Treelike snarks

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Abstract

We study snarks whose edges cannot be covered by fewer than five perfect matchings. Esperet and Mazzuoccolo found an infinite family of such snarks, generalising an example provided by Hägglund. We construct another infinite family, arising from a generalisation in a different direction. The proof that this family has the requested property is computer-assisted. In addition, we prove that the snarks from this family (we call them *treelike snarks*) have circular flow number $\phi_C(G) \geqslant 5$ and admit a 5-cycle double cover.

Keywords: Snark; excessive index; circular flow number; cycle double cover

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1 Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted. Most of our terminology is standard; for further definitions and notation not explicitly stated in the paper, please refer to [2].

By Vizing's theorem, the edge chromatic number of every cubic graph is either three or four. In order to study the class of cubic graphs with edge chromatic number equal to four, it is usual to exclude certain trivial modifications. Thus, a *snark* (cf. e.g. [14]) is defined as a bridgeless cubic graph with edge chromatic number equal to four that contains no circuits of length at most four and no non-trivial 3-edge cuts.

There is a vast literature on snarks and their properties — see, e.g., [1, 4, 16, 17, 18]. The interested reader will find an introduction to the field in, e.g., [14] or [23].

A *perfect matching*, or *1-factor*, in a graph G is a regular spanning subgraph of degree 1. In this context, a *cover* of G is a set of perfect matchings of G such that each edge of G belongs to at least one of the perfect matchings. Following the terminology introduced in [3], the *excessive index of* G, denoted by $\chi'_e(G)$, is the least integer k such that the edge-set of G can be covered by k perfect matchings. Note that the excessive index is sometimes also called *perfect matching index* (see [9]).

The main source of motivation for the above notion is the conjecture of Berge which asserts that the excessive index of any cubic bridgeless graph is at most 5. As proved recently by the third author [19], this conjecture is equivalent to the famous conjecture of Berge and Fulkerson [10] that the edge-set of every bridgeless cubic graph can be covered by six perfect matchings, such that each edge is covered precisely twice.

It is NP-complete to decide for a cubic bridgeless graph G whether $\chi'_e(G) = 3$, since this property is equivalent to 3-edge-colourability. Similarly, Esperet and Mazzuoccolo [7] proved that it is NP-complete to decide whether $\chi'_e(G) \leqslant 4$, as well as to decide whether $\chi'_e(G) = 4$.

As for cubic bridgeless graphs G with $\chi'_e(G) \geqslant 5$, it was asked by Fouquet and Vanherpe [9] whether the Petersen graph was the only such graph that is cyclically 4-edge-connected. Hägglund [12] constructed another example (of order 34) and asked for a characterisation of such graphs [12, Problem 3]. Esperet and Mazzuoccolo [7] generalized Hägglund's example to an infinite family.

We now outline the structure of the present paper, referring to the later sections for the necessary definitions. In Section 3, we construct a family of graphs called *treelike snarks*. We prove that they are indeed snarks and (in Section 5) that their excessive index is greater than or equal to five. We thus expand the known family of snarks of excessive index $\geqslant 5$, with a different generalization of Hägglund's example than the one found by Esperet and Mazzuoccolo in [7]. The proof relies on the use of a computer to determine a certain set of patterns (see Sections 4 and 8).

In Section 6, we recall the definition of the circular flow number and the 5-Flow Conjecture of Tutte. We show that treelike snarks are, in a sense, also critical for this conjecture — namely, their circular flow number is greater than or equal to five.

Section 7 is devoted to cycle double covers. Since it is known that any cubic graph G that is a counterexample to the Cycle Double Cover Conjecture satisfies $\chi'_e(G) \geqslant 5$, it is natural to ask whether treelike snarks admit cycle double covers. We show that this is indeed the case. In fact, using a new general sufficient condition for the existence of a 5-cycle double cover, we show that treelike snarks satisfy the 5-Cycle Double Cover Conjecture of Preissmann [20] and Celmins [5].

2 Preliminaries

For a given graph G, the vertex set of G is denoted by $V(G)$, and its edge set by $E(G)$. Each edge is viewed as composed of two *half-edges* (that are *associated* to each other) and we let $E(v)$ denote the set of half-edges incident with a vertex v.

A *join* in a graph G is a set $J \subseteq E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. In the literature, the terms *postman join* or *parity subgraph* have essentially the same meaning. Throughout this paper, we will be dealing with cubic graphs, so *joins* will be spanning subgraphs where each vertex has degree 1 or 3.

As usual, e.g., in the theory of nowhere-zero flows, we define a *cycle* in a graph G to be any subgraph $H \subseteq G$ such that each vertex of H has even degree in H. Thus, a cycle need not be connected. A *circuit* is a connected 2-regular graph. In a cubic graph, a cycle is a disjoint union of circuits and isolated vertices.

Observation 1. *A subgraph $H \subseteq G$ is a cycle in G if and only if $E(G) - E(H)$ is a join.*

An *edge-cut* (or just *cut*) in G is a set $T \subseteq E(G)$ such that $G-T$ has more components than G, and T is inclusionwise minimal with this property. A cut is *trivial* if it consists of all edges incident with a particular vertex. A *bridge* is a cut of size 1. A graph is *bridgeless* if it contains no bridge (note that with this definition, a bridgeless graph may be disconnected). A graph G is said to be *k-edge-connected* (where $k \geqslant 1$) if G is connected and contains no cut of size at most $k-1$, i.e. G is such that its edge-connectivity $k'(G) \geqslant k$. A set S of edges of a graph G is a *cyclic edge cut* if $G-S$ has two components each of which contains a cycle. We say that a graph G is *cyclically k-edge-connected* if $|E(G)| > k$ and each cyclic edge cut of G has size at least k.

A *cover* (or *covering*) of a graph G is a family $\mathcal{F}$ of subgraphs of G, not necessarily edge-disjoint, such that $\bigcup_{F \in \mathcal{F}} E(F) = E(G)$. A $(1,2)$-*cover* is a cover in which each edge appears at most twice.

