + R_M\left(f_j^{(M+1)}(x_0, x(\epsilon))\right) \right] + R_K^{f_\epsilon}(\epsilon, x(\epsilon)) \\ &= \sum_{j=0}^{K} \epsilon^j \left[\sum_{\gamma \le M} \frac{D^\gamma f_j(x_0)}{\gamma!} \left(\sum_{k=1}^{N} \epsilon^k x_k + R_N^x(\epsilon) \right)^\gamma + R_M\left(f_j^{(M+1)}(x_0, x(\epsilon))\right) \right] \\ &\quad + R_K^{f_\epsilon}(\epsilon, x(\epsilon)). \end{aligned} \tag{8}$$

The estimates on R_M, $R_K^{f_\epsilon}$ and R_N^x have been given in Lemma 1, resp., after (4), resp. (7).

By Newton's formula, we have that, $\forall\, \gamma \in \mathbb{N}_0$, the following holds

$$\left(\sum_{j=1}^{N} \epsilon^j x_j + R_N^x(\epsilon) \right)^\gamma = \sum_{*}^{\gamma} \frac{\gamma!}{\gamma_1! \ldots \gamma_{N+1}!} \epsilon^{\gamma_1 + 2\gamma_2 + \cdots + N\gamma_N} x_1^{\gamma_1} \ldots x_N^{\gamma_N} (R_N^x(\epsilon))^{\gamma_{N+1}}, \tag{9}$$

where we have used the notation

$$\sum_{*}^{\gamma} = \sum_{\substack{\gamma_1, \ldots, \gamma_{N+1} = 0 \\ \gamma_1 + 2\gamma_2 + \cdots + N\gamma_N + \gamma_{N+1} = \gamma}}^{\gamma};$$

hence, using (9) to rewrite (8) we obtain the following.

Lemma 2 *If, for $0 \le \epsilon < \epsilon_0$, $\epsilon \mapsto x(\epsilon)$ is in $C^{N+1}(I)$, $I = [0, \epsilon_0]$, and $\epsilon \mapsto f_\epsilon(y)$ is $C^{K+1}(\mathbb{R})$ in $\epsilon \in I$ and for any $y \in \mathbb{R}$, $y \mapsto f_\epsilon(y)$ is in C^{M+1}, the following expansion in powers of ϵ holds:*

$$\begin{aligned} f_\epsilon(x(\epsilon)) = \sum_{j=0}^{K} \epsilon^j \Bigg[&\sum_{\gamma=0}^{M} \frac{D^\gamma f_j(x_0)}{\gamma!} \sum_{*}^{\gamma} \frac{\gamma!}{\gamma_1! \ldots \gamma_{N+1}!} \epsilon^{\gamma_1 + 2\gamma_2 + \cdots + N\gamma_N} x_1^{\gamma_1} \ldots x_N^{\gamma_N} (R_N^x(\epsilon))^{\gamma_{N+1}} \\ &+ R_M\left(f_j^{(M+1)}(x_0, x(\epsilon))\right) \Bigg] + R_K^{f_\epsilon}(\epsilon, x(\epsilon)), \end{aligned} \tag{10}$$

The estimates for the remainders are as follows:

$$|R_N^x(\epsilon)| \le \tilde{C}_N(0)\epsilon^{N+1},$$
$$R_M\left(f_j^{(M+1)}(x_0, x(\epsilon))\right) \le \tilde{C}_M(x_0)|x - x_0|^{M+1},$$
$$\sup_{x,\epsilon} |R_K^{f_\epsilon}(\epsilon, x)| \le C_{K,f},$$

with $\tilde{C}_N(0)$*,* $\tilde{C}_M(x_0)$ *and* $C_{K,f}$ *independent of* ϵ*.*

Taking Eq. (10) into account, we can group all the terms with the same power $k \in \mathbb{N}_0$ of ϵ. Calling $[f_\epsilon(x(\epsilon))]_k$ the coefficient of ϵ^k and using $k = j + \gamma$ with $j = 0, \ldots, K$, $\gamma_1 + 2\gamma_2 + \cdots + N\gamma_N = \gamma$ with $\gamma = 0, \ldots, M$, we have the following, see, Albeverio and Smii (2013).

Proposition 1 *Let* $x(\epsilon)$ *be as in* (6)*; let* f_ϵ *as in* (4) *with* $f_j \in C^{M+1}$*,* $j = 0, \ldots, K$*. Then,*

$$f_\epsilon(x(\epsilon)) = \sum_{k=0}^{K+M} \epsilon^k \left[f_\epsilon(x(\epsilon))\right]_k + R_{K+M}(\epsilon),$$

with $|R_{K+M}(\epsilon)| \le C_{K+M}\epsilon^{K+M+1}$*, for some constant* $C_{K+M} \ge 0$*, independent of* ϵ*,* $0 \le \epsilon \le \epsilon_0$*, and coefficients* $[f_\epsilon(x(\epsilon))]_k$ *defined by*

$$[f_\epsilon(x(\epsilon))]_0 = f_0(x_0);$$
$$[f_\epsilon(x(\epsilon))]_1 = Df_0(x_0)x_1 + f_1(x_0);$$
$$[f_\epsilon(x(\epsilon))]_2 = Df_0(x_0)x_2 + \frac{1}{2}D^2 f_0(x_0)x_1^2 + Df_1(x_0)x_1 + f_2(x_0);$$
$$[f_\epsilon(x(\epsilon))]_3 = Df_0(x_0)x_3 + \frac{1}{6}D^3 f_0(x_0)x_1^3 + Df_1(x_0)x_2 + Df_2(x_0)x_1 + D^2 f_1(x_0)x_1^2 + f_3(x_0).$$

The general case has the following form:

$$\begin{aligned}[f_\epsilon(x(\epsilon))]_k &= Df_0(x_0)x_k + \frac{1}{k!}D^k f_0(x_0)x_1^k + f_k(x_0) \\ &\quad + B_k^f(x_0, x_1, \ldots, x_{k-1}),\ k = 1, \ldots, K + M \end{aligned} \tag{11}$$

where B_k^f *is a real function depending on* $(x_0, x_1, \ldots, x_{k-1})$ *only.*

Remark 1 We observe that $[f_\epsilon(x(\epsilon))]_k$ depends linearly on x_k, non-linearly in the inhomogeneity involving the coefficients x_j, $0 \le j \le k-1$ in (6). If $x(\epsilon)$ satisfies (6) and both μ^ϵ and σ^ϵ have the properties of the function f_ϵ in (4), then the coefficients $\mu^\epsilon(x(\epsilon))$ and $\sigma^\epsilon(x(\epsilon))$ on the right-hand side of (1) can be rewritten in powers of ϵ, for $0 \le \epsilon \le \epsilon_0$, as follows:

$$\mu^\epsilon(x(\epsilon)) = \sum_{k=0}^{K_\mu+M_\mu} \left[\mu^\epsilon(x(\epsilon))\right]_k \epsilon^k + R^\mu_{K_\mu+M_\mu}(\epsilon);$$
$$\sigma^\epsilon(x(\epsilon)) = \sum_{k=0}^{K_\sigma+M_\sigma} \left[\sigma^\epsilon(x(\epsilon))\right]_k \epsilon^k + R^\sigma_{K_\sigma+M_\sigma}(\epsilon);$$

where the natural numbers K_α and M_α, $\alpha = \mu, \sigma$ depend on the functions μ^ϵ, resp. σ^ϵ, and

$$|R^\alpha_{K_\alpha+M_\alpha}(\epsilon)| \leq C_{K_\alpha+M_\alpha}\epsilon^{K_\alpha+M_\alpha+1},$$

for some constants $C_{K_\alpha+M_\alpha}$ depending on C_j, $j = 0, \ldots, K_\alpha + M_\alpha$ but independent of ϵ, and $0 \leq \epsilon \leq \epsilon_0$.

2.2 The asymptotic character of the expansion of the solution X^ϵ_t of the SDE in powers of ϵ

Theorem 1 *Let us assume that the coefficients α^ϵ, $\alpha = \mu, \sigma$, of the stochastic differential equation (1) are in $C^{K_\alpha}(I)$ as functions of ϵ, $\epsilon \in I = [0, \epsilon_0]$, $\epsilon_0 > 0$, and in $C^{M_\alpha}(\mathbb{R})$ as functions of x. Let us also assume that α^ϵ are such that there exists a solution X^ϵ_t in the probabilistic strong, resp. weak sense of (1) and that the recursive system of random differential equations*

$$\mathrm{d}X^j_t = \left[\mu^\epsilon\left(X^\epsilon_t\right)\right]_j \mathrm{d}t + \left[\sigma^\epsilon\left(X^\epsilon_t\right)\right]_j \mathrm{d}L_t, \quad j = 0, 1, \ldots, N, \quad t \geq 0,$$

has a unique solution.

Then, there exists a sequence $\epsilon_n \in (0, \epsilon_0]$, $\epsilon_0 > 0$, $\epsilon_n \downarrow 0$ as $n \to \infty$ such that $X^{\epsilon_n}_t$ has an asymptotic expansion in powers of ϵ_n, up to order N, in the following sense:

$$X^{\epsilon_n}_t = X^0_t + \epsilon_n X^1_t + \cdots + \epsilon_n^N X^N_t + R_N(\epsilon_n, t),$$

with

$$\mathrm{st} - \lim_{\epsilon_n \downarrow 0} \frac{\sup_{s\in[0,t]} |R_N(\epsilon_n, s)|}{\epsilon_n^{N+1}} \leq C_{N+1},$$

for some deterministic $C_{N+1} \geq 0$, independent of $\epsilon \in I$, where st-lim *stands for the limit in probability.*

Proof We proceed by slightly modifying the proof in Albeverio and Smii (2013) since we have to take care of the presence of the explicit dependence on ϵ of the drift coefficient.

We shall use the fact that

$$T_N(\epsilon, t) := \frac{\left[X^\epsilon_t - \sum_{j=0}^N \epsilon^j X^j_t\right]}{\epsilon^{N+1}}, \ \epsilon \in (0, \epsilon_0],$$

satisfies a random differential equation of the form

$$\epsilon^{N+1} \mathrm{d}T_N(\epsilon, t) = A^{\mu^\epsilon}_{N+1}\left(X^0_t, \ldots, X^N_t, R^N(t,\epsilon)\right)\mathrm{d}t + A^{\sigma^\epsilon}_{N+1}\left(X^0_t, \ldots, X^N_t, R^N(t,\epsilon)\right)\mathrm{d}L_t,$$

with coefficients $A^{\alpha^\epsilon}_{N+1}$, $\alpha = \mu, \sigma$ given by

$$A^{\alpha^\epsilon}_{N+1}(y_0, y_1, \ldots, y_N, y) = \left[\alpha^\epsilon\left(\sum_{j=0}^{N} \epsilon^j y_j + \epsilon^{N+1} y\right) - \sum_{j=0}^{N} \epsilon^j \alpha_j(y_0, y_1, \ldots, y_N)\right],$$

with α_j, $j = 0, 1, \ldots, N$ the expansion coefficients of α^ϵ in powers of $\epsilon \in I$.

By Taylor's theorem, one proves

$$\frac{1}{\epsilon^{N+1}} \sup_{s\in[0,t]} |A^{\alpha^\epsilon}_{N+1}\left(X^0_s, \ldots, X^N_s, R^N_s(\epsilon)\right)| \leq C_{N+1}, \quad \epsilon \in (0, \epsilon_0],$$

for some $C_{N+1} \geq 0$, independent of ϵ, $0 \leq \epsilon \leq \epsilon_0$.

From this, one deduces that one can find a sequence $\epsilon_n \to 0$ as $n \to \infty$ s.t.

$$\text{st-lim}_{\substack{\epsilon_n \downarrow 0 \\ n\to\infty}} \frac{1}{\epsilon_n^{N+1}} \sup_{s\in[0,t]} |A^{\alpha^{\epsilon_n}}_{N+1}\left(X^0_s, \ldots, X^N_s, R^N_s(\epsilon_n)\right)|$$

exists and it is bounded by C_{N+1}.

Under some assumptions on μ^ϵ, σ^ϵ and L, it follows then from a theorem by Skorohod, on the continuous dependence of solutions of SDE's on the coefficients, see, e.g., Gihman and Skorokhod (1972), that

$$\text{st-lim}_{\substack{\epsilon_n \downarrow 0 \\ n\to\infty}} \sup_{s\in[0,t]} |T_N(\epsilon_n, s)|$$

exists and it is bounded by C_{N+1}, which proves the result.

See Albeverio and Smii (2013) for more details. □

Remark 2 It is worth to mention that if L_t is a standard Brownian motion, then results of the type stated in Theorem 1, have been already obtained, exploiting Malliavin calculus' techniques, see, e.g., Takahashi (1999) and Watanabe (1987). Recently, see Shiraya and Takahashi (2017), results of Takahashi (1999) have been partially extended to consider the case when the small perturbation parameter ϵ is also in the jump component and then expand the related SDE around $\epsilon = 0$, even if in a different way compared to our approach. We further generalize previous setting considering a small parameter which fully enters in the volatility term multiplying the stochastic noise L_t, possibly with a polynomial dependence. We would like to underline that, w.r.t. the settings studied in Takahashi (1999) and Watanabe (1987), we retrieve the driving SDE there analyzed and then obtain the same expansion. Let us also recall that asymptotic expansions in the case of L_t with jumps have been also discussed, by using PDE methods, in Benhamou et al. (2009) and Matsuoka et al. (2004), where the coefficients appearing in the expansion for the option price are expressed in terms of the *Geeks*, while in Pagliarani et al. (2013), PDE and Fourier transformation methods are used to handle an expansion of the solution of the Kolmogorov equation associated with processes with stochastic volatility and general jump terms. Expansions in terms of nested systems of linearized SDE's also occur in Fouque et al. (2000) and Takahashi and Yamada (2012).

Remark 3 It can be seen that in general the kth equation for X_t^k in Theorem 1 is a non-homogeneous linear equation in X_t^k, but with random coefficients depending on $X_t^0, \dots, X_t^{k-1}$ and with a random inhomogeneity depending on X_t^k. Thus, it has the general form

$$\begin{aligned} \mathrm{d}X_t^k &= f_k\left(X_t^0, \dots, X_t^{k-1}\right) X_t^k \mathrm{d}t + g_k\left(X_t^0, \dots, X_t^{k-1}\right) \mathrm{d}t \\ &\quad + \tilde{g}_k\left(X_t^0\right) \mathrm{d}L_t + h_k\left(X_t^0, \dots, X_t^{k-1}\right) X_t^k \mathrm{d}L_t, \end{aligned} \tag{12}$$

for some continuous functions $f_k, g_k, \tilde{g}_k$ and h_k.

Let us now look at a particular case

Example 1 Let us consider the linear case, that is, let $\mu^\epsilon = (a + \epsilon b)x$ and $\sigma^\epsilon = (\sigma_0 + \epsilon\sigma_1)x$ with a, b, σ_0 and σ_1 some real constants. Applying Proposition 1, we get

$$\begin{aligned} X_t^0 &= x_0 + \int_0^t a X_s^0 \mathrm{d}s + \int_0^t \sigma_0 X_s^0 \mathrm{d}L_s, \\ X_t^1 &= \int_0^t a X_s^1 \mathrm{d}s + \int_0^t b X_s^0 \mathrm{d}s + \int_0^t \sigma_1 X_s^0 \mathrm{d}L_t + \int_0^t \sigma_0 X_s^1 \mathrm{d}L_t, \\ X_t^k &= \int_0^t a X_s^k \mathrm{d}s + \int_0^t b X_s^{k-1} \mathrm{d}s + \int_0^t \sigma_1 X_s^{k-1} \mathrm{d}L_s + \int_0^t \sigma_0 X_t^k \mathrm{d}L_t, k \geq 2. \end{aligned} \tag{13}$$

If we consider the special case of Remark 1 where $\mu^\epsilon(x) = ax + b$, independent of ϵ, $\sigma^\epsilon(x) = cx + \epsilon\tilde{d}x$, for some real constants a, b, c and $\tilde{d}$, independent of ϵ, and where the Lévy process is taken to be a standard Brownian motion, $L_t = W_t$, then by Eq. (11) we have that X_t^k satisfies a linear equation with constant coefficients for any $k \in \mathbb{N}$; thus, applying standard results, see, e.g., Arnold (1974), an explicit solution for X_t^k can be retrieved.

