STABILITY IN AN OVERDETERMINED PROBLEM FOR THE GREEN'S FUNCTION

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ABSTRACT. In the plane, we consider the problem of reconstructing a domain from the normal derivative of its Green's function (with fixed pole) relative to the Dirichlet problem for the Laplace operator. By means of the theory of conformal mappings, we derive stability estimates of Hölder type.

1. INTRODUCTION

The study of overdetermined boundary problems in partial differential equations finds its motivations in many areas of mathematics, such as inverse and free boundary problems, isoperimetric inequalities and optimal design. As in Serrin's seminal paper [15], in many such problems the analysis is mainly focused on the (spherical) symmetry of the domain considered.

In recent years, several authors have commenced to analyse the stability of the aforementioned symmetric configurations in presence of approximate (boundary) data ([1], [4], [5], [12], [13]); see also the work on quantitative isoperimetric inequalities ([6], [10]).

In [1], a logarithmic estimate of approximate (spherical) symmetry is deduced for a quite general semilinear overdetermined problem. From the proof, based on an ingenious adaptation of Serrin's moving-planes argument, it is clear that the logarithmic character of the stability estimate is due to the use of Harnack inequality. Such a drawback appears to be inherent in the method employed and cannot even be removed by considering simpler nonlinearities.

An improved estimate – of Hölder type, but only for the torsion problem – has been derived in [4] by combining Pohožaev integral identity and some geometric inequalities.

In the present paper, we will tackle a more detailed study of the stability in the plane by exploiting the theory of conformal mappings as we have already done in [2] for the study of symmetries, with the aim of deriving optimal estimates.

As in [2], we will work on a case study: in a planar bounded domain Ω with boundary $\partial\Omega$ of class $C^{1,\alpha}$, we shall consider the problem

$$-\Delta u = \delta_{\zeta_o} \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = \varphi \qquad \text{on } \partial\Omega. \tag{1.3}$$

where ν is the *interior* normal direction to $\partial\Omega$, δ_{ζ_o} is the Dirac delta centered at a given point $\zeta_o \in \Omega$ and $\varphi : \partial\Omega \to \mathbb{R}$ is a positive given function of arclength, measured from a reference point on $\partial\Omega$.

Problem (1.1)-(1.3) should be interpreted as follows: find a domain Ω whose Green's function u with pole at ζ_o has gradient with values on the boundary that fit those of the given function φ. This problem has some analogies to a model for the Hele-Shaw flow, as presented in [8] and [14].

In [2], we estabilished a connection between φ and Ω by using conformal mappings: chosen two distinct points ζ_b and ζ_o and a number $\alpha \in (0,1)$, we introduced the set

$$\mathscr{O} = \{\Omega \subseteq \mathbb{C} : \Omega \text{ open, bounded, simply connected, } C^{1,\alpha}, \ \zeta_o \in \Omega, \ \zeta_b \in \partial\Omega\}$$

and the class of functions

$$\mathscr{F} = \{f \in C^{1,\alpha}(\overline{D}, \mathbb{C}) : f \text{ one-to-one, analytic, } f(0) = \zeta_o, \ f(1) = \zeta_b\},$$

where D is the open unit disk.

By the Riemann Mapping Theorem $\mathscr{O}$ and $\mathscr{F}$ are in one-to-one correspondence. In [2, Theorem 2.2], we proved that the operator $\mathcal{T}$ that to each $f \in \mathscr{F}$ associates the interior normal derivative $\mathcal{T}(f)$ on $\partial\Omega$ of the solution of (1.1)-(1.2) is injective: an $f \in \mathscr{F}$ is uniquely determined by the formula

$$(1.4) \qquad f'(z) = e^{i\gamma} \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{1}{2\pi\varphi(\Phi^{-1}(t))} dt\right\}, \quad z \in D,$$

where

$$(1.5) \qquad \varphi(s) = \mathcal{T}(f)(s), \quad \Phi(s) = 2\pi \int_0^s \varphi(\sigma) d\sigma, \quad s \in [0, |\partial\Omega|];$$

here, the constant γ is normalized by the condition

$$(1.6) \qquad \int_0^1 f'(t) dt = \zeta_b - \zeta_o.$$

Notice that (1.4) can be obtained as a consequence of two classical facts: the connection between the Green's function and conformal mappings in simply connected planar domains and the Schwarz integral formula for the disk (see, e. g., [3], [7], [9]). By means of (1.4)-(1.6), in [2] we obtained results relating the symmetry of Ω to certain invariance properties of φ.

Here, by using the same ideas, we deduce stability results both near the disk and near any simply connected domain. Two typical results that better illustrate our work follow.