Recall that the following lemma is very useful when studying edge-colourability of cubic graphs (see, for instance, Lemma B.1.3 in [24]).

Lemma 2 (Parity Lemma). *Let G be a cubic graph and let $c : E(G) \to \{1,2,3\}$ be a 3-edge-colouring of G. Then, for every edge-cut T in G,*

$$|T \cap c^{-1}(i)| \equiv |T| \pmod 2$$

for $i \in \{1,2,3\}$.

3 Treelike snarks

As noted in Section 2, we view each edge of a graph as composed of two half-edges. We now extend the notion of a graph by allowing for *loose* half-edges that do not form part of any edge; the resulting structures will be called *generalised graphs*. A generalised graph is *cubic* if each vertex is incident with three half-edges.

We define a *fragment* F as a generalised cubic graph with exactly five loose half-edges, ordered in a sequence (see Figure 1). The *Petersen fragment* F_0 is the fragment with loose half-edges $(a_1, \ldots, a_5)$ obtained from the Petersen graph (see Figure 2b) as follows:

- in the Petersen graph, remove a vertex x, keeping the half-edges a_3, a_4, a_5 incident with its neighbours y, z, t, respectively,
- subdivide the two edges incident with y,
- and add half edges a_1, a_2 to the new vertices of the subdivision (see Figures 2a, 2b) in any one of the two ways.

This fragment F_0 will be particularly important throughout the paper.

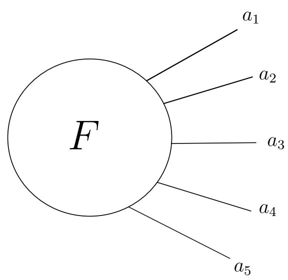
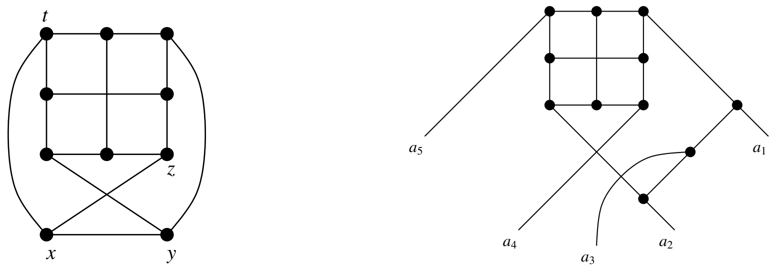


Figure 1: A fragment.



(a) The Petersen graph. (b) The Petersen fragment F_0.

Figure 2: Constructing the Petersen fragment.

A *Halin graph* is a plane graph consisting of a planar representation of a tree without degree 2 vertices, and a circuit on the set of its leaves (cf., e.g., [13, 6]).

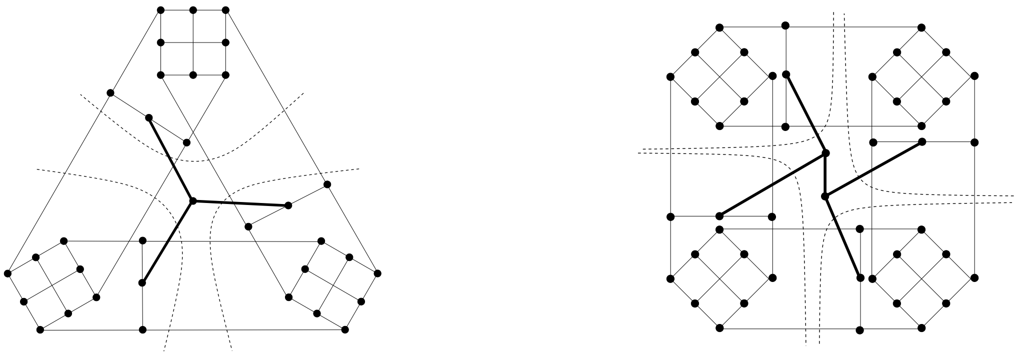
Let H_0 be a cubic Halin graph consisting of the tree T_0 and the circuit C_0. The *treelike snark* $G(T_0, C_0)$ is obtained by the following procedure:

- for each leaf ℓ of T_0, we add a copy F_0^ℓ of the Petersen fragment F_0 with loose half-edges $(a_1, \dots, a_5)$ and attach the half-edge a_3 to ℓ,

- for each leaf ℓ of T_0 and its successor ℓ' with respect to a fixed direction of C_0, if F_0^ℓ has loose half-edges $(a_1, \dots, a_5)$ and $F_0^{\ell'}$ has loose half-edges $(a'_1, \dots, a'_5)$, then we join a_4 to a'_2 and a_5 to a'_1, obtaining new edges.

If there is no danger of a confusion, we abbreviate $G(T_0, C_0)$ to $G(T_0)$.

Some small examples of treelike snarks are shown in Figure 3.



(a) Windmill 1 [7, Figure 7]. (b) Blowup($Prism, C_4$) [12, Figure 7].

Figure 3: Small treelike snarks.

Let F_1 and F_2 be fragments with half edges $(a_1, a_2, a_3, a_4, a_5)$ and $(a'_1, a'_2, a'_3, a'_4, a'_5)$ respectively. We define the *sum* $F_1 + F_2$ of F_1 and F_2 as the fragment obtained from F_1 and F_2 by the following operations:

- joining the half-edges a_5 to a'_1 and a_4 to a'_2 into edges,

- adding a new vertex adjacent to a_3, a'_3 and a new half-edge a''_3,

with half-edges $(a_1, a_2, a''_3, a'_4, a'_5)$ (see Figure 4a).

