On obstacle numbers

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Abstract

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison (2010). Mukkamala et al. (2012) show that there exist graphs with n vertices having obstacle number in $\Omega(n/\log n)$. In this note, we up this lower bound to $\Omega(n/(\log\log n)^2)$. Our proof makes use of an upper bound of Mukkamala et al. on the number of graphs having obstacle number at most h in such a way that any subsequent improvements to their upper bound will improve our lower bound.

1 Introduction

The obstacle number is a new graph parameter introduced by Alpert, Koch, and Laison [2]. Let $G = (V, E)$ be a graph, let $\varphi : V \to \mathbb{R}^2$ be a one-to-one mapping of the vertices of G onto $\mathbb{R}^2$ (hereafter called a *drawing* of G), and let S be a set of connected subsets of $\mathbb{R}^2$. The pair (φ, S) is an *obstacle representation* of G when, for every pair of vertices $u, w \in V$, the edge uw is in E if and only if the closed line segment with endpoints $\varphi(u)$ and $\varphi(w)$ does not intersect any *obstacle* in S. An obstacle representation (φ, S) is an

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h-obstacle representation if $|S| = h$. The *obstacle number* of a graph G, denoted by $\mathrm{obs}(G)$, is the minimum value of h such that G has an h-obstacle representation.[1]

Note that obstacle representations of planar graphs using few obstacles often require drawings of those graphs that are far from crossing free. For example, any crossing-free drawing of the 5×5 grid, $G_{5\times 5}$ shown in the left part of Figure 1 requires at least one obstacle in each of the sixteen internal faces (each of which has at least four sides).

It is somewhat surprising, therefore, that $G_{5\times 5}$ has obstacle number 1. The obstacle representation, illustrated on the right part of Figure 1 was given to us by Fabrizio Frati. In this figure, the single obstacle is drawn as a shaded region. Since at least one obstacle is clearly necessary to represent any graph other than a complete graph, this proves that $\mathrm{obs}(G_{5\times 5}) = 1$. (A similar drawing can be used to show that the $a \times b$, grid graph has obstacle number 1, for any integers $a, b > 1$.)

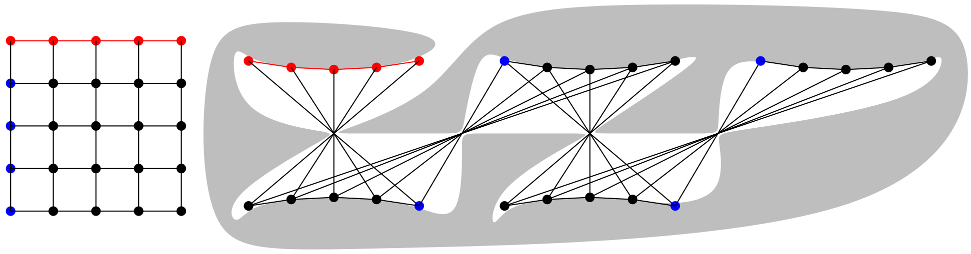

Figure 1: The 5×5 grid graph has obstacle number 1.

Since their introduction, obstacle numbers have generated significant research interest [4, 5, 6, 7, 8, 9, 10]. A fundamental—and far from answered—question about obstacle numbers is that of determining the *worst-case obstacle number*,

$$\mathrm{obs}(n) = \max\{\mathrm{obs}(G) : G \text{ is a graph with } n \text{ vertices}\} \ ,$$

of a graph with n vertices.

For a graph $G = (V, E)$, we call elements of $\binom{V}{2} \setminus E$ *non-edges* of G. The worst-case obstacle number $\mathrm{obs}(n)$ is obviously upper bounded by $\binom{n}{2} \in O(n^2)$ since, by mapping the vertices of G onto a point set in sufficiently general position, one can place a small obstacle—even a single point—on the mid-point of each non-edge of G. No upper bound on $\mathrm{obs}(n)$ that is asymptotically better than $O(n^2)$ is known.

