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FIRST ORDER CORRECTION FOR THE CHARACTERISTIC FUNCTION OF A MULTIDIMENSIONAL AND MULTISCALE STOCHASTIC VOLATILITY MODEL

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Abstract: The present work generalizes the results obtained in [3] to a $d > 1$ dimensional setting. In particular we give the first order asymptotic correction for the characteristic function of the log-return of a multidimensional asset price process whose volatility is driven by two diffusion processes on two different time scales. We consider a fast mean reverting process with reverting scale $\frac{1}{\epsilon}$ and a slow mean reverting process with scale δ, and we perform the expansion for the associated characteristic function, at maturity time $T > 0$, in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$. Latter result, according, e.g., to [2, 4, 9, 12], can be exploited to numerically analyze the fair price of a structured option written on $d > 1$ assets.

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1. Introduction

We consider a multi-scale market model in dimension $d > 1$ which is character-

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ized by two stochastic volatility factors acting on different time scales and driven by two one dimensional diffusion process. Such type of stochastic model has been first studied by Kabanov and Pergamenshchikov, see [10, Ch.4], starting from the Vasil'eva theorem, and then extensively applied to financial markets by Fouque et.al. in [5, 7]. The present work extends [3] to the multidimensional scenario. In particular we focus our attention on the following class of multi-scale, multidimensional, stochastic volatility models defined by:

$$\begin{cases} d\mathbf{S}_t = r\mathbf{S}_t dt + \sigma(Y_t, Z_t)\mathbf{S}_t d\bar{\mathbf{B}}_t \\ dY_t = \frac{1}{\epsilon} b(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\nu(Y_t)d\bar{W}_t^y \\ dZ_t = \delta c(Z_t)dt + \sqrt{\delta}\mu(Z_t)d\bar{W}_t^z \end{cases}, \tag{1}$$

where $t \in [0, T]$, $T > 0$ being the expiration time of our investment, Y_t is a one-dimensional fast mean reversion process, Z_t is a one-dimensional slow mean reversion process, see [5, 7] for details, and $\mathbf{S}_t := \left(S_t^1, \ldots, S_t^d\right)^T$ is the $d-$dimensional, real valued stochastic process which takes into account the time behaviour of a set of $d > 1$ underlying assets. The parameter ϵ, resp. δ, describes the fast, resp. slow, time scale of fluctuations for the diffusion Y_t, resp. for the diffusion Z_t. The stochastic process $\bar{\mathbf{B}}_t := \left(\bar{B}_t^1, \ldots, \bar{B}_t^d\right)^T$ is a d-dimensional Brownian motion, under the *real world* probability measure $\mathbb{P}$, and it is characterized by a $d \times d$ positive definite correlation matrix $\tilde{\rho} := (\tilde{\rho}_{ij})_{0 \le i,j \le d}$ with $\tilde{\rho}_{ij} := d\left\langle \bar{B}^i, \bar{B}^j \right\rangle_t$. Further we assume that the *instantaneous interest rate* r is a positive constant, while the functions $\sigma : \mathbb{R}^2 \to \mathbb{R}^d$, b, ν, c and μ, are assumed to be sufficiently smooth to apply the *Feynman-Kac* theorem. For the financial validity of such an approach see, e.g., [3] and references therein.

2. Probabilistic and Financial Setting

The probabilistic setting for the system in (1) is given by $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P}\right)$, where $(\mathcal{F}_t)_{t \ge 0}$ is the natural filtration generated by the $d + 2$ stochastic process $\mathcal{S}_t = (\mathbf{S}_t, Y_t, Z_t)$, while $\mathbb{P}$ represents the so called associated *real world* measure.

Standard financial assumptions, see, e.g., [5, Sect.4.1] and [13, Sect.5.4], allow us to deal with an arbitrage free market, hence there exists, at least, one *risk-free* probability measure $\mathbb{Q}$. Therefore applying the Girsanov theorem and the Itô-Doeblin lemma for the stochastic process $\mathbf{X}_t = \left(X_t^1, \ldots X_t^d\right)^T$, where