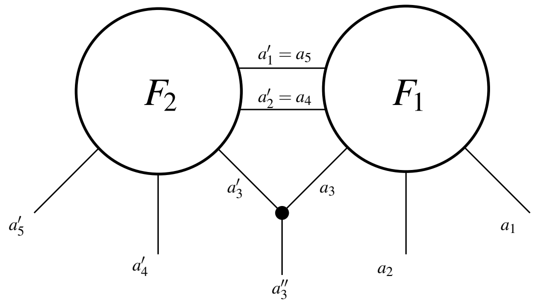
The *fusion* of F_1 with F_2 is the cubic graph obtained from the union of F_1 and F_2 by joining each pair of half-edges a_i, a'_{6-i}, where $1 \leqslant i \leqslant 5$ (see Figure 4b).

Observe that any treelike snark $G(T_0, C_0)$ may be obtained as a fusion of two finite sums of Petersen fragments (with appropriate bracketings, determined by T_0). In fact, one of the sums can be taken to be just a Petersen fragment.

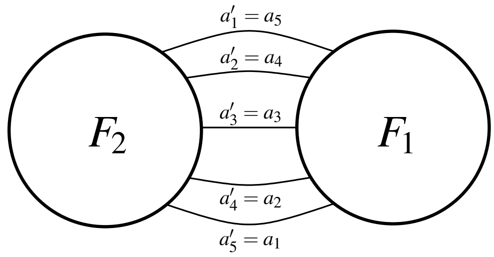
The following properties of the Petersen fragment F_0, arising from its symmetries, will be useful later.

(i) The half-edges a_1 and a_2 are symmetrically equivalent, in the sense that reversing their order leads to an isomorphic fragment (with the notion of isomorphism defined in the natural way). We will call these half-edges the *right spokes*.

(ii) The half-edges a_4 and a_5 are symmetrically equivalent. We will call these the *left spokes*.



(a) Fragment sum $F_1 + F_2$.



(b) Fusion of F_1 and F_2.

Figure 4

(iii) The endvertices of the half-edges a_1, a_3 and a_2 form a path of length two (in that order), which we will call the *special path.* The half-edge a_3 may also be referred to as the *central spoke.*

The *distance* between two half-edges is the length of a shortest path connecting their (unique) vertices.

The following proposition justifies the phrase 'treelike snark':

Proposition 3. *Treelike snarks are snarks.*

Proof. Let $G(T_0) = G(T_0, C_0)$ be a treelike snark. By construction, $G(T_0)$ is cubic. As noted above, $G(T_0)$ is a fusion of a Petersen fragment with a finite sum of Petersen fragments F_0^ℓ, where ℓ ranges over the leaves of T_0.

We prove that $G(T_0)$ has girth 5. In each Petersen fragment F_0 there is a 5-circuit (a circuit of length 5) but no shorter circuits. Suppose that $G(T_0)$ contains a circuit C of length at most 4 traversing some of the loose half-edges $(a_1, \dots, a_5)$ of a Petersen fragment F_0^ℓ. The distance from each left spoke a_4, a_5 to any other half-edge of F_0 is at least 3, hence C traverses no left spoke of F_0^ℓ. By the structure of $G(T_0)$ as a sum of fragments, right spokes of F_0^ℓ are joined to left spokes of some other Petersen fragment, so no right spoke of F_0^ℓ is traversed by C either. The only remaining loose half-edge of F_0^ℓ is a_3, a contradiction. Thus, $G(T_0)$ has girth 5.

We claim that $G(T_0)$ is cyclically 4-edge-connected. Since $G(T_0)$ is clearly 3-edge-connected, it suffices to show that there is no nontrivial 3-edge-cut. This is clearly true of the Petersen fragment has no nontrivial 3-edge-cut, and one can easily check that neither the sum nor the fusion with a Petersen fragment introduce a nontrivial 3-edge-cut.

To determine the chromatic index of $G(T_0)$, suppose $G(T_0)$ admits a 3-edge-colouring. Denote by b_1 and b_2 the two edges of a Petersen fragment F_0^ℓ which are incident with a_1 and a_2, respectively, and are not edges of the special path. If F_0^ℓ has left spokes of different colours, then by Lemma 2, the same two colours also appear on b_1 and b_2, and it is easy to derive a 3-edge-colouring of the Petersen graph, a contradiction. Therefore, we can assume that in each Petersen fragment, the two left spokes are given the same colour. By construction, the right spokes of F_0^ℓ are left spokes of another Petersen fragment, so they also share a colour (in general, different from that of the left spokes). Suppose that a Petersen fragment has the two left spokes of colour a and the two right spokes of colour b. Then $a = b$, since by Lemma 2, neither a nor b can appear exactly twice in the 5-cut separating the fragment from the rest of the graph. In addition, the lemma implies that all central spokes share the same colour a, a contradiction since at least two of them are adjacent edges. Thus, the chromatic index of $G(T_0)$ is 4. □

4 Patterns

The notion of a perfect matching is easily extended from graphs to generalised graphs: it is simply a set M of edges and loose half-edges such that each vertex is incident with exactly one element of M. We will be interested in $(1, 2)$-covers of a given fragment by 4 perfect matchings. To describe the 'behaviour' of a cover on the loose half-edges $a_1, \dots, a_5$ of a fragment F, we introduce the following definitions.

A *pattern* π is a sequence of five subsets of $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ of size 1 or 2 each, such that each symbol from $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ appears in an odd number of the subsets in π. Examples of patterns are A A AB AC AD or AB AC AD BD BD (we omit both the set brackets and the parentheses enclosing a sequence).

Observe that any $(1, 2)$-cover of a fragment by 4 perfect matchings determines a pattern in a natural way. For instance, if a cover by perfect matchings A, B, C, D is such that each of the loose half-edges $a_1, \dots, a_5$ is contained in A, and in addition, a_3, a_4, a_5 are contained in B, C and D respectively, then the corresponding pattern is A A AB AC AD.

The set of all patterns determined by $(1, 2)$-covers of a fragment F by 4 perfect matchings is called the *pattern set of* F and denoted by $\Pi(F)$.

