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Mild solutions to the dynamic programming equation for stochastic optimal control problems ☆

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ABSTRACT

We show via the nonlinear semigroup theory in $L^1(\mathbb{R})$ that the 1-D dynamic programming equation associated with a stochastic optimal control problem with multiplicative noise has a unique mild solution in a sense to be made precise.

1. Introduction

Consider the following stochastic optimal control problem

$$\underline{\text{Minimize}} \quad \mathbb{E}\left\{\int_0^T \Big(g\big(X(t)\big)+h\big(u(t)\big)\Big)\,dt+g_0\big(X(T)\big)\right\}, \tag{1}$$

subject to $u \in \mathcal{U}$ and to state equation

$$\begin{cases} dX = f(X)\,dt + \sqrt{u}\,\sigma(X)\,dW, & \text{for } t \in (0,T) \\ X(0) = X_0 \end{cases} \tag{2}$$

where $\mathcal{U}$ is the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $u : (0,T) \to \mathbb{R}^+ = [0,+\infty]$ and $W : \mathbb{R} \to \mathbb{R}$ is an 1-D Wiener process in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, provided the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Here $X_0 \in \mathbb{R}$, while $X : [0,T] \to \mathbb{R}$ is the strong solution to (2). We would like to underline that the studied optimization problem is related to the so called stochastic volatility models, used in the financial framework, whose relevance has raised exponentially during last years. In fact such models, contrarily to the constant volatility ones as, e.g., the standard Black and Scholes approach, the Vasicek interest rate model, or the Cox–Ross–Rubinstein model, allow to consider the more realistic situation of volatility levels changing in time. As an example, the latter is the case of the Heston model, see Heston (1993), where the variance is assumed to be a stochastic process following a Cox–Ingersoll–Ross (CIR) dynamic, see Cordoni and Di Persio (2013) or Cox, Ingersoll and Ross (1985) and references therein for more recent related techniques, as well as the case of the Constant Elasticity of Variance (CEV) model, see Cox (1975), where the volatility is expressed by a power of the underlying level, which is often referred as a local stochastic volatility model. Other interesting examples, which is the object of our ongoing research particularly from the numerical point of view, include the Stochastic Alpha, Beta, Rho (SABR) model, see, e.g., Hagan, Lesniewski, and Woodward (2015), and models which are used to estimate the stochastic volatility by exploiting directly markets data, as happens using the GARCH approach and its variants. Within latter frameworks and due to several macroeconomic crises that have affected different (type of) financial markets worldwide, governments decided to become *active players of the game*, as, e.g., in the recent case of the *Volatility Control Mechanism* (VCM) established for the securities, resp. for the derivatives, market established in August 2016, resp. in January 2017, within the Hong Kong Stock Exchange (HKEX) framework, see, e.g., Stein (2006) and Stein (2012) and references therein for other applications and examples. It should be said however that problems of the form (1)–(2) are relevant in other applications as well.

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Hypotheses:

(i) $h:\mathbb{R}\to\mathbb{R}$ is convex, continuous and $h(u)\ge\alpha_1|u|^2,\forall u\in\mathbb{R}$, for some $\alpha_1>0$.
(ii) $f\in C_b^2(\mathbb{R}), f''\in L^1(\mathbb{R}), g, g_0\in W^{2,\infty}(\mathbb{R})$.
(iii) $\sigma\in C_b^1(\mathbb{R})$, and

$$|\sigma(x)|\ge\rho>0,\quad \forall x\in\mathbb{R}. \tag{3}$$

In the following H^* is the Legendre conjugate of function

$$H(u)=h(u)+I_{[0,\infty)}(u)=\begin{cases}h(u) & \text{if } u\ge 0\,,\\ +\infty & \text{otherwise.}\end{cases}$$

Namely,

$$H^*(p)=\sup\{p\,u-H(u)\;:\;u\in\mathbb{R}\},\quad \forall p\in\mathbb{R}. \tag{4}$$

We have $\partial H^*(p)=(\partial h+N_{[0,\infty)})^{-1}p$, where ∂ is the subdifferential symbol, and $N_{[0,\infty)}$ is the normal cone to $[0,\infty)$. This yields

$$0\le\partial H^*(p)\le C(|p|+1)\,,\ \forall p\in\mathbb{R}\,. \tag{5}$$

We denote also by j the potential of H^*, that is

$$j(r)=\int_0^r H^*(p)\,dp,\quad \forall r\in\mathbb{R}.$$

The dynamic programming equation corresponding to the stochastic optimal control problem (1) is given by (see, e.g., Fleming & Rishel, 2012; Øksendal, 2003),

$$\begin{cases}\varphi_t(t,x)+\min\limits_u\left\{\frac{1}{2}\sigma^2\varphi_{xx}(t,x)\,u+H(u)\right\}\\ \qquad +f(x)\,\varphi_x(t,x)+g(x)=0,\quad \forall t\in[0,T],x\in\mathbb{R}\\ \varphi(T,x)=g_0(x),\quad x\in\mathbb{R},\end{cases}$$

or equivalently

$$\begin{cases}\varphi_t(t,x)-H^*\big(-\frac{1}{2}\sigma^2\varphi_{xx}(t,x)\big)+f(x)\,\varphi_x(t,x)\\ \qquad\qquad +g(x)=0,\quad \forall(t,x)\in[0,T]\times\mathbb{R}\\ \varphi(T,x)=g_0(x),\quad x\in\mathbb{R}\,.\end{cases} \tag{6}$$

Moreover, if φ is a smooth solution to (1) the associated feedback controller

$$u(t)=\arg\min_u\left\{\frac{1}{2}\,\sigma^2\,\varphi_{xx}\big(t,X(t)\big)\,u+H(u)\right\}, \tag{7}$$

is optimal for problem (1).

