

Feedback optimal controllers for the Heston model

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Abstract

We prove the existence of an optimal feedback controller for a stochastic optimization problem constituted by a variation of the Heston model, where a stochastic input process is added in order to minimize a given performance criterion. The stochastic feedback controller is found by solving a nonlinear backward parabolic equation for which one proves the existence and uniqueness of a martingale solution.

1 Introduction

Stochastic volatility models (SVMs) are widely used within a plethora of financial settings, spanning from the risk sector, to the interest rates one, from econometric problems, to insurance ones, see, e.g., [7, 8, 10, 13, 17, 23], and references therein. In fact, SVMs allow for a finer analysis of relevant time series as they appear in the real-world financial arenas. Indeed, daily return data series show two peculiarities among different types of assets in different markets and in different periods, namely the *volatility clustering phenomenon* and the *fat-tailed and highly peaked distribution* relative to the normal distribution, as assumed, e.g., in the celebrated Black and Scholes model.

Volatility clustering refers to the fact that *large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes* (see [19]). The latter means that, over a significant time window, it can be noted the presence of both high volatility period and low volatility one, separately, rather than a constant average level of volatility persisting over time.

Such peculiarities can be captured by SVMs because they are characterized by a volatility term which is itself a stochastic process. As an example, consider the following price dynamic for given asset S , driven by a Geometric Brownian motion

$$dS_t = \mu_t S_t dt + \sqrt{\nu_t} S_t dW_t \quad (1.1)$$

where

$$d\nu_t = k_t dt + \sigma_t dB_t \quad (1.2)$$

and where W and B denote two Brownian motions with correlation ρ . In this case, the volatility of the price is no longer a deterministic function of S , but it is itself randomly distributed. Different choices for (1.2) allow for a multitude of models that properly represent different financial data, see, e.g, [4, 6, 15, 21, 22]. In this context we focus our attention on the Heston model, firstly introduced in [14] to price European bond and currency options, aiming at generalize and overcome the biases of the Black ad Scholes model. We recall that the Heston model assumes that the asset price follows the dynamic in (1.1), where ν follows the Ornstein-Uhlenbeck process

$$d\nu_t = k(\theta - \nu_t) dt + \sigma\sqrt{\nu_t} dB_t ,$$

explaining the volatility smile. In what follows we focus our attention on a controlled version of the Heston model, namely adding a control component w.r.t. to the volatility term appearing in (1.1). This approach is mainly intended to take in account exogenous influence of external financial actors within a given investment setting. This is the case of the possible action of a Central Bank which aims at minimizing the probability of abrupt changes within markets where a relevant group of banks are exposed. In this scenario, the role played by the Central Bank can be realized putting money in the market by means of actions. Recently, such type of wide breath action has been concretized by the European Central Bank which has adopted the so called quantitative easing monetary policy. In this case, the volatility control has been implemented buying a predetermined amount of financial assets emitted mainly from (national) commercial banks . This results in: a rise of the prices of the interested financial assets, and a raise of interested assets prices lowering their yield and simultaneously increasing the money supply, with the final result of a drastic reduction of the volatility terms due to the instability of some European regions, in general, and concerning particular, highly exposed, banks of well determined states.

Analogously one can see the aforementioned control issue from the point of view of maximizing the expected discounted utility of consumption and terminal wealth, as, e.g., made in [5] w.r.t. a fixed finite investment horizon. On the other hand, as pointed out in [18], solving the portfolio problem for Heston's stochastic

volatility model may lead that partial equilibrium can be obtained only under a specific condition on the model parameters, particularly w.r.t. the specification of the market price of risk, which means that the market price of risk has to be specified with great care.

The main aim of this paper is to study an optimization problem where the performance function we want to minimize depends on a stochastic process whose dynamic is modeled through a *controlled* variation of the Heston model. Namely, we assume that a controller is added in the volatility component in eq. (1.1). The present work is structured as follows: in Section 2 we give the necessary details to introduce the problem, also providing the well-posedness of this stochastic model; in Section 3 we introduce the associated Hamilton-Jacobi equation which is reduced to a nonlinear parabolic equation on $(0, \infty) \times (0, \infty)$. The main result of this paper is the existence and uniqueness of solutions for this equation. In Section 4 we derive the existence of an optimal controller in feedback form.

2 The Heston control model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and W_1, W_2 be Brownian motions. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by $W := (W_1, W_2)$. Consider the controlled stochastic system

$$\begin{cases} dX_1 = \mu X_1 dt + X_1 \sqrt{u X_2} dW_1, & t \in (0, T) \\ dX_2 = k(\theta - X_2) dt + \sigma \sqrt{X_2} dW_2, & t \in (0, T) \\ X_1(0) = X_1^0, \quad X_2(0) = X_2^0, \end{cases} \quad (2.1)$$

where $X_i^0 \geq 0$ $i = 1, 2$ and W_1, W_2 are 1D-Brownian motions. Here $\mu, \kappa, \sigma, \theta$ are positive constants.

We note that (2.1) extends the classical Heston model, by adding a controlled component in the volatility of the primary stochastic process X_1 . The control parameter u is a $(\mathcal{F}_t)_{t \geq 0}$ -adapted stochastic process $u : [0, T] \rightarrow \mathbb{R}$ which is assumed to take values in the interval $U = [a, b]$, where $0 < a < b < \infty$.

We associate with the control system (2.1) a performance function

$$J(u; X_1^0, X_2^0) = \mathbb{E} \int_0^T X_1^2(t) f(X_1(t), u(t)) dt + \mathbb{E} g(X_1(T)) \quad (2.2)$$

and denote by \mathcal{U}_0 the class of stochastic control processes $u : [0, T] \rightarrow U$.

The function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to satisfy the following hypotheses.

- (i) f is continuous on \mathbb{R}^2 and for each $x \in \mathbb{R}$, $u \mapsto f(x, u)$ is convex. Moreover, $\inf\{f(x, u); u \in [a, b]\} = 0$ for all $x \in \mathbb{R}$.