Using a computer program implemented in C, we have determined the pattern sets for each of the following fragments: F_0, $F_0 + F_0$, $(F_0 + F_0) + F_0$, $F_0 + (F_0 + F_0)$, $(F_0 + F_0) + (F_0 + F_0)$. The results are summarised in Section 8. One obvious conclusion from these data is that

$$\Pi(F_0 + (F_0 + F_0)) \neq \Pi((F_0 + F_0) + F_0).$$

This somewhat surprising lack of associativity is illustrated in Figure 5.

A further observation about the results presented in Section 8 is given by the following lemma:

Lemma 4. *Let F_0 be the Petersen fragment. The following inclusions hold:*

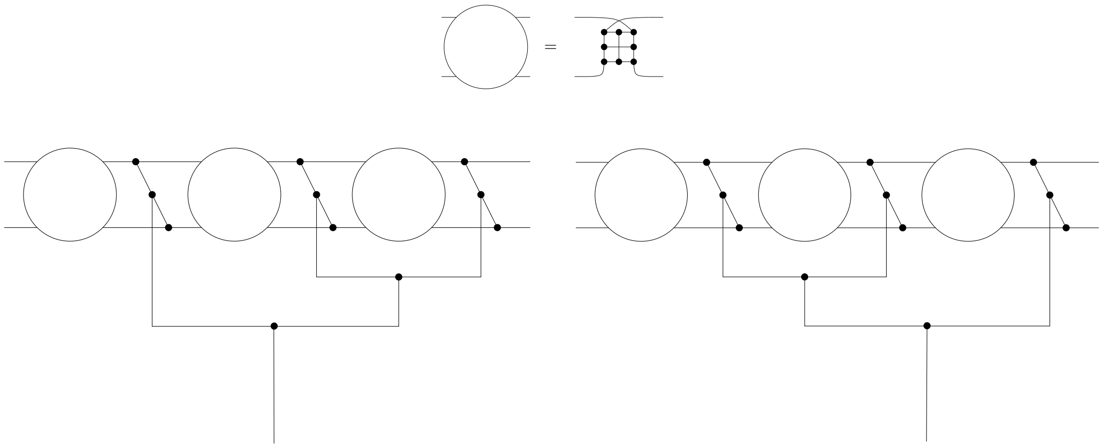
(i) $\Pi(F_0 + (F_0 + F_0)) \subset \Pi(F_0 + F_0)$,

(ii) $\Pi((F_0 + F_0) + (F_0 + F_0)) \subset \Pi((F_0 + F_0) + F_0)$.

Proof. By inspection of the lists in Section 8. □

5 Excessive index

Theorem 5. *Treelike snarks have excessive index at least* 5.



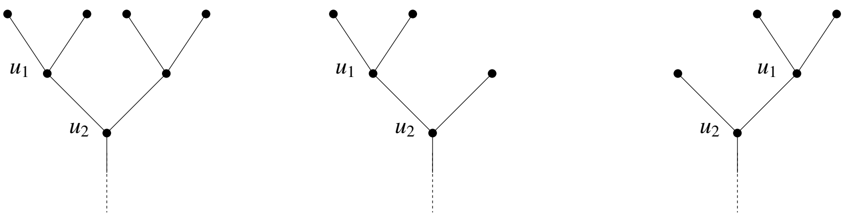
(a) The fragment $F_0 + (F_0 + F_0)$. (b) The fragment $(F_0 + F_0) + F_0$.

Figure 5: Lack of associativity in sums of Petersen fragments.

Proof. Let $G(T_0)$ be a treelike snark with underlying tree T_0. We proceed by induction on the order of T_0. If $T_0 = K_{1,3}$, then by [12, Sec. 3] and [7, Theorem 6], the treelike snark $G(T_0)$ has excessive index 5.

Suppose the statement is true for all treelike snarks with $|V(T_0)| < n$, where $n > 4$.

Suppose further that the underlying tree T_0 has n vertices, and let $U = u_1, u_2, \ldots, u_t$ be a longest path of vertices of degree 3 in T_0. Then the part of T_0 around the endvertex u_1 of U coincides with one of the possibilities in Figure 6.



(a) (b) (c)

Figure 6: Possible cases in the proof of Theorem 5. Any vertex represented as a leaf is a leaf of T_0.

We show that case (a) reduces to case (b). Replace the fragment $(F_0+F_0)+(F_0+F_0)$ corresponding to the part of T_0 shown in Figure 6(a) by $(F_0+F_0)+F_0$, obtaining a cubic graph G'. Suppose that the excessive index of $G(T_0)$ is less than or equal to 4. Then it is, in fact, equal to 4 (as we know that $G(T_0)$ is a snark). Consider a $(1,2)$-cover of $G(T_0)$ by 4 perfect matchings. By Lemma 4(ii), the pattern induced on the loose half-edges of $(F_0 + F_0) + (F_0 + F_0)$ can be extended to a $(1,2)$-cover by 4 perfect matchings of $(F_0 + F_0) + F_0$, and hence to that of G'. This contradicts the induction hypothesis, as the latter implies that the excessive index of G' is greater than 4. Consequently, the excessive index of $G(T_0)$ is greater than 4.

In a similar way, we can reduce case (c) using Lemma 4(i). Although the lemma implies no direct reduction for case (b), Figure 7 shows that by expressing $G(T_0)$ as a sum

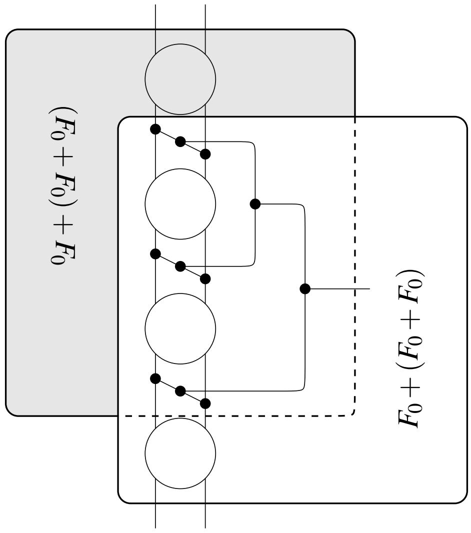


Figure 7: Moving from case (b) to case (c) in the proof of Theorem 5.

