Asymptotic shape and the speed of propagation of continuous-time continuous-space birth processes

Viktor Bezborodov *[1], Luca Di Persio †[1], Tyll Krueger ‡[2], Mykola Lebid §[3], and Tomasz Ożański ¶[2]

[1] The University of Verona
[2]Wroclaw University of Technology
[3](D-BSSE) ETH Zürich

July 17, 2017

Abstract

We formulate and prove a shape theorem for a continuous-time continuous-space stochastic growth model under certain general conditions. Similarly to the classical lattice growth models the proof makes use of the subadditive ergodic theorem. A precise expression for the speed of propagation is given in the case of a truncated free branching birth rate.

Mathematics subject classification: 60K35, 60J80.

1 Introduction

Shape theorems have a long history. Richardson [Ric73] proved the shape theorem for the Eden model. Since then, shape theorems have been proven in various settings, most notably for first passage percolation and permanent and non-permanent growth models. Garet and Marchand [GM12] not only prove a shape theorem for the contact process in random environment, but also have a nice overview of existing results.

Most of literature is devoted to discrete-space models. A continuous-space first passage percolation model was analyzed by Howard and Newman [HN97], see also references therein. A shape theorem for a continuous-space growth model was proven by Deijfen [Dei03], see also

*Email: viktor.bezborodov@univr.it
†Email: luca.dipersio@univr.it
‡Email: tyll.krueger@pwr.wroc.pl
§Email: mykola.lebid@bsse.ethz.ch
¶Email: tomasz.ozanski@pwr.edu.pl
Keywords: *shape theorem, spatial birth process, growth model*

Gouéré and Marchand [GM08]. Our model is naturally connected to that model, see the end of Section 2.

Questions addressed in this article are motivated not only by probability theory but also by studies in natural sciences. In particular, one can mention a demand to incorporate spatial information in the description and analysis of 1) ecology 2) bacteria populations 3) tumor growth 4) epidemiology 5) phylogenetics among others, see e.g. [WBP+15], [TSH+13], [VDPP15], and [TM15]. Authors often emphasize that it is preferable to use the continuous-space spaces R^2 and R^3 as the basic, or 'geographic' space, see e.g. [VDPP15]. More on connections between theoretical studies and applications can be found in [MW03].

The paper is organized as follows. In Section 2 we describe the model and formulate our results, which are proven in Sections 3 and 4. Technical results, in particular on the construction of the process, are collected in the Section 5.

2 The model, assumptions and results

We consider a growth model represented by a continuous-time continuous-space Markov birth process. Let Γ_0 be the collection of finite subsets of $\mathbb{R}^{\mathrm{d}}$,

$$\Gamma_0(\mathbb{R}^{\mathrm{d}}) = \{\eta \subset \mathbb{R}^{\mathrm{d}} : |\eta| < \infty\},$$

where $|\eta|$ is the number of elements in η. Γ_0 is also called the *configuration space*, or the *space of finite configurations*.

The evolution of the spatial birth process on $\mathbb{R}^{\mathrm{d}}$ admits the following description. Let $\mathscr{B}(X)$ be the Borel σ-algebra on the Polish space X. If the system is in state $\eta \in \Gamma_0$ at time t, then the probability that a new particle appears (a "birth") in a bounded set $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$ over time interval $[t; t + \Delta t]$ is

$$\Delta t \int_B b(x,\eta)dx + o(\Delta t),$$

and with probability 1 no two births happen simultaneously. Here $b : \mathbb{R}^{\mathrm{d}} \times \Gamma_0 \to \mathbb{R}_+$ is some function which is called the *birth rate*. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x,\eta)dx$. We note that it is conventional to call the function b the 'birth rate', even though it is not a rate in the usual sense (as in for example 'the Poisson process (N_t) has unit jumps at rate 1 meaning that $\frac{P\{N_{t+\Delta t} - N_t = 1\}}{\Delta t} = 1$ as $\Delta t \to 0$') but rather a version of the Radon–Nikodym derivative of the rate with respect to the Lebesgue measure.

Remark 2.1. We characterize the birth mechanism by the birth rate $b(x,\eta)$ at each spatial position. Oftentimes the birth mechanism is given in terms of contributions of individual particles: a particle at y, $y \in \eta$, gives a birth at x at rate $c(x,y,\eta)$ (often $c(x,y,\eta) = \gamma(y,\eta)k(y,x)$, where $\gamma(y,\eta)$ is the proliferation rate of the particle at y, whereas the dispersion kernel $k(y,x)$ describes the distribution of the offspring), see e.g. Fournier and Méléard [FM04]. As long as we are not interested in the induced genealogical structure, the two ways of describing the process are equivalent under our assumptions. Indeed, given c, we may set

$$b(x,\eta) = \sum_{y\in\eta} c(x,y,\eta), \tag{1}$$

or, conversely, given b, we may set

$$c(y,x,\eta) = \frac{g(x-y)}{\sum\limits_{y\in\eta} g(x-y)} b(x,\eta), \tag{2}$$

where $g : \mathbb{R}^{\mathrm{d}} \to (0,\infty)$ is a continuous function. Note that b is uniquely determined by c, but not vice versa.

We equip Γ_0 with the σ-algebra $\mathscr{B}(\Gamma_0)$ induced by the sets

$$\mathbf{Ball}(\eta,r) = \left\{\zeta \in \Gamma_0 \middle| |\eta| = |\zeta|, \mathrm{dist}(\eta,\zeta) < r\right\}, \quad \eta \in \Gamma_0, r > 0, \tag{3}$$

where $\mathrm{dist}(\eta,\zeta) = \min\left\{\sum\limits_{i=1}^{|\eta|} |x_i - y_i| \middle| \eta = \{x_1,...,x_{|\eta|}\}, \zeta = \{y_1,...,y_{|\eta|}\}\right\}$. For more detail on configuration spaces see e.g. Röckner and Schied [RS99] or Kondratiev and Kutovyi [KK06]. In particular, the dist above coincides with the restriction to the space of finite configurations of the metric ρ used in [RS99], and the σ-algebra $\mathscr{B}(\Gamma_0)$ introduced above coincides with the σ-algebra from [KK06].

We say that a function $f : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ has an exponential moment if there exists $\theta > 0$ such that

$$\int_{\mathbb{R}^{\mathrm{d}}} e^{\theta|x|} f(x)dx < \infty.$$

Of course, if f has an exponential moment, then automatically $f \in L^1(\mathbb{R}^{\mathrm{d}})$.

Assumptions on b. We will need several assumptions on the birth rate b.

Condition 2.2 (Sublinear growth)**.** The birth rate b is measurable and there exists a function $a : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}_+$ with an exponential moment such that

$$b(x,\eta) \le \sum_{y\in\eta} a(x-y). \tag{4}$$

Condition 2.3 (Monotonicity)**.** For all $\eta \subset \zeta$,

$$b(x,\eta) \le b(x,\zeta), \quad x \in \mathbb{R}^{\mathrm{d}}. \tag{5}$$

The previous condition ensures attractiveness, see below.

Condition 2.4 (Rotation and translation invariance)**.** The birth rate b is translation and rotation invariant: for every $x,y \in \mathbb{R}^{\mathrm{d}}$, $\eta \in \Gamma_0$ and $M \in \mathrm{SO(d)}$,

$$b(x+y,\eta+y) = b(x,\eta),$$
$$b(Mx,M\eta) = b(x,\eta).$$

Here $\mathrm{SO(d)}$ is the orthogonal group of linear isometries on $\mathbb{R}^{\mathrm{d}}$, and for a Borel set $B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}})$ and $y \in \mathbb{R}^{\mathrm{d}}$,

$$B + y = \{z \mid z = x + y,\ x \in B\}$$
$$MB = \{z \mid z = Mx,\ x \in B\}.$$

Condition 2.5 (Non-degeneracy)**.** Let there exist $c_0, r > 0$ such that

$$b(x,\eta) \geq c_0 \quad \text{wherever } \min_{y\in\eta} |x-y| \leq r. \tag{6}$$

Remark 2.6. Condition 2.5 is used to ensure that the system grows at least linearly. The condition could be weakened for example as follows:

For some $r_2 > r_1 \geq 0$ *and all* $x, y \in \mathbb{R}^{\mathrm{d}}$,

$$b(y,\{x\}) \geq c_0 \mathbb{1}\{r_1 \leq |x-y| \leq r_2\}.$$

Respectively, the proof would become more intricate.

Remark 2.7. If b is like in (7) and f has polynomial tails, then the result of Durrett [Dur83] suggests that we should expect a superlinear growth. This is in contrast with Deijfen's model, for which Gouéré and Marchand [GM12] give a sharp condition on the distribution of the outbursts for linear or superlinear growth.

Examples of a birth rate are

$$b(x,\eta) = \lambda \sum_{y\in\eta} f(|x-y|), \tag{7}$$

and

$$b(x,\eta) = k \wedge \left(\lambda \sum_{y\in\eta} f(|x-y|)\right), \tag{8}$$

where λ, k are positive constants and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, non-negative, non-increasing function with compact support.

We denote the underlying probability space by $(\Omega, \mathscr{F}, P)$. Let $\mathscr{A}$ be a sub-σ-algebra of $\mathscr{F}$. A random element A in Γ_0 is $\mathscr{A}$-measurable if

$$\Omega \ni \omega \to A = A(\omega) \in \Gamma_0 \tag{9}$$

is a measurable map from the measure space $(\Omega, \mathscr{A})$ to $(\Gamma_0, \mathscr{B}(\Gamma_0))$. Such an A will also be called an $\mathscr{A}$*-measurable finite random* set.

The birth process will be obtained as a unique solution to a certain stochastic equation. The construction and the proofs of key properties, such as the rotation invariance and the strong Markov property, are given in Section 5. We place the construction toward the end because it is rather technical and the methods used there do not shed much light on the ideas of the proofs of our main results. Denote by $(\eta_t^{s,A})_{t\geq s} = (\eta_t^{s,A}, t \geq s)$ the process started at time $s \geq 0$ from an $\mathscr{S}_s$-measurable finite random set A. Here $(\mathscr{S}_s)_{s\geq 0}$ is a filtration of σ-algebras to which $(\eta_t^{s,A})_{t\geq s}$ is adapted; it is introduced after (74). Furthermore, $(\eta_t^{s,A})_{t\geq s}$ is a strong Markov process with respect to $(\mathscr{S}_s)_{s\geq 0}$ - see Proposition 5.8.

The construction method we use has the advantage that the stochastic equation approach resembles graphical representation (see e.g. Durrett [Dur88] or Liggett [Lig99]) in the fact that it preserves monotonicity: if $s \geq 0$ and a.s. $A \subset B$, A and B being $\mathscr{S}_s$-measurable finite random sets, then a.s.

$$\eta_t^{s,A} \subset \eta_t^{s,B}, \quad t \geq s. \tag{10}$$

This property is proven in Lemma 5.10 and is often refered to as *attractiveness*.

The process started from a single particle at $\mathbf{0}$ at time zero will be denoted by $(\eta_t)_{t\geq 0}$; thus, $\eta_t = \eta_t^{0,\{\mathbf{0}\}}$. Let

$$\xi_t := \bigcup_{x\in\eta_t} B(x,r) \tag{11}$$

and similarly

$$\xi_t^{s,A} := \bigcup_{x\in\eta_t^{s,A}} B(x,r),$$

where $B(x,r)$ is the closed ball of radius r centered at x (recall that r appears in (6)).

The following theorem represents the main result of the paper.

Theorem 2.8. *There exists $\mu > 0$ such that for all $\varepsilon \in (0,1)$ a.s.*

$$(1-\varepsilon)B(\mathbf{0},\mu^{-1}) \subset \frac{\xi_t}{t} \subset (1+\varepsilon)B(\mathbf{0},\mu^{-1}) \tag{12}$$

for sufficiently large t.