Preliminarily, we introduce some more notations. Given some positive constants L, m, M_0 and M_1, we define two classes of functions:

$$\mathscr{G}_0^L = \{\varphi \in C^{0,\alpha}(\mathbb{R}) : \ \varphi \text{ is } L\text{-periodic}, \ \varphi \geq m, \ \|\varphi\|_{0,\alpha,[0,L]} \leq M_0\},$$

$$\mathscr{G}_1^L = \{\varphi \in \mathscr{G}_0^L \cap C^{1,\alpha}(\mathbb{R}) : \ \|\varphi\|_{1,\alpha,[0,L]} \leq M_1\}.$$

For the definitions of the relevant Hölder norms, we refer the reader to Section 2.

Theorem 1.1. *Let $\Omega \in \mathscr{O}$ be with perimeter L.*

Assume that $B(\zeta_o, \rho)$ and $B(\zeta_o, R)$ are the largest disk contained in Ω and the smallest disk containing Ω, centered at ζ_o, respectively.

Let φ be the interior normal derivative on $\partial\Omega$ (as function of the arclength) of the solution of (1.1)-(1.2) and set $C = \frac{1}{2\pi\rho}$.

Then, if $\varphi \in \mathscr{G}_0^L$, there exists a constant K, depending on α, ρ and M_0, such that

$$R - \rho \leq K\|\varphi - C\|_{0,\alpha,[0,L]}.$$

Theorem 1.1 can be considered an analogue of [1, Theorem 1] and [4, Theorem 1.2]. Notice that here we obtain Lipschitz stability. In the following result, we give a stability estimate involving the Hausdorff distance d_H of any two bounded simply connected domains (for the definition of d_H, we refer the reader to Section 3).

Theorem 1.2. *Let Ω_1 and $\Omega_2 \in \mathcal{O}$ be domains with the same perimeter L and f_1 and f_2 the corresponding conformal mappings in $\mathscr{F}$. Suppose that $\mathcal{T}(f_1), \mathcal{T}(f_2) \in \mathscr{G}_1^L$.*

Then, up to rotations around ζ_o, we have that

$$d_H(\Omega_1, \Omega_2) \le K \|\mathcal{T}(f_1) - \mathcal{T}(f_2)\|_{1,0,[0,L]}^\alpha,$$

where the constant K, whose expression can be deduced from the proof, depends on α, m, M_1 and L.

Theorem 1.2 seems to be new. Compare the Lipschitz stability obtained in Theorem 1.1 to the Hölder-type estimate obtained in Theorem 1.2 (see Section 3 for the details). In Subsection 3.4 we will also present a more general version of Theorem 1.2.

2. Some useful notations and results

In what follows, D will always be the open unit disk in $\mathbb{C}$ centered at 0.

Let $\varphi : I \to \mathbb{R}$ be a function defined on an interval $I \subseteq \mathbb{R}$. We denote

$$\|\varphi\|_{\infty,I} = \sup_I |\varphi|, \quad [\varphi]_{k,\alpha,I} = \sup_{\substack{x,y \in I \\ x \ne y}} \frac{|\varphi^{(k)}(x) - \varphi^{(k)}(y)|}{|x - y|^\alpha},$$

where $k = 0, 1, ..., 0 < \alpha \le 1$ and $\varphi^{(k)}$ is the k-th derivative of φ, when defined. Moreover, we set:

$$\|\varphi\|_{k,\alpha,I} = \sum_{j=0}^k \|\varphi^{(k)}\|_{\infty,I} + [\varphi]_{k,\alpha,I}; \quad \|\varphi\|_{k,0,I} = \sum_{j=0}^k \|\varphi^{(k)}\|_{\infty,I}$$

and

$$C^{k,\alpha}(I) = \{\varphi \in C^0(I) : \|\varphi\|_{k,\alpha,I} < +\infty\}.$$

Let us recall some basic facts (see [7] and [11] for more details). If $\Omega \subseteq \mathbb{C}$ is a simply connected domain bounded by a Jordan curve and $\zeta_o \in \Omega$, then, from the Riemann Mapping Theorem, it follows that Ω is the image of an analytic function $f : D \to \Omega$ which induces a homeomorphism between the closures $\overline{D}$ and $\overline{\Omega}$, has non-zero derivative f' in D and is such that $f(0) = \zeta_o$. An application of Schwarz's Lemma proves that f is unique if it fixes a point of the boundary, say $f(1) = \zeta_b$ for a certain $\zeta_b \in \partial\Omega$. Moreover, if Ω is of class $C^{1,\alpha}$, for a certain $\alpha \in (0,1)$, then, by Kellogg's theorem, we can infer that $f \in C^{1,\alpha}(\overline{D})$.