In the following we denote by f_u the sub-differential of the function $u \mapsto f(\cdot, u)$.

The optimal control problem we consider here is the following:

(P) Minimize $J(u; X_1^0, X_2^0)$ on the set of all stochastic processes $u \in \mathcal{U}_0$ satisfying (2.1).

Our objective is to find an optimal feedback controller $u(t)$ for problem (P) via the Hamilton-Jacobi (dynamic programming) equation.

A first problem regards the well posedness of state system (2.1). It is clear that the strong solution X_2 , if exists, should be found in class of nonnegative processes on $[0, T]$. However since the function $x \rightarrow \sqrt{x}$ is not Lipschitz it is not clear if such a solution exists for all $X_2^0 \geq 0$. However, we have the following existence result for a martingale solution $X = (X_1, X_2)$ to (2.1).

Theorem 2.1. *Assume that*

$$k\theta \geq \frac{1}{2}\sigma^2, \quad X_2^0 \geq 0. \quad (2.3)$$

Then there is at least one martingale solution $X = (X_1, X_2)$ for (2.1) in $(L^2(\Omega; C([0, T])))^2$ such that

$$X_1(t), X_2(t) \geq 0, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (2.4)$$

Moreover, $X_1(t) \geq 0, \forall t \in [0, T], \mathbb{P}\text{-a.s.}$ and

$$\mathbb{E} \sup_{t \in [0, T]} (|X_1(t)|^2 + |X_2(t)|^2) \leq \bar{C}(|X_1^0|^2 + |X_2^0|^2) + CT \quad (2.5)$$

for suitable constants $\bar{C}, C > 0$.

By martingale solution $X = (X_1, X_2)$ to (2.1) we mean an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process X in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ which is pathwise continuous on $[0, T]$ and satisfies the equations

$$\begin{cases} X_1(t) = X_1^0 + \int_0^t \mu X_1(s) ds + \int_0^t X_1(s) \sqrt{u(s)X_2(s)} dW_1(s), & t \in (0, T), \\ X_2(t) = X_2^0 + \int_0^t k(\theta - X_2(s)) ds + \int_0^t \sigma \sqrt{X_2(s)} dW_2(s), & t \in (0, T). \end{cases} \quad (2.6)$$

together with a Wiener process $W = (W_1, W_2)$.

Proof. We approximate the second equation in (2.1) by

$$\begin{aligned} dX_2^\epsilon &= k(\theta - X_2^\epsilon) dt + \sigma \frac{X_2^\epsilon}{\sqrt{|X_2^\epsilon| + \epsilon}} dW_2, \quad t \in (0, T), \\ X_2^\epsilon(0) &= X_2^0, \end{aligned} \quad (2.7)$$

and associate to (2.7) the equation

$$\begin{cases} dX_1^\epsilon = \mu X_1^\epsilon dt + X_1^\epsilon \sqrt{u X_2^\epsilon} dW_1, & t \in (0, T), \\ X_1^\epsilon(0) = X_1^0. \end{cases} \quad (2.8)$$

We shall prove first that under assumption (2.3), we have

$$X_2^\epsilon(t) \geq 0, \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (2.9)$$

Now we apply Itô's formula to function $\varphi(x) = \frac{1}{2}(x^-)^2$ for all $x \in \mathbb{R}$ where $x^- = \max\{0, -x\}$. We have

$$\varphi'(x) = -x^-, \quad \varphi''(x) = H(-x), \quad \forall x \in \mathbb{R},$$

where H is the Heaviside function. We get

$$\begin{aligned} \frac{1}{2}d|(X_2^\epsilon)^-(t)|^2 &= -k|(X_2^\epsilon)^-(t)|^2 dt - k\theta(X_2^\epsilon)^-(t) dt + \sigma \frac{(X_2^\epsilon(t)^-)^2}{\sqrt{|X_2^\epsilon(t)^-| + \epsilon}} dW_2(t) \\ &\quad + \frac{\sigma^2}{2} \frac{((X_2^\epsilon)^-(t))^2}{(X_2^\epsilon)^-(t) + \epsilon} H(-X_2^\epsilon(t)) dt. \end{aligned}$$

By (2.3) this yields

$$\frac{1}{2}\mathbb{E} \left[|(X_2^\epsilon)^-(t)|^2 + k \int_0^t |(X_2^\epsilon)^-(s)|^2 ds \right] \leq \mathbb{E} \left[\int_0^t \left(\frac{\sigma^2}{2} - k\theta \right) |(X_2^\epsilon)^-(s)| ds \right] \leq 0$$

for all $t \in [0, T]$, and therefore $(X_2^\epsilon)^-(t) = 0$ on $(0, T) \times \Omega$ which implies (2.9) as claimed.

Substituting X_2^ϵ in (2.8), we get that $(X_1^\epsilon, X_2^\epsilon)$ is a strong solution to (2.1). Indeed, if we represent X_1^ϵ as

$$X_1^\epsilon(t) = \exp \left(\int_0^t \sqrt{u(s)X_2^\epsilon(s)} dW_1(s) \right) y_\epsilon(t),$$

we have by Itô's formula that

$$\begin{aligned} dX_1^\epsilon(t) &= \exp \left(\int_0^t \sqrt{u(s)X_2^\epsilon(s)} dW_1(s) \right) dy_\epsilon(t) + \\ &\quad + \sqrt{u(t)X_2^\epsilon(t)} \exp \left(\int_0^t \sqrt{u(s)X_2^\epsilon(s)} dW_1(s) \right) y_\epsilon(t) dW_1(t) + \\ &\quad + \frac{1}{2}u(t)X_2^\epsilon(t) \exp \left(\int_0^t \sqrt{u(s)X_2^\epsilon(s)} dW_1(s) \right) y_\epsilon(t) dt \end{aligned}$$

and substituting in (2.8) we get for y_ϵ the random differential equation

$$\begin{aligned}\frac{dy_\epsilon}{dt} &= \left(\mu - \frac{1}{2}uX_2^\epsilon \right) y_\epsilon, \quad t \in (0, T) \\ y(0) &= X_1^0,\end{aligned}$$

which clearly has a unique $(\mathcal{F}_t)_{t \geq 0}$ -adapted solution y_ϵ . Therefore,

$$X_1^\epsilon(t) = \exp \left(\int_0^t \left(\mu - \frac{1}{2}u(s)X_2^\epsilon(s) \right) ds + \int_0^t \sqrt{u(s)X_2^\epsilon(s)} dW_1(s) \right) X_1^0 \quad (2.10)$$

is together with X_2^ϵ the solution to (2.7)-(2.8). The uniqueness of such a solution is immediate. By (2.10) it follows also that $X_1 \geq 0$ as claimed.

