Nonrepetitive Colourings of Planar Graphs with $O(\log n)$ Colours

Vida Dujmović [†] Fabrizio Frati [‡] Gwenaël Joret [§] David R. Wood [¶]

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Abstract

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. The *nonrepetitive chromatic number* of a graph G is the minimum integer k such that G has a nonrepetitive k-colouring. Whether planar graphs have bounded nonrepetitive chromatic number is one of the most important open problems in the field. Despite this, the best known upper bound is $O(\sqrt{n})$ for n-vertex planar graphs. We prove a $O(\log n)$ upper bound.

1 Introduction

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a *k-colouring* of a graph G is a function ψ that assigns one of k colours to each vertex of G. A path $(v_1, v_2, \dots, v_{2t})$ of even order in G is *repetitively* coloured by ψ if $\psi(v_i) = \psi(v_{t+i})$ for all $i \in [1, t] := \{1, 2, \dots, t\}$. A colouring ψ of G is *nonrepetitive* if no path of G is repetitively coloured by ψ. Observe that a nonrepetitive colouring is *proper*, in the sense that adjacent vertices are coloured differently. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer k such that G admits a nonrepetitive k-colouring.

The seminal result in this field is by Thue [19], who in 1906 proved that every path is nonrepetitively 3-colourable. Nonrepetitive colourings have recently been widely studied;

[†]School of Mathematics and Statistics & Department of Systems and Computer Engineering, Carleton University, Ottawa, Canada (vdujmovic@math.carleton.ca). Supported by NSERC and an Endeavour Fellowship from the Australian Government.

[‡]School of Information Technologies, The University of Sydney, Sydney, Australia (brillo@it.usyd.edu.au).

[§]Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium (gjoret@ulb.ac.be). Supported by an Endeavour Fellowship from the Australian Government.

[¶]School of Mathematical Sciences, Monash University, Melbourne, Australia (david.wood@monash.edu). Research supported by the Australian Research Council.

see the surveys [6, 10, 11]. A number of graph classes are known to have bounded nonrepetitive chromatic number. In particular, trees are nonrepetitively 4-colourable [5, 14], outerplanar graphs are nonrepetitively 12-colourable [4, 14], and more generally, every graph with treewidth k is nonrepetitively 4^k-colourable [14]. Graphs with maximum degree Δ are nonrepetitively $O(\Delta^2)$-colourable [3, 8, 10, 13].

Perhaps the most important open problem in the field of nonrepetitive colourings is whether planar graphs have bounded nonrepetitive chromatic number. This question, first asked by Alon et al. [3], has since been mentioned by numerous authors [2, 4, 7, 9–14, 16, 18]. It is widely known that $\pi(G) \in O(\sqrt{n})$ for n-vertex planar graphs[1], and this is the best known upper bound. The best known lower bound is 11, due to Pascal Ochem; see Appendix A. Here we prove a logarithmic upper bound.

Theorem 1. *For every planar graph G with n vertices,*

$$\pi(G) \leqslant 8(1 + \log_{3/2} n) \ .$$

We now explain that the above open problem is solved when restricted to paths of bounded length. For $p \geqslant 1$, a vertex colouring of a graph G is *p-centered* if for every connected subgraph X of G, some colour appears appears exactly once in X, or at least p colours appear in X. In a repetitively coloured path of at most $2p - 2$ vertices, there are at most $p - 1$ colours each appearing at least twice. Thus the colouring is not p-centered. Equivalently, every p-centered colouring is nonrepetitive on paths with at most $2p - 2$ vertices. Nešetřil and Ossona de Mendez [17] proved that for every graph H and integer $p \geqslant 1$, there exists an integer c, such that every graph with no H-minor has a p-centered colouring with c colours. This shows that (with $H = K_5$) for every integer $p \geqslant 1$, there exists an integer c, such that every planar graph has a c-colouring that is nonrepetitive on paths with at most $2p$ vertices. Note that the bound on c in terms of p here is large. It is open whether there is a polynomial function f such that for every integer $k \geqslant 1$ every planar graph G has a $f(k)$-colouring that is nonrepetitive on paths with most $2k$ vertices.

Finally, we mention a class of planar graphs that seem difficult to nonrepetitively colour. Let T be a tree rooted at a vertex r. Let V_i be the set of vertices in T at distance i from r. Draw T in the plane with no crossings. Add a cycle on each V_i in the cyclic order defined by the drawing to create a planar graph G_T. It is open whether $\pi(G_T) \leqslant c$ for some constant c independent of T. Note that this class of planar graphs includes examples with unbounded degree and unbounded treewidth.

