IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received March 8, 2019, accepted March 26, 2019, date of publication April 11, 2019, date of current version April 24, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2909934

Udwadia-Kalaba Approach for Three Link Manipulator Dynamics With Motion Constraints

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This work was supported in part by the National Science Foundation of China under Grant 51765042, Grant 61463031, Grant 61662044, and Grant 61773051.

ABSTRACT Aiming to dynamic modeling of a three-link manipulator subjected to motion constraints, a novel explicit approach to the dynamical equations based on Udwadia–Kalaba (UK) theory is established. The motion constraints on the three-link manipulator can be regarded as external constraints of the system. However, it is not easy to obtain explicit equations for the dynamic modeling of constrained systems. For a multibody system subjecting to motion constraints, it is common to introduce Lagrange multipliers, but obtaining an explicit dynamical equation using traditional Lagrange multipliers is difficult. In order to obtain such equations more simply, motion constraints are handled using the UK equation. Compared with the Lagrange method, the UK approach can simplify the analysis and solution of a constrained system, without the need to introduce additional auxiliary variables to solve the Lagrange equation. Based on a more real-life nominal system (whose parameters are known) model considering the uncertain environment, this paper develops a nonlinear controller that satisfies the required trajectory. This controller allows the nonlinear nominal system to track the desired trajectory exactly without linearizations or approximations. These continuous controllers compensate extra force to eliminate the errors caused by uncertainties. The controllers are based on a generalization of sliding surfaces. Error bounds on tracking caused by uncertainties are analytically obtained. The numerical results show the simplicity and efficacy of the proposed methodology, and the reliability of the error bounds.

INDEX TERMS Manipulator, Udwadia-Kalaba theory, dynamic modeling, trajectory tracking, constrained mechanics, uncertain systems control, generalized sliding mode control.

I. INTRODUCTION

The main methods currently used for dynamic modeling of manipulators with motion constraints are the Newton–Euler method [1], Lagrange's method [2], [3], and Kane's method [4]. The Newton–Euler method describes motion and force through the use of vectors. In the modeling procedure, every component of the mechanism is isolated and the corresponding Newton–Euler equations are established. Calculations of this method is quite efficient, but it is hard to use this approach when attempting to design control systems for manipulators. For a multibody system with motion constraints, Lagrange's method can be used, generally with the introduction of Lagrange multipliers [5], which is a widely used technique for constrained systems. However, controlling these multipliers is difficult, and the approach is not very well suited for symbolic considerations. Kane's method combines the advantages of vector mechanics and analytical mechanics, with a generalized rate being used as an independent variable in the equations of motion of the system. The fundamental vector projection of the main force and the inertial force of the system is extended directly to derive the equations of motion. However, with this method, these dynamical equations cannot be obtained in the appropriate analytical form for a constrained mechanical system. As is well known, the generation of dynamical equations for constrained systems in symbolic form has a number of advantages with regard to issues of both control and mechanical design [6]. In addition, controller design relies on analytical model of the mechanical system under control, so it is important to find an appropriate, and accurate, dynamical model.

The associate editor coordinating the review of this manuscript and approving it for publication was Congduan Li.

In 1992, Udwadia and Kalaba [7]–[10] derived a basic equation of motion for describing constraint dynamics, the Udwadia–Kalaba equation, which is obtained by using Gauss's principle rather than the more commonly used principles of Lagrange, Hamilton, Gibbs, and Appell [11]. This equation takes constraints into account in the dynamical equation and involves the generalized Moore–Penrose inverse [12]. Therefore, it provides a simple and general explicit equation of motion for constrained mechanical systems without the need for Lagrange multipliers [13]. This relatively simple approach allows detailed dynamical analysis of such systems and should improve fundamental understanding of constrained motion in multibody dynamics. It has been used to solve problems in the dynamical analysis of multibody systems [14]–[16], flexible systems [17], and control of mechanical systems [18]–[21], including tethered satellites [22] and parallel [23]–[25], industrial [26], [27], underwater [28], and mobile [29] robots. This method has the advantage that it is possible to obtain both the constraining force required to control the manipulator and the dynamical equations in an explicit form that is easy to implement in a computer program [30].

To deal with the control problem of the manipulator, valuable control methods have been proposed. For example, Deng et al. [31] addressed the output feedback tracking control problem of a category of multiple input and output nonlinear systems subjecting to time-varying input delay and additive bounded disturbances based on backstepping design approach. To track the issue of uncertainties of nonlinear system, Yao et al [32] designed an active disturbance rejection adaptive controller for tracking control of nonlinear systems with both parametric and uncertain nonlinearities. Unlike most control methods for the tracking control problem of manipulator, this paper does not make any linearization or approximations and obtains exact, analytical solutions to the tracking control problem of manipulator in which the desired trajectory can be any (suitably smooth) arbitrarily prescribed function of time. In this paper, in order to obtain explicit control torque to control the three-link manipulator to move on the desired trajectory, an analytical dynamical model of a three-link manipulator is established by using UK equation. Considering there are always uncertainties in the description of any real-life dynamical systems.

Although the uncertain real-life system can't be fully-known, but the uncertainties are bounded. We refer the best assessment of a given real-life uncertain system as the "nominal system". By using UK equation, we obtain the closed-form control force needed to track the constraint trajectory requirements for the nominal system model. Once the nominal system model gets fixed, no linearizations/approximations are made in the description of the dynamics. Then, we augment this nonlinear controller by an additional additive controller based on sliding mode control, which can provide a general approach to the control of the nonlinear uncertain system, and leads to a closed-form nonlinear controller that satisfies the desired constraint trajectory requirements with error bounds.

This paper is organized as follows. In section II, we introduce the Udwadia-Kalaba theory. In section III, by using UK equation obtained the joint control force, based on nominal system, we adopt a nonlinear controller to control the uncertain three-link manipulator. In section IV, we carry out numerical simulation to control the three-link manipulator to track the desired trajectory and use Braun's method for comparison. The conclusions of this paper are given in section V.

