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# CONTROLLABILITY OF SOME NONLINEAR SYSTEMS WITH DRIFT VIA GENERALIZED CURVATURE PROPERTIES*

ANTONIO MARIGONDA† AND SILVIA RIGO†

**Abstract.** We discuss the problem of local attainability for finite-dimensional nonlinear control systems with quite general assumptions on the target set. Special emphasis is given to control-affine systems with a possibly nontrivial drift term. To this end, we provide some sufficient conditions ensuring local attainability, which involve geometric properties both of the target itself (such as a notion of generalized curvature), and of the Lie algebra associated with the control system. The main technique used is a convenient representation formula for the power expansion of the distance function along the trajectories, made at points sufficiently near to the target set.

**Key words.** geometric control theory, Clarke's generalized gradient, curvature

**AMS subject classifications.** 49J52, 90C56

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## 1. Introduction.

**1.1. Statement of the problem.** Consider a time-independent control system in $\mathbb{R}^d$ of the form

$$\begin{cases} \dot{y}(t) = f(y(t), u(t)) & \text{for } t > 0, \\ y(0) = x_0, \end{cases}$$

where $u \in \mathscr{U} := \{v : [0, +\infty[ \to U, \text{ measurable}\}$, and $U$ is a given compact subset of $\mathbb{R}^m$, called the set of admissible controls.

Given a closed $S$ of $\mathbb{R}^d$, called the target set, in this work we are interested in the problem of providing sufficient conditions ensuring the *small-time local attainability* (STLA; see Definition 2.19) property of $S$. The STLA property amounts to the existence for every $T > 0$ of a suitable neighborhood of the target set $S$ whose points can be steered to $S$ in time less than $T$ along admissible trajectories of the system (see also section 6 in Chapter 4 of [5]). This problem is crucial in control theory, and is strongly related to many applications.

**1.2. A first order condition for STLA.** Petrov's condition is one of the most common conditions ensuring this property even in the fully nonlinear case (see, among the others, [2] and references therein). For a compact and smooth target, Petrov's condition requires that there exist positive constants $\delta, \mu > 0$ such that for all $x \in S_\delta \setminus S := \{z \in \mathbb{R}^d : d_S(z) < \delta\} \setminus S$ we have

$$\min_{u \in \mathscr{U}} \langle \nabla d_S(x), f(x, u) \rangle \le -\mu,$$

where $\nabla d_S(x)$ is the gradient of the distance function from $S$ evaluated at $x$. In the case of a nonsmooth target, the condition requires the existence of a generalized gradient of $d_S$ satisfying the same condition.

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†Department of Computer Science, University of Verona, Strada Le Grazie 15, I-37134 Verona, Italy (antonio.marigonda@univr.it, vr358036@studenti.univr.it).



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We notice that Petrov's condition is a *first order* condition in the sense that it involves only admissible velocities.

From another point of view, this condition requires $d_S$ to be a sort of Lyapunov function for the system. Indeed, from a geometrical point of view, Petrov's condition states that at every point $x$ of a neighborhood of the target there exists an admissible control $u_x \in U$ such that the corresponding trajectory points *sufficiently toward* the target. Moreover, the scalar product between the admissible velocity $f(x, u_x)$ and the gradient of the distance is uniformly bounded away from zero.

Petrov's condition is very strong, even if it is weaker than *full controllability*, which requires that *every* initial state can be steered to *any* final state in finite time along admissible trajectories. Moreover, it can be also shown that Petrov's condition is equivalent to the Lipschitz continuity of the minimal time function $T$ up to the boundary of $S$ (see [23]). However, it is also very easy to give simple examples where it fails. For instance, in $\mathbb{R}^2$ take $S = \{0\}$ and $(\dot{x}(t), \dot{y}(t)) = (y(t), u(t))$, where $u : \mathbb{R} \to [-1, 1]$ is measurable: Petrov's condition fails on the $x$-axis.

**1.3. Higher order condition for pointwise target.** When Petrov's condition is not satisfied, i.e., the trajectories of the system do not approach the target at the first order, it is natural to search for higher order conditions, which involve higher order terms in a convenient expansion of the trajectory itself. These conditions will be related to some properties of the Lie algebra generated by the family of vector fields associated with the system (see [15] for a complete introduction).

In the early 1960s, Kalman proved the following result. Assume that $f$ is *linear*, i.e., $f(x, u) = Ax + Bu$, where $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, $B \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ are two constant matrices, and $S = \{0\}$. Then the following are equivalent:

1. the system is controllable to the equilibrium point 0, i.e., every point can be steered to the origin in finite time;
2. the matrix $(B|AB|A^2B|\dots|A^{n-1}B)$ has full rank (equals $n$).

The second condition above is the celebrated *Kalman rank condition*, and implies the Hölder continuity of $T$, with exponent depending on the smallest $0 \le k \le n-1$ such that the matrix

$$(B|AB|A^2B|\dots|A^kB)$$

has full rank.

Later, in the 1970s, several generalizations, mainly concerning the case when target set $S$ is an equilibrium point for the system, of this condition to nonlinear systems were proved by several authors among which we recall Hermes, Sussmann, Hörmander, and many others. The simplest result can be formulated as follows: if the linearization of the system is STLA around the target equilibrium point, then the system itself is STLA. All these results involve a suitable expansion around the equilibrium point, and STLA is achieved by imposing some conditions on the Lie algebra generated by the vector fields.

**1.4. Higher order conditions for nonpointwise target.** The aforementioned higher order conditions assume that the target is reduced to a single point. Thus, if the target is not a singleton, a natural choice would be to apply them to each point of its boundary. However, as pointed out in [16], the problem of local attainability of a closed set with respect to the trajectories of a control system cannot be reduced to the problem of small-time local attainability at every point of its boundary. The reason is that the small-time local attainability depends not only on

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the dynamics of the control system, but also on the geometry of the considered closed set. So, it needs a specific study.

Theorem 1.18, p. 235 in [1] gives a condition yielding Hölder continuity of $T$ in the case of nonlinear *symmetric* (driftless) systems:

$$\dot{x}(t) = f(x,u) := \sum_{i=1}^{h} u_i g_i(x), \qquad |u_i| \le 1,$$

for a smooth target $S$, not necessarily reduced to a single equilibrium point. The term "symmetric" comes from the fact that in this case $f(x,-u) = -f(x,u)$, thus the set of trajectories enjoys time reversal symmetry.

The condition requires that if at a point $\bar{x} \in \partial S$ Petrov's condition does not hold, then there exists a vector field $F(\bar{x})$ generated by bracket operations from the vector fields of the family

$$\mathcal{F} := \{f(\cdot,u) : \, u \in \mathcal{U}\}$$

associated with the system such that

$$\langle F(\bar{x}), \nu(\bar{x})\rangle < 0,$$

where $\nu(\bar{x})$ is the normal unit vector to the target $S$ at $\bar{x}$.

Equivalently, there exists a constant $\mu > 0$ such that for every point $\bar{y} \notin S$ in a neighborhood of $\bar{x}$ there exists a vector field $F(\bar{y})$ generated by bracket operations from the vector field of $\mathcal{F}$ such that

$$\langle F(\bar{y}), \nabla d_S(\bar{y})\rangle < -\mu.$$

This condition can be viewed as a Petrov's condition of higher order, and in fact it leads to Hölder continuity of $T$ and no longer to Lipschitz continuity, where the exponent of the modulus of continuity depends again on the order of the Lie brackets involved.

A natural question is whether such a condition can be extended to control systems with drift of the form

$$\dot{x} = f(x) + \sum_{i=1}^{d} u_i g_i(x), \qquad |u_i| \le 1, i = 1 \dots d.$$

It would be very interesting also to relax the assumption on $F(x)$, requiring in a neighborhood of $S$ the existence of a vector field generated by bracket operations from vector fields of $\mathcal{F}$ pointing towards the target, but allowing the scalar product between $F(\bar{y})$ and $\nabla d_S$ to vanish sufficiently slowly when $\bar{y}$ approaches the target (thus no longer necessarily bounded away from zero).

Among recent papers, we should mention [16] and [17], where the problem was studied by assuming the existence for every $x$ near to the target of a suitable selection $y_x(t) \in \mathscr{R}_x(t)$ having a special expansion. In this way, a set of *regular higher order variations to the attainable set* is defined, which is related to the first term of a suitable approximation of admissible trajectories. Then the behavior of the scalar product between their leading term and the normal to the target is analyzed. In [16] this scalar product is required to be negative and bounded away from 0.

In the paper [18] a first attempt to relax this assumption was made, stating a second order condition for general systems with drift for a possibly nonsmooth target

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$S$ satisfying some regularity conditions, where the regularity assumptions on the target automatically hold in the smooth case.

This condition ensures the existence of an admissible trajectory steering every point $x$ of a neighborhood of $S$ (or a neighborhood relative to the reachable set) to $S$ in a finite amount of time $\tilde{T}(x)$, continuously depending on the starting point $x$. In order to make the required estimates, we need the expansion of the distance along such a trajectory. This procedure involves the scalar product of the Lie brackets of the controlled vector fields with generalized gradients of the distance, and must also take into account the effect of a nontrivial drift.

In [17], which follows closely the approach of [16] and [18], the results are generalized by relaxing the regularity assumptions on the target.

**1.5. Our contribution.** The aim of this paper is to extend the result of [18] beyond the second order, considering higher order expansion, and possibly taking into account further geometrical properties of the target itself, e.g., curvature.

In Example 5.21 we will present a situation in which the main results of both [16] and [17] do not apply, while our results can be used. However our results cannot be considered a full generalization of [16] and [17], since we assume more regularity on the target set.

We underline also that the authors of [16] and [17] do not take into account any curvature property of the target, since their hypotheses on the regularity of the target are very mild.

To better illustrate the kind of results we are going to prove, let us sketch a simplified version of one of our main results (see Theorem 5.10).

THEOREM 1.1. *Given a compact set* $U \subseteq \mathbb{R}^m$, $f : \mathbb{R}^d \times U \to \mathbb{R}^d$, *and a compact target set* $S \subseteq \mathbb{R}^d$ *of class* $C^2$, *consider the control system* $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x_0$, *with* $u : \mathbb{R} \to U$ *measurable. Let* $\delta, L, C, \alpha > 0$, *and* $k \in \mathbb{N} \setminus \{0\}$ *be constants with* $0 \le \alpha k < 1$ *and* $0 < 1/k - \alpha \le 1$. *Assume that for any* $x \in S_\delta := \{p \in \mathbb{R}^d : d_S(p) := \mathrm{dist}(p, S) < \delta\}$ *and for any* $0 \le t \le 1$ *there exist a point* $y_x(t) \in \mathbb{R}^d$ *and* $v_k^x \in \mathbb{R}^d$ *such that* $\|v_k^x\| \le L$, $y_x(0) = x$, $t \mapsto y_x(t)$ *is continuous,* $\langle v_k^x, \nabla d_S(x)\rangle \le C d_S^\alpha(x)$, *and*

$$\left\| y_x(t) - x - \frac{v_k^x}{k!} t^k \right\| \le L t^{k+1}.$$

*Then STLA holds and, more precisely, the minimum time function* $T$ *is Hölder continuous with exponent* $\min\{1 - \alpha k, 1/k - \alpha\}$ *in* $S_\delta \setminus S$.

Roughly speaking, the theorem states that if from every point $x$ near the target we can approximately reach a point $y$ nearer to the target itself, with a suitable relation between the rate of decrease of the distance (which depends on the scalar product between the gradient of the distance and the nonzero terms in the expansion) and the time needed to reach the point, then STLA holds. Indeed, this result holds true not only for control-affine systems but also for general ones. The problem of providing necessary and sufficient conditions in order to ensure the existence of the desired expansion is still open. Indeed, it is clear that full controllability would be sufficient, but in this case the result would become trivial. We give two sufficient conditions for control-affine systems in Lemma 4.9.

The second main goal of the paper is to study the role of the curvature of the target in controllability issues. Consider for example the simple system $(\dot{x}(t), \dot{y}(t)) = (0, u(t))$, where $u(t) \in [-1, 1]$ and target $S = \mathbb{R} \setminus B(0, 1)$. It is clear that given $(x, y) \in B(0, 1)$, we have $T(x, y) = \sqrt{1 - x^2} - |y|$, which is Hölder-$\frac{1}{2}$ continuous.

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However, we have that at every point of the form $(x,0)$ with $0<|x|<1$, the reachable set from $(x,0)$ is a segment orthogonal to $\nabla d_S(x,0)$. Thus the Petrov condition and also higher order Petrov's conditions will fail. Nevertheless, we will prove that negative curvature of the target will generate a second order effect which allows us to reach the target and have good estimates on the minimum time function even in this case.

The paper is structured as follows: in section 2, we fix the notation, recall some notions of nonsmooth analysis such as generalized gradients and semiconcave functions, and review basic concepts of control theory. In section 3 we introduce a notion of generalized curvature that will be used later to improve controllability conditions. In section 4 we focus on control-affine systems with drift. In section 5 we state and prove the main results about higher order sufficient conditions for STLA to a possibly nonsmooth target $S$, providing some illustrative examples. Finally, section 6 is devoted to conclusions and still open problems.

## 2. Preliminaries and notation.

In this section we will fix the notation we will use and recall some fundamental results that will be used throughout the paper.

Our main reference for this section will be Chapter 1 of [5].

### 2.1. General notation.

Definition 2.1. *Let $K$ be a closed subset of $\mathbb{R}^d$, $S\subset\mathbb{R}^d$, $x=(x_1,\dots,x_d)\in K$, $y=(y_1,\dots,y_d)\in\mathbb{R}^d$, $r>0$, $X$ be a vector space, and $f:X\to\mathbb{R}\cup\{+\infty\}$ be a function.*

*We denote by*

| | |
|---|---|
| $\langle x,y\rangle := \sum_{i=1}^{d} x_i y_i$ | *the* scalar product *in* $\mathbb{R}^d$; |
| $\|x\| := \sqrt{\langle x,x\rangle}$ | *the* Euclidean norm *in* $\mathbb{R}^d$; |
| $\partial S$, $\mathrm{int}(S)$, $\overline{S}$ | *the* topological boundary, interior *and* closure *of* $S$; |
| $\mathrm{diam}(S) := \sup\{\|z_1-z_2\| : z_1,z_2\in S\}$ | *the* diameter *of* $S$; |
| $\mathbb{B}^d := \{w\in\mathbb{R}^d : \|w\|<1\}$ | *the* open unit ball *(centered at the origin)*; |
| $\mathbb{S}^{d-1} := \{w\in\mathbb{R}^d : \|w\|=1\}=\partial\mathbb{B}^d$ | *the* unit sphere *(centered at the origin)*; |
| $B(y,r) := \{z\in\mathbb{R}^d : \|z-y\|<r\} = y+r\mathbb{B}^d$ | *the* open ball *centered at* $y$ *of radius* $r$; |
| $d_K(y) := \mathrm{dist}(y,K) = \min\{\|z-y\| : z\in K\}$ | *the* distance *of* $y$ *from* $K$; |
| $d^\sharp_K(y) := 2d_K(y) - d_{\partial K}(y)$ | *the* signed distance *of* $y$ *from* $K$; |
| $\pi_K(y) := \{z\in K : \|z-y\| = d_K(y)\}$ | *the* set of projections *of* $y$ *onto* $K$; |
| $\mathrm{co}(S) := \bigcap_{\substack{C \text{ convex} \\ C\supseteq S}} C$ | *the* convex hull *of* $S$; |
| $S^c := \mathbb{R}^d\setminus S$ | *the* complement *of* $S$; |
| $S_\delta = B(S,\delta) := \{y\in\mathbb{R}^d : d_S(y)<\delta\}$ | *the* $\delta$-neighborhood *of* $S$; |
| $N^P_K(x)$ | *the* proximal normal cone to $K$ at $x$ *(see Definition* 2.5*)*; |
| $\mathrm{dom}\, f := \{x\in X : f(x)<+\infty\}$ | *the* domain *of* $f$; |
| $\mathrm{epi}\, f := \{(x,\beta)\in X\times\mathbb{R} : x\in\mathrm{dom}\, f,\ \beta\ge f(x)\}$ | *the* epigraph *of* $f$; |
| $\mathrm{hypo}\, f := \{(x,\alpha)\in X\times\mathbb{R} : x\in\mathrm{dom}\, f,\ \alpha\le f(x)\}$ | *the* hypograph *of* $f$; |

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| | |
|---|---|
| $\partial_P f(x)$, $\partial^P f(x)$ | *the* proximal subdifferential *and the* proximal superdifferential *of $f$ at $x$ (see Definition* 2.6*);* |
| $\partial f(x)$ | *the* Clarke generalized gradient *of $f$ at $x$ (see Definition* 2.7*);* |
| $T(\cdot)$ | *the* minimum time function *(see Definition* 2.18*);* |
| $\nu(\cdot)$ | *the* Gauss map *(see Definition* 3.1*);* |
| $\mathcal{S}_x$ | *the* shape operator *(see Definition* 3.1*);* |
| $\lfloor r \rfloor := \max\{n \in \mathbb{Z} : n \leq r\}$ | *the* integer part of $r$. |

*If $\pi_K(y) = \{\xi\}$, i.e., it is a singleton, we will identify it with its unique element and write $\pi_K(y) = \xi$.*

Definition 2.2. *If $X$ is a topological vector space, we say that $f : X \to \mathbb{R} \cup \{+\infty\}$ is* lower semicontinuous *if* epi $f$ *is closed in $X \times \mathbb{R}$ with respect to the product topology on $X \times \mathbb{R}$, i.e.,*

$$\liminf_{y \to x} f(y) \geq f(x).$$

*A function $g : X \to \mathbb{R} \cup \{-\infty\}$ is called* upper semicontinuous *if $-g$ is lower semicontinuous.*

Definition 2.3. *Given the Banach spaces $X$ and $Y$, $U \subseteq X$, $V \subseteq Y$ be open, a function $f : U \to V$ is said to be a* Lipschitz continuous function ($f \in \mathrm{Lip}(U)$) *if there exists $C > 0$, called a* Lipschitz constant, *such that for every $x_1, x_2 \in U$*

$$\|f(x_1) - f(x_2)\|_Y \leq C\|x_1 - x_2\|_X.$$

*A function $f : U \to V$ is called* locally Lipschitz continuous ($f \in \mathrm{Lip}_{\mathrm{loc}}(U)$) *if it is Lipschitz continuous on every compact set of $U$.*

*Similarly, given $0 < \alpha < 1$, $f$ is said to be a* Hölder continuous function of exponent $\alpha$ *if there exists $C > 0$, such that for every $x_1, x_2 \in U$*

$$\|f(x_1) - f(x_2)\|_Y \leq C\|x_1 - x_2\|_X^{\alpha}.$$

We recall the following classical result on regularity of Lipschitz functions on finite-dimensional spaces (the proof can be found, e.g., in Corollary 4.19 p. 148 of [5]).

Theorem 2.4 (Rademacher's theorem). *Let $X$ be a finite-dimensional Banach space, $U \subseteq X$ be open, and $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then $f$ is differentiable a.e.*

**2.2. Nonsmooth analysis.** We recall now a generalized concept of a normal vector to possibly nonsmooth closed sets.

Definition 2.5 (proximal normals). *Let $K$ be a closed subset of $\mathbb{R}^d$. A vector $v$ is called a* proximal normal *to $K$ at $x \in K$ if there exists $\sigma = \sigma(v, x) \geq 0$ such that*

$$\langle v, y - x\rangle \leq \sigma\|v\|\|y - x\|^2$$

*for every $y \in K$. The set of all proximal normals to $K$ at $x$ will be denoted by $N_K^P(x)$ and called* the proximal normal cone to $K$ at $x$. *We recall that when $K$ is closed and convex, we can take $\sigma = 0$ in the above definition, whence the proximal normal cone at $x$ reduces to the* normal cone at $x$ in the sense of convex analysis, *namely, the set of vectors $v \in \mathbb{R}^d$ such that $\langle v, y - x\rangle \leq 0$ for all $y \in K$.*

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When the closed set $K$ is taken to be the epigraph of a lower semicontinuous function, we can use the above definition to define a geometric object generalizing the classical differential.