2 Proof of Theorem 1

A *layering* of a graph G is a partition $V_0, V_1, \dots, V_p$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $|i - j| \leqslant 1$. Each set V_i is called a *layer*. The following lemma by Kündgen and Pelsmajer [14] will be useful.

[1] One can prove this bound using a naive application of the Lipton-Tarjan planar separator theorem.

Lemma 2 ([14]). *For every layering of a graph G, there is a (not necessarily proper) 4-colouring of G such that for every repetitively coloured path $(v_1, v_2, \dots, v_{2t})$, the subpaths $(v_1, v_2, \dots, v_t)$ and $(v_{t+1}, v_{t+2}, \dots, v_{2t})$ have the same layer pattern.*

A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G, such that $G = G_1 \cup G_2$. In particular, there is no edge of G between $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$.

Lemma 3. *Fix $\varepsilon \in (0, 1)$ and $c \geqslant 1$. Let G be a graph with n vertices. Fix a layering $V_0, V_1, \dots, V_p$ of G. Assume that, for every set $B \subseteq V(G)$, there is a separation (G_1, G_2) of G such that:*

- *each layer V_i contains at most c vertices in $V(G_1) \cap V(G_2) \cap B$, and*
- *both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $(1 - \varepsilon)|B|$ vertices in B.*

Then $\pi(G) \leqslant 4c(1 + \log_{1/(1-\varepsilon)} n)$.

Proof. Run the following recursive algorithm COMPUTE$(V(G), 1)$.

> COMPUTE(B, d)
>
> 1. If $B = \emptyset$ then exit.
> 2. Let (G_1, G_2) be a separation of G such that each layer V_i contains at most c vertices in $V(G_1) \cap V(G_2) \cap B$, and both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $(1 - \varepsilon)|B|$ vertices in B.
> 3. Let $\text{depth}(v) := d$ for each vertex $v \in V(G_1) \cap V(G_2) \cap B$.
> 4. For $i \in [1, p]$, injectively label the vertices in $V_i \cap V(G_1) \cap V(G_2) \cap B$ by $1, 2, \dots, c$. Let $\text{label}(v)$ be the label assigned to each vertex $v \in V_i \cap V(G_1) \cap V(G_2) \cap B$.
> 5. COMPUTE$((V(G_1) - V(G_2)) \cap B, d + 1)$
> 6. COMPUTE$((V(G_2) - V(G_1)) \cap B, d + 1)$

The recursive application of COMPUTE determines a rooted binary tree T, where each node of T corresponds to one call to COMPUTE. Associate each vertex whose depth and label is computed in a particular call to COMPUTE with the corresponding node of T. (Observe that the depth and label of each vertex is determined exactly once.)

Colour each vertex v by $(\text{col}(v), \text{depth}(v), \text{label}(v))$, where col is the 4-colouring from Lemma 2. Suppose on the contrary that $(v_1, v_2, \dots, v_{2t})$ is a repetitively coloured path in G. By Lemma 2, $(v_1, v_2, \dots, v_t)$ and $(v_{t+1}, v_{t+2}, \dots, v_{2t})$ have the same layer pattern. In addition, $\text{depth}(v_i) = \text{depth}(v_{t+i})$ and $\text{label}(v_i) = \text{label}(v_{t+i})$ for all $i \in [1, t]$. Let v_i and v_{t+i} be vertices in this path with minimum depth. Since v_i and v_{t+i} are in the same layer and have the same label, these two vertices were not labelled at the same step of the algorithm. Let x and y be the two nodes of T respectively associated with v_i and v_{t+i}. Let z be the least common ancestor of x and y in T. Say node z corresponds to call COMPUTE(B, d). Thus v_i and v_{t+i} are in B (since if a vertex v is in B in the call to COMPUTE associated with some node q of T, then v is in B in the call to COMPUTE associated with each ancestor of q in T). Let (G_1, G_2) be the separation in COMPUTE(B, d). Since $\text{depth}(v_i) = \text{depth}(v_{t+i}) > d$, neither v_i nor v_{t+i} are in $V(G_1) \cap V(G_2)$. Since z is the least common ancestor of x and y, without loss of generality, $v_i \in V(G_1) - V(G_2)$ and $v_{t+i} \in V(G_2) - V(G_1)$. Thus some vertex v_j in the subpath $(v_{i+1}, v_{i+2}, \dots, v_{t+i-1})$ is in $V(G_1) \cap V(G_2)$. If $v_j \in B$ then $\text{depth}(v_j) = d$. If $v_j \notin B$ then $\text{depth}(v_j) < d$. In both cases, $\text{depth}(v_j) < \text{depth}(v_i) = \text{depth}(v_{t+i})$, which contradicts the choice of v_i and v_{t+i}. Hence there is no repetitively coloured path in G.