II. UDWADIA–KALABA THEORY

Consider an unconstrained mechanical system, moving under the influence of gravity alone, described by n generalized coordinates $q := [q_1, q_2, \cdots, q_n]^T$, and with equations of motion expressed in Newtonian or Lagrangian form as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \tag{1}$$

with initial condition

$$q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0. \tag{2}$$

Here t is the independent variable, M is an $n \times n$ matrix that can be either positive-semidefinite ($M \geq 0$) or positive-definite ($M > 0$) at each instant of time. $\dot{q}$ is the $n \times 1$ velocity vector, $\ddot{q}$ is the $n \times 1$ acceleration vector, and $Q(q,\dot{q},t)$, called the given force, collects together the normal and Coriolis inertial terms and the applied forces related to q, $\dot{q}$, and t. From (1), when q, $\dot{q}$, and t are known, the acceleration can be obtained as follows:

$$a(q,\dot{q},t) := M^{-1}(q,t)\,Q(q,\dot{q},t). \tag{3}$$

It is assumed that the constrained form of this system can be described by $m = m_1 + m_2$ equations

$$\varphi_i(q,t) = 0, \quad i = 1, 2, \ldots, m_1 \tag{4}$$

and

$$\psi_j(q,\dot{q},t) = 0, \quad j = 1, 2, \ldots, m_2, \tag{5}$$

where φ is an m_1 vector and ψ is an m_2 vector. Equations (4) and (5) include all the usual varieties of holonomic or nonholonomic constraints. Differentiating (4) twice with respect to time and (5) once yield a set of constraint equations in matrix form as [33]

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{6}$$

where the constraint matrix $A(q,\dot{q},t)$ is an $m \times n$ matrix and $b(q,\dot{q},t)$ is an $m \times 1$ vector.

When the system is constrained, an additional set of forces act on the manipulator system, and the equation of motion of this constrained manipulator system can then be written as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t), \tag{7}$$

where $Q^c(q,\dot{q},t)$ is an $n \times 1$ vector, which is present because of the additional constraint force and satisfies the constraint conditions.

In Lagrangian mechanics, when the constraints are ideal, $Q^c(q, \dot{q}, t)$ is governed by the usual D'Alembert principle. However, the constraints can also be nonideal, and the constrained system can be subject to both ideal and nonideal constraints at the same time, under that $Q^c(q, \dot{q}, t)$ can be written as

$$Q^c(q, \dot{q}, t) = Q^c_{\text{id}}(q, \dot{q}, t) + Q^c_{\text{nid}}(q, \dot{q}, t), \tag{8}$$

where $Q^c_{\text{id}}(q, \dot{q}, t)$ is the ideal constraint force and $Q^c_{\text{nid}}(q, \dot{q}, t)$ the nonideal constraint force.

Assuming that the virtual displacement [34] is ν, the work done by the ideal constraint force is zero, i.e.,

$$\nu^T Q^c_{\text{id}} = 0, \tag{9}$$

while the work done by the nonideal constraint force $Q^c_{\text{nid}}(q, \dot{q}, t)$ is nonzero. i.e.,

$$\nu^T Q^c_{\text{nid}} \neq 0 \tag{10}$$

Udwadia and Kalaba showed that the ideal constraint force is given by

$$Q^c_{\text{id}}(q, \dot{q}, t) = M^{\frac{1}{2}} B^+ \left(b - AM^{-1}Q\right) \tag{11}$$

and the nonideal constraint force by

$$Q^c_{\text{nid}}(q, \dot{q}, t) = M^{\frac{1}{2}} B^+ \left(I - B^+B\right) M^{-\frac{1}{2}} c, \tag{12}$$

where the matrix $B = AM^{-\frac{1}{2}}$ and the superscript "+" indicates the Moore–Penrose inverse matrix. The vector c is a known vector, which can be obtained experimentally or by observation of a given mechanical system.

From (7), (8), (11), and (12), we can get the general equation describing the dynamics of the constrained system

$$M\ddot{q} = Q + M^{\frac{1}{2}} B^+ \left(b - AM^{-1}Q\right) + M^{\frac{1}{2}} B^+ \left(I - B^+B\right) M^{-\frac{1}{2}} c. \tag{13}$$

If the work done by constraint forces under virtual displacements is zero, then $Q^c_{\text{nid}} = 0$, then the general equation of the constrained system can be simplified to

$$M\ddot{q} = Q + M^{\frac{1}{2}} B^+ \left(b - AM^{-1}Q\right). \tag{14}$$

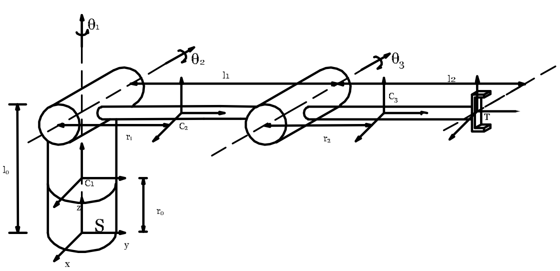


FIGURE 1. Three-link manipulator.

III. DYNAMICS OF A THREE-LINK MANIPULATOR

A. UNCONSTRAINED MANIPULATOR DYNAMICS

In order to calculate the dynamics of the three-link manipulator in Figure 1, to each link we attach a frame C_i at the center of mass and aligned with the principal axes of inertia of the link [35]. The coordinates of the manipulator are denoted by $\theta = [\theta_1, \theta_2, \theta_3]^T$.

We choose a reference configuration of the manipulator, i.e., the configuration where all the joint variables are 0, and let ξ_1, ξ_2, ξ_3 denote the joint twists in this configuration expressed in the global coordinate frame S. Then the joint twist corresponding to a screw is

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}, \tag{15}$$

where the axis of rotation is $\omega \in R^3$, $\|\omega\| = 1$, and $q \in R^3$ is a point on the axis, so

$$\begin{cases} \xi_1 = \begin{bmatrix} -\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ \xi_2 = \begin{bmatrix} -\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ l_0 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \\ \xi_3 = \begin{bmatrix} -\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ l_1 \\ l_0 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \end{cases} \tag{16}$$

The transformation between the frames C_i and base frame at $\theta = 0$ is given by

$$\begin{cases} g_{SC_1}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}, \\ g_{SC_2}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ r_1 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}, \\ g_{SC_3}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ l_1 + r_2 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}. \end{cases} \tag{17}$$

The forward kinematics about the center of mass are given by

$$g_{SC_i}(\theta) = e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_i\theta_i} g_{SC_i}(0), \tag{18}$$

with

$$\begin{cases} e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_2 & S_2 & -l_0 S_2 \\ 0 & -S_2 & C_2 & l_0(1-C_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ e^{\hat{\xi}_3\theta_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_3 & S_3 & l_1(1-C_3) - l_0 S_3 \\ 0 & -S_3 & C_3 & l_0(1-C_3) + l_1 S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{cases} \tag{19}$$