Remark 2.9. Let us note that the statement of Theorem 2.8 does not depend on our choice for the radius in (11) to be r; we could just as well take any positive constant, for example

$$\bigcup_{x\in\eta_t} B(x,1)$$

In particular, μ in (12) does not depend on r.

The proof of Theorem 2.8 and the outline of the proof are given in Section 3. It is common to write the ball radius as the reciprocate μ^{-1}, probably because μ comes up in the proof as the limiting value of a certain sequence of random variables after applying the subadditive ergodic theorem; see e.g. Durrett [Dur88] or Deijfen [Dei03]. We decided to keep the tradition not only for historic reasons, but also because μ comes up as a certain limit in our proof too, even though we do not obtain μ directly from the subadditive ergodic theorem. The value μ^{-1} is called the *speed of propagation*. The subadditive ergodic theorem is a cornerstone in the majority of shape theorem proofs, and our proof relies on it.

Formal connection to Deijfen's model. The model introduced in [Dei03] with deterministic outburst radius, that is, when in the notation of [Dei03] the distribution of ourbursts F is the Dirac measure: $F = \delta_R$ for some $R \geq 0$, can be identified with

$$\zeta_t^R = \bigcup_{x\in\eta_t} B(x,R)$$

for the birth process (η_t) with birth rate

$$b(x,\eta) = \mathbb{1}\{\exists y \in \eta : |x-y| \leq R\}.$$

Explicit growth speed for a particular model. The precise evaluation of speed appears to be a difficult problem. For a general one dimensional branching random walk the speed of propagation is given by Biggins [Big95]. An overview of related results for different classes of models can be found in Auffinger, Damron, and Hanson [ADH15].

Here we give the speed for a model with interaction.

Theorem 2.10. *Let* $\mathrm{d}=1$ *and*

$$b(x,\eta)=2\wedge\left(\sum_{y\in\eta}\mathbb{1}\{|x-y|\le 1\}\right). \tag{13}$$

Then the speed of propagation is given by

$$\mu^{-1}=\frac{144\ln(3)-144\ln(2)-40}{25}\approx 0.73548... \tag{14}$$

Section 4 contains the proof of Theorem 2.10.

3 Proof of Theorem 2.8

Outline of the proof. The proof can roughly be divided into three parts. In the first part we show that the system grows not faster than linearly, which is the content of Proposition 3.1. The proof of Proposition 3.1 relies on Lemma 5.10, which allows a comparison of birth processes with different rates, and on the results on the spread of the supercritical branching random walk by Biggins [Big95].

In the second part we show that the system grows at least linearly. Strictly speaking, in this part we only give exponential estimates on the probability of certain linearly growing balls not to be filled with the particles of our system (Lemma 3.5) as opposed to an a.s. statement about the entire trajectory as in Proposition 3.1. This is however sufficient for our purposes. The main ingredients here are exponential estimates for the Eden model (or first passage percolation model), comparison of the Eden model with our process, and once again Lemma 5.10. The Eden model is described on page 7.

In the third part, the most technical in our opinion, we actually prove the theorem using the previous two parts. We define a specially designed collection of stopping times $\{T_\lambda(x), x\in\mathbb{R}^{\mathrm{d}}\}$ and $\{T_\lambda(x,y), x,y\in\mathbb{R}^{\mathrm{d}}\}$ depending on an additional parameter $\lambda>0$ (see (24) and (25)). The strong Markov property of (η_t) (Proposition 5.8 and Corollary 5.9) allows us to apply Liggett's subadditive ergodic theorem to show that for any $x\in\mathbb{R}^{\mathrm{d}}$, $(T_\lambda(tx))_{t\ge 0}$ grows linearly with t ((32) and Lemma 3.8). We then move on to prove that the limit $\lim\limits_{t\to\infty}\frac{T_\lambda(tx)}{t}$ does not depend on x (Lemma 3.9) and is strictly positive (Lemma 3.10). The bulk of the final part of the proof of Theorem 2.8 is contained in Lemmas 3.12 and 3.13, where we show the necessary a.s. inclusions dropping λ along the way.

Proposition 3.1. *There exists* $C_{upb}>0$ *such that a.s. for large* t*,*

$$\eta_t\subset B(\mathbf{0},C_{upb}t) \tag{15}$$

Remark. The index 'upb' hints on 'upper bound'.

Proof. It is sufficient to show that for $\mathbf{e}=(1,0,...,0)\in\mathbb{R}^{\mathrm{d}}$ there exists $C>0$ such that a.s. for large t

$$\max\{\langle x,\mathbf{e}\rangle : x\in\eta_t\}\subset Ct. \tag{16}$$

Indeed, if (16) holds, then by Proposition 5.7 it is true if we replace $\mathbf{e}$ with any other unit vector along any of the 2d directions in $\mathbb{R}^{\mathrm{d}}$, and hence (15) holds too.

For $z \in \mathbb{R}$, $y = (y_1, ..., y_{\mathrm{d}-1}) \in \mathbb{R}^{\mathrm{d}-1}$ we define $z \circ y$ to be the concatenation $(z, y_1, ..., y_{\mathrm{d}-1}) \in \mathbb{R}^{\mathrm{d}}$. In this proof we denote by $(\bar{\eta}_t)$ the birth process with $\bar{\eta}_0 = \eta_0$ and the birth rate given by the right hand side of (4), namely

$$\bar{b}(x, \eta) = \sum_{y \in \eta} a(x - y). \tag{17}$$

Since $b(x, \eta) \le \bar{b}(x, \eta)$, $x \in \mathbb{R}^{\mathrm{d}}$, $\eta \in \Gamma_0$, we have by Lemma 5.10 a.s. $\eta_t \subset \bar{\eta}_t$ for all $t \ge 0$. Thus, it is sufficient to prove the proposition for $(\bar{\eta}_t)$. The process $(\bar{\eta}_t)$ with rate (17) is in fact a continuous-time continuous-space branching random walk (for an overview of branching random walks and related topics, see e.g. Shi [Shi15]). Denote by $\bar{\eta}_t^{\mathbf{e}}$ the element-wise projection of $\bar{\eta}_t$ onto the line determined by $\mathbf{e}$; that is $\bar{\eta}_t^{\mathbf{e}} = \{x \in \mathbb{R}^1 \mid x = \langle y, \mathbf{e} \rangle \text{ for some } y \in \eta_t\}$. The process $(\bar{\eta}_t^{\mathbf{e}})$ is itself a branching random walk, and by Corollary 2 in Biggins [Big95], the position of the rightmost particle $X_t^{\mathbf{e}}$ of $(\bar{\eta}_t^{\mathbf{e}})$ at time t satisfies

$$\lim_{t \to \infty} \frac{X_t^{\mathbf{e}}}{t} \to \gamma \tag{18}$$

for a certain $\gamma \in (0, \infty)$. The conditions from the Corollary 2 from [Big95] are satisfied because of Condition 2.2. Indeed, $(\bar{\eta}_t^{\mathbf{e}})$ is the branching random walk with the birth kernel

$$\bar{a}^{\mathbf{e}}(z) = \int\limits_{y \in \mathbb{R}^{\mathrm{d}-1}} a(z \circ y) dy,$$

that is, $(\bar{\eta}_t^{\mathbf{e}})$ is the a birth process on $\mathbb{R}^1$ with the birth rate

$$\bar{b}(x, \eta) = \sum_{y \in \eta} \bar{a}^{\mathbf{e}}(x - y), \qquad x \in \mathbb{R}, \ \eta \in \Gamma_0(\mathbb{R}).$$

Note that $a^{\mathbf{e}}(z) = a(z)$ if $\mathrm{d} = 1$. Hence, in the notation of [Big95] for $\theta < 0$

$$m(\theta, \phi) = \int\limits_{\mathbb{R} \times \mathbb{R}_+} e^{-\theta z} e^{-\phi \tau} \bar{a}^{\mathbf{e}}(z) dz d\tau = \frac{1}{\phi} \int\limits_{\mathbb{R}} e^{-\theta |z|} \bar{a}^{\mathbf{e}}(z) dz = \frac{1}{\phi} \int\limits_{\mathbb{R}} e^{-\theta |z|} dz \int\limits_{y \in \mathbb{R}^{\mathrm{d}-1}} a(z \circ y) dy$$

$$= \frac{1}{\phi} \int\limits_{\mathbb{R}^{\mathrm{d}}} e^{-\theta |\langle x, \mathbf{e} \rangle|} a(x) dx \le \frac{1}{\phi} \int\limits_{\mathbb{R}^{\mathrm{d}}} e^{-\theta |x|} a(x) dx,$$

and thus $\alpha(\theta) < \infty$ for a negative θ satisfying $\int\limits_{\mathbb{R}^{\mathrm{d}}} e^{-\theta x} a(x) dx < \infty$ (the functions $m(\theta, \phi)$ and $\alpha(\theta)$ are defined in [Big95] at the beginning of Section 3).

Since (16) follows from (18), the proof of the proposition is now complete. □

Next, using a comparison with the Eden model (see Eden [Ede61]), we will show that the system grows not slower than linearly (in the sense of Lemma 3.5 below). The Eden model is a model of tumor growth on the lattice $\mathbb{Z}^{\mathrm{d}}$. The evolution starts from a single particle at the origin. A site once occupied stays occupied forever. A vacant site becomes occupied at rate $\lambda > 0$ if at least one of its neighbors is occupied. Let us mention that this model is closely related to the first passage percolation model, see e.g. Kesten [Kes87] and Auffinger, Damron, and Hanson [ADH15]. In fact, the two models coincide if the passage times ([Kes87]) have exponential distribution.

For $z = (z_1, ..., z_d) \in \mathbb{Z}^d$, let $|z|_1 = \sum_{i=1}^{d} |z_i|$.

Lemma 3.2. *Consider the Eden model starting from a single particle at the origin. Then there exists a constant $\tilde{C} > 0$ such that for every $z \in \mathbb{Z}^d$ and time $t \geq \frac{4e^2}{\lambda^2(e-1)^2} \vee \tilde{C}|z|_1$,*

$$P\{z \text{ is vacant at } t\} \leq e^{-\sqrt{t}}. \tag{19}$$

Proof. Let σ_z be the time when z becomes occupied. Let v be a path on the integer lattice of length $m = \text{length}(v)$ starting from $\mathbf{0}$ and ending in z, so that $v_0 = \mathbf{0}$, $v_m = z$, $v_i \in \mathbb{Z}^d$ and $|v_i - v_{i-1}| = 1$, $i = 1, ..., m$. Define $\sigma(v)$ as the time it takes for the Eden model to move along the path v; that is, if $v_0, ..., v_j$ are occupied, then a birth can only occur at v_{j+1}. By construction $\sigma(v)$ is distributed as the sum of length(v) independent unit exponentials (the so called passage times; see e.g. [Kes87] or [ADH15]). We have

$$\sigma_z = \inf\{\sigma(v) : v \text{ is a path from } \mathbf{0} \text{ to } z\}.$$

Hence σ_z is dominated by the sum of $|z|_1$ independent unit exponentials, say $\sigma_z \leq Z_1 + ... + Z_{|z|_1}$.

We have the equality of the events

$$\{z \text{ is vacant at } t\} = \{\sigma_z > t\}.$$

Note that $Ee^{\lambda(1-\frac{1}{e})Z_1} = e$. Using Chebyshev's inequality $P\{Z > t\} \leq Ee^{\lambda(1-\frac{1}{e})(Z-t)}$, we get

$$\begin{aligned} P\{\sigma_z > t\} &\leq P\{Z_1 + ... + Z_{|z|_1} > t\} \leq E\exp\{\lambda(1 - \frac{1}{e})(Z_1 + ... + Z_{|z|_1} - t)\} \\ &= \left[Ee^{\lambda(1-\frac{1}{e})Z_1}\right]^{|z|_1} e^{-\lambda(1-\frac{1}{e})t} = e^{|z|_1} e^{-\lambda(1-\frac{1}{e})t}. \end{aligned}$$

Since

$$\frac{1}{2}\lambda(1 - \frac{1}{e})t \geq \sqrt{t},$$

for $t \geq \frac{4e^2}{\lambda^2(e-1)^2}$, we may take $\tilde{C} = \frac{2e}{\lambda(e-1)}$. □

We now continue to work with the Eden model.

