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BACKWARD STOCHASTIC VOLTERRA INTEGRAL EQUATION APPROACH TO STOCHASTIC DIFFERENTIAL UTILITY

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Abstract: In the present paper we find the solution for the stochastic differential utility problem introduced by [2] using a backward stochastic Volterra integral differential approach. In particular we generalize results already obtained in literature passing from global to local Lipschitz assumption for the drift component.

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1. Introduction

In [2] Duffie and Epstein introduced the concept of *stochastic differential utility* (SDU) in the form

$$Y(t) = \mathbb{E}\left[\int_t^T e^{-r(s-t)} g(s, Y(s))ds + e^{-r(T-t)}\xi \mid \mathscr{F}_t\right], \quad t \in [0, T], \tag{1}$$

where r is a positive parameter called *discount rate*, $T > 0$ is considered as the *maturity time* or *expiring date* for a certain tipe of financial investment, ξ is a given *terminal condition*, namely $Y(T) = \xi$, which turns to be a random variable with respect to $\mathscr{F}_T$, where $\{\mathscr{F}\}_{t\in[0,T]}$ is the $\sigma - algebra$ containing all the informations, we shall give more details later, for the process $\{Y(t)\}_{t\in[0,T]}$ up to terminal time T. The main idea behind previous approach is based on an extension of the recursive utility method in continuous time. Indeed, in the discrete case, the problem is to find

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 1 (2014)

a suitable utility function which satisfies a certain recursive relation with respect to a given consumption plan $C = \{C(t)\}_{t\in I}$, where $I := \{t_1, t_2, \ldots\}$ is a given, countable, set of times.

In the continuous case $C = \{C(t)\}_{t\in\mathbb{R}^+}$ and we define the associated value process $Y(t) := U(\{C(s) : s \geq t\})$ with respect to the utility function U, so that $U(C(T)) =: \xi$ represents the so called *utility of terminal consumption*. Note that previous problem, i.e. eq. (1), is equivalent to the following Backward Stochastic Differential Equation (BSDE)

$$\begin{cases} dY(t) = [-rY(t) + g(t, Y(t))]\, dt - Z(t)dW(t), \quad t \in [0, T], \\ Y(T) = \xi, \end{cases} \tag{2}$$

where a second, $\mathscr{F}_t - adapted$, stochastic process Z appears, being the second unknown for the following nonlinear BSDE-type problem, first introduced by Pardoux and Peng, see [6]

$$\begin{cases} dY(t) = -g(t, Y(t), Z(t))dt + Z(t)dW(t), \quad t \in [0, T], \\ Y(T) = \xi, \end{cases} \tag{3}$$

hence (2) is a specialization of (3). In particular one can show that, if the function g is Lipschitz w.r.t. its second and third argument, then there exists a unique, adapted, solution which is in fact a couple of stochastic processes, (Y, Z), moreover we refer to, e.g., [4], for results concerning the study of (3) with non Lipschitz coefficients. BSDE-theory is strictly related to the theory of Backward Stochastic Volterra Integral Equation (BSVIE), see, e.g., [5]. As an example, equation (2) can be reinterpreted in the following BSVIE-form

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T], \tag{4}$$

which turns to be a particular case of the following nonlinear BSVIE for $t \in [0, T]$

$$Y(t) = \xi + \int_t^T g(t, s, Y(s), Z(t, s))ds + \int_t^T [h(t, s, Y(s)) + Z(t, s)]\, dW(s), \tag{5}$$

for which Lin in [3] gave an existence and uniqueness result under global Lipschitz condition on the drift.

Our aim is to exploit latter interpretation and recent results developed in [1] for BSVIE under local Lipschitz condition on the drift component, to determine the existence and uniqueness for the solution of the associated *Stochastic Differential Utility* (SDU) problem.

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2. BSVIE Formulation for the Stochastic Differential Utility Problem