Let us describe this in the case where we have a set of K coupled linear stochastic equations with random coefficients of the form

$$\begin{cases} \mathrm{d}X_t = [A(t)X_t + f(t)]\,\mathrm{d}t + \sum_{i=1}^m [B_i(t)X_t + g_i(t)]\,\mathrm{d}W_t^i, \\ X_0^k = x_0^k \in \mathbb{R}, \quad t \geq 0 \end{cases} \tag{14}$$

where A and B_i are $K \times K$ matrices and f and g_i $\mathbb{R}^K$-valued deterministic functions. Let us underline that, in a financial setting, A and B can be considered as the percentage rate at which drift and volatility change, whereas f and g play the role of inhomogeneous terms. All the coefficients A, B, f and g are assumed to be measurable. The solution of Eq. (14) is then given by

$$X_t = \Phi(t)\left[x_0 + \int_0^t \Phi^{-1}(s)\left(f(s) - \sum_{i=1}^m B_i(s)g_i(s)\right)\mathrm{d}s + \sum_{i=1}^m \int_0^t g_i(s)\mathrm{d}W_s^i\right] \tag{15}$$

where $\Phi(t)$ is the fundamental $K \times K$ matrix solution of the corresponding homogeneous equation, i.e., it is the solution of the problem

$$\begin{cases} \mathrm{d}\Phi(t) = A(t)\Phi(t)\mathrm{d}t + \sum_{i=1}^m B_i(t)\Phi(t)\mathrm{d}W_t^i, \\ \Phi(0) = I,\ t \geq 0, \end{cases}, \tag{16}$$

being I the unit $K \times K$ matrix.

Remark 4 In the case where $K = 1$, we have that Φ reduces to a scalar and is given by

$$\Phi(t) = \exp\left\{\int_0^t \left(A(s) - \frac{1}{2}B^2(s)\right)\mathrm{d}s + \int_0^t B(s)\mathrm{d}W_s\right\}.$$

Still in the case $K = 1$ but with a more general noise, i.e., W_t in Eq. (14) replaced by a Lévy process composed by a Brownian motion plus W_t a jump component expressed by $\tilde{N}$, Eq. (16) is replaced by

$$\begin{cases} \mathrm{d}\Phi(t) = A(t)\Phi(t)\mathrm{d}t + B(t)\Phi(t)\mathrm{d}W_t + \int_{\mathbb{R}_0} \Phi(t_-)C(t,x)\tilde{N}(\mathrm{d}t,\mathrm{d}x), \\ \Phi(0) = I, \quad t \geq 0 \end{cases} \tag{17}$$

with A, B, C Lipschitz and with at most linear growth, and where $\tilde{N}(\mathrm{d}t,\mathrm{d}x)$ is a Poisson compensated random measure to be understood in the following sense: $\tilde{N}(t, A) := N(t, A) - t\nu(A)$ for all $A \in \mathcal{B}(\mathbb{R}_0)$, $0 \notin \bar{A}$, with $\bar{A}$ the closure of A, N being a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ and $\nu(A) := \mathbb{E}(N(1, A))$, while $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ and $\int_{\mathbb{R}_0}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$, ν is the Lévy measure to $\tilde{N}$, see, e.g., Applebaum (2009), Imkeller et al. (2009), Mandrekar and Rüdiger (2015).

Denoting then Eq. (17) for short as

$$\mathrm{d}\Phi(t) = \Phi(t_-)\mathrm{d}X(t), \tag{18}$$

with

$$\mathrm{d}X(t) = A(t)\mathrm{d}t + B(t)\mathrm{d}W_t + \int_{\mathbb{R}_0} C(t,x)\tilde{N}(\mathrm{d}t,\mathrm{d}x), \tag{19}$$

we have then that the solution to Eq. (18) is explicitly given, in terms of the coefficients and noise, and the solution of Eq. (19), by

$$\Phi(t) = \exp\left\{1 + \int_0^t \left(A(s) - \frac{1}{2}B^2(s)\right) \mathrm{d}s + \int_0^t B(s)\mathrm{d}W_s + \int_{\mathbb{R}_0} C(s,x)\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right\} \prod_{0<s\leq t} (1+\Delta X_s)\,\mathrm{e}^{-\Delta X_s}, \tag{20}$$

where $\Delta X(s) := X_s - X_{s_-}$ is the jump at time $s \in (0,t]$. The stochastic process (20) is called *Doléans-Dade exponential* (or stochastic exponential), and it is usually denoted by $\Phi(t) = \mathcal{E}(X_t)$. The *Doléans-Dade exponential* has a wide use in finance since it is the natural extension to the Lévy case of the standard geometric Brownian motion, see, e.g., Arnold (1974) and Gardiner (2004) for a more extensive treatment of the fundamental solution of the homogeneous equation for systems of linear SDE's and Applebaum (2009) for more details on the *Doléans-Dade exponential*.

3 Corrections around the Black–Scholes price (with Brownian, resp. Brownian plus jumps)

We shall study an asset S_t^ϵ evolving according to the particular stochastic differential equation (SDE) governing the Black–Scholes (BS) model, with the possible addition of some driving term determined by a compound Poisson process, see, e.g., Black and Scholes (1973), Shreve (2004), resp. Benhamou et al. (2009), Merton (1976) and Albeverio et al. (2006). Our aim is to apply the theory developed in Sect. 2 in order to give corrections around the price given by the BS model for an option with terminal payoff Φ written on the underlying S_t^ϵ (Φ is a given real-valued function assumed here to be sufficiently smooth). In particular, if we consider the return process defined as $X_t^\epsilon := \log S_t^\epsilon$, S_t^ϵ being supposed to be strictly positive, at least almost surely, we have that the price $P(t,T)$ at time t of the option with final payoff Φ with maturity time T, $0 \leq t \leq T$, is given by

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{r(T-t)}\Phi(X_T)\middle|\mathcal{F}_t\right], \tag{21}$$

where $\mathbb{Q}$ is a relevant equivalent martingale measure, called in financial application *risk-neutral measure*, $\mathbb{E}^{\mathbb{Q}}[\cdot|\cdot]$ the corresponding conditional expectation given the σ-algebra $\mathcal{F}_t$ at time t associated with the underlying Brownian motion; $r > 0$ is the constant interest rate. We refer to, e.g., Black and Scholes (1973), Brigo and Mercurio (2006), Cox et al. (1985), Filipovic (2009), Kim and Kunitomo (1999), Shreve (2004), for a general introduction to option pricing. We stress that, defining the stock price as

$$\mathrm{d}S(t) = S(t_-)\mathrm{d}L(t),$$

for a general Lévy process L, as remarked in Eqs. (18)–(20), its solution can be given in terms of the *Doléans-Dade exponential* as $S(t) = \mathcal{E}(X_t)$. In particular, using the notation in Eq. (20) and requiring that

$$\inf\{\Delta L(t),\ t > 0\} > -1 \quad \text{a.s.},$$

then Z is almost surely finite and positive, see, e.g., Applebaum (2009, Proposition 5.1.1, pp. 247–248). Moreover, by Theorem 1 and using Lemma 2, we have that $\Phi(X_t^\epsilon)$ has an asymptotic expansion, in the sense of Theorem 1, in powers of $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$, with the following form

$$\Phi(X_t^\epsilon) = \sum_{k=0}^{H} \epsilon^k \left[\Phi(X_t^\epsilon)\right]_k + R_H(\epsilon, t), \tag{22}$$

where

$$\sup_{s\in[0,t]} |R_H(\epsilon, s)| \leq C_{H+1}(t)\epsilon^{H+1},$$

for any $H \in \mathbb{N}$, and the coefficients can be computed from the expansion coefficients of X_t^ϵ, as discussed in Sect. 2.

More concretely, we will deal with two particular cases. In the first case, we have an asset S^ϵ evolving according to a geometric Brownian motion with a small perturbation in the diffusion. Namely, the asset evolves, in a risk-neutral setting, according to

$$\begin{cases} \mathrm{d}S_t^\epsilon = S_t^\epsilon \left[(\sigma_0 + \epsilon\sigma_1 \bar{f}(S_t^\epsilon))\mathrm{d}W_t\right], \\ s_0 = s_0, t \geq 0 \end{cases} \tag{23}$$

where $\sigma_0 \neq 0$ and σ_1 are real constants, $s_0 > 0$, W_t is a $\mathbb{Q}$-Brownian motion adapted to the filtration $(\mathcal{F}_t)_t$ and $\bar{f}(S_t^\epsilon) := f(X_t^\epsilon)$ with f a given smooth function on $\mathbb{R}$. In particular, the existence and uniqueness of a strong solution to Eq. (23) follow under the general assumption of $\bar{f} \in C^1$ from McKean (1969, Problem 3.3.2). For the sake of simplicity, we have assumed both σ_0 and σ_1 to be time independent, the generalization to time-dependent functions being almost straightforward, without computational problems w.r.t. derivation of the results developed in what follows.

Suppose, for all $t \geq 0$, $S_t^\epsilon > 0$ a.s. (which is the case if ϵ is sufficiently small). Applying Itô's lemma to $X_t^\epsilon := \log S_t^\epsilon$, we end up with the following evolution for X_t^ϵ, the return of the asset price

$$\begin{aligned} X_t^\epsilon = x_0 - \int_0^t \left[\frac{\sigma_0^2}{2} + \epsilon\sigma_0\sigma_1 f(X_s^\epsilon) + \epsilon^2 \frac{\sigma_1^2 f(X_s^\epsilon)^2}{2}\right] \mathrm{d}s \\ + \int_0^t \left[\sigma_0 + \epsilon\sigma_1 f(X_s^\epsilon)\right] \mathrm{d}W_s, \end{aligned} \tag{24}$$

where we have set $x_0 := \log s_0$.

Applying the results obtained in Sect. 2 and expanding Eq. (24) to the second order in ϵ, we get

$$\begin{aligned} X_t^0 &= x_0 - \frac{\sigma_0^2}{2}t + \sigma_0 W_t, \quad \text{with law} \quad \mathcal{N}\left(x_0 + \mu t, \sigma_0^2 t\right), \\ X_t^1 &= -\int_0^t \sigma_0\sigma_1 f(X_s^0)\mathrm{d}s + \int_0^t \sigma_1 f\left(X_s^0\right)\mathrm{d}W_s, \\ X_t^2 &= -\int_0^t \left(\frac{\sigma_1^2 f(X_s^0)^2}{2} + 2\sigma_0\sigma_1 f'\left(X_s^0\right)X_s^1\right)\mathrm{d}s + \int_0^t \sigma_1 f'\left(X_s^0\right)X_s^1\mathrm{d}W_s, \end{aligned} \tag{25}$$

where $\mathcal{N}\left(-\frac{\sigma_0^2}{2}t, \sigma_0^2 t\right)$ denotes the Gaussian distribution of mean $-\frac{\sigma_0^2}{2}t$ and variance $\sigma_0^2 t$, f' the derivative of f.

The second model we will deal with, following Merton (1976) and Benhamou et al. (2009), is the previous one with an addition of a small compound Poisson process

$$Z_t = \sum_{i=1}^{N_t} J_i,$$

with N_t a standard Poisson process with intensity $\lambda > 0$ and $(J_i)_{i=1,\dots,N_t}$ being independent normally distributed random variables, namely such that

$$J_i \text{ has law } \mathcal{N}(\gamma, \delta^2),$$

for some $\gamma \in \mathbb{R}$ and $\delta > 0$.

We thus have that the Lévy measure $\nu(\mathrm{d}z)$ of Z reads as

$$\nu(\mathrm{d}z) = \frac{\lambda}{\sqrt{2\pi}\delta}\mathrm{e}^{-\frac{(z-\gamma)^2}{2\delta^2}}\mathrm{d}z, \quad z \in \mathbb{R},$$

and the cumulant function of Z is

$$\kappa(\zeta) = \lambda\left(\mathrm{e}^{\gamma\zeta + \frac{\delta^2\zeta^2}{2}} - 1\right).$$

In particular, we assume the asset S^ϵ to evolve according to a geometric Lévy process with a small perturbation in the diffusion. Namely, the asset evolves, in a risk-neutral setting, according to

$$\begin{cases} \mathrm{d}S_t^\epsilon = S_t^\epsilon\left[(\sigma_0 + \epsilon\sigma_1\bar{f}(S_t^\epsilon))\mathrm{d}W_t + \epsilon\sum_{i=1}^{N_t} J_i\right], \\ S_0^\epsilon = s_0 > 0, \ t \geq 0, \end{cases} \tag{26}$$

Again the existence and uniqueness of a strong solution to Eq. (26) can be obtained by arguments similar to the ones used in McKean (1969, Problem 3.3.2) together with the properties of $\sum_{i=1}^{N_t} J_i$.

Proceeding as above and applying Itô's lemma to $X_t^\epsilon := \log S_t^\epsilon$, we have that the log-return process X_t^ϵ evolves according to

$$X_t^\epsilon = x_0 - \int_0^t \left[\frac{\sigma_0^2}{2} + \epsilon \sigma_0 \sigma_1 f(X_s^\epsilon) + \epsilon^2 \frac{\sigma_1^2 f(X_s^\epsilon)^2}{2} \right] \mathrm{d}s + \epsilon \lambda t \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1 \right) + \int_0^t \left(\sigma_0 + \epsilon \sigma_1 f\left(X_s^\epsilon\right)\right) \mathrm{d}W_s + \epsilon \sum_{i=1}^{N_t} J_i, \tag{27}$$

for $\epsilon \in I = [0, \epsilon_0]$, $\epsilon_0 > 0$.

In the present case, it is more tricky to deal with the risk-neutral probability measure $\mathbb{Q}$. Under suitable assumptions on the coefficients and noise, one can assure the existence (but not necessarily the uniqueness) of an equivalent probability measure $\mathbb{Q}$. We will assume the process (27) to evolve under a risk-neutral measure $\mathbb{Q}$, see, e.g., Applebaum (2009).

In particular, we will use two specific forms for the function f that is an exponential function and a polynomial function. The former is of special interest for its general application to integral transforms, such as Fourier or Laplace transforms, see, e.g., Sect. 3.1, Remark 6. The latter mimics a polynomial volatility process [these types of processes have been widely used in finance since they can be easily implemented, see, e.g., Carr et al. (2013) and reference therein].

3.1 A correction given by an exponential function

Let us consider the first model described by Eqs. (24) and (25), i.e., an asset S^ϵ evolving according to a geometric Brownian motion under the unique risk-neutral probability measure $\mathbb{Q}$, recalling that $X_t^\epsilon = \log S_t^\epsilon$. Let us first look at the particular case $f(x) = \mathrm{e}^{\alpha x}$, for some $\alpha \in \mathbb{R}$. We take into account the particular case of an exponential function due to the fact that it can be easily extended to the much more general case where the function f can be written as a Fourier transform or a Laplace transform of some bounded measure on the real line, as it will be further discussed in Remark 6. We then get the following proposition.

Proposition 2 *Let us consider the SDE (24) in the particular case where $f(x) = \mathrm{e}^{\alpha x}$, for some $\alpha \in \mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, $\sigma_0 \in \mathbb{R}_0$.*

Then, the following expansion $X_t^\epsilon = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + R_2(\epsilon, t)$ holds, where the coefficients are given by

$$\begin{aligned} X_t^0 &= x_0 - \frac{\sigma_0^2}{2} t + \sigma_0 W_t, \quad \text{with law} \quad \mathcal{N}\left(x_0 \frac{\sigma_0^2}{2} t, \sigma_0^2 t \right); \\ X_t^1 &= \int_0^t K_\alpha \mathrm{e}^{\alpha X_s^0} \mathrm{d}s + \frac{\sigma_1}{\alpha \sigma_0} \left(\mathrm{e}^{\alpha X_t^0} - 1 \right); \\ X_t^2 &= C_\alpha^1 \int_0^t \mathrm{e}^{2\alpha X_s^0} \mathrm{d}s + C_\alpha^2 \mathrm{e}^{\alpha X_t^0} \int_0^t \mathrm{e}^{\alpha X_s^0} \mathrm{d}s + C_\alpha^3 \int_0^t \mathrm{e}^{\alpha X_s^0} \mathrm{d}s \\ &\quad + C_\alpha^4 \int_0^t \mathrm{e}^{\alpha X_s^0} \int_0^s \mathrm{e}^{\alpha X_r^0} \mathrm{d}r \mathrm{d}s + C_\alpha^5 \mathrm{e}^{2\alpha X_t^0} + C_\alpha^6 \mathrm{e}^{\alpha X_t^0} + C_\alpha^7, \end{aligned} \tag{28}$$

with

$$K_\alpha := \sigma_1\left(\frac{\sigma_0}{2}-\frac{\alpha\sigma_0}{2}-\sigma_0\right),\ C_\alpha^1 := -\sigma_1^2\left(\frac{5}{2}-\frac{1}{2}+\alpha+\frac{K_\alpha}{\sigma_0\sigma_1}\right),\ C_\alpha^2 := K_\alpha\frac{\sigma_1}{\sigma_0},$$
$$C_\alpha^3 := -\sigma_1^2\left(\frac{1}{2}+\frac{\alpha}{2}+2\right),\ C_\alpha^4 := -K_\alpha\sigma_1\alpha\left(2\sigma_0-\frac{\sigma_0}{2}+\frac{\alpha\sigma_0}{2}\right),$$
$$C_\alpha^5 := \frac{\sigma_1^2}{2\alpha\sigma_0^2},\ C_\alpha^6 := -\frac{\sigma_1^2}{\alpha\sigma_0^2},\ C_\alpha^7 := \frac{\sigma_1^2}{2\alpha\sigma_0^2}.$$

Furthermore, $R_2(\epsilon,t)$ satisfies the bound

$$\mathrm{st}-\lim_{\epsilon_n\downarrow 0}\frac{\sup_{s\in[0,t]}|R_2(\epsilon,s)|}{\epsilon_n^3}\le C_3,$$

for some subsequence $\epsilon_n\downarrow 0$ and with some constant $C_3\ge 0$.