We set $\psi=\varphi_x$ and replace Eq. (6) by

$$\begin{cases}\psi_t(t,x)-H^*\big(-\frac{1}{2}\sigma^2\,\psi_x(t,x)\big)_x+(f(x)\,\psi(t,x))_x\\ \qquad\qquad +g'(x)=0,\quad \forall(t,x)\in[0,T]\times\mathbb{R}\\ \psi(T,x)=g_0'(x),\quad x\in\mathbb{R}\,.\end{cases} \tag{8}$$

Up to our knowledge, in literature the rigorous treatment of existence theory for equations of this form has been shown so far within the theory of viscosity solutions only. (See, e.g., Crandall, Ishii, & Lions, 1992.) However, the known existence results for viscosity solutions are not directly applicable in the present case. Here we shall exploit a different approach, namely we use a suitable transformation aiming at reducing (8) to a one dimensional Fokker–Planck equation which is then treated as a nonlinear Cauchy problem in $L^1(\mathbb{R})$. As regards the non-degenerate hypothesis (3) it will be later on dispensed by assuming more regularity on function σ. (See Section 4.)

1.1. Notation and basic results

We shall use the standard notation for functional spaces on $\mathbb{R}$. In particular $C_b^k(\mathbb{R})$ is the space of functions $y:\mathbb{R}\to\mathbb{R}$, differentiable of order k and with bounded derivatives until order k. By $L^p(\mathbb{R})$, $1\le p\le\infty$, we denote the classical space of Lebesgue-measurable p-integrable functions on $\mathbb{R}$ with the norm $\|\cdot\|_p$ and by $H^k(\mathbb{R}^n)$, $W^{k,p}(\mathbb{R}^n)$, $k=1,2$, the standard Sobolev spaces on $\mathbb{R}^n$, $n=1,2$. Denote by $\langle\cdot,\cdot\rangle_2$ the scalar product of $L^2(\mathbb{R})$. We set also $y_x=y'=\partial y/\partial x$, $y_t=\partial y/\partial t$, $y_{xx}=\partial^2y/\partial x^2$, for $x\in\mathbb{R}$. By $\mathcal{D}'(\mathbb{R}^n)$ we denote the space of Schwartz distributions on $\mathbb{R}^n$.

Definition 1.1 (*Accretive Operator*). Given a Banach space X, a nonlinear operator A from X to itself, with domain $D(A)$, is said to be *accretive* if $\forall u_i\in D(A),\forall v_i\in A\,u_i$, $i=1,2$, there exists $\eta\in J(u_1-u_2)$ such that

$${}_X\langle v_1-v_2,\eta\rangle_{X'}\ge 0\,, \tag{9}$$

where X' is the dual space of X, ${}_X\langle\cdot,\cdot\rangle_{X'}$ is the duality pairing and $J:X\to X'$ is the *duality mapping* of X. (See, e.g., Barbu, 2010.)

An accretive operator A is said to be *m-accretive* if $R(\lambda I+A)=X$ for all (equivalently some) $\lambda>0$, while it is said to be *quasi-m-accretive* if there is $\lambda_0\in\mathbb{R}$ such that $\lambda_0 I+A$ is m-accretive.

We refer to Barbu (2010) for basic results on m-accretive operators in Banach spaces and the corresponding associated Cauchy problem.

2. Existence results

We set

$$y(t,x)=-\psi_x(T-t,x),\quad \forall t\in[0,T],x\in\mathbb{R}, \tag{10}$$

and we rewrite Eq. (8) as

$$\begin{cases}y_t(t,x)-\Big(H^*\big(\frac{\sigma^2}{2}\,y(t,x)\big)\Big)_{xx}-f''(x)\psi(T-t,x)\\ \qquad -2f'(x)\psi_x(T-t,x)-f(x)\psi_{xx}(T-t,x)=-g''(x),\\ \qquad\qquad\qquad \text{in }(0,T)\times\mathbb{R}\\ y(0,x)=-g_0''(x),\quad x\in\mathbb{R}.\end{cases} \tag{11}$$

We set

$$\Phi(z)(x)=\int_{-\infty}^x z(\xi)\,d\xi\,,\quad z\in L^1(\mathbb{R}). \tag{12}$$

Then by (10) we have

$$\psi(t,x)=-\Phi\big(y(T-t,x)\big)\,,\quad \forall t\in[0,T]. \tag{13}$$

Setting

$$B\,y=-f''\Phi(y)-2f'y,\quad \forall y\in L^1(\mathbb{R})\,, \tag{14}$$

and taking into account that $f'\in L^\infty(\mathbb{R}), f''\in L^1(\mathbb{R})$, we obtain for the operator B the estimate

$$\|B\,y\|_1\le C\,\|y\|_1,\quad \forall y\in L^1(\mathbb{R})\,. \tag{15}$$

Therefore Eq. (11) can be rewritten as follows

$$\begin{cases}y_t-\Big(H^*\big(\frac{\sigma^2}{2}\,y\big)\Big)_{xx}-f\,y_x+B\,y=g_1,\quad \text{in }[0,T]\times\mathbb{R}\\ y(0)=y_0\in L^1(\mathbb{R})\end{cases}, \tag{16}$$

where $y_0=-g_0''$ and $g_1=-g''$ in $\mathcal{D}'(\mathbb{R})$.

Definition 2.1. The function $y:[0,T]\times\mathbb{R}\to\mathbb{R}$ is said to be a *mild* solution to Eq. (16) if $y\in C([0,T];L^1(\mathbb{R}))$ and

$$y(t)=\lim_{\epsilon\to 0}y_\epsilon(t)\text{ in }L^1(\mathbb{R}),\quad \forall t\in[0,T]\,, \tag{17}$$

$$y_\epsilon(t)=y_\epsilon^i,\ \text{for } t\in[i\epsilon,(i+1)\epsilon],\ i=0,1,\dots,\left[\frac{T}{\epsilon}\right]-1=\left[\frac{T}{\epsilon}\right], \tag{18}$$

$$\frac{1}{\epsilon}(y_\epsilon^{i+1}-y_\epsilon^i)-\Big(H^*(\frac{\sigma^2}{2}y_\epsilon^{i+1})\Big)''$$
$$-f(y_\epsilon^{i+1})'+By_\epsilon^{i+1}=g_1,\quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{19}$$

$y_\epsilon^0=y_0,\ y_\epsilon^i\in L^1(\mathbb{R}),\ i=0,1,\ldots,N.$

See Barbu (2010, p. 127) for general definition of mild solution to nonlinear Cauchy problem in Banach space.

