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VARIATIONAL APPROXIMATION OF FUNCTIONALS DEFINED ON 1-DIMENSIONAL CONNECTED SETS: THE PLANAR CASE*

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Abstract. In this paper we consider variational problems involving 1-dimensional connected sets in the Euclidean plane, such as the classical Steiner tree problem and the irrigation (Gilbert–Steiner) problem. We relate them to optimal partition problems and provide a variational approximation through Modica–Mortola type energies proving a Γ-convergence result. We also introduce a suitable convex relaxation and develop the corresponding numerical implementations. The proposed methods are quite general and the results we obtain can be extended to n-dimensional Euclidean space or to more general manifold ambients, as shown in the companion paper [M. Bonafini, G. Orlandi, and E. Oudet, *Variational Approximation of Functionals Defined on* 1*-Dimensional Connected Sets in* $\mathbb{R}^n$, preprint, 2018].

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1. Introduction.

Connected 1-dimensional structures play a crucial role in very different areas like discrete geometry (graphs, networks, spanning, and Steiner trees), structural mechanics (crack formation and propagation), and inverse problems (defects identification, contour segmentation), etc. The modeling of these structures is a key problem both from the theoretical and the numerical points of view. Most of the difficulties encountered in studying such 1-dimensional objects are related to the fact that they are not canonically associated to standard mathematical quantities. In this article we plan to bridge the gap between the well-established methods of multiphase modeling and the world of 1-dimensional connected sets or networks. Whereas we strongly believe that our approach may lead to new points of view in quite different contexts, we restrict here our exposition to the study of two standard problems in the Calculus of Variations which are, respectively, the classical Steiner Tree Problem (STP) and the Gilbert–Steiner problem (also called the irrigation problem).

The (STP) [22] can be described as follows: given N points $P_1, \dots, P_N$ in a metric space X (e.g., X a graph, with P_i assigned vertices) find a connected graph $F \subset X$ containing the points P_i and having minimal length. Such an optimal graph F turns out to be a tree and is thus called a *Steiner Minimal Tree* (SMT). In case $X = \mathbb{R}^d$, $d \geq 2$, endowed with the Euclidean ℓ^2 metric, one refers often to the *Euclidean* or *geometric* (STP), while for $X = \mathbb{R}^d$ endowed with the ℓ^1 (Manhattan) distance or for X contained in a fixed grid $\mathcal{G} \subset \mathbb{R}^d$, one refers to the *rectilinear* (STP). Here we will

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adopt the general metric space formulation of [31]: given a metric space X, and given a compact (possibly infinite) set of terminal points $A \subset X$, find

$$\text{(STP)} \qquad \inf\{\mathcal{H}^1(S),\, S \text{ connected},\, S \supset A\},$$

where $\mathcal{H}^1$ indicates the 1-dimensional Hausdorff measure on X. Existence of solutions for (STP) relies on Golab's compactness theorem for compact connected sets, and it holds true also in generalized cases (e.g., $\inf \mathcal{H}^1(S)$, $S \cup A$ connected).

The Gilbert–Steiner problem, or α-irrigation problem [10, 37] consists of finding a network S along which flow unit masses located at the sources $P_1, \dots, P_{N-1}$ to the target point P_N. Such a network S can be viewed as $S = \cup_{i=1}^{N-1} \gamma_i$, with γ_i a path connecting P_i to P_N, corresponding to the trajectory of the unit mass located at P_i. To favor branching, one is led to consider a cost to be minimized by S which is a sublinear (concave) function of the mass density $\theta(x) = \sum_{i=1}^{N-1} \mathbf{1}_{\gamma_i}(x)$, i.e., for $0 \leq \alpha \leq 1$, find

$$(I_\alpha) \qquad \inf \int_S |\theta(x)|^\alpha d\mathcal{H}^1(x).$$

Notice that (I_1) corresponds to the Monge optimal transport problem, while (I_0) corresponds to (STP). As for (STP), a solution to (I_α) is known to exist and the optimal network S turns out to be a tree [10].

Problems like (STP) or (I_α) are relevant for the design of optimal transport channels or networks connecting given endpoints, for example, the optimal design of net routing in VLSI circuits in the case $d = 2, 3$. The (STP) has been widely studied from the theoretical and numerical points of view in order to efficiently devise constructive solutions, mainly through combinatoric optimization techniques. Finding an SMT is known to be an NP hard problem (and even NP complete in certain cases); see, for instance, [6, 7] for a comprehensive survey on PTAS algorithms for (STP).

The situation in the Euclidean case for (STP) is theoretically well understood: given N points $P_i \in \mathbb{R}^d$ an SMT connecting them always exists, the solution being, in general, not unique (think, for instance, of symmetric configurations of the endpoints P_i). The SMT is a union of segments connecting the endpoints, possibly meeting at $120°$ in at most $N-2$ further branch points, called *Steiner points*.

Nonetheless, the quest of computationally tractable approximating schemes for (STP) and for (I_α) has recently attracted a lot of attention in the Calculus of Variations community. In particular (I_α) has been studied in the framework of optimal branched transport theory [10, 16], while (STP) has been interpreted as, respectively, a size minimization problem for 1-dimensional connected sets [27, 20], or even a Plateau problem in a suitable class of vector distributions endowed with some algebraic structure [27, 24], to be solved by finding suitable *calibrations* [25]. Several authors have proposed different approximations of those problems, whose validity is essentially limited to the planar case, mainly using a phase field based approach together with some coercive regularization; see, e.g., [13, 19, 29, 12].

Our aim is to propose a variational approximation for (STP) and for the Gilbert–Steiner irrigation problem (in the equivalent formulations of [37, 23]) in the Euclidean case $X = \mathbb{R}^d$, $d \geq 2$. In this paper we focus on the planar case $d = 2$ and prove a Γ-convergence result (see Theorem 3.12 and Proposition 3.11) by considering integral functionals of Modica–Mortola type [26]. In the companion paper [11] we rigorously prove that certain integral functionals of Ginzburg–Landau type (see [1]) yield a variational approximation for (STP) and (I_α) valid in any dimension $d \geq 3$. This approach is related to the interpretation of (STP) and (I_α) as a mass minimization problem in a cobordism class of integral currents with multiplicities in a suitable normed group as studied by Marchese and Massaccesi in [24, 23] (see also [27] for the planar case). Our method is quite general and may be easily adapted to a variety of situations (e.g., in manifolds or more general metric space ambients, with densities or anisotropic norms, etc.).

The plan of the paper is as follows: in section 2 we reformulate (STP) and (I_α) as a suitable modification of the optimal partition problem in the planar case. In section 3, we state and prove our main Γ-convergence results, respectively Proposition 3.11 and Theorem 3.12. Inspired by [18], in section 4 we introduce a convex relaxation of the corresponding energies. In section 5 we present our approximating scheme for (STP) and for the Gilbert–Steiner problem and illustrate its flexibility in different situations, showing how our convex formulation is able to recover multiple solutions, whereas Γ-relaxation detects any locally minimizing configuration. Finally, in section 6 we propose some examples and generalizations that are extensively studied in the companion paper [11].

2. Steiner problem for Euclidean graphs and optimal partitions. In this section we describe some optimization problems on Euclidean graphs with fixed endpoints set A, like (STP) or irrigation type problems, following the approach of [24, 23], and we rephrase them as optimal partition type problems in the planar case $\mathbb{R}^2$.

2.1. Rank one tensor valued measures and acyclic graphs. For $M > 0$, we consider Radon measures Λ on $\mathbb{R}^d$ with values in the space of matrices $\mathbb{R}^{d\times M}$. For each $i = 1, \dots, M$ we define as Λ_i the vector measure representing the ith column of Λ, so that we can write $\Lambda = (\Lambda_1, \dots, \Lambda_M)$. The total variation measures $|\Lambda_i|$ are defined as usual with respect to (w.r.t.) the Euclidean structure on $\mathbb{R}^d$, while we set $\mu_\Lambda = \sum_{i=1}^M |\Lambda_i|$. Thanks to the Radon–Nikodym theorem we can find a matrix valued density function $p(x) = (p_1(x), \dots, p_M(x))$, with entries $p_{ki} \in L^1(\mathbb{R}^d, \mu_\Lambda)$ for all $k = 1, \dots, d$ and $i = 1, \dots, M$, such that $\Lambda = p(x)\mu_\Lambda$ and $\sum_{i=1}^M |p_i(x)| = 1$ for μ_Λ-almost everywhere (a.e.) $x \in \mathbb{R}^d$ (where on vectors of $\mathbb{R}^d$ $|\cdot|$ denotes the Euclidean norm). Whenever p is a rank one matrix μ_Λ-a.e. we say that Λ is a rank one tensor valued measure and we write it as $\Lambda = \tau \otimes g \cdot \mu_\Lambda$ for a μ_Λ-measurable unit vector field τ in $\mathbb{R}^d$ and $g\colon \mathbb{R}^d \to \mathbb{R}^M$ satisfying $\sum_{i=1}^M |g_i| = 1$.

Given $\Lambda \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^{d\times M})$ and a function $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^{d\times M})$, with $\varphi = (\varphi_1, \dots, \varphi_M)$, we have

$$\langle \Lambda, \varphi \rangle = \sum_{i=1}^M \langle \Lambda_i, \varphi_i \rangle = \sum_{i=1}^M \int_{\mathbb{R}^d} \varphi_i \, d\Lambda_i,$$

and fixing a norm Ψ on $\mathbb{R}^M$, one may define the Ψ-mass measure of Λ as

$$|\Lambda|_\Psi(B) := \sup_{\substack{\omega \in C_c^\infty(B;\mathbb{R}^d) \\ h \in C_c^\infty(B;\mathbb{R}^M)}} \left\{ \langle \Lambda, \omega \otimes h \rangle \,, \quad |\omega(x)| \le 1 \,, \; \Psi^*(h(x)) \le 1 \right\} , \tag{2.1}$$

for $B \subset \mathbb{R}^d$ open, where Ψ^* is the dual norm to Ψ w.r.t. the scalar product on $\mathbb{R}^M$, i.e.,

$$\Psi^*(y) = \sup_{x \in \mathbb{R}^M} \langle y, x \rangle - \Psi(x).$$

Denote $||\Lambda||_\Psi = |\Lambda|_\Psi(\mathbb{R}^d)$ to be the Ψ-mass norm of Λ. In particular one can see that μ_Λ coincides with the measure $|\Lambda|_{\ell^1}$, which from now on will be denoted as $|\Lambda|_1$, and any rank one measure Λ may be written as $\Lambda = \tau \otimes g \cdot |\Lambda|_1$ so that $|\Lambda|_\Psi = \Psi(g)|\Lambda|_1$. Along the lines of [24] we will rephrase the Steiner and Gilbert–Steiner problems as the optimization of a suitable Ψ-mass norm over a given class of rank one tensor valued measures.

Let $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$, be a given set of N distinct points, with $N > 2$. We define the class $\mathcal{G}(A)$ as the set of *acyclic graphs* L connecting the endpoints set A such that L can be described as the union $L = \cup_{i=1}^{N-1} \lambda_i$, where λ_i are simple rectifiable curves with finite length having P_i as initial point and P_N as final point, oriented by $\mathcal{H}^1$-measurable unit vector fields τ_i satisfying $\tau_i(x) = \tau_j(x)$ for $\mathcal{H}^1$-a.e. $x \in \lambda_i \cap \lambda_j$ (i.e., the orientation of λ_i is coherent with that of λ_j on their intersection).