Proof The proof consists in a repeated application of the Itô formula and the stochastic Fubini theorem.

In fact, substituting $f(x)=\mathrm{e}^{\alpha x}$ into system (25) we immediately obtain

$$\begin{aligned}
X_t^0 &= x_0\mu t+\sigma_0 W_t, \quad \text{with law} \quad \mathcal{N}\left(x_0+\mu t,\sigma_0^2 t\right);\\
X_t^1 &= -\int_0^t \sigma_0\sigma_1\mathrm{e}^{\alpha X_s^0}\mathrm{d}s+\int_0^t\sigma_1\mathrm{e}^{\alpha X_s^0}\mathrm{d}W_s;\\
X_t^2 &= -\int_0^t\left(\frac{\sigma_1^2}{2}\mathrm{e}^{2\alpha X_s^0}+2\sigma_0\sigma_1\alpha\mathrm{e}^{\alpha X_s^0}X_s^1\right)\mathrm{d}s+\int_0^t\sigma_1\alpha\mathrm{e}^{\alpha X_s^0}X_s^1\mathrm{d}W_s.
\end{aligned} \tag{29}$$

To compute X_t^1, we apply Itô's lemma to the function $g(X_t^0)=\mathrm{e}^{\alpha X_t^0}$ to get

$$\mathrm{e}^{\alpha X_t^0}=1+\int_0^t\left(\mathrm{e}^{\alpha X_s^0}\alpha\mu+\frac{\alpha^2}{2}\sigma_0^2\mathrm{e}^{\alpha X_s^0}\right)\mathrm{d}s+\int_0^t\mathrm{e}^{\alpha X_s^0}\alpha\sigma_0\mathrm{d}W_s. \tag{30}$$

Expressing the latter integral involving $\mathrm{d}W_s$ by the other terms in Eq. (30) and substituting it in the stochastic integral of X_t^1 in the system (29), we get the result for X_t^1 in Eq. (28).

In order to derive the expression for X_t^2, we use again Itô's lemma, in particular Eq. (30), getting from (29)

$$\begin{aligned}
X_t^2 &= -\int_0^t\left(\frac{\sigma_1^2}{2}\mathrm{e}^{2\alpha X_s^0}+2\sigma_0\sigma_1\alpha\mathrm{e}^{\alpha X_s^0}X_s^1\right)\mathrm{d}s+\int_0^t\alpha\sigma_1\mathrm{e}^{\alpha X_s^0}X_s^1\mathrm{d}W_s\\
&= -\int_0^t\sigma_1^2\left(2\alpha+\frac{1}{2}\right)\mathrm{e}^{2\alpha X_s^0}\mathrm{d}s+\int_0^t 2\alpha\sigma_1^2\mathrm{e}^{\alpha X_s^0}\mathrm{d}s-\int_0^t\int_0^s 2K_\alpha\sigma_1\sigma_0\alpha\mathrm{e}^{\alpha X_s^0}\mathrm{e}^{\alpha X_r^0}\mathrm{d}r\mathrm{d}s\\
&\quad+\underbrace{\int_0^t\frac{\sigma_1^2\alpha}{\sigma_0}\mathrm{e}^{2\alpha X_s^0}\mathrm{d}W_s}_{(1)}-\underbrace{\int_0^t\frac{\alpha\sigma_1^2}{\sigma_0}\mathrm{e}^{\alpha X_s^0}\mathrm{d}W_s}_{(2)}+\underbrace{\int_0^t K_\alpha\alpha\sigma_1\mathrm{e}^{\alpha X_s^0}\int_0^s\mathrm{e}^{\alpha X_r^0}\mathrm{d}r\mathrm{d}W_s}_{(3)}.
\end{aligned}$$

For the terms (1) and (2), we use Eq. (30), resp. Itô's lemma applied to the function $g(X_t^0) = \mathrm{e}^{2\alpha X_t^0}$, as before to replace the stochastic integral by an integral against Lebesgue measure. In order to treat the term (3), we use the stochastic Fubini theorem, see, e.g., Theorem 6.2 in Filipovic (2009), to get

$$(3) = \frac{K_\alpha \sigma_1}{\sigma_0} \int_0^t \int_0^s \alpha \sigma_0 \mathrm{e}^{\alpha X_s^0} \mathrm{e}^{\alpha X_r^0} \mathrm{d}r \mathrm{d}W_s = \frac{K_\alpha \sigma_1}{\sigma_0} \int_0^t \mathrm{e}^{\alpha X_r^0} \int_r^t \alpha \sigma_0 \mathrm{e}^{\alpha X_s^0} \mathrm{d}W_s \mathrm{d}r.$$

Using the expression for the integral in $\mathrm{d}W_s$ coming from (30), we then get

$$\begin{aligned}(3) &= \frac{K_\alpha \sigma_1}{\sigma_0} \int_0^t \mathrm{e}^{\alpha X_r^0} \int_r^t \alpha \sigma_0 \mathrm{e}^{\alpha X_s^0} \mathrm{d}W_s \mathrm{d}r \\ &= \frac{K_\alpha \sigma_1}{\sigma_0} \mathrm{e}^{\alpha X_t^0} \int_0^t \mathrm{e}^{\alpha X_s^0} \mathrm{d}s - \frac{K_\alpha \sigma_1}{\sigma_0} \int_0^t \mathrm{e}^{2\alpha X_s^0} \mathrm{d}s - \frac{K_\alpha \sigma_1}{\sigma_0} \left(\alpha \mu + \frac{\alpha^2 \sigma_0^2}{2} \right) \\ &\quad \times \int_0^t \int_0^s \mathrm{e}^{\alpha X_s^0} \mathrm{e}^{\alpha X_r^0} \mathrm{d}r \mathrm{d}s.\end{aligned}$$

Substituting now everything into the original system (29), rearranging and grouping the integrals of the same type, we get the desired result in (28).

The estimate on the remainder is a consequence of Theorem 1. □

Remark 5 Our aim in Proposition 2 is to discuss in detail a particular choice of volatility function around the Black–Scholes one. We obtain explicit formulae for the expansion coefficients, keeping control of the remainder. This expansion can be seen as a particular, but more explicit, case of the one discussed in Takahashi (1999, Proposition 2.1).

Remark 6 The particular choice of $f(x) = \mathrm{e}^{\alpha x}$ can easily be extended to any real function which can be written as a Fourier transform, resp. Laplace transform, $f(x) = \int_{\mathbb{R}_0} \mathrm{e}^{ixy} \varrho(\mathrm{d}\alpha)$, resp. $f(x) = \int_{\mathbb{R}_0} \mathrm{e}^{\alpha x} \varrho(\mathrm{d}\alpha)$, of some positive measure ϱ on $\mathbb{R}_0$ (e.g., a symmetric probability measure) resp. which has finite Laplace transform. Formula (28) holds with $K_\alpha \mathrm{e}^{\alpha X_\tau^0}$ replaced by $\int_{\mathbb{R}_0} K_\alpha \mathrm{e}^{i\alpha X_\tau^0} \varrho(\mathrm{d}\alpha)$, resp. $\int_{\mathbb{R}_0} K_\alpha \mathrm{e}^{\alpha X_\tau^0} \varrho(\mathrm{d}\alpha)$, which are finite if, e.g., $\int_{\mathbb{R}_0} |K_\alpha| \varrho(\mathrm{d}\alpha) < \infty$, resp. ϱ has, e.g., compact support. In fact, Eq. (30) gets replaced by

$$\begin{aligned}\int_{\mathbb{R}_0} \mathrm{e}^{\alpha X_t^0} \varrho(\mathrm{d}\alpha) = 1 + \int_{\mathbb{R}} \left[\int_0^t \left(\mathrm{e}^{\alpha X_s^0} \alpha \mu + \frac{\alpha^2}{2} \sigma_0^2 \mathrm{e}^{\alpha X_s^0} \right) \mathrm{d}s \right] \varrho(\mathrm{d}\alpha) \\ + \int_{\mathbb{R}} \left[\int_0^t \mathrm{e}^{\alpha X_s^0} \alpha \sigma_0 \mathrm{d}W_s \right] \varrho(\mathrm{d}\alpha).\end{aligned} \tag{31}$$

By repeating the steps used before and exploiting again the Stochastic Fubini's theorem, we get the statements in Proposition 2 extended to these more general cases.

If we assume the payoff function $x \mapsto \Phi(x)$ to be smooth, $x \in \mathbb{R}_+$, we can expand $\Phi(X_t^\epsilon)$ in powers of ϵ using the formulae in Proposition 1. Then, exploiting Eq. (22) with $H = 1$, i.e., stopping at the first order, we get

$$\Phi\left(X_t^\epsilon\right) = \Phi(X_t^0) + \epsilon\Phi'(X_t^0)X_t^1 + R_1(\epsilon, t), \tag{32}$$

with $\sup_{s\in[0,t]} |R_1(\epsilon, s)| \leq \tilde{C}(s)\epsilon^2$, for some $\tilde{C}$ independent of ϵ (Φ' is the derivative of Φ).

Calling Φ_1 the terms on the r.h.s. in Eq. (32) minus the reminder term $R_1(\epsilon, t)$, we get that the corresponding corrected fair price $Pr^1(0; T)$, up to the first order in ϵ, of an option written on the underlying $S_t^\epsilon := e^{X_t^\epsilon}$ at time $t = 0$ with maturity T, reads as follows:

$$\begin{aligned} Pr^1(0; T) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi_1(X_T^\epsilon)\right] = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T^0) + \epsilon\Phi'(X_T^0)X_T^1\right] \\ &= Pr_{BS} + \epsilon e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi'(X_T^0)X_T^1\right], \end{aligned} \tag{33}$$

where Pr_{BS} stands for the standard B–S price with underlying $S_t^0 := e^{X_t^0}$, see, e.g., Black and Scholes (1973).

This formula yields thus, for a smooth payoff function, the corrected price up to the first order, with an error term related to the "full price" and bounded in modulus by $C_2\epsilon^2$ for a constant $C_2 \geq 0$ independent of ϵ.

Remark 7 It is worth to recall that typical payoff functions usually fail to be smooth, as in the case of European call options where $\Phi(x) = (e^x - K)^+$, $K > 0$ being the strike price. In particular, latter case is characterized by a point of non-differentiability at $e^X = K$. To avoid such an issue, one can consider a smoothed version of the payoff function, namely $\Phi_h := \Phi * \rho_h$, with ρ_h a smooth kernel s.t. $\Phi_h \to \Phi$ as $h \to \infty$, in distributional sense. As an example, taking ρ_h to be a smooth C^∞ *mollifier*, properties of the convolution product imply that also Φ_h inherits the smooth C^∞ regularity. Then, choosing a suitable calls of mollifier, the desired convergence $\Phi_h \to \Phi$ as $h \to \infty$ is obtained, see, e.g., Showalter (2010). As a byproduct of such an approach, we have that the smoothed payoff function Φ_h, see Eq. (33), is well defined. In particular, the first derivative appearing in Eq. (33) is given by a regularized version of $\mathbb{1}_{[x>\ln K]}(x)$. Heuristically, interchanging the limits involved in the expansion with the regularization removing, we can look at $Pr^1(0, T)$ as given by (33), also in the case of the payoff function $\Phi(x) = (e^x - K)^+$, $x \in \mathbb{R}$. Namely, we can considered we have that it approximates the price, with $\Phi'(x) = \mathbb{1}_{[x>\ln K]}(x)$ given as above. In fact, it can be notice that, being $\Phi \in C^1(\mathbb{R}\backslash\{\ln K\})$, we have that its derivative on the domain of differentiability corresponds to $\mathbb{1}_{[x>\ln K]}(x)$. We stress that the above approach should ideally deal with distributional coefficients, using methods of see Watanabe (1987), and also to handle distributional Φ, see Takahashi and Yamada (2012).

We have the following result.

Proposition 3 *Let us consider the particular case of an European call option Φ with payoff given by $\Phi(X_T^\epsilon) = \max\{e^{X_t^\epsilon} - K, 0\} =: \left(e^{X_t^\epsilon} - K\right)_+$, K being the strike price. Then, the approximated price up to the first-order $Pr^1(0; T)$, in the sense of Remark 7, is explicitly given by*

$$Pr^1(0;T) = P_{BS} + \epsilon \mathcal{K}_1 s_0^{\alpha+1} I_1(s,T,\alpha) - \epsilon \mathcal{K}_2 s_0 N(d_1) \\ + \epsilon \mathcal{K}_3 s_0^{\alpha+1} N(d(2\alpha+1)), \tag{34}$$

with $N(x)$ the cumulative function of the standard Gaussian distribution and

$$d(\alpha) = \frac{1}{\sigma_0\sqrt{T}}\left(log\frac{s_0}{K} + \left(r - \frac{\sigma_0^2}{2}\alpha\right)T\right), \quad d_1 := d(1), \quad d_2 := (d_1 + \sigma_0\sqrt{T}),$$

$$\mathcal{K}_1 = K_\alpha e^{-\frac{\sigma_0^2}{2}T}, \quad \mathcal{K}_2 = \frac{\sigma}{\alpha\sigma_0}, \quad \mathcal{K}_3 = \frac{\sigma_1}{\alpha\sigma_0} e^{\frac{\sigma_0^2}{2}T\alpha(\alpha+1)+\alpha r T},$$

$$I_1(s,T,\alpha) = \int_0^T e^{\alpha\mu s}\int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left\{x+y>\sqrt{T}d_2\right\}} e^{\sigma_0 x} e^{(1+\alpha)\sigma_0 y}\phi(x,0,T-s)\phi(y,0,s)\mathrm{d}x\mathrm{d}y\mathrm{d}s,$$

where we have denoted by $\phi(x;\mu,\sigma)$ the density function of the normal distribution with mean μ and variance σ; P_{BS} denotes the usual B–S price with underlying $S_t^0 = e^{X_t^0}$.

Proof Given the exponential function $f(x) = e^{\alpha x}$, where $\alpha \in \mathbb{R}$, the approximated price up to the first-order $Pr^1(0;T)$ of an European call option with payoff function $\Phi(X_T^\epsilon) = \left(e^{X_T^\epsilon} - K\right)_+$ is

$$\begin{aligned} Pr^1(0;T) &= P_{BS} + \epsilon e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi'(X_T^0)X_T^1\right] \\ &= P_{BS} + \epsilon e^{-rT}\left\{\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{[X_0^T>\ln(K)]}e^{X_T^0}\int_0^T K_\alpha e^{\alpha X_s^0}\mathrm{d}s\right]+\right. \\ &\left. -\mathcal{K}_2\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{[X_0^T>\ln(K)]}e^{X_T^0}\right] + \mathcal{K}_2\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{[X_0^T>\ln(K)]}e^{X_T^0}e^{\alpha X_T^0}\right]\right\}, \end{aligned} \tag{35}$$

where P_{BS} is the standard B–S price with underlying $S_t^0 = e^{X_t^0}$.