Here $S_i = \sin\theta_i$, $C_i = \cos\theta_i$, $S_{ij} = \sin(\theta_i + \theta_j)$, and $C_{ij} = \cos(\theta_i + \theta_j)$. To calculate the manipulator inertia matrix, the body velocity of the center of mass of the ith link is described as

$$V^b_{SC_i} = J^b_{SC_i}(\theta)\dot{\theta}, \tag{20}$$

where $J^b_{SC_i}(\theta)$ is a configuration-dependent $6 \times n$ body manipulator Jacobian matrix corresponding to $g_{SC_i}(\theta)$:

$$J^b_{SC_i}(\theta) = [\,\xi_1^{+} \cdots \xi_i^{+}\ 0 \cdots 0\,], \tag{21}$$

where $\xi_j^{+} = \mathrm{Ad}^{-1}_{\left(e^{\hat{\xi}_j\theta_j}\ldots e^{\hat{\xi}_i\theta_i} g_{SC_i}(0)\right)} \xi_j\ (j \le i)$ is the jth instantaneous joint twist relative to the ith link frame C_i. Calculating of the body Jacobian yields

$$\begin{cases} J_1 = J^b_{SC_1}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ J_2 = J^b_{SC_2}(0) = \begin{bmatrix} -r_1 C_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & -1 & 0 \\ -S_2 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix}, \\ J_3 = J^b_{SC_3}(0) = \begin{bmatrix} -l_2 C_2 - r_2 C_{23} & 0 & 0 \\ 0 & l_1 S_3 & 0 \\ 0 & -r_2 - l_1 C_3 & -r_2 \\ 0 & -1 & -1 \\ -S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \end{bmatrix}. \end{cases} \tag{22}$$

With this choice of link frames, the link inertia matrices have the general form [36]

$$M_i = \mathrm{diag}\left(m_i, m_i, m_i, I_{x_i}, I_{y_i}, I_{z_i}\right), \tag{23}$$

where m_i is the mass of the object and I_{x_i},I_{y_i}, and I_{z_i} are the moments of inertia about the x-, y-, and z-axes of the ith link frame.

The kinetic energy of the ith link is

$$T(\theta, \dot{\theta}) = \frac{1}{2}\left(V^b_{SC_i}\right)^T M_i V^b_{SC_i} = \frac{1}{2}\dot{\theta}^T J_i^T(\theta) M_i J_i(\theta)\dot{\theta}. \tag{24}$$

Now the total kinetic energy can be written as

$$T(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}, \tag{25}$$

where

$$M(\theta) = \sum_{i=1}^{n} J_i^T(\theta) M_i J_i(\theta) \tag{26}$$

is the manipulator inertia matrix. The body manipulator Jacobian $J^b_{SC_i}(\theta)$ is given by (22). The potential energy of the linkage typically consists of the sum of the gravitational potential energies of each link. Let $h_i(\theta)$ denote the height of the center of mass of the ith link. The potential energy of the link is $V_i(\theta) = m_i g h_i(\theta)$, and the potential energy of the linkage is

$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta). \tag{27}$$

$h_i(\theta)$ can be found using the forward kinematics mapping: substituting (17) and (19) into (18) gives

$$\begin{cases} h_1(\theta) = r_0, \\ h_2(\theta) = l_0 - r_1 \sin\theta_2, \\ h_3(\theta) = l_0 - r_1\sin\theta_2 - r_2\sin(\theta_2 + \theta_3). \end{cases} \tag{28}$$

The Lagrangian for the manipulator is the difference between the kinetic and potential energies: for typical manipulators, the Lagrangian function is

$$L(\theta, \dot{\theta}) = T(\theta, \dot{\theta}) - V(\theta) = \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j - V(\theta_i), \tag{29}$$

where $T(\theta, \dot{\theta})$ is the kinetic energy and $V(\theta)$ the potential energy of the system. The Lagrange equations describing the dynamics for each generalized coordinates are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \Upsilon_i, \tag{30}$$

where Υ_i represents the actuator torque and other nonconservative generalized forces acting on the ith joint. Using (30), we have

$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{d}{dt}\left(\sum_{j=1}^{n} M_{ij}(\theta)\dot{\theta}_j\right) \\ \qquad = \sum_{j=1}^{n}\left(M_{ij}(\theta)\ddot{\theta}_j + \dot{M}_{ij}(\theta)\dot{\theta}_j\right), \\ \frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial M_{kj}(\theta)}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j - \frac{\partial V(\theta)}{\partial \theta_i}. \end{cases} \tag{31}$$

The $\dot{M}_{ij}(\theta)$ term can now be expanded in terms of partial derivatives to yield

$$\sum_{j=1}^{n} M_{ij}(\theta)\ddot{\theta}_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_k - \frac{1}{2}\frac{\partial M_{ij}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j\right) + \frac{\partial V(\theta)}{\partial \theta_i} = \Upsilon_i. \tag{32}$$

Equation (32) can be rewritten as

$$M(\theta)\ddot{\theta} + C\left(\theta,\dot{\theta}\right)\dot{\theta} + G\left(\theta,\dot{\theta}\right) = \tau, \tag{33}$$

where τ is the vector of actuator torques, $C\left(\theta,\dot{\theta}\right)$ is the Coriolis matrix for the manipulator, with components

$$C_{ij}\left(\theta,\dot{\theta}\right) = \sum_{k=1}^{n}\Gamma_{ijk}\dot{\theta}_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)\dot{\theta}_k, \tag{34}$$

and

$$G_i\left(\theta,\dot{\theta}\right) = \frac{\partial V(\theta)}{\partial \theta_i} \tag{35}$$

includes gravity terms and other forces acting at the joints. By substituting (22) and (23) into (26) gives

$$M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3. \tag{36}$$

The components of M are given by

$$\begin{cases} M_{11} = I_{y_2}S_2^2 + I_{y_3}S_{23}^2 + I_{z_1} + I_{z_2}C_2^2 + I_{z_3}C_{23}^2 \\ +m_2 r_1^2 C_2^2 + m_3\left(l_1C_2 + r_2C_{23}\right)^2, \\ M_{12} = M_{21} = 0, \\ M_{22} = I_{x_2} + I_{x_3} + m_3 l_1^2 + m_2 r_1^2 + m_3 r_2^2 + 2m_3 l_1 r_2 r_3, \\ M_{13} = M_{31} = 0, \\ M_{23} = M_{32} = I_{x_3} + m_3 r_2^2 + m_3 l_1 r_2 r_3, \\ M_{33} = I_{x_3} + m_3 r_2^2. \end{cases} \tag{37}$$