Let $\mathscr{S} := \left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space, where $0 < T < +\infty$ is considered as the *maturity time* of our *consumption-utility plan*, and $\{W\}_{t\in[0,T]}$ is a one-dimensional Brownian motion determining the filtration $\mathscr{F}_t = \{\mathscr{F}\}_{t\in[0,T]}$. The stochastic approach to the recursive utility problem in continuous, but finite, time $[0,T]$, where $C = \{C(t)\}_{t\in\mathbb{R}^+}$ is a consumption plan, U is a utility function, and $Y(t) := U(\{C(s) : s \geq t\})$ is the associated (stochastic) value process, adapted to $\mathscr{F}_t$, such that $U(C(T)) =: \xi$ represents the *utility of terminal consumption*, leads to the SDU solution given in [2]. In particular we are left to consider the following stochastic differential equation in integral form

$$Y(t) = \xi + \int_t^T g(C(s), Y(S))ds, \quad t \in [0,T], \tag{6}$$

where C is adapted to $\mathscr{F}_t$, hence Y turns out to be a uniformly integrable martingale, with respect to with respect to $\mathscr{F}_t$, with $Y(0) = 0$, so that, by the *Martingale representation theorem*, there exists an adapted stochastic process $Z = Z(t)$, $t \in [0,T]$, such that

$$Y(t) = U(C(T)) + \int_t^T g(C(s), Y(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0,T], \tag{7}$$

analogous of equation (1) given in Sec.1. Moreover the BSDE (7) is equivalent to the BSVIE (4), special case of the nonlinear BSVIE (5), both given in Sec. 1. Hence the SDU problem can be solved finding a couple of adapted stochastic processes (Y, Z) solution of a properly simplified equation of the type (5). In particular, in what follows we shall assume that ξ be a $\mathscr{F}_T$-measurable, square integrable random variable and that the drift g satisfies the following conditions $\forall N \in \mathbb{N}^+$, $\mid y \mid \leq N$, $\mid y' \mid \leq N$, $z, z' \in \mathbb{R}$, and a.e. $s \in [0,T]$

(C1) $\mid g(s,y,z) \mid \leq K(1+ \mid y \mid + \mid z \mid)^\alpha, \mathbb{P} - a.s$,

(C2) $\exists\ K > 0$, $\alpha \in [0,1)$ and $L_N > 0$, such that

$$\mid g(s,y,z) - g(s,y',z' \mid \leq L_N \mid y - y' \mid + K \mid z - z' \mid \ .$$

Moreover we set the following definition

Definition 1. A solution for the SDU problem in BSVIE form (4), with drift g and final condition given by the *utility of terminal consumption* $C(U(T)) =: \xi$, is a pair of square integrable (w.r.t. $\mathbb{P} \times \mathscr{L}$, $\mathscr{L}$ being the usual Lebesgue measure on $\mathbb{R}$), $\mathscr{F}_t-$adapted processes $(Y, Z) = \{Y(t), Z(t)\}_{t\in[0,T]}$ which solve (4).

With previous assumptions, the following result holds, see [1] for details,

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 1 (2014)

Theorem 2. *If assumptions (C1) and (C2) are satisfied, and*

$$\lim_{N\to+\infty} \frac{e^{(2L_N+2L_N^2)T}}{(2L_N+2L_N^2)N^{2(1-\alpha)}} = 0\,,$$

then there exists a unique solution to the SDU problem defined by eq.(1)*, analogously by eq.* (4)*.*

Proof. Concerning the uniqueness we suppose that two different solutions, in the sense given before, (Y,Z) and (Y',Z') exists, then we define $\Delta Y(s) := Y(s) - Y'(s)$, $\Delta Z(s) = Z(s) - Z'(s)$ and $\Delta g(s) := g(s,Y(s),Z(s)) - g(s,Y'(s),Z'(s))$, hence we have

$$\Delta Y(t) + \int_t^T \Delta g(s)ds + \int_t^T \Delta Z(s)dW(s) = 0\,, \tag{8}$$

and by Lemma 2.1 in [3] we have to estimate the following expectation

$$\mathbb{E} \mid \Delta Y(s) \mid^2 + \mathbb{E}\int_t^T \mid \Delta Z(s) \mid^2 ds = \sum_{i=1}^5 I_i\,, \tag{9}$$

where the (integral) terms $I_i, i = 1,\dots,5$, can be controlled by the estimating procedure given in [1], but requiring just the assumptions (C1) and (C2).

The proof for the existence relies on the same concepts, once we have defined, for every $m \le n \in \mathbb{N}^+$, the processes Y_m, Z_m, Y_n, Z_n following the definitions given in Lemma 3.1 and Lemma 3.2 of [1]. In particular we have the following expression

$$\mathbb{E} \mid Y_n(s) - Y_m(s) \mid^2 + \mathbb{E}\int_t^T \mid Z_n(s) - Z_m(s) \mid^2 ds\,, \tag{10}$$

which can be estimated using Lemma 3.2 of [1], together with standard application of both Hölder and Chebychev inequalities, concluding the proof. □

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