For $L \in \mathcal{G}(A)$, if we identify the curves λ_i with the vector measures $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$, all of the information concerning this acyclic graph L is encoded in the rank one tensor valued measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, where the $\mathcal{H}^1$-measurable vector field $\tau \in \mathbb{R}^d$ carrying the orientation of the graph L satisfies $\operatorname{spt} \tau = L$, $|\tau| = 1$, $\tau = \tau_i$ $\mathcal{H}^1$-a.e. on λ_i, and the $\mathcal{H}^1$-measurable vector map $g \colon \mathbb{R}^d \to \mathbb{R}^{N-1}$ has components g_i satisfying $g_i \cdot \mathcal{H}^1 \llcorner L = \mathcal{H}^1 \llcorner \lambda_i = |\Lambda_i|$, with $|\Lambda_i|$ the total variation measure of the vector measure $\Lambda_i = \tau \cdot \mathcal{H}^1 \llcorner \lambda_i$. Observe that $g_i \in \{0, 1\}$ a.e. for any $1 \leq i \leq N-1$ and, moreover, that each Λ_i verifies the property

$$\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}. \tag{2.2}$$

DEFINITION 2.1. *Given any graph $L \in \mathcal{G}(A)$, we call the above constructed $\Lambda_L \equiv \Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical rank one tensor valued measure representation of the acyclic graph L.*

To any compact connected set $K \supset A$ with $\mathcal{H}^1(K) < +\infty$, i.e., to any candidate minimizer for (STP), we may associate in a canonical way an acyclic graph $L \in \mathcal{G}(A)$ connecting $\{P_1, \dots, P_N\}$ such that $\mathcal{H}^1(L) \leq \mathcal{H}^1(K)$ (see, e.g., Lemma 2.1 in [24]). Given such a graph $L \in \mathcal{G}(A)$ canonically represented by the tensor valued measure Λ, the measure $\mathcal{H}^1 \llcorner L$ corresponds to the smallest positive measure dominating $\mathcal{H}^1 \llcorner \lambda_i$ for $1 \leq i \leq N-1$. It is thus given by $\mathcal{H}^1 \llcorner L = \sup_i \mathcal{H}^1 \llcorner \lambda_i = \sup_i |\Lambda_i|$, the supremum of the total variation measures $|\Lambda_i|$. We recall that, for any nonnegative $\psi \in C_c^0(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \psi \, d\left(\sup_i |\Lambda_i|\right) = \sup \left\{ \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \varphi_i \, d|\Lambda_i| \, , \ \varphi_i \in C_c^0(\mathbb{R}^d), \ \sum_{i=1}^{N-1} \varphi_i(x) \leq \psi(x) \right\}.$$

Remark 2.2 (graphs as G-currents). In [24], the rank one tensor measure $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ identifying a graph in $\mathbb{R}^d$ is defined as a *current* with coefficients in the group $\mathbb{Z}^{N-1} \subset \mathbb{R}^{N-1}$. For $\omega \in \mathcal{D}^1(\mathbb{R}^d)$ a smooth compactly supported differential 1-form and $\vec{\varphi} = (\varphi_1, \dots, \varphi_{N-1}) \in [\mathcal{D}(\mathbb{R}^d)]^{N-1}$ a smooth test (vector) function, one sets

$$\begin{aligned} \langle \Lambda, \omega \otimes \vec{\varphi} \rangle &:= \int_{\mathbb{R}^d} \langle \omega \otimes \vec{\varphi}, \tau \otimes g \rangle \ d\mathcal{H}^1 \llcorner L = \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i g_i \, d\mathcal{H}^1 \llcorner L \\ &= \sum_{i=1}^{N-1} \int_{\mathbb{R}^d} \langle \omega, \tau \rangle \varphi_i \, d|\Lambda_i| \, . \end{aligned}$$

Moreover, fixing a norm Ψ on $\mathbb{R}^{N-1}$, one may define the Ψ-mass of the current Λ as it is done in (2.1). In [24] the authors show that classical integral currents, i.e.,

$G = \mathbb{Z}$, are not suited to describe (STP) as a mass minimization problem: for example, minimizers are not ensured to have connected support.

2.2. Irrigation type functionals. In this section we consider functionals defined on acyclic graphs connecting a fixed set $A = \{P_1, \ldots, P_N\} \subset \mathbb{R}^d$, $d \geq 2$, by using their canonical representation as rank one tensor valued measures, in order to identify the graph with an *irrigation plan* from the point sources $\{P_1, \ldots, P_{N-1}\}$ to the target point P_N. We focus here on suitable energies in order to describe the irrigation problem and the (STP) in a common framework as in [24, 23]. We observe, moreover, that the irrigation problem with one point source (I_α) introduced by Xia [37], in the equivalent formulation of [23], approximates the (STP) as $\alpha \to 0$ in the sense of Γ-convergence (see Proposition 2.4).

Consider on $\mathbb{R}^{N-1}$ the norms $\Psi_\alpha = |\cdot|_{\ell^{1/\alpha}}$ (for $0 < \alpha \leq 1$) and $\Psi_0 = |\cdot|_{\ell^\infty}$. Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the canonical representation of an acyclic graph $L \in \mathcal{G}(A)$, so that we have $|\tau| = 1$, $g_i \in \{0, 1\}$ for $1 \leq i \leq N-1$, and hence $|g|_\infty = 1$ $\mathcal{H}^1$-a.e. on L. Let us define for such Λ and any $\alpha \in [0, 1]$ the functional

$$\mathcal{F}^\alpha(\Lambda) := ||\Lambda||_{\Psi_\alpha} = |\Lambda|_{\Psi_\alpha}(\mathbb{R}^d).$$

Observe that, by (2.1),

$$\mathcal{F}^0(\Lambda) = \int_{\mathbb{R}^d} |\tau||g|_\infty \, d\mathcal{H}^1 \llcorner L = \mathcal{H}^1(L)$$

and

$$\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau||g|_{1/\alpha} \, d\mathcal{H}^1 \llcorner L = \int_L |\theta|^\alpha d\mathcal{H}^1 \,, \tag{2.3}$$

where $\theta(x) = \sum_i g_i(x)^{1/\alpha} = \sum_i g_i(x) \in \mathbb{Z}$, and $0 \leq \theta(x) \leq N-1$. We thus recognize that minimizing the functional $\mathcal{F}^\alpha$ among graphs L connecting $P_1, \ldots, P_{N-1}$ to P_N solves the irrigation problem (I_α) with unit mass sources $P_1, \ldots, P_{N-1}$ and target P_N (see [23]), while minimizing $\mathcal{F}^0$ among graphs L with endpoints set $\{P_1, \ldots, P_N\}$ solves (STP) in $\mathbb{R}^d$.

Since both $\mathcal{F}^\alpha$ and $\mathcal{F}^0$ are mass type functionals, minimizers do exist in the class of rank one tensor valued measures. The fact that the minimization problem within the class of canonical tensor valued measures representing acyclic graphs has a solution in that class is a consequence of compactness properties of Lipschitz maps (more generally by the compactness theorem for G-currents [24]; in $\mathbb{R}^2$ it follows alternatively by the compactness theorem in the SBV class [5]). Actually, existence of minimizers in the canonically oriented graph class in $\mathbb{R}^2$ can be deduced as a by-product of our convergence result (see Proposition 3.11 and Theorem 3.12) and in $\mathbb{R}^d$, for $d > 2$, by the parallel Γ-convergence analysis contained in the companion paper [11].

Remark 2.3. A minimizer of $\mathcal{F}^0$ (resp., $\mathcal{F}^\alpha$) among tensor valued measures Λ representing admissible graphs corresponds necessarily to the canonical representation of a minimal graph, i.e., $g_i \in \{0, 1\}$ for all $1 \leq i \leq N-1$. Indeed, since $g_i \in \mathbb{Z}$, if $g_i \neq 0$, we have $|g_i| \geq 1$, hence $g_i \in \{-1, 0, 1\}$ for minimizers. Moreover, if $g_i = -g_j$ on a connected arc in $\lambda_i \cap \lambda_j$, with λ_i going from P_i to P_N and λ_j going from P_j to P_N, this implies that $\lambda_i \cup \lambda_j$ contains a cycle and Λ cannot be a minimizer. Hence, up to reversing the orientation of the graph, $g_i \in \{0, 1\}$ for all $1 \leq i \leq N-1$.

We conclude this section by observing in the following proposition that the (STP) can be seen as the limit of irrigation problems.

PROPOSITION 2.4. *The functional $\mathcal{F}^0$ is the Γ-limit, as $\alpha \to 0$, of the functionals $\mathcal{F}^\alpha$ w.r.t. the convergence of measures.*

Proof. Let $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the canonical representation of an acyclic graph $L \in \mathcal{G}(A)$, so that $|\tau| = 1$ and $g_i \in \{0,1\}$ for all $i = 1, \dots, N-1$. The functionals $\mathcal{F}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |g|_{1/\alpha} d\mathcal{H}^1 \llcorner L$ generate a monotonic decreasing sequence as $\alpha \to 0$, because $|g|_p \le |g|_q$ for any $1 \le q < p \le +\infty$, and, moreover, $\mathcal{F}^\alpha(\Lambda) \to \mathcal{F}^0(\Lambda)$ because $|g|_q \to |g|_\infty$ as $q \to +\infty$. Then, by elementary properties of Γ-convergence (see, for instance, Remark 1.40 of [15]) we have $\mathcal{F}^\alpha \xrightarrow{\Gamma} \mathcal{F}^0$. ∎

2.3. Acyclic graphs and partitions of $\mathbb{R}^2$. This section is dedicated to the 2-dimensional case. The aim is to provide an equivalent formulation of (STP) and (I_α) in terms of an optimal partition type problem. The equivalence of (STP) with an optimal partition problem has been already studied in the case where $P_1, \dots, P_N$ lie on the boundary of a convex set; see, for instance, [3, 4] and Remark 2.10.

To begin we state a result saying that two acyclic graphs having the same endpoints set give rise to a partition of $\mathbb{R}^2$, in the sense that their oriented difference corresponds to the orthogonal distributional gradient of a piecewise integer valued function having bounded total variation, which in turn determines the partition (see [5]). This is actually an instance of the constancy theorem for currents or Poincaré's lemma for distributions (see [21]).

LEMMA 2.5. *Let $\{P, R\} \subset \mathbb{R}^2$, and let λ, γ be simple rectifiable curves from P to R oriented by $\mathcal{H}^1$-measurable unit vector fields τ', τ''. Define as above $\Lambda = \tau' \cdot \mathcal{H}^1 \llcorner \lambda$ and $\Gamma = \tau'' \cdot \mathcal{H}^1 \llcorner \gamma$.*

Then there exists a function $u \in SBV(\mathbb{R}^2; \mathbb{Z})$ such that, denoting Du and $Du^\perp$, respectively, the measures representing the gradient and the orthogonal gradient of u, we have $Du^\perp = \Gamma - \Lambda$.

Proof. Consider simple oriented polygonal curves λ_k and γ_k connecting P to R such that the Hausdorff distance to, respectively, λ and γ is less than $\frac{1}{k}$ and the length of λ_k (resp., γ_k) converges to the length of λ (resp., γ). We can also assume without loss of generality (w.l.o.g.) that λ_k and γ_k intersect only transversally in a finite number of points $m_k \ge 2$. Let τ'_k, τ''_k be the $\mathcal{H}^1$-measurable unit vector fields orienting λ_k, γ_k and define the measures $\Lambda_k = \tau'_k \cdot \mathcal{H}^1 \llcorner \lambda_k$ and $\Gamma_k = \tau''_k \cdot \mathcal{H}^1 \llcorner \gamma_k$.

For a given $k \in \mathbb{N}$ consider the closed polyhedral curve $\sigma_k = \lambda_k \cup \gamma_k$ oriented by $\tau_k = \tau'_k - \tau''_k$ (i.e., we reverse the orientation of γ_k). For every $x \in \mathbb{R}^2 \setminus \sigma_k$ let us consider the index of x w.r.t. σ_k (or winding number) and denote it as

$$u_k(x) = \mathrm{Ind}_{\sigma_k}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\sigma_k} \frac{dz}{z - x}.$$

By Theorem 10.10 in [33], the function u_k is integer valued and constant in each connected component of $\mathbb{R}^2 \setminus \sigma_k$ and vanishes in the unbounded one. Furthermore, for a.e. $x \in \sigma_k$ we have

$$\lim_{\varepsilon \to 0^+} u_k(x + \varepsilon \tau_k(x)^\perp) - \lim_{\varepsilon \to 0^-} u_k(x + \varepsilon \tau_k(x)^\perp) = 1;$$

i.e., u_k has a jump of $+1$ whenever crossing σ_k from "right" to "left" (cf. [32, Lemma 3.3.2]). This means that

$$Du_k^\perp = -\tau_k \cdot \mathcal{H}^1 \llcorner \sigma_k = \Gamma_k - \Lambda_k.$$

Thus, $|Du_k|(\mathbb{R}^2) = \mathcal{H}^1(\sigma_k)$ and $\|u_k\|_{L^1(\mathbb{R}^2)} \leq C|Du_k|(\mathbb{R}^2)$ by Poincaré's inequality in BV. Hence $u_k \in SBV(\mathbb{R}^2;\mathbb{Z})$ is an equibounded sequence in norm, and by the Rellich compactness theorem there exists a subsequence still denoted u_k converging in $L^1(\mathbb{R}^2)$ to a $u \in SBV(\mathbb{R}^2;\mathbb{Z})$. Taking into account that we have $Du_k^\perp = \Gamma_k - \Lambda_k$, we deduce, in particular, that $Du^\perp = \Gamma - \Lambda$ as desired. ∎

Remark 2.6. Let $A \subset \mathbb{R}^2$ as above. For $i = 1, \dots, N-1$, let γ_i be the segment joining P_i to P_N, denote by $\tau_i = \frac{P_N - P_i}{|P_N - P_i|}$ its orientation, and identify γ_i with the vector measure $\Gamma_i = \tau_i \cdot \mathcal{H}^1 \llcorner \gamma_i$. Then $G = \cup_{i=1}^{N-1} \gamma_i$ is an acyclic graph connecting the endpoints set A and $\mathcal{H}^1(G) = (\sup_i |\Gamma_i|)(\mathbb{R}^2)$.