(and fusion) of fragments in a different way, case (b) is transformed into case (c), which is reduced as before. The proof is thus complete. □

6 Circular flow number

We recall the notion of circular nowhere-zero r-flow, first introduced in [11]. Let $G = (V, E)$ be a graph.

Given a real number $r \geqslant 2$, a *circular nowhere-zero r-flow* (r-CNZF for short) in G is an assignment $f : E \to [1, r-1]$ and an orientation D of G, such that f is a flow in D. That is, for every vertex $x \in V$, $\sum_{e \in E^+(x)} f(e) = \sum_{e \in E^-(x)} f(e)$, where $E^+(x)$, respectively $E^-(x)$, are the sets of edges directed from, respectively toward, x in D.

The *circular flow number* $\phi_c(G)$ of G is the infimum of the set of numbers r for which G admits an r-CNZF. If G has a bridge then no r-CNZF exists for any r, and we define $\phi_c(G) = \infty$.

A circular nowhere-zero *modular-r*-flow (r-MCNZF), is an analogue of an r-CNZF, where the additive group of real numbers is replaced by $\mathbb{R}/r\mathbb{Z}$. We would like to stress that, given an r-MCNZF f, the direction of an edge e can be always reversed and f transformed into another r-MCNZF, where $f(e) \in \mathbb{R}/r\mathbb{Z}$ is replaced by $-f(e) \in \mathbb{R}/r\mathbb{Z}$.

The following result is well-known and implicitly proved also in Tutte's original work on integer flows [22].

Proposition 6 ([22]). *The existence of a circular nowhere-zero r-flow in a graph G is equivalent to that of an r-MCNZF.*

The outstanding 5-Flow Conjecture is equivalent to the statement that the circular flow number of no bridgeless graph is greater than 5. In [8], the authors present some general methods for constructing graphs (in particular snarks) with circular flow number at least 5. By a direct application of the main results in [8], as we have mentioned in the Introduction, one can deduce that all (few) known snarks with excessive index 5 have circular flow number at least 5. In other words, if a snark is "critical" with respect to Berge-Fulkerson's Conjecture, then it seems to be critical also for the 5-flow Conjecture. The converse is false, as shown by the snark G of order 28, found by Máčajová and Raspaud [18]: it has $\phi_C(G) = 5$ and $\chi'_e(G) = 4$.

In this section, we furnish a further element in the direction of the previous observation, by proving that also all treelike snarks have circular flow number at least 5 (cf. Theorem 9).

In order to prove the main theorem of this section, we need to briefly recall some notions and results introduced in [8]: we will not present them in the most general setting, but just as needed for the purpose of this paper. For a general presentation, we refer the interested reader to the original paper.

First of all, as a direct consequence of Proposition 6 we have the following:

Proposition 7 ([8])**.** *For any graph G, $\phi_c(G) < 5$ if and only if there exists an 5-MCNZF f in G, such that $f : E \to (1, 4)$.*

The notion of a 2-pole is crucial in this setting. A *2-pole* $G_{u,v}$ consists of a graph G and two of its vertices, u and v. The vertices u and v are the *terminals* of $G_{u,v}$. The *open 5-capacity* $CP_5(G_{u,v})$ of $G_{u,v}$ is a subset of $\mathbb{R}/5\mathbb{Z}$, defined by adding to G a new edge e joining u to v, and setting

$$CP_5(G_{u,v}) = \{f(e) \mid f \text{ is a modular flow in } G \cup e \text{ and } f : E(G) \to (1, 4)\}.$$

The following properties hold (see [8]):

(i) The open 5-capacity of a single edge $[u, v]$ is the open interval $(1, 4)$.

(ii) The open 5-capacity of $P^-_{u,v}$ (where P^- is the Petersen graph minus an edge uv) is the interval $(4, 1)$ in $\mathbb{R}/5\mathbb{Z}$.

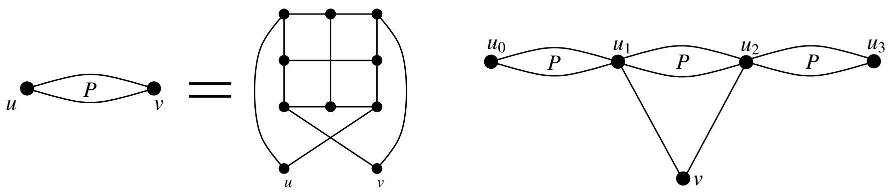


Figure 8: The 2-pole $P^-_{u,v}$ and the subgraph described in Lemma 8.

Now we describe a forbidden configuration for a graph whose circular flow number is less than 5.