The components of Γ are given by

$$\begin{cases} \Gamma_{112} = \left(I_{y_2} - I_{z_2} - m_2 r_1^2\right)C_2S_2 \\ +\left(I_{y_3} - I_{z_3}\right)C_{23}S_{23} - m_3\left(l_1C_2 + r_2C_{23}\right)\left(l_1S_2 + r_2S_{23}\right), \\ \Gamma_{113} = \left(I_{y_3} - I_{z_3}\right)C_{23}S_{23} - m_3 r_2 S_{23}\left(l_1C_2 + r_2C_{23}\right), \\ \Gamma_{121} = \left(I_{y_2} - I_{z_2} - m_2 r_1^2\right)C_2S_2 \\ +\left(I_{y_3} - I_{z_3}\right)C_{23}S_{23} - m_3\left(l_1C_2 + r_2C_{23}\right)\left(l_1S_2 + r_2S_{23}\right), \\ \Gamma_{131} = \left(I_{y_3} - I_{z_3}\right)C_{23}S_{23} - m_3 r_2 S_{23}\left(l_1C_2 + r_2C_{23}\right), \\ \Gamma_{131} = \left(I_{y_3} - I_{z_3}\right)C_{23}S_{23} - m_3 r_2 S_{23}\left(l_1C_2 + r_2C_{23}\right), \\ \Gamma_{211} = \left(-I_{y_2} + I_{z_2} + m_2 r_1^2\right)C_2S_2 + \left(-I_{y_3} + I_{z_3}\right)C_{23}S_{23} \\ +m_3\left(l_1C_2 + r_2C_{23}\right)\left(l_1S_2 + r_2S_{23}\right), \\ \Gamma_{233} = -l_1 m_3 r_2 S_3, \\ \Gamma_{232} = -l_1 m_3 r_2 S_3, \\ \Gamma_{233} = -l_1 m_3 r_2 S_3, \\ \Gamma_{311} = \left(-I_{y_3} + I_{z_3}\right)C_{23}S_{23} + m_3 r_2 S_{23}\left(l_1C_2 + r_2C_{23}\right), \\ \Gamma_{322} = l_1 m_3 r_2 S_3. \end{cases} \tag{38}$$

Substituting (27) and (28) into (35), we get

$$G\left(\theta,\dot{\theta}\right) = \frac{\partial V}{\partial \theta} = \begin{bmatrix} 0 \\ -\left(m_2 g r_1 + m_3 g l_1\right)C_2 - m_3 r_2 C_{23} \\ -m_3 g r_2 C_{23} \end{bmatrix}. \tag{39}$$

B. CONSTRAINTS ON MOTION

The forward kinematic mapping of the three-link manipulator is given by

$$g_{ST}(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{ST}(0), \tag{40}$$

where

$$g_{ST}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}. \tag{41}$$

Substituting (19) and (41) into (40), we get

$$g_{ST}(\theta) = \begin{bmatrix} C_1 & -S_1C_{23} & -S_1S_{23} & -S_1\left(l_1C_2 + l_2C_{23}\right) \\ S_1 & C_1C_{23} & C_1S_{23} & C_1\left(l_1C_2 + l_2C_{23}\right) \\ 0 & -S_{23} & C_{23} & l_0 - l_1S_2 - l_2S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

Based on (42), the position of the end-effector can be written as

$$\begin{cases} x = -\sin\theta_1\left(l_1\cos\theta_2 + l_2\cos\left(\theta_2 + \theta_3\right)\right), \\ y = \cos\theta_1\left(l_1\cos\theta_2 + l_2\cos\left(\theta_2 + \theta_3\right)\right), \\ z = l_0 - l_1\sin\theta_2 - l_2\sin\left(\theta_2 + \theta_3\right). \end{cases} \tag{43}$$

The task description is given by the end-effector trajectory as constraint with [37]

$$\begin{cases} x = r_{dX}(t) \\ y = r_{dY}(t) \\ z = r_{dZ}(t) \end{cases} \tag{44}$$

in which

$$\begin{cases} r_{dX}(t) = \eta \cos(4\pi t)\cos(2\pi t) + x_0 \\ r_{dY}(t) = \eta \cos(4\pi t)\sin(2\pi t) + y_0 \\ r_{dZ}(t) = \eta \cos(4\pi t)\sin(2\pi t) + z_0 \end{cases} \tag{45}$$

Then, on differentiating (44) with respect to time t twice and combining the result with (6), the constraints can be written in matrix form as

$$A = \begin{bmatrix} -C_1(l_1C_2 + l_2C_{23}) & S_1(l_1S_2 + l_2S_{23}) & l_2S_1S_{23} \\ -S_1(l_1C_2 + l_2C_{23}) & -C_1(l_1S_2 + l_2S_{23}) & l_2C_1S_{23} \\ 0 & -l_1C_2 - l_2C_{23} & l_2C_{23} \end{bmatrix} \tag{46}$$

and

$$b = [b_1, b_2, b_3]^T \tag{47}$$

in which

$$\begin{aligned} b_1 &= (l_1C_2 + l_2C_{23})S_1\dot{\theta}_1^2 + 2C_1(\dot{\theta}_3 l_2S_{23} + \dot{\theta}_2(l_1S_2 \\ &\quad + l_2S_{23}))\dot{\theta}_1 + 2\eta\pi^2(\cos(2\pi t) + 9\cos(6\pi t)) \\ &\quad + S_1(l_1C_2\dot{\theta}_2^2 + (\dot{\theta}_2 + \dot{\theta}_3)^2 l_2S_2S_3), \\ b_2 &= -C_1(l_1C_2 + l_2C_{23})\dot{\theta}_1^2 - 2S_1(-\dot{\theta}_2 - (\dot{\theta}_2 + \dot{\theta}_3)l_2S_{23})\dot{\theta}_1 \\ &\quad + C_1(-l_1C_2\dot{\theta}_2^2 - (\dot{\theta}_2 + \dot{\theta}_3)^2 l_2C_{23}) \\ &\quad + 20\eta\pi^2\cos(4\pi t)\sin(2\pi t) \\ &\quad + 16\eta\pi^2\cos(2\pi t)\sin(4\pi t), \\ b_3 &= S_2\dot{\theta}_2^2 + (\dot{\theta}_2 + \dot{\theta}_3)^2 l_2S_{23} + 20\eta\pi^2\cos(2\pi t)\sin(2\pi t) \\ &\quad + 16\eta\pi^2\cos(2\pi t)\sin(4\pi t). \end{aligned} \tag{48}$$