Given the set of terminal points $A = \{P_1, \dots, P_N\} \subset \mathbb{R}^2$, let us fix some $G \in \mathcal{G}(A)$ (for example, the one constructed in Remark 2.6). For any acyclic graph $L \in \mathcal{G}(A)$, denoting Γ (resp., Λ) the canonical tensor valued representation of G (resp., L), by means of Lemma 2.5 we have

$$\mathcal{H}^1(L) = \int_{\mathbb{R}^2} \sup_i |\Lambda_i| = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i| \tag{2.4}$$

for suitable $u_i \in SBV(\mathbb{R}^2;\mathbb{Z})$, $1 \leq i \leq N-1$. Thus, using the family of measures $\Gamma = (\Gamma_1, \dots, \Gamma_{N-1})$ of Remark 2.6, we are led to consider the minimization problem for $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ for the functional

$$F^0(U) = |DU^\perp - \Gamma|_{\Psi_0}(\mathbb{R}^2) = \int_{\mathbb{R}^2} \sup_i |Du_i^\perp - \Gamma_i|. \tag{2.5}$$

PROPOSITION 2.7. *There exists* $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ *such that*

$$F^0(U) = \inf_{V \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})} F^0(V).$$

Moreover, $\operatorname{spt} U \subset \Omega = \{x \in \mathbb{R}^2 \,:\, |x| < 10 \max_i |P_i|\}$.

Proof. Observe first that for any $U \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ with $F^0(U) < \infty$, we can find $\tilde{U}$ such that (s.t.) $F^0(\tilde{U}) \leq F^0(U)$ and $\operatorname{spt} \tilde{U} \subset \Omega$. Indeed, consider $r = 8 \max_i |P_i|$, $\chi = \mathbf{1}_{B_r(0)}$, and $\tilde{U} = (\chi u_1, \dots, \chi u_{N-1})$. One has, for $1 \leq i \leq N-1$,

$$\int_{\mathbb{R}^2 \setminus B_r(0)} |D\tilde{u}_i| = \int_{\partial B_r(0)} |u_i^+|,$$

where u_i^+ is the trace on $\partial B_r(0)$ of u_i restricted to $B_r(0)$, and

$$\begin{aligned}\int_{\mathbb{R}^2} |D\tilde{u}_i^\perp - \Gamma_i| &= \int_{B_r(0)} |Du_i^\perp - \Gamma_i| + \int_{\partial B_r(0)} |u_i^+| \\ &\leq \int_{B_r(0)} |Du_i^\perp - \Gamma_i| + \int_{\mathbb{R}^2 \setminus B_r(0)} |Du_i| = \int_{\mathbb{R}^2} |Du_i^\perp - \Gamma_i|\end{aligned}$$

for any $i = 1, \dots, N-1$, i.e., $F^0(\tilde{U}) \leq F^0(U)$.

Consider now a minimizing sequence $U^k \in SBV(\mathbb{R}^2;\mathbb{Z}^{N-1})$ of F^0. We may suppose w.l.o.g. $\operatorname{spt}(U^k) \subset \Omega$, so that, for any $1 \leq i \leq N-1$,

$$|Du_i^k|(\Omega) \leq |Du_i^k - \Gamma_i|(\Omega) + \mathcal{H}^1(G) \leq F^0(U^k) + \mathcal{H}^1(G) \leq 3\mathcal{H}^1(G)$$

for k sufficiently large. Hence U^k is uniformly bounded in BV by Poincaré's inequality on Ω, so that it is compact in $L^1(\Omega;\mathbb{R}^{N-1})$ and, up to a subsequence, $U^k \to U$ a.e., whence $U \in SBV(\Omega;\mathbb{Z}^{N-1})$, $\operatorname{spt} U \subset \Omega$, and U minimizes F^0 by lower semicontinuity of the norm. ∎

We have already seen that to each acyclic graph $L \in \mathcal{G}(A)$ we can associate a function $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $\mathcal{H}^1(L) = F^0(U)$. On the other hand, for minimizers of F^0, we have the following proposition.

Proposition 2.8. *Let $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ be a minimizer of F^0; then there exists an acyclic graph $L \in \mathcal{G}(A)$ connecting the terminal points $P_1, \dots, P_N$ and such that $F^0(U) = \mathcal{H}^1(L)$.*

Proof. Let $U = (u_1, \dots, u_{N-1})$ be a minimizer of F^0 in $SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$, and denote $\Lambda_i = \Gamma_i - Du_i^{\perp}$. Observe that each Du_i has no absolutely continuous part w.r.t. the Lebesgue measure (indeed, u_i is piecewise constant being integer valued) and so $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$ for some 1-rectifiable set λ_i and $\mathcal{H}^1$-measurable vector field τ_i. Since we have $\operatorname{div} \Lambda_i = \delta_{P_i} - \delta_{P_N}$, λ_i necessarily contains a simple rectifiable curve λ_i' connecting P_i to P_N (use, for instance, the decomposition theorem for rectifiable 1-currents in cyclic and acyclic parts, as it is done in [23], or the Smirnov decomposition of solenoidal vector fields [35]).

Consider now the canonical rank one tensor measure Λ' associated to the acyclic subgraph $L' = \lambda_1' \cup \dots \cup \lambda_{N-1}'$ connecting $P_1, \dots, P_{N-1}$ to P_N. Then by Lemma 2.5, there exists $U' = (u_1', \dots, u_{N-1}') \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ such that $Du_i'^{\perp} = \Gamma_i - \Lambda_i'$ and, in particular, $F^0(U') = \mathcal{H}^1(L') \leq \mathcal{H}^1(L) \leq F^0(U)$. We deduce $\mathcal{H}^1(L') = \mathcal{H}^1(L)$, hence $L' = L$, L is acyclic, and $H^1(L) = F^0(U)$. $\square$

Remark 2.9. We have shown the relationship between (STP) and the minimization of F^0 over functions in $SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$, namely

$$\inf\{F^0(U) \,:\, U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})\} = \inf\{\mathcal{F}^0(\Lambda_L) \,:\, L \in \mathcal{G}(\{P_1, \dots, P_N\})\}.$$

A similar connection can be made between the α-irrigation problem (I_α) and minimization over $SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ of

$$F^\alpha(U) = |DU^\perp - \Gamma|_{\Psi_\alpha}(\mathbb{R}^2), \tag{2.6}$$

namely we have

$$\inf\{F^\alpha(U) \,:\, U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})\} = \inf\{\mathcal{F}^\alpha(\Lambda_L) \,:\, L \in \mathcal{G}(\{P_1, \dots, P_N\})\},$$

where $\mathcal{F}^\alpha$ is defined in (2.3). Indeed, given a norm Ψ on $\mathbb{R}^{N-1}$ and $F^\Psi(U) = |DU^\perp - \Gamma|_\Psi(\mathbb{R}^2)$ for $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$, the proofs of Propositions 2.7 and 2.8 carry over to this general context: there exists $U \in SBV(\mathbb{R}^2; \mathbb{Z}^{N-1})$ realizing $\inf F^\Psi$, with $\operatorname{spt} U \subset \Omega$ and $DU^\perp - \Gamma = \Lambda_L$ with $\Lambda_L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ the canonical representation of an acyclic graph $L \in \mathcal{G}(\{P_1, \dots, P_N\})$.

Remark 2.10. In the case $P_1, \dots, P_N \in \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ a convex set, we may choose $G = \cup_{i=1}^{N-1} \gamma_i$ with γ_i connecting P_i to P_N and $\operatorname{spt} \gamma_i \subset \partial\Omega$. We deduce by (2.4) that for any acyclic graph $L \in \mathcal{G}(A)$

$$\mathcal{H}^1(L) = \int_\Omega \sup_i |Du_i^\perp|$$

for suitable $u_i \in SBV(\Omega; \mathbb{Z})$ such that (in the trace sense) $u_i = 1$ on $\gamma_i \subset \partial\Omega$ and $u_i = 0$ elsewhere in $\partial\Omega$, $1 \leq i \leq N-1$. We recover here an alternative formulation of the optimal partition problem in a convex planar set Ω as studied, for instance, in [3] and [4].

The aim of the next section is then to provide an approximation of minimizers of the functionals F^α (and more generally F^Ψ) through minimizers of more regular energies of Modica–Mortola type.

3. Variational approximation of F^α. In this section we state and prove our main results, namely Proposition 3.11 and Theorem 3.12, concerning the approximation of minimizers of F^α through minimizers of Modica–Mortola type functionals, in the spirit of Γ-convergence.

3.1. Modica–Mortola functionals for functions with prescribed jump. In this section we consider Modica–Mortola functionals for functions having a prescribed jump part along a fixed segment in $\mathbb{R}^2$ and we prove compactness and lower-bounds for sequences having a uniform energy bound. Let $P, Q \in \mathbb{R}^2$, and let s be the segment connecting P to Q. We denote by $\tau_s = \frac{Q-P}{|Q-P|}$ its orientation and define $\Sigma_s = \tau_s \cdot \mathcal{H}^1 \llcorner s$. Up to rescaling, suppose $\max(|P|, |Q|) = 1$, and let $\Omega = B_{10}(0)$ and $\Omega_\delta = \Omega \setminus (B_\delta(P) \cup B_\delta(Q))$ for $0 < \delta \ll |Q - P|$. We consider the Modica–Mortola type functionals

$$F_\varepsilon(u, \Omega_\delta) = \int_{\Omega_\delta} e_\varepsilon(u)\, dx = \int_{\Omega_\delta} \varepsilon |Du^\perp - \Sigma_s|^2 + \frac{1}{\varepsilon} W(u)\, dx, \tag{3.1}$$

defined for $u \in H_s = \{u \in W^{1,2}(\Omega_\delta \setminus s) \cap SBV(\Omega_\delta) \ : \ u|_{\partial\Omega} = 0\}$, where W is a smooth nonnegative 1-periodic potential vanishing on $\mathbb{Z}$ (e.g., $W(u) = \sin^2(\pi u)$). Define $H(t) = 2\int_0^t \sqrt{W(\tau)}\, d\tau$ and $c_0 = H(1)$.

Remark 3.1. Notice that any function $u \in H_s$ with $F_\varepsilon(u, \Omega_\delta) < \infty$ has necessarily a prescribed jump $u^+ - u^- = +1$ across $s \llcorner \Omega_\delta$ in the direction $\nu_s = -\tau_s^\perp$ in order to erase the contribution of the measure term Σ_s in the energy. We thus have the decomposition

$$Du^\perp = \nabla u^\perp \mathcal{L}^2 + Ju^\perp = \nabla u^\perp \mathcal{L}^2 + \Sigma_s \llcorner \Omega_\delta,$$

where $\nabla u \in L^2(\Omega_\delta)$ is the absolutely continuous part of Du w.r.t. the Lebesgue measure $\mathcal{L}^2$, and $Ju = (u^+ - u^-)\nu_s \cdot \mathcal{H}^1 \llcorner s = \nu_s \cdot \mathcal{H}^1 \llcorner s$.

Remark 3.2. Notice that we cannot work directly in Ω with F_ε due to summability issues around the points P and Q for the absolutely continuous part of the gradient; indeed, there are no functions $u \in W^{1,2}(\Omega \setminus s)$ such that $u^+ - u^- = 1$ on s. To avoid this issue one could consider variants of the functionals $F_\varepsilon(\cdot, \Omega)$ by relying on suitable smoothings $\Sigma_{s,\epsilon} = \Sigma_s * \eta_\varepsilon$ of the measure Σ_s, with η_ε a symmetric mollifier.

PROPOSITION 3.3 (compactness). *For any sequence $\{u_\varepsilon\}_\varepsilon \subset H_s$ such that $F_\varepsilon(u_\varepsilon, \Omega_\delta) \le C$, there exists $u \in SBV(\Omega_\delta; \mathbb{Z})$ such that (up to a subsequence) $u_\varepsilon \to u$ in $L^1(\Omega_\delta)$.*

Proof. By Remark 3.1 we have $Du_\varepsilon^\perp = \nabla u_\varepsilon^\perp \mathcal{L}^2 + \Sigma_s \llcorner \Omega_\delta$, and using the classical Modica–Mortola trick one has

$$\begin{aligned} C &\ge \int_{\Omega_\delta} \varepsilon |Du_\varepsilon^\perp - \Sigma_s|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx \\ &= \int_{\Omega_\delta} \varepsilon |\nabla u_\varepsilon^\perp|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx \ge 2\int_{\Omega_\delta} \sqrt{W(u_\varepsilon)} |\nabla u_\varepsilon|\, dx. \end{aligned}$$

Recall that $H(t) = 2\int_0^t \sqrt{W(\tau)}\, d\tau$ and $c_0 = H(1)$. By the chain rule, we have

$$\begin{aligned} |D(H \circ u_\varepsilon)|(\Omega_\delta) &= 2\int_{\Omega_\delta} \sqrt{W(u_\varepsilon)} |\nabla u_\varepsilon|\, dx + \int_s \left(H(u_\varepsilon^+) - H(u_\varepsilon^-)\right) d\mathcal{H}^1(x) \\ &\le C + c_0 \mathcal{H}^1(s). \end{aligned}$$