$X^i_t := \log S^i_t$, $i = 1, \dots, d$, we get

$$\begin{cases} d\mathbf{X}_t = \left(r - \frac{1}{2}\sigma^2\right) dt + f(Y_t, Z_t) d\mathbf{B}_t \\ dY_t = \left[\frac{1}{\epsilon} b\left(Y_t\right) - \frac{1}{\sqrt{\epsilon}} \nu\left(Y_t\right) \Lambda_1\left(Y_t, Z_t\right)\right] dt + \frac{1}{\sqrt{\epsilon}} \nu\left(Y_t\right) dW^y_t \\ dZ_t = \left[\delta c\left(Z_t\right) - \sqrt{\delta}\mu\left(Z_t\right) \Lambda_2\left(Y_t, Z_t\right)\right] dt + \sqrt{\delta}\mu\left(Z_t\right) dW^z_t \end{cases}, \tag{2}$$

where Λ_1 and Λ_2 are the combined market prices of volatility risk determining the risk-neutral pricing measure $\mathbb{Q}$, see, e.g., [5, Ch.2], and the Brownian motions' correlation matrix $\rho := (\rho_{ij})_{0 \le i,j \le d}$ is given by $\rho_{ij} := d\left\langle B^i, B^j\right\rangle_t$. Note that since ρ is a symmetric, positive definite, hence nonsingular, matrix, then, e.g. by Cholesky decomposition, we can always think the asset-process $\mathbf{S}_t$ to be driven by a d uncorrelated Brownian motions. In order to have the well-posedness of the parabolic problems we will deal with, see, e.g. problem (5) below, with a slight abuse of notation, we define $t := T - t$ to be the *backward time*. Therefore t will denote the *time to maturity*, thus $t = 0$ will be the *maturity time*. Moreover we assume the process Y_t to be mean-reverting and to admit a unique invariant distribution in the following sense

Definition 2.1 (Invariant distribution)**.** Let $Y := \{Y_t : t \in [0, T]\}$ be a given stochastic process, then an initial distribution Φ for Y_0 is an *invariant distribution* for Y iff

$$\frac{d}{dt}\mathbb{E}\left[g\left(Y_t\right)\right] = \frac{d}{dt}\int \mathbb{E}\left[g\left(Y_t\right) \middle| Y_0 = y\right] \Phi\left(dy\right) = 0 \qquad \text{for any } g \in C_b(\mathbb{R}), \forall t > 0\,. \tag{3}$$

Previous assumptions are not financially restrictive, indeed they hold if the stochastic volatility factors are driven by Ornstein-Uhlenbeck (OU) processes, as e.g. in the Cox-Ingersoll-Ross (CIR) setting.

Let Φ be the invariant distribution of the stochastic process Y in (2), then we define

$$L^2_Y := \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |f(y)|^2 \Phi(y) < \infty \right\} . \tag{4}$$

In our setting it is natural, see e.g. [5], to require that for any $t \in [0, T]$, the *spot volatility* $\sigma : \mathbb{R}^2 \to \mathbb{R}^d$ of $\mathbf{X} = \left(X^1_t, \dots X^d_t\right)^T$ in (2) is driven by the two volatility processes Y_t and Z_t and its restriction to the first argument is a positive and smooth function belonging to L^2_Y.

In what follows, we will approximate the characteristic function of the log-normal value of the underlying assets $\mathbf{S}$ at expiring time T, i.e. the random variable $\mathbf{X}_T$. We underline that the study of φ allows to numerically treat the fair price of an option written on the underlying $\mathbf{X}$, see, e.g., [2, 4, 9, 12].

In what follows, if not otherwise stated, all the expectations are computed with respect to the risk neutral measure $\mathbb{Q}$. Let us consider, for any $t \in [0,T)$, the process $\{\mathcal{W}_s\}_{s\in[t,T]} = \{(\mathbf{W}_s, W^y_s, W^z_s)\}_{s\in[t,T]}$, such that $\mathcal{W}$ is conditioned to start from $\bar{x} := (x,y,z)$, at time $t \in [0,T)$, then we define

$$\varphi(u;(s,\bar{x})) = \mathbb{E}\left[e^{i(u\cdot\mathbf{X}_T)}\middle|\mathcal{W}_s = \bar{x}\right] =: \mathbb{E}^{s,\bar{x}}\left[e^{i(u\cdot\mathbf{X}_T)}\right] .$$