By keeping in mind the identification of the classes $\mathcal{O}$ and $\mathscr{F}$ introduced in Section 1, let us recall some formulas from [2], which will be useful in the sequel. Let $\mathcal{T}$ be the operator that associates to each f in $\mathscr{F}$ the interior normal derivative $\frac{\partial u}{\partial \nu}$ — as function of the arclength s, which will be measured counterclockwise on $\partial\Omega$ and starting from ζ_b — of the solution of (1.1)-(1.2). We can define parametrically the values of $\mathcal{T}(f)(s)$, $s \in [0, |\partial\Omega|]$, by

$$(2.1) \qquad s = \int_0^\theta |f'(e^{it})| dt, \quad T(f) = \frac{1}{2\pi |f'(e^{i\theta})|}, \quad \theta \in [0, 2\pi].$$

Observe that $\mathcal{T}(f)$ is of class $C^{0,\alpha}$ and satisfies the compatibility conditions

$$\int_0^{|\partial\Omega|} \mathcal{T}(f)(s)ds = 1, \quad \mathcal{T}(f) > 0 \text{ on } [0, |\partial\Omega|].$$

From (2.1), it descends the relation

$$2\pi\mathcal{T}(f)(s(\theta))s'(\theta) = 1, \quad \theta \in [0, 2\pi],$$

which, once integrated, together with (2.1), gives

$$(2.2) \qquad s(\theta) = \Phi^{-1}(\theta), \quad |f'(e^{i\theta})| = \frac{1}{2\pi\mathcal{T}(f)(\Phi^{-1}(\theta))}, \quad \theta \in [0, 2\pi],$$

where Φ^{-1} is the inverse of the function Φ defined in (1.5).

3. Stability estimates

For $\Omega_j \in \mathcal{O}$, we fix $\zeta_j \in \partial\Omega_j$ and let f_j be the mapping in $\mathcal{F}$ (with $\zeta_b = \zeta_j$) corresponding to Ω_j $(j = 1, 2)$. From (1.4) we know that

$$(3.1) \qquad f_j'(z) = e^{i\gamma_j} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{1}{2\pi\mathcal{T}(f_j)(\Phi_j^{-1}(t))} dt \right\}, \quad z \in D,$$

for some $\gamma_j \in \mathbb{R}$, where Φ_j^{-1} is the inverse of the function Φ_j defined by (1.5) with f replaced by f_j $(j = 1, 2)$.

We are going to estimate how far the domains Ω_1 and Ω_2 are from one other (up to rotations), depending on an appropriate norm of the difference of the functions $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$.

3.1. A preliminary estimate.
All our estimates will be based on Theorem 3.1 below, where a bound of the norm

$$\|f_1 - f_2\|_{1,0,\partial D}$$

is given in terms of the Hölder norm of the difference between the composite functions $\mathcal{T}(f_1) \circ \Phi_1^{-1}$ and $\mathcal{T}(f_2) \circ \Phi_2^{-1}$, which are defined on ∂D and not on $\partial\Omega_1$ and $\partial\Omega_2$. Later on, we shall convert such a bound into estimates involving the functions $\mathcal{T}(f_j)$ $(j = 1, 2)$ only.

To this end, let us list here two estimates of Hölder seminorms which will be useful in the sequel. Let us define

$$(3.2) \qquad \psi_j = \mathcal{T}(f_j) \circ \Phi_j^{-1} \quad (j = 1, 2),$$

and

$$(3.3) \qquad h = \log\psi_1 - \log\psi_2.$$

If $\mathcal{T}(f_j) \in \mathcal{G}_0^{L_j}$, then

$$(3.4) \qquad [\psi_j]_{0,\alpha,[0,2\pi]} \leq \frac{[\mathcal{T}(f_j)]_{o,\alpha,[0,L_j]}}{(2\pi m)^\alpha} \leq \frac{M_0}{(2\pi m)^\alpha} \quad (j = 1, 2),$$

and

$$(3.5) \qquad [h]_{0,\alpha,[0,2\pi]} \leq C_1\|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} + C_2[\psi_1 - \psi_2]_{0,\alpha,[0,2\pi]},$$

where

$$C_1 = \frac{M_0^2}{(2\pi)^\alpha m^{\alpha+3}}, \quad C_2 = \frac{M_0}{m^2}.$$

These two estimates follow from the general fact that, if ξ and η are real-valued functions defined on intervals in $\mathbb{R}$, then

$$[\xi \circ \eta] \leq [\xi]_{0,\alpha}[\eta]_{0,1}^{\alpha},$$

and from some algebraic identities.

