A Generalized Winternitz Theorem

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Abstract

We prove that, for every simple polygon P having $k \geq 1$ reflex vertices, there exists a point $q \in P$ such that every half-polygon that contains q contains nearly $1/2(k+1)$ times the area of P. We also give a family of examples showing that this result is the best possible.

Keywords: Winternitz' Theorem, Centerpoint Theorem, Polygons

1 Introduction

Winternitz' Theorem [1, pp. 54–55] is a classic theorem in convex geometry that has been rediscovered many times [4, 9, 12, 13, 15]. Winternitz' Theorem states that, for any convex polygon P, there exists a point $q \in P$ such that any halfspace that contains q contains at least 4/9 of the area of P. The dissection of a triangle into 9 similar triangles shown in Figure 1 can easily be used to show that the bound of 4/9 is tight when P is a triangle.

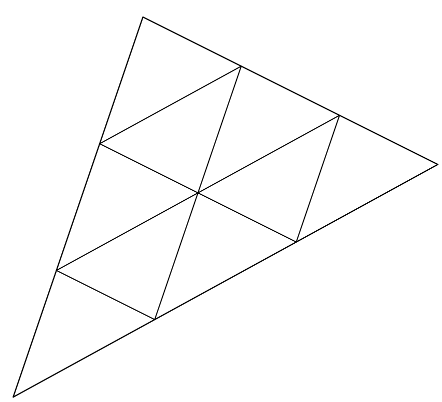

Figure 1: A triangle has maximum halfspace depth 4/9.

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In this paper, we consider a generalization of Winternitz' Theorem to the case when P is a simple polygon. A *chord* of a simple polygon P is a closed line segment whose interior is contained in the interior of P and whose endpoints are on the boundary of P. If c is a chord of P then $P \setminus c$ has two components P^+ and P^-. We call the closure of these polygons *half-polygons* of P. We define the *depth* of a point $q \in P$ as

$$\delta_P(q) = \min\{\mathrm{area}(h \cap P) : h \text{ is a half-polygon of } P \text{ that contains } q\} \ .$$

Winternitz' Theorem states that, if P is convex then there exists a point $q \in P$ with $\delta_P(q) \geq (4/9)\,\mathrm{area}(P)$.

Winternitz' Theorem is closely related to the *Centerpoint Theorem* [10, 14] which states that for any set S of n points in $\mathbb{R}^2$ there exists a point $q \in \mathbb{R}^2$ such that every closed halfplane that contains q contains at least $n/3$ points of S. The Centerpoint Theorem is easily derived from Helly's Theorem [3] by considering all halfplanes that contain at least $2n/3$ points of S and taking q to be in their common intersection.

Helly's Theorem also holds for half-polygons of P. In particular, if $P_1, \ldots, P_n$ are half-polygons of P and $P_i \cap P_j \cap P_k \neq \emptyset$ for any $1 \leq i < j < k \leq n$ then $\bigcap_{i=1}^{n} P_i \neq \emptyset$. Therefore one might expect that there always exists a point q with $\delta_P(q)$ greater than or equal to some constant fraction of area(P), independent of the number of reflex vertices in P. However, this intuition turns out to be false.

Theorem 1. *For any $\epsilon > 0$ and any simple polygon P with $k \geq 1$ reflex vertices, there exists a point $q \in P$ such that $\delta_P(q) \geq \mathrm{area}(P)/2(k+1) - \epsilon$.*

The lower bound of Theorem 1 is essentially the best possible:

Theorem 2. *For every integer $k \geq 1$ and every $\epsilon > 0$, there exists a polygon P with k reflex vertices, such that no point in P has depth greater than* $\mathrm{area}(P)/2(k+1) + \epsilon$.

Our results continue an existing line of research relating the combinatorial and computational properties of polygons to the number of their reflex vertices. Hurtado and Noy [7] give tight upper and lower bounds on the number of triangulations of a polygon as a function of the number of its reflex vertices. Hurtado, Noy, and Urrutia [5] prove that the diameter of the flip graph of triangulations of a polygon is $O(n+k^2)$. Bose *et al* [2] show that the computational complexity of computing ham-sandwich cuts in simple polygons is $\Theta(n \log k)$. Hertel and Mehlhorn [6] give a simple $O(n \log k)$ time algorithm for triangulating a simple polygon. Keil [8] gives an $O(k^2 n \log n)$ time algorithm for finding an optimal convex partitioning of a simple polygon. The above results, and those of the current paper, illustrate the importance of the number of reflex vertices as a parameter when studying combinatorial and computational properties of simple polygons.