By Itô's formula we see that

$$\begin{aligned}\frac{1}{2} |X_2^\epsilon(t)|^2 &= \frac{1}{2} |X_2^0|^2 + \int_0^t k(\theta - X_2^\epsilon(s))X_2^\epsilon(s) ds + \frac{\sigma^2}{2} \int_0^t \frac{(X_2^\epsilon(s))^2}{X_2^\epsilon(s) + \epsilon} ds + \\ &\quad + \int_0^t \frac{\sigma(X_2^\epsilon(s))^2}{\sqrt{X_2^\epsilon(s) + \epsilon}} dW_2(s)\end{aligned}$$

and this yield via Burkholder-Davis-Gundy Theorem (see e.g. [9])

$$\mathbb{E} \sup\{|X_2^\epsilon(t)|^2 ; t \in [0, T]\} \leq C_1(1 + |X_2^0|^2).$$

We have also by (2.7)

$$\mathbb{E} [|X_2^\epsilon(t) - X_2^\epsilon(s)|^2] \leq C\mathbb{E} \left[\int_s^t (1 + |X_2^\epsilon(r)|^2) dr \right] \leq C_2(t - s) \quad \forall s, t \in [0, T].$$

Similarly by (2.8), we get

$$\begin{aligned}\mathbb{E} \sup\{|X_1^\epsilon(t)|^2 ; t \in [0, T]\} &\leq C_3(1 + |X_1^0|^2), \\ \mathbb{E} [|X_1^\epsilon(t) - X_1^\epsilon(s)|^2] &\leq C_4(t - s) \quad \forall s, t \in [0, T].\end{aligned}$$

Hence we have

$$\mathbb{E} \sup\{|X^\epsilon(t)|^2 ; t \in [0, T]\} \leq C_5(1 + |X_1^0|^2 + |X_2^0|^2), \quad (2.11)$$

$$\mathbb{E} [|X^\epsilon(t) - X^\epsilon(s)|^2] \leq C_6(t - s) \quad \forall s, t \in [0, T]. \quad (2.12)$$

for $X^\epsilon = (X_1^\epsilon, X_2^\epsilon)$.

We set $\nu_\epsilon = \mathcal{L}(X^\epsilon)$, there is, $\nu_\epsilon(\Gamma) = \mathbb{P}(X^\epsilon \in \Gamma)$ for each Borelian set $\Gamma \subset (C([0, T]; \mathbb{R}))^2$ and note that $\{\nu_\epsilon\}_{\epsilon > 0}$ is tight in $C([0, T]; \mathbb{R}^2)$. This means that for

each $\delta > 0$ there is a complete set $\Gamma \subset (C([0, T]; \mathbb{R}^2))$ such that $\nu_\epsilon(\Gamma^c) \leq \delta$ for all $\epsilon > 0$. We take Γ of the form

$$\Gamma_{r,\gamma} = \{y \in C([0, T]; \mathbb{R}^2) : |y(t)| \leq r, \forall t \in [0, T], \\ |y(t) - y(s)| \leq \gamma |t - s|^{\frac{1}{2}}, \forall t, s \in [0, T]\}.$$

Clearly by the Ascoli-Arzelà theorem, $\Gamma_{r,\gamma}$ is compact in $C([0, T]; \mathbb{R}^2)$. On the other hand, by (2.11)-(2.12) and the well known inequality

$$\rho \mathbb{P}[|Y| \geq \rho] \leq \mathbb{E}|Y|, \quad \forall \rho > 0$$

it follows that there are r, γ independent of ϵ such that $\nu_\epsilon(\Gamma_{r,\gamma}^c) \leq \delta$ as claimed. Then by the Skorohod's theorem there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $\tilde{X}, \tilde{X}^\epsilon$ such that $\mathcal{L}(\tilde{X}^\epsilon) = \mathcal{L}(X^\epsilon)$ and $\tilde{\mathbb{P}}$ -a.e. $\omega \in \tilde{\Omega}$,

$$\tilde{X}_\epsilon \rightarrow \tilde{X} \quad \text{in } C([0, T]; \mathbb{R}^2).$$

Then we may pass to limit in (2.7)-(2.8) and see that $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ satisfies system (3.2) in the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ for a new pair $\tilde{W} = (\tilde{W}_1, \tilde{W}_2)$ of Wiener processes in this space. See [3] for details. This completes the proof of existence of a martingale solution. Clearly (2.4),(2.5) hold for this solution which in the following we shall denote again $X = (X_1, X_2)$, $W = (W_1, W_2)$ and $(\Omega, \mathcal{F}, \mathbb{P})$. \square

3 The dynamic programming equation

Now we are going to study the Hamilton Jacobi equation for problem (P) and design on this basis an optimal closed loop (feedback) controller u . To this end we associate to problem (P) the optimal value function $V : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$

$$V(t, x, y) = \inf_{u \in \mathcal{U}_0} \left\{ \mathbb{E} \int_t^T X_1^2(s) f(X_1(s), u(s)) ds + \mathbb{E}[g(X_1(T))] \right\} \quad (3.1)$$

subject to the control system

$$\begin{cases} dX_1(s) = \mu X_1(s) ds + X_1 \sqrt{u(s) X_2(s)} dW_1(s), & s \in (t, T), \\ dX_2(s) = k(\theta - X_2(s)) ds + \sigma \sqrt{X_2(s)} dW_2(s), & s \in (t, T), \\ X_1(t) = x, \quad X_2(t) = y. \end{cases} \quad (3.2)$$