By the *Feynman-Kac* theorem, φ solves the following deterministic problem

$$\begin{cases} \mathcal{L}^{\epsilon,\delta}\varphi(u;(t,\bar{x})) = 0 \\ \varphi(u;(0,\bar{x})) = e^{\mathbf{i}u\cdot\mathbf{x}} \end{cases} , \tag{5}$$

with

$$\mathcal{L}^{\epsilon,\delta} := \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3 ,$$

where the operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are defined as follows

$$\begin{cases} \mathcal{L}_0 := \frac{1}{2}\beta^2(y)\partial_{yy} + \alpha(y)\partial_y \\ \mathcal{L}_1 := \beta(y)\left[\sum_{i,j=1}^d \Sigma_{ij}(y,z)\rho_{jy}\partial_{x_iy} - \Lambda_1(y,z)\partial_y\right] \\ \mathcal{L}_2 := \partial_t + \frac{1}{2}\sum_{i,j=1}^d \Sigma^2_{ij}(y,z)\partial_{x_ix_j} + \sum_{i=1}^d\left(r_i - \frac{1}{2}\sigma_i^2(y,z)\right)\partial_{x_i} \end{cases} , \tag{6}$$

$$\begin{cases} \mathcal{M}_1 := g(z)\left[\sum_{i,j=1}^d \Sigma_{ij}(y,z)\rho_{jz}\partial_{x_iz} - \Lambda_2(y,z)\partial_z\right] \\ \mathcal{M}_2 := \frac{1}{2}g^2(z)\partial_{zz} + c(z)\partial_z \\ \mathcal{M}_3 := \beta(y)\rho_{yz}g(z)\partial_{yz} \end{cases} , \tag{7}$$

hence $\varphi(u;(T,\bar{x}))$ is the characteristic function of the random variable $\mathbf{X}_T$.

3. Characteristic Function and its Expansion

In what follows we consider the first order approximation $\tilde{\varphi}$ of φ with respect to its formal expansion $\sum_{i,j\geq 0}(\sqrt{\epsilon})^i\left(\sqrt{\delta}\right)^j\varphi_{i,j}$, for suitable functions $\varphi_{i,j}$, hence computing the terms $\varphi_0 := \varphi_{0,0}$, $\varphi^\epsilon_{1,0} := \sqrt{\epsilon}\varphi_{1,0}$, $\varphi^\delta_{0,1} := \sqrt{\delta}\varphi_{0,1}$, therefore

$$\tilde{\varphi} := \varphi_0 + \varphi^\epsilon_{1,0} + \varphi^\delta_{0,1} . \tag{8}$$

The functions $\varphi_0, \varphi^\epsilon_{1,0}, \varphi^\delta_{0,1}$, will be obtained expanding first the function φ in powers of $\sqrt{\delta}$ and then in powers of $\sqrt{\epsilon}$. The $\sqrt{\delta} - expansion$ gives us

$$\varphi = \varphi^\epsilon_0 + \sqrt{\delta}\varphi^\epsilon_1 + \delta\varphi^\epsilon_2 + \dots , \tag{9}$$

then, substituting (9) into (5), we get

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_0 + \sqrt{\delta}\left[\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_1 + \left(\mathcal{M}_1 + \frac{\mathcal{M}_3}{\sqrt{\epsilon}}\right)\varphi^\epsilon_0\right] + \dots = 0 \quad ,$$

and equating all the terms to 0, we obtain that φ^ϵ_0, resp. φ^ϵ_1, is the unique solution to the problem

$$\begin{cases} \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_0 = 0 \\ \varphi^\epsilon_0(u;(0,\bar{x})) = e^{\mathbf{i}u\cdot\mathbf{x}} \end{cases} , \tag{10}$$

resp. to the problem

$$\begin{cases} \left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\varphi^\epsilon_1 = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)\varphi^\epsilon_0 \\ \varphi^\epsilon_1(u;(0,\bar{x})) = 0 \end{cases} . \tag{11}$$

The $\sqrt{\epsilon} - expansion$ of φ^ϵ_0 gives us

$$\varphi^\epsilon_0 = \varphi_0 + \sqrt{\epsilon}\varphi_{1,0} + \epsilon\varphi_{2,0} + \epsilon^{3/2}\varphi_{3,0} + \dots , \tag{12}$$

and inserting (12) in eq. (10), we have

$$\begin{aligned} &\frac{1}{\epsilon}\mathcal{L}_0\varphi_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0\varphi_{1,0} + \mathcal{L}_1\varphi_0\right) + 1\left(\mathcal{L}_0\varphi_{2,0} + \mathcal{L}_1\varphi_{1,0} + \mathcal{L}_2\varphi_0\right) \\ &+ \sqrt{\epsilon}\left(\mathcal{L}_0\varphi_{3,0} + \mathcal{L}_1\varphi_{2,0} + \mathcal{L}_2\varphi_{1,0}\right) + \dots = 0 \end{aligned} .$$