We also have $(H\circ u_\varepsilon)|_{\partial\Omega}=0$ since u_ε vanishes on $\partial\Omega$, so that, by Poincaré's inequality, $\{H\circ u_\varepsilon\}_\varepsilon$ is an equibounded sequence in $BV(\Omega_\delta)$, thus compact in $L^1(\Omega_\delta)$. In particular, there exists $v\in L^1(\Omega_\delta)$ such that, up to a subsequence, $H\circ u_\varepsilon\to v$ in $L^1(\Omega_\delta)$ and pointwise a.e. Since H is a strictly increasing continuous function with $c_0(t-1)\le H(t)\le c_0(t+1)$ for any $t\in\mathbb{R}$, then H^{-1} is uniformly continuous and $|H^{-1}(t)|\le c_0^{-1}(|t|+1)$ for all $t\in\mathbb{R}$. Hence, up to a subsequence, the family $\{u_\varepsilon\}_\varepsilon\subset L^1(\Omega_\delta)$ is pointwise convergent a.e. to $u=H^{-1}(v)\in L^1(\Omega_\delta)$. By Egoroff's theorem, for any $\sigma>0$ there exists a measurable $E_\sigma\subset\Omega_\delta$, with $|E_\sigma|<\sigma$, such that $u_\varepsilon\to u$ uniformly in $\Omega_\delta\setminus E_\sigma$. Then, taking into account that $|t|\le c_0^{-1}(|H(t)|+1)$ for all $t\in\mathbb{R}$, we have

$$\begin{aligned}||u_\varepsilon-u||_{L^1(\Omega_\delta)}&\le ||u_\varepsilon-u||_{L^1(\Omega_\delta\setminus E_\sigma)}+\int_{E_\sigma}(|u_\varepsilon|+|u|)\,dx\\&\le|\Omega|\,||u_\varepsilon-u||_{L^\infty(\Omega_\delta\setminus E_\sigma)}+2c_0^{-1}|E_\sigma|+c_0^{-1}\int_{E_\sigma}(|H\circ u_\varepsilon|+|v|)\,dx,\end{aligned}$$

and for ε, σ small enough the right-hand side can be made arbitrarily small thanks to the uniform integrability of the sequence $\{H\circ u_\varepsilon\}_\varepsilon$. Hence $u_\varepsilon\to u$ in $L^1(\Omega_\delta)$. Furthermore, by Fatou's lemma we have

$$\int_{\Omega_\delta}W(u)\,dx\le\liminf_{\varepsilon\to0}\int_{\Omega_\delta}W(u_\varepsilon)\,dx\le\liminf_{\varepsilon\to0}\varepsilon F_\varepsilon(u_\varepsilon,\Omega_\delta)=0,$$

whence $u(x)\in\mathbb{Z}$ for a.e. $x\in\Omega_\delta$. Finally, we have

$$c_0|Du|(\Omega_\delta)=|D(H\circ u)|(\Omega_\delta)\le\liminf_{\varepsilon\to0}|D(H\circ u_\varepsilon)|(\Omega_\delta)\le C+c_0\mathcal{H}^1(s),$$

i.e., $u\in SBV(\Omega_\delta;\mathbb{Z})$. ◻

PROPOSITION 3.4 (lower-bound inequality). *Let $\{u_\varepsilon\}_\varepsilon\subset H_s$, and let $u\in SBV(\Omega_\delta;\mathbb{Z})$ such that $u_\varepsilon\to u$ in $L^1(\Omega_\delta)$. Then*

$$\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,\Omega_\delta)\ge c_0|Du^\perp-\Sigma_s|(\Omega_\delta).\tag{3.2}$$

Proof. Step 1. Let us prove first that for any open ball $B\subset\Omega_\delta$ we have

$$\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,B)\ge c_0|Du^\perp-\Sigma_s|(B).\tag{3.3}$$

We distinguish two cases, according to whether $B\cap s=\emptyset$ or not. In the first case we have

$$F_\varepsilon(u_\varepsilon,B)=\int_B\varepsilon|Du_\varepsilon^\perp|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\,dx.$$

Reasoning as in the proof of Proposition 3.3,

$$c_0|Du|(B)=|D(H\circ u)|(B)\le\liminf_{\varepsilon\to0}|D(H\circ u_\varepsilon)|(B)\le\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,B),$$

and (3.3) follows.

In the case $B\cap s\ne\emptyset$, we follow the arguments of [8] and consider $u_0=\mathbf{1}_{B^+}$, where $B^+=\{z\in B\setminus s\,:\,(z-z_0)\cdot\nu_s>0\}$ for $z_0\in B\cap s$ and $\nu_s^\perp=\tau_s$, so that $Du_0^\perp=\Sigma_s\llcorner B$. Letting $v_\varepsilon=u_\varepsilon-u_0$ we have $Dv_\varepsilon^\perp=Du_\varepsilon^\perp-\Sigma_s=\nabla u_\varepsilon^\perp\mathcal{L}^2$, with $\nabla u_\varepsilon\in L^2(B)$ and $W(v_\varepsilon)=W(u_\varepsilon)$ on B by 1-periodicity of the potential W. Hence

$$F_\varepsilon(u_\varepsilon,B)=\int_B\varepsilon|Dv_\varepsilon|^2+\frac{1}{\varepsilon}W(v_\varepsilon)\,dx.$$

Let $v = u - u_0$; we then have

$$c_0|Du^\perp - \Sigma_s|(B) = c_0|Dv|(B) \le \liminf_{\varepsilon\to 0} \int_B \varepsilon|Dv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon)\,dx = \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, B),$$

and (3.3) follows.

Step 2. Since $|Du^\perp - \Sigma_s|$ is a Radon measure, one has

$$|Du^\perp - \Sigma_s|(\Omega_\delta) = \sup\left\{\sum_j |Du^\perp - \Sigma_s|(B_j)\right\}, \tag{3.4}$$

where the supremum is taken among all finite collections $\{B_j\}_j$ of pairwise disjoint open balls such that $\cup_j B_j \subset \Omega_\delta$. Applying (3.3) to each B_j and summing over j we have

$$\begin{aligned} c_0 \sum_j |Du^\perp - \Sigma_s|(B_j) &\le \sum_j \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, B_j) \\ &\le \liminf_{\varepsilon\to 0} \sum_j F_\varepsilon(u_\varepsilon, B_j) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, \Omega_\delta), \end{aligned}$$

which gives (3.2) thanks to (3.4). □

Remark 3.5. The proof of Proposition 3.4 can be easily adapted to prove a weighted version of (3.2): in the same hypothesis, for any nonnegative $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\delta} \varphi e_\varepsilon(u_\varepsilon)\,dx \ge c_0 \int_{\Omega_\delta} \varphi\, d|Du^\perp - \Sigma_s|.$$

Remark 3.6. Proposition 3.4 holds true also in case the measure Σ_s is associated to oriented simple polyhedral (or even rectifiable) finite length curves joining P to Q.

3.2. The approximating functionals F_ε^Ψ. We now consider Modica–Mortola approximations for Ψ-mass functionals such as F^α. Let $A = \{P_1, \dots, P_N\}$ be our set of terminal points, and let $\Psi\colon \mathbb{R}^{N-1} \to [0, +\infty)$ be a norm on $\mathbb{R}^{N-1}$. For any $i \in \{1, \dots, N-1\}$ let $\Gamma_i = \tau_i \cdot \mathcal{H}^1 \llcorner \gamma_i$ be the measure defined in Remark 2.6. Without loss of generality suppose $\max_i(|P_i|) = 1$ and define $\Omega = B_{10}(0)$ and $\Omega_\delta = \Omega \setminus \cup_i B_\delta(P_i)$ for $0 < \delta \ll \min_{ij} |P_i - P_j|$. Let

$$H_i = \{u \in W^{1,2}(\Omega \setminus \gamma_i) \cap SBV(\Omega) \,:\, u|_{\partial\Omega} = 0\}, \quad H = H_1 \times \cdots \times H_{N-1}, \tag{3.5}$$

and for $u \in H_i$ define

$$e_\varepsilon^i(u) = \varepsilon|Du^\perp - \Gamma_i|^2 + \frac{1}{\varepsilon}W(u). \tag{3.6}$$

Denote $\vec{e}_\varepsilon(U) = (e_\varepsilon^1(u_1), \dots, e_\varepsilon^{N-1}(u_{N-1}))$ and consider the functionals

$$F_\varepsilon^\Psi(U, \Omega_\delta) = |\vec{e}_\varepsilon(U)\,dx|_\Psi(\Omega_\delta), \tag{3.7}$$

or equivalently, thanks to (2.1),

$$F_\varepsilon^\Psi(U, \Omega_\delta) = \sup_{\varphi \in C_c^\infty(\Omega_\delta; \mathbb{R}^{N-1})} \left\{ \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i e_\varepsilon^i(u_i)\,dx, \quad \Psi^*(\varphi(x)) \le 1 \right\}. \tag{3.8}$$

The previous compactness and lower-bound inequality for functionals with a single prescribed jump extend to F_ε^Ψ as follows.

PROPOSITION 3.7 (compactness). *Given* $\{U_\varepsilon\}_\varepsilon \subset H$ *such that* $F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq C$, *there exists* $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ *such that (up to a subsequence)* $U_\varepsilon \to U$ *in* $[L^1(\Omega_\delta)]^{N-1}$.

Proof. For each $i = 1, \ldots, N-1$, by definition of F_ε^Ψ we have

$$\int_{\Omega_\delta} e_\varepsilon^i(u_{\varepsilon,i})\,dx \leq \Psi^*(e_i) F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq C\Psi^*(e_i),$$

and the result follows applying Proposition 3.3 componentwise. ◻

PROPOSITION 3.8 (lower-bound inequality). *Let* $\{U_\varepsilon\}_\varepsilon \subset H$ *and* $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ *such that* $U_\varepsilon \to U$ *in* $[L^1(\Omega_\delta)]^{N-1}$. *Then*

$$\liminf_{\varepsilon \to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \geq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta). \tag{3.9}$$

Proof. Fix $\varphi \in C_c^\infty(\Omega_\delta; \mathbb{R}^{N-1})$ with $\varphi_i \geq 0$ for any $i = 1, \ldots, N-1$ and $\Psi^*(\varphi(x)) \leq 1$ for all $x \in \Omega_\delta$. By Remark 3.5 we have

$$\begin{aligned} c_0 \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i\, d|Du_i^\perp - \Gamma_i| &\leq \sum_{i=1}^{N-1} \liminf_{\varepsilon \to 0} \int_{\Omega_\delta} \varphi_i e_\varepsilon^i(u_{\varepsilon,i})\,dx \\ &\leq \liminf_{\varepsilon \to 0} \sum_{i=1}^{N-1} \int_{\Omega_\delta} \varphi_i e_\varepsilon^i(u_{\varepsilon,i})\,dx \leq \liminf_{\varepsilon \to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta), \end{aligned}$$

and taking the supremum over φ we get (3.9). ◻

We now state and prove a version of an upper-bound inequality for the functionals F_ε^Ψ which will enable us to deduce the convergence of minimizers of F_ε^Ψ to minimizers of $F^\Psi(U, \Omega_\delta) = c_0|DU^\perp - \Gamma|_\Psi(\Omega_\delta)$, for $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$.

PROPOSITION 3.9 (upper-bound inequality). *Let* $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ *be a rank one tensor valued measure canonically representing an acyclic graph* $L \in \mathcal{G}(A)$, *and let* $U = (u_1, \ldots, u_{N-1}) \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ *such that* $Du_i^\perp = \Gamma_i - \Lambda_i$ *for any* $i = 1, \ldots, N-1$. *Then there exists a sequence* $\{U_\varepsilon\}_\varepsilon \subset H$ *such that* $U_\varepsilon \to U$ *in* $[L^1(\Omega_\delta)]^{N-1}$ *and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta). \tag{3.10}$$

Proof. Step 1. We consider first the case $\Lambda_i = \tau_i \cdot \mathcal{H}^1 \llcorner \lambda_i$ with λ_i a polyhedral curve transverse to γ_i for any $1 \leq i < N$. Then the support of the measure Λ is an acyclic polyhedral graph (oriented by τ and with normal $\nu = \tau^\perp$) with edges $E_0, \ldots, E_M$ and vertices $\{S_0, \ldots, S_\ell\} \nsubseteq (\cup_i \gamma_i) \cap \Omega_\delta$ such that $E_k = [S_{k_1}, S_{k_2}]$ for suitable indices $k_1, k_2 \in \{0, \ldots, \ell\}$. Denote also $g^k = g|_{E_k} \in \mathbb{R}^{N-1}$ and recall $g_i^k \in \{0, 1\}$ for all $1 \leq i < N$. By finiteness there exist $\eta > 0$ and $\alpha \in (0, \pi/2)$ such that given any edge E_k of that graph the sets

$$V^k = \{x \in \mathbb{R}^2, \operatorname{dist}(x, E_k) < \min\{\eta,\, \cos(\alpha) \cdot \operatorname{dist}(x, S_{k_1}),\, \cos(\alpha) \cdot \operatorname{dist}(x, S_{k_2})\}\}$$

are disjoint and their union forms an open neighborhood of $\cup_i \lambda_i \backslash \{S_0, \ldots, S_\ell\}$ (choose, for instance, α such that 2α is smaller than the minimum angle realized by two edges and then pick η satisfying $2\eta \tan\alpha < \min_j \mathcal{H}^1(E_j)$).