Lemma 3.3. *For the Eden model starting from a single particle at the origin, there are constants $c_1, t_0 > 0$ such that*

$$P\{\text{there is a vacant site in } B(0, c_1 t) \cap \mathbb{Z}^d \text{ at } t\} \leq e^{-\sqrt[4]{t}}, t \geq t_0 \tag{20}$$

Proof. By the previous lemma for $c_1 < \frac{1}{\tilde{C}}$,

$$\begin{aligned} &P\{\text{there is a vacant site in } B(0, c_1 t) \cap \mathbb{Z}^d \text{ at } t\} \\ &\leq \sum_{z \in B(0,c_1 t) \cap \mathbb{Z}^d} P\{z \text{ is vacant at } t\} \end{aligned}$$

$$\leq |B(0, c_1 t)| e^{-\sqrt{t}},$$

where $|B(0, c_1 t)|$ is the number of integer points (that is, points whose coordinates are integers) inside $B(0, c_1 t)$. It remains to note that $|B(0, c_1 t)|$ grows only polynomially fast in t. □

Definition 3.4. Let the growth process $(\alpha_t)_{t\geq 0}$ be a $\mathbb{Z}_+^{\mathbb{Z}^{\mathrm{d}}}$-valued process with

$$\alpha(z) \to \alpha(z) + 1 \quad \text{at rate } \lambda \mathbb{1}\Big\{ \sum_{\substack{y \in \mathbb{Z}^{\mathrm{d}}: \\ |z-y| \leq 1}} \alpha(y) > 0 \Big\}, \quad z \in \mathbb{Z}^{\mathrm{d}},\ \alpha \in \mathbb{Z}_+^{\mathbb{Z}^{\mathrm{d}}},\ \sum_{y \in \mathbb{Z}^{\mathrm{d}}} \alpha(y) < \infty, \tag{21}$$

where $\lambda > 0$.

Clearly, Lemma 3.3 also applies to $(\alpha_t)_{t\geq 0}$, since it dominates the Eden process. Recall that r appears in (6), and (ξ_t) is defined in (11).

Lemma 3.5. *There are $c, s_0 > 0$ such that*

$$P\{B(\mathbf{0}, cs) \not\subset \xi_s\} \leq e^{-\sqrt[4]{s}}, \quad s \geq s_0. \tag{22}$$

Proof. For $x \in \mathbb{R}^{\mathrm{d}}$ let $z_x \in \frac{r}{2\mathrm{d}}\mathbb{Z}^{\mathrm{d}}$ be uniquely determined by $x \in z_x + (-\frac{r}{4\mathrm{d}}, \frac{r}{4\mathrm{d}}]^{\mathrm{d}}$. Recall that c_0 appears in Condition 2.5. Define

$$\bar{b}(x, \eta) = c_0 \mathbb{1}\{z_x \sim z_y \text{ for some } y \in \eta\}, \tag{23}$$

where $z_x \sim z_y$ means that z_x and z_y are neighbors on $\frac{r}{2\mathrm{d}}\mathbb{Z}^{\mathrm{d}}$. Let $(\bar{\eta}_t)_{t\geq 0}$ be the birth process with birth rate $\bar{b}$. Note that by (6) for every $\eta \in \Gamma_0$,

$$\bar{b}(x, \eta) \leq b(x, \eta), \quad x \in \mathbb{R}^{\mathrm{d}},$$

hence a.s. $\bar{\eta}_t \subset \eta_t$ by Lemma 5.10, $t \geq 0$. Then the 'projection' process defined by

$$\bar{\bar{\eta}}_t(z) = \sum_{x \in \bar{\eta}_t} \mathbb{1}\{x \in z + (-\frac{r}{4\mathrm{d}}, \frac{r}{4\mathrm{d}}]^{\mathrm{d}}\}, \quad z \in \frac{r}{2\mathrm{d}}\mathbb{Z}^{\mathrm{d}},$$

is the process $(\alpha_t)_{t\geq 0}$ from Definition 3.4 with $\lambda = c_0 \left(\frac{r}{2\mathrm{d}}\right)^{\mathrm{d}}$ and the 'geographic' space $\frac{r}{2\mathrm{d}}\mathbb{Z}^{\mathrm{d}}$ instead of $\mathbb{Z}^{\mathrm{d}}$, that is, taking values in $\mathbb{Z}_+^{\frac{r}{2\mathrm{d}}\mathbb{Z}^{\mathrm{d}}}$ instead of $\mathbb{Z}_+^{\mathbb{Z}^{\mathrm{d}}}$. Since $\bar{\bar{\eta}}_t(z_x) > 0$ implies that $x \in \xi_t$, the desired result follows from Lemma 3.3 and the fact that Lemma 3.3 also applies to $(\alpha_t)_{t\geq 0}$. □

Notation and conventions. In what follows for $x, y \in \mathbb{R}^{\mathrm{d}}$ we define

$$[x, y] = \{z \in \mathbb{R}^{\mathrm{d}} \mid z = tx + (1-t)y, t \in [0, 1]\}.$$

We call $[x, y]$ an interval. Similarly, open or half-open intervals are defined, for example

$$(x, y] = \{z \in \mathbb{R}^{\mathrm{d}} \mid z = tx + (1-t)y, t \in (0, 1]\}.$$

We also adopt the convention $B(x, 0) = \{x\}$.

For $x \in \mathbb{R}^{\mathrm{d}}$ and $\lambda \in (0,1)$ we define a stopping time $T_\lambda(x)$ (here and below, all stopping times are considered with respect to the filtration $(\mathscr{S}_t)$ introduced after (74)) by

$$T_\lambda(x) = \inf\{t > 0 : |\eta_t \cap B(x, \lambda|x|)| > 0\}, \tag{24}$$

and for $x, y \in \mathbb{R}^{\mathrm{d}}$, we define

$$T_\lambda(x,y) = \inf\left\{t > T_\lambda(x) : |\eta_t^{T_\lambda(x),\{z_\lambda(x)\}} \cap B(y + z_\lambda(x) - x, \lambda|y-x|)| > 0\right\} - T_\lambda(x), \tag{25}$$

where $z_\lambda(x)$ is uniquely defined by $\{z_\lambda(x)\} = \eta_{T_\lambda(x)} \cap B(x, \lambda|x|)$. Note that $\{z_\lambda(x)\}$ is a $\mathscr{S}_{T_\lambda(x)}$-measurable finite random set. Also, $T_\lambda(\mathbf{0}) = 0$ and $T_\lambda(x,x) = 0$ for $x \in \mathbb{R}^{\mathrm{d}}$. To reduce the number of double subscripts, we will sometimes write $z(x)$ instead of $z_\lambda(x)$.

Since for $q \geq 1$

$$\big\{x_1 + x_2 : x_1 \in B(x, \lambda|x|), x_2 \in B((q-1)x, \lambda(q-1)|x|)\big\} = B(qx, \lambda q|x|),$$

we have by attractiveness (recall (10))

$$\begin{aligned} T_\lambda(qx) &\leq T_\lambda(x) + \Big(\inf\{t > 0 : |\eta_t^{T_\lambda(x), \eta_{T_\lambda(x)}} \cap B(qx, \lambda q|x|)| > 0\} - T_\lambda(x)\Big) \\ &\leq T_\lambda(x) + \Big(\inf\{t > 0 : |\eta_t^{T_\lambda(x),\{z_\lambda(x)\}} \cap B(z_\lambda(x) + (q-1)x, \lambda(q-1)|x|)| > 0\} - T_\lambda(x)\Big), \end{aligned}$$

that is,

$$T_\lambda(qx) \leq T_\lambda(x) + T_\lambda(x, qx), \qquad x \in \mathbb{R}^{\mathrm{d}} \setminus \{\mathbf{0}\}. \tag{26}$$

Note that by the strong Markov property (Proposition 5.8 and Corollary 5.9),

$$T_\lambda(x, qx) \overset{(d)}{=} T_\lambda((q-1)x). \tag{27}$$

The following elementary lemma is used in the proof of Lemma 3.7.

Lemma 3.6. *Let $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ be two* d*-dimensional balls.*

(i) There exists a constant $c_{ball}(\mathrm{d}) > 0$ depending on d *only such that if B_1 and B_2 are two balls in $\mathbb{R}^{\mathrm{d}}$ and $x_1 \in B_2$ then*

$$Vol(B_1 \cap B_2) \geq c_{ball}(\mathrm{d})\big(Vol(B_1) \wedge Vol(B_2)\big), \tag{28}$$

where $Vol(B)$ is the d*-dimensional volume of B. (ii) The intersection $B_1 \cap B_2$ contains a ball of radius r_3 provided that*

$$2r_3 \leq (r_1 + r_2 - |x_1 - x_2|) \wedge r_1 \wedge r_2.$$

Proof. (i) Without loss of generality we can assume that $r_1 \leq r_2$. Indeed, if $r_1 > r_2$, then $x_2 \in B_1$, so we can swap B_1 and B_2. Let $B_1' = B(x_1', r_1)$ be the shifted ball B_1 with $x_1' = x_1 + r_1 \frac{x_2 - x_1}{|x_2 - x_1|}$ (see Figure 1). The intersection $B_1' \cap B_1$ is a subset of B_2 and is a union of two identical d-dimensional hyperspherical caps with height $\frac{r_1}{2}$. Using the standard formula for the volume of a hyperspherical cap, we see that we can take

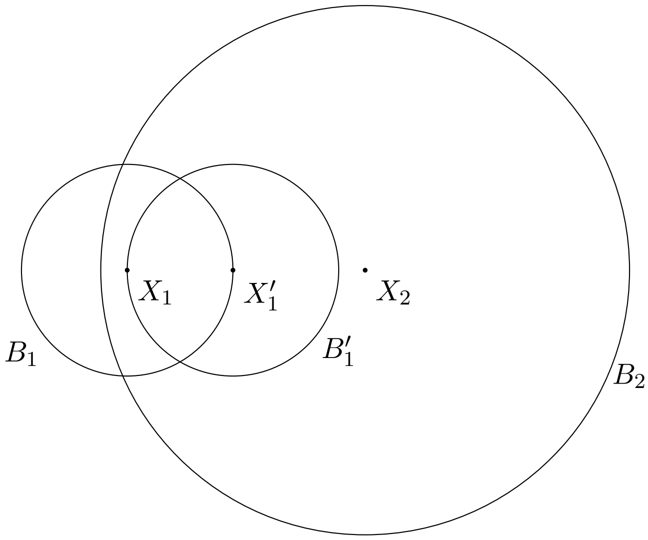

Figure 1: for Lemma 3.6 (i)

$$c_{\text{ball}}(\mathrm{d}) = \frac{V(B_1' \cap B_1)}{V(B_1)} = 2\frac{\Gamma(\frac{d}{2}+1)}{\sqrt{\pi}\Gamma(\frac{d+1}{2})} \int\limits_0^{\frac{\pi}{3}} \sin^d(s) ds.$$

(ii) We have $B_3 \subset B_1 \cap B_2$, where $B_3 = B(x_3, r_3)$ and x_3 is the middle point of the interval $[x_1, x_2] \cap B_1 \cap B_2$.