Observe that the maximum depth is at most $1 + \log_{1/(1-\varepsilon)} n$. Therefore the number of colours is at most $4c(1 + \log_{1/(1-\varepsilon)} n)$. □

We now show that a result by Lipton and Tarjan [15] implies the condition in Lemma 3 for planar graphs.

Lemma 4. *Let r be a vertex in a connected planar graph G. For $i \geqslant 0$, let V_i be the set of vertices at distance i from r. Then $V_0, V_1, \dots, V_p$ is a layering of G. For every set $B \subseteq V(G)$, there is a separation (G_1, G_2) of G such that:*

- *each layer V_i contains at most two vertices in $V(G_1) \cap V(G_2) \cap B$,*
- *both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $\frac{2}{3}|B|$ vertices in B.*

Proof. Let T be a breath-first spanning tree in G starting at r. Thus, for each vertex v, the distance between v and r in T equals the distance between v and r in G.

Lipton and Tarjan [15, Lemma 2] proved that for every vertex weighting of G (with non-negative weights totalling at most 1), there is an edge $vw \in E(G) - E(T)$, such that if C is the 'fundamental' cycle consisting of vw and the two paths from v and w back to their least common ancestor in T, then the vertices inside C have total weight at most $\frac{2}{3}$, and the vertices outside C have total weight at most $\frac{2}{3}$.

Apply this result with each vertex in B weighted $\frac{1}{|B|}$, and each vertex in $V(G) - B$ weighted 0. Let G_1 and G_2 be the subgraphs of G induced by C and the vertices inside C and outside C respectively. Then (G_1, G_2) is a separation. The total weight of $V(G_1) - V(G_2)$ equals the number of vertices in $(V(G_1) - V(G_2)) \cap B$. Hence $V(G_1) - V(G_2)$, and by symmetry $V(G_2) - V(G_1)$, contains at most $\frac{2}{3}|B|$ vertices in B.

Since T is breadth-first, the paths from v and w back to their least common ancestor in T each contain at most one vertex from each layer V_i. Hence, each layer V_i contains at most two vertices in $V(G_1) \cap V(G_2) \cap B$. □

Lemmas 3 and 4 together prove Theorem 1 (by adding edges to make G connected).

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A Lower Bounds

Barát and Varjú [4] constructed a planar graph G with $\pi(G) \geqslant 10$. Pascal Ochem [private communication] observed that this lower bound can be improved to 11 by adapting a construction due to Albertson et al. [1] as follows. Barát and Varjú [4] constructed an outerplanar graph H with $\pi(H) \geqslant 7$. Let G be the following planar graph. Start with a path $P = (v_1, \dots, v_{22})$. Add two adjacent vertices x and y that both dominate P. Let each vertex v_i in P be adjacent to every vertex in a copy H_i of H. Suppose on the contrary that G is nonrepetitively 10-colourable. Without loss of generality, x and y are respectively coloured 1 and 2. A vertex in P is *redundant* if its colour is used on some other vertex in P. If no two adjacent vertices in P are redundant then at least 11 colours appear exactly once on P, which is a contradiction. Thus some pair of consecutive vertices v_i and v_{i+1} in P are redundant. Without loss of generality, v_i and v_{i+1} are respectively coloured 3 and 4. If some vertex in $H_i \cup H_{i+1}$ is coloured 1 or 2, then since v_i and v_{i+1} are redundant, with x or y we have a repetitively coloured path on 4 vertices. Now assume that no vertex in $H_i \cup H_{i+1}$ is coloured 1 or 2. If some vertex in H_i is coloured 4 and some vertex in H_{i+1} is coloured 3, then with v_i and v_{i+1}, we have a repetitively coloured path on 4 vertices. Thus no vertex in H_i is coloured 4 or no vertex in H_{i+1} is coloured 3. Without loss of generality, no vertex in H_i is coloured 4. Since v_i dominates H_i, no vertex in H_i is coloured 3. We have proved that no vertex in H_i is coloured $1, 2, 3$ or 4, which is a contradiction, since $\pi(H_i) \geqslant 7$. Therefore $\pi(G) \geqslant 11$.