Let us first compute the integral

$$\epsilon e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{[X_T^0>\ln(K)]}e^{X_T^0}\int_0^T K_\alpha e^{\alpha X_s^0}\mathrm{d}s\right]$$

By means of Fubini Theorem, we can exchange the expectation with respect to the integration in time so that we obtain

$$\epsilon e^{-rT}K_\alpha\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{[X_t^0>\ln(K)]}e^{X_T^0}e^{\alpha X_s^0}\right]\mathrm{d}s. \tag{36}$$

From the definition of X_T^0 and X_s^0, for every fixed $0 < s < T$, we have

$$X_T^0 = x_0 + \mu T + \sigma_0 W_T,$$
$$X_s^0 = x_0 + \mu s + \sigma_0 W_s,$$

are two correlated random variables, by means of the Wiener processes involved. By algebraic manipulation let us define $W_T = W_T - W_s + W_s$, where $X := W_T - W_s$ is $\mathcal{N}(0, T-s)$ independent with respect to W_s. Then, $X_T^0 = x_0 + \mu T + \sigma_0 X + \sigma_0 W_s$ and (36) becomes

$$\epsilon \mathrm{e}^{-rT} K_\alpha \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\sigma_0 X + \sigma_0 W_s > \ln(\frac{K}{s_0}) - \mu T\right\}} \mathrm{e}^{(1+\alpha)x_0 + \mu T} \mathrm{e}^{\alpha \mu s} \mathrm{e}^{\sigma_0 X} \mathrm{e}^{(1+\alpha)\sigma_0 W_s}\right] \mathrm{d}s$$
$$= \epsilon \mathrm{e}^{-rT} K_\alpha s_0^{(1+\alpha)} \mathrm{e}^{rT} \mathrm{e}^{-\frac{\sigma_0^2}{2}T}$$
$$\times \int_0^T \mathrm{e}^{\alpha \mu s} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\sigma_0 X + \sigma_0 W_s > \ln\left(\frac{K}{s_0}\right) - \mu T\right\}} \mathrm{e}^{\sigma_0 X} \mathrm{e}^{(1+\alpha)\sigma_0 W_s}\right] \mathrm{d}s.$$

The expectation with respect to the risk-neutral measure can be exchanged with the time integration. Moreover, by exploiting the independence of X and W_s, we get the final result

$$\epsilon K_\alpha s_0^{(1+\alpha)} \mathrm{e}^{-\frac{\sigma_0^2}{2}T} \int_0^T \mathrm{e}^{\alpha \mu s} \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left\{x+y>-\sqrt{T}d_2\right\}} \mathrm{e}^{\sigma_0 x} \mathrm{e}^{(1+\alpha)\sigma_0 y} \phi(x, 0, T-s)\phi(y, 0, s) \mathrm{d}x \mathrm{d}y \mathrm{d}s$$
$$= \epsilon s_0^{(1+\alpha)} \mathcal{K}_1 \int_0^T \mathrm{e}^{\alpha \mu s} \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left\{x+y>-\sqrt{T}d_2\right\}} \mathrm{e}^{\sigma_0 x} \mathrm{e}^{(1+\alpha)\sigma_0 y} \phi(x, 0, T-s)\phi(y, 0, s) \mathrm{d}x \mathrm{d}y \mathrm{d}s.$$

Then, we have from the definition of X_T^0

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{[X_0^T > \ln(K)]} \mathrm{e}^{X_T^0}\right] &= \int_{x>-d_2} \mathrm{e}^{x_0 + \mu T + \sigma_0 \sqrt{T} x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-x^2}{2}} \mathrm{d}x \\ &= s_0 \mathrm{e}^{rT} \mathrm{e}^{-\frac{\sigma_0^2}{2}T} \int_{x>-d_2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\left(\frac{x}{\sqrt{2}} - \frac{\sigma_0\sqrt{T}}{\sqrt{2}}\right)^2} \mathrm{e}^{\frac{\sigma_0^2 T}{2}} \mathrm{d}x \\ &= s_0 \mathrm{e}^{rT} \int_{x>-d_2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\left(\frac{x}{\sqrt{2}} - \frac{\sigma_0\sqrt{T}}{\sqrt{2}}\right)^2} \mathrm{d}x. \end{aligned} \tag{37}$$

By setting $y = x - \sigma_0\sqrt{T}$, the integral in (68) reads as

$$\mathbb{E}\left[\mathbb{1}_{[X_0^T > \ln(K)]} \mathrm{e}^{X_T^0}\right] = s_0 \mathrm{e}^{rT} \int_{y>-d_1} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}x = s_0 \mathrm{e}^{rT} N(d_1). \tag{38}$$

Eventually by multiplying by $-\epsilon \mathrm{e}^{-rT} \mathcal{K}_2$, we obtain

$$-\epsilon \mathrm{e}^{-rT} \mathcal{K}_2 \mathbb{E}\left[\mathbb{1}_{[X_0^T > \ln(K)]} \mathrm{e}^{X_T^0}\right] = -\epsilon \mathcal{K}_2 s_0 N(d_1) \tag{39}$$

Let us now compute the last term in the bracket { } in (35). We have

$$\begin{aligned}\mathcal{K}_2\mathbb{E}\left[\mathbb{1}_{[X_0^T>\ln(K)]}\mathrm{e}^{X_T^0}\mathrm{e}^{\alpha X_T^0}\right] &= \mathcal{K}_2\int_{x_0+\mu T+\sigma_0\sqrt{T}x>\ln(K)}\mathrm{e}^{(1+\alpha)\left(x_0+\mu T+\sigma_0\sqrt{T}x\right)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-x^2}{2}}\mathrm{d}x\\ &= \mathcal{K}_2\int_{x>-d_2}\mathrm{e}^{(1+\alpha)(x_0+\mu T)}\mathrm{e}^{(1+\alpha)\sigma_0\sqrt{T}x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-x^2}{2}}\mathrm{d}x \qquad (40)\\ &= \mathcal{K}_2 s_0^{(1+\alpha)}\mathrm{e}^{(1+\alpha)rT}\mathrm{e}^{-(1+\alpha)\frac{\sigma_0^2}{2}T}\int_{x>-d_2}\mathrm{e}^{(1+\alpha)\sigma_0\sqrt{T}x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-x^2}{2}}\mathrm{d}x\end{aligned}$$

The integrand function can be recast as

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{(1+\alpha)\sigma_0\sqrt{T}x}\mathrm{e}^{\frac{-x^2}{2}} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\left(\frac{x}{\sqrt{2}}-\frac{(1+\alpha)\sigma_0\sqrt{T}}{\sqrt{2}}\right)^2}\mathrm{e}^{\frac{\sigma_0^2}{2}(1+\alpha)^2T}.$$

By the change of variable $x \mapsto y = x - (1+\alpha)\sigma_0\sqrt{T}$, the domain of integration becomes

$$\begin{aligned}y > -d_2 - (1+\alpha)\sigma_0 T &= -\frac{1}{\sigma_0\sqrt{T}}\left(\ln\left(\frac{K}{s_0}\right) - rT + \sigma_0^2/2 - (1+\alpha)\sigma_0^2 T\right)\\ &= -\frac{1}{\sigma_0\sqrt{T}}\left(\ln\left(\frac{K}{s_0}\right) + rT + \frac{\sigma_0^2}{2}(2\alpha+1)T\right)\\ &= -d(2\alpha+1).\end{aligned}$$

Therefore, (40) becomes

$$\begin{aligned}&\mathcal{K}_2\mathbb{E}\left[\mathbb{1}_{[X_0^T>\ln(K)]}\mathrm{e}^{X_T^0}\mathrm{e}^{\alpha X_T^0}\right]\\ &= \mathcal{K}_2 s_0^{(1+\alpha)}\mathrm{e}^{(1+\alpha)rT}\mathrm{e}^{-(1+\alpha)\frac{\sigma_0^2}{2}T}\mathrm{e}^{\frac{\sigma_0^2}{2}(1+\alpha)^2T}\int_{y>-d(2\alpha+1)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-y^2}{2}}\mathrm{d}y\\ &= \mathcal{K}_2 s_0^{(1+\alpha)}\mathrm{e}^{(1+\alpha)rT}\mathrm{e}^{\alpha(1+\alpha)\frac{\sigma_0^2}{2}T}N(d(2\alpha+1))\end{aligned}$$

Eventually by multiplying by $\epsilon\mathrm{e}^{-rT}$, we get

$$\begin{aligned}\mathcal{K}_2\mathbb{E}\left[\mathbb{1}_{[X_0^T>\ln(K)]}\mathrm{e}^{X_T^0}\mathrm{e}^{\alpha X_T^0}\right] &= \mathcal{K}_2 s_0^{(1+\alpha)}\mathrm{e}^{\alpha rT}\mathrm{e}^{\alpha(1+\alpha)\frac{\sigma_0^2}{2}T}N(d(2\alpha+1))\\ &= \epsilon\mathcal{K}_3 s_0^{(1+\alpha)}N(d(2\alpha+1))\end{aligned}$$ □

By Proposition 3, we have that the explicit computation of the corrected fair price is reduced to a numerical evaluation of a deterministic integral, which might be more efficient than directly simulating the random variables involved.

Remark 8 Our result in Proposition 3 covers the case of a perturbation around the classical Black–Scholes model. This is different in this sense from the one discussed in Takahashi (1999).

Remark 9 We could have also considered the second order perturbation $Pr^2(0;T)$ around the BS price. This is given by

$$Pr^2(0;T) = Pr^1(0;T) + \epsilon^2 \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T^0)' X_T^2\right] + \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T^0)''\left(X_T^1\right)^2\right],$$

with Pr^1 the up to first-order price in Eq. (33). For the particular case of a European call option, we have that $\Phi'' = \delta(X - \log K)\mathrm{e}^X + \mathbb{1}_{[X>\log K]}\mathrm{e}^X$, with δ the Dirac measure at the origin. Thus, the correction up to the second order of the BS price for a European call option reads

$$\begin{aligned} Pr^2(0;T) = Pr^1 &+ \epsilon^2 \mathcal{K}_4 s_0^{2\alpha+1} I_1(s,T,2\alpha) \\ &+ \epsilon^2 \mathcal{K}_5 s_0^{2\alpha+1} I_2(s,T) + \epsilon^2 \mathcal{K}_6 s_0^{\alpha+1} I_1(s,T,\alpha) \\ &+ \epsilon^2 \mathcal{K}_7 s_0^{2\alpha+1} I_3(r,s,T) + \epsilon^2 \mathcal{K}_8 s_0^{2\alpha+1} N\left(d(-3-4\alpha)\right) \\ &+ \epsilon^2 \mathcal{K}_9 s_0^{\alpha+1} N\left(d(-1-2\alpha)\right) + \epsilon^2 \mathcal{K}_{10} s_0 N(d(1)), \end{aligned} \tag{41}$$

with Pr^1 as in Eq. (34), the notations as in Proposition 3 and

$$\begin{aligned} \mathcal{K}_4 &= \left(C_\alpha^1 + 2K_\alpha \frac{\sigma_1}{\alpha\sigma_0}\right)\mathrm{e}^{-\frac{\sigma_0^2}{2}T},\ \mathcal{K}_5 = C_\alpha^2 \mathrm{e}^{\alpha r T - \frac{\sigma_0^2}{2}(\alpha+1)T}, \\ \mathcal{K}_6 &= \left(C_\alpha^3 + 2K_\alpha \frac{\sigma_1}{\alpha\sigma_0}\right)\mathrm{e}^{-\frac{\sigma_0^2}{2}T},\ \mathcal{K}_7 = \left(C_\alpha^4 + 2K_\alpha^2\right)\mathrm{e}^{-\frac{\sigma_0^2}{2}T}, \\ \mathcal{K}_8 &= C_\alpha^5 \mathrm{e}^{\frac{\sigma_0^2}{2}T\alpha(2\alpha+1)+2\alpha r T},\ \mathcal{K}_9 = \left(C_\alpha^6 + \frac{\sigma_1}{\alpha\sigma_0}\right)\mathrm{e}^{\frac{\sigma_0^2}{2}T\alpha(\alpha+1)+\alpha r T}, \\ \mathcal{K}_{10} &= \left(C_\alpha^7 - \frac{\sigma_1}{\alpha\sigma_0}\right), \\ I_2(s,T) &= \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left[x+y>-\sqrt{T}d(1)\right]} \mathrm{e}^{\alpha\mu s+(2\alpha+1)\sigma_0 y+(\alpha+1)\sigma_0 x} \\ &\quad \times \phi(x;y,T-s)\phi(y;0,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s, \\ I_3(r,s,T) &= \int_0^T \int_0^s \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left[x+y+z>-\sqrt{T}d(1)\right]} \mathrm{e}^{\alpha\mu(s+r)+\sigma_0 x+(1+\alpha)\sigma_0 y+(1+2\alpha)\sigma_0 z} \\ &\quad \times \phi(x;y,T-s)\phi(y;z,s-r)\phi(z;0,r)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}s, \end{aligned}$$

3.1.1 Numerical results concerning the pricing formula in Proposition 3.

We will now use the techniques based on the *multi-element Polynomial Chaos Expansion* (PCE) approach, to show the accuracy of the above derived approximated pricing formula in Proposition 3.

In what follows, we will numerically compute the first-order correction of the price of an European call option, whose payoff function is $\left(\mathrm{e}^{X_T^\epsilon} - K\right)_+$. In particular, we focus our attention on the second summand of

Table 1 Numerical values of the parameters employed in further computations

Sigma1	r	Alpha	T	K	Sigmazero	$s0$	Epsilon
0.15	0.03	0.1	0.5	50	0.15	90	0.15
			0.75	75	0.25	100	0.1
			1	90	0.35	110	0.06
			1.25	100			0.03
			1.5	110			0.01
			1.75	125			
			2				

$$Pr_1(0;T) = P_{BS} + \epsilon \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi'(X_T^0)X_T^1\right]. \tag{42}$$

Also, X_T^0 and X_T^1 are defined as in Proposition 2.

The expectation is computed by means of the standard Monte Carlo method, using 10,000 independent realization, and by means of the multi-element PCE, see, e.g., Bonollo et al. (2015), Crestaux et al. (2009), Ernst et al. (2012), Peccati and Taqqu (2011) and references therein, for a detailed introduction to such a method. Indeed, the random variable of interest is

$$\mathbb{1}_{\{X_0^T(\omega)>\ln(K)\}}\exp(X_T^0)X_T^1.$$

For both methods, we will use the available analytical expression of X_T^0 and X_T^1, depending on the function $f(x)$. In what follows, $D := \{X_0^T(\omega) > \ln(K)\}$.

In particular exploiting the linearity of the expectation and the definition of the two random variables involved, (42) becomes

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0}\int_0^T K_\alpha \mathrm{e}^{\alpha X_s^0}\mathrm{d}s\right] + K_2\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0}\mathrm{e}^{\alpha X_T^0}\right] - K_2\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0}\right] \tag{43}$$

Then, we perform a *multi-element* PCE approximation of each random variable in (43), setting the degree of the approximation to be $p = 15$, since the degree of precision reached for such approximation seems to be sufficient. For higher degree, the computational costs increase as well as numerical fluctuations; as witnessed exploiting the *Non-Intrusive Spectral Projection* (NISP) toolbox developed within the *Scilab* open-source software for mathematics and engineering sciences becomes relevant for multi-element approximation. It is worth to mention that *multi-element* PCE is nothing else that a PCE focused on D. Moreover, the global statistics are given by D, scaled by means of the weight w.

The numerical values of the parameters are gathered in Table 1.

Figures 1, 2 and 3 report percentage error between analytical and PCE approximations of the above pricing equation, for different strike prices, maturity and ϵ; also similarly, the same results are collected in Fig. 4 regarding relative error over ϵ grouped by maturity.

	PCE					
	Strike (K)					
T	50	75	90	100	110	125
0,5	0,0025%	0,0018%	0,0010%	0,0003%	0,0005%	0,0016%
0,75	0,0038%	0,0028%	0,0017%	0,0008%	0,0002%	0,0017%
1	0,0052%	0,0037%	0,0024%	0,0014%	0,0002%	0,0016%
1,25	0,0065%	0,0048%	0,0033%	0,0020%	0,0007%	0,0013%
1,5	0,0079%	0,0058%	0,0041%	0,0027%	0,0013%	0,0010%
1,75	0,0092%	0,0069%	0,0050%	0,0035%	0,0019%	0,0005%
2	0,0106%	0,0079%	0,0059%	0,0043%	0,0026%	0,0000%
0,5	0,0027%	0,0021%	0,0015%	0,0010%	0,0003%	0,0007%
0,75	0,0041%	0,0031%	0,0024%	0,0017%	0,0009%	0,0005%
1	0,0055%	0,0042%	0,0033%	0,0024%	0,0015%	0,0001%
1,25	0,0069%	0,0054%	0,0042%	0,0033%	0,0022%	0,0004%
1,5	0,0084%	0,0065%	0,0052%	0,0041%	0,0029%	0,0009%
1,75	0,0098%	0,0077%	0,0062%	0,0050%	0,0037%	0,0015%
2	0,0113%	0,0089%	0,0072%	0,0059%	0,0045%	0,0022%
0,5	0,0025%	0,0023%	0,0019%	0,0015%	0,0010%	0,0001%
0,75	0,0043%	0,0034%	0,0029%	0,0023%	0,0017%	0,0005%
1	0,0058%	0,0046%	0,0039%	0,0032%	0,0025%	0,0011%
1,25	0,0073%	0,0058%	0,0049%	0,0042%	0,0033%	0,0017%
1,5	0,0088%	0,0070%	0,0060%	0,0051%	0,0041%	0,0024%
1,75	0,0103%	0,0083%	0,0071%	0,0061%	0,0050%	0,0032%
2	0,0119%	0,0096%	0,0082%	0,0071%	0,0059%	0,0040%

Fig. 1 Percentage error for PCE and analytical estimation—error w.r.t. strike price K

In particular, Figs. 5, 6 and 7 show that, as expected, the relative error increases with and volatility; nonetheless, the relative error remains low. Same reasoning holds increasing the maturity T.

Also varying ϵ, it emerges how the error decreases for smaller values of ϵ, in accord with the small noise expansion nature of the pricing formula.