More is known about lower bounds on $\mathrm{obs}(n)$. Alpert, Koch, and Laison [2] initially show that the worst-case obstacle number is $\Omega\left(\sqrt{\log n / \log\log n}\right)$ and posed as an open problem the question of determining if $\mathrm{obs}(n) \in \Omega(n)$. Mukkamala et al. [7] showed that $\mathrm{obs}(n) \in \Omega(n/\log^2 n)$ and Mukkamala et al. [6] later increased this to $\mathrm{obs}(n) \in$

[1]Note that this definition of obstacle representation is more generous than that of Alpert, Koch, and Laison [2], which requires that the obstacles be polygonal and that the set of points determined by vertices of the obstacles and the image of φ not contain 3 collinear points. Since the current paper proves a lower bound on the obstacle number, this lower bound also applies to the original definition.

$\Omega(n/\log n)$. In the current paper, we up the lower bound again by proving the following theorem:

Theorem 1. *For every integer $n > 0$, $\mathrm{obs}(n) \in \Omega(n/(\log\log n)^2)$, that is, there exists a sequence, $\langle G_n\rangle_{n=1}^{\infty}$, such that G_n is a graph with n vertices and such that $\mathrm{obs}(G) \in \Omega(n/(\log\log n)^2)$.*

The proof of Theorem 1 makes use of an upper bound of Mukkamala et al. [6, Theorem 1] on the number of graphs having obstacle number at most h in such a way that any subsequent improvements on their upper bound will result in an improved lower bound on $\mathrm{obs}(n)$.

Although some aspects of our proof are a little technical, the main idea is quite simple: Mukkamala et al. [6] show that, with probability at least $1 - 2^{-\Omega(n^2)}$, a random graph on n vertices has obstacle number at least $\Omega(n/(\log n)^2)$. Our proof trades off a lower probability for a higher obstacle number. When all is said and done, our proof shows that, with probability at least $1 - 2^{-\Omega(n\log n)}$, a random graph on n vertices has obstacle number at least $\Omega(n/(\log\log n)^2)$.

2 The Proof

Our proof strategy is an application of the probabilistic method [1]. We fix an arbitrary ordering, π, on the vertices of an Erdős–Rényi random graph, $G = G_{n,\frac{1}{2}}$. We then show that it is very unlikely that there is an obstacle representation, (φ, S) of G such that $|S| \in o(n/(\log\log n)^2)$ and the lexicographic ordering of the points assigned to vertices by φ agrees with the ordering given by π. Here, "very unlikely" means that this occurs with probability $p < 1/n!$. Since there are only $n!$ possible orderings of G's vertices, we then apply the union bound which shows that with probability $1-pn! > 0$, there is no obstacle representation of G using $o(n/(\log\log n)^2)$ obstacles, that is, $\mathrm{obs}(G) \in \Omega(n/(\log\log n)^2)$.

2.1 A Random Graph with a Fixed Ordering

We make use of the following theorem, due to Mukkamala, Pach, and Pálvölgyi [6, Theorem 1] about the number of n-vertex graphs with obstacle number at most h:

Theorem 2 (Mukkamala, Pach, and Pálvölgyi 2012)**.** *For any $h \geqslant 1$, the number of graphs having n vertices and obstacle number at most h is at most $2^{O(hn\log^2 n)}$.*

Recall that an Erdős-Rényi random graph $G_{n,\frac{1}{2}}$ is a graph with n vertices and each pair of vertices is chosen as an edge or non-edge with equal probability and independently of every other pair of vertices [3]. The following lemma shows that, for random graphs, a fixed drawing is *very* unlikely to yield an obstacle representation with few obstacles. Recall that the *lexicographic ordering*, $\prec$, for points in the plane is defined as

$$(x_1, y_1) \prec (x_2, y_2) \text{ iff } x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2) \ .$$