We shall assume $x > 0$, $y \geq 0$ and that conditions (2.3) holds. Then by Theorem 2.1 there is a martingale solution $X = (X_1, X_2)$ such that $X_1(s) \geq 0$,

$X_2(s) \geq 0, \forall s \in [0, T], \mathbb{P}$ -a.s. We consider the Hamilton-Jacobi equation associated with problem (P), namely

$$\begin{aligned} \varphi_t(t, x, y) + \mu x \varphi_x(t, x, y) + k(\theta - y) \varphi_y(t, x, y) + \frac{1}{2} \sigma^2 y \varphi_{yy}(t, x, y) + \\ + x^2 G(x, y, \varphi_{xx}(t, x, y)) = 0, \quad t \in [0, T], \quad x, y \in \mathbb{R}, \\ \varphi(T, x, y) = g(x), \quad \forall x \in \mathbb{R}, \end{aligned} \quad (3.3)$$

where $G: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$G(x, y, z) = \min_u \left\{ \frac{1}{2} u y z + f(x, u); u \in [a, b] \right\}, \quad \forall x, y, z \in \mathbb{R}. \quad (3.4)$$

It is well known (see e.g. [11, 20]) that if φ is a smooth (of class $C^{1,2}$ for instance) solution to (3.3) then the feedback controller

$$u(t) = \phi(t, X_1(t), X_2(t)), \quad t \in [0, T], \quad (3.5)$$

where

$$\phi(t, x, y) = \arg \min_u \left\{ \frac{1}{2} u y \varphi_{xx}(t, x, y) + f(x, u); u \in [a, b] \right\}, \quad (3.6)$$

is optimal in problem (P).

It should be mentioned that in general an equation of the form (3.3) does not have a classical solution. The best one can expect is a viscosity solution (see e.g. [12]) which under our conditions is not unique and also it is not smooth enough to provide a feedback controller ϕ of the form (3.6). However, as shown below we can reduce (3.3) by a simple argument to a nonlinear parabolic equation for which one can prove the existence and uniqueness of a strong solution. Indeed, by setting

$$p(t, x, y) = \varphi_x(t, x, y), \quad \forall t \in [0, T], \quad x, y \in \mathbb{R} \quad (3.7)$$

and differentiating in x , we transform (3.3) in the second order nonlinear parabolic equation

$$\begin{cases} p_t(t, x, y) + \mu(x p(t, x, y))_x + k(\theta - y) p_y(t, x, y) + \frac{1}{2} \sigma^2 y p_{yy}(t, x, y) + \\ \quad + (x^2 G(x, y, p_x(t, x, y)))_x = 0, \quad \forall t \in [0, T], \quad x, y \in \mathbb{R}, \\ p(T, x, y) = g_x(x), \quad \forall x \in \mathbb{R}, \quad y \in \mathbb{R}, \end{cases} \quad (3.8)$$

with natural boundary conditions at $x = \pm\infty, y = \pm\infty$. In terms of p the feedback controller ϕ in (3.5) is expressed as

$$\phi(t, x, y) = \arg \min_u \left\{ \frac{1}{2} u y p_x(t, x, y) + f(x, u); u \in [a, b] \right\}. \quad (3.9)$$

Taking into account that by (2.4) the state X_2 is in the half plane $y \geq 0$, we see that the flow $t \mapsto (X_1(t), X_2(t))$ leaves invariant the domain $Q = \{(x, y) \in \mathbb{R}^2; 0 \leq y < \infty\}$ and so equation (3.8) can be treated on this domain. For simplicity we shall restrict ourselves to the domain

$$Q = \{x \in \mathbb{R}, \rho < y < M\} = \mathbb{R} \times (\rho, M)$$

where M is sufficient large, but finite. In other words, we shall consider equation (3.8) on domain $(0, T) \times Q$ with boundary value condition on ∂Q

$$p(t, x, \rho) = 0; \quad p(t, x, M) = 0, \quad \forall x \in \mathbb{R}, \quad t \in [0, T]. \quad (3.10)$$

By estimate (2.5) we see that for M large enough the above stochastic flow remains with a high probability in Q and so we may treat problem (3.8) in such a bounded domain. We set $H = L^2(Q)$ with the standard norm $\|\cdot\|_H$ and define the space

$$V = \{z \in H \cap H_{\text{loc}}^1(Q); xz_x, z_y \in L^2(Q); z(x, \rho) = z(x, M) = 0, \quad \forall x \in \mathbb{R}\} \quad (3.11)$$

where z_x, z_y are taken in sense of distributions on Q . The space V is a Hilbert space with the norm

$$\|z\|_V = \left(\int_Q (z^2 + x^2 z_x^2 + z_y^2) dx dy \right)^{\frac{1}{2}}, \quad \forall z \in V.$$

We have $V \subset H$ algebraically and topologically and denote by V^* the dual space of V having H as pivot space. Denote by ${}_{V^*}(\cdot, \cdot)_V$ the duality (V, V^*) and by $\|\cdot\|_{V^*}$ the dual norm of V^* . On $H \times H$, ${}_{V^*}(\cdot, \cdot)_V$ is just the scalar product of H .