DEFINITION 2.6 (proximal subdifferential). *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous map, $x \in \mathrm{dom}(f)$, and $\zeta \in \mathbb{R}^d$. We say that $\zeta$ is a* proximal subgradient *of $f$ at $x$ if $(\zeta, -1) \in N^P_{\mathrm{epi}(f)}(x, f(x))$. The possibly empty set of all proximal subgradients of $f$ at $x$ will be denoted by $\partial_P f(x)$ and called the* proximal subdifferential *of $f$ at $x$. The following* proximal inequality formula *gives another characterization of $\partial_P f(x)$:*

$$\zeta \in \partial_P f(x) \text{ iff there exist } \sigma, \eta > 0 \text{ s.t. } f(y) \geq f(x) + \langle \zeta, y - x\rangle - \sigma \|y - x\|^2$$

*for all $y \in B(x, \eta)$.*

*Symmetrically, it is possible to define the* proximal superdifferential $\partial^P g(x)$ *of an upper semicontinuous function $g : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ by taking*

$$\partial^P g(x) = \{\zeta \in \mathbb{R}^d : (\zeta, 1) \in N^P_{\mathrm{hypo}\, g}(x)\}.$$

*In this case the proximal inequality formula becomes*

$$\zeta \in \partial^P f(x) \text{ iff there exist } \sigma, \eta > 0 \text{ s.t.} f(y) \leq f(x) + \langle \zeta, y - x\rangle + \sigma \|y - x\|^2$$

*for all $y \in B(x, \eta)$.*

DEFINITION 2.7 (Clarke's generalized gradient). *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $f : \Omega \to \mathbb{R}$ be locally Lipschitz continuous, and $x \in \Omega$. We recall that* Clarke's generalized gradient *of $f$ at $x$ is given by*

$$\partial f(x) := \mathrm{co}\left\{ \lim_{k\to\infty} \nabla f(y_k) : \{y_k\}_{k\in\mathbb{N}} \subset \Omega,\, \exists \nabla f(y_k), \text{ and } y_k \to x \right\}.$$

*Notice that, by Rademacher's theorem, we have $\partial f(x) \neq \emptyset$ for all $x \in \Omega$.*

The multidimensional version of Clarke's generalized gradient is Clarke's generalized Jacobian.

DEFINITION 2.8. *Let $\Omega$ be an open subset of $\mathbb{R}^d$, $F : \Omega \to \mathbb{R}^m$ be a Lipschitz continuous map. We denote the components of $F$ by $f_i$, $i = 1, \ldots, m$, i.e.,*

$$F(x_1, \ldots, x_d) = (f_1(x_1, \ldots, x_d), \ldots, f_d(x_1, \ldots, x_d)).$$

*The* generalized Jacobian $\partial F(x)$ *of $F$ at $x \in \Omega$ is the set*

$$\partial F(x) := \mathrm{co}\left\{ \lim_{i\to\infty} \mathrm{Jac}\, F(x_i) : x_i \to x \text{ and } F \text{ is differentiable at } x_i \right\}.$$

*In the case $m = 1$, we identify the $1 \times d$ matrix in the right-hand side with a $d$-dimensional vector.*

We recall the following properties of the generalized Jacobian, referring the reader to Propositions 2.6.2, 2.6.4, 2.6.5 in [4] for a proof.

PROPOSITION 2.9. *Using the same notation as Definition* 2.8, *the following hold:*

1. *$\partial F(x)$ is a compact convex subset of $\mathbb{R}^{m\times d}$, bounded by the ball centered at the origin of radius $K := \|(K_1, \ldots, K_m)\|$, where $K_i$ is the Lipschitz constant of $f_i$, $i = 1, \ldots, m$.*
2. *$\partial F(x)$ is contained in the set of all matrices whose $i$th row belongs to $\partial f_i(x)$ for every $i = 1, \ldots, m$.*

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3. *Given $v \in \mathbb{R}^d$, $w \in \mathbb{R}^m$, we have that*

$$\partial F(x)v = \tilde{\partial} F(x)v, \qquad (\partial F(x))^T w = (\tilde{\partial} F(x))^T w,$$

*where*

$$\tilde{\partial} F(x) := \mathrm{co}\left\{\lim_{i\to\infty} \mathrm{Jac}\, F(x_i) : x_i \to x,\ x_i \notin \mathcal{N},\ \text{and } F \text{ is differentiable at } x_i\right\},$$

*and $\mathcal{N}$ is an arbitrary set of Lebesgue measure zero.*

4. *$\partial F$ is an upper semicontinuous multifunction: if $\{(x_i, M_i)\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^d \times \mathrm{Mat}_{m\times d}$ is a sequence such that $x_i \to x$, $M_i \to M$, and $M_i \in \partial F(x_i)$, then $M \in \partial F(x)$.*
5. *We have that*

$$F(x) - F(y) \in \mathrm{co}\left\{Z(y-x) : Z \in \partial F(tx + (1-t)y) : t \in [0,1]\right\}.$$

Now we recall the definition of a particular class of functions generalizing convex functions. Our main reference is [2].

DEFINITION 2.10 (semiconcavity). *A continuous function $v : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^d$ is called* locally semiconcave *in $\Omega$ if, for any compact convex set $\mathcal{K} \subset \Omega$ there exists $c = c(\mathcal{K}) \geq 0$ such that*

$$\lambda v(x) + (1-\lambda)v(y) - v(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)c\|x-y\|^2 \tag{2.1}$$

*for any $x, y \in \mathcal{K}$, $\lambda \in [0,1]$. The constant $c = c(\mathcal{K})$ appearing in* (2.1) *is called a semiconcavity constant for $v$ on $\mathcal{K}$. Given a compact convex set $\mathcal{K} \subseteq \Omega$, the minimal constant $c = c(\mathcal{K})$ such that* (2.1) *is satisfied for any $x, y \in \mathcal{K}$, $\lambda \in [0,1]$ will be called the* sharp *semiconcavity constant of $v(\cdot)$ on $\mathcal{K}$, and will be denoted by $c_{\mathcal{K}} \geq 0$.*

*We say that $u$ is* locally semiconvex *if $-u$ is locally semiconcave. If $c_{\mathcal{K}}$ can be chosen independently of $\mathcal{K}$, we will omit the adjective "locally."*

Semiconcave functions enjoy some remarkable properties, summarized in the following.

PROPOSITION 2.11. *Let $v : \Omega \to \mathbb{R}$ be a function. Then*

1. *$v$ is semiconcave if and only if there exists $c > 0$ such that $v(x) - \frac{c\|x\|^2}{2}$ is concave in every convex subset of $\Omega$;*
2. *if $v : \Omega \to \mathbb{R}$ is both semiconcave and semiconvex, then $v \in C^{1,1}(\Omega)$;*
3. *let $v : \Omega \to \mathbb{R}$ be semiconvex. Then $v$ is locally Lipschitz in $\Omega$ and $\partial_P v(x) = \partial v(x)$ at every $x \in \Omega$. In particular, the subdifferentials $\partial_P v(x) \neq \emptyset$ at each point. If $v$ is semiconcave, the same results hold, with superdifferentials instead of subdifferentials;*
4. *if $v$ is semiconcave, then it is twice differentiable a.e. in the domain.*

Alternative characterizations of semiconcave functions can be given (see [2]).

PROPOSITION 2.12. *Let $\Omega$ be an open convex subset of $\mathbb{R}^d$, $v : \Omega \to \mathbb{R}$ be a function, and $c \geq 0$. The following are equivalent:*

1. *$v$ is semiconcave and $c$ is a semiconcavity constant for $v$;*
2. *for every $p \in \partial^P v(x)$ we have*

$$v(y) - v(x) \leq \langle p, y-x\rangle + c\|y-x\|^2;$$

3. *for any $w \in \mathbb{R}^d$ such that $\|w\| = 1$ we have $\partial^2_{ww} v \leq 2c$ in the sense of distributions in $\Omega$.*

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The prototype of a semiconcave function is the *distance function* from a closed subset $S$ of $\mathbb{R}^d$, i.e. the function $d_S : \mathbb{R}^d \to \mathbb{R}$ defined as

$$d_S(x) = \min\{\|y - x\| : y \in S\}.$$

The following semiconcavity result is proved, e.g., in Proposition 3.2 in [2].

PROPOSITION 2.13. *Let $S \subset \mathbb{R}^d$ be a closed set. Then*

(i) *the distance function $d_S$ is locally semiconcave in $\mathbb{R}^d \setminus S$, more precisely, $d_S$ is semiconcave on each open bounded set $A$ whose closure is contained in $\mathbb{R}^d \setminus S$ with constant $1/d_S(A)$, where*

$$d_S(A) = \inf\{d_S(y) : y \in A\};$$

(ii) *if there exists $\rho > 0$ such that the following condition (called* $\rho$-internal sphere condition*) holds,*

$$\forall x \in S \quad \exists x_0 \in S : x \in \overline{B(x_0, \rho)} \subset S, \tag{2.2}$$

*then $d_S$ is semiconcave also in $\mathbb{R}^d \setminus \operatorname{int} S$, i.e., it is semiconcave up to the boundary of $S$.*

*Remark* 2.14. The above result states that for every closed set $S \subseteq \mathbb{R}^d$ the distance function is locally semiconcave in $\mathbb{R}^d \setminus S$ and the constant of semiconcavity in general blows up as we consider domains approaching the boundary of $S$. If $S$ satisfies the $\rho$-internal sphere condition, the constants of semiconcavity in each domain of $\mathbb{R}^d \setminus S$ are bounded, i.e., for every $x, y \in \mathbb{R}^d \setminus S$ we have

$$d_S(y) - d_S(x) \le \langle \nabla d_S(x), y - x\rangle + \frac{1}{\rho}\|y - x\|^2.$$

In particular, if $\partial S$ is a smooth hypersurface of class $C^2$ property (2.2) holds locally, i.e., for every $R > 0$ with $S \cap B(0, R) \neq \emptyset$ there exists $\rho = \rho_R > 0$ such that

$$\forall x \in S \cap B(0, R) \quad \exists x_0 \in S : x \in \overline{B(x_0, \rho_R)} \subset S. \tag{2.3}$$

### 2.3. Control theory.

DEFINITION 2.15 (control system). *Let $U$ be a compact subset of $\mathbb{R}^m$, called* the set of admissible controls, *$f : \mathbb{R}^d \times U \to \mathbb{R}^d$ be a function continuous in the variable $u$ and Lipschitz continuous in the variable $x \in \mathbb{R}^d$, uniformly w.r.t. $u \in U$. We are interested in* Carathéodory solutions *of the system*

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)),\ t > 0, \\ x(0) = \bar{x} \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

*where $u(\cdot)$, which is called an* open-loop control function*, belongs to the set $\mathscr{U}$ of* admissible control functions*, defined as*

$$\mathscr{U} := \{u : [0, +\infty[ \to U,\ \ measurable\}.$$

*Given $\tau > 0$ and $u(\cdot) \in \mathscr{U}$, an* admissible trajectory *of system* (2.4) *generated by $u(\cdot)$ and defined on $[0, \tau]$ is a function $x : [0, \tau] \to \mathbb{R}^d$ satisfying for all $t \in [0, \tau]$*

$$x(t) = \bar{x} + \int_0^t f(x(s), u(s))\, ds.$$

*In particular, $x(\cdot)$ is an absolutely continuous function satisfying the initial condition $x(0) = \bar{x}$ and the ordinary differential equation in* (2.4) *for a.e. $t \in [0, \tau]$.*

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*By the regularity of* $f$ *and the compactness of* $U$*, for every* $\bar{x} \in \mathbb{R}^d$ *and* $u(\cdot) \in \mathscr{U}$*, for any* $\tau > 0$ *there exists a unique solution of* (2.4) *defined on* $[0, \tau]$ *(see, e.g., section* 5 *of Chapter* 2 *in* [1]*). A relaxed version of this result will be presented in Lemma* 5.1.

DEFINITION 2.16 (reachable set). *Given the system* (2.4) *and* $t \geq 0$*, we define the reachable set* $\mathscr{R}_{\bar{x}}(t)$ *in time* $t$ *from* $\bar{x}$ *by setting*

$$\mathscr{R}_{\bar{x}}(t) := \{y \in \mathbb{R}^d : \text{ there exists a trajectory of (2.4) with } x(0) = \bar{x} \text{ and } x(t) = y\}.$$

We will be interested in the following particular case of (2.4).

DEFINITION 2.17 (control-affine system). *We say that system* (2.4) *is* control affine *if the dynamics assumes the following special form*

$$\begin{cases} \dot{x}(t) = f(x(t)) + \displaystyle\sum_{i=1}^{N} u_i(t) g_i(x(t)), & t > 0, \\ x(0) = \bar{x} \in \mathbb{R}^d. \end{cases}$$

*The function* $f : \mathbb{R}^d \to \mathbb{R}^d$ *is called the* drift term*, while* $g_i : \mathbb{R}^d \to \mathbb{R}^d$ *are the* controlled vector fields*. If* $f = 0$ *the system is called* driftless *or* symmetric*.*

The main object of our study is the following.

DEFINITION 2.18 (minimum time function). *Consider the control system* (2.4) *and let* $S$ *be a closed subset of* $\mathbb{R}^d$*, called* the target set*. Given* $\bar{x} \in \mathbb{R}^d$ *and* $u(\cdot) \in \mathscr{U}$*, we consider the trajectory* $x(\cdot)$ *of* (2.4) *generated by* $u(\cdot)$*. Let* $I$ *be its maximum interval of definition, define* $I^+ = I \cap [0, +\infty[$*, and set*

$$\tau(\bar{x}, u(\cdot)) = \begin{cases} +\infty & \text{if } \displaystyle\lim_{s \to t^-} x(s) \notin S \text{ for every } t \in \overline{I^+}, \\ \inf \left\{ t \in \overline{I^+} : \displaystyle\lim_{s \to t^-} x(s) \in S \right\} & \text{otherwise.} \end{cases}$$

*The* minimum time function $T : \mathbb{R}^d \to [0, +\infty]$ *is defined as*

$$T(x) = \inf_{u(\cdot) \in \mathscr{U}} \tau(x, u(\cdot)).$$

The property in which we are interested is the following (see also [16], [17]).

DEFINITION 2.19 (STLA). *We say that* $S$ *is* STLA *for the system* (2.4) *if for any* $T > 0$ *there exists an open neighborhood* $U_T \subseteq \mathbb{R}^d$ *of* $S$ *such that* $T(x) \leq T$ *for all* $x \in U_T$*.*

*Remark* 2.20. We will deal mainly with the case in which there exists $\delta > 0$ and a continuous increasing function $\omega : [0, +\infty[ \to [0, +\infty[$ with $\omega(p) = 0$ iff $p = 0$ such that in $S_\delta$ we have $T(x) \leq \omega(d_S(x))$. In this case, by the properties of $\omega(\cdot)$, for any $T > 0$ there exists $0 < r < \delta$ such that $\omega(s) < T$ for any $s \in [0, r]$. Thus if we set $U_T = S_r$ we obtain $T(x) \leq \omega(r) \leq T$ for any $x \in S_r$.

Another controllability property for the system (2.4) which turns out to be strictly related to STLA is the following.

DEFINITION 2.21 (small-time locally controllable). *We say that the system* (2.4) *is* small-time locally controllable *if for any* $T > 0$ *and* $x_0 \in \mathbb{R}^d$ *we have that* $\mathscr{R}_{x_0}(T)$ *contains a neighborhood of* $x_0$*.*

Similarly, a control system is defined to be small-time locally controllable on a target $S$ if

$$\operatorname{int}\left(\{y \in \mathbb{R}^d : T(y) < T\}\right) \supseteq S \text{ for all } T > 0. \tag{2.5}$$

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It is clear that the notion of small-time controllability on the target given in (2.5) is equivalent to STLA: given $T > 0$,

1. assume that (2.5) holds. Then we define $U_T = \mathrm{int}(\{y \in \mathbb{R}^d : T(y) < T\})$ which by (2.5) is an open neighborhood of $S$ and satisfies $T(y) \le T$ for all $y \in U_T$, hence STLA holds;
2. assume that STLA holds. Then we have the existence of an open set $U_{T/2}$ such that

$$S \subseteq U_{T/2} \subseteq \mathrm{int}(\{y : T(y) \le T/2\}) \subseteq \mathrm{int}(\{y : T(y) < T\}),$$

thus (2.5) is satisfied.

**3. Generalized curvature.** In this section we will define a notion of curvature for a class of sets whose boundary is not assumed necessarily to be of class $C^2$.

Indeed, we recall that our objective is to steer every point $x_0$ belonging to a suitable neighborhood of the target $S$ to the target itself in finite time. To this end, we will construct a Cauchy sequence of points $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^d$ and times $\{t_i\}_{i\in\mathbb{N}} \subseteq ]0, +\infty[$ with the following properties:

1. $x_{i+1} \in \mathscr{R}_{x_i}(t_i)$, i.e., every point $x_{i+1}$ of the sequence can be reached from its predecessor $x_i$ in time $t_i$ by an admissible trajectory;
2. $d_S(x_i)$ is strictly decreasing and $d_S(x_i) \to 0$, i.e., the $x_i$ are approaching $S$;
3. $\sum_i t_i < +\infty$, i.e., the process will take a finite amount of time.

The results that we will prove in this section will be used mainly regarding issue (2) above, providing a suitable second order quadratic expansion of the distance function.

The semiconcavity property of the distance stated in Proposition 2.13, and in particular the semiconcavity inequality of Remark 2.14, seem to be natural tools to give a *quantitative* estimate of the decreasing of the distance from the target, even in a nonsmooth setting in which the distance fails to be differentiable at every point.

We recall the following classical definitions (see, e.g., [13]).

Definition 3.1. *Let $\Omega \subset \mathbb{R}^d$ an open bounded set whose boundary $S$ is an hypersurface of class $C^2$, oriented with unit normal external to $\Omega$. For every $x \in S$, we denote by $\nu_x \in \mathbb{R}^d$ the positive unit normal to $S$ at $x$ (i.e., the external unit normal to $\Omega$ at $x$). The* Gauss map *$\nu : S \to \mathbb{S}^{d-1}$ of $S$ associates with every $x \in S$ the vector $\nu_x$. If the map $\nu(\cdot)$ is differentiable at $x$, its differential $d\nu(x)$ at $x \in S$ is called the* shape operator *(or* Weingarten map*) $\mathcal{S}_x$. If we represent $\mathcal{S}_x$ as a square matrix of order $d$, we have that its eigenvalues are the classical* principal curvatures *of $S$ at $x$, while its determinant is the* Gaussian curvature *of $S$ at $x$.*

We start from a simple remark giving a useful geometric interpretation of the sharp semiconcavity constant of the distance to the target set.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ an open bounded set whose boundary $S$ is a hypersurface of class $C^2$, oriented with unit normal external to $\Omega$. Then all the principal curvatures of $S$ at each point are bounded above by twice the sharp semiconcavity constant of $d_S$.*

*Proof.* Since $S$ is smooth, we have that the *signed distance* $d^\sharp_{\overline{\Omega}}(\cdot)$ from $\overline{\Omega}$ (see Definition 2.1) is of class $C^2$ near $S$, hence $x \mapsto \mathcal{S}_x$ is continuous on $S$. Indeed, for every $x \in S$, $v, w \in \mathbb{R}^d$ we have, according to Definition 3.1,

$$\nu(x) = \nabla d^\sharp_{\overline{\Omega}}(x), \qquad \langle\langle \mathcal{S}_x, v'\rangle, w'\rangle = \left\langle \left\langle \nabla^2 d^\sharp_{\overline{\Omega}}(x), v \right\rangle, w \right\rangle,$$

where $v', w'$ are the projection on the tangent space to $S$ at $x$ of $v, w$, respectively.