Theorem 3.1. *Given $\Omega_j \in \mathcal{O}$ ($j = 1, 2$), suppose that the arclength is measured counterclockwise on $\partial\Omega_j$ starting from $\zeta_j \in \partial\Omega_j$ and assume that f_j is the function in $\mathcal{F}$ (with $\zeta_b = \zeta_j$) corresponding to Ω_j ($j = 1, 2$).*

Let ψ_j be defined by (3.2) and suppose that $\mathcal{T}(f_j) \in \mathscr{G}_0^{L_j}$ ($j = 1, 2$). Then, up to rotations around ζ_o, we have that

$$(3.6) \qquad \|f_1 - f_2\|_{1,0,\partial D} \leq K\|\psi_1 - \psi_2\|_{0,\alpha,\partial D},$$

where K, whose expression can be deduced from the proof, is a constant depending on α, m and M_0.

Proof. Up to a rotation around ζ_o, we can assume that in (3.1) $\gamma_1 = \gamma_2 = \gamma$. Let us set

$$\beta_j(z) = \arg f_j'(z), \quad z \in D \quad (j = 1, 2).$$

It is clear that

$$
\begin{aligned}
|f_1'(z) - f_2'(z)| &= \left| |f_1'(z)|e^{i\beta_1(z)} - |f_2'(z)|e^{i\beta_2(z)} \right| \\
&\leq \left| |f_1'(z)| - |f_2'(z)| \right| + |f_2'(z)| \left| e^{i\beta_1(z)} - e^{i\beta_2(z)} \right|,
\end{aligned}
$$

and hence

$$(3.7) \qquad |f_1'(z) - f_2'(z)| \leq \left| |f_1'(z)| - |f_2'(z)| \right| + |f_2'(z)||\beta_1(z) - \beta_2(z)|,$$

since

$$\left| e^{i\beta_1(z)} - e^{i\beta_2(z)} \right| \leq |\beta_1(z) - \beta_2(z)|.$$

Thus, by keeping in mind (3.1) and writing $z = re^{i\theta}$, it turns out that

$$
\begin{aligned}
\beta_1(z) - \beta_2(z) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{2r\sin(\theta - t)}{1 + r^2 - 2r\cos(\theta - t)} h(t)dt \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{2r\sin t}{1 + r^2 - 2r\cos t}[h(\theta + t) - h(\theta - t)]dt,
\end{aligned}
$$

where h is defined as in (3.3). Since

$$0 \leq \frac{2r\sin t}{1 + r^2 - 2r\cos t} \leq \frac{1}{\tan\frac{t}{2}}, \quad t \in [0, \pi],$$

and $h \in C^{0,\alpha}[0, 2\pi]$, we get

$$(3.8) \qquad |\beta_1(z) - \beta_2(z)| \leq \frac{[h]_{0,\alpha,[0,2\pi]}}{2\pi}\int_0^{\pi} \frac{(2t)^{\alpha}}{\tan\frac{t}{2}}dt,$$

and the integral converges.

On the other hand,

$$|f_j'(z)| = \exp\left\{ \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t)} \log\frac{1}{2\pi\psi_j(t)}dt \right\}, \quad z \in D \quad (j = 1, 2);$$

thus,

$$|f_2'(z)| \leq \frac{1}{2\pi m}, \tag{3.9}$$

and

$$\||f_1'(z)| - |f_2'(z)|\| \leq \frac{1}{2\pi m^2}\|\psi_1 - \psi_2\|_{\infty,[0,2\pi]}, \tag{3.10}$$

since both ψ_1 and ψ_2 are bounded below by m.

From (3.7), (3.8), (3.9) and (3.10), we infer that

$$|f_1'(z) - f_2'(z)| \leq \frac{1}{2\pi m^2}\|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} + \frac{c_\alpha}{m}[h]_{0,\alpha,[0,2\pi]},$$

where

$$c_\alpha = \frac{2^\alpha}{4\pi^2}\int_0^\pi t^\alpha \cot\frac{t}{2}dt.$$

Notice that

$$|f_1(e^{i\theta}) - f_2(e^{i\theta})| = \left|\int_0^1 \frac{d}{dt}[f_1(te^{i\theta}) - f_2(te^{i\theta})]dt\right| \leq \|f_1' - f_2'\|_{\infty,\partial D};$$

in order to obtain (3.6), we write

$$\|f_1 - f_2\|_{\infty,\overline{D}} \leq \frac{1}{2\pi m^2}\|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} + \frac{c_\alpha}{m}[h]_{0,\alpha,[0,2\pi]}$$

and we estimate $[h]_{0,\alpha,[0,2\pi]}$ in terms of $\|\psi_1 - \psi_2\|_{0,\alpha,[0,2\pi]}$, by using (3.5). $\qquad\square$