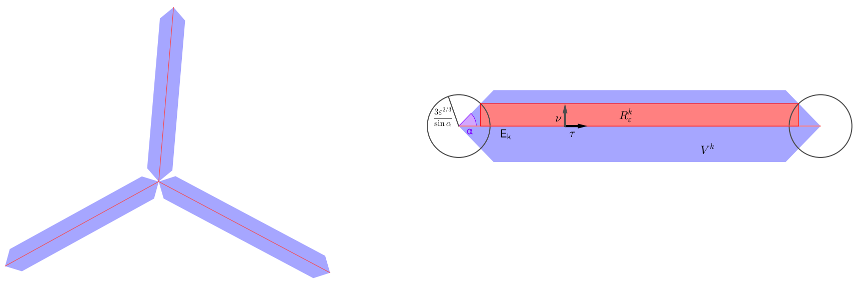

FIG. 1. *Typical shape of the sets V_k (left) and general construction involved in the definition of R^k_ε (right).*

For $0 < \varepsilon \ll \delta$, let $B^m_\varepsilon = \left\{x \in \mathbb{R}^2 \,:\, |x - S_m| < \frac{3\varepsilon^{2/3}}{\sin\alpha}\right\}$, $B_\varepsilon = \cup_m B^m_\varepsilon$, and define $R^k_\varepsilon \subset V^k$ as (see Figure 1)

$$R^k_\varepsilon = \{y + t\nu \,:\, y \in E_k,\, \min\{\mathrm{dist}(y, S_{k_1}), \mathrm{dist}(y, S_{k_2})\} > 3\varepsilon^{2/3}\cot(\alpha),\, 0 < t \le 3\varepsilon^{2/3}\}.$$

Let φ_0 be the optimal profile for the 1-dimensional Modica–Mortola functional, which solves $\varphi_0' = \sqrt{W(\varphi_0)}$ on $\mathbb{R}$ and satisfies $\lim_{\tau\to-\infty}\varphi_0(\tau) = 0$, $\lim_{\tau\to\infty}\varphi_0(\tau) = 1$, and $\varphi_0(0) = 1/2$. Let us define $\tau_\varepsilon = \varepsilon^{-1/3}$, $r^+_\varepsilon = \varphi_0(\tau_\varepsilon)$, $r^-_\varepsilon = \varphi_0(-\tau_\varepsilon)$, and

$$\tilde\varphi_\varepsilon(\tau) = \begin{cases} 0, & \tau < -\tau_\varepsilon - r^-_\varepsilon, \\ \tau + \tau_\varepsilon + r^-_\varepsilon, & -\tau_\varepsilon - r^-_\varepsilon \le \tau \le -\tau_\varepsilon, \\ \varphi_0(\tau), & |\tau| \le \tau_\varepsilon, \\ \tau - \tau_\varepsilon + r^+_\varepsilon, & \tau_\varepsilon \le \tau \le \tau_\varepsilon + 1 - r^+_\varepsilon, \\ 1, & \tau > \tau_\varepsilon + 1 - r^+_\varepsilon. \end{cases}$$

Observe that $(1 - r^+_\varepsilon)$ and r^-_ε are $o(1)$ as $\varepsilon \to 0$. For $x = y + t\nu \in R^k_\varepsilon$ let us define $\varphi_\varepsilon(x) = \tilde\varphi_\varepsilon\left(\frac{t}{\varepsilon} - \tau_\varepsilon - r^-_\varepsilon\right)$, so that, as $\varepsilon \to 0$,

$$\begin{aligned} &\int_{R^k_\varepsilon\cap\Omega_\delta} \varepsilon|D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)\,dx \le \mathcal{H}^1(E_k\cap\Omega_\delta)\int_{-\tau_\varepsilon - r^-_\varepsilon}^{2\tau_\varepsilon - r^-_\varepsilon} |D\tilde\varphi_\varepsilon(\tau)|^2 + W(\tilde\varphi_\varepsilon(\tau))\,d\tau + o(1) \\ &\le \mathcal{H}^1(E_k\cap\Omega_\delta)\int_{-\tau_\varepsilon}^{\tau_\varepsilon} 2\varphi_0'(\tau)\sqrt{W(\varphi_0(\tau))}\,d\tau + o(1) \le c_0\mathcal{H}^1(E_k\cap\Omega_\delta) + o(1). \end{aligned}$$

Define, for $x \in \Omega_\delta \setminus B_\varepsilon$,

$$u_{\varepsilon,i}(x) = \begin{cases} u_i(x) + \varphi_\varepsilon(x) - 1 & \text{if } x \in (R^k_\varepsilon \setminus B_\varepsilon) \cap \Omega_\delta \text{ whenever } E_k \subset \lambda_i, \\ u_i(x) & \text{elsewhere on } \Omega_\delta \setminus B_\varepsilon, \end{cases}$$

and on $B_\varepsilon \cap \Omega_\delta$ define $u_{\varepsilon,i}$ to be a Lipschitz extension of $u_{\varepsilon,i}|_{\partial(B_\varepsilon\cap\Omega_\delta)}$ with the same Lipschitz constant, which is of order $1/\varepsilon$. Note that $u_{\varepsilon,i}$ has the same prescribed jump as u_i across γ_i, and thus $F^\Psi_\varepsilon(U_\varepsilon, \Omega_\delta) < \infty$. Moreover, $u_{\varepsilon,i} \to u_i$ in $L^1(\Omega_\delta)$.

Observe now that if E_k is contained in $\lambda_i \cap \lambda_j$, then by construction

$$e^i_\varepsilon(u_{\varepsilon,i}) = e^j_\varepsilon(u_{\varepsilon,j}) = \varepsilon|D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon)$$

on $\tilde{R}^k_\varepsilon = (R^k_\varepsilon \cap \Omega_\delta) \setminus B_\varepsilon$. Let $\varphi = (\varphi_1, \dots, \varphi_{N-1})$, with $\varphi_i \geq 0$ and $\Psi^*(\varphi) \leq 1$; we deduce

$$\begin{aligned}
\int_{\Omega_\delta} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx &\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx + \int_{B_\varepsilon \cap \Omega_\delta} \sum_i \varphi_i e^i_\varepsilon(u_{\varepsilon,i})\, dx \\
&\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \sum_i \varphi_i g^k_i \left(\varepsilon |D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right) dx + \int_{B_\varepsilon \cap \Omega_\delta} \Psi(\vec{e}_\varepsilon(U_\varepsilon))\, dx \\
&\leq \sum_{k=1}^{\ell} \int_{\tilde{R}^k_\varepsilon} \Psi(g^k) \left(\varepsilon |D\varphi_\varepsilon|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right) dx + C\varepsilon^{1/3} \\
&\leq \sum_{k=1}^{\ell} \Psi(g^k)(c_0 \mathcal{H}^1(E_k \cap \Omega_\delta) + o(1)) + C\varepsilon^{1/3} \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + o(1)
\end{aligned}$$

as $\varepsilon \to 0$. In view of (3.8) we have

$$F^\Psi_\varepsilon(U_\varepsilon, \Omega_\delta) \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + o(1),$$

and conclusion (3.10) follows.

Step 2. Let us now consider the case $\Lambda_L \equiv \Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, $L = \cup_i \lambda_i$, and the λ_i are not necessarily polyhedral. Let $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $DU^\perp = \Gamma - \Lambda_L$. We rely on Lemma 3.10 below to secure a sequence of acyclic polyhedral graphs $L_n = \cup_i \lambda^n_i$, λ^n_i transverse to γ_i, and s.t. the Hausdorff distance $d_H(\lambda^n_i, \lambda_i) < \frac{1}{n}$ for all $i = 1, \dots, N-1$, and $|\Lambda_{L_n}|_\Psi(\Omega_\delta) \leq |\Lambda_L|_\Psi(\Omega_\delta) + \frac{1}{n}$. Let $U^n \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $(DU^n)^\perp = \Gamma - \Lambda_{L_n}$. In particular, $U^n \to U$ in $[L^1(\Omega_\delta)]^{N-1}$, and by Step 1 we may construct a sequence U^n_ε s.t. $U^n_\varepsilon \to U^n$ in $[L^1(\Omega_\delta)]^{N-1}$ and

$$\begin{aligned}
\limsup_{\varepsilon \to 0} F^\Psi_\varepsilon(U^n_\varepsilon, \Omega_\delta) &\leq c_0 |(DU^n)^\perp - \Gamma|_\Psi(\Omega_\delta) = c_0 |\Lambda_{L_n}|_\Psi(\Omega_\delta) \\
&\leq c_0 |\Lambda_L|_\Psi(\Omega_\delta) + \frac{c_0}{n} = c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta) + \frac{c_0}{n}.
\end{aligned}$$

We deduce

$$\limsup_{n \to \infty} F^\Psi_{\varepsilon_n}(U^n_{\varepsilon_n}, \Omega_\delta) \leq c_0 |DU^\perp - \Gamma|_\Psi(\Omega_\delta)$$

for a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$. Conclusion (3.10) follows. ☐

LEMMA 3.10. *Let $L \in \mathcal{G}(A)$, $L = \cup_{i=1}^{N-1} \lambda_i$, be an acyclic graph connecting $P_1, \dots, P_N$. Then for any $\eta > 0$ there exists $L' \in \mathcal{G}(A)$, $L' = \cup_{i=1}^{N-1} \lambda'_i$, with λ'_i a simple polyhedral curve of finite length connecting P_i to P_N and transverse to γ_i, such that the Hausdorff distance $d_H(\lambda_i, \lambda'_i) < \eta$ and $|\Lambda_{L'}|_\Psi(\mathbb{R}^2) \leq |\Lambda_L|_\Psi(\mathbb{R}^2) + \eta$, where Λ_L and $\Lambda_{L'}$ are the canonical tensor valued representations of L and L'.*

Proof. Since $L \in \mathcal{G}(A)$, we can write $L = \cup_{m=1}^M \zeta_m$, with ζ_m simple Lipschitz curves such that, for $m_i \neq m_j$, $\zeta_{m_i} \cap \zeta_{m_j}$ is either empty or reduces to one common endpoint. Let $\Lambda_L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ be the rank one tensor valued measure canonically representing L, and let $d_m = \Psi(g(x))$ for $x \in \zeta_m$. The d_m are constants because by construction g is constant over each ζ_m. Consider now a polyhedral approximation $\tilde{\zeta}_m$ of ζ_m having its same endpoints, with $d_H(\tilde{\zeta}_m, \zeta_m) \leq \eta$, $\mathcal{H}^1(\tilde{\zeta}_m) \leq \mathcal{H}^1(\zeta_m) + \eta/(CM)$ (C to be fixed later) and, for $m_i \neq m_j$, $\tilde{\zeta}_{m_i} \cap \tilde{\zeta}_{m_j}$ is either empty or reduces to one common endpoint. Observe that whenever ζ_m intersects some γ_i, such a $\tilde{\zeta}_m$ can be constructed in order to intersect γ_i transversally in a finite number of points. Define

$L' = \cup_{m=1}^M \tilde{\zeta}_m$, and let $\Lambda_{L'} = \tau' \otimes g' \cdot \mathcal{H} \llcorner L'$ be its canonical tensor valued measure representation. Then, by construction $\Psi(g'(x)) = d_m$ for any $x \in \tilde{\zeta}_m$, hence

$$|\Lambda_{L'}|_\Psi(\mathbb{R}^2) = \sum_{m=1}^M d_m \mathcal{H}^1(\tilde{\zeta}_m) \leq \sum_{m=1}^M d_m \left(\mathcal{H}^1(\zeta_m) + \frac{\eta}{CM}\right) \leq |\Lambda_L|_\Psi(\mathbb{R}^2) + \eta,$$

provided $C = \max\{\Psi(g) : g \in \mathbb{R}^{N-1}, g_i \in \{0,1\} \text{ for all } i = 1, \dots, N-1\}$. Finally, we remark that $d_H(L, L') < \eta$ by construction. ∎

Thanks to the previous propositions we are now able to prove the following proposition.

PROPOSITION 3.11 (convergence of minimizers). *Let $\{U_\varepsilon\}_\varepsilon \subset H$ be a sequence of minimizers for F_ε^Ψ in H. Then (up to a subsequence) $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$, and $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ is a minimizer of $F^\Psi(U, \Omega_\delta) = c_0|DU^\perp - \Gamma|_\Psi(\Omega_\delta)$ in $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$.*

Proof. Let $V \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ such that $DV^\perp = \Gamma - \Lambda$, where Λ canonically represents an acyclic graph $L \in \mathcal{G}(A)$, and let $V_\varepsilon \in H$ such that $\limsup_{\varepsilon\to 0} F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)$. Since $F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta)$, by Proposition 3.7 there exists $U \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ s.t. $U_\varepsilon \to U$ in $[L^1(\Omega_\delta)]^{N-1}$, and by Proposition 3.8 we have

$$F^\Psi(U, \Omega_\delta) \leq \liminf_{\varepsilon\to 0} F_\varepsilon^\Psi(U_\varepsilon, \Omega_\delta) \leq \limsup_{\varepsilon\to 0} F_\varepsilon^\Psi(V_\varepsilon, \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)\,.$$

Given a general $V \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$, we can proceed like in Remark 2.9 and find V' such that $DV'^\perp = \Gamma - \Lambda_{L'}$ with L' acyclic, and $F^\Psi(V', \Omega_\delta) \leq F^\Psi(V, \Omega_\delta)$. The conclusion follows.