Lemma 3.7. *For every $x \in \mathbb{R}^{\mathrm{d}}$ and $\lambda > 0$ there exist $A_{x,\lambda}, q_{x,\lambda} > 0$ such that*

$$P\{T_\lambda(x) > s\} \le A_{x,\lambda} e^{-q_{x,\lambda}\sqrt[4]{s}}, \quad s \ge 0. \tag{29}$$

Proof. Let

$$\tau_x = \inf\{s > 0 : x \in \xi_s\}$$

(recall that (ξ_t) is defined in (11)), that is τ_x is the moment when the first point in the ball $B(x, r)$ appears. By Lemma 3.5 for $s \ge s_0 \vee \frac{|x|}{c}$

$$P\{\tau_x > s\} \le P\{x \notin \xi_s\} \le P\{B(\mathbf{0}, |x|) \nsubseteq \xi_s\} \le P\{B(\mathbf{0}, cs) \nsubseteq \xi_s\} \le e^{-\sqrt[4]{s}}. \tag{30}$$

In the case $r \le \lambda |x|$ we have a.s. $T_\lambda(x) \le \tau_x$, and the statement of the lemma follows from (30) since for $s \ge s_0 \vee \frac{|x|}{c}$

$$P\{T_\lambda(x) > s\} \le P\{\tau_x > s\} \le e^{-\sqrt[4]{s}}.$$

Let us now consider the case $r > \lambda |x|$. Denote by $\bar{x} \in B(x, r)$ the place where the particle is born at τ_x. For $t \ge 0$ on $\{t > \tau_x\}$ we have

$$\int\limits_{y \in B(x, \lambda|x|)} b(y, \eta_t) dy \ge \int\limits_{y \in B(x, \lambda|x|)} b(y, \{\bar{x}\}) dy \ge \int\limits_{y \in B(x, \lambda|x|)} c_0 \mathbb{1}\{y \in B(\bar{x}, r)\} dy,$$

so that by Lemma 3.6 on $\{t > \tau_x\}$

$$\int\limits_{y\in B(x,\lambda|x|)} b(y,\eta_t)dy \geq \int\limits_{y\in B(x,\lambda|x|)} c_0 \mathbb{1}\{y\in B(\bar{x},r)\}dy$$

$$= c_0 \mathrm{Vol}(B(x,\lambda|x|)\cap B(\bar{x},r)) \geq c_0 c_{\mathrm{ball}}(\mathrm{d})\mathrm{Vol}(B(x,\lambda|x|)) = c_0 c_{\mathrm{ball}}(\mathrm{d}) V_{\mathrm{d}} \lambda^{\mathrm{d}}|x|^{\mathrm{d}},$$

where $V_{\mathrm{d}} = \mathrm{Vol}(B(\mathbf{0},1))$, hence

$$P\{T_\lambda(x) - \tau_x > s'\} \leq P\{\inf\{t>0 : \eta_t^{\tau_x,\{\bar{x}\}} \cap B(x,r) \neq \varnothing\} - \tau_x > s'\} \leq e^{-c_0 c_{\mathrm{ball}}(\mathrm{d}) V_{\mathrm{d}} \lambda^{\mathrm{d}} |x|^{\mathrm{d}} s'}.$$

Combining this with (30) yields the desired result. □

Let us fix an $x \in \mathbb{R}^{\mathrm{d}}$, $x \neq \mathbf{0}$, and define for $k, n \in \mathbb{N}$, $k < n$,

$$s_{k,n} = T_\lambda(kx, nx). \tag{31}$$

Note that the random variables $s_{k,n}$ are integrable by Lemma 3.7. The conditions of Liggett's subadditive ergodic theorem, see [Lig85], are satisfied here. Indeed, condition (1.7) in [Lig85] is ensured by (26), while conditions (1.8) and (1.9) in [Lig85] follow from (27) and the strong Markov property of (η_t) (Proposition 5.8 and Corollary 5.9). Thus, there exists $\mu_\lambda(x) \in [0,\infty)$ such that a.s. and in L^1,

$$\frac{s_{0,n}}{n} \to \mu_\lambda(x). \tag{32}$$

Lemma 3.8. *Let $\lambda > 0$. For every $x \neq \mathbf{0}$,*

$$\lim_{t\to\infty} \frac{T_\lambda(tx)}{t} = \mu_\lambda(x). \tag{33}$$

Proof. We know that for every $x \in \mathbb{R}^{\mathrm{d}} \setminus \{\mathbf{0}\}$

$$\lim_{n\to\infty} \frac{T_\lambda(nx)}{n} = \mu_\lambda(x). \tag{34}$$

Denote $\sigma_n = \inf\limits_{y\in[nx,(n+1)x]} T_\lambda(y)$. Since there are only a finite number of particles born in a bounded time interval, this infinum is achieved. So, let $\tilde{z}_n$ be such that $\eta_{\sigma_n} \setminus \eta_{\sigma_n -} = \{\tilde{z}_n\}$. By definition of σ_n, the set

$$\{y \in [nx,(n+1)x] \mid \tilde{z}_n \in B(y,\lambda|y|)\}$$

is not empty. $\{\tilde{z}_n\}$ is an $\mathscr{S}_{\sigma_n}$-measurable finite random set, so we can apply Corollary 5.9 here.

Define now another stopping time

$$\tilde{\sigma}_n = \inf\{t>0 : \xi_t^{\sigma_n,\{\tilde{z}_n\}} \supset B(\tilde{z}_n, \lambda|x| + |x| + 2r)\}.$$

Let us show that

$$\sup_{y\in[nx,(n+1)x]} T_\lambda(y) \leq \tilde{\sigma}_n. \tag{35}$$

For any $y \in [nx,(n+1)x]$,

$$|y - \tilde{z}_n| \leq |\tilde{z}_n - nx| \vee |\tilde{z}_n - (n+1)x| \leq \lambda(n+1)|x| + |x|.$$

Therefore the intersection of the balls $B(\tilde{z}_n, \lambda|x| + |x| + 2r)$ and $B(y, \lambda|y|)$ contains a ball $\tilde{B}$ of radius r by Lemma 3.6, (ii), since

$$\lambda|x| + |x| + 2r + \lambda|y| - \lambda(n+1)|x| - |x| \geq \lambda|x| + 2r + \lambda n|x| - \lambda(n+1)|x| = 2r.$$

Since the radius of $\tilde{B}$ is r and $\xi_{\tilde{\sigma}_n}^{\sigma_n, \{\tilde{z}_n\}} \supset B(\tilde{z}_n, \lambda|x| + |x| + 2r) \supset \tilde{B}$,

$$\eta_{\tilde{\sigma}_n}^{\sigma_n, \{\tilde{z}_n\}} \cap \tilde{B} \neq \varnothing,$$

and hence

$$\eta_{\tilde{\sigma}_n} \cap \tilde{B} \neq \varnothing. \tag{36}$$

Since $\tilde{B} \subset B(y, \lambda|y|)$ for all $y \in [n|x|, (n+1)|x|]$, (36) implies (35).

For $q \geq \big(\lambda|x| + |x| + 2r\big) \vee cs_0$, by Lemma 3.5

$$P\{\tilde{\sigma}_n - \sigma_n \geq \frac{q}{c}\} = P\{B(\tilde{z}_n, \lambda|x| + |x| + 2r) \not\subseteq \xi_{\frac{q}{c}+\sigma_n}^{\sigma_n, \{\tilde{z}_n\}}\}$$
$$\leq P\{B(\tilde{z}_n, q) \not\subseteq \xi_{\frac{q}{c}+\sigma_n}^{\sigma_n, \{\tilde{z}_n\}}\} \leq e^{-\sqrt[4]{\frac{q}{c}}},$$

hence

$$P\{\tilde{\sigma}_n - \sigma_n \geq q'\} \leq e^{-\sqrt[4]{q'}}, \quad q' \geq \big(\frac{\lambda|x| + |x| + 2r}{c}\big) \vee s_0. \tag{37}$$

By the Borel–Cantelli lemma

$$P\{\tilde{\sigma}_n - \sigma_n > \sqrt{n} \text{ for infinitely many } n\} = 0,$$

and since $\sigma_n \leq T_\lambda(nx) \leq \tilde{\sigma}_n$, a.s. for large n

$$\tilde{\sigma}_n < T_\lambda(nx) + \sqrt{n}$$

and

$$\sigma_n \geq T_\lambda(nx) - \sqrt{n}.$$

By (35)

$$\limsup_{n\to\infty} \frac{\sup\limits_{y\in[nx,(n+1)x]} T_\lambda(y)}{n} \leq \limsup_{n\to\infty} \frac{\tilde{\sigma}_n}{n} \leq \limsup_{n\to\infty} \frac{T_\lambda(nx) + \sqrt{n}}{n} \leq \mu_\lambda(x),$$

and

$$\liminf_{n\to\infty} \frac{\inf\limits_{y\in[nx,(n+1)x]} T_\lambda(y)}{n} = \liminf_{n\to\infty} \frac{\sigma_n}{n} \geq \limsup_{n\to\infty} \frac{T_\lambda(nx) - \sqrt{n}}{n} \geq \mu_\lambda(x).$$

□

Lemma 3.9. *The ratio* $\frac{\mu_\lambda(x)}{|x|}$ *in* (32) *does not depend on* x, $x \neq \mathbf{0}$.

Proof. First let us note that for every $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and every $q > 0$,

$$\mu_\lambda(x) = \frac{\mu_\lambda(qx)}{q} \tag{38}$$

by Lemma 3.8.

On the other hand, if $|x| = |y|$ then by Proposition 5.7

$$\mu_\lambda(x) = \mu_\lambda(y), \tag{39}$$

since the distribution of (η_t) is invariant under rotations and we can consider $\mu_\lambda(x)$ as a functional acting on the trajectory $(\eta_t)_{t\geq 0}$. The statement of the lemma follows from (38) and (39). □

Set

$$\mu_\lambda := \frac{\mu_\lambda(x)}{|x|}, \quad x \neq \mathbf{0}.$$

As λ decreases, $T_\lambda(x)$ increases and therefore μ_λ increases too. Denote

$$\mu = \lim_{\lambda \to 0+} \mu_\lambda. \tag{40}$$

Lemma 3.10. *The constants μ_λ and μ are strictly positive: $\mu_\lambda > 0$, $\mu > 0$.*

Proof. By Proposition 3.1 for x with large $|x|$,

$$\eta_{\frac{(1-\lambda)|x|}{C_{\mathrm{upb}}}} \subset B(\mathbf{0}, (1-\lambda)|x|),$$

hence for every $\lambda \in (0,1)$ for x with large $|x|$

$$T_\lambda(x) \geq \frac{(1-\lambda)|x|}{C_{\mathrm{upb}}}.$$

Thus,

$$\mu_\lambda \geq \frac{(1-\lambda)}{C_{\mathrm{upb}}}$$

and

$$\mu = \lim_{\lambda \to 0+} \mu_\lambda \geq \frac{1}{C_{\mathrm{upb}}}.$$

□

Lemma 3.11. *Let $q, R > 0$. Suppose that for all $\varepsilon \in (0,1)$ a.s. for sufficiently large $n \in \mathbb{N}$*

$$\frac{\eta_{qn}}{qn} \subset (1+\varepsilon)B(\mathbf{0}, R) \qquad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_{qn}}{qn} \right). \tag{41}$$

Then for all $\varepsilon \in (0,1)$ a.s. for sufficiently large $t \geq 0$

$$\frac{\eta_t}{t} \subset (1+\varepsilon)B(\mathbf{0}, R) \qquad \left((1-\varepsilon)B(\mathbf{0}, R) \subset \frac{\xi_t}{t} \text{ respectively} \right).$$

Proof. We consider the first case only – the proof of the other one is similar. Since $\varepsilon \in (0,1)$ is arbitrary, (41) implies that for all $\tilde{\varepsilon} \in (0,1)$ a.s. for large $n \in \mathbb{N}$,

$$\frac{\eta_{q(n+2)}}{qn} \subset (1+\tilde{\varepsilon})B(\mathbf{0}, R).$$

Since a.s. $(\eta_t)_{t\geq 0}$ is monotonically growing, it is sufficient to note that

$$\frac{\eta_t}{t} \subset (1+\varepsilon)B(\mathbf{0}, R) \quad \text{if} \quad \frac{\eta_{\left\lceil \frac{t}{q} \right\rceil q + q}}{\left\lfloor \frac{t}{q} \right\rfloor q} \subset (1+\varepsilon)B(\mathbf{0}, R).$$

□

Recall that c is a constant from Lemma 3.5.