Since $S$ is compact and $C^2$, it follows from Proposition 2.13 that $d_S$ is semiconcave up to the boundary of $\Omega$ . Let $C > 0$ be a semiconcavity constant for $d_S$. We recall

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from Proposition 2.12 that $\partial^2_{ww} d_S \le 2C$ in the sense of distributions, but since $d_S$ is smooth sufficiently near to $S$, we have that this inequality holds pointwise at every $y \notin \bar\Omega$ sufficiently near to $S$. For every $x \in S$ and $w \in \mathbb{R}^d$, we write $w = \mu\nu(x) + \lambda w'$ with $\mu, \lambda \in \mathbb{R}$, $\langle \nu(x), w'\rangle = 0$, and $\|w'\| = 1$. Thus, since $\nabla^2 d_S$ is continuous and $\nabla^2 d^\sharp_{\Omega}(y) = \nabla^2 d_S(y)$ for every $y \notin \bar\Omega$, we have

$$\lambda^2 \langle\langle \mathcal{S}_x, w'\rangle, w'\rangle = \left\langle \left\langle \nabla^2 d^\sharp_{\Omega}(x), w \right\rangle, w \right\rangle = \lim_{\substack{y \to x\\ y \notin \bar\Omega}} \langle\langle \nabla^2 d_S(y), w\rangle, w\rangle \le 2C(\lambda^2 + \mu^2).$$

By the arbitrariness of $\lambda, \mu, w$, this means that all the eigenvalues of $\mathcal{S}_x$, i.e., all the principal curvatures of $S$ at $x$, are bounded above by $2C$, and since $C$ was an arbitrary semiconcavity constant for $d_S$, the proof is concluded. ▯

Lemma 3.2 links semiconcavity to curvature in the smooth hypersurface case, leading to an interpretation of the sharp semiconcavity constant of the distance as an *upper bound* for principal curvatures, even in the nonsmooth case. Nevertheless, even in the smooth case, the *signs* of the principal curvatures turn out to play an important role when we have to consider second order phenomena. This is illustrated by the following simple example.

*Example* 3.3. The ground space is $\mathbb{R}^2$. Consider the system

$$\begin{cases} \dot x_1(t) = u(t) \in [-1, 1],\\ \dot x_2(t) = 0. \end{cases}$$

Take $S := \mathbb{R}^2 \setminus B((0,0),1)$. Given $x = (x_1, x_2) \in B((0,0),1)$ we have $d_S(x) = 1 - |x|$, moreover, for $x \in B((0,0),1) \setminus \{(0,0)\}$, we have

$$\nabla d_S(x) = -\frac{x}{|x|}, \qquad D^2 d_S(x) = \frac{1}{|x|^3}\begin{pmatrix} -x_2^2 & x_1x_2\\ x_1x_2 & -x_1^2 \end{pmatrix}.$$

Let $-1 < \alpha < -1/2$ and set $\bar x = (0, \alpha)$. We have $\nabla d_S(\bar x) = (0,1)$ and $D^2 d_S(\bar x) = \left(\begin{smallmatrix} \alpha^{-1} & 0\\ 0 & 0\end{smallmatrix}\right)$. The eigenvalues of $D^2 d_S(\bar x)$ are $\alpha^{-1} < 0$ and 0, hence we can choose $K = 0$ in the semiconcavity inequality (this follows also from the fact that since $S$ is the complement of a convex set, it enjoys the internal sphere condition with arbitrary large $\rho$).

If $t \mapsto x(t) = (x_1(t), x_2(t)) \in B((0,0),1)$ is an admissible trajectory for the system satisfying $x(0) = \bar x$, using Taylor's expansion we get

$$\begin{aligned} d_S(x(t)) &= 1 - \sqrt{x_1^2(t) + \alpha^2} = 1 - |\alpha| - \frac{x_1^2(t)}{2|\alpha|} + O(x_1(t)^4)\\ &= d_S(\bar x) - \frac{x_1^2(t)}{2|\alpha|} + O(x_1^4(t)), \end{aligned}$$

where $|O(x_1^4(t))| \le \frac{6}{|\alpha|^5} x_1^4(t)$.

In this case, the semiconcavity inequality yields just the much weaker estimate

$$d_S(x(t)) - d_S(\bar x) \le \langle \nabla d_S(\bar x), x(t) - \bar x\rangle + K\|x(t) - \bar x\|^2 = 0.$$

We conclude that starting from $\bar x$ we can approach the target at second order thanks to (negative) curvature properties of the target. This fact cannot be deduced from the semiconcavity inequality only. However, a strong smoothness assumption on the target was crucial in order to obtain second order derivatives of the distance, while the semiconcavity inequality holds for a much wider class of closed sets, including also nonsmooth sets (e.g.,the complement of nonsmooth convex sets).

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**3.1. Sets with positive reach and generalized curvature.** In order to improve the estimate given by the semiconcavity inequality taking into account, more carefully, curvature effects, we focus our attention on sets belonging to an intermediate class of smoothness between general closed sets and sets with boundary of class $C^2$. The sets belonging to this class, called *sets with positive reach*, enjoy a strong *regularity property* of the proximal normals, which results in certain regularity properties of the boundary. The class of sets with positive reach was introduced first by Federer in [10], and then was studied by several authors, both in finite and infinite dimensions. We refer the reader to [6], [7], and [9], for a recent survey on this subject.

DEFINITION 3.4 (Federer). *Let $A \subset \mathbb{R}^d$, $a \in A$. We denote by $\mathrm{Unp}(A)$ the (possibly empty) set of all those points $x \in \mathbb{R}^d$ for which there exists a unique point of $A$ closest to $x$. Then the projection map $\pi_A : \mathrm{Unp}(A) \to A$, which associates with $x \in \mathrm{Unp}(A)$ the unique $a \in A$ such that $d_A(x) = \|x - a\|$, is well defined for all $x \in \mathrm{Unp}(A)$. We set*

$$\begin{aligned} \mathrm{reach}(A, a) &:= \sup\{r \geq 0 : \, B(a, r) \subseteq \mathrm{Unp}(A)\}, \\ \mathrm{reach}(A) &:= \inf\{\mathrm{reach}(A, a) : \, a \in A\}. \end{aligned}$$

Classical results on closed sets and sets with positive reach are collected in the following.

THEOREM 3.5 (Federer). *For every nonempty closed subset $A$ of $\mathbb{R}^d$ the following statements hold, with $d(\cdot) = d_A(\cdot)$, $\pi(\cdot) = \pi_A(\cdot)$, $U = \mathrm{Unp}(A)$:*

1. *$\|d(x) - d(y)\| \leq \|x - y\|$ whenever $x, y \in \mathbb{R}^d$.*
2. *If $x \in \mathbb{R}^d \setminus A$ and $d(\cdot)$ is differentiable at $x$, then $x \in U$ and*

$$\mathrm{grad}\ \mathrm{d}(x) = \frac{x - \pi(x)}{d(x)}.$$

3. *$\pi(\cdot)$ is continuous.*
4. *$d(\cdot)$ is continuously differentiable on $\mathrm{int}(U \setminus A)$ and $d^2$ is continuously differentiable on $\mathrm{int}\, U$ with*

$$\mathrm{grad}\, d^2(x) = 2[x - \pi(x)]$$

*for $x \in \mathrm{int}\, U$.*

5. *If $x \in U$, $a = \pi(x)$, $\mathrm{reach}(A, a) > 0$, and $b \in A$, then*

$$\langle x - a, b - a\rangle \leq \frac{\|x - a\|}{2\mathrm{reach}(A, a)} \|a - b\|^2,$$

*i.e., for every $x \in U$ we have $x - \pi(x) \in N_A^P(x - \pi_A(x))$ and we can take $\sigma = \frac{1}{2\mathrm{reach}(A, \pi_A(x))}$ in the proximal normal inequality.*

6. *Assume that $0 < r < \mathrm{reach}\, A$. Then $\pi_A(\cdot)$ is Lipschitz continuous on $A_r \setminus A$, where $A_r := \{y \in \mathbb{R}^d : \, d_A(y) \leq r\}$ with constant $r/(\mathrm{reach}\, A - r)$.*

For a set $A$ with positive reach, we notice that $\nabla d_A(x) \in N_A^P(\pi_A(x))$ for every $x \in \mathrm{Unp}(A) \setminus A$, but, since the distance function is just $C^{1,1}$, we have that the second order differential of $d_A$ is defined just a.e. in a neighborhood of $A$. Our aim is to replace it by a suitable construction employing Clarke's generalized Jacobian of $\nabla d_A$, which is Lipschitz continuous in $\mathrm{Unp}(A) \setminus A$.

We are ready now to formulate a quite simple and natural notion of generalized curvature as follows.

DEFINITION 3.6 (generalized curvature). *Let $A$ be a closed subset of $\mathbb{R}^d$ with positive reach, $x, y \in \mathbb{R}^d \setminus A$ such that $tx + (1 - t)y \in \mathrm{Unp}(A) \setminus A$ for every $t \in [0, 1]$.*

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*We define the following* generalized curvature operator:

$$\mathscr{K}(x,y) := \frac{1}{2}\mathrm{co}\,\{\langle\langle Z, y-x\rangle, y-x\rangle :\, Z \in \partial\nabla d_A(x+t(y-x)),\, t\in[0,1]\}\,,$$

*where $\partial\nabla d_A(p)$ is taken in the sense of the generalized Jacobian of the Lipschitz map $\nabla d_A(\cdot)$ at $p$.*

*Remark* 3.7. If we assume that the boundary of $A$ is a smooth hypersurface, and that $x$ and $y$ are sufficiently near to $\partial A$, we can replace the generalized Jacobian with the classical one, and consider

$$\frac{\mathscr{K}(x,y)}{\|y-x\|^2} = \frac{1}{2}\mathrm{co}\left\{\left\langle\left\langle \mathrm{Hess}\, d_A(x+t(y-x)), \frac{y-x}{\|y-x\|}\right\rangle, \frac{y-x}{\|y-x\|}\right\rangle :\, t\in[0,1]\right\}.$$

In particular, for every $v \in \mathbb{S}^{n-1}$ we have

$$\lim_{t\to 0^+}\frac{\mathscr{K}(x,x+tv)}{t^2} = \frac{1}{2}\left\langle\langle \mathrm{Hess}\, d_A(x), v\rangle, v\right\rangle = \frac{1}{2}\langle\langle \mathcal{S}_{\pi_A(x)}, v\rangle, v\rangle,$$

where $\mathcal{S}_{\pi_A(x)}$ is the shape operator at $\pi_A(x)$.

The main motivation of Definition 3.6 is illustrated by the following.

PROPOSITION 3.8. *Let $A$ be a closed subset of $\mathbb{R}^d$ with positive reach, $x, y \in \mathbb{R}^d \setminus A$ such that $tx + (1-t)y \in \mathrm{Unp}(A) \setminus A$ for every $t \in [0,1]$. We have the following generalized Taylor formula:*

$$d_A(y) - d_A(x) - \langle \nabla d_A(x), y-x\rangle \in \mathscr{K}(x,y).$$

*In particular, given $\mu > 0$, $0 < \delta < d_A(x)$, if we assume that there exists $v \in \mathbb{S}^{d-1}$ such that for every $p \in B(x,\delta)$ we have*

$$\langle\langle Z, v\rangle, v\rangle < -\mu \text{ for all } Z \in \partial\nabla d_A(p),$$

*then for every $t \in ]0,\delta[$*

$$d_A(x+tv) \le d_A(x) + t\langle \nabla d_A(x), v\rangle - t^2\mu.$$

*Remark* 3.9. The above result can be interpreted as follows. Suppose that we can choose $v \in (\nabla d_A(x))^\perp$ in the above proposition. Then the above result states that if we move from $x$ in direction $v$ for a short time, we are still approaching the target at the second order, due to curvature effects.

Proof of Proposition 3.8 is based on the following lemma, which is a special case of Theorem 2.3 in [14], to which we refer the reader for a proof.

LEMMA 3.10. *Let $\Omega$ be an open convex subset of $\mathbb{R}^d$, $f : \Omega \to \mathbb{R}^m$ be a function of class $C^{1,1}$. Then for every $x, y \in \Omega$ we have that*

$$f(y) - f(x) - \langle \nabla f(x), y-x\rangle$$

*belongs to the set*

$$\partial\nabla f[x,y] := \frac{1}{2}\mathrm{co}\,\{\langle\langle Z, y-x\rangle, y-x\rangle :\, Z \in \partial\nabla f(x+s(y-x)),\, s\in[0,1]\}\,.$$

*Proof of Proposition* 3.8. We apply Lemma 3.10 to $d_A$ choosing as $\Omega$ an open tubular neighborhood of $\{tx + (1-t)y :\, t\in[0,1]\}$. $\square$

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We present now a variation of Example 3.3, where we are able to explicitly compute the generalized curvature operator.

*Example* 3.11. The ground space is $\mathbb{R}^2$. Consider the system

$$\begin{cases} \dot{x}(t) = u(t) \in [-1,1], \\ \dot{y}(t) = 0. \end{cases}$$

Define

$$g(x) := \begin{cases} -\sqrt{1-x^2} & \text{for } -1 \le x \le 0, \\ 1-\sqrt{4-x^2} & \text{for } 0 < x \le 2, \\ +\infty & \text{for } x < -1 \text{ or } x > 2, \end{cases}$$

and set $S := \operatorname{hypo} g$. We have that $\operatorname{reach}(S) = 1$. Consider now

$$U := (\mathbb{R}^2 \setminus S) \cap \{y < -1/2\} \subseteq \operatorname{Unp}(S),$$

which can be written as $U = U^+ \cup U^- \cup U^0$, where $U^+ := U \cap \{x > 0\}$, $U^- := U \cap \{x < 0\}$, and $U^0 := U \cap \{x = 0\} = \{0\} \times ]-1, -1/2[$.

As in Example 3.3, the sharp semiconcavity constant of $d_S(\cdot)$ is 0 since $S$ is the complement of an open convex set, hence satisfies the internal sphere condition at every point of the boundary with arbitrary large radius.

By elementary computations we have

$$d_S(x,y) = \begin{cases} 1-\sqrt{x^2+y^2} & \text{if } (x,y) \in U^-, \\ 2-\sqrt{x^2+(y-1)^2} & \text{if } (x,y) \in U^+, \\ 1-|y| & \text{if } (x,y) \in U^0. \end{cases}$$

$$\nabla d_S(x,y) = \begin{cases} -\dfrac{(x,y-1)}{\sqrt{x^2+(y-1)^2}} & \text{for } (x,y) \in U^+, \\ -\dfrac{(x,y)}{\sqrt{x^2+y^2}} & \text{for } (x,y) \in U^-, \\ (0,1) & \text{for } (x,y) \in U^0. \end{cases}$$

$$D^2 d_S(x,y) = \begin{cases} \dfrac{1}{(x^2+y^2)^{3/2}} \begin{pmatrix} -y^2 & xy \\ xy & -x^2 \end{pmatrix} & \text{for } (x,y) \in U^-, \\ \dfrac{1}{(x^2+(y-1)^2)^{3/2}} \begin{pmatrix} -(y-1)^2 & x(y-1) \\ x(y-1) & -x^2 \end{pmatrix} & \text{for } (x,y) \in U^+. \end{cases}$$

If we consider an admissible trajectory $t \mapsto (x(t), y(t))$ starting from $U^0$ at $t_0 = 0$, we have $\langle \nabla d_S(0, y(0)), (x(t), 0) \rangle = 0$, so the semiconcavity inequality yields no strong information about approaching the target.

It is clear that $d_S(\cdot) \in C^1(U) \cap C^2(U^+ \cup U^-)$; however, it turns out that $\nabla d_S(\cdot)$ is not differentiable on $U^0$ and hence its *classical* Hessian cannot be defined on the whole of $U$. Thus the method used in Example 3.3 cannot be used.

Since we have that $\nabla d_S$ is Lipschitz continuous on $U$ it make sense to consider Clarke's generalized Jacobian $\partial \nabla d_S(x,y)$ at *every* point of $U$.

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In particular, since $d_S \in C^2(U^+ \cup U^-)$, we have that $\partial\nabla d_S(x,y) = D^2 d_S(x,y)$ for every $(x,y) \in U^+ \cup U^-$. Noticing that

$$\lim_{\substack{(x',y')\to(0,y)\\(x,y)\in U^-}} D^2 d_S(x,y) = \frac{1}{y}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\lim_{\substack{(x',y')\to(0,y)\\(x,y)\in U^+}} D^2 d_S(x,y) = \frac{1}{y-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\partial\nabla d_S(0,y) = \left\{\frac{\lambda - y}{y-y^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in [0,1]\right\}.$$

Given $(x,y) \in U^+$, we compute the generalized curvature operator

$$\begin{aligned}
&\mathscr{K}\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \frac{1}{2}\mathrm{co}\left(\left\{\left\langle\left\langle Z, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle : Z \in \partial\nabla d_S(0,y)\right\}\right. \\
&\qquad\qquad \left.\cup \left\{\left\langle\left\langle D^2 d_S(\lambda x, y), \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle, \begin{pmatrix} x \\ 0 \end{pmatrix}\right\rangle : \lambda \in ]0,1]\right\}\right) \\
&= \frac{1}{2}\mathrm{co}\left(\left[\frac{x^2}{y}, \frac{x^2}{y-1}\right] \cup \left]\frac{x^2}{y-1}, -\frac{x^2(y-1)^2}{(x^2+(y-1)^2)^{3/2}}\right]\right) \\
&= \left[\frac{x^2}{2y}, -\frac{x^2(y-1)^2}{2\,(x^2+(y-1)^2)^{3/2}}\right].
\end{aligned}$$

Recalling that since $-1 < y < -1/2$ we have $2 < 9/4 < (y-1)^2 < 4$, and that $x^2 < 1$, we can give the following estimate:

$$\mathscr{K}\left((0,y),(x,y)\right) \subseteq [-x^2, -x^2/10].$$

According to Proposition 3.8, this means that if $t \mapsto (x(t), y(t))$ is an admissible trajectory starting at $t_0 = 0$ from $U^0$, and such that $(x(t), y(t)) \in U^+$, we have

$$d_S(x(t), y(t)) - d_S(0, y(0)) \le -\frac{1}{10}x^2(t),$$

i.e., we are approaching the target at the second order due to curvature effects. Similar computations can be performed for the case in which $(x(t), y(t)) \in U^-$.

**3.2. Comparison with existing notions of generalized curvature.** In Chapter 13 of [22] properties of different notions of second order generalized differentials for several classes of functions are studied. We recall that our main motivation for introducing the generalized curvature was to obtain a suitable *quadratic approximation* of the distance function $d_S(\cdot)$ around each point $\bar{x}$ sufficiently near to $S$.

In this spirit, according to Theorem 13.2 of [22], given a lower semicontinuous function $f : \mathbb{R}^d \to \mathbb{R}$, $v_{\bar{x}} \in \mathbb{R}^d$, and $A_{\bar{x}} \in \mathrm{Mat}_{d\times d}(\mathbb{R})$, we have that $f$ has the following second order expansion at $\bar{x}$,

$$f(x) = f(\bar{x}) + \langle v_{\bar{x}}, x - \bar{x}\rangle + \langle\langle A_{\bar{x}}, x - \bar{x}\rangle, x - \bar{x}\rangle + o_{\bar{x}}(|x - \bar{x}|^2), \tag{3.1}$$

if and only if $\partial_L f(\bar{x}) = \{v_{\bar{x}}\}$ and

$$\emptyset \ne \partial_L f(x) \subseteq v_{\bar{x}} + A_{\bar{x}}(x - \bar{x}) + o_{\bar{x}}(|x - \bar{x}|)\mathbb{B}^d,$$

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where $\partial_L f(x)$ is defined in Definition 8.3 of [22] as

$$\partial_L f(x) := \left\{ v \in \mathbb{R}^d : \exists \left\{ \begin{array}{l} x_n \to x \\ v_n \to v \end{array} \right. \quad \text{s.t.} \quad \liminf_{\substack{y \to x_n \\ y \neq x_n}} \frac{f(y) - f(x_n) - \langle v_n, y - x_n \rangle}{|y' - y_n|} \geq 0 \, \forall n \in \mathbb{N} \right\}$$

(actually in [22] $\partial_L f$ is denoted by $\partial f$, but we use this symbol for a different purpose).

If we want to apply this result to the distance function $d_S$, we obtain that the assumption for $S$ to be a set with positive reach is sharp. More precisely we have the following result.