3.2 A correction given by an exponential function and jumps

In what follows, we extend the results in Sect. 3.1 to the second model in Sect. 3. In particular, we will consider a correction up to the first order around the BS price (for a European call option) where both diffusive and jump perturbations are taken into account. We consider an asset whose return evolves according to Eq. (27) and consider as before the particular case where $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{R}_0$. Carrying out the asymptotic expansion in powers of ϵ, $0 \leq \epsilon \leq \epsilon_0$, and stopping it at the second order, we get the following proposition:

Fig. 2 Percentage error for PCE and analytical estimation—error w.r.t. σ

s0	epsilon	PCE SIGMA		
s0	epsilon	0,15	0,25	0,35
90	0,010	0,00%	0,00%	0,00%
90	0,030	0,01%	0,01%	0,01%
90	0,060	0,01%	0,01%	0,02%
90	0,100	0,01%	0,02%	0,03%
90	0,150	0,01%	0,02%	0,03%
100	0,010	0,00%	0,00%	0,01%
100	0,030	0,01%	0,01%	0,01%
100	0,060	0,01%	0,02%	0,02%
100	0,100	0,01%	0,02%	0,03%
100	0,150	0,02%	0,03%	0,04%
110	0,010	0,00%	0,00%	0,01%
110	0,030	0,01%	0,01%	0,02%
110	0,060	0,01%	0,02%	0,03%
110	0,100	0,02%	0,03%	0,04%
110	0,150	0,02%	0,03%	0,04%

				PCE epsilon				
Strile (K)	s0	sigma	T	0,01	0,03	0,06	0,1	0,15
100	90	0,15	0,5	0,000%	0,000%	0,000%	0,000%	0,000%
100	90	0,15	0,75	0,001%	0,001%	0,001%	0,002%	0,002%
100	90	0,15	1	0,001%	0,002%	0,003%	0,003%	0,003%
100	90	0,15	1,25	0,002%	0,004%	0,004%	0,005%	0,005%
100	90	0,15	1,5	0,003%	0,005%	0,006%	0,007%	0,007%
100	90	0,15	1,75	0,004%	0,006%	0,008%	0,009%	0,010%
100	90	0,15	2	0,004%	0,008%	0,010%	0,012%	0,013%
100	100	0,15	0,5	0,001%	0,002%	0,002%	0,002%	0,002%
100	100	0,15	0,75	0,002%	0,003%	0,004%	0,004%	0,004%
100	100	0,15	1	0,002%	0,004%	0,006%	0,006%	0,007%
100	100	0,15	1,25	0,003%	0,006%	0,008%	0,009%	0,009%
100	100	0,15	1,5	0,004%	0,008%	0,010%	0,012%	0,012%
100	100	0,15	1,75	0,005%	0,010%	0,013%	0,015%	0,016%
100	100	0,15	2	0,006%	0,012%	0,016%	0,018%	0,019%
100	110	0,15	0,5	0,001%	0,003%	0,003%	0,004%	0,004%
100	110	0,15	0,75	0,002%	0,004%	0,006%	0,006%	0,007%
100	110	0,15	1	0,003%	0,006%	0,008%	0,009%	0,010%
100	110	0,15	1,25	0,004%	0,008%	0,011%	0,013%	0,014%
100	110	0,15	1,5	0,005%	0,010%	0,014%	0,016%	0,017%
100	110	0,15	1,75	0,006%	0,013%	0,017%	0,020%	0,022%
100	110	0,15	2	0,007%	0,015%	0,020%	0,024%	0,026%

Fig. 3 Percentage error for PCE and analytical estimation—error w.r.t. ϵ

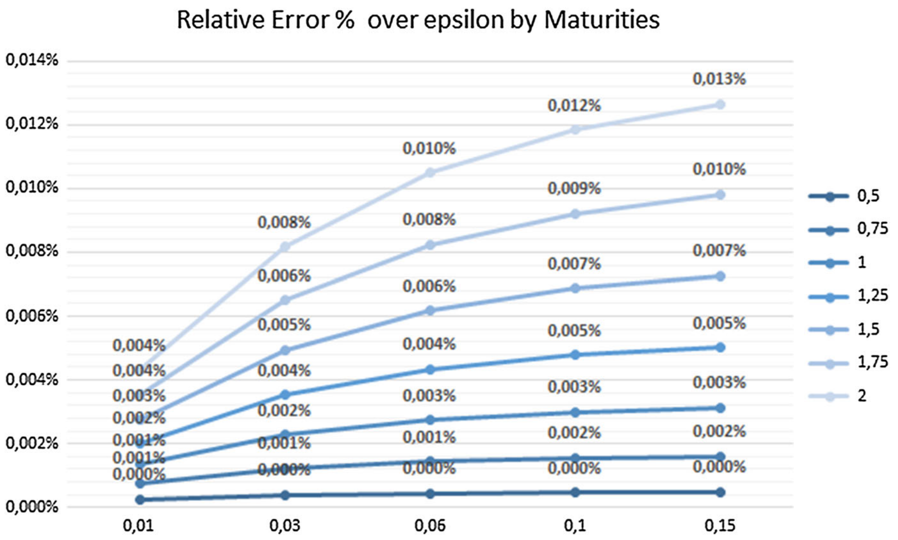

Fig. 4 Percentage error for PCE and analytical estimation grouped by maturity—error w.r.t. ϵ

				PCE					
				Strike (K)					
s0	sigma	epsilon	T	50	75	90	100	110	125
90	0,15	0,01	0,5	0,003%	0,002%	0,002%	0,001%	0,000%	0,001%
90	0,15	0,01	0,75	0,004%	0,004%	0,003%	0,002%	0,002%	0,000%
90	0,15	0,01	1	0,006%	0,005%	0,004%	0,004%	0,003%	0,002%
90	0,15	0,01	1,25	0,008%	0,007%	0,006%	0,006%	0,005%	0,003%
90	0,15	0,01	1,5	0,009%	0,009%	0,008%	0,007%	0,007%	0,005%
90	0,15	0,01	1,75	0,011%	0,011%	0,010%	0,010%	0,009%	0,007%
90	0,15	0,01	2	0,013%	0,013%	0,012%	0,012%	0,011%	0,010%
100	0,15	0,01	0,5	0,003%	0,002%	0,002%	0,002%	0,001%	0,000%
100	0,15	0,01	0,75	0,004%	0,004%	0,003%	0,003%	0,002%	0,001%
100	0,15	0,01	1	0,006%	0,005%	0,005%	0,005%	0,004%	0,003%
100	0,15	0,01	1,25	0,008%	0,007%	0,007%	0,006%	0,006%	0,005%
100	0,15	0,01	1,5	0,010%	0,009%	0,009%	0,008%	0,008%	0,006%
100	0,15	0,01	1,75	0,012%	0,011%	0,011%	0,010%	0,010%	0,009%
100	0,15	0,01	2	0,014%	0,013%	0,013%	0,012%	0,012%	0,011%
110	0,15	0,01	0,5	0,003%	0,003%	0,002%	0,002%	0,002%	0,001%
110	0,15	0,01	0,75	0,005%	0,004%	0,004%	0,003%	0,003%	0,002%
110	0,15	0,01	1	0,006%	0,006%	0,005%	0,005%	0,005%	0,004%
110	0,15	0,01	1,25	0,008%	0,007%	0,007%	0,007%	0,006%	0,005%
110	0,15	0,01	1,5	0,010%	0,009%	0,009%	0,009%	0,008%	0,007%
110	0,15	0,01	1,75	0,012%	0,011%	0,011%	0,011%	0,010%	0,009%
110	0,15	0,01	2	0,014%	0,013%	0,013%	0,013%	0,012%	0,012%

Fig. 5 Percentage error for PCE and analytical estimation—error w.r.t. strike price K

Proposition 4 *Let us assume X_t^ϵ evolves according to* Eq. (27) *with $f(x) = e^{\alpha x}$, for some $\alpha \in \mathbb{R}$, then we have the asymptotic expansion up to the second order in powers of ϵ, $0 \le \epsilon \le \epsilon_0$, $X_t^\epsilon = X_t^0 + \epsilon X_t^1 + \epsilon^2 X_t^2 + R_2(\epsilon, t)$, where the coefficients are given by*

Fig. 6 Percentage error for PCE and analytical estimation—error w.r.t. σ

s0	epsilon	PCE SIGMA		
s0	epsilon	0,15	0,25	0,35
90	0,010	0,006%	0,010%	0,016%
90	0,030	0,012%	0,024%	0,042%
90	0,060	0,016%	0,036%	0,068%
90	0,100	0,019%	0,046%	0,091%
90	0,150	0,021%	0,053%	0,111%
100	0,010	0,007%	0,010%	0,016%
100	0,030	0,014%	0,025%	0,041%
100	0,060	0,019%	0,038%	0,068%
100	0,100	0,023%	0,048%	0,093%
100	0,150	0,025%	0,056%	0,114%
110	0,010	0,007%	0,010%	0,016%
110	0,030	0,015%	0,025%	0,041%
110	0,060	0,021%	0,039%	0,069%
110	0,100	0,026%	0,051%	0,094%
110	0,150	0,029%	0,060%	0,116%

				PCE epsilon				
Strile (K)	s0	sigma	T	0,01	0,03	0,06	0,1	0,15
100	90	0,15	0,5	0,001%	0,002%	0,002%	0,002%	0,002%
100	90	0,15	0,75	0,002%	0,004%	0,004%	0,005%	0,005%
100	90	0,15	1	0,004%	0,006%	0,008%	0,008%	0,009%
100	90	0,15	1,25	0,006%	0,010%	0,012%	0,013%	0,014%
100	90	0,15	1,5	0,007%	0,013%	0,017%	0,018%	0,019%
100	90	0,15	1,75	0,010%	0,017%	0,022%	0,024%	0,026%
100	90	0,15	2	0,012%	0,022%	0,028%	0,031%	0,033%
100	100	0,15	0,5	0,002%	0,003%	0,003%	0,003%	0,004%
100	100	0,15	0,75	0,003%	0,005%	0,006%	0,007%	0,007%
100	100	0,15	1	0,005%	0,008%	0,010%	0,011%	0,012%
100	100	0,15	1,25	0,006%	0,012%	0,015%	0,016%	0,018%
100	100	0,15	1,5	0,008%	0,015%	0,020%	0,022%	0,024%
100	100	0,15	1,75	0,010%	0,020%	0,026%	0,029%	0,031%
100	100	0,15	2	0,012%	0,024%	0,032%	0,036%	0,039%
100	110	0,15	0,5	0,002%	0,004%	0,005%	0,005%	0,005%
100	110	0,15	0,75	0,003%	0,006%	0,008%	0,009%	0,010%
100	110	0,15	1	0,005%	0,010%	0,012%	0,014%	0,015%
100	110	0,15	1,25	0,007%	0,013%	0,017%	0,020%	0,021%
100	110	0,15	1,5	0,009%	0,017%	0,023%	0,026%	0,028%
100	110	0,15	1,75	0,011%	0,021%	0,029%	0,033%	0,036%
100	110	0,15	2	0,013%	0,026%	0,035%	0,041%	0,045%

Fig. 7 Percentage error for PCE and analytical estimation—error w.r.t. ϵ

$$
\begin{aligned}
X_t^0 &= x_0 + \mu t + \sigma_0 W_t, \quad \text{with law } \mathcal{N}\left(x_0 + \mu t, \sigma_0^2 t\right);\\
X_t^1 &= \int_0^t K_\alpha e^{\alpha X_s^0} ds + \frac{\sigma_1}{\alpha\sigma_0}\left(e^{\alpha X_t^0} - 1\right) + \lambda t\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right) + \sum_{i=1}^{N_t} J_i;\\
X_t^2 &= C_\alpha^1 \int_0^t e^{2\alpha X_s^0} ds + C_\alpha^2 e^{\alpha X_t^0} \int_0^t e^{\alpha X_s^0} ds\\
&\quad + C_\alpha^3 \int_0^t e^{\alpha X_s^0} ds + C_\alpha^4 \int_0^t e^{\alpha X_s^0} \int_0^s e^{\alpha X_r^0} dr ds\\
&\quad + C_\alpha^5 e^{2\alpha X_t^0} + C_\alpha^6 e^{\alpha X_t^0} + C_\alpha^7 + C_\alpha^8 \lambda\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right)\nu(dx) \int_0^t s e^{\alpha X_s^0} ds\\
&\quad - t e^{\alpha X_t^0} \lambda\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right) + \frac{\sigma_1}{\sigma_0}\lambda\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right)\int_0^t e^{\alpha X_s^0} ds\\
&\quad + C_9^\alpha \int_0^t \sum_{i=1}^{N_s} J_i e^{\alpha X_s^0} ds + \frac{\sigma_1}{\sigma_0} e^{\alpha X_t^0} \sum_{i=1}^{N_t} J_i - \frac{\sigma_1}{\sigma_0} \sum_{i=1}^{N_t} J_i \int_0^t e^{\alpha X_s^0} ds,
\end{aligned}
\tag{44}
$$

with the constants as in Proposition 2 *and*

$$
C_\alpha^8 = \frac{\sigma_1}{\sigma_0}\alpha\mu + \frac{\sigma_0\sigma_1}{2}\alpha^2 - 2\sigma_0\sigma_1\alpha, \quad C_\alpha^9 = 2\sigma_0\sigma_1\alpha - \frac{\sigma_1}{\sigma_0}\alpha\mu - \frac{\sigma_0\sigma_1}{2}\alpha^2.
$$

Proof The proof follows from Proposition 2 just taking into account the presence of the Poisson random measure terms and applying Itô's lemma, together with the stochastic Fubini theorem. □

Remark 10 As mentioned in Remark 6, it is easy to extend Proposition 4 and formula (33) to the case where $f(x) = e^{\alpha x}$ is replaced by $\int_{\mathbb{R}_0} e^{i\alpha x} \varrho(d\alpha)$, resp. $\int_{\mathbb{R}_0} e^{\alpha x} \varrho(d\alpha)$, with assumptions corresponding to those in Remark 6.

Proposition 5 *Let us consider the model described by* (27) *in the particular case of an European call option* Φ *with payoff given by* $\Phi(X_T^\epsilon) = \left(e^{X_t^\epsilon} - K\right)_+$. *Then, the approximated price up to the first-order* $Pr_\nu^1(0;T)$, *in the sense explained in* Remark 7, *is explicitly given by*

$$
Pr_\nu^1(0;T) = Pr^1 + \epsilon T s_0 N(d_1)\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right) + \epsilon T s_0 N(d_1)\,\delta\lambda,
$$

where Pr^1 *is the corrected fair price up to the first order as given in* Eq. (34) *(the notations are as* Proposition 3*).*

Proof The proof is analogous of the proof of Proposition 3 adding the jump process. The claim follows then from the independence of the jump process and of the Brownian motion together with the fact that $\mathbb{E}\left[\sum_{i=1}^{N_t} J_i\right] = \delta T \lambda$ as consequence of the definition of J_i in Sect. 3.1. □

3.2.1 Numerical results concerning the pricing formula in Proposition 4

We consider numerically the model discussed in Proposition 5, assuming that the J_i is independent and normally distributed random variable

$$J_i \sim \mathcal{N}(\gamma, \delta^2) \quad \gamma = 0.05, \quad \delta = 0.02,$$

and $\lambda = 2$. In particular, we are aiming at numerically computing the expectations in the second summand of (42), which in the present case reads

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} \int_0^T K_\alpha e^{\alpha X_s^0} ds\right] + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} e^{\alpha X_T^0}\right] - K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0}\right] \\ &\quad + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} \lambda T \left(e^{\gamma + \frac{\delta^2}{2}} - 1\right)\right] + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} \sum_{i=1}^{N_T} J_i\right]. \end{aligned} \tag{45}$$

By means of independence of the jumps and $\mathbb{E}_t\left[\sum_{i=1}^{N_T} J_i\right] = \lambda T \delta$, we get

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} \int_0^T K_\alpha e^{\alpha X_s^0} ds\right] + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} e^{\alpha X_T^0}\right] - K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0}\right] \\ &\quad K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0} \lambda T \left(e^{\gamma + \frac{\delta^2}{2}} - 1\right)\right] + K_2 \lambda T \delta \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_T^0}\right]. \end{aligned} \tag{46}$$

We are going to compute (64) by *multi-element* PCE approximations.

Figures 5, 6 and 7 report percentage error between analytical and PCE approximations of above pricing equation, for different strike prices, maturity and ϵ; also similarly, the same results are collected in Fig. 8 regarding relative error over ϵ grouped by maturity. Also parameters are shown in Table 1

Conclusion similar to the one drawn in Sect. 3 can be derived.

3.3 A correction given by a polynomial function

Let us consider Eq. (24) with f a polynomial correction, namely $f(x) = \sum_{i=0}^{N} \alpha_i x^i$, with $\alpha_i \in \mathbb{R}$ and $N \in \mathbb{N}_0$. We then get the following proposition.