Definition 3.1. *The function $p : [0, T] \times Q \rightarrow \mathbb{R}$ is called weak solution to problem (3.8)-(3.10) if the following conditions hold*

$$p \in C([0, T]; H) \cap L^2([0, T]; V), \quad \frac{dp}{dt} \in L^2([0, T]; V^*), \quad (3.12)$$

$$\begin{aligned} \frac{d}{dt} \int_Q p(t, x, y) \psi(x, y) dx dy + \int_Q (\mu(xp(t, x, y))_x + k(\theta - y)p_y(t, x, y)) \psi(x, y) dx dy \\ - \frac{\sigma^2}{2} \int_Q p_y(t, x, y) (y\psi(x, y))_y dx dy - \\ - \int_Q x^2 G(x, y, p_x(t, x, y)) \psi_x(x, y) dx dy = 0 \quad \forall \psi \in V, \quad a.e. \quad t \in [0, T], \end{aligned} \quad (3.13)$$

$$p(T, x, y) = g_x(x), \quad \forall (x, y) \in Q. \quad (3.14)$$

Taking into account (3.7) and Definition 3.1 we say that φ is a weak solution to (3.3) if

$$\begin{aligned} \varphi &\in L^2([0, T]; L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R})), \quad \varphi_x \in C([0, T]; H) \cap L^2([0, T]; V), \\ \frac{d\varphi_x}{dt} &\in L^2([0, T]; V^*), \\ \frac{d}{dt} \int_Q \varphi_x(t, x, y) \psi(x, y) dx dy &+ \int_Q (\mu(x\varphi_x(t, x, y))_x + k(\theta - y)\varphi_{xy}(t, x, y)) \psi(x, y) dx dy \\ &- \frac{\sigma^2}{2} \int_Q \varphi_{xy}(t, x, y) (y\psi(x, y))_y dx dy - \\ &- \int_Q x^2 G(x, y, \varphi_{xx}(t, x, y)) \psi_x(x, y) dx dy = 0 \quad \forall \psi \in V, \quad \text{a.e. } t \in [0, T], \end{aligned} \quad (3.15)$$

$$\begin{aligned} \varphi(T, x, y) &= g(x), \quad \forall (x, y) \in Q, \\ \varphi_x(t, x, \rho) &= \varphi_x(t, x, M) = 0 \quad \forall x \in \mathbb{R}, t \in [0, T]. \end{aligned} \quad (3.16)$$

Clearly, if p is a weak solution to (3.8)-(3.10) then by (3.7) the function

$$\varphi(t, x, y) = \int_{-\infty}^x p(t, \xi, y) d\xi, \quad (t, x, y) \in [0, T] \times \mathbb{R} \times (0, M)$$

is a weak solution to (3.3) and conversely if φ is a weak solution to (3.3) then p given by (3.7) is a weak solution to (3.8). It should be said that φ is unique until an additive function $\tilde{\varphi} = \tilde{\varphi}(t, y)$.

We have

Theorem 3.1. *Let $g_x \in L^2(\mathbb{R})$. Then there is a unique weak solution p to problem (3.8)-(3.10).*

Proof. We can rewrite problem (3.8)-(3.10) as the backward infinite dimensional Cauchy problem

$$\begin{aligned} \frac{d}{dt} p(t) - Ap(t) &= 0, \quad \text{a.e. } t \in (0, T), \\ p(T) &= g_x, \end{aligned} \quad (3.17)$$

where $A : V \rightarrow V^*$ is the nonlinear operator defined by

$$\begin{aligned} (Az, \psi) &= - \int_Q (\mu(xz)_x + k(\theta - y)z_y) \psi(x, y) dx dy + \frac{\sigma^2}{2} \int_Q z_y (y\psi(x, y))_y dx dy + \\ &+ \int_Q x^2 G(x, y, z_x) \psi_x(x, y) dx dy, \quad t \in [0, T], \quad z, \psi \in V. \end{aligned} \quad (3.18)$$

In order to apply the standard existence theory for the Cauchy problem (3.18) (see e.g. [1, p. 177]) we need to check the following properties for $A(t)$.

(I) There is $\alpha_1 \geq 0$ such that

$${}_{V^*}(Az - A\bar{z}, z - \bar{z})_V \geq -\alpha_1 \|z - \bar{z}\|_H^2, \quad \forall z, \bar{z} \in V.$$

(II) There is $\alpha_2 > 0$ such that

$$\|Az\|_{V^*} \leq \alpha_2 \|z\|_V, \quad \forall z \in V, t \in (0, T).$$

(III) There are $\alpha_3 > 0$, $\alpha_4 \geq 0$ such that

$${}_{V^*}(A(t)z, z)_V \geq \alpha_3 \|z\|_V^2 - \alpha_4 \|z\|_H^2, \quad \forall z \in V, t \in (0, T).$$

To check (I) we note that by (3.4) we have

$$\begin{aligned} G(x, y, z) &= -\sup_u \left\{ -\frac{1}{2}uyz - f(x, u); u \in [a, b] \right\} \\ &= -\tilde{f}^* \left(x, -\frac{1}{2}yz \right) \quad \forall (x, y) \in Q, z \in \mathbb{R}, \end{aligned} \quad (3.20)$$

where $\tilde{f}^*(x, v)$ is the convex conjugate of function $v \xrightarrow{\tilde{f}} f(x, v) + I_{[a, b]}(v)$ that is $\tilde{f}^*(x, q) = \sup_v \{qv - \tilde{f}(x, v); v \in [a, b]\}$ and $I_{[a, b]}(v) = 0$ if $v \in [a, b]$, $I_{[a, b]} = +\infty$ otherwise. Given a convex, lower semicontinuous function $h: \mathbb{R} \rightarrow]-\infty, +\infty]$ denote by $\partial h(v) = h_v(v)$ the subdifferential of h at v , that is

$$\partial h(v) = \{ \eta \in \mathbb{R} : \eta(v - \bar{v}) \geq h(v) - h(\bar{v}) \quad \forall \bar{v} \in \mathbb{R} \},$$

and recall that if h^* is the conjugate of h then $\partial h^*(q) = (\partial h)^{-1}(q)$, $\forall q \in \mathbb{R}$ and so $h_q^* = \partial_q h^*(\alpha q) = \alpha \partial h^*(\alpha q)$, $\forall \alpha \in \mathbb{R}$, $q \in \mathbb{R}$. This yields

$$G_z(x, y, z) = \frac{1}{2}y\tilde{f}_v^* \left(x, -\frac{1}{2}yz \right), \quad \forall (x, y) \in Q, z \in \mathbb{R}, \quad (3.21)$$

where G_z is the subdifferential of function $z \mapsto G(x, y, z)$. On the other hand, we have

$$\tilde{f}_v^*(x, v) = (f_u(x, \cdot) + N_{[a, b]})^{-1}(v), \quad \forall v \in \mathbb{R} \quad (3.22)$$

where $N_{[a, b]}(v) \subset 2^{\mathbb{R}}$ is the normal cone to $[a, b]$ in v , that is

$$N_{[a, b]}(v) = \begin{cases} \mathbb{R}^- & \text{if } v = a, \\ 0 & \text{if } a < v < b, \\ \mathbb{R}^+ & \text{if } v = b. \end{cases} \quad (3.23)$$