We have

Theorem 2.2. *Under hypotheses (i)–(iii) Eq. (11) has a unique mild solution y. Assume further that* $j(\frac{\sigma^2}{2}y_0)\in L^1(\mathbb{R})$. *Then* $j(\frac{\sigma^2}{2}y)\in L^\infty([0,T];L^1(\mathbb{R}))$ *and* $\Big(H^*(\frac{\sigma^2}{2}y)\Big)_x\in L^2([0,T]\times\mathbb{R})$.

Theorem 2.2 will be proven by using the standard existence theory for the Cauchy problem in Banach spaces with nonlinear quasi-*m*-accretive operators. Now taking into account that for $y\in C([0,T];L^1(\mathbb{R}))$ Eq. (12) uniquely defines the function $\psi\in C([0,T];W^{1,\infty}(\mathbb{R}))$, by Theorem 2.2 we obtain the following existence result for the dynamic programming Eq. (1).

Theorem 2.3. *Under hypothesis (i)–(iii) there is a unique mild solution*

$$\psi\in C\big([0,T];W^{1,\infty}(\mathbb{R})\big) \tag{20}$$

to Eq. (8). Moreover, if $j(-\frac{\sigma^2}{2}g_0'')\in L^1(\mathbb{R})$ *and* $j(\lambda u)\le c_\lambda j(u)$, $\forall u\in\mathbb{R},\lambda>0$, *then* $H^*\big(-\frac{\sigma^2}{2}\psi_x(T-t,x)\big)\in L^2([0,T]\times\mathbb{R})$.

According to Definition 2.1 and (13), by *mild* solution ψ to Eq. (8), we mean a function $\psi\in C([0,T];W^{1,\infty}(\mathbb{R}))$ defined by

$$\psi(t)=\lim_{\epsilon\to 0}\psi_\epsilon(t)\text{ in } W^{1,\infty}(\mathbb{R}),\quad \forall t\in[0,T], \tag{21}$$

$$\psi_\epsilon(t)=-\Phi(y_\epsilon^i),\quad t\in[T-(i+1)\epsilon,T-i\epsilon], \tag{22}$$

for $i=0,1,\ldots,N=\left[\frac{T}{\epsilon}\right]$ and $\{y_\epsilon^i\}$ is the solution to (19).

In particular, the *mild* solution ψ to Eq. (8) is in $W^{1,\infty}(\mathbb{R})$ for each $t\in[0,T]$. Therefore, the feedback controller (7) where $\varphi_{xx}=\psi_x$ is well defined on $[0,T]$.

Remark 2.4. The principal advantage of Theorem 2.2 compared with standard existence results expressed in terms of viscosity solutions is the regularity of ψ and the fact that the optimal feedback controller can be computed explicitly by the finite difference scheme (21)–(22). This will be treated in a forthcoming paper.

3. Proof of Theorem 2.2

The idea is to write equation (16) as a Cauchy problem of the form

$$\begin{cases}\dfrac{dy}{dt}+Ay+By=g_1, & \text{in } [0,T]\\ y(0)=y_0\end{cases}, \tag{23}$$

in the space $L^1(\mathbb{R})$, where A is a suitable nonlinear quasi-*m*-accretive operator. The operator $A:D(A)\subset L^1(\mathbb{R})\to L^1(\mathbb{R})$ is defined as follows

$$Ay=-\Big(H^*(\frac{\sigma^2}{2}y)\Big)''-fy'\quad \text{in } \mathcal{D}'(\mathbb{R}),\ \forall y\in D(A), \tag{24}$$

$$D(A)=\Big\{y\in L^1(\mathbb{R})\,:\,Ay\in L^1(\mathbb{R})\Big\},$$

where the derivatives are taken in the $\mathcal{D}'(\mathbb{R})$ sense.

Lemma 3.1. *For each* $\eta\in L^1(\mathbb{R})$ *and* $\lambda\ge\lambda_0=\|f'\|_\infty$ *there exists a unique solution* $y=y(\eta)$ *to equation*

$$\lambda y+Ay=\eta. \tag{25}$$

Moreover, it holds

$$\|y(\eta)-y(\bar\eta)\|_1\le(\lambda-\lambda_0)^{-1}\|\eta-\bar\eta\|_1, \tag{26}$$

$\forall\eta,\bar\eta\in L^1(\mathbb{R}),\lambda>\lambda_0$, *hence A turns to be quasi-m-accretive in* $L^1(\mathbb{R})$.

Proof of Lemma 3.1. Assume first that $\eta\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. For each $\nu>0$ consider the equation

$$\lambda y-\nu y''-\Big(H^*(\frac{\sigma^2}{2}y)\Big)''+\nu H^*(\frac{\sigma^2}{2}y)-fy'=\eta, \tag{27}$$

in $\mathcal{D}'(\mathbb{R})$. Equivalently,

$$(\lambda-\nu^2)\Big(\nu I-\frac{d^2}{dx^2}\Big)^{-1}y+H^*(\frac{\sigma^2}{2}y)+\nu y$$
$$-\Big(\nu I-\frac{d^2}{dx^2}\Big)^{-1}(fy')=\Big(\nu I-\frac{d^2}{dx^2}\Big)^{-1}\eta, \tag{28}$$