Let us focus on the case $\Psi = \Psi_\alpha$, where $\Psi_\alpha(g) = |g|_{1/\alpha}$ for $0 < \alpha \leq 1$ and $\Psi_0(g) = |g|_\infty$, and denote $F_\varepsilon^0 \equiv F_\varepsilon^{\Psi_0}$ and $F_\varepsilon^\alpha \equiv F_\varepsilon^{\Psi_\alpha}$. For $U = (u_1, \dots, u_{N-1}) \in H$ we have

$$F_\varepsilon^0(U, \Omega_\delta) = \int_{\Omega_\delta} \sup_i e_\varepsilon^i(u_i)\,dx, \qquad F_\varepsilon^\alpha(U, \Omega_\delta) = \int_{\Omega_\delta} \left(\sum_{i=1}^{N-1} e_\varepsilon^i(u_i)^{1/\alpha}\right)^\alpha dx, \tag{3.11}$$

and

$$F^0(U, \Omega_\delta) := c_0|DU^\perp - \Gamma|_{\Psi_0}(\Omega_\delta) \quad \text{and} \quad F^\alpha(U, \Omega_\delta) := c_0|DU^\perp - \Gamma|_{\Psi_\alpha}(\Omega_\delta), \tag{3.12}$$

which are the localized versions of (2.5) and (2.6).

THEOREM 3.12. *Let $\{P_1, \dots, P_N\} \subset \mathbb{R}^2$ such that $\max_i |P_i| = 1$, $0 < \delta \ll \max_{ij} |P_i - P_j|$, $\Omega = B_{10}(0)$, and $\Omega_\delta = \Omega \setminus (\cup_i B_\delta(P_i))$. For $0 \leq \alpha \leq 1$ and $0 < \varepsilon \ll \delta$, denote $F_\varepsilon^{\alpha,\delta} \equiv F_\varepsilon^\alpha(\cdot, \Omega_\delta)$ and $F^{\alpha,\delta} \equiv F^\alpha(\cdot, \Omega_\delta)$, with $F_\varepsilon^\alpha(\cdot, \Omega_\delta)$ (resp., $F^\alpha(\cdot, \Omega_\delta)$) defined in* (3.11) *(resp.,* (3.12)*).*

(i) *Let $\{U_\varepsilon^{\alpha,\delta}\}_\varepsilon$ be a sequence of minimizers for $F_\varepsilon^{\alpha,\delta}$ on H, with H defined in* (3.5)*. Then, up to subsequences, $U_\varepsilon^{\alpha,\delta} \to U^{\alpha,\delta}$ in $[L^1(\Omega_\delta)]^{N-1}$ as $\varepsilon \to 0$, with $U^{\alpha,\delta} \in SBV(\Omega_\delta; \mathbb{Z}^{N-1})$ a minimizer of $F^{\alpha,\delta}$ on $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$. Furthermore, $F_\varepsilon^{\alpha,\delta}(U_\varepsilon^{\alpha,\delta}) \to F^{\alpha,\delta}(U^{\alpha,\delta})$.*

(ii) *Let $\{U^{\alpha,\delta}\}_\delta$ be a sequence of minimizers for $F^{\alpha,\delta}$ on $SBV(\Omega_\delta; \mathbb{Z}^{N-1})$. Up to subsequences we have $U^{\alpha,\delta} \to U^\alpha|_{\Omega_\eta}$ in $[L^1(\Omega_\eta)]^{N-1}$ as $\delta \to 0$ for every fixed η sufficiently small, with $U^\alpha \in SBV(\Omega; \mathbb{Z}^{N-1})$ a minimizer of F^α on $SBV(\Omega; \mathbb{Z}^{N-1})$, and F^α defined in* (2.5), (2.6)*. Furthermore, $F^{\alpha,\delta}(U^{\alpha,\delta}) \to F^\alpha(U^\alpha)$.*

Proof. In view of Proposition 3.11 it remains to prove item (ii). The sequence $\{U^{\alpha,\delta}\}_\delta$ is equibounded in $BV(\Omega_\eta)$ uniformly in η, hence $U^{\alpha,\delta} \to U$ in $[L^1(\Omega_\eta)]^{N-1}$ for all $\eta > 0$ sufficiently small, with $U^\alpha \in SBV(\Omega;\mathbb{Z}^{N-1})$ and $F^{\alpha,\eta}(U^\alpha) \le \liminf_{\delta\to 0} F^{\alpha,\eta}(U^{\alpha,\delta})$ by lower semicontinuity of $F^{\alpha,\eta}$. On the other hand, let $\bar{U}^\alpha$ be a minimizer of F^α on $SBV(\Omega;\mathbb{Z}^{N-1})$. We have $F^{\alpha,\eta}(U^{\alpha,\delta}) \le F^{\alpha,\delta}(U^{\alpha,\delta})$ for any $\delta < \eta$, and by minimality, $F^{\alpha,\delta}(U^{\alpha,\delta}) \le F^{\alpha,\delta}(\bar{U}^\alpha) \le F^\alpha(\bar{U}^\alpha) \le F^\alpha(U^\alpha)$. This proves (ii). ◻

4. Convex relaxation. In this section, we propose convex positively 1-homogeneous relaxations of the irrigation type functionals $\mathcal{F}^\alpha$ for $0 \le \alpha < 1$ so as to include the (STP) corresponding to $\alpha = 0$ (notice that the case $\alpha = 1$ corresponds to the well-known Monge–Kantorovich optimal transportation problem w.r.t. the Monge cost $c(x,y) = |x-y|$).

More precisely, we consider relaxations of the functional defined by

$$\mathcal{F}^\alpha(\Lambda) = \|\Lambda\|_{\Psi_\alpha} = \int_{\mathbb{R}^d} |g|_{1/\alpha}\, d\mathcal{H}^1 \llcorner L$$

if Λ is the canonical representation of an acyclic graph L with terminal points $\{P_1,\ldots,P_N\} \subset \mathbb{R}^d$, so that, in particular, according to Definition 2.1, we can write $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ with $|\tau| = 1$, $g_i \in \{0,1\}$. For any other $d \times (N-1)$-matrix valued measure Λ on $\mathbb{R}^d$ we set $\mathcal{F}^\alpha(\Lambda) = +\infty$.

As a preliminary remark we observe that, since we are looking for positively 1-homogeneous extensions, any candidate extension $\mathcal{R}^\alpha$ satisfies

$$\mathcal{R}^\alpha(c\Lambda) = |c|\mathcal{F}^\alpha(\Lambda)$$

for any $c \in \mathbb{R}$ and Λ of the form $\tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ as above. As a consequence we have that $\mathcal{R}^\alpha(-\Lambda) = \mathcal{R}^\alpha(\Lambda)$, where $-\Lambda$ represents the same graph L as Λ but only with reversed orientation.

4.1. Extension to rank one tensor measures. First of all let us discuss the possible positively 1-homogeneous convex relaxations of $\mathcal{F}^\alpha$ on the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\Lambda|_1$, where $|\tau| = 1$, $g \in \mathbb{R}^{N-1}$ (cf. section 2.1). For a generic rank one tensor valued measure $\Lambda = \tau \otimes g \cdot |\Lambda|_1$, we can consider extensions of the form

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Psi^\alpha(g)\, d|\Lambda|_1$$

for a convex positively 1-homogeneous Ψ^α on $\mathbb{R}^{N-1}$ (i.e., a norm) verifying

$$\begin{aligned} &\Psi^\alpha(g) = |g|_{1/\alpha} && \text{if } g_i \in \{0,1\} \text{ for all } i = 1,\ldots,N-1,\\ &\Psi^\alpha(g) \ge |g|_{1/\alpha} && \text{for all } g \in \mathbb{R}^{N-1}. \end{aligned} \tag{4.1}$$

One possible choice is represented by $\Psi^\alpha(g) = |g|_{1/\alpha}$ for all $g \in \mathbb{R}^{N-1}$, while sharper relaxations are given for $\alpha > 0$ by

$$\Psi^\alpha_*(g) = \left(\sum_{1\le i\le N-1} |g_i^+|^{1/\alpha}\right)^\alpha + \left(\sum_{1\le i\le N-1} |g_i^-|^{1/\alpha}\right)^\alpha, \tag{4.2}$$

and for $\alpha = 0$ by

$$\Psi^0_*(g) = \sup_{1\le i\le N-1} g_i^+ - \inf_{1\le i\le N-1} g_i^-, \tag{4.3}$$

with $g_i^+ = \max\{g_i, 0\}$ and $g_i^- = \min\{g_i, 0\}$. In particular, Ψ_*^α represents the maximal choice within the class of extensions Ψ^α satisfying

$$\Psi^\alpha(g) = |g|_{1/\alpha} \quad \text{if } g_i \geq 0 \text{ for all } i = 1, \dots, N-1.$$

Indeed, for $\alpha > 0$, $g \in \mathbb{R}^{N-1}$, and $g^\pm = (g_1^\pm, \dots, g_{N-1}^\pm)$, we have

$$\begin{aligned}\Psi^\alpha(g) &\leq \Psi^\alpha(g^+ + g^-) = 2\Psi^\alpha\left(\frac{1}{2}g^+ + \frac{1}{2}g^-\right) \leq 2\left(\frac{1}{2}\Psi^\alpha(g^+) + \frac{1}{2}\Psi^\alpha(g^-)\right) \\ &= \Psi^\alpha(g^+) + \Psi^\alpha(g^-) = |g^+|_{1/\alpha} + |g^-|_{1/\alpha} = \Psi_*^\alpha(g).\end{aligned}$$

The interest in optimal extensions Ψ^α on rank one tensor valued measures relies on the so-called calibration method as a minimality criterion for Ψ^α-mass functionals, as it is done, in particular, in [24] for (STP) using the (optimal) norm Ψ_*^0.

According to the convex extensions Ψ^α and Ψ^0 considered, when it comes to finding minimizers of, respectively, $\mathcal{R}^\alpha$ and $\mathcal{R}^0$ in suitable classes of weighted graphs with prescribed fluxes at their terminal points, or more generally in the class of rank one tensor valued measures having divergence prescribed by (2.2), the minimizer is not necessarily the canonical representation of an acyclic graph. Let us consider the following example, where the minimizer contains a cycle.

Example 4.1. Consider the (STP) for $\{P_1, P_2, P_3\} \subset \mathbb{R}^2$. We claim that a minimizer of $\mathcal{R}^0(\Lambda) = \int_{\mathbb{R}^2} |g|_\infty \, d|\Lambda|_1$ within the class of rank one tensor valued Radon measures $\Lambda = \tau \otimes g \cdot |\Lambda|_1$ satisfying (2.2) is supported on the triangle $L = [P_1, P_2] \cup [P_2, P_3] \cup [P_1, P_3]$, hence its support is not acyclic and such a minimizer is not related to any optimal Steiner tree. Denoting τ the global orientation of L (i.e., from P_1 to P_2, P_1 to P_3, and P_2 to P_3) we actually have as minimizer

(4.4)
$$\Lambda = \tau \otimes \left(\left[\frac{1}{2}, -\frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_1, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_2] + \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathcal{H}^1 \llcorner [P_3, P_1]\right).$$

The proof of the claim follows from Remark 4.2 and Lemma 4.3.

Remark 4.2 (calibrations). A way to prove the minimality of $\Lambda = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$ within the class of rank one tensor valued Radon measures satisfying (2.2) is to exhibit a *calibration* for Λ, i.e., a matrix valued differential form $\omega = (\omega_1, \dots, \omega_{N-1})$, with $\omega_j = \sum_{i=1}^d \omega_{ij} dx_i$ for measurable coefficients ω_{ij}, such that

- $d\omega_j = 0$ for all $j = 1, \dots, N-1$;
- $\|\omega\|_* \leq 1$, where $\|\cdot\|_*$ is the dual norm to $\|\tau \otimes g\| = |\tau| \cdot |g|_\infty$, defined as

$$\|\omega\|_* = \sup\{\tau^t \, \omega \, g \ : \ |\tau| = 1, \ |g|_\infty \leq 1\};$$

- $\langle \omega, \Lambda \rangle = \sum_{i,j} \tau_i \omega_{ij} g_j = |g|_\infty$ pointwise, so that

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda).$$

In this way for any competitor $\Sigma = \tau' \otimes g' \cdot |\Sigma|_1$ we have $\langle \omega, \Sigma \rangle \leq |g'|_\infty$, and moreover, $\Sigma - \Lambda = DU^\perp$ for $U \in BV(\mathbb{R}^2; \mathbb{R}^{N-1})$, hence

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda - \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, DU^\perp \rangle = \int_{\mathbb{R}^2} \langle d\omega, U \rangle = 0\,.$$

It follows that

$$\mathcal{R}^0(\Sigma) \geq \int_{\mathbb{R}^2} \langle \omega, \Sigma \rangle = \int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda)\,,$$

i.e., Λ is a minimizer within the given class of competitors.

Let us construct a calibration $\omega = (\omega_1, \omega_2)$ for Λ in the general case $P_1 \equiv (x_1, 0)$, $P_2 \equiv (x_2, 0)$, and $P_3 \equiv (0, x_3)$, with $x_1 < 0$, $x_1 < x_2$, and $x_3 > 0$.