Lemma 1. *Let* $G = (V, E)$ *be an Erdős–Rényi random graph* $G_{n,\frac{1}{2}}$*, let* $v_1, \ldots, v_n$ *be an ordering of the vertices in* V *that is independent of the choices of edges in* G*, and let* (φ, S) *be an obstacle representation of* G *using the minimum number of obstacles subject to the constraint that*

$$\varphi(v_1) \prec \varphi(v_2) \prec \cdots \varphi(v_n) \ ,$$

where $\prec$ *denotes the lexicographic ordering of points. Then, for any constant* $c > 0$*,*

$$\Pr\{|S| \in \Omega(n/(\log\log n)^2)\} \geqslant 1 - e^{-cn\log n} \ .$$

Proof. Fix some integer $k = k(n) \in \omega_n(1)$ to be specified later and first consider the subgraph G_0 of G induced by the vertices $v_1, \ldots, v_k$. Applying Theorem 2 with $n = k$ and $h = \alpha k/\log^2 k$, we obtain

$$\Pr\{\mathrm{obs}(G_0) \leqslant \alpha k/\log^2 k\} \leqslant \frac{2^{O(\alpha k^2)}}{2^{\binom{k}{2}}} \leqslant e^{-\beta k^2} \ , \tag{1}$$

where $\beta > 0$ for a sufficiently small constant $\alpha > 0$, and sufficiently large k. Note that, if $\mathrm{obs}(G_0) \geqslant h$, then, in the obstacle representation (φ, S), there must be at least $h - 1$ obstacles of S that are contained in the convex hull of $\varphi(v_1), \ldots, \varphi(v_k)$; this is because the obstacle representation (φ, S) can be turned into an obstacle representation of G_0 by merging all obstacles that are not contained in the convex hull of $\varphi(v_1), \ldots, \varphi(v_k)$.

Let $m = \lfloor n/k \rfloor$ and notice that the preceding argument applies to any subset $V_i = \{v_{ki+1}, \ldots, v_{(k+1)i}\}$ of vertices, for any $i \in \{0, \ldots, m-1\}$. That is, Equation (1) shows that, with probability at least $1 - 2^{-\Omega(k^2)}$, the obstacle number of the subgraph G_i induced by V_i is $\Omega(k/\log^2 k)$. If this occurs, then S has $\Omega(k/\log^2 k)$ obstacles that are completely contained in the convex hull of V_i. In particular, the obstacles contained in the convex hull of V_i are different from the obstacles contained in the convex hull of V_j, for all $j \neq i$.

We are proving a lower bound on the number of obstacles, so we are worried about the case where the number of convex hulls that do *not* contain at least $\alpha k/\log^2 k$ obstacles exceeds m/e.[2] The number of convex hulls, M, not containing at least $\alpha k/\log^2 k$ obstacles is dominated by a binomial$(m, e^{-\beta k^2})$ random variable. Using Chernoff's bound on the tail of a binomial random variable,[3] we have that

$$\begin{aligned}
\Pr\{M \geqslant m/e\} &= \Pr\{M \geqslant (1+\delta)\mu\} \qquad \text{(where } \mu = me^{-\beta k^2} \text{ and } \delta = e^{\beta k^2 - 1} - 1\text{)} \\
&\leqslant \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu \\
&\leqslant \left(\frac{e^{e^{\beta k^2}}}{(e^{\beta k^2 - 1})^{e^{\beta k^2 - 1}}}\right)^{me^{-\beta k^2}}
\end{aligned}$$

[2]Euler's constant $e = \lim_{n\to\infty}(1 - 1/n)^n$ is just a convenient constant to use here.

[3]Chernoff's Bound: For any binomial(m, p) random variable, B, any $\delta > 0$ and $\mu = mp$,

$$\Pr\{B \geqslant (1+\delta)\mu\} \leqslant \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu \ .$$

$$
\begin{aligned}
&= \left(\frac{e^{e^{\beta k^2}}}{e^{(\beta k^2 - 1)e^{\beta k^2 - 1}}} \right)^{me^{-\beta k^2}} \\
&= \frac{e^m}{e^{m(\beta k^2-1)e^{\beta k^2-1}e^{-\beta k^2}}} \\
&= \frac{e^m}{e^{m(\beta k^2-1)/e}} \\
&= e^{-\Omega(mk^2)} \enspace .
\end{aligned}
$$

Taking $k = c' \log n$, for a sufficiently large constant, c', and recalling that $m = \lfloor n/k \rfloor$, we obtain the desired result. In particular,

$$|S| \geqslant \Omega\left(\left(k/\log^2 k\right) \cdot (m - m/e)\right) = \Omega\left(n/(\log\log n)^2\right)$$

with probability at least

$$1 - e^{-\Omega(mk^2)} = 1 - e^{-\Omega(c' n \log n)} \geqslant 1 - e^{-cn \log n} \enspace ,$$

for all n greater than some sufficiently large constant n_0. For $n \in \{1, \ldots, n_0\}$, the lemma is trivially satisfied since $|S| \geqslant 0$ with probability $1 \geqslant 1 - e^{-cn\log n}$. □

2.2 Finishing Up

For completeness, we now spell out the proof of Theorem 1.