The remainder of the paper is organized as follows: In Section 2 a proof of Theorem 1 is given. Section 3 presents a family of simple polygons that prove Theorem 2.

2 The Lower Bound

For simplicity, we will prove a discrete version of Theorem 1 that is a polygonal analog of the Centerpoint Theorem. In the discrete version, we are given a polygon P and a finite set of points N in the interior of P, such that no point of N is collinear with 2 vertices of P. We call N a *general set of points* in P. The N-*depth* of a point $q \in P$ is defined as

$$\delta_{P,N}(q) = \min\{|h \cap N| : h \text{ is a half-polygon of } P \text{ that contains } q\} \ .$$

The following claim generalizes the Centerpoint Theorem. Also, by taking the point set N to be (sufficiently close to) the vertices of a (sufficiently dense) grid, the claim establishes Theorem 1.

Claim 1. *Let P be a simple polygon having $k \geq 1$ reflex vertices and let N be a general set of points in P. Then there exists a point $q \in P$ such that $\delta_{P,N}(q) \geq |N|/2(k+1)$.*

Proof. Refer to Figure 2 for what follows. Divide polygon P into at most $k+1$ convex sub-polygons by iteratively adding a chord on each reflex vertex so that it becomes a convex vertex in each of the two subpolygons generated. Let P^* be a convex sub-polygon that contains at least $|N|/(k+1)$ points of N. Note that P^* contains at most $k' \leq k$ edges $e_1, \ldots, e_{k'}$ that are not edges of P. For each such edge, e_i, define Q_i as the half polygon of P bounded by the chord of P that contains the edge e_i and that does not contain P^*. Observe that $\bigcup_{i=1}^{k'} Q_i$ contains $P \setminus P^*$. In particular, the union of the Q_i contain all the point of N that are not contained in P.

Let Q be any of the Q_i, for $1 \leq i \leq k'$, that maximizes $|Q_i \cap N|$. Observe that $|(P^* \cup Q) \cap N| \geq 2|N|/(k+1)$. We will show how to find a point q in $P^* \cup Q$ such that

$$\delta_{P^* \cup Q, N \cap (P^* \cup Q)}(q) \geq |N|/2(k+1) \ .$$

The Claim then follows from the fact that $P^* \cup Q \subseteq P$ and $N \cap (P^* \cup Q) \subseteq N$, so that $\delta_{P,N}(q) \geq \delta_{P^* \cup Q, N \cap (P^* \cup Q)}(q)$.

Let $r_1 r_2$ be a maximal line segment that is on the boundary of both P^* and Q. Define $r_1' r_2'$ to be a chord of P^* parallel to $r_1 r_2$ and that separates exactly $|N|/(k+1)$ points of $N \cap P^*$ from $r_1 r_2$. The chord $r_1' r_2'$ separates $P^* \cup Q$ into two sub-polygons, P' and Q', where $P' \subseteq P^*$. Observe that $|Q' \cap N| \geq |P' \cap N| = |N|/(k+1)$

The point, q, of high depth we are searching for will be on the segment $r_1' r_2'$. The remainder of the proof uses a fairly standard technique that can be used, for example, to prove the Planar Ham Sandwich Theorem [11]. However, unlike most applications of this technique we do not have the continuity that is usually required to use this technique. We therefore take special care to explain it in detail.

For $0 < t < 1$, let $q_t = (1-t)r_1' + t r_2'$. Let C_t be the chord of $P' \cup Q'$ that contains q_t and that bisects $P' \cap N$. If $|P' \cap N|$ is odd, the C_t is unique and

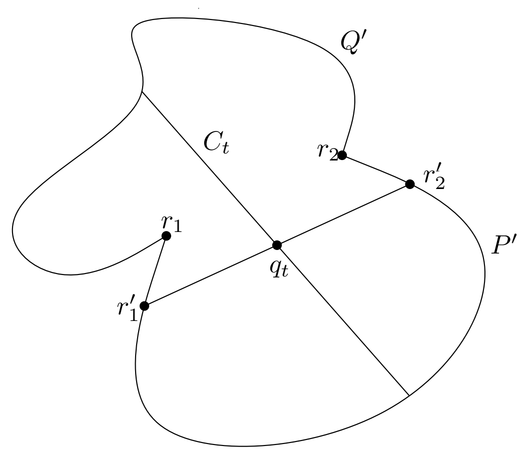

Figure 2: The Proof of Claim 1.

always contains a point of N. Otherwise, we can make C_t unique by defining it to be equidistant from the nearest points of $P' \cap N$ on its left and right.