and $z=\Big(\nu I-\frac{d^2}{dx^2}\Big)^{-1}y$ is defined by equation

$$\nu z-z''=y,\quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{29}$$

We shall get the solution y to (25) by proving first the existence of (28) for all $\nu>0$ and letting $\nu\to 0$. Note that by Hypothesis (ii) the operator $\Gamma y=(\lambda-\nu^2)\big(\nu I-\frac{d^2}{dx^2}\big)^{-1}y-\big(\nu I-\frac{d^2}{dx^2}\big)^{-1}(fy')+\nu y$ is linear continuous in $L^2(\mathbb{R})$ and by (29) we have that

$$\langle z,y\rangle_2=\nu\|z\|_2^2+\|z'\|_2^2, \tag{30}$$

$$\begin{aligned}-\Big\langle\Big(\nu I-\frac{d^2}{dx^2}\Big)^{-1}(fy'),y\Big\rangle_2&=-\langle(fy)'-f'y,z\rangle_2\\&=\langle y,f'z+fz'\rangle_2\\&\le\|f'\|_\infty\|y\|_2\|z\|_2\\&\quad+\|f\|_\infty\|y\|_2\|z'\|_2.\end{aligned} \tag{31}$$

Here $\|\cdot\|_2$ and $\langle\cdot,\cdot\rangle_2$ are the norm and the scalar product in $L^2(\mathbb{R})$, respectively, and by $\|\cdot\|_p$, $1\le p\le\infty$ we denote the norm of $L^p(\mathbb{R})$. We note that Hypothesis (i) and Eq. (4) imply that the function H^* is continuous, monotonically non-decreasing, and

$$H^*(0)=0,\quad 0\le H^*(v)\le C_1v^2,\quad \forall v\in\mathbb{R}. \tag{32}$$

Furthermore, by (29)–(31), we have

$$\begin{aligned}\langle\Gamma y,y\rangle_2&=\nu\|y\|_2^2+(\lambda-\nu^2)\langle y,z\rangle_2-\langle fy',z\rangle_2\\&\ge\nu\|y\|_2^2+(\lambda-\nu^2)(\nu\|z\|_2^2+\|z'\|_2^2)\\&\quad-\|y\|_2(\|f'\|_\infty\|z\|_2+\|f\|_\infty\|z'\|_2)\\&\ge\nu\|y\|_2^2+(\lambda-\nu^2)(\nu\|z\|_2^2+\|z'\|_2^2)\\&\quad-C(f)\|y\|_2(\|z\|_2+\|z'\|_2).\end{aligned}$$

The latter yields

$$\langle\Gamma y,y\rangle_2\ge\frac{\nu}{2}\|y\|_2^2,\quad \lambda\ge C\left(\frac{1}{\nu}+\nu^2\right),\forall\nu>0, \tag{33}$$

where C is dependent on ν. By assumption (3) we have that the operator $y\to\mathcal{H}(y)\equiv H^*\big(\frac{\sigma^2}{2}y\big)$ is maximal monotone in $L^2(\mathbb{R})$, hence, by (33), Γ is maximal monotone and coercive, i.e. positively definite, therefore we have

$$R(\Gamma+\mathcal{H})=L^2(\mathbb{R}),$$

for $\lambda \geq \lambda^* = C(\frac{1}{\nu}+\nu^2)$. Consequently, for each $\nu > 0$ and $\lambda \geq \lambda^*$, Eq. (28) (equivalently Eq. (27)) has a unique solution $y=y_{\lambda,\nu}\in L^2(\mathbb{R})$, with $H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\in L^2(\mathbb{R})$.

We have also $z_{\lambda,\nu}-z''_{\lambda,\nu}\in L^2(\mathbb{R})$, so that $z_{\lambda,\nu}=\left(\nu I-\frac{d^2}{dx^2}\right)^{-1} y_{\lambda,\nu}\in H^2(\mathbb{R})$.

Note that by (30), $\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}(fy')\in H^1(\mathbb{R})$. Hence by (28) we have $\nu y_{\lambda,\nu}+H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\in H^1(\mathbb{R})$ and since $(\alpha I+H^*)^{-1}\in \mathrm{Lip}(\mathbb{R})$ $\forall\alpha>0$, we infer that $y_{\lambda,\nu}\in H^1(\mathbb{R})$.

It is worth to mention that by (27), we have

$$\lambda\left\|y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta)\right\|_1\leq \left\|f'\right\|_\infty\left\|y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta)\right\|_1+\|\eta-\bar\eta\|_1$$

$\forall\eta,\bar\eta\in L^1(\mathbb{R})$, so that

$$\left\|y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta)\right\|_1\leq\frac{1}{\lambda-\lambda_0}\|\eta-\bar\eta\|_1\,,\quad\forall\eta,\bar\eta\in L^1(\mathbb{R}),\tag{34}$$

for $\lambda\geq\max(\lambda_0,\lambda^*)$ and where $\lambda_0=\left\|f'\right\|_\infty$. To get (34), we simply multiply the equation

$$\begin{aligned}&\lambda(y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta))-\nu(y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta))''\\&+\nu\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\eta)\right)-H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\bar\eta)\right)\right)\\&-\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\eta)\right)-H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\bar\eta)\right)\right)''\\&+f(y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta))'=\eta-\bar\eta\end{aligned}$$

by $\zeta\in L^\infty(\mathbb{R})$,

$$\begin{aligned}\zeta&\in\mathrm{sgn}(y_{\lambda,\nu}(\eta)-y_{\lambda,\nu}(\bar\eta))\\&=\mathrm{sgn}\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\eta)\right)-H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}(\bar\eta)\right)\right),\end{aligned}$$

where $\mathrm{sgn}\,r=\frac{r}{|r|}$ for $r\neq0$, $\mathrm{sgn}\,0=[-1,1]$ and we integrate on $\mathbb{R}$, taking into account that

$$-\int_{\mathbb{R}}y''\,\mathrm{sgn}\,y\,dx\geq0\,,\quad\forall y\in H^1(\mathbb{R}),$$

$$\int_{\mathbb{R}}fy'\,\mathrm{sgn}\,y\,dx=\int_{\mathbb{R}}f|y|'dx=-\int_{\mathbb{R}}f'\,|y|\,dy.$$