Lemma 3.12. *Let $\varepsilon \in (0,1)$. Then a.s.*

$$(1-\varepsilon)B(\mathbf{0},\mu^{-1}) \subset \frac{\xi_m}{m} \tag{42}$$

for large m of the form $m = (1+\frac{\lambda\mu_\lambda^{-1}}{c})n$, $n \in \mathbb{N}$.

Proof. Let $\lambda = \lambda_\varepsilon > 0$ be chosen so small that

$$(1-\varepsilon)\mu^{-1} \le \frac{1-\frac{\varepsilon}{2}}{1+\frac{\lambda\mu_\lambda^{-1}}{c}}\mu_\lambda^{-1}. \tag{43}$$

Such a λ exists since

$$\lim_{\lambda\to 0+}\frac{\mu_\lambda^{-1}}{1+\frac{\lambda\mu_\lambda^{-1}}{c}} = \mu^{-1}.$$

Choose a finite sequence of points $\{x_j, j = 1,...,N\}$ such that $x_j \in (1-\frac{\varepsilon}{2})B(\mathbf{0},\mu_\lambda^{-1})$ and

$$\bigcup_j B(x_j,\frac{\varepsilon}{4}c) \supset (1-\frac{\varepsilon}{2})B(\mathbf{0},\mu_\lambda^{-1}).$$

Let $\delta > 0$ be so small that $(1+\delta)(1-\frac{\varepsilon}{2}) \le (1-\frac{\varepsilon}{4})$. Since a.s.

$$\frac{T_\lambda(nx_j)}{n|x_j|} \to \mu_\lambda,$$

for large n for every $j \in \{1,...,N\}$

$$T_\lambda(nx_j) \le n|x_j|(1+\delta)\mu_\lambda \le n(1-\frac{\varepsilon}{2})(1+\delta) \le n(1-\frac{\varepsilon}{4}), \tag{44}$$

so that the system reaches the ball $B(nx_j,\lambda n|x_j|)$ before the time $n(1-\frac{\varepsilon}{4})$. Let Q_n be the random event

$$\{T_\lambda(nx_j) \le n(1-\frac{\varepsilon}{4}) \text{ for } j = 1,...,N\} = \{\eta_{n(1-\frac{\varepsilon}{4})} \cap B(nx_j,\lambda n|x_j|) \ne \varnothing, \text{ for } j = 1,...,N\}.$$

Note that $P(Q_n) \to 1$ by (44), and even

$$P\{\bigcup_{m\in\mathbb{N}}\bigcap_{i=m}^{\infty} Q_i\} = 1. \tag{45}$$

In other words, a.s. for large i all Q_i occur.

Let $\bar z(nx_j)$ be defined as $z(nx_j)$ on Q_n and as nx_j on the complement $\Omega \setminus Q_n$ (recall that $z(x) = z_\lambda(x)$, $x \in \mathbb{R}^{\mathrm{d}}$, was defined after (25)). The set $\{\bar z(nx_j)\}$ is a finite random $\mathscr{S}_{n(1-\frac{\varepsilon}{4})}$-measurable set.

Using Lemma 3.5, we will show that after an additional time interval of length $(\frac{\varepsilon}{4}+\frac{\lambda\mu_\lambda^{-1}}{c})n$ the entire ball $(1-\frac{\varepsilon}{2})nB(\mathbf{0},\mu_\lambda^{-1})$ is covered by (ξ_t), that is, a.s. for large n

$$(1-\frac{\varepsilon}{2})nB(\mathbf{0},\mu_\lambda^{-1}) \subset \xi_{n(1-\frac{\varepsilon}{4})+(\frac{\varepsilon}{4}+\frac{\lambda\mu_\lambda^{-1}}{c})n} = \xi_{n+\frac{\lambda n\mu_\lambda^{-1}}{c}}. \tag{46}$$

Indeed, since

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda|x_j|n) \subset B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n),$$

the series

$$\sum_{n\in\mathbb{N}} P\{B(nx_j, c\frac{\varepsilon}{4}n) \not\subset \xi_{n+\frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}),\{\bar{z}(nx_j)\})} \text{ for some } j\}$$

$$\leq \sum_{n\in\mathbb{N}} P\{B(\bar{z}(nx_j), c\frac{\varepsilon}{4}n + \lambda\mu_\lambda^{-1}n) \not\subset \xi_{n+\frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}),\{\bar{z}(nx_j)\})} \text{ for some } j\}$$

converges by Lemma 3.5, thus a.s. for large n,

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset \xi_{n+\frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}),\{\bar{z}(nx_j)\})}, \quad j = 1, ..., N. \tag{47}$$

By (45) a.s. for large n

$$B(nx_j, c\frac{\varepsilon}{4}n) \subset \xi_{n+\frac{\lambda\mu_\lambda^{-1}n}{c}}^{(n(1-\frac{\varepsilon}{4}),\{z(nx_j)\})}, \quad j = 1, ..., N. \tag{48}$$

Hence the choice of $\{x_j,\ j = 1, ..., N\}$ and (48) yield (46). Because of our choice of λ,

$$(1-\varepsilon)nB(\mathbf{0}, \mu^{-1}) \subset \frac{(1-\frac{\varepsilon}{2})}{(1+\frac{\lambda\mu_\lambda^{-1}}{c})} nB(\mathbf{0}, \mu_\lambda^{-1}),$$

which in conjunction with (46) implies that (42) holds a.s. for large m of the form $(1+\frac{\lambda\mu_\lambda^{-1}}{c})n$, where $n \in \mathbb{N}$. □

Lemma 3.13. *Let $\varepsilon \in (0,1)$. Then a.s. for large $n \in \mathbb{N}$*

$$\frac{\eta_n}{n} \subset (1+\varepsilon)B(\mathbf{0}, \mu^{-1}). \tag{49}$$

Proof. Let $\lambda = \lambda_\varepsilon > 0$ be so small that

$$(1+\frac{\varepsilon}{2})B(\mathbf{0}, \mu_\lambda^{-1}) \subset (1+\varepsilon)B(\mathbf{0}, \mu^{-1}) \tag{50}$$

Let $q \in (\varepsilon, \infty)$ and A be the annulus

$$A := (1+q)B(\mathbf{0}, \mu_\lambda^{-1}) \setminus (1+\frac{1}{2}\varepsilon)B(\mathbf{0}, \mu_\lambda^{-1}), \tag{51}$$

and $\{x_j, j = 1, ..., N\}$ be a finite sequence such that $x_j \in A$ and

$$\bigcup_j B(x_j, \lambda|x_j|) \supset A.$$

Define $F := \{\eta_n \cap nA \neq \varnothing \text{ infinitely often}\}$. On F there exists a (random) $i \in \{1, ..., N\}$ such that the intersection

$$\eta_n \cap nB(x_i, \lambda|x_i|) \tag{52}$$

is non-empty infinitely often. Define also

$$F_i := \{\eta_n \cap nB(x_i, \lambda|x_i|) \neq \varnothing \text{ infinitely often}\} \tag{53}$$

Note that $F \subset \bigcup_{i=1}^{N} F_i$.

On F_i we have

$$T_\lambda(nx_i) \le n$$

infinitely often, hence our choice of A implies

$$\liminf_{n\to\infty} \frac{T_\lambda(nx_i)}{n|x_i|} \le \liminf_{n\to\infty} \frac{n}{(1+\frac{1}{2}\varepsilon)\mu_\lambda^{-1} n} = \mu_\lambda \frac{1}{(1+\frac{1}{2}\varepsilon)}.$$

The last inequality and Lemma 3.8 imply that $P(F_i) = 0$ for every $i \in \{1, ..., N\}$. Hence $P(F) = 0$ too. Setting $q = 2\mu_\lambda C_{upb} + 1$, so that the radius of the ball on the left-hand side of (50)

$$q\mu_\lambda^{-1} > 2C_{upb},$$

by Proposition 3.1 and the definition of F we get a.s. for large n,

$$\frac{\eta_n}{n} \subset (1+\frac{1}{2}\varepsilon)B(\mathbf{0}, \mu_\lambda^{-1}) \tag{54}$$

and the statement of the lemma follows from (50) and (54). □

Proof of Theorem 2.8. The theorem follows from Lemmas 3.11, 3.12, and 3.13. Note that

$$\frac{\xi_n}{n} \subset (1+\varepsilon)B(\mathbf{0}, \mu^{-1}). \tag{55}$$

is obtained from Lemma 3.13 by replacing ε in (49) with $\frac{\varepsilon}{2}$.

□

4 Proof of Theorem 2.10

We precede the proof of Theorem 2.10 with an auxiliary lemma about Markovian functionals of a general Markov chain.

Let $(S, \mathscr{B}(S))$ be a Polish (state) space. Consider a (time-homogeneous) Markov chain on $(S, \mathscr{B}(S))$ as a family of probability measures on S^∞. Namely, on the measurable space $(\bar\Omega, \mathscr{F}) = (S^\infty, \mathscr{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s\in S}$ such that for the coordinate mappings

$$X_n : \bar\Omega \to S,$$
$$X_n(s_1, s_2, ...) = s_n,$$

the process $X := \{X_n\}_{n\in\mathbb{Z}_+}$ is a Markov chain such that for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$

$$P_s\{X_{n+m_j} \in A_j, j = 1, ..., l \mid \mathscr{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, ..., l\}.$$

Here $A_j \in \mathscr{B}(S)$, $m_j \in \mathbb{N}$, $l \in \mathbb{N}$, $\mathscr{F}_n = \sigma\{X_1, ..., X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathscr{B}(S) \to [0, 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S, \ A \in \mathscr{B}(S).$$

Consider a transformation of the chain X, $Y_n = f(X_n)$, where $f : S \to \mathbb{R}$ is a Borel-measurable function. Here we will give sufficient conditions for $Y = \{Y_n\}_{n\in\mathbb{Z}_+}$ to be a Markov chain. A very similar question was discussed by Burke and Rosenblatt [BR58] for discrete space Markov chains.

Lemma 4.1. *Assume that for any bounded Borel function $h : S \to S$*

$$E_s h(X_1) = E_q h(X_1) \textit{ whenever } \; f(s) = f(q), \tag{56}$$

Then Y is a Markov chain.

Remark. Condition (56) is the equality of distributions of X_1 under two different measures, P_s and P_q.