Lemma 8. *Let (u_0, u_1, u_2, u_3) be a path in a graph G, along vertices of degree 3 such that there exists a vertex v adjacent both to u_1 and u_2. Let G' be the graph obtained from G by substituting each of the three edges $[u_i, u_{i+1}]$ for $i = 0, 1, 2$ with a 2-pole $P^-_{u_i,u_{i+1}}$. Then $\phi_c(G') \geqslant 5$.*

Proof. For the sake of a contradiction, assume $\phi_c(G') < 5$. By Proposition 7, G' admits a 5-MCNZF f', such that $f' : E \to (1,4)$. The flow f' of G' induces a flow f on the original graph G with values in $(1,4)$ for all edges but $[u_i, u_{i+1}]$ which have values in $(4,1)$ (since the corresponding 2-pole has open 5-capacity $(4,1)$ as previously remarked). Conversely, any such flow of G corresponds to a flow of G' with values in $(1,4)$. Hence, we have to prove that G cannot have a flow as the one described above. Assume (u_0, u_1, u_2, u_3) is a directed path in G from u_0 to u_3. If this is not the case, we can reverse some edge of the path for obtaining a new valid flow of G. For the same reason, we can assume $[v, u_1], [v, u_2]$ are directed from v and the third edge in v, say e, is directed towards v (recall that both $(1,4)$ and $(4,1)$ are symmetric subsets of $\mathbb{R}/5\mathbb{Z}$). Since the flow values of $[v, u_1], [v, u_2]$ belong to $(1,4)$, we cannot have values of f on two consecutive edges of the path (u_0, u_1, u_2, u_3) in the same unit interval: hence, the values of f along the path (u_0, u_1, u_2, u_3) are alternating between the two intervals $(4,0)$ and $(0,1)$. Furthermore, the values of f on $[v, u_1]$ and $[v, u_2]$ are one in $(1,2)$ and the other in $(3,4)$. Finally, the value of f on e is the sum of the values on $[v, u_1]$ and $[v, u_2]$ and so it is in $(4,1)$, that is a contradiction since e has open 5-capacity $(1,4)$. $\square$

Note that the graph G' constructed in Lemma 8 is not cubic since some vertices have degree more than 3. More precisely, all vertices u_i have degree 5 in G'. However, it is well known that the expansion of a vertex x to a subgraph X (see Figure 9) does not decrease the circular flow number of a graph, since each flow in the expansion can be naturally reduced to a flow of the original graph. Thus, by performing a series of expansions, we can transform G' into a cubic graph without decreasing the circular flow number.

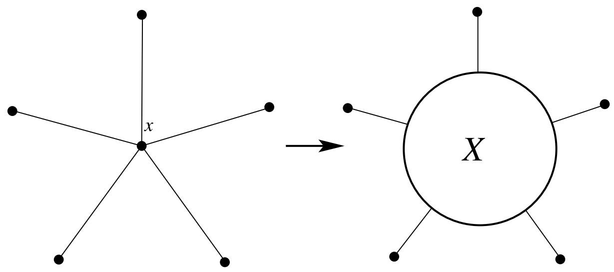


Figure 9: Expansion of a vertex x to an arbitrary subgraph X.

By choosing the starting graph and the expansions in a suitable way, we obtain the following result for treelike snarks:

Theorem 9. *Treelike snarks have circular flow number at least* 5.

Proof. Let $G(T_0, C_0)$ be a treelike snark; consider the corresponding cubic Halin graph H_0 consisting of a tree T_0 and a circuit C_0 on its leaves. In H_0, substitute each edge of C_0 with a 2-pole $P^{-}_{u,v}$. The resulting graph H'_0 contains the configuration described in Lemma 8, and therefore its circular flow number is greater than or equal to 5. Expanding the terminals of every 2-pole as depicted in Figure 10, we obtain $G(T_0, C_0)$. As argued above, vertex expansions do not decrease the circular flow number, so the theorem follows. $\square$

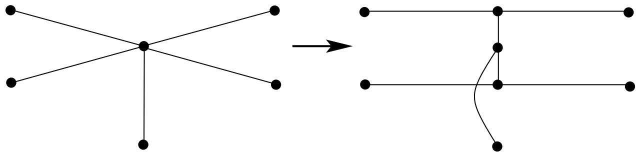


Figure 10: Vertex expansion to obtain treelike snarks.

7 Cycle double covers

In this section, we investigate the properties of treelike snarks with respect to cycle double covers. Recall that the notion of a $(1,2)$-cover was defined in Section 2. A 2-*packing* of a cubic graph G a set of joins such that each edge of G belongs to at most two of the joins.

Hou, Lai and Zhang [15] have recently proved the following equivalent formulation of the 5-CDC Conjecture.

Theorem 10 ([15], Theorem 3.3)**.** *Let G be a cubic graph. The following statements are equivalent:*

- *G has a 5-cycle double cover,*
- *G has a $(1,2)$-cover by 4 joins.*

As a consequence of Theorem 10, every cubic graph with excessive index 4 admits a 5-cycle double cover, as already directly proved by Steffen in [21]. Hence, looking for a possible counterexample for the 5-Cycle Double Cover Conjecture, one should look into the class of snarks with excessive index at least 5.

In what follows, we introduce a new general sufficient condition for a cubic graph to admit a 5-cycle double cover (Theorem 13) and we show how it can be easily used to prove the existence of a 5-cycle double cover for every treelike snark (Theorem 16).

If C is a circuit in a cubic graph G, we let G_C denote the multigraph obtained by successively contracting each edge of C to a vertex.

Lemma 11. *Let G be a cubic graph and let C be a circuit of G. Then any join J' of G_C can be extended to a join J of G. Moreover, the previous extension can be performed in two distinct ways.*

Proof. Let the vertices of C be denoted by $v_0, \ldots, v_t$ in order. For $i \in \{0, \ldots, t\}$, let w_i be the unique neighbour of v_i such that v_iw_i is not an edge of C. Furthermore, let v denote the vertex of G_C corresponding to the contracted circuit C.

Assume that J' is a given join of G_C. Let I be the set of all i such that $0 \leqslant i \leqslant t$ and the edge vw_i of G_C does not belong to J'. Since J' is a join of G', $|I|$ is even, say $|I| = 2r$. Let us write $I = \{i_0, \ldots, i_{2r-1}\}$, where $i_0 < \ldots < i_{2r-1}$. We extend J' to a join J of G as follows:

- an edge of G not incident with C belongs to J if and only if the corresponding edge of G_C belongs to J',

- an edge $[v_i, w_i]$ of G belongs to J if and only if the corresponding edge $[v, w_i]$ belongs to J',
- an edge $[v_i, v_{i+1}]$ (indices taken modulo t) of C belongs to J if and only if the relation $i_{2\ell} \leqslant i < i_{2\ell+1}$ holds for some non-negative integer ℓ.