For a rigorous proof of these relations we replace $\mathrm{sgn}\,y$ by $\chi_\delta(y)$, where χ_δ is a smooth approximation of signum function, while $\delta\to0$, see , e.g., Barbu (2010), p. 115. If $\eta\in L^1(\mathbb{R})$ and $\{\eta_n\}_{n=1}^\infty\subset L^1(\mathbb{R})\cap L^2(\mathbb{R})$ is strongly convergent to $\eta\in L^1(\mathbb{R})$, we can proceed as above to obtain for the corresponding solution y_n to (27) the estimate (34), namely, for all $\lambda>\max(\lambda^*,\lambda_0)$

$$\|y_n-y_m\|_1\leq(\lambda-\lambda_0)^{-1}\|\eta_n-\eta_m\|_1\,.$$

Hence there exists $y\in L^1(\mathbb{R})$ such that

$$y_n\to y\quad\text{in } L^1(\mathbb{R})\text{ as } n\to\infty\,.\tag{35}$$

By (28), we have

$$\begin{aligned}&(\lambda-\nu^2)\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}y_n+H^*\left(\frac{\sigma^2}{2}y_n\right)+\nu y_n\\&\quad-\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}(fy_n')=\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}\eta_n\,.\end{aligned}\tag{36}$$

By (12) and (29), we have

$$\|z_n\|_{W^{1,\infty}(\mathbb{R})}\leq\|\nu z_n-y_n\|_1\leq(\nu+1)\|y_n\|_1\,.\tag{37}$$

Let $\theta_n:=\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}(fy_n')$,
that is $\nu\theta_n-\theta_n''=fy_n'=(fy_n)'-f'y_n$ in $\mathcal{D}'(\mathbb{R}^n)$. Equivalently

$$\begin{aligned}&\nu\left(\theta_n(x)+\int_0^x fy_n d\xi\right)-\left(\theta_n(x)+\int_0^x fy_n d\xi\right)''\\&=\nu\int_0^x fy_n d\xi-f'y_n\,.\end{aligned}\tag{38}$$

By accretivity of operator $z\to-z''$ in $L^1(\mathbb{R})$, this yields

$$\begin{aligned}&\left\|\nu\theta_n+\nu\int_0^x fy_n d\xi\right\|_1\leq\nu\left\|\int_0^x fy_n d\xi\right\|_1+\|f'y_n\|_1\,,\\&\leq\nu\|f\|_\infty\|y_n\|_1+\|f'\|_\infty\|y_n\|_1\end{aligned}\tag{39}$$

and then

$$\nu\|\theta_n\|_1\leq\left((\nu+1)\|f\|_\infty\|y_n\|_1+\|f'\|_\infty\right)\|y_n\|_1.$$

On the other hand, by (38), we have

$$\begin{aligned}&\left\|\theta_n+\int_0^x fy_n d\xi\right\|_{W^{1,\infty}(\mathbb{R})}\leq\left\|\nu\theta_n+\nu\int_0^x fy_n d\xi\right\|_1\\&+\left\|\nu\int_0^x fy_n d\xi-f'y_n\right\|_1\leq\nu\|\theta\|_1\\&+\left(2\nu\|f\|_\infty+\|f'\|_\infty\right)\|y\|_1\,.\end{aligned}\tag{40}$$

Hence (see e.g. Benilan, Brezis, and Crandall (1975)),

$$\|\theta_n\|_{W^{1,\infty}(\mathbb{R})}\leq\left((3\nu+1)\|f\|_\infty+2\|f'\|_\infty\right)\|y_n\|_1\,.$$

This yields

$$\left\|\left(\nu I-\frac{d^2}{dx^2}\right)^{-1}(fy_n')\right\|_\infty\leq C\,\|y_n\|_1\leq\frac{C_1}{\lambda-\lambda_0}\|\eta_n\|_1\,,\tag{41}$$

and therefore, by (36), we derive the estimate

$$\left\|H^*\left(\frac{\sigma^2}{2}y_n\right)+\nu y_n\right\|_\infty\leq C\,\|y_n\|_1\leq\frac{C_1}{\lambda-\lambda_0}\|\eta_n\|_1\,.$$

Since, by hypothesis (i) $H^*(v)v\geq0$, $\forall v\in\mathbb{R}$, the latter implies that

$$\left\|H^*\left(\frac{\sigma^2}{2}y_n\right)\right\|_\infty+\nu\,\|y_n\|_\infty\leq\frac{C_1}{\lambda-\lambda_0}\|\eta_n\|_1\,,\quad\forall n\,,\tag{42}$$

where C_1 is still independent of n as well as of ν.

By (35) and (42), it follows that (eventually on a subsequence),

$$H^*\left(\frac{\sigma^2}{2}y_n\right)\overset{n\to\infty}{\longrightarrow}H^*\left(\frac{\sigma^2}{2}y\right),\ \text{a.e. in }\mathbb{R}\tag{43}$$

and weakly in $L^2(\mathbb{R})$, and therefore $y=y_{\lambda,\nu}\in L^\infty(\mathbb{R})\cap L^2(\mathbb{R})$ solves (27). Furthermore, by (34) and (42), we have

$$\left\|y_{\lambda,\nu}\right\|_1+\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_\infty+\nu\left\|y_{\lambda,\eta}\right\|_\infty\leq\frac{C_1}{\lambda-\lambda_0}\|\eta\|_1\,,\tag{44}$$

$\forall\lambda>\max(\lambda^*,\lambda_0)$, where C_1 is independent of ν. We also obtain that inequality (34) holds for solution $y_{\lambda,\nu}$ to (27), with $\eta\in L^1(\mathbb{R})$ only. Now we are going to extend the solution $y_{\lambda,\nu}$ to (27) for all