Equating the term $\epsilon^{-\mathbf{1}}$ and $\sqrt{\epsilon^{-\mathbf{1}}}$ to 0 we recover the independence of φ_0 and $\varphi_{1,0}$ from y, and the next order term gives

$$\mathcal{L}_2\varphi_0 + \mathcal{L}_1\varphi_{1,0} + \mathcal{L}_0\varphi_{2,0} = \mathcal{L}_2\varphi_0 + \mathcal{L}_0\varphi_{2,0} = 0 , \tag{13}$$

that is a Poisson equation for the function $\varphi_{2,0}$ in the y variable.

Proposition 3.1. *The Poisson equation* (13) *has a solution if and only if*

$$\langle\mathcal{L}_2\varphi_0\rangle = \langle\mathcal{L}_2\rangle\varphi_0 = 0 \quad \textit{with} \quad \langle f\rangle := \int f(y)\Phi(dy) ,$$

where Φ*, see def.* (2.1)*, is the invariant distribution of the process* Y_t*.*

Proof. Let $\mathcal{L}$ be the infinitesimal generator of the process Y_t and let $\mathbf{P}_t$ be the Markov transition semigroup for the process Y_t, see e.g. [5] for details. Then $\frac{d}{dt}\mathbf{P}_t f(y) = \mathcal{L}\mathbf{P}_t f(y)$ and, exploiting (3), we get

$$0 = \frac{d}{dt}\int \mathbb{E}^y\left[f\left(Y_t\right)\right]\Phi\left(dy\right) = \int \mathcal{L}\mathbf{P}_t f(y)\Phi(dy) = \int \mathbf{P}_t f(y)\mathcal{L}^*\Phi(dy)\,, \tag{14}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ with respect to the scalar product in the space L^2_Y defined in (4). Since eq. (14) has to be satisfied for any f, we have that the existence of an invariant distribution Φ implies the centering condition $\mathcal{L}^*\Phi = 0$. Averaging (13) with respect to Φ, integrating by parts, using the centering condition and since $\lim\limits_{y\to\pm\infty}\Phi(y) = \lim\limits_{y\to\pm\infty}\Phi'(y) = 0$, we get

$$\langle\mathcal{L}_2\varphi_0\rangle = -\langle\mathcal{L}_0\varphi_{2,0}\rangle = -\int\left(\mathcal{L}_0\varphi_{2,0}\right)\Phi(y)dy = \int \varphi_{2,0}\left(\mathcal{L}_0^*\Phi(y)\right)dy = 0\,. \quad \square$$

Exploiting Prop. (3.1), we are left with the following problem

$$\begin{cases}\langle\mathcal{L}_2\rangle\,\varphi_0 = 0 \\ \varphi_0(u;(0,\bar{x})) = e^{\mathbf{i}u\cdot\mathbf{x}}\end{cases}, \tag{15}$$

where

$$\begin{aligned}\langle\mathcal{L}_2\rangle &= -\partial_t + \frac{1}{2}\sum_{i,j=1}^{d} h_{ij}^2\left\langle\sigma_i^2\right\rangle\partial_{x_i x_j} + \sum_{i=1}^{d}\left(r_i - \frac{1}{2}\left\langle\sigma_i^2\right\rangle\right)\partial_{x_i} \\ &= -\partial_t + \frac{1}{2}\sum_{i,j=1}^{d}\bar{\varsigma}_{ij}^2\partial_{x_i x_j} + \sum_{i=1}^{d}\left(r_i - \bar{\sigma}_i^2\right)\partial_{x_i}\,,\end{aligned} \tag{16}$$

where $\bar{\sigma}^2 = \left(\bar{\sigma}_1^2,\dots,\bar{\sigma}_d^2\right)$ is a d dimensional vector, also called *average effective volatility*, whose components are defined as follows

$$\bar{\sigma}_i^2 := \bar{\sigma}_i^2(z) = \left\langle\sigma_i^2(y,z)\right\rangle = \int \sigma_i^2(y,z)\Phi(dy)\,,$$