3.2. **Stability near a disk.** As we pointed out in [2], the disk is the only domain whose Green's function has constant normal derivative on the boundary. More precisely, in our notations, the mappings

$$f_C(z) = \zeta_o + \frac{e^{i\gamma}}{2\pi C}z, \quad z \in \overline{D}, \tag{3.11}$$

with $\gamma \in \mathbb{R}$, are the only elements $\mathscr{F}$ such that $\mathcal{T}(f) = C$. The next result specifies how far from f_C is a mapping $f \in \mathscr{F}$ if $\mathcal{T}(f)$ is not constant.

Theorem 3.2. *Let f_C be given by (3.11) for some constants $C \in [m, M_0]$ and $\gamma \in \mathbb{R}$ and let Ω be in $\mathscr{O}$ with perimeter L.*

If $\mathcal{T}(f) \in \mathscr{G}_0^L$, then

$$\|f - f_C\|_{1,0,\partial D} \leq K\left[1 + \frac{1}{(2\pi m)^\alpha}\right]\|\mathcal{T}(f) - C\|_{0,\alpha,[0,L]},$$

where K is the constant of Theorem 3.1.

Proof. Theorem 3.1 gives that

$$\|f - f_C\|_{1,0,\partial D} \leq K\|\psi - C\|_{0,\alpha,[0,2\pi]},$$

where $\psi = \mathcal{T}(f) \circ \Phi^{-1}$; it remains to estimate the right-hand side of the latter inequality by the Hölder norm of $\mathcal{T}(f) - C$. This is readily achieved by observing that

$$\|\psi - C\|_{\infty,[0,2\pi]} = \|\mathcal{T}(f) - C\|_{\infty,[0,L]}$$

and, from (3.4), that

$$[\psi - C]_{0,\alpha,[0,2\pi]} \leq \frac{1}{(2\pi m)^\alpha}[\mathcal{T}(f) - C]_{0,\alpha,[0,L]}.$$

$\qquad\square$

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let f_C be defined as in (3.11); then $\rho = |f_C(e^{i\theta}) - \zeta_o|$ for every $\theta \in [0, 2\pi]$.

Now, notice that

$$R = \max_{0 \le \theta \le 2\pi} |f(e^{i\theta}) - \zeta_o|;$$

therefore, if $\theta_o \in [0, 2\pi]$ maximizes $|f(e^{i\theta}) - \zeta_o|$, we have:

$$R - \rho = |f(e^{i\theta_o}) - \zeta_o| - |f_C(e^{i\theta_o}) - \zeta_o| \le |f(e^{i\theta_o}) - f_C(e^{i\theta_o})|.$$

Thus, the conclusion plainly follows from Theorem 3.2. □

Remark 3.3. In order to compare this result with [1, Theorem 1] and [4, Theorem 1.2], we observe that the proof of Theorem 1.1 is straightforward also if we replace $B(\zeta_o, \rho)$ and $B(\zeta_o, R)$ by the largest disk contained in Ω and for the smallest disk containing Ω (not necessarily centered in ζ_o).

3.3. **Domains with same perimeter.** We want to estimate the right-hand side of (3.6) in terms of some suitable distance between the functions $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$. In this subsection, we shall start by considering the case of two domains with the same perimeter. Differently from Subsection 3.2, it seems that in this case we cannot avoid to require that $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$ are of class $C^{1,\alpha}$.

Theorem 3.4. *Given Ω_1, $\Omega_2 \in \mathcal{O}$, both with perimeter that equals L, let f_1 and f_2 be the conformal mappings in $\mathcal{F}$ corresponding to Ω_1 and Ω_2, respectively.*

If $\mathcal{T}(f_1)$, $\mathcal{T}(f_2) \in \mathcal{G}_1^L$, then, up to domains' rotations around ζ_o, we have that

$$(3.12) \qquad \|f_1 - f_2\|_{1,0,\partial D} \le K \left\{ \|\varphi_1 - \varphi_2\|_{\infty,[0,L]}^{\alpha} + \|\varphi_1' - \varphi_2'\|_{\infty,[0,L]} \right\},$$

where $\varphi_j = \mathcal{T}(f_j)$ ($j = 1, 2$) and the constant K depends on α, m, M_1 and L and can be deduced from the proof.