Proof. For the natural filtrations of the processes X and Y we have an inclusion

$$\mathscr{F}_n^X \supset \mathscr{F}_n^Y, \quad n \in \mathbb{N}, \tag{57}$$

since Y is a function of X. For $k \in \mathbb{N}$ and bounded Borel functions $h_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, ..., k$,

$$\begin{gathered} E_s\left[\prod_{j=1}^k h_j(Y_{n+j}) \mid \mathscr{F}_n^X\right] = E_{X_n}\prod_{j=1}^k h_j(f(X_j)) = \\ \int_S Q(x_0, dx_1)h_1(f(x_1))\int_S Q(x_1, dx_2)h_2(f(x_2))...\int_S Q(x_{n-1}, dx_n)h_n(f(x_n))\Big|_{x_0=X_n} \end{gathered} \tag{58}$$

To transform the last integral, we introduce a new kernel: for $y \in f(S)$ chose $x \in S$ with $f(x) = y$, and then for $B \in \mathscr{B}(\mathbb{R})$ define

$$\overline{Q}(y, B) = Q(x, f^{-1}(B)). \tag{59}$$

The expression on the right-hand side does not depend on the choice of x because of (56). To make the kernel $\overline{Q}$ defined on $\mathbb{R} \times \mathscr{B}(\mathbb{R})$, we set

$$\overline{Q}(y, B) = \mathbb{1}_{\{0\in B\}}, \; y \notin f(S).$$

Then, setting $z_n = f(x_n)$, we obtain from the change of variables formula for the Lebesgue integral that

$$\int_S Q(x_{n-1}, dx_n)h_n(f(x_n)) = \int_{\mathbb{R}} \overline{Q}(f(x_{n-1}), dz_n)h_n(z_n).$$

Likewise, setting $z_{n-1} = f(x_{n-1})$, we get

$$\begin{gathered} \int_S Q(x_{n-2}, dx_{n-1})h_n(f(x_{n-1}))\int_S Q(x_{n-1}, dx_n)h_n(f(x_n)) = \\ \int_S Q(x_{n-2}, dx_{n-1})h_n(f(x_{n-1}))\int_{\mathbb{R}} \overline{Q}(f(x_{n-1}), dz_n)h_n(z_n) = \\ \int_{\mathbb{R}} \overline{Q}(f(x_{n-2}), dz_{n-1})h_n(z_{n-1})\int_{\mathbb{R}} \overline{Q}(z_{n-1}, dz_n)h_n(z_n). \end{gathered}$$

Proceeding further, we obtain

$$\int_S Q(x_0, dx_1) h_1(f(x_1)) \int_S Q(x_1, dx_2) h_2(f(x_2)) ... \int_S Q(x_{n-1}, dx_n) h_n(f(x_n)) =$$

$$\int_{\mathbb{R}} \overline{Q}(z_0, dz_1) h_1(z_1) \int_{\mathbb{R}} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{R}} \overline{Q}(z_{n-1}, dz_n) h_n(z_n),$$

where $z_0 = f(x_0)$.

Thus,

$$E_s \left[\prod_{j=1}^{k} h_j(Y_{n+j}) \mid \mathscr{F}_n^X \right] =$$

$$\int_{\mathbb{R}} \overline{Q}(f(X_0), dz_1) h_1(z_1) \int_{\mathbb{R}} \overline{Q}(z_1, dz_2) h_2(z_2) ... \int_{\mathbb{R}} \overline{Q}(z_{n-1}, dz_n) h_n(z_n).$$

This equality and (57) imply that Y is a Markov chain. □

Remark 4.2. From the proof it follows that $\overline{Q}$ is the transition probability kernel for the chain $\{f(X_n)\}_{n \in \mathbb{Z}_+}$.

Remark 4.3. Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space because the property of a process to be a Markov chain depends on its distribution only.

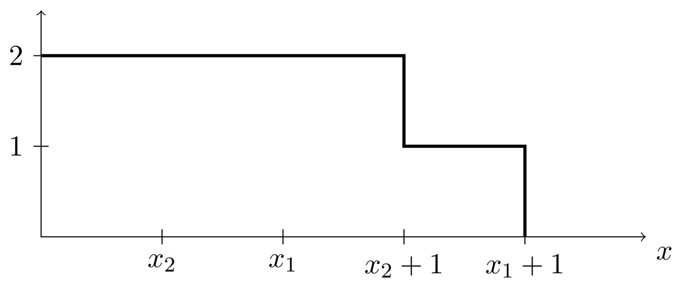

Figure 2: The plot of $b(\cdot, \eta_t)$.

Proof of Theorem 2.10. Without any loss of generality, we will consider the speed of propagation in one direction only, say toward $+\infty$. Let $x_1(t)$ and $x_2(t)$ denote the positions of the rightmost particle and the second rightmost particle, respectively ($x_2(t) = 0$ until first two births occurs inside $(0, +\infty)$). Let us observe that $b(x, \eta_t) \equiv 2$ on $(0, x_2(t) + 1]$, and $X = (x_1(t), x_2(t))$ is a continuous-time pure jump Markov process on $\{(x_1, x_2) \mid x_1 \geq x_2 \geq 0, x_1 - x_2 \leq 1\}$ with transition densities

$$\begin{aligned}
(x_1, x_2) &\to (v, x_1) \quad \text{at rate } 1, \quad v \in (x_2 + 1, x_1 + 1]; \\
(x_1, x_2) &\to (v, x_1) \quad \text{at rate } 2, \quad v \in (x_1, x_2 + 1]; \\
(x_1, x_2) &\to (x_1, v) \quad \text{at rate } 2, \quad v \in (x_2, x_1].
\end{aligned} \tag{60}$$

(to be precise, the above is true from the moment the first birth inside $\mathbb{R}_+$ occurs).

Furthermore, $z(t) := x_1(t) - x_2(t)$ satisfies

$$E\{f(z(t+\delta)) \mid x_1(t) = x_1, x_2(t) = x_2\} = E\{f(z(t+\delta)) \mid x_1(t) = x_1 + h, x_2(t) = x_2 + h\}$$

for every $h > 0$ and every Borel bounded function f. In other words, transition rates of $(z(t))_{t\geq 0}$ are entirely determined by the current state of $(z(t))_{t\geq 0}$. Therefore, by Lemma 4.1, $(z(t))_{t\geq 0}$ is itself a pure jump Markov process on $[0,1]$ (Lemma 4.1 ensures that the embedded Markov chain of $(z(t))_{t\geq 0}$ is indeed a discrete-time Markov process). The transition densities of $(z(t))_{t\geq 0}$ are

$$\begin{aligned} q(x,y) &= 4\mathbb{1}\{y \leq x\} + 2\mathbb{1}\{x \leq y \leq 1-x\} + \mathbb{1}\{y \geq 1-x\}, \quad x \leq \frac{1}{2}, y \in [0,1], \\ q(x,y) &= 4\mathbb{1}\{y \leq 1-x\} + 3\mathbb{1}\{1-x \leq y \leq x\} + \mathbb{1}\{y \geq x\}, \quad x \geq \frac{1}{2}, y \in [0,1]. \end{aligned} \tag{61}$$

Note that the total jump rate out of x is $q(x) := \int_0^1 q(x,y)dy = 2+x$. The process $(z(t))_{t\geq 0}$ is a regular Harris recurrent Feller process with the Lebesgue measure on $[0,1]$ being a supporting measure (see e.g. [Kal02, Chapter 20]). Hence a unique invariant measure exists and has a density g with respect to the Lebesgue measure. The equation for g is

$$\int_0^1 q(x,y)g(x)dx = q(y)g(y). \tag{62}$$

Set

$$f(x) = g(x)q(x)\left(\int_0^1 g(y)q(y)dy\right)^{-1}, \quad x \in [0,1].$$

It is clear that f is again a density (as an aside we point out that f is the density of invariant distribution of the embedded Markov chain of $(z(t))_{t\geq 0}$). Equation (62) becomes

$$f(y) = \int_0^1 \frac{q(x,y)}{q(x)} f(x)ds,$$

which after some calculations transforms into

$$f(y) = 2\int_0^{\frac{1}{2}} \frac{f(x)dx}{2+x} + 2\int_y^{\frac{1}{2}} \frac{f(x)dx}{2+x} + 3\int_{\frac{1}{2}}^1 \frac{f(x)dx}{2+x} + \int_{\frac{1}{2}}^{1-y} \frac{f(x)dx}{2+x}, \quad y \leq \frac{1}{2}, \tag{63}$$

$$f(y) = \int_0^{\frac{1}{2}} \frac{f(x)dx}{2+x} + \int_0^{1-y} \frac{f(x)dx}{2+x} + \int_{\frac{1}{2}}^1 \frac{f(x)dx}{2+x} + 2\int_y^1 \frac{f(x)dx}{2+x}, \quad y \geq \frac{1}{2}. \tag{64}$$

Differentiating (63), (64) with respect to y, we find that f solves the equation

$$\frac{df}{dx}(x) = -2\frac{f(x)}{2+x} - \frac{f(1-x)}{3-x}, \quad x \in [0,1]. \tag{65}$$

Let

$$\varphi(x) := \left[(2+x)^2(3-x)^2\right]f(x), \quad x \in [0,1].$$

Then (65) becomes

$$(3-x)\frac{d\varphi}{dx}(x) + 2\varphi(x) + \varphi(1-x) = 0, \quad x \in [0,1]. \tag{66}$$

Looking for solutions to (66) among polynoms, we find that $\varphi(x) = c(4-3x)$ is a solution. By direct substitution we can check that

$$f(x) = \frac{c(4-3x)}{(2+x)^2(3-x)^2} \quad x \in [0,1] \tag{67}$$

solves (63)-(64). The constant $c > 0$ can be computed, but is irrelevant for our purposes. Hence, after some more computation,

$$g(x) = \frac{36(4-3x)}{(2+x)^3(3-x)^2}, \quad x \in [0,1]. \tag{68}$$

Note that we do not prove analytically that equation (63), (64) has a unique solution. However, uniqueness for non-negative integrable solutions follows from the uniqueness of the invariant distribution for $(z(t))_{t\geq 0}$. Let l be the Lebesgue measure on $\mathbb{R}$. By an ergodic theorem for Markov processes, see e.g. [Kal02, Theorem 20.21 (i)], for any $0 \leq p < p' \leq 1$,

$$\lim_{t\to\infty} \frac{l\{s : z(s) \in [p,p'], 0 \leq s \leq t\}}{t} \to \int_p^{p'} g(x)dx. \tag{69}$$

Conditioned on $z(t) = z$, the transition densities of $x_1(t)$ are

$$\begin{aligned} x_1 &\to x_1 + v \quad \text{at rate } 2, \quad v \in (0, 1-z]; \\ x_1 &\to x_1 + v \quad \text{at rate } 1, \quad v \in (1-z, 1]. \end{aligned} \tag{70}$$

Hence by (68) the speed of propagation is

$$\int_0^1 g(z)dz \left[\int_0^{1-z} 2ydy + \int_{1-z}^1 ydy\right] = \int_0^1 g(z)(1-z+\frac{1}{2}z^2)dz = \frac{144\ln(3) - 144\ln(2) - 40}{25}. \qquad \square$$

Remark 4.4. We see from the proof that the speed can be computed in a similar way for the birth rates of the form

$$b_k(x,\eta) = k \wedge \left(\sum_{y\in\eta} \mathbb{1}\{|x-y| \leq 1\}\right), \tag{71}$$

where $k \in (1,2)$. However, the computations quickly become unwieldy.

5 The construction and properties of the process

Here we proceed to construct the process as a unique solution to a stochastic integral equation. First such a scheme was carried out by Massoulié [Mas98]. This method can be deemed an analog of the construction from graphical representation. We follow here [Bez15].

Remark 5.1. Of course, the process starting from a fixed initial condition we consider here can be constructed as the minimal jump process (pure jump type Markov processes in the terminology of [Kal02]) as is done for example in [EW03]. Note however that we use coupling of infinitely many processes starting at different time points from different initial conditions, so we here employ another method.

Recall that

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

and the σ-algebra on Γ_0 was introduced in (3). To construct the family of processes $(\eta_t^{q,A})_{t\geq q}$, we consider the stochastic equation with Poisson noise

$$|\eta_t \cap B| = \int\limits_{(q,t]\times B\times[0,\infty)} \mathbb{1}_{[0,b(x,\eta_{s-})]}(u) N(ds, dx, du) + |\eta_q \cap B|, \quad t \geq q, \; B \in \mathscr{B}(\mathbb{R}^d), \tag{72}$$

where $(\eta_t)_{t\geq q}$ is a cadlag Γ_0-valued solution process, N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$, the mean measure of N is $ds \times dx \times du$. We require the processes N and η_0 to be independent of each other. Equation (72) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathscr{B}(\mathbb{R}^d)$ and $t \geq q$. In the integral on the right-hand side of (72), x is the location and s is the time of birth of a new particle. Thus, the integral over B from q to t represents the number of births inside B which occurred before t.