Let Q'_t denote the component of $Q' \setminus C_t$ that contains r'_1 and let $\overline{Q}'_t = Q' \setminus Q'_t$. Observe that, for all sufficiently small $\epsilon > 0$, $Q'_\epsilon \cap N = \emptyset$ and $Q'_{1-\epsilon} = Q' \cap N$. Furthermore, $|Q'_t \cap N|$ is an increasing function of t. Therefore, there is some value t^*, $0 < t^* < 1$, such that, for all $\delta > 0$, $|Q'_{t^*+\delta} \cap N| \geq |N|/2(k+1)$ and $|\overline{Q}'_{t^*-\delta} \cap N| \geq |N|/2(k+1)$.

We claim that $\delta_{N,P}(q_{t^*}) \geq |N|/2(k+1)$. To see why this is so, observe that C_{t^*} partitions P' into two half-polygons, P'_1 and P'_2, each of which contains $|N|/2(k+1)$ points. Any half-polygon that contains q_{t^*} but does not contain either P'_1 or P'_2 must contain at least one of $Q'_{t^*+\delta}$ or $\overline{Q}'_{t^*-\delta}$ for some $\delta > 0$. Therefore, $\delta_{N,O}(q_{t^*}) \geq |N|/2(k+1)$. $\square$

3 The Upper Bound

Next we proceed with the proof of Theorem 2.

Proof (of Theorem 2). Refer to Figure 3. Our construction is parameterized by a value $c < 1/2$. The construction begins by constructing a spiral, with $k+1$ segments $s_1, \ldots, s_{k+1}$, where segment s_i has length $1 + \lceil i/2 \rceil c$ and creates an angle of $\pi/2$ with s_{i+1}. Next, we expand the segments $s_1, \ldots, s_k$ inwards so that each segment s_i becomes a rectangle R_i of the same length as s_i, but whose area is c. It is easy to verify that the union of these rectangles is a simple polygon with k reflex vertices. Furthermore, the area of the intersection of any two rectangles R_i and $R_i + 1$ is at most c^2. Finally, we replace each reflex vertex with two convex vertices and one reflex vertex as shown in Figure 3.b. Suppose the reflex vertex v occurs at the intersection of a horizontal rectangle H and

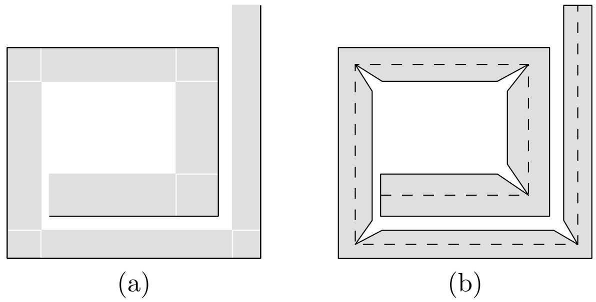

Figure 3: The construction for the proof of Theorem 2 with $c = 1/4$.

a vertical rectangle V. Then the location of the vertex is chosen so that its y-coordinate bisects H and its x-coordinate bisects V. By choosing the two convex vertices sufficiently close together, this decreases the area of P by at most δ for any constant $\delta > 0$. Denote the resulting simple polygon by P.

Consider the path, shown in Figure 3.b, that passes through every reflex vertex and nearly bisects R_1 and R_{k+1}. This path partitions P into $k+2$ pieces. One of these pieces has area at most $c(k+1)/2$ and the other $k+1$ pieces have area at most $c/2$. Each of the small pieces is a half-polygon of P, so any point q contained in such a piece has $\delta_P(q) \leq c/2$. On the other hand, any point contained in the large piece is also contained in a half-polygon of p whose area is at most $c/2$. Therefore, $\delta_P(q) \leq c/2$ for any $q \in P$. Finally, observe that the area of P is at least

$$\mathrm{area}(P) \geq (k+1)c - k(c^2 + \delta) \geq (k+1)(c - c^2 - \delta)$$

Therefore,

$$\frac{\delta_P(q)}{\mathrm{area}(P)} \leq \left(\frac{1}{2(k+1)}\right)\left(\frac{1}{1 - c - \delta/c}\right) .$$

Selecting $\delta = c^2$ and c sufficiently small completes the proof. □

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