Proof. From (2.1) and (2.2), we have that

$$\frac{\theta}{2\pi} = \int_0^{\Phi_1^{-1}(\theta)} \varphi_1(\sigma) d\sigma = \int_0^{\Phi_2^{-1}(\theta)} \varphi_2(\sigma) d\sigma, \quad \theta \in [0, 2\pi],$$

and hence

$$\int_{\Phi_1^{-1}(\theta)}^{\Phi_2^{-1}(\theta)} \varphi_1(\sigma) d\sigma = \int_0^{\Phi_2^{-1}(\theta)} [\varphi_1(\sigma) - \varphi_2(\sigma)] d\sigma.$$

Thus,

$$(3.13) \qquad |\Phi_1^{-1}(\theta) - \Phi_2^{-1}(\theta)| \le \frac{L}{m} \|\varphi_1 - \varphi_2\|_{\infty,[0,L]}, \quad \theta \in [0, 2\pi],$$

since φ_1, $\varphi_2 \in \mathcal{G}_1^L$.

Let ψ_j be the functions defined in (3.2). We now estimate $\|\psi_1 - \psi_2\|_{\infty,[0,2\pi]}$ and $[\psi_1 - \psi_2]_{0,\alpha,[0,2\pi]}$. It is clear that

$$|\psi_1(\theta) - \psi_2(\theta)| \le$$
$$|\varphi_1(\Phi_1^{-1}(\theta)) - \varphi_1(\Phi_2^{-1}(\theta))| + |\varphi_1(\Phi_2^{-1}(\theta)) - \varphi_2(\Phi_2^{-1}(\theta))| \le$$
$$[\varphi_1]_{0,\alpha,[0,L]} |\Phi_1^{-1}(\theta) - \Phi_2^{-1}(\theta)|^{\alpha} + \|\varphi_1 - \varphi_2\|_{\infty,[0,L]},$$

and hence, from (3.13), we obtain the inequality

$$(3.14) \qquad \|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} \le \left[M_1 \left(\frac{L}{m} \right)^\alpha + (2M_1)^{1-\alpha} \right] \|\varphi_1 - \varphi_2\|_{\infty,[0,L]}^\alpha,$$

since $\varphi_1, \varphi_2 \in \mathscr{G}_1^L$.

Next, by Lagrange's Theorem, we have:

$$\begin{aligned}
|(\psi_1 - \psi_2)(\theta) - (\psi_1 - \psi_2)(\hat\theta)| &\le \|\psi_1' - \psi_2'\|_{\infty,[0,2\pi]} |\theta - \hat\theta| \\
&\le (2\pi)^{1-\alpha} \|\psi_1' - \psi_2'\|_{\infty,[0,2\pi]} |\theta - \hat\theta|^\alpha;
\end{aligned}$$

thus,

$$(3.15) \qquad [\psi_1 - \psi_2]_{0,\alpha,[0,2\pi]} \le (2\pi)^{1-\alpha} \|\psi_1' - \psi_2'\|_{\infty,[0,2\pi]}.$$

In order to estimate the right-hand side of the latter inequality, we notice that

$$\psi_j'(\theta) = \frac{\varphi_j'(\Phi_j^{-1}(\theta))}{2\pi\varphi_j(\Phi_j^{-1}(\theta))} \quad (j = 1, 2)$$

and, by setting $s_j = \Phi_j^{-1}(\theta)$, we write:

$$\begin{aligned}
2\pi|\psi_1'(\theta) - \psi_2'(\theta)| &\le \\
\frac{|\varphi_1'(s_1) - \varphi_1'(s_2)|}{\varphi_1(s_1)} &+ \left| \frac{1}{\varphi_1(s_1)} - \frac{1}{\varphi_2(s_2)} \right| \varphi_1'(s_2) + \frac{|\varphi_1'(s_2) - \varphi_2'(s_2)|}{\varphi_2(s_2)} \\
&\le \frac{M_1}{m} |s_1 - s_2|^\alpha + \frac{M_1}{m^2} \|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} + \frac{1}{m} \|\varphi_1' - \varphi_2'\|_{\infty,[0,L]},
\end{aligned}$$

since $\varphi_1, \varphi_2 \in \mathscr{G}_1^L$. By (3.13) and (3.14), we then obtain

$$(3.16) \quad 2\pi\|\psi_1' - \psi_2'\|_{\infty,[0,2\pi]} \le \frac{M_1}{m} \left(\frac{L}{m} \right)^\alpha \|\varphi_1 - \varphi_2\|_{\infty,[0,L]}^\alpha$$

$$+ \frac{M_1}{m^2} \left\{ M_1 \left(\frac{L}{m} \right)^\alpha + (2M_1)^{1-\alpha} \right\} \|\varphi_1 - \varphi_2\|_{\infty,[0,L]}^\alpha + \frac{1}{m} \|\varphi_1' - \varphi_2'\|_{\infty,[0,L]}.$$