C. DYNAMIC EQUATIONS AND CONSTRAINT TORQUE

We have obtained the dynamical model and kinematic constraints of the three-link manipulator, and now we impose the additional motion constraints on the unconstrained system. The explicit dynamical equation including motion constraints can be written as:

$$\begin{aligned} M(\theta)\ddot{\theta} &= -C(\theta,\dot{\theta})\dot{\theta} - G(\theta,\dot{\theta}) + M^{\frac{1}{2}}\left(AM^{-\frac{1}{2}}\right)^+ \\ &\quad \times \left(b - AM^{-1}\left(-C(\theta,\dot{\theta})\dot{\theta} - G(\theta,\dot{\theta})\right)\right). \end{aligned} \tag{49}$$

According to the Udwadia–Kalaba equation, the constraint force, which represents the inverse dynamics of the manipulator, can be written in the form

$$Q^c = M^{\frac{1}{2}}\left(AM^{-\frac{1}{2}}\right)^+\left(b - AM^{-1}\left(-C(\theta,\dot{\theta})\dot{\theta} - G(\theta,\dot{\theta})\right)\right). \tag{50}$$

D. END-EFFECTOR TRAJECTORY CONTROL

The constraints must be satisfied at each instant of time, including the initial time, while in practice, due to various factors, it is usually quite difficult for the initial conditions to satisfy the constraint equations, for this reason, many researches use famous Baumgarte's method [38] to correct numerical drift when the initial conditions are incompatible with the constrained equations, see refs [6], [39]–[41]. However, the introduced parameters, α and β must be carefully selected, since the selection can make the reformulated problem stiff. For this reason, Lin and Huang [42] presented a stabilization parameters analysis method in digital control theory. Floes et al [43] presented a parametric study on the Baumgarte stabilization method, the influence of the stabilization parameters, integration method, time step and quality of the initial conditions on the dynamic response of planar constrained multibody systems. Braun and Goldfarb [44] proposed an approach which is based on Udwadia and Kalaba equation, and corrected it to enforce both the second order derivative of holonomic constraints and the first order derivative. But none of the above methods consider the uncertainties of the system. By using the UK equation. Udwadia et al [22], [33], [45]–[48] addressed the control problem of nonlinear multibody mechanical systems with uncertainties. Here, we extend the control methodology proposed in ref [46] to control the uncertain three-link manipulator system.

The end-effector trajectory can be regarded as constraint with $f(\theta(t)) - \mathbb{X}_d(t) = \mathbb{X}(t) - \mathbb{X}_d(t) = 0$, where $\mathbb{X} = f(\theta)$ denotes the forward kinematics [49], $\mathbb{X}_d(t)$ is the desired workspace trajectory. We make use of the differential forward kinematics,

$$\begin{aligned} \dot{\mathbb{X}} &= J(\theta)\dot{\theta}, \\ \ddot{\mathbb{X}} &= J(\theta)\ddot{\theta} + \dot{J}(\theta)\dot{\theta}. \end{aligned} \tag{51}$$

We bring this equation into the form (6) with

$$\begin{aligned} A(\theta,\dot{\theta}) &= J, \\ b(\theta,\dot{\theta},t) &= \ddot{\mathbb{X}}_d - \dot{J}\dot{\theta}. \end{aligned} \tag{52}$$

Based on existence of uncertainties, this control force $Q^c(t)$ obtained from Eq.(50) needs to be compensated for these uncertainties. In order to ensure that the real-life system without an already known model tracks the trajectory requirements of the nominal system [47], the constrained equation of motion, Eq.(49) is replaced with

$$M_a(q_c,t)\ddot{q}_c = Q_a(q_c,t) + Q^c(t) + Q^u(q_c,\dot{q}_c,t), \tag{53}$$

where q_c is the generalized coordinate n-vector of the controlled system and Q^u is the additional control force n-vector, which is a function of q_c, $\dot{q}_c$ and t that compensates for the fact that the model is known only imprecisely. M_a is the actual mass matrix of the real-life system, which is a function of q_c and t. The actual given force vector is Q_a. Since uncertainties in the mass of a mechanical system have perhaps the most pervasive effect on its response, we consider below the situation wherein the uncertainty in our description of the three-link manipulator system resides in our lack of exact knowledge of the mass of each body. Pre-multiplying both sides of Eq.(53) by M_a^{-1}, the acceleration of this controlled system can then be expressed as

$$\ddot{q}_c = M_a(q_c, t)^{-1}(Q_a(q_c, t) + Q^c(t) + Q^u(q_c, \dot{q}_c, t)). \quad (54)$$

The tracking error between the actual and the nominal system can be represented

$$e_{\mathbb{X}a}(t) = \mathbb{X} - \mathbb{X}_d, \quad \dot{e}_{\mathbb{X}a}(t) = \dot{\mathbb{X}} - \dot{\mathbb{X}}_d = A(\dot{\theta} - \dot{\theta}_d) = A\dot{e}_a(t) \quad (55)$$

where e_a represents the error of joint angle. Using the equations of motion of the controlled nominal system (Eq.(49)) and the controlled actual system (Eq.(53)), we get

$$\ddot{e}_a = M_a^{-1}(Q_a + Q^c) - M^{-1}(Q + Qc) + M_a^{-1}Q^u. \quad (56)$$

Defining

$$\delta\ddot{q} = M_a^{-1}(Q_a + Q^c) - M^{-1}(Q + Q^c), \quad (57)$$

where M is the mass matrix of the nominal system. By substituting Eq.(57) into Eq.(56) we can get

$$\ddot{e}_a = \delta\ddot{q} + M_a^{-1}Q^u \quad (58)$$

A sliding surface is defined as,

$$s(t) = \dot{e}_{\mathbb{X}a}(t) + ke_{\mathbb{X}a}(t) \quad (59)$$

where $k > 0$ is an arbitrary positive number. When the actual system can be restricted to stay on the sliding nominal system exactly, since they both start out with same surface $s = 0$, when the actual system is restricted on the sliding nominal system exactly, it can track the trajectories of the nominal system exactly. However, since we want a smooth function (instead of the traditionally used sigum function and saturation functions), we can only ensure that the actual system stays within a small region around the origin $\Omega_\epsilon := \{s \in R^n \big| \|s\| \leq \epsilon\}$ can be made arbitrarily small, as will be seen shortly [46]. The method requires the computation of the following estimates

$$(i) \quad \lambda_{min} := \min\{eigenvalues \quad of \quad AM_a^{-1}A^+\},$$
$$(ii) \quad \beta \geq \frac{\|A\|\|\delta\ddot{q}\| + (\|\dot{A}\| + k\|A\|)\|\dot{e}_a\|}{\lambda_{min}}, \quad \forall t > 0 \quad (60)$$