Let us assume for convenience that $q = 0$. We will make the following assumption on the initial condition:

$$E|\eta_0| < \infty. \tag{73}$$

We say that the process N is *compatible* with an increasing, right-continuous and complete filtration of σ-algebras $(\mathscr{F}_t, t \geq 0)$ if N is adapted, that is, all random variables of the type $N(\bar{T}_1, U)$, $\bar{T}_1 \in \mathscr{B}([0;t])$, $U \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are $\mathscr{F}_t$-measurable, and all random variables of the type $N(t+h, U) - N(t, U)$, $h \geq 0, U \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R}_+)$, are independent of $\mathscr{F}_t$, $N(t, U) = N([0;t], U)$.

Definition 5.2. A *(weak) solution* of equation (72) is a triple $((\eta_t)_{t\geq 0}, N)$, $(\Omega, \mathscr{F}, P)$, $(\{\mathscr{F}_t\}_{t\geq 0})$, where

(i) $(\Omega, \mathscr{F}, P)$ is a probability space, and $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing, right-continuous and complete filtration of sub-σ-algebras of $\mathscr{F}$,

(ii) N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity $ds \times dx \times du$,

(iii) η_0 is a random $\mathscr{F}_0$-measurable element in Γ_0 satisfying (73),

(iv) the processes N and η_0 are independent, N is compatible with $\{\mathscr{F}_t\}_{t\geq 0}$,

(v) $(\eta_t)_{t\geq 0}$ is a cadlag Γ_0-valued process adapted to $\{\mathscr{F}_t\}_{t\geq 0}$, $\eta_t\big|_{t=0} = \eta_0$,

(vi) all integrals in (72) are well-defined,

$$E\int\limits_0^t ds \int\limits_{\mathbb{R}^d} b(x, \eta_{s-}) < \infty, \quad t > 0,$$

(vii) equality (72) holds a.s. for all $t \in [0, \infty]$ and all Borel sets B.

Let

$$\mathscr{S}_t^0 = \sigma\{\eta_0, N([0,q] \times B \times C),\\ q \in [0,t], B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}}), C \in \mathscr{B}(\mathbb{R}_+)\}, \tag{74}$$

and let $\mathscr{S}_t$ be the completion of $\mathscr{S}_t^0$ under P. Note that $\{\mathscr{S}_t\}_{t\geq 0}$ is a right-continuous filtration (see Remark 6.2).

Definition 5.3. A solution of (72) is called *strong* if $(\eta_t)_{t\geq 0}$ is adapted to $(\mathscr{S}_t, t \geq 0)$.

Remark 5.4. In the definition above we considered solutions as processes indexed by $t \in [0,\infty)$. The reformulations for the case $t \in [0,T]$, $0 < T < \infty$, are straightforward. This remark also applies to many of the results below.

Definition 5.5. We say that *joint uniqueness in law* holds for equation (72) with an initial distribution ν if any two (weak) solutions $((\eta_t), N)$ and $((\eta'_t), N')$ of (72), $Law(\eta_0) = Law(\eta'_0) = \nu$, have the same joint distribution:

$$Law((\eta_t), N) = Law((\eta'_t), N').$$

Theorem 5.6. *Pathwise uniqueness, strong existence and joint uniqueness in law hold for equation* (72)*. The unique solution is a Markov process.*

Proof. Without loss of generality assume that $P\{\eta_0 \neq \varnothing\} = 1$. Define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf\{t > 0 : \int\limits_{(\sigma_n, \sigma_n+t]\times B\times[0,\infty)} \mathbb{1}_{[0,b(x,\zeta_{\sigma_n})]}(u) N(ds,dx,du) > 0\} + \sigma_n, \quad \sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^{\mathrm{d}} : N(\{\sigma_{n+1}\} \times \{x\} \times [0, b(x, \zeta_{\sigma_n})]) > 0\}$. The points z_n are uniquely determined a.s. Furthermore, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s by (6). We define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n, \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathscr{F}_{\sigma_n}$-measurable. By direct substitution we see that $(\zeta_t)_{t\geq 0}$ is a strong solution to (72) on the time interval $t \in [0, \lim\limits_{n\to\infty} \sigma_n)$. Although we have not defined what is a solution, or a strong solution, on a random time interval, we do not discuss it here. Instead we are going to show that

$$\lim_{n\to\infty} \sigma_n = \infty \quad \text{a.s.} \tag{75}$$

The process $(\zeta_t)_{t\in[0, \lim\limits_{n\to\infty}\sigma_n)}$ has the Markov property, because the process N has the strong Markov property and independent increments. Indeed, conditioning on $\mathscr{I}_{\sigma_n}$,

$$E\left[\mathbb{1}_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x\in B\}} \mid \mathscr{I}_{\sigma_n}\right] = \frac{\int\limits_B b(x,\zeta_{\sigma_n})dx}{\int\limits_{\mathbb{R}^{\mathrm{d}}} b(x,\zeta_{\sigma_n})dx},$$

thus the chain $\{\zeta_{\sigma_n}\}_{n\in Z_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n\in Z_+}$, $\sigma_{n+1}-\sigma_n$ are distributed exponentially:

$$E\{\mathbb{1}_{\{\sigma_{n+1}-\sigma_n>a\}} \mid \{\zeta_{\sigma_n}\}_{n\in Z_+}\} = \exp\{-a\int\limits_{\mathbb{R}^{\mathrm{d}}} b(x,\zeta_{\sigma_n})dx\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1})\int\limits_{\mathbb{R}^{\mathrm{d}}} b(x,\zeta_{\sigma_n})dx$ constitute an independent of $\{\zeta_{\sigma_n}\}_{n\in Z_+}$ sequence of independent unit exponentials. Theorem 12.18 in [Kal02] implies that $(\zeta_t)_{t\in[0,\lim\limits_{n\to\infty}\sigma_n)}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t\in[0,\lim\limits_{n\to\infty}\sigma_n)}$ is given by

$$c(\alpha) = \int\limits_{\mathbb{R}^{\mathrm{d}}} b(x,\alpha)dx.$$

Condition 2.2 implies that $c(\alpha)\le ||a||_1\cdot|\alpha|$, where $||a||_1 = ||a||_{L^1(\mathbb{R}^{\mathrm{d}})}$. Consequently,

$$c(\zeta_{\sigma_n}) \le ||a||_1 \cdot |\zeta_{\sigma_n}| = ||a||_1\cdot|\eta_0| + n||a||_1.$$

We see that $\sum_n \frac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [Kal02] implies that $\sigma_n\to\infty$.

We have proved the existence of a strong solution. The uniqueness follows by induction on jumps of the process. Namely, let $(\tilde{\zeta}_t)_{t\ge0}$ be a solution to (72). From (vii) of Definition 5.2 and the equality

$$\int\limits_{(0,\sigma_1)\times\mathbb{R}^{\mathrm{d}}\times[0,\infty]} \mathbb{1}_{[0,b(x,\eta_0)]}(u)N(ds,dx,du) = 0,$$

it follows that $P\{\tilde{\zeta}$ has a birth before $\sigma_1\} = 0$. At the same time, the equality

$$\int\limits_{\{\sigma_1\}\times\mathbb{R}^{\mathrm{d}}\times[0,\infty]} \mathbb{1}_{[0,b(x,\eta_0)]}(u)N(ds,dx,du) = 1,$$

which holds a.s., yields that $\tilde{\zeta}$ too has a birth at the moment σ_1, and in the same point of space at that. Therefore, $\tilde{\zeta}$ coincides with ζ up to σ_1 a.s. Similar reasoning shows that they coincide up to σ_n a.s., and, since $\sigma_n\to\infty$ a.s.,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t\ge 0\} = 1.$$

Thus, pathwise uniqueness holds. Joint uniqueness in law follows from the functional dependence between the solution to the equation and the 'input' η_0 and N. □

Proposition 5.7. *If b is rotation invariant, then so is (η_t).*

Proof. It is sufficient to note that $(M_{\mathrm{d}}\eta_t)$, where $M_{\mathrm{d}}\in \mathrm{SO(d)}$, is the unique solution to (72) with N replaced by $M_{\mathrm{d}}^{-1}N$ defined by

$$M_{\mathrm{d}}^{-1}N([0,q]\times B\times C) = N([0,q]\times M_{\mathrm{d}}^{-1}B\times C), \quad q\ge0, B\in\mathscr{B}(\mathbb{R}^{\mathrm{d}}), C\in\mathscr{B}(\mathbb{R}_+).$$

$M_{\mathrm{d}}^{-1}N$ is a Poisson point process with the same intensity, therefore by uniqueness in law $(M_{\mathrm{d}}\eta_t) \overset{d}{=} (\eta_t)$.

Proposition 5.8. ***(The strong Markov property)*** *Let τ be an $(\mathscr{S}_t, t \geq 0)$-stopping time and let $\tilde{\eta}_0 \overset{d}{=} \eta_\tau$. Then*

$$(\eta_{\tau+t}, t \geq 0) \overset{d}{=} (\tilde{\eta}_t, t \geq 0). \tag{76}$$

Furthermore, for any $\mathscr{D} \in \mathscr{B}(D_{\Gamma_0}[0,\infty))$,

$$P\{(\eta_{\tau+t}, t \geq 0) \in \mathscr{D} \mid \mathscr{S}_\tau\} = P\{(\eta_{\tau+t}, t \geq 0) \in \mathscr{D} \mid \eta_\tau\};$$

that is, given η_τ, $(\eta_{\tau+t}, t \geq 0)$ is conditionally independent of $(\mathscr{S}_t, t \geq 0)$.

Proof. Note that

$$|\eta_{\tau+t} \cap B| = \int\limits_{(\tau,\tau+t]\times B\times[0,\infty)} \mathbb{1}_{[0,b(x,\eta_{s-})]}(u) N(ds,dx,du) + |\eta_\tau \cap B|, \quad t \geq 0,\ B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}}).$$

Since the unique solution is adapted to the filtration generated by the noise and initial condition, the conditional independence follows, and (76) follows from the uniqueness in law. We rely here on the strong Markov property of the Poisson point process, see Proposition 6.1 below. □

Corollary 5.9. *Let τ be an $(\mathscr{S}_t, t \geq 0)$-stopping time and $\{y\}$ be an $\mathscr{S}_\tau$- measurable finite random singleton. Then*

$$(\eta^{\tau,\{y\}}_{\tau+t} - y)_{t\geq 0} \overset{(d)}{=} (\eta_t)_{t\geq 0}.$$

Proof. This is a consequence of Theorem 5.6 and Proposition 5.8. □

Consider two growth processes $(\zeta^{(1)})_t$ and $(\zeta^{(2)})_t$ defined on the common probability space ans satisfying equations of the form (72),

$$|\zeta^{(k)}_t \cap B| = \int\limits_{(q,t]\times B\times[0,\infty)} \lambda \mathbb{1}_{[0,b_k(x,\zeta^{(k)}_{s-})]}(u) N(ds,dx,du) + |\zeta^{(k)}_q \cap B|, \quad k = 1,2. \tag{77}$$

Assume that and the rates b_1 and b_2 satisfy the conditions of imposed on b in Section 2. Let $(\zeta^{(k)}_t)_{t\in[0,\infty)}$ be the unique strong solutions.

Lemma 5.10. *Assume that a.s. $\zeta^{(1)}_0 \subset \zeta^{(2)}_0$, and for any two finite configurations $\eta^1 \subset \eta^2$,*

$$b_1(x,\eta^1) \leq b_2(x,\eta^2), \quad x \in \mathbb{R}^{\mathrm{d}}. \tag{78}$$

Then a.s.

$$\zeta^{(1)}_t \subset \zeta^{(2)}_t, \quad t \in [0,\infty). \tag{79}$$

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be the ordered sequence of the moments of births for $(\zeta^{(1)}_t)$, that is, $t \in (\sigma_n)_{n\in\mathbb{N}}$ if and only if $|\zeta^{(1)}_t \setminus \zeta^{(1)}_{t-}| = 1$. It suffices to show that for each $n \in \mathbb{N}$, σ_n is a moment of birth for $(\zeta^{(2)}_t)_{t\in[0,\infty)}$ too, and the birth occurs at the same place. We use induction on n.