To prove the last assertion of the lemma, we can obtain a different join from J by taking the symmetric difference with C (that is, removing from J all of its edges contained in C, and adding to J all edges of C not in J). □

Recall that an edge of a graph is *pendant* if it is incident with a vertex of degree 1. Let J be a join of a graph G and $c : E(J) \to \{1, 2, 3\}$ be a proper 3-edge-colouring of J. The colouring c is said to be *congruent* if $|S| \equiv |S \cap c^{-1}(i)| \pmod 2$ for any set of pendant edges of J forming an edge-cut of G and for $i = 1, 2, 3$.

Roughly speaking, a 3-edge colouring of a join is congruent if it satisfies the condition of the Lemma 2 on every set of pendant edges incident with a circuit in $E(G) \setminus J$.

Lemma 12. *A cubic graph G admits a connected join with a congruent 3-edge-colouring if and only if it admits a 2-packing of 4 joins, at least one of which is connected.*

Proof. Assume that a join J_4 is connected and admits a congruent 3-edge-colouring c. In order to prove our assertion, we need to construct further joins J_1, J_2, J_3 of G such that each edge is contained in at most two of $J_1, \ldots, J_4$. We choose J_1, J_2 and J_3 using Lemma 11 in such a way that $J_i \cap J_4 = \{e \in J_4 : c(e) = i\}$ for each $i \in \{1, 2, 3\}$. We need to modify $\{J_1, \ldots, J_4\}$ in order to obtain a 2-packing. Clearly, every edge of J_4 is covered exactly twice by the four joins. The complement of J_4 is a family of disjoint circuits. Let C be a circuit in $E(G) \setminus J_4$ and note that either all edges of C are covered an even number of times or (all of them) an odd number of times by $\{J_1, J_2, J_3\}$. In the latter case, if there is an edge of C covered three times, then we shift the selection of one join, say J_1, on C according to the second part of Lemma 11. In this way, every edge of C is covered 0 or 2 times as desired. By repeating the same process on each circuit of $E(G) \setminus J_4$, we obtain a join J_1' such that $\{J_1', J_2, J_3, J_4\}$ is a 2-packing and J_4 connected.

Conversely, suppose $\{J_1, J_2, J_3, J_4\}$ a 2-packing of joins with J_4 connected. We have to prove that J_4 has a congruent 3-edge-colouring. If an edge is incident to a vertex of degree 3 of J_4, then it is covered exactly twice: it belongs to J_4 and to exactly one of the joins J_1,J_2 and J_3. But, since J_4 is connected, every edge of J_4 has at least one end-vertex which has degree 3 in J_4. Hence, $J_1 \cap J_4$, $J_2 \cap J_4$ and $J_3 \cap J_4$ induce a 3-edge-colouring of J_4. Moreover, the 3-edge-colouring is congruent since Lemma 2 holds for the intersection of any join with an edge-cut of G. □

Theorem 13. *Let G be a cubic graph. If G admits a connected join with a congruent 3-edge-colouring, then it has a 5-cycle double cover.*

Proof. By Lemma 12, we have that G admits a 2-packing $\mathcal{J}$ of 4 joins with at least one of them connected, say J_4. Since J_4 is connected, for every edge e uncovered in $\mathcal{J}$, there exist a circuit C_e such that $e \in C_e$ and all other edges of C_e belong to J_4. Construct J_4'

as the symmetric difference of J_4 and all cycles C_e. Every edge of J_4 is still covered in J'_4 at least once, since it belongs to one of the joins J_1,J_2 and J_3, while every edge not covered in $\mathcal{J}$ belongs to J'_4, hence the set $\{J_1, J_2, J_3, J'_4\}$ is a $(1, 2)$-covering by 4 joins and by Theorem 10, G has a 5-cycle double cover. □

Corollary 14. *Let G be a cubic graph. If G admits a 3-edge-colourable connected join J with a connected complement in G, then G has a 5-cycle double cover.*

Proof. The complement of J is connected if and only if it is a unique circuit of G. The unique edge-cut of G with pendant edges of J is the entire set of all pendant edges, hence Lemma 2 holds. It follows that any 3-edge-colouring of J is congruent and the assertion follows by previous theorem. □

Now we prove that every treelike snark has a 5-cycle double cover.

Let H be the 5-sunlet, that is the graph on 10 vertices obtained by attaching 5 pendant edges to a 5-circuit (see Figure 11). In order to prove Theorem 16, we will make use of the following lemma.

Lemma 15. *Let H be the 5-sunlet. If we prescribe the colours of any two non-consecutive pendant edges of H, we can complete this prescription to a proper 3-edge-colouring of H.*

Proof. The desired colouring can be easily obtained with a suitable permutation of the colouring c' in Figure 11. □

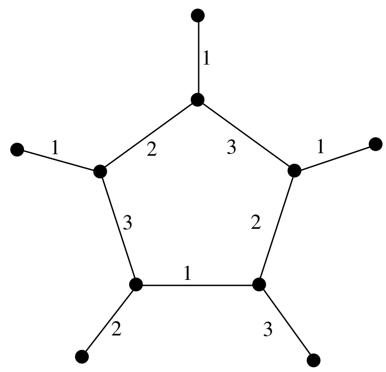


Figure 11: A proper 3-edge-colouring c' of the 5-sunlet.

Theorem 16. *Treelike snarks admit a 5-cycle double cover.*

Proof. By Corollary 14 and Theorem 13, it is sufficient to prove that every treelike snark admits a connected 3-edge-colourable join which is the complement of a circuit. Let $G = G(T_0, C_0)$ be a treelike snark. Recall that for each leaf ℓ of T_0, there is a Petersen fragment F_0^ℓ; suppose that its loose half-edges are denoted by $(a_1^\ell, \ldots, a_5^\ell)$. In addition, let P^ℓ denote the unique path of length 3 connecting the endvertices of a_1^ℓ and a_5^ℓ.