$\lambda > \lambda_0$. To this end we set $G_\lambda^\nu = (\Gamma + \mathcal{H})y$, rewriting (27) as follows $G_\lambda^\nu = \eta$. For every $\lambda > 0$, we can equivalently write this as

$$y = (G_{\lambda+\delta}^\nu)^{-1}(\eta) + \delta(G_{\lambda+\delta}^\nu)^{-1}(\eta)\,. \tag{45}$$

By (34) we also have

$$\|(G_{\lambda+\delta}^\nu)^{-1}\|_{L(L^1(\mathbb{R}),L^1(\mathbb{R}))} \le \frac{1}{\lambda - \lambda_0},$$

then, by contraction principle, (45) has a unique solution $y = y_{\lambda,\nu} \in L^1(\mathbb{R})$, for all $\lambda > \lambda_0$. Estimate (44) extends for all $\lambda > \lambda_0$. In order to complete the proof of Lemma 3.1, we are going to let $\nu \to 0$ in Eq. (27) which holds for all $\lambda > \lambda_0$. To this end, we assume first that $\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and extend afterward by density the convergence result to all of $\eta \in L^2(\mathbb{R})$. By multiplying (27) by $y_{\lambda,\nu}$ and $H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)$, respectively, we obtain after integration on $\mathbb{R}$,

$$\lambda \|y_{\lambda,\nu}\|_2^2 + \nu\|y'_{\lambda,\nu}\|_2^2 \le \int_{\mathbb{R}} f\, y_{\lambda,\nu}y'_{\lambda,\nu}\,dx + \langle \eta, y_{\lambda,\nu}\rangle_2$$
$$\le \frac{1}{2}\|f'\|_\infty \|y_{\lambda,\nu}\|_2^2 + \|\eta\|_2\|y_{\lambda,\nu}\|_2$$

and

$$\frac{2\lambda}{\sigma^2}\left\|j\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1 + \left\|\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right)'\right\|_2^2 + \nu\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_2^2$$
$$\le \left\langle \eta, H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\rangle_2 + \frac{2}{\sigma^2}\int_{\mathbb{R}} f\left(j\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right)' dx$$
$$\le \|\eta\|_\infty\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1 + \frac{2}{\sigma^2}\|f'\|_\infty\left\|j\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1 .$$

Taking into account (32) and that $\|y_{\lambda,\nu}\|_2 \le \left(\lambda - \frac{1}{2}\|f'\|_\infty\right)^{-1}\|\eta\|_2$, we get

$$\left\|j\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1 + \left\|\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right)'\right\|_2^2$$
$$+\ \nu\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_2^2 \le C\left(\|\eta\|_2^2 + \|\eta\|_2\|\eta\|_\infty + 1\right),$$

for all $\lambda \ge \lambda_1 > 0$, for some $\lambda_1 > 0$ independent of λ and ν. We have therefore the estimate

$$\begin{aligned}&\|y_{\lambda,\nu}\|_2^2 + \nu\|y'_{\lambda,\nu}\|_2^2 + \left\|j\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1\\ &+ \left\|\left(H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right)'\right\|_2^2 + \nu\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_2^2\\ &\le C\left(\|\eta\|_2^2 + \|\eta\|_\infty^2 + 1\right), \quad \forall \lambda \ge \lambda_1 .\end{aligned} \tag{46}$$

Here and everywhere in the following C is a positive constant independent of λ and ν.

Now taking into account (32) we have also by (46) that

$$\left\|H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\|_1^2 \le C\left(\|\eta\|_2^2 + \|\eta\|_\infty^2 + 1\right), \quad \forall \lambda \ge \lambda_1 . \tag{47}$$

By (46)–(47) it follows that $\{y_{\lambda,\nu}\}_{\nu>0}$ is weakly compact in $L^2(\mathbb{R})$ and for each $\phi \in C_0^\infty(\mathbb{R})$, $\phi \ge 0$, $\left\{\phi H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right)\right\}$ is bounded in $W^{1,1}(\mathbb{R})$. Hence on a subsequence $\{\nu\} \to 0$, we have

$$\begin{aligned}&y_{\lambda,\nu} \to y \text{ weakly in } L^2(\mathbb{R})\\ &\phi H^*\left(\frac{\sigma^2}{2}y_{\lambda,\nu}\right) \to \xi \text{ strongly in } L^2(\mathbb{R}).\end{aligned} \tag{48}$$

Taking into account that the map $z \to \phi H^*\left(\frac{\sigma^2}{2}z\right)$ is maximal monotone in $L^2(\mathbb{R})$ we infer that $\xi = \phi H^*\left(\frac{\sigma^2}{2}y\right)$, a.e. in $\mathbb{R}$.

We note also that as seen earlier, we have by (27) that

$$\lambda\|y_{\lambda,\nu}\|_1 \le \|\eta\|_1 + \langle f y'_{\lambda,\nu}, \operatorname{sgn} y_{\lambda,\nu}\rangle_2 \le \|\eta\|_1 + \|f'\|_\infty\|y_{\lambda,\nu}\|_1 .$$

Hence

$$\|y_{\lambda,\nu}\|_1 \le \left(\lambda - \|f'\|_1\right)^{-1}\|\eta\|_1, \ \forall \nu > 0$$

and so by (48) we see that $y \in L^1(\mathbb{R})$. Now letting $\nu \to 0$ in (27) it follows by (48) that

$$\lambda\, y - \left(H^*\left(\frac{\sigma^2}{2}y\right)\right)'' - f\, y' = \eta \text{ in } \mathcal{D}'(\mathbb{R}),$$

for $\lambda > \lambda_1$, λ_1 independent of $\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, by (34), the map $\eta \to y$ is Lipschitz in $L^1(\mathbb{R})$, with Lipschitz constant $(\lambda - \lambda_1)^{-1}$, then y solves (25) for all $\eta \in L^1(\mathbb{R})$, and (26) follows. This completes the proof of Lemma 3.1. □