Therefore, (3.12) easily follows from (3.14) and from (3.15) together with the latter inequality. $\qquad\square$

We recall that the *Hausdorff distance* between two compact subsets A and B of R^n is defined as

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\},$$

where

$$\rho(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

Proof of Theorem 1.2. As usual, let f_j be the mapping in $\mathscr{F}$ corresponding to Ω_j $(j = 1, 2)$. Thus,

$$\begin{aligned}
\rho(\Omega_1, \Omega_2) &= \sup_{\zeta_1 \in \Omega_1} \inf_{\zeta_2 \in \Omega_2} |\zeta_1 - \zeta_2| \\
&= \sup_{0 \le \theta_1 \le 2\pi} \inf_{0 \le \theta_2 \le 2\pi} |f_1(e^{i\theta_1}) - f_2(e^{i\theta_2})|,
\end{aligned}$$

and hence

$$\rho(\Omega_1, \Omega_2) \le \sup_{0 \le \theta_1 \le 2\pi} |f_1(e^{i\theta_1}) - f_2(e^{i\theta_1})| \le \|f_1 - f_2\|_{1,0,\partial D}.$$

The conclusion then follows from Theorem 3.4. $\qquad\square$

3.4. **Domains with different perimeters.** If Ω_1 and Ω_2 have different perimeters, say L_1 and L_2, the functions $\mathcal{T}(f_1)$ and $\mathcal{T}(f_2)$ are defined on different intervals, $[0, L_1]$ and $[0, L_2]$, and we cannot compare their values directly. Thus, we rescale them: if $\varphi_j = \mathcal{T}(f_j)$, we set

$$(3.17) \qquad \hat{\varphi}_j(s) = \varphi_j\left(\frac{L_j}{L}s\right), \quad s \in [0, L], \quad \text{where} \quad L = \frac{L_1 + L_2}{2} \quad (j = 1, 2).$$

The functions $\hat{\varphi}_j$ are now defined on a common interval.

Theorem 3.5. *Let Ω_1 and Ω_2 be domains in $\mathscr{O}$ with perimeters L_1 and L_2, respectively, such that*

$$0 < p \le L_1, L_2 \le P$$

for some constants p and P. Let f_1, $f_2 \in \mathscr{F}$ be as usual and assume that (3.17) holds.

If $\varphi_j = \mathcal{T}(f_j) \in \mathscr{G}_1^{L_j}$ $(j = 1, 2)$, then, up to domains' rotations around ζ_o, we have that

$$\|f_1 - f_2\|_{1,0,\partial D} \le K\left\{\left(\frac{|L_1 - L_2|}{P} + \frac{\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]}}{M_1}\right)^\alpha + \|\hat{\varphi}_1' - \hat{\varphi}_2'\|_{\infty,[0,L]}\right\},$$

where the constant K depends on α, m, M_1, p and P and its expression can be deduced from the proof.

Proof. We preliminary notice that

$$(3.18) \qquad \|\hat{\varphi}_j\|_{\infty,[0,L]} = \|\varphi_j\|_{\infty,[0,L_j]}, \quad [\hat{\varphi}_j]_{0,\alpha,[0,L]} = \left(\frac{L_j}{L}\right)^\alpha [\varphi_j]_{0,\alpha,[0,L_j]}$$

and

$$(3.19) \qquad \|\hat{\varphi}_j'\|_{\infty,[0,L]} = \frac{L_j}{L}\|\varphi_j'\|_{\infty,[0,L_j]}, \quad [\hat{\varphi}_j']_{0,\alpha,[0,L]} = \left(\frac{L_j}{L}\right)^{\alpha+1}[\varphi_j']_{0,\alpha,[0,L_j]}.$$

The proof will proceed as the one of Theorem 3.4, with some variations. The following notations and formulas will be useful:

$$\hat{s}_j(\theta) = \frac{L}{L_j}\Phi_j^{-1}(\theta), \quad \psi_j(\theta) = \hat{\varphi}_j(\hat{s}_j(\theta)) \quad (j = 1, 2).$$