Proposition 6 *Let us consider the case of the B–S model corrected by a nonlinear term given by* (24) *with* $f(x) = \sum_{i=0}^{N} \alpha_i x^i$, *for some* $\alpha_i \in \mathbb{R}$, *then the expansion coefficients for the solution* X_t^ϵ *of* (24) *up to the second order are given by the system*

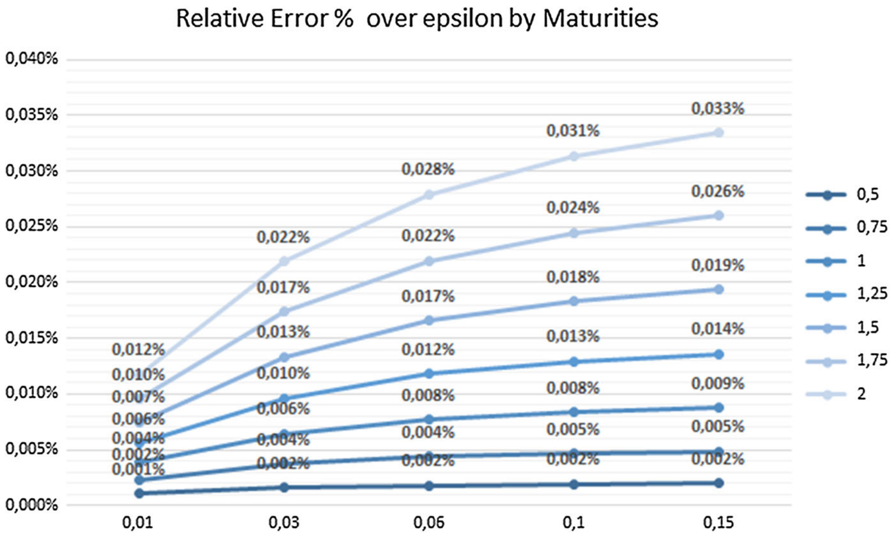

Fig. 8 Percentage error for PCE and analytical estimation grouped by maturity—error w.r.t. ϵ

$$
\begin{aligned}
X_t^0 &= x_0 + \mu t + \sigma_0 W_t, \quad \text{with law } \mathcal{N}\left(x_0 + \mu t, \sigma_0^2 t\right); \\
X_t^1 &= \sum_{i=1}^{N} \tilde{K}_i (X_t^0)^{i+1} - \sum_{i=0}^{N} \int_0^t K_i (X_s^0)^i \mathrm{d}s + \sigma_1 \alpha_0 W_t; \\
X_t^2 &= \sum_{k=1}^{2N+1} C_k^1 (X_t^0)^k - \sum_{k=1}^{2N+1} \int_0^t C_k^2 (X_s^0)^k \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \sum_{j=0}^{N} \int_0^t \int_0^s C_{i,j}^3 (X_s^0)^{i-1} (X_r^0)^j \mathrm{d}r \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \sum_{j=0}^{N} (X_t^0)^i \int_0^s C_{i,j}^4 (X_r^0)^j \mathrm{d}r.
\end{aligned}
\tag{47}
$$

where the constants are given by

$$
K_i = \begin{cases} \sigma_0 \sigma_1 \alpha_i + \frac{\sigma_1}{\sigma_0} \mu \alpha_i + \frac{\sigma_0 \sigma_1}{2} \alpha_{i+1} (i+1), & i \neq 0, i \neq N, \\ \sigma_0 \sigma_1 \alpha_0 + \frac{\sigma_0 \sigma_1 \alpha_1}{2}, & i = 0, \\ \sigma_0 \sigma_1 \alpha_N + \frac{\sigma_1}{\sigma_0} \mu \alpha_N, & i = N, \end{cases} \qquad \tilde{K}_i = \frac{\sigma_1}{\sigma_0} \frac{\alpha_i}{(i+1)},
$$

$$
C_k^1 = \gamma_k^1 + \gamma_k^2 + \gamma_k^3,
$$

where

$$\gamma_k^1 = \begin{cases} \sum_{k=i+j+1} \mu i \alpha_i + \frac{\sigma_1}{\sigma_0} - \frac{\sigma_0}{2}(i+j+1), & k \neq 1, k \neq 2N, \\ \frac{\sigma_0}{2}, & k = 0, \\ \mu \frac{\sigma_1}{\sigma_0} N \alpha_N \left(\sigma_0 \sigma_1 \alpha_N + \frac{\sigma_1}{\sigma_0} \mu \alpha_N \right), & k = 2N, \end{cases}$$

$$\gamma_k^2 = \begin{cases} \left(\frac{(-1)^k + 1}{2} \frac{\sigma_1^2}{2} \right) \alpha_k^2, & \text{if } 1 \leq k \leq N, \\ 0, & \text{otherwise} \end{cases} .$$

$$\gamma_k^3 = \sum_{i+j=k-1} 2\sigma_0 \sigma_1 \alpha_i i \tilde{K}_j,$$

$$C_{i,j}^3 = - \begin{cases} \frac{\sigma_1}{\sigma_0} \alpha_1 K_0, & \text{if } i = 1, \ \ j = 0, \\ \frac{\sigma_1}{\sigma_0} i \alpha_i K_j + \frac{\sigma_0 \sigma_1}{2} i \alpha_i K_j (i-1), & \text{otherwise.} \end{cases}$$

$$C_{i,j}^4 = \frac{\sigma_1}{\sigma_0} \alpha_i K_j,$$

Proof The proof consists in a series of applications of Itô's formula and stochastic Fubini theorem, see, e.g., Filipovic (2009), Theorem 6.2. In fact, substituting $f(x) = \sum_{i=0}^N \alpha_i x^i$ into system (25) we obtain

$$\begin{aligned} X_t^0 &= x_0 + \mu t + \sigma_0 W_t, \quad \text{with law} \quad \mathcal{N}\left(x_0 + \mu t, \sigma_0^2 t\right); \\ X_t^1 &= -\int_0^t \sigma_0 \sigma_1 \left(\sum_{i=0}^N \alpha_i (X_s^0)^i \right) \mathrm{d}s + \int_0^t \sigma_1 \left(\sum_{i=0}^N \alpha_i (X_s^0)^i \right) \mathrm{d}W_s; \\ X_t^2 &= -\int_0^t \frac{\sigma_1^2}{2} \left(\sum_{i=0}^N \alpha_i (X_s^0)^i \right)^2 + 2\sigma_1 \left(\sum_{i=0}^N \alpha_i (X_s^0) \right)' X_s^1 \mathrm{d}s \\ &\quad + \int_0^t \sigma_1 \left(\sum_{i=0}^N \alpha_i (X_s^0) \right)' X_s^1 \mathrm{d}W_s. \end{aligned} \tag{48}$$

To compute X_t^1 obtaining Eq. (47), we apply Itô's lemma to the function $g(X_t^0) = \alpha_{i+1}(X_t^0)^{i+1}$ to get

$$\begin{aligned} (X_t^0)^{i+1} &= \int_0^t \left(\mu(i+1)(X_s^0)^i \frac{1}{2} \sigma_0^2 i(i+1)(X_s^0)^{i-1} \right) \mathrm{d}s \\ &\quad + \int_0^t (X_s^0)^i (i+1) \sigma_0 \mathrm{d}W_s. \end{aligned} \tag{49}$$

Then, summing up we obtain

$$\sum_{i=1}^{N}\int_0^t (X_s^0)^i (i+1)\sigma_0 \mathrm{d}W_s = \sum_{i=1}^{N}(X_s^0)^{i+1} + \\ -\sum_{i=1}^{N}\int_0^t \left(\mu(i+1)(X_s^0)^i + \frac{1}{2}\sigma_0^2 i(i+1)(X_s^0)^{i-1}\right)\mathrm{d}s. \tag{50}$$

Substituting now Eq. (50) into X^1 in Eq. (48), we obtain the following

$$X_t^1 = \sum_{i=1}^{N}\frac{\sigma_1}{\sigma_0}\frac{\alpha_i}{(i+1)}(X_t^0)^{i+1} + \\ -\sum_{i=1}^{N}\int_0^t \sigma_0\sigma_1\alpha_i(X_s^0)^i - \sum_{i=1}^{N}\int_0^t \mu(i+1)\frac{\sigma_1\alpha_i}{\sigma_0(i+1)}(X_s^0)^i + \\ -\sum_{i=1}^{N}\int_0^t \frac{1}{2}\sigma_0^2 i(i+1)\frac{\sigma_1\alpha_i}{\sigma_0(i+1)}(X_s^0)^{i-1}\mathrm{d}s,$$

and rearranging the terms we then get the desired result in (47) for X_t^1.

Substituting the expression of X_t^1 into X_t^2, we obtain

$$X_t^2 = -\sum_{i=1}^{N}\int_0^t \frac{\sigma_1^2}{2}\alpha_i^2(X_s^0)^{2i}\mathrm{d}s - \sum_{i,j=1}^{N}\int_0^t 2\sigma_0\sigma_1\alpha_i i\tilde{K}_j(X_s^0)^{i-1}(X_s^0)^{j+1}\mathrm{d}s \\ = \sum_{j=0}^{N}\sum_{i=1}^{N}\int_0^t\int_0^s 2\sigma_0\sigma_1\alpha_i i K_j(X_s^0)^{i-1}(X_r^0)^j \mathrm{d}r\mathrm{d}s \\ + \sum_{i,j=1}^{N}\int_0^t \sigma_1\alpha_i i K_j(X_s^0)^{i-1}(X_s^0)^{j+1}\mathrm{d}Ws \\ -\sum_{j=0}^{N}\sum_{i=1}^{N}\int_0^t\int_0^s \sigma_1\alpha_i i K_j(X_s^0)^{i-1}(X_r^0)^j\mathrm{d}r\mathrm{d}Ws.$$

Exploiting again the stochastic Fubini theorem from Eq. (50) and grouping the terms with the same powers, we obtain (47). □

Proposition 7 *Let us consider the particular case of* $N = 1$, *i.e., a linear perturbation, namely* $f(x) = \alpha_0 + \alpha_1 x$, $\alpha_i \in \mathbb{R}$, $i = 0, 1$. *Then, the terms up to the first order in* Eq. (47) *read*

$$X_t^0 = x_0 + \mu t + \sigma_0 W_t, \\ X_t^1 = \beta_1 t + \beta_2 t^2 + \beta_3 W_t + \beta_4 W_t^2 + \beta_5 t W_t - \int_0^t \beta_6 W_s \mathrm{d}s, \tag{51}$$

with

$$\beta_1 = -\sigma_0\sigma_1\alpha_0 - \sigma_0\sigma_1\alpha_1 x_0 - \frac{\sigma_0\sigma_1\alpha_1}{2},$$
$$\beta_2 = -\frac{\sigma_0\sigma_1\alpha_1\mu}{2}, \quad \beta_3 = \alpha_1\sigma_0 + x_0\sigma_1\alpha_1,$$
$$\beta_4 = \frac{\sigma_0\sigma_1\alpha_1}{2}, \quad \beta_5 = \sigma_1\alpha_1\mu, \quad \beta_6 = \sigma_1\alpha_1\mu + \sigma_0^2\sigma_1\alpha_1.$$

The first-order correction (in the sense discussed in Remark 7*) of the price of an European call option* Φ *with payoff given by* $\Phi(X_T^\epsilon) = \left(e^{X_T^\epsilon} - K\right)_+$ *is explicitly given by*

$$\begin{aligned} Pr^1(0;T) = P_{BS} &+ \epsilon s_0(\beta_1 + \sigma_0\beta_3 + \beta_4)TN(d_1) + \epsilon s_0(\beta_2 + \sigma_0^2\beta_4)T^2 N(d_1) \\ &+ \epsilon s_0(\beta_3 + 2\sigma_0\beta_4 T + T\beta_5)\sqrt{T}\phi(-d_1) - \epsilon s_0\beta_4 T d_1\phi(d_1) \\ &+ \epsilon s_0 T^2\beta_5\sigma_0 T^2 N(d_1) - \epsilon s_0 e^{+\frac{\sigma_0^2}{2}T}\beta_6 I(s,T), \end{aligned} \tag{52}$$

where the notation is as in Proposition 3 *and we have denoted for short by* $\phi(x)$ *the density function of the standard Gaussian law and we have set*

$$I(s,T) = \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\left[x+y>-\sqrt{T}d_1\right]} e^{\sigma_0(x+y)} y\phi(x;0,T-s)\phi(y;0,s)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

Proof Let us consider the linear function $f(x) = \alpha_0 + \alpha_1 x$, where $\alpha_0, \alpha_1 \in \mathbb{R}$. The approximated price up to the first-order $Pr^1(0;T)$ of an European call option with payoff function $\Phi(X_T^\epsilon) = \left(e^{X_T^\epsilon} - K\right)_+$ is

$$Pr^1(0;T) = P_{BS} + \epsilon e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(X_T^0)X_T^1\right] \tag{53}$$

where P_{BS} is the standard B–S price with underlying $s_0(t) = e^{X_t^0}$.

In particular, we have that X_T^0 and X_1^T are defined as

$$X_T^0 = x_0 + \mu T + \sigma_0 W_T \tag{54}$$
$$X_T^1 = \beta_1 T + \beta_2 T^2 + \beta_3 W_T + \beta_4 W_T^2 + \beta_5 T W_T - \beta_6 \int_0^T W_s \mathrm{d}s. \tag{55}$$

By linearity of the expectation, (53) becomes, collecting the terms with coefficients β_3 and β_5,

$$
\begin{aligned}
Pr^1(0;T) = P_{BS} + \epsilon \mathrm{e}^{-rT}\Big\{ & \mathbb{E}^{\mathbb{Q}}\left[\beta_1 T \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] \\
& + \mathbb{E}^{\mathbb{Q}}\left[\beta_2 T^2 \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] \\
& + \mathbb{E}^{\mathbb{Q}}\left[\beta_{3,5}^T W_T \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] \\
& + \mathbb{E}^{\mathbb{Q}}\left[\beta_4 W_T^2 \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] \\
& + \mathbb{E}^{\mathbb{Q}}\left[\beta_6 \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0} \int_0^T W_s \mathrm{d}s\right]\Big\},
\end{aligned} \tag{56}
$$

with $\beta_{3,5}^T := \beta_3 + T\beta_5$.

From the definition of X_T^0, we have that

$$
\epsilon \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\beta_1 T \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] = \epsilon T \beta_1 s_0 N(d_1),
$$

and as above we have

$$
\epsilon \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\beta_2 T^2 \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] = \epsilon T^2 \beta_2 s_0 N(d_1)
$$

Concerning the third term in (56), we have that

$$
\begin{aligned}
& \beta_{3,5}^T \mathbb{E}^{\mathbb{Q}}\left[W_T \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] \\
& = \beta_{3,5}^T s_0 \mathrm{e}^{rT} \mathrm{e}^{-\frac{\sigma_0^2}{2}T} \sqrt{T} \int_{x > -d_2} \mathrm{e}^{\sigma_0 \sqrt{T} x} x \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x \\
& = \beta_{3,5}^T s_0 \mathrm{e}^{rT} \mathrm{e}^{-\frac{\sigma_0^2}{2}T} \sqrt{T} \int_{x > -d_2} x \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\left(\frac{x}{\sqrt{2}} - \frac{\sigma_0\sqrt{T}}{\sqrt{2}}\right)^2} \mathrm{e}^{\frac{\sigma_0^2}{2}T} \mathrm{d}x,
\end{aligned}
$$

and by setting $y = x - \sigma_0\sqrt{T}$, we get that the r.h.s. is given by

$$
\begin{aligned}
& \beta_{3,5}^T s_0 \mathrm{e}^{rT} \sqrt{T} \int_{y > -d_1} \left(\sigma_0\sqrt{T} + y\right) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y \\
& \quad = \beta_{3,5}^T T s_0 \mathrm{e}^{rT} \sigma_0 N(d_1) - \beta_{3,5}^T \sqrt{T} s_0 \mathrm{e}^{rT} \left[\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}}\right]_{-d_1}^{+\infty} \\
& \quad = \beta_{3,5}^T T s_0 \mathrm{e}^{rT} \sigma_0 N(d_1) + \beta_{3,5}^T \sqrt{T} s_0 \mathrm{e}^{rT} \phi(-d_1, 0, 1).
\end{aligned}
$$

Hence, the third term in (56) reads

$$
\epsilon \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\beta_3 W_T \mathbb{1}_{\{X_0^T > \ln(K)\}} \mathrm{e}^{X_T^0}\right] = \epsilon \beta_3 T \sigma_0 s_0 N(d_1) + \epsilon \beta_3 s_0 \sqrt{T} \phi(-d_1, 0, 1).
$$

Exploiting the definition of X_T^0 occurring in the fourth term in (56), as well as similar algebraic computation as in the previous previous section, we get

$$\mathbb{E}^{\mathbb{Q}}\left[\beta_4 W_T^2 \mathbb{1}_{\{X_0^T > \ln(K)\}} e^{X_T^0}\right] = \beta_4 s_0 e^{rT} e^{-\frac{\sigma_0^2}{2}T} \int_{x>-d_2} T x^2 e^{\sigma_0\sqrt{T}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
$$= \beta_4 s_0 e^{rT} T \int_{y>-d_1} (y + \sigma_0\sqrt{T})^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Developing the square and using the linearity property of the integral, we get that the r.h.s. is equal to

$$\int_{y>-d_1} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \int_{y>-d_1} 2\sigma_0\sqrt{T} y \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$
$$+ \int_{y>-d_1} \sigma_0^2 T \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$
$$= \int_{y>-d_1} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + 2\sigma_0\sqrt{T}\phi(-d_1, 0, 1) + \sigma_0^2 T N(d_1).$$

The first term is computed using integration by parts,

$$\int_{y>-d_1} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = -d_1\phi(d_1) + N(d_1);$$

therefore,

$$\epsilon e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\beta_4 W_T^2 \mathbb{1}_{\{X_0^T > \ln(K)\}} e^{X_T^0}\right]$$
$$= \epsilon\beta_4 s_0 T\left(-d_1\phi(d_1) + N(d_1) + 2\sigma_0\sqrt{T}\phi(-d_1, 0, 1) + \sigma_0^2 T N(d_1)\right).$$

To compute the fifth term in (56), we use Fubini theorem to exchange the expectation with the integral with respect to time, getting

$$\beta_6 \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{X_0^T > \ln(K)\}} e^{X_T^0} W_s ds\right]. \tag{57}$$

For every fixed $s \in [0, T]$, W_s and W_T, the latter is included in X_0^T by its very definition, are Gaussian random variable jointly distributed. Therefore, exploiting basic properties of Brownian motion we can recast them by means of a sum of independent random variable, namely

$$W_s = Y \sim \mathcal{N}(0, s),$$
$$W_T = W_T - W_s + W_s = X + Y.$$

In particular, $X \sim \mathcal{N}(0, T - s)$ and it is independent with respect to Y. Thus, (57) reads

$$\beta_6 \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\{x+y>-\sqrt{T}d_2\}} e^{x_0+\mu T} e^{\sigma_0(x+y)} y \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2(T-s)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2s}} ds$$
$$= \beta_6 s_0 e^{rT} e^{-\frac{\sigma_0^2}{2}T} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{1}_{\{x+y>-\sqrt{T}d_2\}} e^{\sigma_0(x+y)} y\phi(x;0;T-s)\phi(y,0,s) dx dy ds,$$

and the claim follows. □

3.3.1 Numerical results concerning the pricing formula in Proposition 7

Let us consider the case of the B–S model corrected by a linear term given as in Proposition 7 by $f(x) = \alpha_0 + \alpha_1 x$. We compute the first-order correction of the price of an European call option with $\Phi(X_T^\epsilon) = (e^{X_T^\epsilon - K})_+$ as payoff function, according to Proposition 7.