In the above relations, $\| * \|$ denotes the L_2 norm, the simple closed form expression for the additional control force can be written as

$$Q^u(t) = -\beta A^+(s/\epsilon) \quad (61)$$

In this expression, ϵ is a positive number, which is chosen to meet desired tracking tolerances. The tracking errors of workspace are guaranteed to be bounded:

$$e_{\mathbb{X}a}(t) \leq \frac{\epsilon}{k}, \qquad \dot{e}_{\mathbb{X}a}(t) \leq 2\epsilon \quad (62)$$

Thus, as seen in Eq.(62), decreasing the value of ϵ will shrink the region Ω_ϵ and reduce the upper bound of errors of tracking. The Proof of this approach(based on [46]) to tracking control of the end-effector of three-link manipulator's position and velocity as following

Proof: Noticing the definition of the sliding manifold in Eq.(59), combine Eqs.(51),(52),and (58), we can write the time derivative of the sliding manifold as,

$$\begin{aligned} \dot{s}(t) &= \ddot{e}_{\mathbb{X}a}(t) + k\dot{e}_{\mathbb{X}a}(t) \\ &= \overbrace{A\ddot{e}_a + \dot{A}\dot{e}_a}^{\ddot{e}_{\mathbb{X}a}(t)} + k\underbrace{A\dot{e}_a}_{\dot{e}_{\mathbb{X}a}(t)} \\ &= A(\delta\ddot{q} + M_a^{-1}Q^u) + \dot{A}\dot{e}_a + kA\dot{e}_a \end{aligned} \quad (63)$$

Considering the Lyapunov function

$$V_a = \frac{1}{2}s^T s, \quad (64)$$

whose derivative of the trajectories of the dynamical system can be given as

$$\begin{aligned} \dot{V}_a &= s^T\dot{s} \\ &= s^T(A(\delta\ddot{q} + M_a^{-1}Q^u) + \dot{A}\dot{e}_a + kA\dot{e}_a) \\ &= s^T(A(\delta\ddot{q} - M_a^{-1}\beta A^+(s/\epsilon)) + \dot{A}\dot{e}_a + kA\dot{e}_a) \\ &= s^T(A\delta\ddot{q} - \beta AM_a^{-1}A^+(\frac{s}{\epsilon}) + \dot{A}\dot{e}_a + kA\dot{e}_a) \end{aligned} \quad (65)$$

Observing that $s^TAM_a^{-1}A^+s \geq \lambda_{min}\|s\|^2$, we have

$$\begin{aligned} \dot{V}_a &\leq \|s\|\big(\|A\|\|\delta\ddot{q}\| - \beta\lambda_{min}\frac{\|s\|}{\epsilon} + (\|\dot{A}\| + k\|A\|)\|\dot{e}_a\|\big) \\ &= \|s\|\|A\|(\|\delta\ddot{q}\| - \beta\lambda_{min}\frac{\|s\|}{\|A\|\epsilon} + \frac{\|\dot{A}\| + k\|A\|}{\|A\|}\|\dot{e}_a\|) \end{aligned} \quad (66)$$

The region Ω_ϵ is defined such that $\|s\| \leq \epsilon$, and we have $\|s\|/\epsilon > 1$ outside Ω_ϵ. Hence outside Ω_ϵ, the right-hand side of Eq.(66) is strictly negative when $\beta \geq \frac{\|A\|\|\delta\ddot{q}\|+(\|\dot{A}\|+k\|A\|)\|\dot{e}_a\|}{\lambda_{min}}$. Since the controlled actual system starts inside the region Ω_ϵ, it will stay within this attracting region and never escape from it.

Inside the region Ω, $\|s\| \leq \epsilon$ and hence,

$$|s| \leq \epsilon. \quad (67)$$

From Eq.(59), we have

$$|\dot{e}_{\mathbb{X}a}(t) + ke_{\mathbb{X}a}(t)| < \epsilon \quad (68)$$

This inequality can be alternatively expressed as,

$$-\epsilon \leq \dot{e}_{\mathbb{X}a}(t) + ke_{\mathbb{X}a}(t) \leq \epsilon \quad (69)$$

which can further be written as

$$-\epsilon - ke_{\mathbb{X}a}(t) \leq \dot{e}_{\mathbb{X}a}(t) \leq \epsilon + ke_{\mathbb{X}a}(t). \quad (70)$$

If we can prove that $e_{\mathbb{X}a}(t)\dot{e}_{\mathbb{X}a}(t) < 0$ (which is the derivative of the Lyapunov function $\frac{1}{2}e_{\mathbb{X}a}e_{\mathbb{X}a}$) is outside a

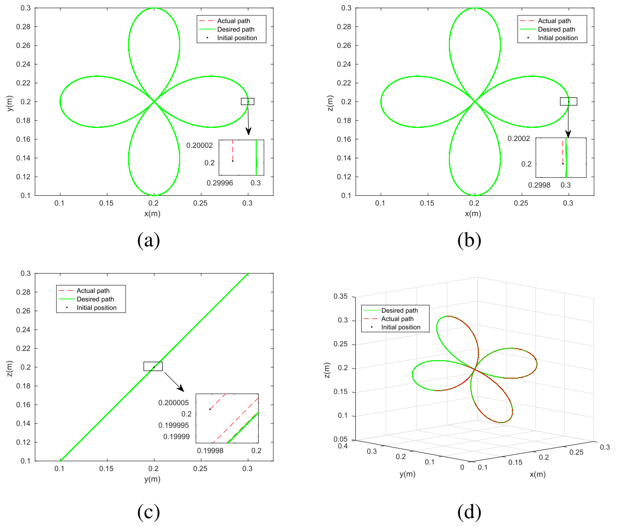


FIGURE 2. **The end-effector of the three-link manipulator tracks the Rhodonea path in Cartesian space. (a) Trajectory of the three-link manipulator in *xy* plane. (b) Trajectory of the three-link manipulator in *xz* plane. (c) Trajectory of the three-link manipulator in a *yz* plane. (d) Trajectory of the three-link manipulator in 3*D* space.**

region L_ϵ, we can then give a conclusion that the region L_ϵ is an attracting region. Defining L_ϵ as,

$$L_\epsilon := \{e_{\mathbb{X}a} \in R\big||e_{\mathbb{X}a}| \leq \frac{\epsilon}{k}\}, \tag{71}$$

there are two possibilities in which $e_{\mathbb{X}a}$, could lie outside L_ϵ.