Here we deal only with the base case, the induction step is done in the same way. Assume that

$$\zeta^{(1)}_{\sigma_1} \setminus \zeta^{(1)}_{\sigma_1-} = \{x_1\}.$$

The process $(\zeta^{(1)})_{t\in[0,\infty)}$ satisfies (77), therefore $N(\{x\}\times[0,b_k(x_1,\zeta^{(1)}_{\sigma_1-})])=1$. Since

$$\zeta^{(1)}_{\sigma_1-}=\zeta^{(1)}_0\subset\zeta^{(2)}_0\subset\zeta^{(2)}_{\sigma_1-},$$

by (78)

$$N_1(\{x\}\times\{\sigma_1\}\times[0,b_k(x_1,\zeta^{(2)}_{\sigma_1-})])=1,$$

hence

$$\zeta^{(2)}_{\sigma_1}\setminus\zeta^{(2)}_{\sigma_1-}=\{x_1\}.$$

□

6 Appendix. The strong Markov property of a Poisson point process

We need the strong Markov property of a Poisson point process. Denote $X:=\mathbb{R}^{\mathrm{d}}\times\mathbb{R}_+$ (compare the proof of Proposition 5.8), and let l be the Lebesgue measure on X. Consider a a Poisson point process N on $\mathbb{R}_+\times X$ with intensity measure $dt\times l$. Let N be compatible with a right-continuous complete filtration $\{\mathscr{F}_t\}_{t\geq 0}$, and τ be a finite a.s. $\{\mathscr{F}_t\}_{t\geq 0}$-stopping time . Introduce another Point process $\overline{N}$ on $\mathbb{R}_+\times X$,

$$\overline{N}([0;s]\times U)=N((\tau;\tau+s]\times U),\quad U\in\mathscr{B}(X)).$$

Proposition 6.1. *The process $\overline{N}$ is a Poisson point process on $\mathbb{R}_+\times X$ with intensity $dt\times l$, independent of $\mathscr{F}_\tau$.*

Proof. To prove the proposition, it suffices to show that

(i) for any $b>a>0$ and open bounded $U\subset X$, $\overline{N}((a;b),U)$ is a Poisson random variable with mean $(b-a)l(U)$, and

(ii) for any $b_k>a_k>0$, $k=1,...,m$, and any open bounded $U_k\subset X$, such that $((a_i;b_i)\times U_i)\cap((a_j;b_j)\times U_j)=\varnothing$, $i\neq j$, the collection $\{\overline{N}((a_k;b_k)\times U_k)\}_{k=1,m}$ is a sequence of independent random variables, independent of $\mathscr{F}_\tau$.

Indeed, $\overline{N}$ is determined completely by values on sets of type $(b-a)\beta(U)$, a,b,U as in (i), therefore it must be an independent of $\mathscr{F}_\tau$ Poisson point process if (i) and (ii) hold.

Let τ_n be the sequence of $\{\mathscr{F}_t\}_{t\geq 0}$-stopping times, $\tau_n=\frac{k}{2^n}$ on $\{\tau\in(\frac{k-1}{2^n};\frac{k}{2^n}]\}$, $k\in\mathbb{N}$. Then $\tau_n\downarrow\tau$ and $\tau_n-\tau\leq\frac{1}{2^n}$. Note that the stopping times τ_n take countably many values only. The process N satisfies the strong Markov property for τ_n: the processes $\overline{N}_n$, defined by

$$\overline{N}_n([0;s]\times U):=N((\tau_n;\tau_n+s]\times U),$$

are Poisson point processes, independent of $\mathscr{F}_{\tau_n}$. To prove this, take k with $P\{\tau_n=\frac{k}{2^n}\}>0$ and note that on $\{\tau_n=\frac{k}{2^n}\}$, $\overline{N}_n$ coincides with process the Poisson point process $\tilde{N}_{\frac{k}{2^n}}$ given by

$$\tilde{N}_{\frac{k}{2^n}}([0;s]\times U):=N\bigg((\frac{k}{2^n};\frac{k}{2^n}+s]\times U)\bigg),\quad U\in\mathscr{B}(\mathbb{R}^{\mathrm{d}}).$$

Conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is again a Poisson point process, with the same intensity. Furthermore, conditionally on $\{\tau_n = \frac{k}{2^n}\}$, $\tilde{N}_{\frac{k}{2^n}}$ is independent of $\mathscr{F}_{\frac{k}{2^n}}$, hence it is independent of $\mathscr{F}_\tau \subset \mathscr{F}_{\frac{k}{2^n}}$.

To prove (i), note that $\overline{N}_n((a;b) \times U) \to \overline{N}((a;b) \times U)$ a.s. and all random variables $\overline{N}_n((a;b) \times U)$ have the same distribution, therefore $\overline{N}((a;b) \times U)$ is a Poisson random variable with mean $(b-a)\lambda(U)$. The random variables $\overline{N}_n((a;b) \times U)$ are independent of $\mathscr{F}_\tau$, hence $\overline{N}((a;b) \times U)$ is independent of $\mathscr{F}_\tau$, too. Similarly, (ii) follows. □

Remark 6.2. We assumed in Proposition 6.1 that there exists an increasing, right-continuous and complete filtration $\{\mathscr{S}_t\}_{t\geq 0}$ compatible with N. Let us show that such filtrations exist.

Introduce the natural filtration of N,

$$\bar{\mathscr{S}}_t^0 = \sigma\{N_k(C, B), B \in \mathscr{B}(\mathbb{R}^{\mathrm{d}}), C \in \mathscr{B}([0;t])\},$$

and let $\bar{\mathscr{S}}_t$ be the completion of $\bar{\mathscr{S}}_t^0$ under P. Then N is compatible with $\{\bar{\mathscr{S}}_t\}$. We claim that $\{\bar{\mathscr{S}}_t\}_{t\geq 0}$, defined in such a way, is right-continuous (this may be regarded as an analog of Blumenthal's 0-1 law). Indeed, as in the proof of Proposition 6.1, we can check that $\tilde{N}_a$ is independent of $\bar{\mathscr{S}}_{a+}$. Since $\bar{\mathscr{S}}_\infty = \sigma(\tilde{N}_a) \vee \bar{\mathscr{S}}_a$, $\sigma(\tilde{N}_a)$ and $\bar{\mathscr{S}}_a$ are independent and $\bar{\mathscr{S}}_{a+} \subset \bar{\mathscr{S}}_\infty$, we see that $\bar{\mathscr{S}}_{a+} \subset \bar{\mathscr{S}}_a$. Thus, $\bar{\mathscr{S}}_{a+} = \bar{\mathscr{S}}_a$.

Acknowledgement

Luca Di Persio would like to acknowledge the Informatics Department at the University of Verona for having funded the project "Stochastic differential equations with jumps in Mathematical Finance: applications to pricing, hedging and dynamic risk measure's problems" and the Gruppo Nazionale per l'analisi matematica, la Probabilità e le loro applicazioni (GNAMPA). Viktor Bezborodov (VB) is supported by the Department of Computer Science at the University of Verona. VB was also partially supported by the DFG through the SFB 701 "Spektrale Strukturen und Topologische Methoden in der Mathematik" and by the European Commission under the project STREVCOMS PIRSES-2013-612669. Mykola Lebid (ML) is supported by the Department of Biosystems Science and Engineering at the ETH Zürich. VB, ML, TK and TO are grateful for the support of the ZIF Cooperation group "Multiscale modelling of tumor evolution, progression and growth", and Wroclaw University of Science and Technology, Faculty of Electronics. TK is also supported by the National Science Center in Poland (NCN) through grant 2013/11/B/HS4/01061: "Agent based modeling of innovation diffusion".

References

[ADH15] A. Auffinger, M. Damron, and J. Hanson. 50 years of first passage percolation. 2015. https://arxiv.org/abs/1511.03262.

[Bez15] V. Bezborodov. Spatial birth-and-death Markov dynamics of finite particle systems. 2015. http://arxiv.org/abs/1507.05804.

[Big95] J. D. Biggins. The growth and spread of the general branching random walk. *Ann. Appl. Probab.*, 5(4):1008–1024, 1995.

[BR58] C. J. Burke and M. Rosenblatt. A Markovian function of a Markov chain. *E Ann. Math. Statist*, 29:1112–1122, 1958.

[Dei03] M. Deijfen. Asymptotic shape in a continuum growth model. *Adv. in Appl. Probab.*, 35(2):303–318, 2003.

[Dur83] R. Durrett. Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete*, 62(2):165–170, 1983.

[Dur88] R. Durrett. *Lecture notes on particle systems and percolation.* The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1988.

[Ede61] M. Eden. A two-dimensional growth process. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. IV*, pages 223–239. Univ. California Press, Berkeley, Calif., 1961.

[EW03] A. Eibeck and W. Wagner. Stochastic interacting particle systems and nonlinear kinetic equations. *Ann. Appl. Probab.*, 13(3):845–889, 2003.

[FM04] N. Fournier and S. Méléard. A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, 14(4):1880–1919, 2004.

[GM08] J.-B. Gouéré and R. Marchand. Continuous first-passage percolation and continuous greedy paths model: linear growth. *Ann. Appl. Probab.*, 18(6):2300–2319, 2008.

[GM12] O. Garet and R. Marchand. Asymptotic shape for the contact process in random environment. *Ann. Appl. Probab.*, 22(4):1362–1410, 2012.

[HN97] C. D. Howard and C. M. Newman. Euclidean models of first-passage percolation. *Probab. Theory Related Fields*, 108(2):153–170, 1997.

[Kal02] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications. Springer-Verlag, second edition, 2002.

[Kes87] H. Kesten. Percolation theory and first-passage percolation. *Ann. Probab.*, 15(4):1231–1271, 1987.

[KK06] Y. G. Kondratiev and O. V. Kutoviy. On the metrical properties of the configuration space. *Math. Nachr.*, 279(7):774–783, 2006.

[Lig85] T. M. Liggett. An improved subadditive ergodic theorem. *Ann. Probab.*, 13(4):1279–1285, 1985.

[Lig99] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324. Springer-Verlag, Berlin, 1999.

[Mas98] L. Massoulié. Stability results for a general class of interacting point processes dynamics, and applications. *Stochastic Process. Appl.*, 75(1):1–30, 1998.

[MW03] J. Moller and R. Waagepetersen. *Statistical inference and simulation for spatial point processes*. CRC Press, 2003.

[Ric73] D. Richardson. Random growth in a tessellation. *Proc. Cambridge Philos. Soc.*, 74:515–528, 1973.

[RS99] Michael Röckner and Alexander Schied. Rademacher's theorem on configuration spaces and applications. *J. Funct. Anal.*, 169(2):325–356, 1999.

[Shi15] Z. Shi. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham, 2015. Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[TM15] D. Tartarini and E. Mele. Adult stem cell therapies for wound healing: Biomaterials and computational models. *Frontiers in bioengineering and biotechnology*, 3, 2015.

[TSH+13] K. Treloar, M. Simpson, P. Haridas, K. Manton, D. Leavesley, S. McElwain, and R. Baker. Multiple types of data are required to identify the mechanisms influencing the spatial expansion of melanoma cell colonies. *BMC systems biology*, 7(1):1, 2013.

[VDPP15] B. Vo, C. Drovandi, A. Pettitt, and G. Pettet. Melanoma cell colony expansion parameters revealed by approximate bayesian computation. *PLoS Comput Biol*, 11(12):e1004635, 2015.

[WBP+15] B. Waclaw, I. Bozic, M.E. Pittman, R.H. Hruban, B Vogelstein, and M.A. Nowak. A spatial model predicts that dispersal and cell turnover limit intratumour heterogeneity. *Nature. Research letter*, 525:261–277, 2015.