This yields

$$\tilde{f}_v^*(x, v) \in [a, b], \quad \forall x, v \in \mathbb{R}, \quad (3.24)$$

and since $y \in [0, M]$, $a > 0$, by (3.21) we see that $G_z \geq 0$ and so the function $z \rightarrow G(x, y, z)$ is monotonically nondecreasing. By (3.19) it follows via integrations by part that (I) holds with some suitable α_1 .

To prove (II) we note that

$$\int_Q x^2 G(x, y, z_x) \psi_x(x, y) dx dy \leq \|\psi\|_V \left(\int_Q |xG(x, y, x)|^2 dx dy \right)^{\frac{1}{2}},$$

while by (3.21) and (3.24) we have that for all $(x, y) \in Q$, $v \in \mathbb{R}$,

$$|G(x, y, v)| = |vG_v(x, y, \xi_v)| \leq \frac{1}{2}Mb|v|,$$

because by hypothesis (i) and (3.20) it follows that $G(x, y, 0) = 0$. Hence $\forall \psi, z \in V$

$$\left| \int_Q x^2 G(x, y, z_x) \psi_x(x, y) dx dy \right| \leq \frac{1}{2}Mb \int_Q x^2 |z_x| |\psi_x| dx dy \leq C \|\psi\|_V \|z\|_V.$$

Similarly, we have

$$\left| \int_Q z_y (y\psi)_y dx dy \right| \leq C \|\psi\|_V \|z\|_V \quad \forall \psi, z \in V$$

and by (3.19) it follows (II).

Finally, taking into account that by (3.21)-(3.22), $G_z(x, y, \xi_z) \geq \frac{1}{2}ay > \frac{1}{2}a\rho$, we have

$$\int_Q x^2 G(x, y, z_x) z_x dx dy = \int_Q x^2 G_z(x, y, \xi_z) z_x^2 dx dy \geq \frac{1}{2}a\rho \int_Q x^2 z_x^2 dx dy. \quad (3.25)$$

We note also the inequalities

$$\begin{aligned} \int_Q z_y (yz)_y dx dy &= \int_Q \left(y|z_y|^2 + \frac{1}{2}(z^2)_y \right) dx dy \geq \\ &\geq \int_Q (y|z_y|^2) dx dy \geq \rho \int_Q (|z_y|^2) dx dy \end{aligned}$$

and

$$\begin{aligned} - \int_Q ((xz)_x + k(\theta - y)z_y) z dx dy &= - \int_Q \left(\frac{1}{2}x(z^2)_x + z^2 + \frac{k}{2}(\theta - y)(z^2)_y \right) dx dy \\ &\geq C \int_Q (z^2 + y^2) dx dy. \end{aligned}$$

Here we have integrated by parts. Together with (3.25) the latter implies (III) as claimed.

Then we infer that the Cauchy problem (3.18) has a unique solution p which satisfies (3.12) and this completes the proof of the theorem. \square

An alternative approach to treat equation (3.18) is the so called *semigroup approach* we briefly present below.

We set $q(t) = p(T - t)$ and rewrite (3.18) as the forward Cauchy problem

$$\frac{dq}{dt} + Aq = 0, \quad t \in (0, T), \quad (3.26)$$

$$q(0) = g_x. \quad (3.27)$$

Define the operator $A_H : \mathcal{D}(A_H) \subset H \rightarrow H$ as

$$A_H q = Aq, \quad \forall q \in \mathcal{D}(A_H) = \{q \in V : Aq \in H\}.$$

By (I)-(III) it follows via Minty-Browder theory (see e.g. [1]) that the operator A_H is quasi-m-accretive in $H \times H$, that is, there is $\eta_0 \in \mathbb{R}$ such that

$$(A_H q - A_H \bar{q}, q - \bar{q}) \geq -\eta_0 \|q - \bar{q}\|_H^2, \quad \forall q, \bar{q} \in \mathcal{D}(A_H),$$

and

$$R(\eta I + A_H) = H, \quad \forall \eta > \eta_0,$$

where I is the identity operator and R indicates the range. As a matter of fact by (I)-(III) it follows that $R(\eta I + A_H) \supset V^*$ which clearly implies the latter. Then, by the standard existence theory for the Cauchy problem associated with non-linear quasi-m-accretive operators in Hilbert spaces, if $g_x \in \mathcal{D}(A_H)$ then there is a unique strong solution $q \in W^{1,\infty}([0, T]; H) = \{q \in L^\infty([0, T], H); \frac{dq}{dt} \in L^\infty([0, T], H)\}$ to equation (3.26), that is

$$\begin{aligned} q &\in L^2([0, T], V), \quad Aq(t) \in L^\infty([0, T]; H), \\ \frac{d^+}{dt} q(t) + A_H(q(t)) &= 0, \quad t \in [0, T[, \\ q(0) &= g_x = q_0. \end{aligned} \quad (3.28)$$

Moreover, q is expressed by the exponential Crandall&Liggett formula

$$q(t) = \lim_{n \rightarrow \infty} \left(1 + \frac{t}{n} A\right)^{-n} q_0 \quad (3.29)$$

uniformly on $[0, T]$ (see e.g. [1], p. 139). This means that the solution q is the limit of the finite difference scheme

$$q(t) = \lim_{h \rightarrow 0} q_h(t) \quad \forall t \in [0, T]$$

where

$$\begin{aligned} q_h(t) &= q_h^i \quad t \in [ih, (i+1)h), \\ q_h^{i+1} + hAq_h^{i+1} &= q_h^i, \quad i = 0, 1, \dots, N = \left\lceil \frac{T}{h} \right\rceil - 1, \\ q_h^0 &= q_0. \end{aligned} \quad (3.30)$$

For $q_0 \in H$ the function q given by (3.30) is a generalized mild solution to (3.26). This scheme may be used to numerical approximation of equation (3.8). Moreover, this reveals that for g regular the solution p of equation (3.8) is locally in $H^2(Q)$.