Proposition 3.12. *Given a closed set $S \subseteq \mathbb{R}^d$ and an open set $U \subseteq \mathbb{R}^d$ such that $S \subseteq U$, we have that if $d_S(\cdot)$ possesses a second order expansion around every point $\bar{x} \in U \setminus S$ then $S$ has locally positive reach and $U \subseteq \operatorname{Unp}(S)$.*

*Proof.* According to Theorem 13.2 of [22], we have that $\partial_L d_S$ must be a singleton at every point $\bar{x} \in U \setminus S$. Since $d_S$ is locally semiconcave we have that this implies that $d_S$ is differentiable at $\bar{x}$ by Theorem 3.3.15 in [3]. Hence we have that $d_S$ is differentiable at $\bar{x}$ for every $\bar{x} \in U \setminus S$. According to Proposition 4.4 in [3], this implies that every point of $U \setminus S$ has a unique projection on $S$ and thus $U \subseteq \operatorname{Unp}(S)$. Since $U$ is an open neighborhood of $S$, we conclude that $S$ has locally positive reach. ∎

The generalized curvature operator introduced in Definition 3.6 improves the second order expansion (3.1) giving some *uniformity* for the remainder term.

Another approach to study curvature properties in the nonsmooth setting is suggested after Exercise 13.17, p. 600 of [22] by means of another kind of second order generalized differential applied to the indicator function $I_C : \mathbb{R}^d \to \{0, +\infty\}$ defined as

$$I_C(x) := \left\{ \begin{array}{ll} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C \end{array} \right.$$

for a given set $C \subseteq \mathbb{R}^d$. Clearly, if $I_C$ is finite and classically differentiable at a point $\bar{x}$, it turns out that $I_C(\cdot)$ must be constant around $\bar{x}$, since it must be continuous at $\bar{x}$ and $I_C(y) \in \mathbb{R}$ iff $I_C(y) = 0$. In particular, all classical differentials vanish at $\bar{x}$. The notion of second order generalized differential used by authors is the *second order epi-differential*, defined in Definition 13.6, p. 586 of [22]. An explicit formula for the second order epi-differential of $I_C(\cdot)$ is provided by Exercise 13.17 in the case when $C$ has the special form $C = \{x \in X : F(x) \in D\}$, where $F : \mathbb{R}^d \to \mathbb{R}^m$ is a $C^2$ function, and $X \subseteq \mathbb{R}^d$, $D \subseteq \mathbb{R}^m$ are polyhedral sets, i.e. intersections of a finite family of hyperplanes or half-spaces in $R^d$ and in $\mathbb{R}^m$, respectively. Also, a suitable constraint qualification condition is required to be satisfied involving $D$ and $F$. The following example will show that this notion of generalized curvature differs from the one we introduced.

*Example* 3.13. In $\mathbb{R}^2$, let $h(x) = |x|^{3/2}$. Set $C = \operatorname{epi} h$. Notice that $C$ is convex, hence $\operatorname{reach} C = +\infty$. Thus the generalized curvature operator $\mathscr{K}(x, y)$ can be computed for every $x, y \in \mathbb{R}^2$ such that the segment joining $x$ and $y$ does not intersect $C$. If we take $X = \mathbb{R}^2$, $D = \{0\}$, $F(x, y) = y - h(x)$ we have that $C = \{x \in X : F(x) \in D\}$, but $F$ fails to be of class $C^2(\mathbb{R}^2)$ since $h \in C^{1,1}(\mathbb{R}) \setminus C^2(\mathbb{R})$

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as its second order derivative is not continuous at $x = 0$. Thus the notion of curvature given by the formula in Exercise 13.17, p. 600 of [22] cannot be used.

From a totally different point of view, in [10] were introduced the *curvature measures* as a generalization of Steiner's formula for convex bodies. More precisely, given a set $A \subseteq \mathbb{R}^d$ with $\mathrm{reach}(A) > 0$ it was proved that for each bounded Borel subset $Q$ of $\mathbb{R}^d$ and for $0 \le r < \mathrm{reach}(A)$, the $d$-dimensional measure of

$$\{x \in \mathbb{R}^d : d_A(x) \le r \text{ and } \pi_A(x) \in Q\}$$

is given by a polynomial of degree at most $d$ in $r$, say

$$\sum_{i=0}^{d} r^{d-i}\alpha(d-i)\Phi_i(A,Q),$$

where $\alpha(j)$ is the $j$-dimensional measure of the unit ball in $\mathbb{R}^j$.

The coefficients $\Phi(A,Q)$ are countably additive with respect to $Q$, defining the *curvature measures* $\Phi_i(A,\cdot)$, $i = 0,\dots,d$. If the Hausdorff dimension of $A$ is $k$, then $\Phi_i(A,\cdot) = 0$ for $i > k$, $\Phi_k(A,\cdot)$ is the restriction of the $k$-dimensional Hausdorff measure to $A$, and the measures $\Phi_i(A,\cdot)$ corresponding to $i < k$ depend on second order properties of $A$.

Curvature measures also enjoy some remarkable stability properties, namely, given $\varepsilon > 0$ and a sequence $\{A_h\}_{h\in\mathbb{N}}$ of sets with $\mathrm{reach}(A_h) \ge \varepsilon$ for all $h \in \mathbb{N}$ and such that $A_j \to A$ in the Hausdorff metric, then the associated sequences of curvature measures converge weakly to the curvature measures of the limit set $A$, whose reach is also at least $\varepsilon$. Federer concludes that in this way any set $A$ with positive reach may be approximated in curvature by the solids $\{x \in \mathbb{R}^d : d_A(x) \le s\}$ for $s \to 0^+$.

According to Definition 5.7 in [10], if the numbers $\Phi_i(A,A)$, $i = 0,\dots,d$ are finite (this is true e.g., if $A$ is compact) they can be considered the *total curvatures* of $A$ in the following sense. If $\partial A$ is smooth, then $\Phi_i(A,A)$ is the $i$th mean curvature of $A$, i.e., the integral on $\partial A$ of the $i$th elementary symmetric polynomial of the principal curvatures. In the positive reach case, curvature measures are constructed as the limit as $s \to 0^+$ of the integral of the $i$th mean curvature of the (smooth) sets $\{x \in \mathbb{R}^d : d_A(x) \le s\}$ (Theorem 5.5 in [10]). Thus it is possible to use the generalized curvature operator introduced in Definition 3.6 to give upper and lower bounds on them. However, since we are interested more in a pointwise directional decreasing property of the distance and not in the mean curvature properties of the surface, the use of curvature measures seems not to be natural.

**4. Control-affine systems with drift.** We turn now our attention to control-affine systems. Our problem will be to give a suitable approximation of $\mathscr{R}_t(\bar{x})$ for $t > 0$ sufficiently small. These results will be later applied to particularize the general results of section 5. Throughout this section we will assume that we have a control system in $\mathbb{R}^d$ of the form

$$\begin{cases} \dot{x}(t) = f(x(t)) + \displaystyle\sum_{i=1}^{N} u_i(t) g_i(x(t)), \\ x(0) = \bar{x}, \end{cases} \tag{4.1}$$

where $|u_i(\cdot)| \le 1$, $f, g_i \in \mathrm{Lip}(\mathbb{R}^d)$, $i = 1,\dots,N$ (thus conditions in Definition 2.15 are satisfied).

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We consider now the problem of approaching a target $S$ along admissible trajectories of the control system (4.1) starting from a point $\bar{x}$ sufficiently near to $S$. We are interested to find $t > 0$ and $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ such that $d_S(y_{\bar{x}}(t)) < d_S(\bar{x})$.

If the system is fully controllable this turns out to be trivial. In this case, by definition, for any $t > 0$ we have that $\mathscr{R}_{\bar{x}}(t)$ contains a ball centered at $\bar{x}$. The aim of this section is to give some sufficient condition weaker than controllability, guaranteeing the existence of points $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ for any $t \in [0, \delta_{\bar{x}}[$, $\delta_{\bar{x}} > 0$ sufficiently small, such that $t \mapsto d_S(y_{\bar{x}}(t))$ is strictly decreasing.

DEFINITION 4.1. *Given the system* (4.1), *define the sets*

$$\mathscr{F}(x) := \left\{ f(x) + \sum_{i=1}^{N} u_i g_i(x) : \, u_i \in [-1,1] \right\},$$
$$\tilde{\mathscr{F}}(x) := \left\{ f(x) + \sum_{i=1}^{N} u_i g_i(x) : \, u_i \in \{-1,0,1\} \right\} \subset \mathscr{F}(x).$$

*We recall that since* $\mathrm{co}\,\tilde{\mathscr{F}}(x) \supseteq \mathscr{F}(x)$ *for every* $x \in \mathbb{R}^d$, *every trajectory of the system, which can be written in the form of the differential inclusion* $\dot{x} \in \mathscr{F}(x)$, *can be uniformly approximated by trajectories of the control system* $\dot{x} \in \tilde{\mathscr{F}}(x)$. *Thus in order to study controllability properties of the original system, it is sufficient to study the analogous properties of* $\dot{x} \in \tilde{\mathscr{F}}(x)$, *which are strictly related to the Lie algebra* $\mathrm{Lie}(\tilde{\mathscr{F}})$ *generated by the vector fields of* $\tilde{\mathscr{F}}$.

DEFINITION 4.2 (formal bracket). *We denote by* $\mathrm{Diffeo}(\mathbb{R}^d)$ *the set of all diffeomorphism of* $\mathbb{R}^d$. *Let* $\psi, \varphi \in \mathrm{Diffeo}(\mathbb{R}^d)$ *be two diffeomorphisms. We define their* formal bracket *by setting*

$$[\psi, \varphi](x) := \psi \circ \varphi \circ \psi^{-1} \circ \varphi^{-1}(x).$$

*Since for every* $\psi, \varphi \in \mathrm{Diffeo}(\mathbb{R}^d)$ *we have that* $[\psi, \varphi] \in \mathrm{Diffeo}(\mathbb{R}^d)$, *by iterating the procedure we can construct* formal bracket expressions *by nesting formal brackets of diffeomorphisms. Given a subset* $\mathscr{S} \subseteq \mathrm{Diffeo}(\mathbb{R}^d)$, *we define the* length *(also* order *or* depth*) of nested formal brackets of elements of* $\mathscr{S}$ *by induction. If* $\varphi \in \mathscr{S}$ *is a single diffeomorphism, then* $\mathrm{ord}\,(\varphi) = 1$. *Otherwise, if* $A$ *and* $B$ *are formal bracket expressions of elements of* $\mathscr{S}$, *we set* $\mathrm{ord}\,[A, B] = \mathrm{ord}\,A + \mathrm{ord}\,B$. *We define similarly* $\mathrm{pw}\,(\varphi) = 1$ *if* $\varphi \in \mathscr{S}$, *otherwise we set* $\mathrm{pw}\,[A, B] = 2\mathrm{pw}\,A + 2\mathrm{pw}\,B$ *if* $A$ *and* $B$ *are formal bracket expressions of elements of* $\mathscr{S}$.

DEFINITION 4.3. *Let* $X : \mathbb{R}^d \to \mathbb{R}^d$ *be a locally Lipschitz vector field. Given* $x \in \mathbb{R}^d$, *we denote by* $\phi_t^X(x)$ *or* $\phi^X(t, x)$ *the* flow of $X$ starting from $x$ *the (unique) solution of* $\dot{x}(s) = X(x(s))$, $x(0) = x$ *evaluated at* $s = t$. *We have* $\phi^X(0, x) = x$ *and* $\frac{\partial}{\partial t}\phi^X(t, x) = X(\phi^X(t, x))$.

*For t sufficiently small, it is well known that* $\phi_t^X(\cdot)$ *is a diffeomorphism and that if* $X, Y$ *are two* $C^2$*-smooth vector fields, we have that*

$$\begin{cases} \dfrac{d}{dt}[\phi_t^X, \phi_t^Y](x)_{|t=0} = 0, \\ \dfrac{d^2}{dt^2}[\phi_t^X, \phi_t^Y](x)_{|t=0} = 2[X, Y](x), \end{cases}$$

*where on the right-hand side we have the usual* Lie bracket *of vector fields defined by*

$$[X, Y](x) = \langle \nabla X(x), Y(x) \rangle - \langle \nabla Y(x), X(x) \rangle.$$

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The correspondence between the first nonvanishing derivative at 0 of flows generating the bracket and the order of the Lie bracket is explained in this classical result (see, e.g., Theorem 1 in [20]).

THEOREM 4.4. *Let $k \in \mathbb{N} \setminus \{0\}$, $M$ be a manifold of class $C^{k+1}$, and for $i = 1, \dots, k$ let $\phi^i : \mathbb{R} \times M \supset U_{\phi^i} \to M$ be a smooth map of class $C^{k+1}$ such that*

1. *$U_{\phi^i}$ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$,*
2. *$\phi^i_t$ is a diffeomorphism of class $C^{k+1}$ on its domain,*
3. *$\phi^i_0 = Id_M$ and $\frac{\partial}{\partial t}\phi^i_t\big|_{t=0} = X_i \in \mathrm{Vec}_k(M)$,*

*where $\mathrm{Vec}_k(M)$ is the set of vector fields on $M$ of class $C^k$. Then for each formal bracket expression $B$ of order $k$ (w.r.t. $\mathscr{S} = \{\phi^i : i = 1, \dots, k\}$) we have*

$$\frac{\partial^j}{\partial t^j} B(\phi^1_t, \dots, \phi^k_t)\Big|_{t=0} = 0 \quad \forall 1 \le j < k,$$
$$\frac{1}{k!} \cdot \frac{\partial^k}{\partial t^k} B(\phi^1_t, \dots, \phi^k_t)\Big|_{t=0} = B(X_1, \dots, X_k).$$

DEFINITION 4.5. *Given the system* (4.1), *we will set*

$$\begin{aligned}\mathscr{L} := \big\{&B(X_1, \dots, X_k) : k \in \mathbb{N} \setminus \{0\}, B \text{ is a formal bracket of order } k \text{ w.r.t. } \mathscr{F},\\ &X_j \in C^{k,1} \text{ are vector fields with } X_j(x) \in \tilde{\mathscr{F}}(x) \text{ for every } x,\\ &\exists \tau > 0 \text{ s.t. } t \mapsto B(\phi^1_t, \dots, \phi^k_t)(x) \text{ is an admissible trajectory for } 0 \le t < \tau\big\}.\end{aligned}$$

*Moreover, if $Z \in \mathscr{L}$, we will define $\operatorname{ord} Z$ as the minimum natural number $k > 0$ such that there exists a formal bracket $B$ and $k$ vector fields $X_j \in C^{k,1}$, with $X_j(x) \in \tilde{\mathscr{F}}(x)$ for every $x$, satisfying $Z = B(X_1, \dots, X_k)$. In this case, we set $\operatorname{pw} Z = \min\{\operatorname{pw} B : \operatorname{ord} B = \operatorname{ord} Z\}$.*

An immediate consequence of Theorem 4.4 is the following.

LEMMA 4.6. *Consider the system* (4.1) *and the sets defined in Definitions* 4.1 *and* 4.5. *Let $Z \in \mathscr{L}$, $\operatorname{ord} Z = k$. Then there exists a curve $t \mapsto y_{\bar{x}}(t)$ such that $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ for every $t$ and*

$$y_{\bar{x}}(t) - \bar{x} = Z(\bar{x}) \cdot \frac{t^k}{\operatorname{pw}^k Z} + o(t^k). \tag{4.2}$$

*Moreover, for each compact $K$ there exists $L_K > 0$ such that $|o(t^k)| \le L_K t^{k+1}$ for $\bar{x} \in K$.*

*Proof.* Since $Z$ is in the Lie algebra generated by the elements of $\tilde{\mathscr{F}}$, it can be written as $Z = B(X_1, \dots, X_k)$, where $B(X_1, \dots, X_k)$ is a bracket expression of order $k$ depending on $k$ vector fields $X_i \in \tilde{\mathscr{F}}$ for $1 \le i \le k$. We define $y_{\bar{x}}(\operatorname{pw}^k B \cdot t) = B(\phi^{X_1}_t, \dots, \phi^{X_k}_t)(\bar{x})$. We have by construction that $y_{\bar{x}}(\operatorname{pw}^k B \cdot t) \in \mathscr{R}_{\bar{x}}(\operatorname{pw}^k B)$. Moreover, according to Theorem 4.4, we have the following Taylor expansion in a neighborhood of 0:

$$\begin{aligned}y_{\bar{x}}(\operatorname{pw} B \cdot t) &= \bar{x} + \sum_{j=1}^{k} \frac{1}{j!} \frac{d^j}{dt^j} B(\phi^{X_1}_t, \dots, \phi^{X_k}_t)(\bar{x})\Big|_{t=0} \cdot t^j + o(t^k)\\ &= \bar{x} + B(X_1, \dots, X_k)(\bar{x}) \cdot t^k + o(t^k).\end{aligned}$$

This is the desired expansion up to a reparametrization of the time variable. The last assertion follows from the smoothness of the vector fields. ∎

Lemma 4.6 gives us an approximation of $\mathscr{R}_{\bar{x}}(t)$ using the elements of $\mathscr{L}$. We can give a natural sufficient condition ensuring that there are points of $\mathscr{R}_{\bar{x}}(t)$ whose distance from $S$ is *strictly less* than $d_S(\bar{x})$.

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COROLLARY 4.7. *Given the control system* (4.1), *assume that there exist a neighborhood $V$ of the target $S$, a vector field $Y \in \mathscr{L}$, and a real number $\mu > 0$ such that for every $x \in V \setminus S$ there exists $\zeta_x \in \partial^P d_S(x)$ such that*

$$\langle \zeta_x, Y(x) \rangle < -\mu.$$

*Then given $\bar{x} \in V \setminus S$ there exist $\delta = \delta_{\bar{x}} > 0$ and a curve $t \mapsto y_{\bar{x}}(t)$ defined on* $[0,\delta]$, *such that $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ for any $t \in [0,\delta]$ and $t \mapsto d_S(y_{\bar{x}}(t))$ is strictly decreasing.*

*Proof.* According to Lemma 4.6, there exists a curve $t \mapsto y_{\bar{x}}(t)$ such that $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ satisfying

$$y_{\bar{x}}(t) = \bar{x} + Y(\bar{x}) \cdot \frac{t^k}{\mathrm{pw}^k\, Y} + o(t^k),$$

where $k$ is the order of $Y$. Moreover, we choose $\delta > 0$ so small such that for $0 < t < \delta$

$$y_{\bar{x}}(t) \in B(\bar{x}, d_S(\bar{x})/2) \cap V.$$

In particular, we have for $0 < t < \delta$,

$$\begin{aligned} d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) &\le \langle \zeta_{\bar{x}}, y_{\bar{x}}(t) - \bar{x} \rangle + \frac{2}{d_S(\bar{x})} \|y_{\bar{x}}(t) - \bar{x}\|^2 \\ &= \left\langle \zeta_{\bar{x}}, Y \cdot \frac{t^k}{\mathrm{pw}^k\, Y} + o(t^k) \right\rangle + \frac{2}{d_S(\bar{x})} \left\| Y(\bar{x}) \cdot \frac{t^k}{\mathrm{pw}^k\, Y} + o(t^k) \right\|^2 \\ &\le -\frac{\mu}{\mathrm{pw}^k\, Y} t^k + o(t^k) \end{aligned}$$

by the local semiconcavity of $d_S$ on $B(\bar{x}, \frac{d_S(\bar{x})}{2})$.

By further shrinking $\delta$, we obtain that for every $\bar{x} \in V \setminus S$ there exists $\delta_{\bar{x}} > 0$ and a curve $y_{\bar{x}}(\cdot)$ of points of admissible trajectories such that

$$d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) \le -\frac{\mu}{2\mathrm{pw}^k\, Y} t^k$$

for $0 < t < \delta_{\bar{x}}$, from which the result follows. Notice that if $d_S$ is differentiable at $\bar{x}$ (or, equivalently, if $\bar{x}$ has a unique projection on $S$) then $\zeta_{\bar{x}} = \nabla d_S(\bar{x})$. ∎

*Remark* 4.8. We can repeat the procedure described in Corollary 4.7 starting from $y_{\bar{x}}(\delta_x)$, and in this way we construct a sequence of points belonging to admissible trajectories starting from $\bar{x}$ along which the distance is strictly decreasing. However Corollary 4.7 *does not imply* that the constructed trajectory actually will reach the target in *finite* time, since at each step we have no uniformity properties on $\delta_{\bar{x}}$; further assumptions are needed to grant suitable estimates on $\delta_x$.