Proof of Theorem 2.2 (*Continued*). Coming back to Eq. (23), by Lemma 3.1 and (14), it follows that the operator $A + B$ is quasi-m-accretive in $L^1(\mathbb{R})$. Then by the Crandall & Liggett theorem, see Barbu (2010), p. 147, the Cauchy problem (23) has a unique mild solution $y \in C([0, T]; L^1(\mathbb{R}))$, that is

$$y(t) = \lim_{\epsilon\to 0} y_\epsilon(t) \text{ in } L^1(\mathbb{R}), \forall t \in [0, T],$$

$$y_\epsilon(t) = y_\epsilon^i \text{ for } t \in [i\epsilon, (i+1)\epsilon],\ i = 0, \ldots, N = \left[\frac{T}{\epsilon}\right] - 1$$

$$\frac{1}{\epsilon}(y_\epsilon^{i+1} - y_\epsilon^i) + (A + B)(y_\epsilon^{i+1}) = g_1 \quad i = 0, \ldots, N$$

$$y_\epsilon^0 = y_0 .$$

The function y is a mild solution to (16) in the sense of Definition 2.1.

Assume now that $j(\lambda v) \le c_\lambda j(v)$, $\forall \lambda > 0$, $v \in \mathbb{R}$. Taking into account that $j(v) \le j(2v) - vH^*(v)$, $\forall v \in \mathbb{R}$, this implies that

$$H^*(v)v \le (C_2 - 1)\, j(v)\,, \quad \forall v \in \mathbb{R}\,. \tag{49}$$

Assume also that $j(\frac{\sigma^2}{2}y_0) \in L^1(\mathbb{R})$. Then, if we take in (19), $z^i = \frac{\sigma^2}{2}y_\epsilon^i$, we get

$$\frac{2}{\sigma^2\epsilon}(z^{i+1} - z^i) - (H^*(z^{i+1}))'' - f\left(\frac{2}{\sigma^2}z^{i+1}\right)'$$
$$+ B\left(\frac{2}{\sigma^2}z^{i+1}\right) = g_1.$$

Multiplying by $H^*(z^{i+1})$ and integrating on $\mathbb{R}$ we get

$$\frac{2}{\epsilon}\int_{\mathbb{R}}\frac{1}{\sigma^2}\left(j(z^{i+1}) - j(z^i)\right)dx + \int_{\mathbb{R}}\left(\left(H^*(z^{i+1})\right)'\right)^2 dx$$
$$+\ 2\int_{\mathbb{R}} f\left(\frac{z^{i+1}}{\sigma^2}\right)H^*(z^{i+1})\,dx$$
$$+\ 2\int_{\mathbb{R}} B\left(\frac{z^{i+1}}{\sigma^2}\right)H^*(z^{i+1})\,dx = \int_{\mathbb{R}} g_1\, H^*(z^{i+1})\,dx.$$

Integrating by parts in $\int_{\mathbb{R}} f\left(\frac{z^{i+1}}{\sigma^2}\right)' H^*\left(z^{i+1}\right)dy$, summing up, after some calculation involving (14) and (49), we get the estimate

$$2\int_{\mathbb{R}}\frac{1}{\sigma^2}j(z^{k+1})\,dx + \epsilon\sum_{i=0}^{k}\int_{\mathbb{R}}\left(\left(H^*(z^{i+1})\right)'\right)^2 dx \le C,\ \forall k,$$

which implies the desired conclusion

$$\Big(H^*\Big(\frac{\sigma^2}{2}y\Big)\Big)_x \in L^2((0,T)\times\mathbb{R}),$$
$$j\Big(\frac{\sigma^2}{2}y\Big) \in L^\infty([0,T];L^1(\mathbb{R})). \quad \square$$

4. The degenerate 1-D case

Consider here Eq. (16), that is

$$\begin{cases} y_t - \Big(H^*\big(\frac{\sigma^2}{2}y\big)\Big)_{xx} - f\,y_x + B\,y = g_1, & \text{in } [0,T]\times\mathbb{R} \\ y(0) = y_0 \in \mathbb{R} \end{cases} \tag{50}$$

where σ is assumed to satisfy the condition $\sigma \in C_b^2(\mathbb{R})$ only. Moreover, if we consider, as above, the operator $A : D(A) \subset L^1(\mathbb{R}) \to L^1(\mathbb{R})$, such that

$$A\,y = -\Big(H^*\Big(\frac{\sigma^2}{2}y\Big)\Big)'' - f\,y', \tag{51}$$

$$D(A) = \Big\{y \in L^1(\mathbb{R});\ f\,y' + \Big(H^*\Big(\frac{\sigma^2}{2}y\Big)\Big)'' \in L^1(\mathbb{R})\Big\},$$

we have the following holds

Lemma 4.1. *A is quasi-m-accretive in* $L^1(\mathbb{R})$.

Proof. For each $\epsilon > 0$ we consider the operator

$$A_\epsilon\,y = -\Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}y\Big)\Big)'' - f\,y', \tag{52}$$

which is quasi-m-accretive, seen Lemma 3.1. Hence, for each $\eta \in L^1(\mathbb{R})$ and $\lambda \ge \lambda_0$ the equation

$$\lambda\,y_\epsilon - \Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big)\Big)'' - f\,y_\epsilon' = \eta, \quad \text{in } \mathbb{R}, \tag{53}$$

has a unique solution $y_\epsilon \in L^1(\mathbb{R})$, with $H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big) \in L^\infty(\mathbb{R})$.