We note also that $q_0 \rightarrow q(t)$ (for instance in $H^2(Q)$) is a semigroup of quasi contraction on H , that is

$$\|q(t, q_0) - q(t, \bar{q}_0)\|_H \leq \exp(\eta_0 t) \|q_0 - \bar{q}_0\|_H \quad \forall t \geq 0, \quad q_0, \bar{q}_0 \in H.$$

Now coming back to equation (3.3), we get by Theorem 3.1

Corollary 3.1. *Let $g_x \in L^2(\mathbb{R})$. Then equation (3.3),(3.17) has a weak solution in the sense of (3.15)-(3.16). This solution is unique up to an additive function $\tilde{\varphi} \equiv \tilde{\varphi}(t, y)$.*

Remark 3.1. In literature, dynamic programming equations of the form (3.3) have been treated so far in the framework of viscosity solutions, see, e.g., [11], [12], [14], and, more recently, in [2], where a unique mild solution is derived, via the nonlinear semigroup theory, for the one dimensional dynamic programming equation associated to a stochastic optimal control problem with multiplicative noise. The main advantage of Theorem 3.1 and respectively of Corollary 3.1 is that are obtained here a weak solution φ which is however sufficiently regular to represent the feedback controller u of problem (P) in explicit terms (3.6).

4 The optimal feedback controller

Now coming back to (3.5)-(3.6) one might suspect that the feedback controller u^* defined by

$$u^*(t) = \phi(t, X_1(t), X_2(t)) \quad t \in (0, T), \quad (4.1)$$

where

$$\phi(t, x, y) = \arg \min_{u \in [a, b]} \left\{ \frac{1}{2} u y p_x(t, x, y) + f(x, u) \right\}, \quad \forall (x, y) \in Q, \quad (4.2)$$

is optimal in problem (P) for $(X_1, X_2) \in (0, T) \times Q$, where p is the weak solution to equation (3.8)-(3.10). Since $p_x \in L^2(Q)$, the mapping ϕ is well defined and (see

(3.22)) we have

$$\phi(t, x, y) = (f_u(x, \cdot) + N_{[a,b]})^{-1} \left(-\frac{1}{2} y p_x(t, x, y) \right), \quad \forall t \in [0, T], (x, y) \in \mathcal{Q}, \quad (4.3)$$

where $f_u(x, \cdot)$ is the subdifferential of function $u \mapsto f(x, u)$. The corresponding closed loop system (2.1) is

$$\begin{cases} dX_1 = \mu X_1 dt + X_1 \sqrt{\phi(t, X_1, X_2)} X_2 dW_1, & t \in (0, T) \\ dX_2 = k(\theta - X_2) dt + \sigma \sqrt{X_2} dW_2, & t \in (0, T), \\ X_1(0) = X_1^0, \quad X_2(0) = X_2^0. \end{cases} \quad (4.4)$$

The existence of a strong solution (X_1, X_2) to (4.4) would imply by a standard computation that the map $u^* = \phi(t, X_1, X_2)$ is indeed an optimal feedback controller for problem (P). However the existence of a strong solution for (4.4) is a delicate problem and the best one can expect in this case is a martingale solution. To this end we assume in addition beside above hypotheses that

(ii) $u \mapsto f_u(x, u)$ is strictly monotone.

Theorem 4.1. *Assume that (i),(ii) and (2.3) hold. Then there is a martingale solution (X_1, X_2) to equation (4.4).*

Proof. The proof is similar to that of Theorem 2.1 and so it will be sketched only. Consider the map $\psi: [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$

$$\psi(t, x) = x \sqrt{\phi(t, x, X_2(t)) X_2(t)}, \quad t \in [0, T], x \in \mathbb{R}.$$

By (4.3) it follows that for all $R > 0$, $|x| + |\bar{x}| \leq R$, \mathbb{P} -a.s.

$$\begin{aligned} |\psi(t, x) - \psi(t, \bar{x})| &= \left| x \sqrt{\phi(t, x, X_2(t)) X_2(t)} - \bar{x} \sqrt{\phi(t, \bar{x}, X_2(t)) X_2(t)} \right| \\ &\leq \sqrt{X_2(t)} \left(|x - \bar{x}| \sqrt{\phi(t, x, X_2(t))} + |\bar{x}| \sqrt{|\phi(t, x, X_2(t)) - \phi(t, \bar{x}, X_2(t))|} \right) \\ &\leq C \sqrt{X_2(t)} |x - \bar{x}| + R \sqrt{X_2(t)} \sqrt{L_\phi \frac{1}{2} X_2(t) |p_x(t, x, X_2(t)) + p_x(t, \bar{x}, X_2(t))|} \\ &\leq C \sqrt{X_2(t)} |x - \bar{x}| + R X_2(t) \sqrt{L_\phi} \sqrt{L_R^1} |x - \bar{x}| \\ &\leq C \sqrt{X_2(t)} |x - \bar{x}| + K X_2(t) \sqrt{|x - \bar{x}|} \end{aligned} \quad (4.5)$$

and

$$|\psi(t, x)| \leq C |x| \sqrt{X_2(t)}. \quad (4.6)$$

Indeed, by (ii) the map $(f_u(x, \cdot) + N_{[a,b]})^{-1}$ is Lipschitz on \mathbb{R} while

$$|p_x(t, x) - p_x(t, \bar{x})| \leq L_R^1 |x - \bar{x}|, \quad \text{for } |x| + |\bar{x}| \leq R.$$

The latter is a consequence of high order regularity of solutions to quasilinear parabolic equation (see [16], Theorem 6.1, p 452) which implies that

$$p \in H^{2+\alpha, 1+\frac{\alpha}{2}}((0, T) \times Q_R) \quad \forall R > 0$$

where $\alpha \in (0, 1)$ and $Q_R = (0, R) \times (0, M)$. On the other hand again by (4.3) we see that $\phi(t, x, y) \in [a, b] \forall x, y \in \mathbb{R}, t \in [0, T]$, and so (4.5) follows.