The computation of $\mathscr{L}$ turns out to be quite simple in the driftless case (symmetric systems) as in this case the set of admissible trajectories enjoys *time reversal symmetry*. In particular, we have that $\mathscr{L}$ coincides with all possible nested brackets of the vector fields $g_i(\cdot)$ of suitable order (depending on the smoothness of the field) and, more specifically, if $g_i(\cdot) \in C^\infty$, we have $\mathscr{L} = \mathrm{Lie}(\{g_i\}_{i=1}^N)$.

In the presence of a nontrivial drift term, which breaks the time reversal symmetry of the system, this is no longer true. The computation of elements of $\mathscr{L}$ becomes much more tricky but can be done using repeatedly the classical Baker–Campbell–Hausdorff formula (see, e.g., sections III.4 and III.5 in [12]) and induction.

To cover the example we provided at the end of section 5, we give some explicit computations of this case, giving explicit formulas for some elements of $\mathscr{L}$ in two simple cases.

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LEMMA 4.9 (second order expansion). *Let $X_1, \dots, X_j$ be a finite family of $C^1$-vector fields on $\mathbb{R}^d$, $q \in \mathbb{R}^d$. For $i = 1, \dots, j$ denote by $\phi^{X_i}(s, q)$ the flow of $X_i$ at time $t = s$ starting from $q$ at $t = 0$. Fix $x \in \mathbb{R}^d$, and define by induction $p_0(t) = x$, $p_i(t) = \phi^{X_i}(t, p_{i-1}(t))$ for $0 < i \le j$. Then*

$$\left.\frac{dp_j}{dt}\right|_{t=0} = \sum_{i=1}^{j} X_i(x),$$

$$\left.\frac{d^2p_j}{dt^2}\right|_{t=0} = 2 \sum_{\substack{i,\, h = 1 \\ i < h}}^{j} \nabla X_h(x) \cdot X_i(x) + \sum_{h=1}^{j} \nabla X_h(x) \cdot X_h(x).$$

*In particular, given $f, g, g_1, g_2 \in C^{1,1}(\mathbb{R}^d)$, and $u \in \mathbb{R}$,*

1. *if we define $X_1(x) := f(x) + ug(x)$, $X_2(x) = f(x) - ug(x)$ we have*

$$\left.\frac{d}{dt}\phi^{X_2}(t, \phi^{X_1}(t, x))\right|_{t=0} = 2f(x),$$

$$\left.\frac{d^2}{dt^2}\phi^{X_2}(t, \phi^{X_1}(t, x))\right|_{t=0} = 4\nabla f(x)f(x) + 2u[f, g](x);$$

2. *if we define*

$$X_1(x) = X_7(x) = f(x) + g_1(x), \qquad X_2(x) = X_8(x) = f(x) + g_2(x),$$
$$X_3(x) = X_5(x) = f(x) - g_1(x), \qquad X_4(x) = X_6(x) = f(x) - g_2(x),$$

*then we have*

$$\left.\frac{d}{dt}\phi_8(t, \phi_7(t, \dots))\right|_{t=0} = 8f(x),$$

$$\left.\frac{d^2}{dt^2}\phi_8(t, \phi_7(t, \dots))\right|_{t=0} = 64\nabla f(x)f(x) + 4[g_1, g_2](x).$$

*Proof.* We have

$$p_j(0) = \phi_{X_j}(0, p_{j-1}(0)) = p_{j-1}(0) = \dots = p_0(0) = x,$$

$$\frac{dp_j}{dt} = \frac{d}{dt}\left[\phi_{X_j}(t, p_{j-1})\right] = X_j(\phi_{X_j}(t, p_{j-1})) + \nabla\phi_{X_j}(t, p_{j-1}) \cdot \frac{dp_{j-1}}{dt}$$
$$= X_j(p_j) + \nabla\phi_{X_j}(t, p_{j-1}) \cdot \frac{dp_{j-1}}{dt},$$

$$\left.\frac{dp_j}{dt}\right|_{t=0} = X_j(p_j(0)) + \nabla\phi_{X_j}(0, p_{j-1}(0)) \cdot \frac{dp_{j-1}}{dt}(0)$$
$$= X_j(x) + \mathrm{Id}_{\mathbb{R}^d} \cdot \frac{dp_{j-1}}{dt}(0) = X_j(x) + \frac{dp_{j-1}}{dt}(0) = \sum_{i=1}^{j} X_i(x),$$

where we use induction in the last step.

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The computation of the second order derivative yields

$$\begin{aligned}
\frac{d^2p_j}{dt^2} &= \frac{d}{dt}\left[X_j(p_j) + \nabla\phi_{X_j}(t,p_{j-1})\cdot\frac{dp_{j-1}}{dt}\right]\\
&= \nabla X_j(p_j)\cdot\frac{dp_j}{dt} + \nabla\phi_{X_j}(t,p_{j-1})\cdot\frac{d^2p_{j-1}}{dt^2}\\
&\quad + \left(\nabla\frac{\partial}{\partial t}\phi_{X_j}(t,p_{j-1}) + \nabla^2\phi_{X_j}(t,p_{j-1})\cdot\frac{dp_{j-1}}{dt}\right)\cdot\frac{dp_{j-1}}{dt}\\
&= \nabla X_j(p_j)\cdot\frac{dp_j}{dt} + \nabla\phi_{X_j}(t,p_{j-1})\cdot\frac{d^2p_{j-1}}{dt^2} + \nabla X_j(p_j)\cdot\frac{dp_{j-1}}{dt}\\
&\quad + \nabla^2\phi_{X_j}(t,p_{j-1})\cdot\frac{dp_{j-1}}{dt}\cdot\frac{dp_{j-1}}{dt}.
\end{aligned}$$

If we evaluate at $t=0$, we notice that $\nabla^2\phi_{X_j}(0,x)=0$, and hence by induction again

$$\begin{aligned}
\left.\frac{d^2p_j}{dt^2}\right|_{t=0} &= \nabla X_j(x)\cdot\left(\sum_{i=1}^{j}X_i(x) + \sum_{i=1}^{j-1}X_i(x)\right) + \frac{d^2p_{j-1}}{dt^2}\\
&= 2\sum_{i=1}^{j}\nabla X_j(x)\cdot X_i(x) - \nabla X_j(x)\cdot X_j(x) + \frac{d^2p_{j-1}}{dt^2}\\
&= 2\sum_{\substack{i,\,h\,=\,1\\ i\,\le\,h}}^{j}\nabla X_h(x)\cdot X_i(x) - \sum_{h=1}^{j}\nabla X_h(x)\cdot X_h(x)\\
&= 2\sum_{\substack{i,\,h\,=\,1\\ i\,<\,h}}^{j}\nabla X_h(x)\cdot X_i(x) + \sum_{h=1}^{j}\nabla X_h(x)\cdot X_h(x).
\end{aligned}$$

The two statements (1) and (2) follow from the above computations (see also Lemmas 1 and 2 in [18] for a different approach). $\square$

## 5. Small-time attainability in control systems.

**5. Small-time attainability in control systems.** In this section we prove the main results of the paper.

Throughout this section we consider the following control system

$$\begin{cases} \dot{x}(t) = f(x(t),u(t)), & t>0,\\ x(0)=\bar{x}\in\mathbb{R}^d, \end{cases} \tag{5.1}$$

where $u\in\mathscr{U} := \{u:[0,+\infty[\to U \text{ measurable}\}$, $U$ is a compact subset of $\mathbb{R}^m$, $S\subseteq\mathbb{R}^d$ is a given closed target set, $f:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^d$, $f$ is of class $C^{1,1}_{\mathrm{loc}}((\mathbb{R}^d\setminus S)\times U)$ (in particular we have local existence and uniqueness of the Carathéodory solution for every $\bar{x}\notin S$).

We recall that our aim is to give sufficient conditions in order to find an admissible trajectory of (5.1) steering $\bar{x}$ to $S$ in finite time.

We collect in this lemma some basic properties of solutions of (5.1). The proof can be found, e.g., in Theorem 5.4 in Chapter III, section 5, p. 219 of [1].

LEMMA 5.1. *Let $\delta,m>0$ and assume that $U$ is a compact subset of $\mathbb{R}^m$, $S\subseteq\mathbb{R}^d$ is a given closed target set, $f:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^d$, $f$ is of class $C^{1,1}_{\mathrm{loc}}((\mathbb{R}^d\setminus S)\times U)$, $\|f\|_\infty<m$ on $S_\delta\setminus S$. Given a measurable function $u:[0,+\infty[\to U$, consider the Carathéodory solution of* (5.1), *let $I$ be its (open) maximal interval of existence, and*

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*set* $I^+ = I \cap [0, +\infty[$. *We recall that by assumption we have uniqueness of the trajectory in* $S_\delta \setminus S$. *For any* $t \in \overline{I^+}$, *we have that if* $0 < \lim_{s \to t^-} d_S(x(s)) \le \delta$ *then* $\sup I^+ > t$. *For any* $t \in I^+$ *such that* $x(s) \in S_\delta \setminus S$ *for all* $0 \le s \le t$ *we have* $\|x(t) - \bar{x}\| \le mt$.

DEFINITION 5.2 ($\mathscr{A}$-trajectory). *Let* $\bar{x} \in \mathbb{R}^d$, $T > 0$. *We say that a continuous curve* $y_{\bar{x}} : [0, T] \to \mathbb{R}^d$ *is an* $\mathscr{A}$-trajectory *starting from* $\bar{x}$ *if we have* $y_{\bar{x}}(0) = \bar{x}$ *and* $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t)$ *for any* $t \in [0, T]$ *(see also section* 3.1 *in* [16]*).*

It is clear that every admissible trajectory of (5.1) starting from a point $\bar{x}$ is an $\mathscr{A}$-trajectory starting from the same point, as shown in this classical example.

*Example* 5.3. In $\mathbb{R}^3$, define $X_1(x, y, z) := (1, 0, -y)$ and $X_2(x, y, z) := (0, 1, x)$. Consider the system $\dot{\gamma}(t) = u_1(t) X_1(\gamma(t)) + u_2(t) X_2(\gamma(t))$, where $u_1, u_2 : [0, +\infty[ \to [-1, 1]$ are measurable. Set $\bar{P} = (0, 0, 0)$, and define a smooth curve $y_{\bar{P}}(\cdot)$ by setting

$$y_{\bar{P}}(t) = \left(0, 0, \frac{t^2}{16}\right).$$

For every $t > 0$ we have that

$$\dot{y}_{\bar{P}}(t) \notin \{u_1 X_1(y_{\bar{P}}(t)) + u_2 X_2(y_{\bar{P}}(t)) : u_1, u_2 \in [-1, 1]\}.$$

Hence $y_{\bar{P}}(\cdot)$ is *not* an admissible trajectory for the system starting from the origin.

However, given $t > 0$ and defining the controls

$$\left(u_1^{(t)}(s), u_2^{(t)}(s)\right) = \begin{cases} (1, 0) \text{ for } 0 \le s \le t/4, \\ (0, 1) \text{ for } t/4 \le s \le t/2, \\ (-1, 0) \text{ for } t/2 \le s \le 3t/4, \\ (0, -1) \text{ for } 3t/4 \le s \le t, \end{cases}$$

we have that the trajectory $\bar{\gamma}^{(t)}(\cdot)$ generated by the system using $(u_1^{(t)}(\cdot), u_2^{(t)}(\cdot))$ with initial condition $\bar{\gamma}^{(t)}(0) = 0$ is an admissible trajectory starting from the origin and satisfying $\bar{\gamma}^{(t)}(t) = y_{\bar{P}}(t)$. By the arbitrariness of $t > 0$ in the previous construction, we conclude that $y_{\bar{P}}(\cdot)$ is an $\mathscr{A}$-trajectory starting from the origin.

We now start to consider the behavior of the distance function from $S$ along smooth $\mathscr{A}$-trajectories of (5.1), providing a first result concerning estimates on the distance to $S$ of $y_{\bar{x}}(t)$, $t > 0$ sufficiently small. The main tool will be the semiconcavity estimates of $d_S$ provided in Proposition 2.13. This proposition may be viewed as a quantitative counterpart of Corollary 4.7 for the system (5.1).

*Remark* 5.4. *The following simple fact, that can be easily proved by induction, will be used: given* $\xi_0, \dots, \xi_N \in \mathbb{R}^d$, *we have*

$$\begin{aligned} \left\| \sum_{i=0}^N \xi_i \right\|^2 &= \|\xi_0\|^2 + 2 \sum_{i=1}^N \langle \xi_0, \xi_i \rangle + \sum_{i=1}^N \|\xi_i\|^2 + 2 \sum_{\substack{l,\, j\, =\, 1 \\ i\, <\, j}}^N \langle \xi_l, \xi_j \rangle \\ &= \sum_{i=1}^{\lfloor N/2 \rfloor} \|\xi_i\|^2 + 2 \sum_{i=1}^N \sum_{\substack{l,\, j\, =\, 1 \\ l\, <\, j \\ l + j\, =\, i}}^N \langle \xi_l, \xi_j \rangle \\ &\quad + \sum_{i=1+\lfloor N/2 \rfloor}^N \|\xi_i\|^2 + 2 \sum_{\substack{l,\, j\, =\, 1 \\ l\, <\, j \\ l + j\, >\, N}}^N \langle \xi_l, \xi_j \rangle + \|\xi_0\|^2 + 2 \sum_{i=1}^N \langle \xi_0, \xi_i \rangle. \end{aligned}$$

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PROPOSITION 5.5. *Consider the control system* (5.1). *Let* $\bar{x} \notin S$, $L > 0$, $k \in \mathbb{N} \setminus \{0\}$, *and* $\{v_1^{\bar{x}}, \dots, v_k^{\bar{x}}\} \subseteq \mathbb{R}^d$ *with* $\sum_{i=1}^k |v_i^{\bar{x}}| \le L$. *Denote by* $K_{\bar{x}}$ *a semiconcavity constant of* $d_S(\cdot)$ *on* $B(\bar{x}, d_S(\bar{x})/2)$. *Assume that there exists an* $\mathscr{A}$*-trajectory* $y_{\bar{x}}(\cdot)$ *starting from* $\bar{x}$ *such that for any* $t \ge 0$ *with* $y_{\bar{x}}(t) \in B(\bar{x}, d_S(x)/2)$,

$$\left\| y_{\bar{x}}(t) - \bar{x} - \sum_{i=1}^k \frac{t^i}{i!} v_i^{\bar{x}} \right\| \le L t^{k+1}.$$

*Then for every* $\zeta_{\bar{x}} \in \partial^P d_S(\bar{x})$, $0 \le t \le \min\{\frac{d_S(\bar{x})}{2\|f\|_\infty + 1}, 1\}$, *we have*

$$d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) \le \sum_{i=1}^k C_i(\zeta_{\bar{x}}) t^i + C_\psi t^{k+1}, \tag{5.2}$$

*where* $C_\psi = C_\psi(K_{\bar{x}}, L) \ge 0$ *is a constant independent of* $t$, *and*

$$C_i(\zeta_{\bar{x}}) := \begin{cases} \dfrac{1}{i!}\langle \zeta_{\bar{x}}, v_i^{\bar{x}} \rangle + 2K_{\bar{x}} \displaystyle\sum_{\substack{i,\, j\, =\, 1 \\ j + l = i \\ j < l}}^{k} \frac{1}{l! j!} \langle v_l^{\bar{x}}, v_j^{\bar{x}} \rangle, & i \text{ odd}, \\[2ex] \dfrac{1}{i!}\langle \zeta_{\bar{x}}, v_i^{\bar{x}} \rangle + 2K_{\bar{x}} \displaystyle\sum_{\substack{i,\, j\, =\, 1 \\ j + l = i \\ j < l}}^{k} \frac{1}{l! j!} \langle v_l^{\bar{x}}, v_j^{\bar{x}} \rangle + K_{\bar{x}} \left( \frac{1}{(i/2)!} \right)^2 \|v_{i/2}^{\bar{x}}\|^2, & i \text{ even}. \end{cases} \tag{5.3}$$

*Proof.* Set

$$\eta(t) = y_{\bar{x}}(t) - \bar{x} - \sum_{i=1}^k \frac{t^i}{i!} v_i^{\bar{x}}.$$

According to Lemma 5.1, we have that $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}(t) \subseteq B(\bar{x}, \frac{d_S(\bar{x})}{2})$ for any $0 \le t \le \min\{\frac{d_S(\bar{x})}{2\|f\|_\infty + 1}, 1\}$, so the segment joining $y_{\bar{x}}(t)$ and $\bar{x}$ lies inside $B(\bar{x}, d_S(\bar{x})/2)$, and hence outside $S$. We apply now Remark 5.4 with $\xi_0 = \eta(t)$ and $\xi_i = \frac{t^i}{i!} v_i^{\bar{x}}$, recalling that $\|\zeta_{\bar{x}}\| \le 1$ by the 1-Lipschitz continuity of the distance function and $0 \le t \le 1$, and that by assumption $|v_i^{\bar{x}}| \le L$, to obtain

$$\begin{aligned} &d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) \\ &\quad \le \langle \zeta_{\bar{x}}, y_{\bar{x}}(t) - \bar{x} \rangle + K_{\bar{x}} \|y_{\bar{x}}(t) - \bar{x}\|^2 \\ &\quad \le \langle \zeta_{\bar{x}}, \sum_{i=1}^k \frac{t^i}{i!} v_i^{\bar{x}} + \eta(t) \rangle + K_{\bar{x}} \left\| \sum_{l=1}^k \frac{t^l}{l!} v_l^{\bar{x}} + \eta(t) \right\|^2 \\ &\quad \le \sum_{i=1}^k \frac{t^i}{i!} \langle \zeta_{\bar{x}}, v_i^{\bar{x}} \rangle + K_{\bar{x}} \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{t^{2i}}{(i!)^2} \|v_i^{\bar{x}}\|^2 + 2K_{\bar{x}} \sum_{i=1}^k \sum_{\substack{l,\, j\, =\, 1 \\ l < j \\ l + j = i}}^{k} \frac{t^i}{l! j!} \langle v_l^{\bar{x}}, v_j^{\bar{x}} \rangle \\ &\qquad + \|\eta(t)\| + K_{\bar{x}} \sum_{i=1+\lfloor k/2 \rfloor}^{k} \frac{t^{2i}}{(i!)^2} \|v_i^{\bar{x}}\|^2 + 2K_{\bar{x}} \sum_{\substack{l,\, j\, =\, 1 \\ l < j \\ l + j > k}}^{k} \frac{t^{l+j}}{l! j!} \langle v_l, v_j \rangle \\ &\qquad + K_{\bar{x}} \|\eta(t)\|^2 + 2K_{\bar{x}} \sum_{i=1}^k \frac{t^i}{i!} \langle \eta(t), v_i^{\bar{x}} \rangle. \end{aligned}$$

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Since $\sum_{i=1}^{k}\frac{1}{i!}\le e-1$, we can estimate from above all the terms with order greater than or equal to $k+1$ by $C_\psi t^{i+1}$, where

$$C_\psi := L + K_{\bar{x}}L(L(e-1)+2L(e-1)^2+L+2L(e-1)).$$

Rearranging the terms in the previous inequality, the result follows. □

*Remark* 5.6. For a general closed set, the local sharp semiconcavity constant $c_{\mathcal{K}}$ of $d_S(\cdot)$ on a compact convex subset $\mathcal{K}$ of $\mathbb{R}^d$ disjoint from $S$ can blow up as the minimum distance between elements of $\mathcal{K}$ and $S$ tends to zero. In particular, given $\bar{x}\notin S$, if we take $\mathcal{K}=\overline{B(\bar{x},d_S(\bar{x})/2)}$, then $2/d_S(\bar{x})$ is a semiconcavity constant of $d_S(\cdot)$ on $\mathcal{K}$. However, if the set $S$ satisfies the $\rho$-internal sphere condition, according to Proposition 2.13 we have that $1/\rho$ is a semiconcavity constant for $d_S(\cdot)$ on every compact convex set disjoint from the interior of $S$, thus the sharp semiconcavity constant is uniformly bounded.