Our aim is computing the expectation in (42) in the present case. By the very definition of X_0^T and X_1^T and the form of Φ', it reads as

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_1 T\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_2 T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_3 W_T\right]$$
$$+ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_4 W_T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_5 T W_T\right] - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D e^{X_0^T}\beta_6 \int_0^T W_s ds\right] \tag{58}$$

Each random variable in the brackets is approximated by means of a *multi-element* PCE of degree $p = 15$ and, respectively, by means of standard Monte Carlo methods, using $N = 10{,}000$ independent simulations of the random variable involved.

The accuracy of *PCE* is represented by its absolute error, using as benchmark the analytical value coming from (52). Due to the *Law of Large Numbers*, the accuracy of MC estimation of (33) is provided by its standard error (SE_{MC}). Upon considering $N = 10{,}000$ realizations (Y_j) of the random variable $Y := \Phi'(X_T^0)X_T^1$ inside the expectation in the r.h.s. of Eq. (33), let us compute

$$SE_{MC} = \frac{\hat{\sigma}}{\sqrt{N}} \tag{59}$$

where $\hat{\sigma}^2 = \frac{1}{N-1}\sum_{j=1}^N \left(Y_j - \mu_{MC}\right)^2$ and $\mu_{MC} = \frac{1}{N}\sum_{j=1}^N Y_j$.

Figures 9, 10 and 11 report percentage error between analytical and PCE approximations of above pricing equation, for different strike prices, maturity and ϵ; also similarly, the same results are collected in Fig. 12 regarding relative error over ϵ grouped by maturity. Also parameters are shown in Table 1

Conclusion similar to the one drawn in Sect. 3 can be derived.

3.4 A correction given by a polynomial function and jumps

In the present section, we generalize the results obtained in Sect. 3.3 adding a compensated Poisson random measure. In particular, let us assume that the normal return

			PCE					
			Strike (K)					
sigma	epsilon	Maturity (T)	50	75	90	100	110	125
0,15	0,01	0,50	0,00001%	0,00005%	0,00050%	0,00138%	0,00269%	0,00504%
0,15	0,01	0,75	0,00003%	0,00012%	0,00075%	0,00174%	0,00316%	0,00575%
0,15	0,01	1,00	0,00005%	0,00022%	0,00100%	0,00208%	0,00358%	0,00632%
0,15	0,01	1,25	0,00008%	0,00033%	0,00124%	0,00240%	0,00397%	0,00683%
0,15	0,01	1,50	0,00012%	0,00046%	0,00149%	0,00272%	0,00434%	0,00729%
0,15	0,01	1,75	0,00016%	0,00059%	0,00173%	0,00303%	0,00470%	0,00773%
0,15	0,01	2,00	0,00020%	0,00074%	0,00198%	0,00333%	0,00505%	0,00814%
0,15	0,01	0,50	0,00001%	0,00002%	0,00014%	0,00050%	0,00127%	0,00305%
0,15	0,01	0,75	0,00003%	0,00006%	0,00027%	0,00075%	0,00162%	0,00355%
0,15	0,01	1,00	0,00005%	0,00011%	0,00042%	0,00100%	0,00195%	0,00400%
0,15	0,01	1,25	0,00007%	0,00017%	0,00058%	0,00124%	0,00227%	0,00441%
0,15	0,01	1,50	0,00010%	0,00025%	0,00076%	0,00149%	0,00258%	0,00479%
0,15	0,01	1,75	0,00014%	0,00034%	0,00094%	0,00173%	0,00288%	0,00517%
0,15	0,01	2,00	0,00018%	0,00044%	0,00112%	0,00198%	0,00317%	0,00552%
0,15	0,01	0,50	0,00001%	0,00002%	0,00004%	0,00016%	0,00050%	0,00164%
0,15	0,01	0,75	0,00002%	0,00004%	0,00010%	0,00030%	0,00075%	0,00203%
0,15	0,01	1,00	0,00004%	0,00007%	0,00019%	0,00046%	0,00100%	0,00238%
0,15	0,01	1,25	0,00007%	0,00012%	0,00029%	0,00063%	0,00124%	0,00273%
0,15	0,01	1,50	0,00010%	0,00017%	0,00041%	0,00081%	0,00149%	0,00305%
0,15	0,01	1,75	0,00013%	0,00024%	0,00053%	0,00099%	0,00173%	0,00337%
0,15	0,01	2,00	0,00017%	0,00031%	0,00067%	0,00118%	0,00197%	0,00369%

Fig. 9 Percentage error for PCE and analytical estimation—error w.r.t. strike price K

Fig. 10 Percentage error for PCE and analytical estimation—error w.r.t. σ

		PCE		
		SIGMA		
s0	epsilon	15%	25%	35%
90	0,010	0,00250%	0,00190%	0,00210%
90	0,030	0,00650%	0,00530%	0,00600%
90	0,060	0,01120%	0,00990%	0,01130%
90	0,100	0,01590%	0,01510%	0,01770%
90	0,150	0,02030%	0,02060%	0,02460%
100	0,010	0,00150%	0,00140%	0,00160%
100	0,030	0,00400%	0,00390%	0,00480%
100	0,060	0,00730%	0,00740%	0,00920%
100	0,100	0,01080%	0,01150%	0,01460%
100	0,150	0,01430%	0,01610%	0,02060%
110	0,010	0,00090%	0,00100%	0,00130%
110	0,030	0,00240%	0,00290%	0,00390%
110	0,060	0,00460%	0,00560%	0,00760%
110	0,100	0,00710%	0,00880%	0,01220%
110	0,150	0,00970%	0,01250%	0,01740%

				PCE				
				epsilon				
Strile (K)	s0	sigma	T	0,01	0,03	0,06	0,1	0,15
100	90	0,15	0,5	0,000%	0,000%	0,000%	0,000%	0,000%
100	90	0,15	0,75	0,001%	0,001%	0,001%	0,002%	0,002%
100	90	0,15	1	0,001%	0,002%	0,003%	0,003%	0,003%
100	90	0,15	1,25	0,002%	0,004%	0,004%	0,005%	0,005%
100	90	0,15	1,5	0,003%	0,005%	0,006%	0,007%	0,007%
100	90	0,15	1,75	0,004%	0,006%	0,008%	0,009%	0,010%
100	90	0,15	2	0,004%	0,008%	0,010%	0,012%	0,013%
100	100	0,15	0,5	0,001%	0,002%	0,002%	0,002%	0,002%
100	100	0,15	0,75	0,002%	0,003%	0,004%	0,004%	0,004%
100	100	0,15	1	0,002%	0,004%	0,006%	0,006%	0,007%
100	100	0,15	1,25	0,003%	0,006%	0,008%	0,009%	0,009%
100	100	0,15	1,5	0,004%	0,008%	0,010%	0,012%	0,012%
100	100	0,15	1,75	0,005%	0,010%	0,013%	0,015%	0,016%
100	100	0,15	2	0,006%	0,012%	0,016%	0,018%	0,019%
100	110	0,15	0,5	0,001%	0,003%	0,003%	0,004%	0,004%
100	110	0,15	0,75	0,002%	0,004%	0,006%	0,006%	0,007%
100	110	0,15	1	0,003%	0,006%	0,008%	0,009%	0,010%
100	110	0,15	1,25	0,004%	0,008%	0,011%	0,013%	0,014%
100	110	0,15	1,5	0,005%	0,010%	0,014%	0,016%	0,017%
100	110	0,15	1,75	0,006%	0,013%	0,017%	0,020%	0,022%
100	110	0,15	2	0,007%	0,015%	0,020%	0,024%	0,026%

Fig. 11 Percentage error for PCE and analytical estimation—error w.r.t. ϵ

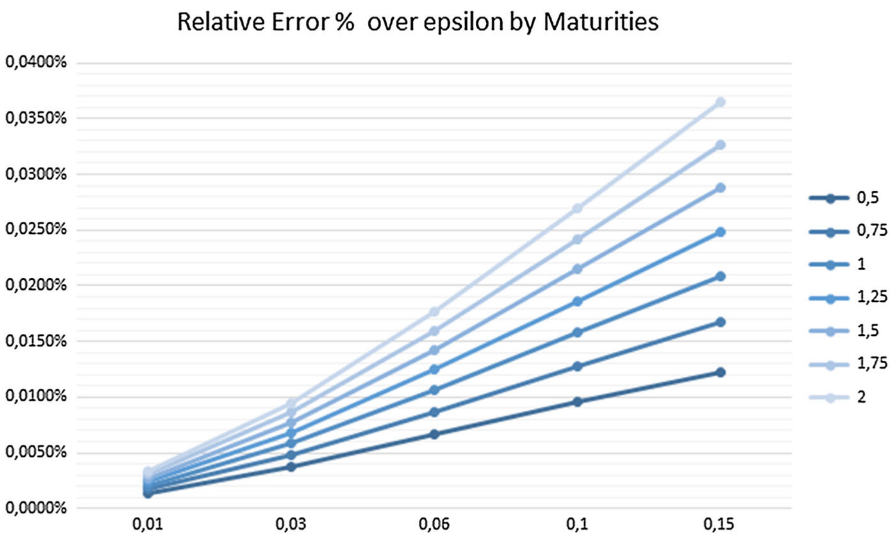

Fig. 12 Percentage error for PCE and analytical estimation grouped by maturity—error w.r.t. ϵ

of the asset price evolves according to Eq. (27) with a polynomial f. Then, we have the following proposition.

Proposition 8 *Let us consider the case of the B–S model with added compensated Poisson noise and corrected by a nonlinear term given by* (27) *with* $f(x) = \sum_{i=0}^{N} \alpha_i x^i$, *for some* $\alpha_i \in \mathbb{R}$, *then the expansion coefficients for the solution* X_t^ϵ *of* (27) *up to the second order are given by the system*

$$\begin{aligned}
X_t^0 &= x_0 + \mu t + \sigma_0 W_t, \quad \text{with law } \mathcal{N}\left(x_0 + \mu t, \sigma_0^2 t\right);\\
X_t^1 &= \sum_{i=1}^{N} \tilde{K}_i (X_t^0)^{i+1} - \sum_{i=0}^{N} \int_0^t K_i (X_s^0)^i \mathrm{d}s + \sigma_1 \alpha_0 W_t - \lambda t \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right) + \sum_{i=1}^{N_t} J_i;\\
X_t^2 &= \sum_{k=1}^{2N+1} C_k^1 (X_t^0)^k - \sum_{k=1}^{2N+1} \int_0^t C_k^2 (X_s^0)^k \mathrm{d}s + \sum_{i=1}^{N}\sum_{j=0}^{N} \int_0^t \int_0^s C_{i,j}^3 (X_s^0)^{i-1} (X_r^0)^j \mathrm{d}r\mathrm{d}s\\
&\quad + \sum_{i=1}^{N}\sum_{j=0}^{N} (X_t^0)^i \int_0^s C_{i,j}^4 (X_r^0)^j \mathrm{d}r + \sum_{i=0}^{N-1} C_i^5 \lambda \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right) \int_0^t s \left(X_s^0\right)^i \mathrm{d}s\\
&\quad - \alpha_{i+1}\sigma_1 t \left(X_t^0\right)^i \lambda \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right)\\
&\quad + \int_0^t \sigma_1 \alpha_1 W_s \mathrm{d}s \lambda \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right) - \sigma_1 t \alpha_1 W_t \lambda \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right)\\
&\quad + \sum_{i=2}^{N} \alpha_i \sigma_1 \lambda \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1\right) \int_0^t \left(X_s^0\right)^i \mathrm{d}s + \sigma_1 \alpha_1 W_t + \sum_{i=2}^{N} \sigma_1 \alpha_i \left(X_s^0\right)^i \sum_{i=1}^{N_t} J_i\\
&\quad - \sum_{i=2}^{N} \sigma_1 \alpha_i \int_0^t \int_{\mathbb{R}_0} \left(X_s^0\right)^i \mathrm{d}s \sum_{i=1}^{N_t} J_i + \sum_{i=0}^{N-1} C_i^5 \int_0^t \sum_{i=1}^{N_s} J_i \left(X_s^0\right)^i \mathrm{d}s
\end{aligned} \tag{60}$$

where the constants are as in Proposition 6 *and*

$$C_i^5 = \begin{cases} \sigma_0^2 \sigma_1 \alpha_2 + 2\sigma_0\sigma_1\alpha_1, & i = 0,\\ \sigma_1 \mu \alpha_{i+1}(i+1) + \frac{\sigma_0^2}{2}(i+2)(i+1) + 4\sigma_0\sigma_1\alpha_{i+1}, & i \neq 0, i \neq N,\\ \alpha_N \sigma_1 N \mu + 2\sigma_0\sigma_1 N \alpha_{N+1}, & i \neq N, \end{cases}$$

and

$$\tilde{K}_i = \frac{\sigma_1}{\sigma_0} \frac{\alpha_i}{(i+1)}.$$

Proof The proof is analogous to the one in Proposition 6 taking into account the compensated Poisson random measure terms and applying Itô's lemma together with the stochastic Fubini theorem. □

Proposition 9 *Let us consider the particular case of* $N = 1$, *i.e., a linear perturbation, namely* $f(x) = \alpha_0 + \alpha_1 x$ *in* Proposition 8. *Then, the terms up to the first order in* Eq. (60) *read*

$$\begin{aligned} X_t^0 &= x_0 + \mu t + \sigma_0 W_t, \\ X_t^1 &= \beta_1 t + \beta_2 t^2 + \beta_3 W_t + \beta_4 W_t^2 \\ &\quad + \beta_5 t W_t - \int_0^t \beta_6 W_s \mathrm{d}s - \lambda t \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1 \right) + \sum_{i=1}^{N_t} J_i, \end{aligned} \tag{61}$$

the constants being as in Proposition 7.