LEMMA 4.3. *Let P_1, P_2, P_3 be defined as above, and let Λ be as in* (4.4). *Consider $\omega = (\omega_1, \omega_2)$ defined as*

$$\begin{aligned} \omega_1 &= \frac{1}{2a}[(x_1+a)dx + x_3dy], \quad \omega_2 = \frac{1}{2a}[(x_1-a)dx + x_3dy], \quad \textit{for } (x,y) \in B_L \\ \omega_1 &= \frac{1}{2b}[(x_2+b)dx + x_3dy], \quad \omega_2 = \frac{1}{2b}[(x_2-b)dx + x_3dy], \quad \textit{for } (x,y) \in B_R \end{aligned}$$

with B_L the left half-plane w.r.t. the line containing the bisector of vertex P_3, B_R the corresponding right half-plane, and $a = \sqrt{x_1^2 + x_3^2}$, $b = \sqrt{x_2^2 + x_3^2}$. The matrix valued differential form ω is a calibration for Λ.

Proof. For simplicity we consider here the particular cases $x_1 = -\frac{1}{2}$, $x_2 = \frac{1}{2}$, and $x_3 = \frac{\sqrt{3}}{2}$ (the general case is similar). For this choice of x_1, x_2, x_3 we have

$$\begin{aligned} \omega_1 &= \frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x,y) \in \mathbb{R}^2,\ x < 0, \\ \omega_1 &= \frac{3}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \omega_2 = -\frac{1}{4}dx + \frac{\sqrt{3}}{4}dy, \quad \text{for } (x,y) \in \mathbb{R}^2,\ x > 0. \end{aligned}$$

The piecewise constant 1-forms ω_i for $i = 1, 2$ are globally closed in $\mathbb{R}^2$ (on the line $\{x = 0\}$ they have continuous tangential component), $\|\omega\|_* \leq 1$ (cf. Remark 4.2), and taking their scalar product with, respectively, $(1, 0) \otimes (1/2, -1/2)$, $(-1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x < 0$, and $(1/2, \sqrt{3}/2) \otimes (1/2, 1/2)$ for $x > 0$ we obtain in all cases $1/2$, i.e., $|g|_\infty$, so that

$$\int_{\mathbb{R}^2} \langle \omega, \Lambda \rangle = \mathcal{R}^0(\Lambda)\,.$$

Hence ω is a calibration for Λ. ∎

Remark 4.4. A calibration always exists for minimizers in the class of rank one tensor valued measures as a consequence of the Hahn–Banach theorem (see, e.g., [24]), while it may be not the case, in general, for graphs with integer or real weights. The classical minimal configuration for (STP) with three endpoints P_1, P_2, and P_3 admits a calibration w.r.t. the norm Ψ_*^0 in $\mathbb{R}^{N-1}$ (see [24]), and hence it is a minimizer for the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Psi_*^0}$ among all real weighted graphs (and all rank one tensor valued Radon measures satisfying (2.2)). It is an open problem to show whether or not a minimizer of the relaxed functional $\mathcal{R}^0(\Lambda) = ||\Lambda||_{\Psi_*^0}$ has integer weights.

4.2. Extension to general matrix valued measures. Let us turn next to the convex relaxation of $\mathcal{F}^\alpha$ for generic $d \times (N-1)$-matrix valued measures $\Lambda = (\Lambda_1, \dots, \Lambda_{N-1})$, where Λ_i, for $1 \leq i \leq N-1$, are the vector measures corresponding to the columns of Λ. As a first step observe that, due to the positively 1-homogeneous request on $\mathcal{R}^\alpha$, whenever $\Lambda = p \cdot \mathcal{H}^1 \llcorner L = \tau \otimes g \cdot \mathcal{H}^1 \llcorner L$, with $|\tau|$ =cte. and $g_i \in \{0, 1\}$, we must have

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} |\tau| |g|_{1/\alpha}\, d\mathcal{H}^1 \llcorner L = \int_{\mathbb{R}^d} \Phi_\alpha(p)\, d\mathcal{H}^1 \llcorner L,$$

with $\Phi_\alpha(p) = |\tau||g|_{1/\alpha}$ defined only for matrices $p \in K_0$ ($+\infty$ otherwise), where

$$K_0 = \{\tau \otimes g \in \mathbb{R}^{d\times(N-1)},\ g_i \in \{0,1\},\quad |\tau| = \text{cte.}\}.$$

Following [18], we look for Φ_α^{**}, the positively 1-homogeneous convex envelope on $\mathbb{R}^{d\times(N-1)}$ of Φ_α. Setting $q = (q_1, \dots, q_{N-1})$, with $q_i \in \mathbb{R}^d$ its columns, we have that the convex conjugate function $\Phi_\alpha^*(q) = \sup\{q\cdot p - \Phi_\alpha(p),\ p \in K_0\}$ is given by

$$\begin{aligned}\Phi_\alpha^*(q) &= \sup\left\{ \tau^t \cdot q \cdot g - |\tau|\cdot|g|_{1/\alpha}\,, \quad |\tau| = \text{cte.}, g = \sum_{i\in J} e_i,\ J \subset \{1,\dots,N-1\}\right\}\\ &= \sup\left\{ c\left[\tau^t\cdot\left(\sum_{j\in J} q_j\right) - |J|^\alpha\right], \quad c\geq 0,\ |\tau| = 1,\ J\subset\{1,\dots,N-1\}\right\}.\end{aligned}$$

Hence Φ_α^* is the indicator function of the convex set

$$K^\alpha = \left\{ q\in\mathbb{R}^{d\times(N-1)},\ \left|\sum_{j\in J} q_j\right| \leq |J|^\alpha \ \text{ for all } J\subset\{1,\dots,N-1\}\ \right\},$$

and in particular, for $\alpha = 0$, it holds (cf. [18]) that

$$K^0 = \left\{ q\in\mathbb{R}^{d\times(N-1)},\ \left|\sum_{j\in J} q_j\right| \leq 1 \ \text{ for all } J\subset\{1,\dots,N-1\}\ \right\}.$$

It follows that Φ_α^{**} is the support function of K^α, i.e., for $p\in\mathbb{R}^{d\times(N-1)}$,

$$\Phi_\alpha^{**}(p) = \sup_{q\in K^\alpha} p\cdot q = \sup\left\{ p\cdot q\,,\ \left|\sum_{j\in J} q_j\right| \leq |J|^\alpha\,,\ J\subset\{1,\dots,N-1\}\ \right\}. \tag{4.5}$$

We are then led to consider, for matrix valued test functions $\varphi = (\varphi_1,\dots,\varphi_{N-1})$, the relaxed functional

$$\mathcal{R}^\alpha(\Lambda) = \int_{\mathbb{R}^d} \Phi_\alpha^{**}(\Lambda) = \sup\left\{ \sum_{i=1}^{N-1}\int_{\mathbb{R}^d} \varphi_i\, d\Lambda_i, \quad \varphi\in C_c^\infty(\mathbb{R}^d; K^\alpha)\right\}.$$

Observe that for Λ a rank one tensor valued measure and $\alpha = 0$ the above expression coincides with the one obtained in the previous section choosing $\Psi^0 = \Psi^0_*$.

In the planar case $d = 2$, consider a $2\times(N-1)$-matrix valued measure $\Lambda = (\Lambda_1,\dots,\Lambda_{N-1})$ such that $\operatorname{div}\Lambda_i = \delta_{P_i} - \delta_{P_N}$. Fix a measure Γ as, for instance, in Remark 2.6. We have $\operatorname{div}(\Lambda - \Gamma) = 0$ in $\mathbb{R}^2$ and by Poincaré's lemma there exists $U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1})$ such that $\Lambda = \Gamma - DU^\perp$. So the relaxed functional reads

$$\mathcal{E}^\alpha(U) = \mathcal{R}^\alpha(\Lambda) \quad \text{for } \Lambda = \Gamma - DU^\perp,\ U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1}). \tag{4.6}$$

The relaxed irrigation problem $(I^\alpha) \equiv \min_{BV}\mathcal{E}^\alpha(U)$ can thus be described in the following equivalent way, according to (4.5): let $q = \varphi$ be any matrix valued test function (with columns $q_i = \varphi_i$ for $1\leq i\leq N-1$); then we have

$$(I^\alpha) \equiv \min_{U\in BV(\mathbb{R}^2;\mathbb{R}^{N-1})} \sup\left\{ \int_{\mathbb{R}^2}\sum_{i=1}^{N-1} (Du_i^\perp - \Gamma_i)\cdot\varphi_i\,, \quad \varphi\in C_c^\infty(\mathbb{R}^2;K^\alpha)\ \right\}.$$

Notice that w.r.t. the similar formulation proposed in [18], here is the presence of an additional "drift" term; moreover, the constraints set K^α is somewhat different.

We now compare the functional $\mathcal{E}^\alpha(U)$ with the actual convex envelope $(F^\alpha)^{**}(U)$ in the space $BV(\mathbb{R}^2;\mathbb{R}^{N-1})$, where we set $F^\alpha(U) = |DU^\perp - \Gamma|_{\ell^{1/\alpha}}(\mathbb{R}^2)$ if $\Gamma - DU^\perp = \Lambda$ canonically represents an acyclic graph, and $F^\alpha(U) = +\infty$ elsewhere in $BV(\mathbb{R}^2;\mathbb{R}^{N-1})$. In the spirit of [18, Proposition 3.1], we have the following.

LEMMA 4.5. *We have* $\mathcal{E}^\alpha(U) \leq (F^\alpha)^{**}(U) \leq (N-1)^{1-\alpha}\mathcal{E}^\alpha(U)$ *for any* $U \in BV(\mathbb{R}^2;\mathbb{R}^{N-1})$ *and any* $0 \leq \alpha < 1$.

Proof. Observe that $\mathcal{E}^\alpha(U) \leq (\mathcal{F}^\alpha)^{**}(U)$ by convexity of $\mathcal{E}^\alpha(U)$. Moreover, whenever $\Lambda = \Gamma - DU^\perp$ canonically represents a graph connecting $P_1, \ldots, P_N$, we have $(F^\alpha)^{**}(U) \leq (F^1)^{**}(U)$ since $F^\alpha(U) \leq F^1(U)$. For $\alpha > 0$, denoting $\Lambda = \Gamma - DU^\perp$, we deduce

$$(F^1)^{**}(U) \leq \sum_{i=1}^{N-1} |\Lambda_i|(\mathbb{R}^d) \leq (N-1)^{1-\alpha} \left(\sum_{i=1}^{N-1} |\Lambda_i|^{1/\alpha} \right)^\alpha (\mathbb{R}^d) \leq (N-1)^{1-\alpha}\mathcal{E}^\alpha(U),$$

and analogously we have $(F^1)^{**}(U) \leq (N-1)\mathcal{E}^0(U)$. ∎

5. Numerical identification of optimal structures.

5.1. Local optimization by Γ-convergence. In this section, we plan to illustrate the use of Theorem 3.12 to identify numerically local minima of the Steiner problem. We base our numerical approximation on a standard discretization of (3.11). Let $\Omega = (0,1)^2$ and assume $\{P_1, \ldots, P_N\} \subset \Omega$; thus, as a standard consequence, the associated Steiner tree is also contained in Ω. Consider a Cartesian grid covering Ω of step size $h = \frac{1}{S}$, where $S > 1$ is a fixed integer. Dividing every square cell of the grid into two triangles, we define a triangular mesh $\mathcal{T}$ associated to Ω and replace each point P_i with the closest grid point.

Fix now Γ_i an oriented vectorial measure absolutely continuous w.r.t. $\mathcal{H}^1$ as in Remark 2.6. Assume for simplicity that Γ_i is supported on γ_i a union of vertical and horizontal segments contained in Ω and covered by the grid associated to the discrete points $\{(kh, lh),\ 0 \leq k, l < S\}$. Notice that such a measure can be easily constructed by considering, for instance, the oriented ℓ^1-spanning tree of the given points.

To mimic the construction in section 3.2, we define the function space

$$H_i^h \equiv P_1(\mathcal{T}, \Omega \setminus \gamma_i) \cap BV(\Omega)$$

to be the set of functions which are globally continuous on $\Omega \setminus \gamma_i$ and piecewise linear on every triangle of $\mathcal{T}$. Moreover, we require that every function of H_i^h has a jump through γ_i of amplitude -1 in the orthogonal direction of the orientation of Γ_i. Observe that H_i^h is a finite-dimensional space of dimension S^2: one element u_i^h can be described by $S^2 + n_i$ parameters and n_i linear constraints describing the jump condition where n_i is the number of grid points covered by γ_i.