When we consider (5.2), for $t>0$ sufficiently small, the sign of the right-hand side is determined by the first nonzero element of $\{C_i(\zeta_{\bar{x}})\}_{i=1}^k$, since we can neglect higher order terms. In particular, if this coefficient is negative, we have that for $t>0$, $t$ sufficiently small, the distance is decreasing along the trajectory.

Now we introduce a sort of higher order Petrov's condition. Our aim is to show that for every $\bar{x}\notin S$ there exists an admissible trajectory of the system and a time $t_{\bar{x}}>0$ such that for $0\le t\le t_{\bar{x}}$ the trajectory is strictly approaching the target $S$, moreover, we will provide a lower estimate on $t_{\bar{x}}$, linking it to $\mu_x$, thus providing the information lacking in Remark 4.8.

LEMMA 5.7. *Consider the control system* (5.1), *assuming that $S$ satisfies a $\rho$-internal sphere condition for a certain $\rho>0$. Assume that there exists $L>0$, $\delta>0$ such that for every $x\in S_\delta\setminus S$ the following property is satisfied:*

*(HP) there exist $\mu_x>0$, $k_x\in\mathbb{N}\setminus\{0\}$, $\{v_1^x,\dots,v_k^x\}\subseteq\mathbb{R}^d$, $\sum_{i=1}^{k_x}|v_i|\le L$, $\zeta_x\in\partial^P d_S(x)$, and an $\mathscr{A}$-trajectory $y_x(\cdot)$ starting from $x$ satisfying for any $y_{\bar{x}}(t)\in S_\delta\setminus S$*

$$\begin{cases} C_i(\zeta_x)\le 0 \text{ for } i=1,\dots,k_x-1 \text{ and } C_{k_x}(\zeta_x)<-\mu_x,\\ \left\| y_{\bar{x}}(t)-\bar{x}-\sum_{i=1}^{k}\frac{t^i}{i!}v_i^{\bar{x}}\right\|\le Lt^{k+1}, \end{cases} \tag{5.4}$$

*where the $C_i(\zeta_x)$ are defined as in Proposition* 5.5.

*Then there exists a constant $c>0$ such that if we set*

$$T_x := \min\left\{1,\frac{\mu_x}{c},\left(\frac{d_S(x)}{\mu_x}\right)^{1/k_x}\right\},$$

*we have that for all $0\le t\le T_x$ such that $y_x(t)\notin S$,*

$$d_S(y_x(t))-d_S(x)\le -\frac{\mu_x}{2}\cdot t^{k_x}.$$

*Proof.* Set $K=1/\rho$. We notice from (5.3) that $|C_i(\zeta_x)|\le L+KL^2(2k^2+1)=:C_1$. Define

$$\tau_x=\sup\{t>0:\ y_x(s)\in S_\delta\setminus S \text{ for every } 0\le s\le t\}.$$

We have $\tau_x>0$ according to Lemma 5.1. Using the same argument as in Proposi-

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tion 5.5, for $y_x(s) \notin S$ we have

$$\begin{aligned} d_S(y_x(s)) - d_S(x) &\le \langle \zeta_x, y_x(s) - x \rangle + K\|y_x(t) - x\|^2 \\ &\le -\mu_x \cdot s^{k_x} + \sum_{i=k_x+1}^{k} C_i(\zeta_x) s^i + s^{k+1} C_\psi \\ &\le -\mu_x \cdot s^{k_x} + C_2 s^{k_x+1}, \end{aligned}$$

where $C_2 = kC_1 + C_\psi > 0$ does not depend on $x$, $t$. Letting $c = 2C_2$, for all $0 \le t \le T_x$ such that $y_x(t) \notin S$ we have

$$d_S(y_x(t)) - d_S(x) \le -\mu_x \cdot t^{k_x} + C_2 t^{k_x+1} \le -\frac{\mu_x}{2} t^{k_x}. \qquad \square$$

*Remark* 5.8. We have that (5.4) reduces to the classical Petrov's condition if we assume that it is satisfied for every $\bar{x}$ in a neighborhood of the target with $k_{\bar{x}} \equiv 1$ and $\mu_{\bar{x}} \equiv \mu > 0$ which do not depend on $\bar{x}$.

We have introduced the higher order Petrov's condition pointwise, so we can consider $\mu_x$ as a function defined in a suitable neighborhood $V$ of $S$ and strictly positive in $V \setminus S$. An interesting case is when we take it to be continuous, satisfying $\mu_{|V\setminus S} > 0$ but allowing $\mu_{|\partial S} = 0$. Geometrically speaking, this case means that we are allowing the (negative) coefficient of the leading term in expansion (5.2) to vanish as we approach the target; the modulus of the component of the speed pointing toward the target becomes *smaller and smaller*. In particular, we may not conclude that the target is reached in finite time, even if the distance is still strictly decreasing along at least one $\mathscr{A}$-trajectory starting from every point of $V \setminus S$.

We will consider in particular the case in which $\mu_x = \mu(d_S(x))$, where $\mu : [0, \delta[ \to [0, +\infty[$ is a nondecreasing continuous function satisfying $\mu(r) \neq 0$ if $r \neq 0$ (but allowing the possibility that $\mu(0) = 0$). This case occurs frequently in applications. We will give sufficient conditions on $\mu$ to ensure for every $x \in B(S, \delta)$ the existence of an admissible trajectory starting from $\bar{x}$ that reaches the target $S$ in finite time. We will use a strategy similar to [18].

*Remark* 5.9. Since we will provide different results for targets satisfying the $\rho$-internal sphere condition and for targets with positive reach, we summarize here the relationships between sets with positive reach, sets satisfying the $\rho$-internal sphere condition, and smooth sets:

1. if $C$ is closed and convex then $\operatorname{reach} C = +\infty$ and $\overline{\mathbb{R}^d \setminus C}$ enjoys the $\rho$-internal sphere condition, for every $\rho > 0$. However $C$ may fail to enjoy the $\rho$-internal sphere condition (an example is given by taking a square in $\mathbb{R}^2$, where the internal sphere property fails at the vertices);
2. if $K$ is has positive reach, then $\overline{\mathbb{R}^d \setminus K}$ enjoys the $\rho$-internal sphere condition, with $\rho = \operatorname{reach} K$;
3. in general, if $\overline{\mathbb{R}^d \setminus K}$ enjoys just the $\rho$-internal sphere condition we have that $K$ may not have positive reach, additional hypotheses are required (see references below);
4. if $K$ is a compact set with $C^{1,1}$ boundary then both $K$ and $\overline{\mathbb{R}^d \setminus K}$ have positive reach (possibly $\operatorname{reach} K \neq \operatorname{reach} \overline{\mathbb{R}^d \setminus K}$).

We refer the reader to [21] and [8] for further details and applications, and to [19] for a generalized version of these results.

Our first main result is the following.

THEOREM 5.10 (controllability result). *Consider the control system* (5.1) *with* $f \in C^{1,1}_{\mathrm{loc}}((\mathbb{R}^d \setminus S) \times U)$. *Let* $\rho, \delta, m > 0$, $k \in \mathbb{N} \setminus \{0\}$ *be constants, and* $\mu : ]0, \delta[ \to [0, 1]$

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*be continuous, nondecreasing, satisfying $\mu(p) > 0$ for any $p > 0$ and*

$$\int_0^{d_S(x)} \frac{dr}{\mu^k(r)} + \int_0^{d_S(x)} \frac{dr}{r^{1-1/k}\mu(r)} < +\infty.$$

*We make the following assumptions:*

1. *the closed target set $S$ satisfies the $\rho$-internal sphere condition;*
2. *$\|f(x,u)\| \le m$ in $(S_{2\delta} \setminus S) \times U$ (automatically satisfied if $\partial S$ is compact);*
3. *there exists $L > 0$ such that for every $x \in S_\delta \setminus S$ there exist $1 \le k_x \le k$, $\{v_1, \dots, v_{k_x}\} \in \mathbb{R}^d$, $\zeta_x \in \partial^P d_S(x)$, and an $\mathscr{A}$-trajectory $y_x(\cdot)$ starting from $x$ such that (5.4) is satisfied with $\mu_x = \mu(d_S(x))$.*

*Then $T(x) \le \omega(d_S(x))$ for every $x \in S_\delta$, where*

$$\omega(s) := C\left(\int_0^s \frac{dr}{\mu^k(r)} + \int_0^s \frac{dr}{r^{1-1/k}\mu(r)}\right)$$

*for a suitable constant $C > 0$, thus STLA holds.*

*If we have also $\|\nabla f(x,u)\| \le m$ in $(S_{2\delta} \setminus S) \times U$, then $T$ is continuous in $S_\delta$ with modulus of continuity bounded by $\omega(s)$.*

*Proof.* Fix $\bar{x} \in S_\delta$. The strategy of the proof will be to define by induction a sequence $\{(t_i, x_i)\}_{i\in\mathbb{N}}$ such that $(t_0, x_0) = (0, \bar{x})$ and $x_{i+1} \in \mathscr{R}_{x_i}(t_{i+1})$, i.e., $x_{i+1}$ can be reached from $x_i$ in time $t_{i+1}$ by an admissible trajectory of the system. Gluing all these trajectories we can construct an admissible trajectory starting from $\bar{x}$ that passes through each $x_i$ at time $\sum_{h=0}^{i} t_h$. Our aim is to show that $\lim_{i\to\infty} d_S(x_i) = 0$ and $\sum_{h=0}^{\infty} t_h < +\infty$, i.e., this admissible trajectory reaches $S$ in finite time.

The basic idea is to apply at each step the estimates of Proposition 5.5 and Lemma 5.7, ensuring that at each step we satisfy their assumptions.

Let $K := 1/\rho$. For every $x \in S_\delta$ we define $T_x$ as in Lemma 5.7. Define by induction a sequence of times and points $\{(t_i, x_i)\}_{i\in\mathbb{N}} \subseteq [0, +\infty[\times\mathbb{R}^d$ as follows. We set $(t_0, x_0) = (0, \bar{x})$ and given $(t_i, x_i) \in [0, +\infty[\times\mathbb{R}^d$, we define

$$(t_{i+1}, x_{i+1}) := \begin{cases} (T_{x_i}, y_{x_i}(T_{x_i})) & \text{if } T(x_i) > T_{x_i}, \\ (T(x_i), \theta_{x_i}(T(x_i))) & \text{if } T(x_i) \le T_{x_i}, \end{cases}$$

where $\theta_{x_i}(\cdot)$ is an optimal trajectory starting from $x_i$. Notice that if $T(x_i) \le T_{x_i}$, then in particular $T(x_i) < +\infty$, so we have that there exists a trajectory of the system such that $\theta_{x_i}(T(x_i)) \in S$. We set also $r_i := d_S(x_i)$.

We have to check the following.

*Claim* (a). If $x_i \in S_\delta$, then $x_{i+1} \in S_\delta$.

*Claim* (b). $\lim_{i\to+\infty} r_i = 0$.

*Claim* (c). $\sum_{i=1}^{+\infty} t_i < +\infty$.

*Proof of Claim* (a). If $t_{i+1} = T(x_i)$ we have $x_{i+1} \in S \subseteq S_\delta$. Otherwise, we have that $y_{x_i}(t) \notin S$ for all $0 \le t \le T_{x_i}$ since $T_{x_i} < T(x_i)$. According to Lemma 5.7, we have that for $0 \le t < t_{i+1}$

$$d_S(y_{x_i}(t)) - r_i \le -\frac{\mu(r_i)}{2} \cdot t^{k_{x_i}} < 0.$$

By passing to the limit for $t \to T_{x_i}^-$, we have $r_{i+1} < r_i$ so $x_{i+1} \in S_\delta$. ☐

*Proof of Claim* (b). If there exists $i \in \mathbb{N}$ such that $r_i = 0$, we have that $r_j = t_j = 0$ for all $j > i$ and the proof is concluded. Assume now that $r_i \neq 0$ for all $i \in \mathbb{N}$. According to Claim (a), the sequence $\{r_i\}_{i\in\mathbb{N}}$ is a strictly decreasing sequence of

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positive numbers, thus there exists $r \geq 0$ such that $\lim_{i\to\infty} r_i = r$. Moreover, for every $i \in \mathbb{N}$

$$r_{i+1} - r_i \leq -\frac{\mu(r_i)}{2} \cdot T_{x_i}^{k_{x_i}} \leq \max\left\{-\frac{\mu^{k_{x_i}+1}(r_i)}{2c^{k_x-1}}, -\frac{r_i\mu(r_i)}{2}, -\frac{\mu(r_i)}{2}\right\} < 0.$$

If we pass to the limit for $i \to +\infty$ and recall the continuity of $\mu(\cdot)$ we obtain

$$\lim_{i\to\infty} \mu(r_i) = \mu(r) = 0.$$

Then necessarily $r = 0$ and the proof of this claim is concluded. $\square$

*Proof of Claim* (c). This claim states that we are going to reach the target in a finite time. The result is trivial if there exists $i$ such that $r_i = 0$, so we assume that $r_i \neq 0$ for all $i \in \mathbb{N}$, i.e., $T(x_i) > T_{x_i}$ for every $i \in \mathbb{N}$. According to the estimate given in Claim (b), and recalling that $1 \leq k_{x_i} \leq k$ and that $T_{x_i} < 1$, we have that

$$\begin{aligned}
r_{i+1} - r_i &\leq -\frac{1}{2}T_{x_i}^{k_{x_i}}\mu(r_i) \leq -\frac{1}{2}T_{x_i}^{k} \leq -\frac{1}{2}T^{k-1}\mu(r_i)T_{x_i}\\
&= -\frac{\mu(r_i)}{2}\left(\min\left\{1, \frac{\mu(r_i)}{c}, \left(\frac{r_i}{\mu(r_i)}\right)^{1/k}\right\}\right)^{k-1} T_{x_i}\\
&= -\min\left\{\frac{\mu(r_i)}{2}, \frac{\mu^k(r_i)}{2c^{k-1}}, \frac{1}{2}r_i^{1-1/k}\mu^{1/k}(r_i)\right\} T_{x_i}\\
&\leq -C_3 \min\left\{\mu^k(r_i), r_i^{1-1/k}\mu(r_i)\right\} T_{x_i},
\end{aligned}$$

where $C_3 > 0$ is a suitable constant independent of $x_i$, $r_i$. This implies that

$$\frac{2(r_i - r_{i+1})}{C_3\mu^k(r_i)} + \frac{2(r_i - r_{i+1})}{C_3 r_i^{1-1/k}\mu(r_i)} \geq T_{x_i}.$$

Since $\mu(\cdot)$ is positive and nondecreasing we have that $p \mapsto \mu^{-k}(p)$ and $p \mapsto (p^{1-\frac{1}{k}}\mu(p))^{-1}$ are nonincreasing, and thus we have

$$T(x) \leq \sum_{i=0}^{\infty} T_{x_i} \leq \frac{2}{C_3}\left(\int_0^{d_S(x)} \frac{dr}{\mu^k(r)} + \int_0^{d_S(x)} \frac{dr}{r^{1-1/k}\mu(r)}\right) < +\infty,$$

which concludes the proof of the claim. $\square$

To conclude the proof of the theorem, we define

$$\omega(s) := \frac{2}{C_3}\left(\int_0^{s} \frac{dr}{\mu^k(r)} + \int_0^{s} \frac{dr}{r^{1-1/k}\mu(r)}\right),$$

and notice that we have that $T(x) \leq \omega(d_S(x))$ for all $x \in S_\delta$.

If $\nabla f$ is bounded in $S_\delta \setminus S$, we can obtain more information by proceeding in a way similar to Proposition 1.6 of Chapter 4, p. 230 in [1], or to Propositions 2 and 3 of [18].

Take $x, y \in S_\delta$ and without loss of generality assume that $T(y) \leq T(x)$. According to the previous estimate, we have that

$$T(y) \leq T(x) \leq \omega(d_S(x)) \leq \omega(\delta),$$

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since by definition $\omega$ is nondecreasing. Let $u_y : [0, T(y)] \to U$ be an optimal control steering $y$ to $\bar{y} \in S$ in time $T(y)$. Consider the solution $\hat{x}(\cdot)$ of $\dot{x}(t) = f(x(t), u_y(t))$, $x(0) = x$, and set $\bar{x} := \hat{x}(T(y))$. By Gronwall's inequality we have that there exist constants $c_1, c_2 > 0$ such that

$$d_S(\bar{x}) \le \|\bar{y} - \bar{x}\| \le e^{c_1 T(y)}\|y - x\| \le c_2\|y - x\|.$$

By the dynamic programming principle,

$$T(x) \le T(y) + T(\bar{x}) \le T(y) + \omega(d_S(\bar{x})) = T(y) + \omega(c_2\|y - x\|),$$

so $|T(x) - T(y)| = T(x) - T(y) \le \omega(c_2\|y - x\|)$, as desired. ∎

*Remark* 5.11. In the above assumption, if we are allowed to choose $\mu(r) = Cr^\alpha$ for a suitable $C, \alpha > 0$ such that $0 \le \alpha k < 1$ and $0 < 1/k - \alpha \le 1$, then $T(x) \le C' d_S^\eta(x)$, where $\eta = \min\{1 - \alpha k, 1/k - \alpha\}$. If the dynamics is Lipschitz, this implies Hölder continuity with exponent $\eta$. Petrov's condition corresponds to $\alpha = 0$ and $k = 1$, i.e., $\mu \equiv$ const., and in this case we have $\omega(r) = cr$ for a suitable positive constant $c > 0$, which yields Lipschitz continuity, as is well known.

Now we want to apply Theorem 5.10 to the affine control systems we considered in (4.1), exploiting Lemma 4.6.

COROLLARY 5.12. *Consider the control-affine system* (4.1). *Let $\rho, \delta, m > 0$, $k \in \mathbb{N} \setminus \{0\}$ be constants, and $\mu : ]0, \delta[ \to [0, 1]$ be continuous, nondecreasing, with $\mu(p) > 0$ for any $p > 0$ and*

$$\int_0^{d_S(x)} \frac{dr}{\mu^k(r)} + \int_0^{d_S(x)} \frac{dr}{r^{1-1/k}\mu(r)} < +\infty.$$

*We make the following assumptions:*

1. *the closed target set $S$ satisfies the $\rho$-internal sphere condition;*
2. *$f, g_j \in C^{k,1}(\mathbb{R}^d)$ with $\|D^i f\|_{L^\infty}$, $\|D^i g_j\|_{L^\infty}$ uniformly bounded in $S_{2\delta}$ by a constant $m > 1$ for $j = 0, \ldots, k+1$;*
3. *for every $x \in S_\delta \setminus S$ there exist $1 \le k_x \le k$, $\zeta_x \in \partial^P d_S(x)$, and $Y^x \in \mathscr{L}$ such that $\operatorname{ord} Y^x = k_x$ and*

$$\langle \zeta_x, Y^x(x) \rangle \le -\mu(d_S(x)).$$

*Then $T$ is continuous in $S_\delta$ with the modulus of continuity bounded by*

$$\omega(s) := C\left(\int_0^s \frac{dr}{\mu^k(r)} + \int_0^s \frac{dr}{r^{1-1/k}\mu(r)}\right)$$

*for a suitable constant $C > 0$.*

*Proof.* According to Lemma 4.6, since $Y^x \in \mathscr{L}$ is of order $k_x$, then there exists an $\mathscr{A}$-trajectory $y_x(\cdot)$ such that

$$y_x(t) - \bar{x} - \frac{t^{k_x}}{k_x!} \cdot \frac{k_x!}{\mathrm{pw}^{k_x}} Y^x(x) = o_x(t^{k_x}).$$

By the smoothness and boundedness assumptions on $f, g_i$ we have that $|o(t^{k_x})| \le m^{k+1} t^{k_x+1}$ and $\|Y^{(\bar{x})}(\bar{x})\| \le m^k$. Moreover, since

$$\langle \zeta_x, Y \rangle \le -\mu(d_S(x)),$$

the assumptions of Theorem 5.10 are satisfied (possibly replacing $\mu(\cdot)$ by $\mu(\cdot)/k!$ and setting $L = (m + 1)^{k+1}$). ∎

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The previous results concern closed target sets satisfying the internal sphere condition property with uniform radius. Now we are going to switch to the case when the target set $S$ has positive reach. In this case, we can refine the rough semiconcavity estimate of Proposition 5.5 with the generalized curvature defined in Definition 3.6.