and $\bar{\varsigma}^2 = \left(\bar{\varsigma}_{ij}^2\right)_{i,j=1,\dots,d}$ is a $d\times d$ matrix with elements, the *corrected average effective volatility* coefficients, defined by

$$\bar{\varsigma}_{ij}^2 := \bar{\varsigma}_{ij}^2(z) := h_{ij}^2\bar{\sigma}^2 = h_{ij}^2\left\langle\sigma_i^2(y,z)\right\rangle = \left\langle\Sigma_{ij}^2(y,z))\right\rangle = \int\Sigma_{ij}^2(y,z)\Phi(dy)\,.$$

By the *Feynman-Kac* theorem we have that φ_0 is the characteristic function of $\mathbf{X}_t, t \in [0,T]$, given $\bar{\sigma}^2$, moreover it can be computed analytically by, e.g., the separation of variables method, obtaining

$$\varphi_0(u;(t,\mathbf{x})) = \exp\left\{\mathbf{i}\left[\mathbf{x} + \left(r - \bar{\sigma}^2\right)t\right]\cdot u - \frac{u\cdot(\bar{\varsigma}^2 u)}{2}t\right\}\ , \tag{17}$$

so that, taking $t = T$, we have

$$\varphi_0(u;(T,\mathbf{x})) = \exp\left\{\mathbf{i}\left[\mathbf{x} + \left(r - \bar{\sigma}^2\right)T\right]\cdot u - \frac{u\cdot(\bar{\varsigma}^2 u)}{2}T\right\}\ ,$$

which agrees with the fact that $\varphi_0(u;(T,\mathbf{x}))$ is the characteristic function of the random variable $X_T^1 \sim \mathcal{N}\left(\mathbf{x} + \left(r - \bar{\sigma}^2\right)T, \bar{\varsigma}^2 T\right)$. In the formal expansion of ϕ_0^ϵ in eq. (12), we have that the term of order $\sqrt{\epsilon}$ is related to the solution of the following Poisson equation in the y variable

$$\mathcal{L}_0\varphi_{3,0} + \mathcal{L}_1\varphi_{2,0} + \mathcal{L}_2\varphi_{1,0} = 0\ ,$$

hence by requiring the centering condition to hold, and by eq. (13), we have

$$\varphi_{2,0} = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)\varphi_0\ .$$

In order to retrieve $\varphi_{1,0}^\epsilon$, we multiply by $\sqrt{\epsilon}$, hence the function $\varphi_{1,0}^\epsilon$ turns to be the classical solution of

$$\begin{cases}\langle\mathcal{L}_2\rangle\,\varphi_{1,0}^\epsilon = \mathcal{A}^\epsilon\varphi_0\\ \varphi_{1,0}^\epsilon(u;(0,x,y)) = 0\end{cases}\ , \tag{18}$$

where $\mathcal{A}^\epsilon := \sqrt{\epsilon}\left\langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\right\rangle$ is computed letting $\phi(y,z)$ to be a solution, with respect to y, of $\mathcal{L}_0\phi(y,z) = f^2 - \bar{\sigma}^2$, i.e.

$$\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right) = \frac{1}{2}\left[\sum_{i,j=1}^{d} h_{ij}^2\phi(y,z)\partial_{x_i x_i} - \sum_{i=1}^{d}\phi(y,z)\partial_{x_i}\right]\ ,$$

hence for $\mathcal{A}^\epsilon$, we have

$$\mathcal{A}^\epsilon = \left(\sum_{i,j,h,k}\mathbf{A}_{i,j,h,k}^{1,\epsilon}(z)\left[h_{ij}^2\partial_{x_i x_j x_k} - \partial_{x_i x_j}\right]\right)$$

$$-\left(\mathbf{A}^{2,\epsilon}(z)\sum_{ij}\left[h_{ij}\partial_{x_i x_j}-\partial_{x_i}\right]\right) \quad , \tag{19}$$

with

$$\mathbf{A}^{1,\epsilon}_{i,j,h,k}(z) := \frac{\rho_{jy}\sqrt{\epsilon}h_{ij}}{2}\left\langle \beta(y)\sigma_i(y,z)\partial_y\phi(y,z)\right\rangle ,$$
$$\mathbf{A}^{2,\epsilon}(z) := \frac{\sqrt{\epsilon}}{2}\left\langle \beta(y)\Lambda_1(y,z)\partial_y\phi(y,z)\right\rangle \ ,$$