Proof of Theorem 1. Let $G = (V, E)$ be an Erdős-Rényi random graph with n vertices with vertex set $V = \{1, \ldots, n\}$. For every obstacle representation (φ, S) of G, there is an ordering on V given by the lexicographic ordering of the points $\{\varphi(v) : v \in V\}$.

By Lemma 1, the probability that a particular such ordering, $v_1, \ldots, v_n$, allows an obstacle representation using $o(n/(\log\log n)^2)$ obstacles is at most $p \leqslant e^{-cn\log n}$ for every constant $c > 0$. In particular, for sufficiently large c, we have $p < 1/n!$. By the union bound the probability that there is any ordering that supports an obstacle representation of G with $o(n/(\log\log n)^2)$ obstacles is at most

$$\hat{p} = p \cdot n! < 1 \enspace .$$

We deduce that there exists some graph, G', with $\mathrm{obs}(G') \in \Omega(n/(\log\log n)^2)$. □

3 Remarks

Our proof of Theorem 1 relates the problem of counting the number of n-vertex graphs with obstacle number at most h to the problem of determining the worst-case obstacle number of a graph with n vertices. Currently, we use Theorem 2 of Mukkamala et al. [7], which proves an upper bound of $e^{O(hn\log^2 n)}$ on the number of n-vertex graphs with obstacle number at most h.

Any improvement on the upper bound for the counting problem will immediately translate into an improved lower bound on the worst-case obstacle number. Let $f(h,k)$ denote the number of k-vertex graphs with obstacle number at most h and let

$$\hat{h}(k) = \max\left\{h : f(h,k) \leqslant 2^{k^2/4}\right\} \quad .$$

The quantity $\hat{h}(k)$ is chosen so that a random graph on k vertices has probability at most $2^{-\Omega(k^2)}$ of having obstacle number less than $\hat{h}(k)$; Theorem 2 shows that $\hat{h}(k) \in \Omega(k/(\log k)^2)$. Our proof of Lemma 1 shows that there exist graphs with obstacle number at least $\Omega(n\hat{h}(c\log n)/(c\log n))$.

We note that our technique gives an improved lower bound until someone is able to prove that $\hat{h}(n) \in \Omega(n)$. At this point, our approach gives a lower bound worse than the trivial lower bound $\hat{h}(n)$.

We conjecture that improved upper bounds on $f(h,n)$ that reduce the dependence on h are the way forward:

Conjecture 1. *$f(h,n) \leqslant 2^{g(n)\cdot o(h)}$, where $g(n) \in O(n\log^2 n)$.*

In support of this conjecture, we observe that an upper bound of the form $f(1,n) \leqslant 2^{g(n)}$ is sufficient to give the crude upper bound $f(h,n) \leqslant 2^{h\cdot g(n)}$ since any graph with an h-obstacle representation is the common intersection of h graphs that each have a 1-obstacle representation. That is, if $\mathrm{obs}(G) \leqslant h$, then there exists $E_1,\ldots,E_h$ such that $G = (V, \bigcap_{i=1}^{h} E_i)$ and $\mathrm{obs}(V,E_i) = 1$ for all $i \in \{1,\ldots,h\}$. This seems like a very crude upper bound in which many graphs are counted multiple times. Conjecture 1 asserts that this argument overestimates the dependence on h.

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A previous draft of this article proved a version Lemma 1 for a fixed drawing, φ, and then went to great lengths to argue that the number of combinatorially distinct drawings was at most $2^{O(n\log n)}$. We are grateful to an anonymous referee who pointed out that the proof of Lemma 1 also holds when only the lexicographic ordering of the vertices is fixed, thereby eliminating the need to bound the number of combinatorially equivalent drawings.

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