PROPOSITION 5.13. *Consider the control system* (5.1). *Let* $\delta > 0$, $L > 0$, $k \in \mathbb{N} \setminus \{0\}$, $\{v_1^{\bar{x}}, \dots, v_k^{\bar{x}}\} \subseteq \mathbb{R}^d$ *with* $\sum_{i=1}^k |v_i^{\bar{x}}| \le L$. *We assume that*

1. $\operatorname{reach} S > 2\delta$;
2. *for every* $\bar{x} \in S_\delta \setminus S$ *there exists an* $\mathscr{A}$*-trajectory* $y_{\bar{x}}(\cdot)$ *starting from* $\bar{x}$ *such that*

$$\left\| y_{\bar{x}}(t) - \bar{x} - \sum_{i=1}^k \frac{t^i}{i!} v_i^{\bar{x}} \right\| \le L t^{k+1}.$$

*Then for every* $0 \le t \le \min\{\frac{d_S(x)}{2\|f\|_\infty + 1}, 1\}$, *we have*

$$d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) \le \max_{\substack{Z \in \partial\nabla d_S(p) \\ p \in B(\bar{x}, d_S(x)/2)}} \sum_{i=1}^k \tilde{C}_{i,Z}(\bar{x}, v_1^{\bar{x}}, \dots, v_k^{\bar{x}}) t^i + C_\psi t^{k+1}, \tag{5.5}$$

*where* $C_\psi$ *is a constant not depending on* $\bar{x}$, *and*

$$\tilde{C}_{i,Z}(\bar{x}, v_1^{\bar{x}}, \dots, v_k^{\bar{x}}) := \begin{cases} \dfrac{1}{i!}\langle \nabla d_S(\bar{x}), v_i^{\bar{x}} \rangle + \displaystyle\sum_{\substack{j,\, l = 1 \\ j + l = i \\ j < l}}^k \frac{1}{l!j!} \langle \langle Z, v_l^{\bar{x}} \rangle, v_j^{\bar{x}} \rangle + \langle \langle Z, v_j^{\bar{x}} \rangle, v_l^{\bar{x}} \rangle, & i \text{ odd}, \\[2ex] \dfrac{1}{i!}\langle \nabla d_S(\bar{x}), v_i^{\bar{x}} \rangle + \dfrac{\langle \langle Z, v_{i/2}^{\bar{x}} \rangle, v_{i/2}^{\bar{x}} \rangle}{[(i/2)!]^2} \\ \qquad + \displaystyle\sum_{\substack{j,\, l = 1 \\ j + l = i \\ j < l}}^k \frac{1}{l!j!} \left( \langle \langle Z, v_l^{\bar{x}} \rangle, v_j^{\bar{x}} \rangle + \langle \langle Z, v_j^{\bar{x}} \rangle, v_l^{\bar{x}} \rangle \right), & i \text{ even}. \end{cases} \tag{5.6}$$

*Proof.* The proof follows exactly the argument of Proposition 5.5 using the estimate proved in Proposition 3.8. Since for every $0 \le t \le \frac{d_S(x)}{2\|f\|_\infty + 1}$ we have $y_{\bar{x}}(t) \in B(\bar{x}, d_S(\bar{x})/2)$, we then have

$$\begin{aligned} &d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) - \langle \nabla d_S(\bar{x}), y_{\bar{x}}(t) - \bar{x} \rangle \\ &\quad \le \max \mathscr{K}(\bar{x}, y_{\bar{x}}(t)) \\ &\quad \le \frac{1}{2} \max_{\substack{Z \in \partial\nabla d_S(p) \\ p \in B(\bar{x}, d_S(\bar{x})/2)}} \langle \langle Z, y_{\bar{x}}(t) - \bar{x} \rangle, y_{\bar{x}}(t) - \bar{x} \rangle. \end{aligned} \tag{5.7}$$

Set

$$\eta(t) = y_{\bar{x}}(t) - \bar{x} - \sum_{i=1}^k \frac{v_i^{\bar{x}}}{i!} t^i,$$

and recall that $\|\eta(t)\| \le L t^{k+1}$ by assumption.

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Given $Z \in \{\partial\nabla d_S(p) :\ p \in \overline{B(\bar{x}, d_S(\bar{x})/2)}\}$, and performing a computation similar to Proposition 5.5 (recalling that $0 \le t \le 1$ and $\|Z\| \le \mathrm{Lip}(\nabla d_S)$), we have

$$
\begin{aligned}
&\langle\langle Z, y_{\bar{x}}(t) - \bar{x}\rangle, y_{\bar{x}}(t) - \bar{x}\rangle \\
&\quad = \sum_{i=1}^{\lfloor k/2\rfloor} \frac{t^{2i}}{(i!)^2}\langle\langle Z, v_i^{\bar{x}}\rangle, v_i^{\bar{x}}\rangle + 2\sum_{i=1}^{k} \sum_{\substack{l,\, j = 1 \\ l < j \\ l + j = i}}^{k} \frac{t^i}{l!j!}\langle\langle Z, v_l^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle + \sum_{i=1+\lfloor k/2\rfloor}^{k} \frac{t^{2i}}{(i!)^2}\langle\langle Z, v_i^{\bar{x}}\rangle, v_i^{\bar{x}}\rangle \\
&\qquad + 2\sum_{\substack{l,\, j = 1 \\ l < j \\ l + j > k}}^{k} \frac{t^{l+j}}{l!j!}\langle\langle Z, v_l\rangle, v_j\rangle + \langle\langle Z, \eta(t)\rangle, \eta(t)\rangle + 2\sum_{i=1}^{k} \frac{t^i}{i!}\langle\langle Z, \eta(t)\rangle, v_i^{\bar{x}}\rangle \\
&\quad \le \sum_{i=1}^{\lfloor k/2\rfloor} \frac{t^{2i}}{(i!)^2}\langle\langle Z, v_i^{\bar{x}}\rangle, v_i^{\bar{x}}\rangle + 2\sum_{i=1}^{k} \sum_{\substack{l,\, j = 1 \\ l < j \\ l + j = i}}^{k} \frac{t^i}{l!j!}\langle\langle Z, v_l^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle \\
&\qquad + \mathrm{Lip}(\nabla d_S)L\left(L\sum_{i=1+\lfloor k/2\rfloor}^{k} \frac{t^{2i}}{(i!)^2} + 2L\sum_{\substack{l,\, j = 1 \\ l < j \\ l + j > k}}^{k} \frac{t^{l+j}}{l!j!} + Lt^{2k+2} + 2\sum_{i=1}^{k}\frac{t^i}{i!}\right).
\end{aligned}
$$

Recalling that $0 \le t \le 1$, we notice that

$$
\begin{aligned}
&\left\langle \nabla d_S(\bar{x}), \sum_{i=1}^{k} \frac{v_i^{\bar{x}}}{i!}t^i + \eta(t)\right\rangle \\
&\qquad \le \sum_{i=1}^{k} \frac{1}{i!}\langle\nabla d_S(\bar{x}), v_i^{\bar{x}}\rangle t^i + Lt^{k+1},
\end{aligned} \tag{5.8}
$$

$$
\begin{aligned}
&\langle\langle Z, \eta(t)\rangle, \eta(t)\rangle \\
&\qquad \le \|Z\|L^2t^{k+1},
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
&2\sum_{i=1}^{k} \frac{t^i}{i!}\langle\langle Z, v_i^{\bar{x}}\rangle, \eta(t)\rangle \\
&\qquad \le 2\|Z\|Lt^{k+1}\sum_{i=1}^{k}\frac{1}{i!} \le 2e\|Z\|Lt^{k+1},
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
&2\sum_{\substack{i,\, j = 1 \\ i < j}}^{k} \frac{t^{i+j}}{i!j!}\langle\langle Z, v_i^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle \\
&\qquad = 2\sum_{\substack{i,\, j = 1 \\ i < j \\ i + j \le k}}^{k} \frac{t^{i+j}}{i!j!}\langle\langle Z, v_i^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle + 2\sum_{\substack{i,\, j = 1 \\ i < j \\ i + j > k}}^{k} \frac{t^{i+j}}{i!j!}\langle\langle Z, v_i^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle \\
&\qquad \le 2\sum_{\substack{i,\, j = 1 \\ i < j \\ i + j \le k}}^{k} \frac{t^{i+j}}{i!j!}\langle\langle Z, v_i^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle + 2t^{k+1}\|Z\|L^2\sum_{\substack{i,\, j = 1 \\ i < j \\ i + j > k}}^{k} \frac{1}{i!j!} \\
&\qquad \le 2\sum_{\substack{i,\, j = 1 \\ i < j \\ i + j \le k}}^{k} \frac{t^{i+j}}{i!j!}\langle\langle Z, v_i^{\bar{x}}\rangle, v_j^{\bar{x}}\rangle + 2t^{k+1}ke^2\|Z\|L^2.
\end{aligned} \tag{5.11}
$$

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Substituting this estimate into inequality (5.7), and taking into account that

$$\langle \nabla d_S(\bar{x}), y_{\bar{x}}(t) - \bar{x} \rangle \leq \sum_{i=1}^{k} \frac{t^i}{i!} \langle \nabla d_S(\bar{x}), v_i^{\bar{x}} \rangle + Lt^{k+1},$$

we end up with

$$\begin{aligned} d_S(y_{\bar{x}}(t)) - d_S(\bar{x}) \leq \max_{\substack{Z \in \partial \nabla d_S(p) \\ p \in B(\bar{x}, d_S(\bar{x})/2)}} & \sum_{i=1}^{k} \frac{t^i}{i!} \langle \nabla d_S(\bar{x}), v_i^{\bar{x}} \rangle + \sum_{i=1}^{k} \frac{t^{2i}}{(i!)^2} \langle \langle Z, v_i^{\bar{x}} \rangle, v_i^{\bar{x}} \rangle \\ & + \sum_{\substack{i,\, j\, =\, 1 \\ i\, <\, j \\ i\, +\, j\, \leq\, k}}^{k} \frac{t^{i+j}}{i!j!} \langle \langle Z, v_i^{\bar{x}} \rangle, v_j^{\bar{x}} \rangle + C_\psi t^{k+1}. \end{aligned}$$

Rearranging the terms, and setting

$$C_\psi = L + \mathrm{Lip}(\nabla d_S) L (L(e-1) + 2L(e-1)^2 + L + 2L(e-1)),$$

we obtain the desired inequality. Notice the similarity between this expression and the corresponding one obtained in Proposition 5.5. ∎

Finally, we give the second main result of the paper, about controllability via generalized curvature properties.

THEOREM 5.14 (controllability via curvature). *Consider the control system* (5.1). *Let* $\rho, \delta, m > 0$, $k \in \mathbb{N} \setminus \{0\}$ *be constants, and* $\mu : ]0, \delta[ \to [0, 1]$ *be a continuous and nondecreasing function, satisfying* $\mu(p) > 0$ *for any* $p > 0$ *and*

$$\int_0^{d_S(x)} \frac{dr}{\mu^k(r)} + \int_0^{d_S(x)} \frac{dr}{r^{1-1/k}\mu(r)} < +\infty.$$

*We make the following assumptions:*

1. $\mathrm{reach}\, S > 2\delta$;
2. $\|f(x,u)\| + \|\nabla f(x,u)\| \leq m$ *in* $(S_{2\delta} \setminus S) \times U$ *(automatically satisfied if* $S$ *is compact);*
3. *there exists* $L > 0$ *such that for every* $x \in S_\delta \setminus S$ *there exist* $1 \leq k_x \leq k$, $\{v_1, \ldots, v_{k_x}\} \in \mathbb{R}^d$, $\sum_{i=1}^{k_x} |v_i| \leq L$, *and an* $\mathscr{A}$*-trajectory* $y_x(\cdot)$ *starting from* $x$ *satisfying*

$$\left\| y_x(t) - \bar{x} - \sum_{i=1}^{k_x} \frac{t^i}{i!} v_i^x \right\| \leq Lt^{k+1},$$

$$(5.12) \qquad \begin{cases} \tilde{C}_{i,Z}(x, v_1^x, \ldots, v_k^x) \leq 0 \text{ for } i = 1, \ldots, k_x - 1, \\ \tilde{C}_{k_x,Z}(x, v_1^x, \ldots, v_k^x) < -\mu(d_S(x)) \end{cases}$$

*for every* $Z \in \{\partial \nabla d_S(p) : p \in B(x, d_S(x)/2)\}$.

*Then* $T$ *is continuous in* $S_\delta$ *with the modulus of continuity bounded by*

$$\omega(s) := C \left( \int_0^s \frac{dr}{\mu^k(r)} + \int_0^s \frac{dr}{r^{1-1/k}\mu(r)} \right)$$

*for a suitable constant* $C > 0$.

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*Proof.* The proof follows the same argument as Theorem 5.10 replacing Proposition 5.5 and the occurrence of $C_i(\cdot)$, by Proposition 5.13 and $\tilde{C}_{i,Z}(\cdot)$. ∎

*Remark* 5.15. We observe that if $\partial S$ is $C^{1,1}$, and hence satisfies both the internal sphere condition and the positive reach property (see Remark 5.9), we have that $\tilde{C}_i(x) \le C_i(\nabla d_S(x))$. In particular, the controllability condition given in terms of $\tilde{C}_i$ is sharper. A situation where strict inequality holds will be presented in Example 5.22.

We state here a special case of Theorem 5.14.

COROLLARY 5.16. *Consider the control system* (5.1). *Let* $\rho, \delta, m > 0$, $k \in \mathbb{N} \setminus \{0\}$ *be constants, and* $\mu : ]0, \delta[ \to [0, 1]$ *be continuous, nondecreasing, satisfying* $\mu(p) > 0$ *for any* $p > 0$ *and*

$$\int_0^{d_S(x)} \frac{dr}{\mu^k(r)} + \int_0^{d_S(x)} \frac{dr}{r^{1-1/k}\mu(r)} < +\infty.$$

*We make the following assumptions:*

1. $\operatorname{reach} S > 2\delta$*;*
2. $\|f(x,u)\| + \|\nabla f(x,u)\| \le m$ *in* $(S_{2\delta} \setminus S) \times U$ *(automatically satisfied if* $S$ *is compact);*
3. *there exists* $L > 0$ *such for every* $x \in S_\delta \setminus S$ *there exist* $1 \le k_x \le k$, $v^x_{k_x} \in \mathbb{R}^d$, $|v^x_{k_x}| \le L$, *and an* $\mathscr{A}$*-trajectory* $y_x(\cdot)$ *starting from* $x$ *satisfying*

$$\left\| y_x(t) - x - \frac{t^k}{k!} v^x_{k_x} \right\| \le L t^{k+1},$$

$$\mathscr{K}\left(p, p + \frac{d_S(x)}{2m} v^x_{k_x}\right) < \mu(d_S(x)) \tag{5.13}$$

*for every* $p \in B(x, d_S(x)/2)$.

*Then* $T$ *is continuous in* $S_\delta$ *with the modulus of continuity bounded by*

$$\omega(s) := C\left(\int_0^s \frac{dr}{\mu^k(r)} + \int_0^s \frac{dr}{r^{1-1/k}\mu(r)}\right)$$

*for a suitable constant* $C > 0$.

We conclude this section by providing some examples illustrating the results.

*Example* 5.17 (Brockett's nonholonomic integrator). The ground space is $\mathbb{R}^3$. Set

$$g_1(x_1, x_2, x_3) := (1, 0, x_2), \qquad g_2(x_1, x_2, x_3) = (0, 1, -x_1),$$

and consider the (driftless) system $\dot{x}(t) = u_1(t) g_1(x) + u_2(t) g_2(x)$, where $u_1(t), u_2(t) \in [-1, 1]$. It is well known that at every point of $\mathbb{R}^3$ we have

$$\dim\left(\operatorname{span}\{g_1(x), g_2(x), [g_1, g_2](x)\}\right) = 3,$$

and hence the system is fully controllable, according to the classical Chow–Rashevskii's theorem (see, e.g., section 0.4 in [11]). We set $k = 2$, i.e., the min order of Lie bracket needed to generate the whole space. Take $S := \overline{B(0,1)}$ and $\delta = 1$. We notice that $\nabla d_S(x) = \dfrac{x}{\|x\|}$ for every $x \in S_\delta = B(0, 2)$, and so

$$\min_{u_1 \in [-1,1]} \langle u_1 g_1(x), \nabla d_S(x) \rangle = -|x_1 + x_2 x_3|,$$

$$\min_{u_2 \in [-1,1]} \langle u_2 g_2(x), \nabla d_S(x) \rangle = -|x_2 - x_1 x_3|,$$

$$\min_{u_1, u_2 \in [-1,1]} \langle [u_1 g_1, u_2 g_2](x), \nabla d_S(x) \rangle = -|2x_3|.$$

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These three values are identically 0 if and only if $x_1 = x_2 = x_3 = 0$. Consequently in $S_\delta \setminus S$ there exists $\mu > 0$ independent of $x$ such that

$$\min_{Y\in\mathcal{L}}\langle Y(x), \nabla d_S(x)\rangle \leq -\mu.$$

Theorem 5.10 applies, yielding $T(x) \leq C\sqrt{d_S(x)}$, i.e., $T(\cdot)$ turns out to be Hölder continuous with exponent $1/2$.

*Remark* 5.18. We notice that if we take the origin $S' = \{(0,0)\}$ as target set for the system of Example 5.17, given a point $z = (x_1, x_2, x_3)$ the corresponding minimum time function $T'(z)$ is equivalent to the Carnot–Carathéodory distance between $z$ and the origin, while $d_{S'}(z) = |z|$ is the Euclidean distance between $z$ and the origin. Applying the well-known Ball–Box theorem (see, e.g., section 0.5 in [11]), we recover also in this case that there exists $C' > 0$ such that $T'(z) \leq C'\sqrt{|z|}$, and thus the Hölder continuity of $T'(\cdot)$ with exponent $1/2$.

A similar situation is given by the following.

*Example* 5.19. The ground space is $\mathbb{R}^2$. Set $S := \overline{B(0,1)}$, $f(x,y) = 10^{-3}(-y,x)$, $g(x,y) = (-y,2x)$, and consider the system

$$\begin{cases} (\dot{x}(t),\dot{y}(t)) = f(x(t),y(t)) + u(t)g(x(t),y(t)), \\ (x(0),y(0)) = (x_0,y_0), \end{cases}$$

where $u(t) \in [-1,1]$.

We notice that $\langle f, \nabla d_S\rangle = 0$, whence $\langle f + ug, \nabla d_S\rangle = u\langle g, \nabla d_S\rangle$. The scalar product vanishes on the axis, so along these lines Petrov's condition cannot be satisfied.

Set $\bar{z} = (\bar{x},\bar{y})$ and $X_1(z) = f(z) + u_1 g(z)$, $X_2(z) = f(z) - u_1 g(z)$, $X_3(z) = f(z)+u_2 g(z)$, where $u_1 := -\mathrm{sign}(-\bar{x}^2+\bar{y}^2) \in \{0,\pm1\}$, and $u_2 := -\mathrm{sign}(\bar{x}\bar{y}) \in \{0,\pm1\}$. We consider the $\mathscr{A}$-trajectories $y_z^{(1)}(t) := \phi_2(t/2, \phi_1(t/2,z))$ and $y_z^{(2)}(t) = \phi_1(t,z)$ starting from $z$. According to Lemma 4.9, we have

$$\begin{cases} y_{\bar{z}}^{(1)}(t) = \bar{z} + \dfrac{t}{4}\cdot f(\bar{z}) + \dfrac{t^2}{4}\left(2\nabla f(\bar{z})f(\bar{z}) + u_1[f,g](\bar{z})\right) + o_1(t^2), \\ y_{\bar{z}}^{(2)}(t) = \bar{z} + (f(\bar{z}) + u_2 g(\bar{z}))t + o_2(t^2). \end{cases} \tag{5.14}$$

We have also $h(z) := [f,g](z) = 10^{-3}(-x,y)$.