Case 1: If $e_{\mathbb{X}a} > \frac{\epsilon}{k} > 0$, then $\epsilon - ke_{\mathbb{X}a} < 0$. From Eq.(70), we then have

$$e_{\mathbb{X}a}\dot{e}_{\mathbb{X}a} \leq e_{\mathbb{X}a}(\epsilon - ke_{\mathbb{X}a}) < 0. \tag{72}$$

Case 2: If $e_{\mathbb{X}a} < -\frac{\epsilon}{k} < 0$, then $\epsilon + ke_{\mathbb{X}a} < 0$. From Eq.(70), we also have

$$e_{\mathbb{X}a}\dot{e}_{\mathbb{X}a} \leq -e_{\mathbb{X}a}(\epsilon + ke_{\mathbb{X}a}) < 0. \tag{73}$$

which further yields

$$|\dot{e}_{\mathbb{X}a}| \leq \epsilon + |ke_{\mathbb{X}a}| \leq 2\epsilon \tag{74}$$

□

IV. NUMERICAL SIMULATIONS

The equation of motion of controlled system given in Eq.(53) is integrated in the MATLAB R2016b under Ubuntu 18.04 LTS, using the ODE15s package, with a relative error tolerance of 10^{-9} and an absolute error tolerance of 10^{-11}.The simulation time is 20 s. The numerical values of the parameters used in simulation are : $m_1 = m_2 = m_3 = 1\ kg$, $l_0 = l_1 = l_2 = 1m$, $r_0 = r_1 = r_2 = \frac{1}{2}m$, $I_{z_1} = I_{x_2} = I_{x_3} = I_{y_2} = I_{y_3} = I_{z_2} = I_{z_3} = 0.1\ kg \cdot m^2$, $g = 9.8m/s^2$,$\eta = 0.1$, $x_0 = y_0 = z_0 = 0.2$, $\Delta m_1 = 0.1\ m_1 \sin(t)$, $\Delta m_2 = 0.1\ m_2 \sin(t)$, $\Delta m_3 = 0.1\ m_3 \sin(t)$. The initial condition of the system given as $\theta(0) = [-0.9828, 0.0308, 2.2332]^T$, $\dot{\theta}(0) = [1.4450, -0.8954, 0.4781]^T$. We choose the control parameters as follows: $k = 10$, $\beta = 10^3$, $\epsilon = 10^{-4}$. For these chosen parameters, we are guaranteed that the tracking errors in position and velocity as given by Eq.(62)

$$e_{\mathbb{X}a}(t) \leq \frac{\epsilon}{k} = 10^{-5}, \qquad \dot{e}_{\mathbb{X}a}(t) \leq 2\epsilon = 2 \times 10^{-4} \tag{75}$$

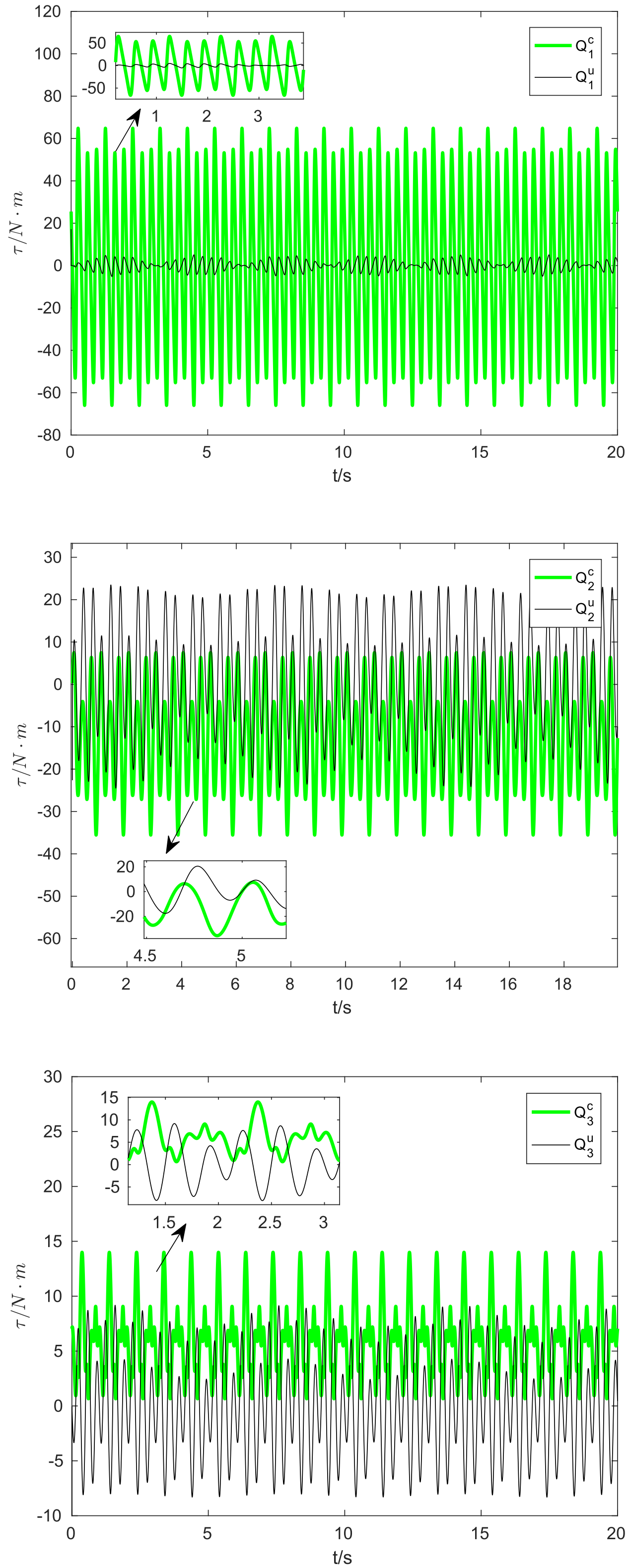
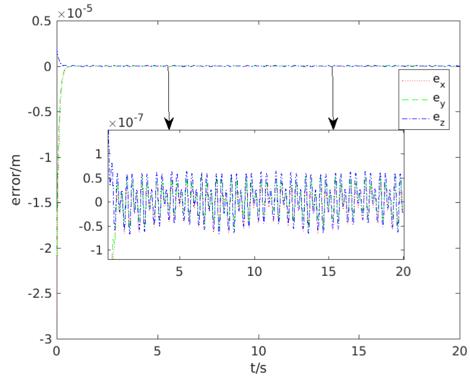
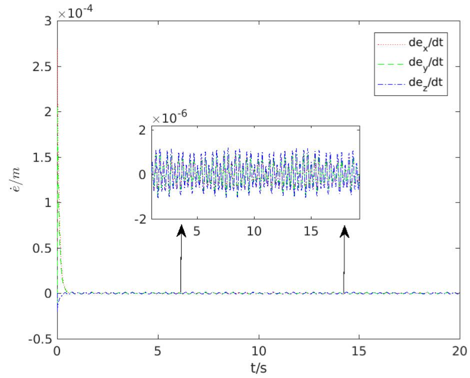