Since

$$\frac{\theta}{2\pi} = \int_0^{\Phi_j^{-1}(\theta)} \varphi_j(\sigma)d\sigma = \frac{L_j}{L}\int_0^{\hat{s}_j(\theta)} \hat{\varphi}_j(\sigma)d\sigma, \quad \theta \in [0, 2\pi] \quad (j = 1, 2),$$

we derive an estimate similar to (3.13):

$$\frac{L_1}{L}m|\hat{s}_1(\theta) - \hat{s}_2(\theta)| \le |L_1 - L_2|\,\|\hat{\varphi}_1\|_{\infty,[0,L]} + L_2\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]};$$

thus,

$$|\hat{s}_1(\theta) - \hat{s}_2(\theta)| \le \frac{M_1}{m}\frac{P^2}{p}\left\{\frac{|L_1 - L_2|}{P} + \frac{\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]}}{M_1}\right\}, \quad \theta \in [0, 2\pi].$$

From now on, we can proceed, by using (3.18) and (3.19), as in the proof of Theorem (3.4), with φ_j and s_j replaced by $\hat{\varphi}_j$ and $\hat{s}_j$, respectively: (3.14) changes into

$$(3.20) \qquad \|\psi_1 - \psi_2\|_{\infty,[0,2\pi]} \le K_1\left\{\frac{|L_1 - L_2|}{P} + \frac{\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]}}{M_1}\right\}^\alpha,$$

where

$$K_1 = M_1 \left[\left(\frac{M_1}{m} \frac{P^3}{p^2} \right)^{\alpha} + 4^{1-\alpha} \right];$$

(3.16) becomes

$$2\pi \|\psi_1' - \psi_2'\|_{\infty,[0,2\pi]} \le K_2 \left\{ \frac{|L_1 - L_2|}{P} + \frac{\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]}}{M_1} \right\}^{\alpha} + \frac{P}{pm} \|\hat{\varphi}_1' - \hat{\varphi}_2'\|_{\infty,[0,L]}$$

where K_2, easy computable, is still a constant depending on α, m, M_1, p and P. The conclusion then follows from (3.20), (3.15) and the latter inequality. $\qquad\square$

By the same arguments used for the proof of Theorem 1.2, Theorem 3.5 yields the following corollary.

Corollary 3.6. *Under the same assumptions of Theorem 3.5, it holds that*

$$d_H(\Omega_1, \Omega_2) \le K \left\{ \left(\|\hat{\varphi}_1 - \hat{\varphi}_2\|_{\infty,[0,L]} + |L_1 - L_2| \right)^{\alpha} + \|\hat{\varphi}_2' - \hat{\varphi}_1'\|_{\infty,[0,L]} \right\},$$

where K is a constant depending on α, m, M_1, on p and P.

References

[1] A. Aftalion, J. Busca, W. Reichel, *Approximate radial symmetry for overdetermined boundary value problems*, Adv. Differential Equations **4** n. 6, 1999, 907–932.

[2] V. Agostiniani, R. Magnanini, *Symmetries in an overdetermined problem for the Green's function*, preprint, arXiv: 0912.1935, to appear on Discrete Contin. Dyn. Syst. Ser. S.

[3] L. V. Ahlfors, "Complex Analysis," McGraw-Hill, New York, 1953.

[4] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, *On the stability of the Serrin problem*, J. Differential Equations **245** n. 6, 2008, 1566–1583.

[5] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, *Stability of radial symmetry for a Monge-Ampère overdetermined problem*, Ann. Mat. Pura Appl. (4) **188** n. 3, 2009, 445–453.

[6] A. Cianchi, *A quantitative Sobolev inequality in BV*, J. Funct. Anal. **237** n. 2, 2006, 466–481.

[7] G. M. Goluzin, "Geometric theory of functions of a complex variable," American Math. Soc., Providence, 1969.

[8] B. Gustafsson and A. Vasil'ev, "Conformal and Potential Analysis in Hele-Shaw Cells," Birkhäuser Verlag, Basel, 2006.

[9] O. D. Kellogg, "Foundations of Potential Theory," Springer-Verlag, Berlin, 1967.

[10] F. Maggi, *Some methods for studying stability in isoperimetric inequality*, Bull. Amer. Math. Soc. (N.S.) **45** n. 3, 2008, 367–408.

[11] A. I. Markushevich, "Theory of functions of a complex variable," Prentice-Hall, Englewood Cliffs, 1965.

[12] P. J. McKenna, W. Reichel, *Gidas-Ni-Niremberg results for finite difference equations: estimates of approximate symmetry*, J. Math. Anal. Appl. **334** n. 1, 2007, 206–222.

[13] E. Rosset, *An approximate Gidas-Ni-Nirenberg theorem*, Math. Methods Appl. Sci. **17** n.13, 1994, 1045–1052.

[14] M. Sakai, "Quadrature Domains," Springer-Verlag, Berlin, 1982.

[15] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43**, 1971, 304–318.

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