Recalling that a semiconcavity constant $K$ for the unit ball is equal to 1, that $\|\nabla d_S(z)\| \leq 1$, $\|f(z)\| \leq 10^{-3}|z|$, and $\|\nabla f(z)\| \leq 1$, with the notation of Theorem 5.10, we have

$$\begin{aligned}
C_1^{(1)}(\nabla d_S(z)) &= \langle\nabla d_S(z), f(z)/4\rangle = 0,\\
C_2^{(1)}(\nabla d_S(z)) &= \frac{1}{2}\langle\nabla d_S(z), \nabla f(z)f(z) + \frac{u_1}{2}[f,g](z)\rangle + \frac{1}{16}\|f(z)\|^2\\
&\leq \frac{1}{2}10^{-6}|z| + \frac{10^{-6}|z|^2}{16} + \frac{1}{4}\langle\nabla d_S(z), u_1[f,g](z)\rangle\\
&= \frac{10^{-6}\sqrt{x^2+y^2}}{2}\left(1 + \frac{\sqrt{x^2+y^2}}{8}\right) - \frac{10^{-3}}{4}\frac{\left|-x^2+y^2\right|}{\sqrt{x^2+y^2}},\\
C_1^{(2)}(\nabla d_S(z)) &= u_2\langle\nabla d_S(z), g(z)\rangle = -|xy|,
\end{aligned}$$

where $C_i^{(h)}(\nabla d_S(z))$, $h = 1,2$, are the coefficients $C_i(\nabla d_S(z))$ computed, respectively, for the expansions of $y_z^{(h)}(\cdot)$, $h = 1,2$.

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Consider now a point $z \in \mathbb{R}^2$ such that $1 \le |z| \le 8$. In polar coordinates, we have $z = (\rho\cos\theta, \rho\sin\theta)$, with $1 \le \rho \le 8$ and

$$\begin{cases} C_1^{(1)}(\nabla d_S(z)) = 0, \\ C_2^{(1)}(\nabla d_S(z)) \le 8 \cdot 10^{-6} - \dfrac{10^{-3}}{4}|\cos 2\theta|, \\ C_1^{(2)}(\nabla d_S(z)) \le -\dfrac{1}{2}|\sin 2\theta|. \end{cases}$$

We can always choose $y_z^1(t)$ or $y_z^2(t)$ in such a way that

$$\max\left\{C_1^{(1)}(z), C_1^{(2)}(z)\right\} \le 0.$$

Moreover, we have

$$\begin{aligned} \min\{C_2^{(1)}, C_1^{(2)}\} &\le 8 \cdot 10^{-6} - \frac{10^{-3}}{4}\max\{|\sin 2\theta|, |\cos 2\theta|\} \\ &= 10^{-3}\left(8 \cdot 10^{-3} - \frac{\sqrt{2}}{8}\right) < -10^{-4}. \end{aligned}$$

So if we take $\mu(\cdot) \equiv 10^{-4}$, the assumptions of Theorem 5.10 are satisfied with $k = 2$. Thus $T(\cdot)$ is 1/2-Hölder continuous.

A more complex situation is described in the following

*Example* 5.20. The ground space is $\mathbb{R}^2$. Set $S := \overline{B(0,1)}$,

$$f(x,y) = 10^{-3}\sqrt[16]{x^2+y^2-1}(-y,x), \qquad g(x,y) = \sqrt[16]{x^2+y^2-1}(1,1),$$

and consider the system

$$\begin{cases} (\dot{x}(t), \dot{y}(t)) = f(x(t),y(t)) + u(t)g(x(t),y(t)), \\ (x(0),y(0)) = (x_0,y_0), \end{cases}$$

where $u(t) \in [-1,1]$.

We notice again that $\langle f, \nabla d_S\rangle = 0$ and hence $\langle f + ug, \nabla d_S\rangle = u\langle g, \nabla d_S\rangle$. Thus $\langle g, \nabla d_S\rangle$ vanishes on the line $x + y = 0$ and so Petrov's condition cannot be satisfied at these points.

We proceed in the same way as Example 5.19, defining $y_z^{(1)}(t)$ and $y_z^{(2)}(t)$ as in (5.14). When we consider higher order terms, we have to compute

$$\begin{aligned} h(x,y) := [f,g](x,y) &= \frac{10^{-3}}{8}\left(\frac{-8x^2 - xy - 9y^2 + 8}{(x^2+y^2-1)^{7/8}}, \frac{9x^2 + xy + 8y^2 - 8}{(x^2+y^2-1)^{7/8}}\right), \\ \langle h(x,y), \nabla d_S(x,y)\rangle &= -10^{-3}\frac{(y-x)\sqrt[8]{x^2+y^2-1}}{\sqrt{x^2+y^2}}, \\ \langle \nabla f(x) f(x), \nabla d_S(x)\rangle &\le 10^{-6}\sqrt[8]{x^2+y^2-1}\sqrt{x^2+y^2}. \end{aligned}$$

If we consider points $z = (x,y)$ with $1 \le |z| \le 2$, recalling that $\|f(z)\| \le 10^{-2} d_S^{1/16}(x,y)$,

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and that $0 \le d_S(z) \le 1$, we have

$$\begin{cases} C_1^{(1)}(\nabla d_S(z)) = 0, \\ C_2^{(1)}(\nabla d_S(z)) \le \dfrac{10^{-5}}{2} d_S^{1/8}(x,y) + \dfrac{1}{16} \cdot 10^{-4} d_S^{1/8}(x,y) - \dfrac{1}{4} 10^{-3} |y-x| d_S^{1/8}(x,y), \\ \qquad\qquad\qquad \le 10^{-3} \left( \dfrac{3}{2} \cdot 10^{-2} - \dfrac{|y-x|}{4} \right) d_S^{1/8}(x,y), \\ C_1^{(2)}(\nabla d_S(z)) \le -d_S^{1/16}(x,y)|x+y| \le -d_S^{1/8}(x,y)|x+y|. \end{cases}$$

As in Example 5.19, we can always choose $y_z^1(t)$ or $y_z^2(t)$ in such a way that

$$\max\left\{ C_1^{(1)}(z), C_1^{(2)}(z) \right\} \le 0,$$

and there exists $c > 0$ such that we have also

$$\min\{C_2^{(1)}, C_1^{(2)}\} \le -\frac{10^{-3}}{4} d_S^{1/8}(x,y) \left( \frac{\sqrt{2}}{2} - \frac{3}{2} \cdot 10^{-2} \right) \le -c d_S^{1/8}(x,y).$$

So if we take $k = 2$, $\mu(r) = cr^{1/8}$, the assumptions of Theorem 5.10 are satisfied, yielding that $T(x) \le C d_S^{3/8}(x)$ (see Remark 5.11 with $\alpha = 1/8$).

We discuss now the example marking the difference between our results and the results in [16] and [17].

*Example* 5.21. The ground space is $\mathbb{R}$. Set $S = ]-\infty, 0]$, thus it satisfies the internal sphere condition and has compact boundary. Define $f(x,u) = \frac{u}{\log x}$ for $0 < x < 1/2$, $u \in [-1,1]$, $f(x) = 0$ for $x \le 0$. We have that $f \in C^{1,1}_{\mathrm{loc}}(S_{1/2} \setminus S) \times [-1,1])$ and without loss of generality, we can extend it to a function $C^{1,1}_{\mathrm{loc}}((\mathbb{R} \setminus S) \times [-1,1])$.

Clearly, for any $0 < \bar{x} < 1/2$ the optimal control corresponds to $u(t) \equiv 1$. We consider the optimal solution $\gamma_{\bar{x}}(t) = \bar{x} + \frac{t}{\log \bar{x}} + o(t)$ corresponding to this choice of control. We notice that if we take $y_{\bar{x}}(t) = \bar{x} + \frac{t}{2\log \bar{x}}$, we have that $\bar{x} > y_{\bar{x}}(t) > \gamma_{\bar{x}}(t)$, thus $y_{\bar{x}}(\cdot)$ is an $\mathscr{A}$-trajectory. We choose $v_1^x = \frac{t}{2 \log \bar{x}}$. Recalling that $d_S(x) = x$ for any $0 < x < 1/2$, we have $\langle \nabla d_S(x), v_1 \rangle \le \mu(d_S(x))$, where $\mu(s) = \frac{1}{2\log s}$. The assumptions of Theorem 5.10 are satisfied, providing the estimate $T(x) \le C(x - x\log x)$ for some $C > 0$ (we recall that $d_S(x) = x$ for $x > 0$). Indeed, we can compute exactly the minimum time function in this case, which turns out to be $T(x) = x - x\log x$.

To compare our result with Theorem 3.1 of [16], consider now any $\mathscr{A}$-trajectory $\sigma_{\bar{x}}(\cdot)$ starting from $\bar{x}$ of the form

$$\sigma_{\bar{x}}(t) = \bar{x} + a(t,\bar{x}) + t^\alpha A(\bar{x}) + o(t^\alpha, \bar{x}), \tag{5.15}$$

where $A(\cdot)$ is a Lipschitz continuous map, $\|a(t,x)\| \le t^s c(x)$ for $s > 0$, and a Lipschitz map $c(\cdot)$ satisfying $c(x) \to 0$ when $d_S(x) \to 0$. Since for small $t$ we have $\mathscr{R}_{\bar{x}}(t) - \bar{x} \subseteq [2t/\log \bar{x}, -2t/\log \bar{x}]$, we have

$$|a(t,\bar{x}) + t^\alpha A(\bar{x}) + o(t^\alpha, \bar{x})| \le \frac{2t}{|\log \bar{x}|}.$$

If we pass to the limit for $\bar{x} \to 0$, divide by $t^\alpha$, and let $t \to 0^+$, we have that $A(0) = 0$. We conclude that Assumption A3 of Theorem 3.1 of [16], requiring the existence of a

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Lipschitz function $b(\cdot)$ such that

$$\max_{\substack{z \in B(0,r) \setminus S \\ \xi \in \pi_S(x) = \{0\}}} \left\langle \frac{x-\xi}{\|x-\xi\|}, A(x) \right\rangle \leq b(z) < 0,$$

is violated, because the right-hand side is greater than or equal to 0, thus the main result in [16] cannot be applied. Theorem 3.4 of [16], based on the same techniques as Theorem 3.1 of [16] and dealing with control-affine systems, requires the vector fields to be analytic and the target to be compact, but does not assume that the target $S$ satisfies any internal sphere condition or positive reach property. So this result seems to be not directly comparable with ours in general. However, in this case, it cannot be applied, too.

In [17] the results of [16] were extended allowing $\langle x - \pi_S(x), A(x)\rangle$ to vanish as $d_S(x) \to 0$. More precisely, Theorem 3.1, which is the main result of [17], requires the existence of $0 \leq \lambda < \frac{2\alpha}{2\alpha-1}$ for the $\mathscr{A}$-trajectory $\sigma_{\bar{x}}(\cdot)$ defined in (5.15) such that

$$\langle x - \pi_S(x), A(x)\rangle \leq -\delta d_S^\lambda(x).$$

Since $A(0) = 0$, by the Lipschitz continuity of $A(\cdot)$ we have $|A(x)| \leq C|x|$, and this implies $\lambda \geq 2$. However, since it is assumed $\alpha \geq 1$, we have $\frac{2\alpha}{2\alpha-1} \leq 2$, thus even Theorem 3.1 in [17] cannot be used.

Finally, we provide an example where the negative curvature of the target plays a distinguished role, thus indicating the difference between Theorem 5.10 and Theorem 5.14.

*Example* 5.22. The ground space is $\mathbb{R}^2$. Set $S := \mathbb{R}^2 \setminus B(0,1)$, $f(x,y) = (0, x^2+y^2)$, $g(x,y) = (0, 2(x^2+y^2))$. Notice that $[f,g] = 0$. We notice that $x(t)$ is constant for every admissible trajectory, so no horizontal shifting is allowed. This implies that for *every* $\mathscr{A}$-trajectory we have that the first component of *any* $v_{(x,y)}^{(i)}$ appearing in item (4) of Theorem 5.10 vanishes. In particular, we notice that $\langle f + ug, \nabla d_S(x,y)\rangle$ vanishes on $]-1,1[\times\{0\}$ for every $u \in [-1,1]$, so Petrov's condition cannot be satisfied on this line. More precisely, we have that

$$\min_{u\in[-1,1]} \langle f + ug, \nabla d_S(x,y)\rangle = -\max_{u\in[-1,1]} y(1+2u)\sqrt{x^2+y^2} \leq -|y|\sqrt{x^2+y^2} \leq 0.$$

Moreover, on this line we have $\nabla d_S(x,0) = (\operatorname{sign} x, 0)$ for every $x \neq 0$, and hence

$$\langle v_{(x,0)}^{(i)}, \nabla d_S(x,0)\rangle = 0.$$

In particular, we have $C_1(\nabla d_S(x,0)) = 0$ and $C_2(\nabla d_S(x,0)) \geq 0$, so the higher order conditions required by Theorem 5.10 also fail.

Fix $0 < \varepsilon < 1$, consider

$$U_1^\varepsilon := \left\{(x,y) \in B(0,1) : \, d_S(x,y) \leq \frac{1}{3}, \, |y| > \frac{\varepsilon}{3}\right\},$$
$$U_2^\varepsilon := \left\{(x,y) \in B(0,1) : \, d_S(x,y) \leq \frac{1}{3}, \, |y| \leq \frac{\varepsilon}{3}\right\},$$

and observe that in $U_1^\varepsilon$ we have

$$\min_{u\in[-1,1]} \langle f + ug, \nabla d_S(x,y)\rangle \leq -\frac{\varepsilon}{9}.$$

We have that $(U_1^\varepsilon \cup U_2^\varepsilon) \setminus S = S_{1/3}$.

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Indeed, we have that $d_S$ is concave on $\mathbb{R}^2 \setminus S$, and by direct calculation we have that

$$\operatorname{Hess} d_S(x,y) = \frac{1}{|(x,y)|^3}\begin{pmatrix} -y^2 & xy \\ xy & -x^2 \end{pmatrix}$$

is smooth and semidefinite negative on $B(0,1)\setminus\{0\} = \mathbb{R}^2 \setminus (S \cup \{0\})$.

Given $z := (x,y), z' := (x',y') \in B(0,1)\setminus\{0\}$ such that $(0,0) \notin \operatorname{co}\{z,z'\}$ we have

$$\begin{aligned}\mathscr{K}(z,z') &= \frac{1}{2}\operatorname{co}\{\langle\langle \operatorname{Hess} d_S(z+\lambda(z-z')), z-z'\rangle, z-z'\rangle, \lambda\in[0,1]\} \\ &= \frac{1}{2}\Big\{\langle\langle Z, z-z'\rangle, z-z'\rangle,\, Z \in \operatorname{co}\{\operatorname{Hess} d_S(z+\lambda(z-z')), \lambda\in[0,1]\}\Big\}\end{aligned}$$

since, according to the smoothness of $d_S(\cdot)$ in $B(0,1)\setminus\{0\}$, we have that the Clarke generalized gradient $\partial\nabla d_S(z+\lambda(z-z'))$ reduces to $\operatorname{Hess} d_S(z+\lambda(z-z'))$ at all $\lambda\in[0,1]$.

If $z'$ can be reached from $z$ in time $t$ by an admissible trajectory, we must have $x = x'$, and if $(0,0)\notin\operatorname{co}\{z,z'\}$, we have

$$\mathscr{K}(z,z') = \frac{1}{2}\operatorname{co}\left\{-\frac{x^2(y'-y)^2}{|(x,y+\lambda(y'-y)|^{3/2}}, \lambda\in[0,1]\right\}.$$

In particular, given $z \in U_2^\varepsilon$ and $z' \in B(z, d_S(z)/2)\cap\mathscr{R}_z(t)$ such that $(0,0)\notin\operatorname{co}\{z,z'\}$, we have that

$$\max\mathscr{K}(z,z') \le -x^2(y'-y)^2 \le -\frac{4-\varepsilon^2}{9}(y'-y)^2,$$

since on $U_2^\varepsilon$ we have $x^2 \ge (4-\varepsilon^2)/9$ and $|z+\lambda(z'-z)|\le 1$.

Take now the admissible trajectory $\gamma$ of the system satisfying $\gamma(0,0) = \bar z := (\bar x,\bar y)$ obtained by using $u(t)\equiv 1$ if $\bar y \ge 0$, and $u(t)\equiv -1$ if $\bar y < 0$.

In this case, we have

$$v_1^{(\bar x,\bar y)} = f(\bar x,\bar y) + ug(\bar x,\bar y) = \begin{cases} 3(0, x^2+y^2) & \text{if } y\ge 0, \\ -(0,\bar x^2+\bar y^2) & \text{if } \bar y < 0,\end{cases}$$

and

$$v_2^{(\bar x,\bar y)} = \nabla(f(\bar x,\bar y)+ug(\bar x,\bar y))\cdot(f(\bar x,\bar y)+ug(\bar x,\bar y)) = \begin{cases} 6\bar y(0,\bar x^2+\bar y^2) & \text{if } \bar y\ge 0, \\ -2\bar y(0,\bar x^2+\bar y^2) & \text{if } \bar y<0.\end{cases}$$

Accordingly, if $(\bar x,\bar y)\in U_2^\varepsilon$, for every $p=(p_x,p_y)\in B((\bar x,\bar y), \frac{d_S(\bar x,\bar y)}{2})$ we have

$$\tilde C_{1,D^2 d_S(p)}(x,y,v_1^{(x,y)}, v_2^{(x,y)}) = \langle\nabla d_S(\bar x,\bar y), v_1^{(\bar x,\bar y)}\rangle \le 0.$$

We have also

$$\langle\nabla d_S(\bar x,\bar y), v_2^{(\bar x,\bar y)}\rangle \le 0,$$

which yields

$$\begin{aligned}\tilde C_{2,D^2 d_S}(x,y,v_1^{(\bar x,\bar y)}, v_2^{(\bar x,\bar y)}) &\le \langle\langle D^2 d_S(p), v_1^{(x,y)}\rangle, v_1^{(x,y)}\rangle \\ &\le -p_x^2(x^2+y^2) \le -p_x^2\frac{4-\varepsilon^2}{9} \le -\frac{4-\varepsilon^2}{81},\end{aligned}$$

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recalling that $|p_x| \geq 1/2$, since

$$1 - d_S(\bar{z}) = |\bar{z}| \leq |p - \bar{z}| + |p| \leq \frac{d_S(\bar{z})}{2} + |p|.$$

Hence

$$1 - \frac{3}{2} d_S(\bar{z}) \leq |p|.$$

Taking $\varepsilon = 1/3$, we obtain that on $S_{1/3}$ the assumptions of Theorem 5.14 are satisfied with $k = 2$ and $\mu = -1/27$. This yields Hölder continuity of $T(\cdot)$ with exponent $1/2$.

**6. Conclusions.** We provided some controllability results both for general control systems and for affine control systems. These results rely on some estimates on the distance function, which depend on the smoothness of the target, together with structural assumptions on the dynamics, in a sort of interplay between the smoothness of the target and the speed of approach, linking them also to the modulus of continuity of the minimum time function $T(\cdot)$ in a neighborhood of the target.

The stated higher order Petrov's condition generalizes the first order Petrov's condition and the results of [18], extending controllability conditions for this class of nonlinear systems to quite general target sets, and do not require the target to be an equilibrium point for the system, as in many known results. In particular, we are able to cover a broad class of affine systems with nontrivial drift, thus extending the result of [1].

The role of (generalized) curvature can be crucial in some cases to have controllability, helping to improve the rough semiconcavity estimate of the distance function. It would be very interesting to substitute the pointwise formula provided with an integral estimate on curvature, in the spirit of the generalized Steiner formula proved by Federer in [10] which involves the generalized curvatures as *measures*. This would lead to conditions for controllability in a *measure theoretic generic* sense, and no longer pointwise.

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