FIGURE 3. **Constraint torque Q^c (control input) and compensate torque Q^u which compensate for uncertainties in describing the actual system.**

The results of the numerical simulations are presented in Figs. 2, 3, 4 and 5. Fig.2 reflects that the end-effector of the three-link manipulator tracks the Rhodonea path in Cartesian



(a)



(b)

FIGURE 4. Tracking error. (a) The error of tracking the nominal system on position. (b) The error of tracking the nominal system on velocity.

space. We can see clearly from Fig.2 that the position is well coincident with the desired trajectory. The required servo joint forces are shown in Fig.3, where Q^c denotes the control input, and Q^u denotes the additional control torques for uncertainties of the actual system. Fig4 shows the error of tracking process of the nominal system. We can ensure the error bound of position to be within 10^{-5} and the error bound of velocity within 2×10^{-4}. By integrated Eq.(54), we can see the actual joint angle and angular velocity is the function of time.

Because it's hard to decide the Baumgarte's parameters, and the selection of inappropriate parameters will cause the instability of numerical integration. For comparison, we use the method proposed by Braun [44] to control the three-link manipulator. The main equation to eliminating constraint drift (here we regard the tracking error as constraint drift) can be given as:

$$\begin{aligned} \dot{q} &= v + M^{-1/2}B^{+}(b_q - Av - \Phi/dt), \\ \dot{v} &= a + M^{-1/2}B^{+}(b_v - Aa - \dot{\Phi}/dt) \end{aligned} \tag{76}$$

in which

$$bq = -\frac{d\Phi}{dt} = \begin{bmatrix} -\eta\pi(\sin(2\pi t) + 3\sin(6\pi t)) \\ \eta\pi(-\cos(2\pi t) + 3\cos(6\pi t)) \\ \eta\pi(-\cos(2\pi t) + 3\cos(6\pi t)) \end{bmatrix} \tag{77}$$

where $v = \dot{q}$ is the constrained velocity, $\dot{v}$ is the constrained acceleration, $\Phi = [x - r_{dX}, y - r_{dY}, z - r_{dZ}]^T$ is the motion

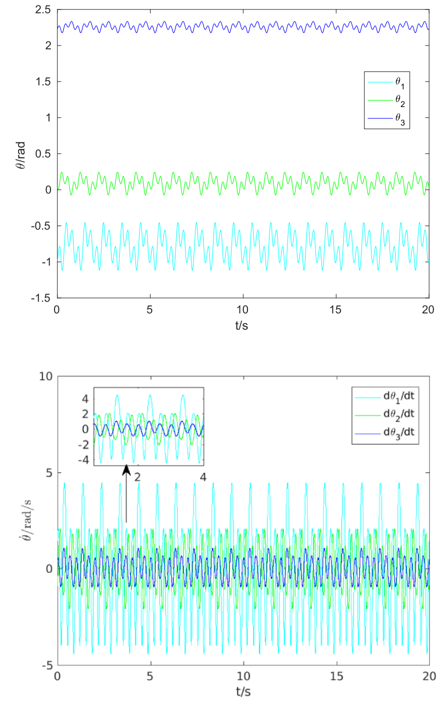
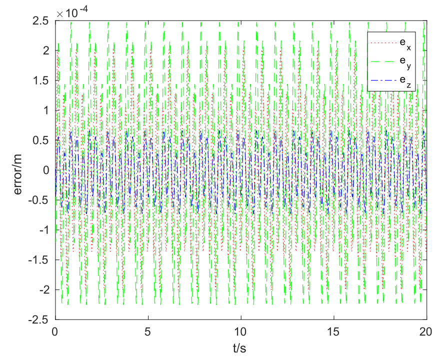
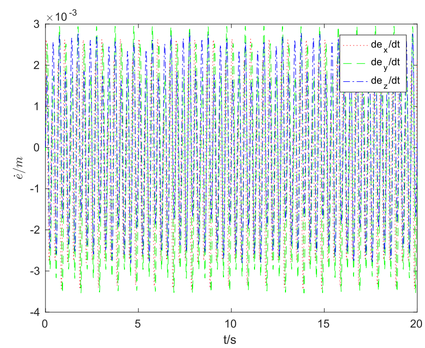


FIGURE 5. Actual joint angle and angular velocity as function of time.



(a)



(b)

FIGURE 6. Braun's method to eliminating the error with uncertainties dynamics parameters. (a) The error of tracking the nominal system on position. (b) The error of tracking the nominal system on velocity.

constraints, bv obtained from Eq.(48), and dt is the integration step. We consider system have uncertain parameters, then the

Eq.(78) is replaced with

$$\dot{q} = v + M^{-1/2}B^{+}(b_q - Av - \Phi/dt),$$
$$\dot{v} = M_a^{-1}Q_a + M^{-1/2}B^{+}(b_v - AM_a^{-1}Q_a - \dot{\Phi}/dt) \quad (78)$$

We solved Eq.(78) with a fourth order fixed step Runge-Kutta method with 10^{-4} time step. The results of the numerical simulations are presented in Fig.6, we can see no error accumulation in motion constraints. The error in position about 3×10^{-4} and the error in velocity about 3×10^{-3}. As shown by the Fig.6, when the dynamics parameters are uncertain, Braun's method still has good performance, but it has no ability to control the error bounds.

V. CONCLUSION

In this paper, problems in the dynamic model and simulation of the constrained mechanical system of a three-link manipulator have been considered. The dynamical equations and the constrained torque are obtained from Udwadia-Kalaba equations, and a simple method for the tracking control problem with desired trajectory (treated as motion constrains) requirements, with regard to model uncertainties, has been developed. We use Braun's method for comparison, although the method proposed by Braun still has good performance when the dynamics parameters are uncertain, but it is not able to control the error bounds. The contribution of our study includes

- We obtain explicit equations for dynamic modeling of constrained three-link manipulator by using UK equation.
- Based on the Udwadia-Kalaba equation, we designed a nonlinear controller which has controllable error bounds for the three-link manipulator systems with uncertain dynamics parameters.
- The obtained controller relies on state-of-art dynamic modelling method rather than on control theory. Error bounds on tracking due to uncertainties are analytically obtained.
- These control parameters can be adjusted to get desired error bounds.

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