Let C be the union of the paths P^ℓ (with ℓ ranging over the leaves of T_0), together with the edges connecting their endvertices (i.e., the edges containing a loose half-edge

a_1^ℓ or a_5^ℓ of some Petersen fragment F_0^ℓ). By the construction of G, D is a circuit. (See Figure 12 where the edges of C are shown dashed.)

Let J be the join obtained as complement of C in G. The join J is connected because there is a path from any vertex of J to a vertex of the tree T_0. We now prove that J is 3-edge-colourable. Let X be the set of all edges in each Petersen fragment F_0^ℓ that are not contained in P^ℓ, do not contain any loose half-edge of F_0^ℓ, and are not incident with a_1^ℓ nor a_2^ℓ. The edges in X are shown bold in Figure 12. By inspecting the figure, we find that $G - X$ is a tree of maximum degree 3; let us denote it by T, Clearly, T admits a proper 3-edge-colouring c. By Lemma 15, whatever are the colours of the two edges of T incident with the unique 5-circuit of X inside any fragment F_0^ℓ, we can extend c to a proper 3-edge-colouring on F_0^ℓ. Eventually, we obtain a proper 3-edge-colouring of the join J. □

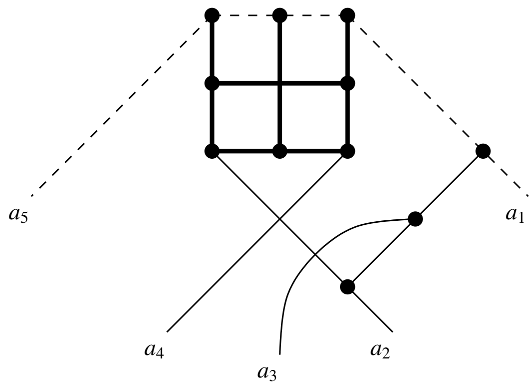


Figure 12: An illustration to the proof of Theorem 16. Edges of the circuit are shown dashed, edges of X are shown bold.

We conclude this section by pointing out a reformulation of Theorem 13 in terms of nowhere-zero flows:

Corollary 17. *Let G be a cubic graph admitting a connected join J. Let G' be the graph obtained from G by contracting each circuit of the complement of J to a vertex. If G' admits a nowhere-zero* 4*-flow, then G admits a* 5*-cycle double cover.*

Proof. It is well known that if G' admits a nowhere-zero 4-flow, then its edges can be coloured with three colours such that the edges of each colour constitute a join. The corresponding colouring of the edges of J is a congruent 3-edge-colouring, so the hypothesis of Theorem 13 is satisfied and the claim follows. □

8 Pattern sets

The notions of a pattern and the pattern set of a fragment were introduced in Section 4. The pattern sets in this section were determined by computer enumeration. We do not list patterns obtained from the listed ones by one or more of the following operations (cf. the discussion above Proposition 3):

- interchanging the first two subsets of $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ in the pattern,
- interchanging the last two subsets of $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ in the pattern,
- permuting the elements of $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$.

8.1 The pattern set of the Petersen fragment

The pattern set of F_0 (42 patterns):

A A AB AC AD	A B CD AB AB	A BC D BC BC
A A AB C D	A B CD AC AC	A BC D BD BD
A AB A AC AD	A B CD C C	A BC D D D
A AB A BC BD	A B CD CD CD	AB AB AB AC AD
A AB AC A AD	A BC A AB BD	AB AC AB AB AD
A AB AC B BD	A BC B AB AD	AB AC AB BC CD
A AB AC C CD	A BC B BC CD	AB AC AD A A
A B AB AB CD	A BC BD A AB	AB AC AD AB AB
A B AB AC BD	A BC BD BC C	AB AC AD AD AD
A B AC A D	A BC BD BD D	AB AC AD B B
A B AC AB BD	A BC D A A	AB AC AD BC BC
A B C A AD	A BC D AB AB	AB AC AD BD BD
A B C C CD	A BC D AD AD	AB AC AD D D
A B CD A A	A BC D B B	AB CD AC AB BC

8.2 The pattern sets of sums of Petersen fragments

The pattern set of $F_0 + F_0$ (18 patterns):

A AB C AB BD	A BC A A D	A BC D AB AB
A AB C AC CD	A BC A AB BD	A BC D AD AD
A B AC AB BD	A BC B AB AD	A BC D B B
A B AC AC CD	A BC B B D	A BC D BC BC
A B C A AD	A BC B BC CD	A BC D BD BD
A B C C CD	A BC D A A	A BC D D D

The pattern set of $F_0 + (F_0 + F_0)$ (9 patterns):
A AB C AB BD
A AB C AC CD
A B AC AB BD
A B AC AC CD
A B C A AD
A B C C CD
A BC A AB BD
A BC B AB AD
A BC B BC CD

The pattern set of $(F_0 + F_0) + F_0$ (25 patterns):

A AB A AC AD
A AB A BC BD
A AB C AB BD
A AB C AC CD
A B AC AB BD
A B AC AC CD
A B C A AD
A B C C CD
A BC A A D
A BC A AB BD
A BC AD AB B
A BC AD AD D
A BC B AB AD
A BC B B D
A BC B BC CD
A BC BD A AB
A BC BD BC C
A BC BD BD D
A BC D A A
A BC D AB AB
A BC D AD AD
A BC D B B
A BC D BC BC
A BC D BD BD
A BC D D D

The pattern set of $(F_0 + F_0) + (F_0 + F_0)$ (10 patterns):
A B AC AB BD
A B AC AC CD
A BC A AB BD
A BC AD AB B
A BC AD AD D
A BC B AB AD
A BC B BC CD
A BC BD A AB
A BC BD BC C
A BC BD BD D

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