We approximate (4.4) by (see (2.7))

$$\begin{aligned} dX_1^\epsilon &= \mu X_1^\epsilon dt - X_1^\epsilon \sqrt{\phi(t, X_1^\epsilon, X_2^\epsilon)} X_2^\epsilon dW_1, \\ dX_2^\epsilon &= k(\theta - X_2^\epsilon) dt + \sigma \frac{X_2^\epsilon}{\sqrt{|X_2^\epsilon| + \epsilon}} dW_2, \\ X_1^\epsilon(0) &= X_1^0, \quad X_2^\epsilon(0) = X_2^0. \end{aligned} \tag{4.7}$$

Arguing as in the proof of Theorem 2.1 it follows by (4.5)-(4.6) that (4.7) has a unique solution $(X_1^\epsilon, X_2^\epsilon)$, $X_1^\epsilon, X_2^\epsilon \geq 0$, \mathbb{P} -a.s. Moreover, one obtains also in this case estimates (2.11)-(2.12) and so by the Skorohod theorem it follows as above the existence of a martingale solution $(\tilde{X}_1, \tilde{X}_2)$ satisfying system (4.4) in a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2)$. \square

Remark 4.1. Roughly speaking Theorem 4.1 amounts to saying that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2)$ where the closed loop system corresponding feedback controller $\tilde{u}^* = \phi(t, \tilde{X}_1, \tilde{X}_2)$ has a solution $(\tilde{X}_1, \tilde{X}_2)$. This means that the feedback controller u^* is admissible in problem (P) though it is not clear if it is optimal. However this is a suboptimal feedback controller. Indeed, if one constructs in a similar way the feedback controller $u_\epsilon = \phi_\epsilon(X_1^\epsilon, X_2^\epsilon)$ for problem (P) with state system (4.7) and ϕ_ϵ is the solution to the corresponding equation (3.15)-(3.17) then u_ϵ is optimal for the approximating optimal control problem and it is convergent in law in the above sense to a controller u^* as found above.

Example 1. Let $f(x, u) \equiv 0$. Then equation (3.8) reduces to

$$\begin{aligned} p_t + \mu(xp)_x + k(\theta - y)p_y + \frac{\sigma^2}{2}yp_{yy} + y(x^2(aH(p_x) + bH(-p_x)))_x &= 0, \quad x \in \mathbb{R}, y \in (0, M), \\ p(T, x, y) &= g_x(x), \end{aligned} \tag{4.8}$$

while by (4.3) the feedback controller u^* (see (4.1)) is given by

$$u^*(t) = \begin{cases} a & \text{if } p_x(t, X_1(t), X_2(t)) > 0, \\ b & \text{if } p_x(t, X_1(t), X_2(t)) < 0, \\ \in (a, b) & \text{if } p_x(t, X_1(t), X_2(t)) = 0. \end{cases} \tag{4.9}$$

In (4.8) H is the Heaviside function.

It should be said however that in this case Theorem 4.1 does not apply and so parabolic equation (4.8) has only a weak solution p in the sense of (3.12)-(3.14) and so p is not sufficiently regular to assume existence of the closed loop system (4.8). However, if for almost all $\omega \in \Omega$ the set $\Sigma = \{t : p_x(t, X_1^*(t), X_2^*(t)) = 0\}$ is finite (as it is expected to be) then the controller u^* is a bang-bang controller with Σ as set of switch points. This fact might lead to a simplification of control problem (P) by replacing the set \mathcal{U}_0 of admissible controllers $u: [0, T] \rightarrow \mathbb{R}$ by

$$\tilde{\mathcal{U}}_0 = \left\{ u: [0, T] \rightarrow \mathbb{R}, \mathcal{F}_t\text{-adapted}, u(t) = \sum_{i=0}^{N-1} v_i \chi_{[t_i, t_{i+1})}(t) \right\}.$$

Here $t_0 = 0 < t_1 < t_2 < \dots < t_N = T$ is a given partition of interval $[0, T]$ while $v_i: \Omega \rightarrow \mathbb{R}$ are \mathcal{F}_{t_i} -measurable functions. More precisely we can consider the stochastic optimal control problem

$$\text{Minimize } \mathbb{E}g(X(T)) \tag{4.10}$$

subject to (2.1) and $u \in \tilde{\mathcal{U}}_0$ where $v_j \in [a, b], \forall j = 0, 1, \dots, N-1$.

5 Concluding comments

In the present paper we have studied an optimization problem where the performance criterion is a function of a variation of the celebrated Heston model, where an additional control process has been taken into account, namely

$$\begin{aligned} & \min_{u \in \mathcal{U}} \mathbb{E} \int_0^T X_1^2(t) f(X_1(t), u(t)) dt + \mathbb{E}g(X_1(T)), \\ & \text{s.t. } \begin{cases} dX_1 = \mu X_1 dt + X_1 \sqrt{u X_2} dW_1, & t \in (0, T) \\ dX_2 = k(\theta - X_2) dt + \sigma \sqrt{X_2} dW_2, & t \in (0, T) \\ X_1(0) = X_1^0, & X_2(0) = X_2^0. \end{cases} \end{aligned}$$

Under suitable conditions, we have established the well posedness of the control problem via an approximation-technique. Moreover, studying the associated Hamilton-Jacobi equation, we have proven that an optimal feedback controller for this problem exists.

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