Also, the first-order correction of the price of an European call option Φ *with payoff given by* $\Phi(X_T^\epsilon) = \left(\mathrm{e}^{X_T^\epsilon} - K\right)_+$ *(in the sense of* Remark 7*) is explicitly given by*

$$Pr^1(0;T) = Pr^1 + \epsilon T s_0 N\left(d(1)\right) \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1 \right) + \epsilon T s_0 N\left(d(1)\right) \delta\lambda, \tag{62}$$

where Pr^1 *is the corrected fair price up to the first order as given in* Eq. (52) *and the notations are as above.*

Proof The proof is similar to the one in Proposition 7. □

3.4.1 Numerical results concerning the pricing formula in Proposition 8

The J_i is assumed to be independent and normally distributed random variables

$$J_i \sim \mathcal{N}(\gamma, \delta^2), \quad \text{for all } i \in \{1, 2, \ldots, N_T\}, \quad \gamma = 0.05, \quad \delta = 0.02,$$

and $\lambda = 2$. In particular, we are aiming at computing the expectation in (42) for the model described in Proposition 8. In the present case, we have that this expectation in equal to

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_1 T\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_2 T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_3 W_T\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_4 W_T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_5 T W_T\right] - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_6 \int_0^T W_s \mathrm{d}s\right] \\ &\quad + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0} \lambda T \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1 \right)\right] + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0} \sum_{i=1}^{N_T} J_i\right]. \end{aligned} \tag{63}$$

By means of the independence of the jumps and $\mathbb{E}\left[\sum_{i=1}^{N_T} J_i\right] = \delta\lambda T$, we can rewrite (63) as

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_1 T\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_2 T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_3 W_T\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_4 W_T^2\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_5 T W_T\right] - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_0^T} \beta_6 \int_0^T W_s \mathrm{d}s\right] \\ &\quad + K_2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0} \lambda T \left(\mathrm{e}^{\gamma + \frac{\delta^2}{2}} - 1 \right)\right] + K_2 \lambda T \delta \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_D \mathrm{e}^{X_T^0}\right]. \end{aligned} \tag{64}$$

				PCE					
				Strike (K)					
s0	sigma	epsilon	Maturity (T)	50	75	90	100	110	125
90	0,15	0,01	0,50	0,000%	0,000%	0,000%	0,001%	0,003%	0,005%
90	0,15	0,01	0,75	0,000%	0,000%	0,001%	0,002%	0,003%	0,006%
90	0,15	0,01	1,00	0,000%	0,000%	0,001%	0,002%	0,004%	0,006%
90	0,15	0,01	1,25	0,000%	0,000%	0,001%	0,002%	0,004%	0,007%
90	0,15	0,01	1,50	0,000%	0,000%	0,001%	0,003%	0,004%	0,007%
90	0,15	0,01	1,75	0,000%	0,001%	0,002%	0,003%	0,005%	0,008%
90	0,15	0,01	2,00	0,000%	0,001%	0,002%	0,003%	0,005%	0,008%
100	0,15	0,01	0,50	0,000%	0,000%	0,000%	0,000%	0,001%	0,003%
100	0,15	0,01	0,75	0,000%	0,000%	0,000%	0,001%	0,002%	0,003%
100	0,15	0,01	1,00	0,000%	0,000%	0,000%	0,001%	0,002%	0,004%
100	0,15	0,01	1,25	0,000%	0,000%	0,001%	0,001%	0,002%	0,004%
100	0,15	0,01	1,50	0,000%	0,000%	0,001%	0,001%	0,003%	0,005%
100	0,15	0,01	1,75	0,000%	0,000%	0,001%	0,002%	0,003%	0,005%
100	0,15	0,01	2,00	0,000%	0,000%	0,001%	0,002%	0,003%	0,005%
110	0,15	0,01	0,50	0,000%	0,000%	0,000%	0,000%	0,000%	0,002%
110	0,15	0,01	0,75	0,000%	0,000%	0,000%	0,000%	0,001%	0,002%
110	0,15	0,01	1,00	0,000%	0,000%	0,000%	0,000%	0,001%	0,002%
110	0,15	0,01	1,25	0,000%	0,000%	0,000%	0,001%	0,001%	0,003%
110	0,15	0,01	1,50	0,000%	0,000%	0,000%	0,001%	0,001%	0,003%
110	0,15	0,01	1,75	0,000%	0,000%	0,001%	0,001%	0,002%	0,003%
110	0,15	0,01	2,00	0,000%	0,000%	0,001%	0,001%	0,002%	0,004%

Fig. 13 Percentage error for PCE and analytical estimation—error w.r.t. strike price K

We shall then compute *multi-element* PCE approximations for this expression.

Figures 13 and 14 report percentage error between analytical and PCE approximations of above pricing equation, for different strike prices, maturity and ϵ; also similarly, the same results are collected in Fig. 15 regarding relative error over ϵ grouped by maturity. Also parameters are shown in Table 1

Conclusion similar to the one drawn in Sect. 3 can be derived.

4 Conclusions

In this work, we have focused our attention on the analysis of the small noise asymptotic expansions for particular classes of local volatility models arising in finance. We have given explicit expressions for the associated coefficients, along with accurate estimates on the remainders. Furthermore, we have provided a detailed numerical analysis, with accuracy comparisons, of the obtained results exploiting the standard Monte Carlo technique as well as the so-called Polynomial Chaos Expansion approach. We would like to underline that our approach allows to consider, other than the well know Gaussian noise component, a realistic stochastic perturbation of jump type.

In a future work, we plan to use the latter extension, along with the described asymptotic expansion techniques, to study particular types of implied volatilities models and further related functionals, as suggested by one of the anonymous reviewers. Such developments will be also the basis for an extensive calibration work on real financial data.

Fig. 14 Percentage error for PCE and analytical estimation—error w.r.t. σ

s0	epsilon	PCE SIGMA		
s0	epsilon	0,15	0,25	0,35
90	0,01	0,0024%	0,0019%	0,0020%
90	0,03	0,0062%	0,0052%	0,0058%
90	0,06	0,0102%	0,0094%	0,0109%
90	0,10	0,0140%	0,0139%	0,0166%
90	0,15	0,0174%	0,0184%	0,0226%
100	0,01	0,0014%	0,0013%	0,0016%
100	0,03	0,0038%	0,0038%	0,0047%
100	0,06	0,0067%	0,0070%	0,0089%
100	0,10	0,0096%	0,0107%	0,0137%
100	0,15	0,0123%	0,0144%	0,0190%
110	0,01	0,0008%	0,0010%	0,0013%
110	0,03	0,0023%	0,0028%	0,0038%
110	0,06	0,0042%	0,0053%	0,0073%
110	0,10	0,0063%	0,0082%	0,0115%
110	0,15	0,0084%	0,0113%	0,0160%

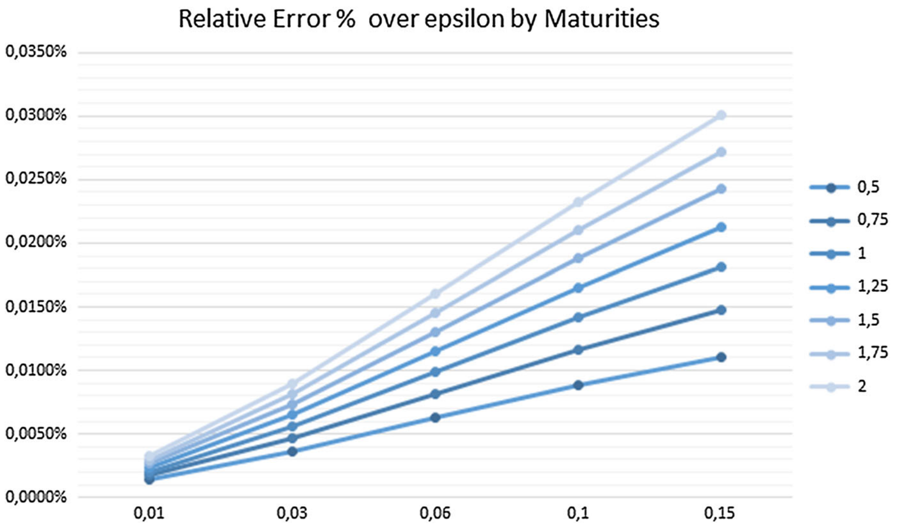

Fig. 15 Percentage error for PCE and analytical estimation grouped by maturity—error w.r.t. ϵ

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Appendix

Polynomial Chaos Expansion

In the present section, we briefly recall the basics and main characteristic of the *Polynomial Chaos Expansion* (PCE) approach. We refer the interested reader to, e.g., Bonollo et al. (2015), Crestaux et al. (2009), Ernst et al. (2012), Peccati and Taqqu (2011) and references therein, for a detailed introduction to such a method, particularly from the financial point of view. The PCE approach allows to approximate a random variable as a linear combination of orthogonal polynomials in order to compute its statistics with a lower computational effort if compared to the one needed by taking into consideration the *full set of information* characterizing it. Before entering into details, we would like to underline that the PCE technique is a generalization of the original Wiener Chaos decomposition, see Wiener (1938).

Let us consider a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the Hilbert space of real-valued random variables $L^2(\Omega, \mathcal{F}, \mathbb{P})$ X defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\mathbb{E}[X^2] = \int_\Omega \big(X(\omega)\big)^2 \mathbb{P}(\mathrm{d}\omega) < +\infty,$$

equipped with the standard scalar product

$$\mathbb{E}[XY] = \langle X, Y\rangle_{\mathbb{P}} = \int_\Omega X(\omega)Y(\omega)\mathbb{P}(\mathrm{d}\omega),$$

and corresponding norm

$$\|X\|^2_{\mathbb{P}} = \mathbb{E}[X^2],$$

with related concept of *mean square convergence* or *strong convergence*.

Among the elements of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, we can identify the class of *basic random variables*, which is used to decompose stochastic quantity of interest as, e.g., the stochastic process solution of a Stochastic Differential Equation (SDE), evaluated at (*horizon*) time $T > 0$. It is worth to mention that not all the functions $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be used to perform aforementioned decomposition. In fact, they have to satisfy, see, e.g., Ernst et al. (2012, Section 3), at least the following two properties

- ξ has finite moments of all orders
- the distribution function $F_\xi(x) := \mathbb{P}(\xi \le x), x \in \mathbb{R}$, of the basic random variables ξ is absolutely continuous, with a probability density function (pdf) denoted by f_ξ.

Let us denote by $\sigma(\xi)$ the σ-algebra generated by the basic random variable ξ; hence, $\sigma(\xi) \subset \mathcal{F}$. If we want to polynomially decompose a given random variable Y in terms of ξ, then Y has to be, at least, measurable with respect to the σ-algebra $\sigma(\xi)$. Exploiting the Doob–Dynkin Lemma, see, e.g., Kallenberg et al. (2006, Lemma 1.13), we have that Y is $\sigma(\xi)$-measurable if for some Borel measurable function

$g : \mathbb{R} \to \mathbb{R}, Y = g(\xi)$. In what follows, without loss of generality, we restrict ourselves to consider the decomposition in $L^2(\Omega, \sigma(\xi), \mathbb{P})$. The basic random variable ξ is assumed to determine a class of orthogonal polynomials $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$, which is called the *generalized polynomial chaos* (gPC) basis. We underline that their orthogonality properties is detected by means of the measure induced by ξ in the image space $(D, \mathcal{B}(D))$, where $D \subset \mathbb{R}$ is the range of ξ and where $\mathcal{B}(D) \subset \mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra associated with D. For each $i, j \in \mathbb{N}$, we have

$$\left\langle \Psi_i, \Psi_j \right\rangle_{\mathbb{P}} = \int_\Omega \Psi_i\left(\xi(\omega)\right) \Psi_j\left(\xi(\omega)\right) \mathrm{d}\mathbb{P}(\omega) = \int_D \Psi_i(x)\Psi_j(x) f_\xi(x)\mathrm{d}x. \tag{65}$$

If ξ has law $\mathcal{N}(0, 1/2)$, namely the centered normal distribution of variance $\frac{1}{2}$, then the related set $\{\Psi_i(x)\}_{i\in\mathbb{N}}$ is represented by the family of non-normalized Hermite polynomials defined on the whole real line, namely $D = \mathbb{R}$, and

$$\begin{cases} \Psi_0(x) &= 1 \\ \Psi_1(x) &= 2x \\ \Psi_2(x) &= 4x^2 - 2 \\ & \vdots \end{cases} \tag{66}$$

Figure 16 provides the graph of the first six orthonormal polynomials, achieved by scaling each Ψ_i in (66) by its norm in $L^2(\Omega, \sigma(\xi), \mathbb{P})$, namely, $\forall i \in \mathbb{N}$, Ψ_i is divided by $\|\Psi_i\|_{\mathbb{P}} := \sqrt{2^i i!}$.

Latter polynomials Ψ_i constitute a maximal system in $L^2(\Omega, \sigma(\xi), \mathbb{P})$; therefore, every random variable $Y \in L^2(\Omega, \sigma(\xi), \mathbb{P})$ can be approximated as follows:

$$Y^{(p)} = \sum_{i=0}^{p} c_i \Psi_i(\xi), \tag{67}$$

for some $p \in \mathbb{N}$ and suitable real coefficients c_i which depend on the random variable Y, see, e.g., Ernst et al. (2012, Section 3.1). We refer to Eq. (67) as the *truncated PCE*, at degree p, of Y. Exploiting previous definitions, taking $i \in \{0, \ldots, p\}$ and considering the orthogonality property of the polynomials $\{\Psi_i(\xi)\}_{i\in\mathbb{N}}$, we have

$$c_i = \frac{1}{\|\Psi_i\|^2_{\mathbb{P}}} \langle Y, \Psi_i \rangle_{\mathbb{P}} = \frac{1}{\|\Psi_i\|^2_{\mathbb{P}}} \langle g, \Psi_i \rangle_{\mathbb{P}}, \tag{68}$$

and, since $Y = g(\xi)$, we also obtain

$$\langle Y, \Psi \rangle_{\mathbb{P}} = \langle g, \Psi \rangle_{\mathbb{P}} = \int_\Omega g(\xi(\omega))\Psi_i(\xi(\omega))\mathrm{d}P(\omega) = \int_{\mathbb{R}} g(x)\Psi_i(x) f_\xi(x)\mathrm{d}x. \tag{69}$$

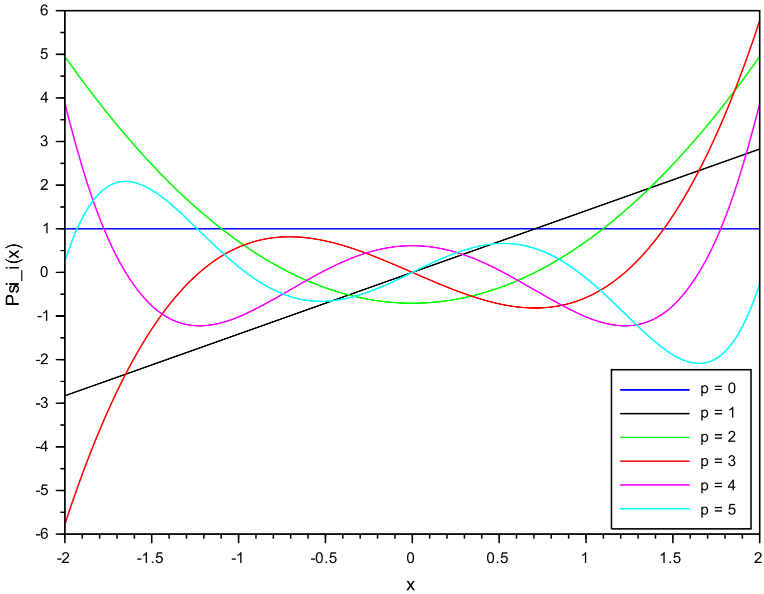

Fig. 16 Hermite normalized polynomials up to degree 5

The convergence rate of the PCE approximation (67) in $L^2(\Omega, \sigma(\xi), \mathbb{P})$ norm is strictly linked to the magnitude of the coefficients of the decomposition. Indeed, by the Parseval identity, we have

$$\|Y\|_{\mathbb{P}}^2 = \sum_{i=0}^{+\infty} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2;$$

furthermore, using the orthogonality property of the Hermite polynomials in $L^2(\Omega, \sigma(\xi), \mathbb{P})$, the norm of (67) is given by

$$\left\| Y^{(p)} \right\|_{\mathbb{P}}^2 = \sum_{i=0}^{p} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2.$$

Exploiting the fundamental properties of the orthogonal projections in Hilbert space, see, e.g., Rudin (1986, Theorem 4.11), we can estimate the mean square error as

$$\left\| Y - Y^{(N)} \right\|_{\mathbb{P}}^2 = \|Y\|_{\mathbb{P}}^2 - \left\| Y^{(N)} \right\|_{\mathbb{P}}^2 = \sum_{i=N+1}^{+\infty} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2, ; \tag{70}$$

thus, the rate of convergence depends on the coefficients. In particular, the PCE of $Y^{(p)}$ approximates the Y-statistics in terms of the c_i coefficients appearing in Eq. (67), e.g., the first two centered moments are determined by

$$\mathbb{E}\left[Y^{(p)}\right] = c_0, \tag{71}$$

$$\mathbb{V}ar\left[Y^{(p)}\right] = \sum_{i=1}^{p} c_i^2 \|\Psi_i\|_{\mathbb{P}}^2. \tag{72}$$

Multi-element decomposition

Concerning the application of the PCE method to the approximation of quantities as in the case of European call options, we have implemented a method called multi-element generalized polynomial chaos (ME-gPC) method, see, e.g., Peccati and Taqqu (2011), and references therein. Without entering into technical details, let us mention that it is an extension of the PCE approach which can be applied to arbitrary probability measures. In particular, see, e.g., Wan and Karniadakis (2005, 2006), the ME-gPC approach can be effectively used to numerically solve S(P)DEs, by decomposing the random inputs, e.g., the Brownian motion, into smaller elements. Each of the latter is then used to define a new random variable, with respect to a conditional probability density function, and a set of orthogonal polynomials defined in terms of the aforementioned random variable. Then, the procedure we have already recalled is applied element by element and, thanks to the convergence of the method, the final result is achieved rearranging, in a suitable way, the ones obtained for each term.

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