Dynamic estimates. As in the proof of Lemma 3.1, we have

$$\lambda\,\|y_\epsilon\|_1 \le \|\eta\|_1 + \|f'\|_1\,\|y_\epsilon\|_1, \quad \forall \epsilon > 0, \tag{54}$$

that is for $\lambda > \|f'\|_\infty$

$$\|y_\epsilon\|_1 \le (\lambda - \|f'\|_1)^{-1}\,\|\eta\|_1, \quad \forall \epsilon > 0. \tag{55}$$

Assume now that $\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then, by (53) we see that for each $M > 0$

$$\lambda\,(y_\epsilon - M) - \Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big) - H^*\Big(\frac{\sigma^2+\epsilon}{2}M\Big)\Big)''$$
$$- f(y_\epsilon - M)' = \eta - \lambda M + \Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}M\Big)\Big)'' = \tilde\eta.$$

Moreover, by (5), we also have

$$\tilde\eta(x) \le \eta - M\lambda + M^2\|(H^*)''\|_\infty\|\sigma\sigma'\|_\infty$$
$$+ M\|(H^*)'\|_\infty\|\sigma\sigma'' + (\sigma')^2\|_\infty \le 0$$

for M and λ large enough (independently of ϵ). This yields

$$\lambda\,\|(y_\epsilon - M)^+\|_1 \le \|f'\|_\infty\,\|(y_\epsilon - M)^+\|_1.$$

Hence $y_\epsilon \le M$ in $\mathbb{R}$ for $\lambda > \|f'\|_\infty$. Similarly, it follows that

$$\lambda\,(y_\epsilon + M) - \Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big) - H^*\Big(-\frac{\sigma^2+\epsilon}{2}M\Big)\Big)''$$
$$+ f(y_\epsilon + M)'$$
$$= \eta + \lambda M + \Big(H^*\Big(-\frac{\sigma^2+\epsilon}{2}M\Big)\Big)''$$
$$= \eta + \lambda M \ge 0,$$

if M is large enough, but independent of ϵ. Therefore, if multiplying the equation by $(y_\epsilon + M)^-$ and integrating on $\mathbb{R}$, we get $\|(y_\epsilon + M)^-\|_1 \ge 0$ which implies $y_\epsilon \ge -M$ in $\mathbb{R}$.

By (53) and the inequality (see Benilan et al., 1975)

$$\left\|\Big(I - \frac{d^2}{dx^2}\Big)^{-1} z\right\|_{W^{1,\infty}(\mathbb{R})} \le \|z\|_1$$

we see that $\Big\{H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big) + \int_0^x f\,y_\epsilon\,dx\Big\}_{\epsilon>0}$ is bounded in $W^{1,\infty}(\mathbb{R})$.

Hence $\Big(H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big)\Big)'$ bounded in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, so that $\Big\{\eta_\epsilon = H^*\Big(\frac{\sigma^2+\epsilon}{2}y_\epsilon\Big)\Big\}$ is compact in $C_b(\mathbb{R})$. It follows that on a subsequence $\epsilon \to 0$, we have

$$y_\epsilon \to y, \quad \text{weakly in all } L^p,\ 1 < p \le \infty,$$
$$\eta_\epsilon \to \zeta, \quad \text{strongly in } C_b(\mathbb{R}),$$

where $\zeta = H^*\Big(\frac{\sigma^2}{2}y\Big)$ in $\mathbb{R}$. Letting $\epsilon \to 0$ in (53), we get

$$\lambda\,y - \Big(H^*\Big(\frac{\sigma^2}{2}y\Big)\Big)'' + f\,y' = \eta, \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Next for $\eta \in L^1(\mathbb{R})$ we choose $\{\eta_n\} \subset L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\eta_n \to \eta$ in $L^1(\mathbb{R})$ and we have

$$\lambda\,y_n - \Big(H^*\Big(\frac{\sigma^2}{2}y_n\Big)\Big)'' + f\,y_n' = \eta_n, \quad \forall n,$$

getting

$$\lambda\,\|y_n - y_m\|_1 \le \|\eta_n - \eta_m\|_1 + \|f'\|_\infty\,\|y_n - y_m\|_1, \ \forall n, m.$$

Hence, for $\lambda > \|f'\|_\infty$ we have for $n \to \infty$

$$y_n \to y, \quad \text{strongly in } L^1(\mathbb{R})$$
$$\Big(H^*\Big(\frac{\sigma^2}{2}y_n\Big)\Big) \to \Big(H^*\Big(\frac{\sigma^2}{2}y\Big)\Big), \quad \text{a.e. in } \mathbb{R}$$
$$f\,y_n' \to f\,y', \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

This yields

$$\lambda\,y - \Big(H\Big(\frac{\sigma^2}{2}y\Big)\Big)'' - f\,y' = \eta, \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Hence for $\lambda \ge \lambda_0$, $y \in L^1(\mathbb{R})$ is the solution to equation $\lambda\,y + A\,y = \eta$ as claimed. As seen earlier this implies that the operator $A + B$ is quasi-m-accretive in $L^1(\mathbb{R})$. $\square$

Then by the existence theorem for the equation

$$\begin{cases} \frac{\partial y}{\partial t} + A\,y + B\,y = 0 \\ y(0) = y_0, \end{cases}$$

we get

Theorem 4.1. *There is a unique mild solution* $y \in C([0,T];L^1(\mathbb{R}))$ *to* Eq. (50).

As in previous case Theorem 4.1 implies via (13) the existence of a mild solution φ to Eq. (1) satisfying (20). We omit the details.

5. Conclusions

In this paper it is shown, via nonlinear semigroup theory in L^1, both the existence and the uniqueness of a mild solution for the dynamic programming equation for stochastic optimal control problem with control in the volatility term. Latter problem is related to the analysis of controlled stochastic volatility models, within the financial frameworks, whose related computational study is the subject of our ongoing research.

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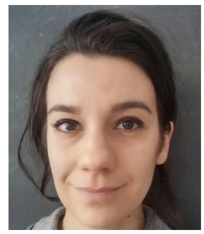

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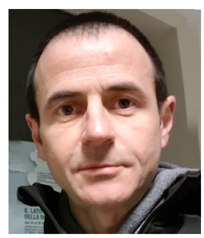

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