Then, we define

$$f_h^i(u_i^h) = h|Du_i^h|^2 + \frac{1}{h}W(u_i^h) \tag{5.1}$$

if $u \in L^1(\Omega)$ is in H_i^h and extend f_h^i by letting $f_h^i(u) = +\infty$ otherwise. Notice that these discrete energy densities do not contain the drift terms Γ_i because the information about the drift has been encoded within the discrete spaces H_i^h, leaving us to deal only with the absolutely continuous part of the gradient (see Remark 3.1). Then, for $U^h = (u_1^h, \ldots, u_{N-1}^h) \in H_1^h \times \cdots \times H_{N-1}^h$ we define

$$G_h^0(U^h) = \int_\Omega \sup_{1 \leq i \leq N-1} f_h^i(u_i^h) \quad \text{and} \quad G_h^\alpha(U^h) = \int_\Omega \left(\sum_{i=1}^{N-1} f_h^i(u_i^h)^{1/\alpha} \right)^\alpha .$$

By a similar strategy we used to prove Theorem 3.12, we still also have convergence of minimizers of G_h^0 (resp., G_h^α) to minimizers of $c_0 F^0$ (resp., $c_0 F^\alpha$) w.r.t. the strong topology of $L^1(\mathbb{R}^2; \mathbb{R}^{N-1})$. Observe that an exact evaluation of the integrals involved in (5.1) is required to obtain this convergence result (an approximation formula can also be used but then a theoretical proof of convergence would require one to study the interaction of the order of approximation with the convergence of minimizers). We point out that this constraint is not critical from a computational point of view since every function u_h^i of finite energy has a constant gradient on every triangle of the mesh. On the other hand, the potential integral can be evaluated formally to obtain an exact estimate of this term w.r.t. the degrees of freedom which describe a function of H_i^h.

Based on these results we performed two different numerical experiments. We first approximated the optimal Steiner trees associated to the vertices of a triangle, a regular pentagon, and a regular hexagon with its center. To obtain the results of Figure 3 we discretized the problem on a grid of size 200×200. In the case of the triangle we used the associated spanning tree to define the measures $(\Gamma_i)_{i=1,2}$. In the case of the pentagon and of the hexagon we used the rectilinear Euclidean Steiner trees computed by the Geosteiner's library (see [36], for instance) to initiate the vectorial measures. We refer the reader to Figure 2 for an illustration of both singular vector fields. We solved the resulting finite-dimensional problem using an interior point solver. Notice that in order to deal with the nonsmooth cost function G_h^0 we had to introduce standard gap variables to get a smooth nonconvex constrained optimization problem. Using [17], we have been able to recover the locally optimal solutions depicted in Figure 3 in less than five minutes on a standard computer. Whereas the results obtained for the triangle and the pentagon describe globally optimal Steiner trees, the result obtained for the hexagon and its center is only a local minimizer.

In a second experiment we focus on simple irrigation problems to illustrate the versatility of our approach. We applied exactly the same approach to the pentagon setting minimizing the functional G_h^α. We illustrate our results in Figure 4 in which we recover the solutions of Gilbert–Steiner problems for different values of α. Observe that for small values of α, as expected by Proposition 2.4, we recover an irrigation network close to an optimal Steiner tree.

5.2. Convex relaxation and multiple solutions. The convex relaxation of Steiner problem (I^0) obtained following [18] reads in our discrete setting as

$$\min_{(u_i^h)_{1 \leq i < N}} \sup_{(\varphi_i^h)_{1 \leq i < N} \in K^0} \frac{h^2}{2} \sum_{t \in \mathcal{T}} \sum_{i=1}^{N-1} (\nabla u_i^h)_t \cdot (\varphi_i^h)_t, \tag{5.2}$$

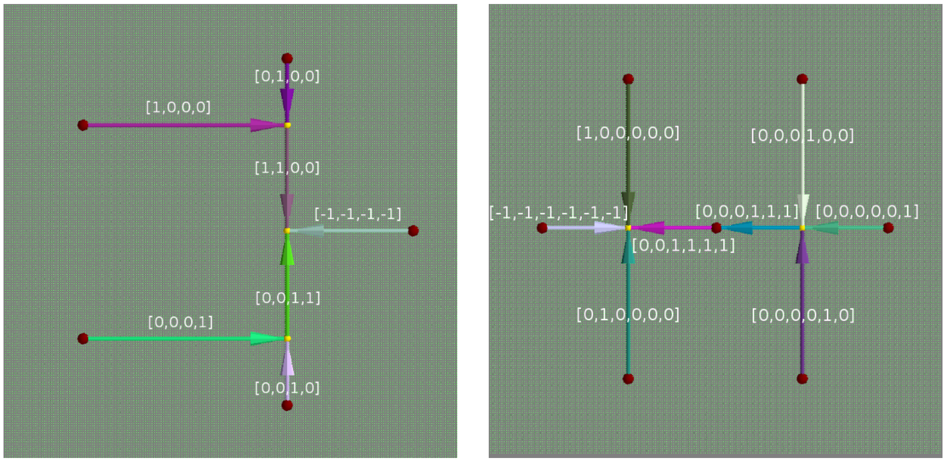

FIG. 2. *Rectilinear Steiner trees and associated vectorial drifts for five and seven points.*

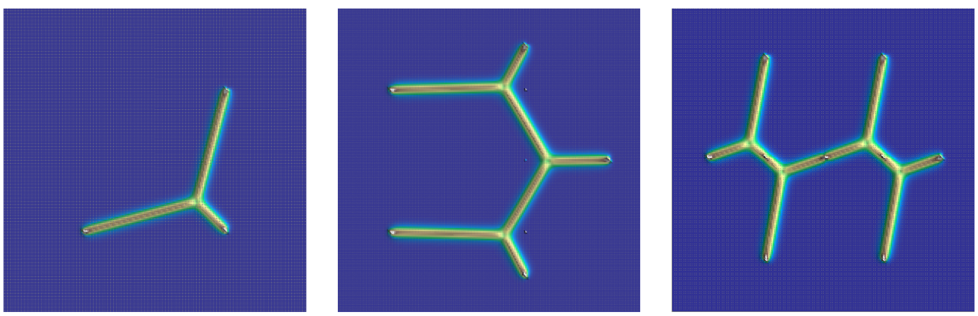

FIG. 3. *Local minimizers obtained by the* Γ*-convergence approach applied to three, five and seven points.*

where

$$K^0 = \Big\{ (\varphi_i^h)_{1\le i<N} \in (\mathbb{R}^{2\mathcal{T}})^{N-1} \,|\, \text{ for all } J \subset \{1,\dots,N-1\}, \text{ for all } t \in \mathcal{T}, \Big| \sum_{j\in J} (\varphi_j^h)_t \Big| \le 1 \Big\} \tag{5.3}$$

and for all $1 \le i < N$, $u_i^h \in H_i^h$. Applying conic duality (see, for instance, Lecture 2 of [9]), we obtain that the optimal vector (u_i^h) solves the following minimization problem:

$$\min_{(u_i^h)_{1\le i<N}\in L,\, (\psi_J^h)_{J\subset\{1,\dots,N-1\}}\in(\mathbb{R}^{2\mathcal{T}})^{2^{N-1}}} \frac{h^2}{2} \sum_{t\in\mathcal{T}} \sum_{J\subset\{1,\dots,N-1\}} |(\psi_J^h)_t|, \tag{5.4}$$

where L is the set of discrete vectors $(u_i^h)_{1\le i<N}$ which satisfy for all $i = 1,\dots,N-1$, for all $t \in \mathcal{T}$,

$$(\nabla u_i^h)_t = \sum_{J\subset\{1,\dots,N-1\},\, i\in J} (\psi_J^h)_t. \tag{5.5}$$

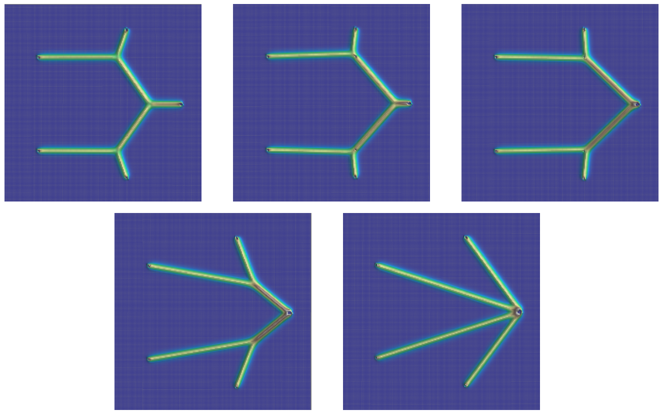

FIG. 4. *Gilbert–Steiner solutions associated to parameters* $\alpha = 0.2$, 0.4, 0.6, 0.8 *and* 1 *(from left to right)*

We solved this convex linearly constrained minimization problem using the conic solver of the library Mosek [28] on a grid of dimension 300×300. Observe that this convex formulation is also well adapted to the, now standard, large scale algorithms of proximal type. We studied four different test cases: the vertices of an equilateral triangle, a square, a pentagon, and finally a hexagon and its center as in previous sections. As illustrated in the left picture of Figure 5, the convex formulation is able to approximate the optimal structure in the case of the triangle. Due to the symmetries of the problems, the three last examples do not have unique solutions. Thus, the result of the optimization is expected to be a convex combination of all solutions whenever the relaxation is sharp, as it can be observed in the second and fourth cases of Figure 5. Notice that we do not expect this behavior to hold for any configuration of points. Indeed, the numerical solution in the third picture of Figure 5 is not supported on a convex combination of global solutions since the density in the middle point is not 0. Whereas the local Γ-convergence approach of previous sections was only able to produce a local minimum in the case of the hexagon and its center, the convexified formulation gives a relatively precise idea of the set of optimal configurations (see the last picture of Figure 5 where we can recognize within the figure the two global solutions).

6. Generalizations. In this article we have focused on the optimization of 1-dimensional structures in the plane in specific, classical cases. A first possible generalization is to consider the same problems w.r.t. more general norms, for instance, anisotropic ones: given $|\cdot|_a$ an anisotropic norm on $\mathbb{R}^d$ and a norm Ψ^α on $\mathbb{R}^{N-1}$ as in section 4.1, one could consider, for a matrix valued measure $\Lambda \in \mathcal{M}(\mathbb{R}^d, \mathbb{R}^{d\times N-1})$, the (Ψ^α, a)-mass measures

$$(6.1)\quad |\Lambda|_{\Psi^\alpha,a}(B) := \sup_{\substack{\omega\in C_c^\infty(B;\mathbb{R}^d)\\ h\in C_c^\infty(B;\mathbb{R}^{N-1})}} \left\{\langle \Lambda, \omega\otimes h\rangle\,,\quad |\omega(x)|_{a^*}\le 1\,,\ (\Psi^\alpha)^*(h(x))\le 1\right\},$$

for $B\subset\mathbb{R}^d$ open, and the corresponding (Ψ^α, a)-mass norm $||\Lambda||_{\Psi^\alpha,a} = |\Lambda|_{\Psi^\alpha,a}(\mathbb{R}^d)$. Then minimizers of $\mathcal{F}_a^\alpha = ||\cdot||_{\Psi^\alpha,a}$ over rank one tensor valued measures representing graphs $L\in\mathcal{G}(A)$ will solve the anisotropic irrigation problem (resp., the anisotropic (STP) in case $\alpha = 0$), in particular, if $|\cdot|_a = |\cdot|_1$, $\mathcal{F}_a^0$ is related to the rectilinear

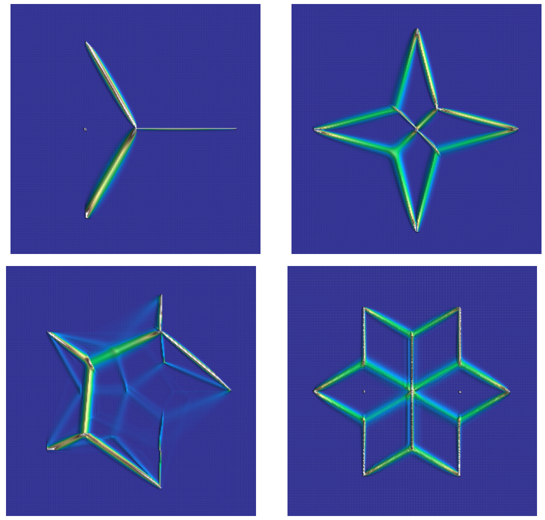

FIG. 5. *Results obtained by convex relaxation for three, four, five, and seven given points*

(STP) in $\mathbb{R}^d$. For $d = 2$, following [14, 30, 2] one may reproduce the Γ-convergence and convex relaxation approach developed here to numerically study the anisotropic problem (6.1). A further step in this direction would consist of considering size or α-mass minimization problems in suitable homology and/or oriented cobordism classes for 1-dimensional chains in manifolds endowed with a Finsler metric.

Another generalization concerns the convex relaxation and the variational approximation of (STP) and (I_α) in the higher dimensional case $d \geq 3$. This is done in the companion paper [11], where we obtain a Γ-convergence result by using functionals of Ginzburg–Landau-type in the spirit of [1] and [34]. Moreover, as in the present paper, we introduce appropriate "local" convex envelopes, discuss calibration principles, and show some numerical simulations.

In parallel to previous theoretical generalizations, we are currently developing numerical approaches adapted to these new formulations. On the one hand, we are studying a large scale approach to solve problems analogous to the conic convexified formulation of section 5.2. Such an extension is definitely required to approximate realistic problems in dimension three and higher. On the other hand, we want to focus on refinement techniques which may decrease dramatically the number of degrees of freedom involved in the optimization process. Observe, for instance, that very few parameters are required to describe exactly a drift such as the ones given in Figure 2. Based on such observations, a sequential localized approach may provide a very precise description of, at least locally, optimal structures.

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