then the action of $\mathcal{A}^\epsilon$ on φ_0, reads as follows

$$\mathcal{A}^\epsilon\varphi_0 = \left[\left(\sum_{i,j,h,k}\mathbf{A}^{1,\epsilon}_{i,j,h,k}(z)\left[h^2_{ij}\mathbf{i}u_iu_ju_k-u_iu_j\right]\right)-\left(\mathbf{A}^{2,\epsilon}(z)\sum_{ij}\left[h_{ij}u_iu_j-\mathbf{i}u_i\right]\right)\right]\varphi_0 ,$$

and we set $-C^\epsilon_u(z) := \mathcal{A}^\epsilon\varphi_0$. In what follows, we will write C^ϵ_u for $C^\epsilon_u(z)$ to underline that C^ϵ_u depends only on the z variable, hence nor on $\mathbf{x}$, neither on y. The following proposition gives an analytic solution for the eq. (18).

Proposition 3.2. *The correction term $\varphi^\epsilon_{1,0}$ which solves*(18) *is given by*

$$\varphi^\epsilon_{1,0} = -t\mathcal{A}^\epsilon\varphi_0 = tC^\epsilon_u\varphi_0 \ , \tag{20}$$

where φ_0 is as in (17)*.*

Proof. In order to show that a solution for the eq. (20) is also a solution for eq. (18), it is enough to note that

$$\langle\mathcal{L}_2\rangle\varphi^\epsilon_{1,0} = \langle\mathcal{L}_2\rangle(tC^\epsilon_u\varphi_0) = C^\epsilon_u\langle\mathcal{L}_2\rangle t\varphi_0 = -C^\epsilon_u\varphi_0 + C^\epsilon_u t\langle\mathcal{L}_2\rangle\varphi_0 = -C^\epsilon_u\varphi_0 = \mathcal{A}^\epsilon\varphi_0 \ ,$$

indeed C^ϵ_u can be taken out of the differential operator, since it does not depend neither on $\mathbf{x}$ nor t, and $\langle\mathcal{L}_2\rangle\varphi_0 = 0$. Moreover, since φ_0 is defined as a Fourier transform, it follows that $C^\epsilon_u\varphi_0$ remains bounded as $t\to 0$ so that $\lim_{t\to 0} tC^\epsilon_u\varphi_0 = 0$ and the initial condition holds. □

Since our aim is to retrieve the characteristic function of $\mathbf{X}_T$, we take $t = T$ in eq. (20), so that, by eq. (17), we get

$$\varphi^\epsilon_{1,0} = -C^\epsilon_u(z)T\varphi_0 \ . \tag{21}$$

The $\sqrt{\epsilon}-$ *expansion* of φ^ϵ_1 is given by

$$\varphi^\epsilon_1 = \varphi_{0,1} + \sqrt{\epsilon}\varphi_{1,1} + \epsilon\varphi_{2,1} + \epsilon^{3/2}\varphi_{3,1} + \dots \ , \tag{22}$$

then substituting expansion (22) into the eq. (11), we get

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_0\varphi_{0,1} + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0\varphi_{1,1} + \mathcal{L}_1\varphi_{0,1} + \mathcal{M}_3\varphi_0\right) & \\ + \left(\mathcal{L}_0\varphi_{2,1} + \mathcal{L}_1\varphi_{1,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 + \mathcal{M}_3\varphi_{1,0}\right) & \\ + \sqrt{\epsilon}\left(\mathcal{L}_0\varphi_{3,1} + \mathcal{L}_1\varphi_{2,1} + \mathcal{L}_2\varphi_{1,1} + \mathcal{M}_1\varphi_{1,0} + \mathcal{M}_3\varphi_{2,0}\right) + \dots & \\ = 0\,. &\end{aligned} \tag{23}$$

The first two terms in equation (23) state the independence from the y variable, whereas the third term gives us

$$\mathcal{L}_0\varphi_{2,1} + \mathcal{L}_1\varphi_{1,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 + \mathcal{M}_3\varphi_{1,0} = \mathcal{L}_0\varphi_{2,1} + \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 = 0\,,$$

which admits a solution iff

$$\langle \mathcal{L}_2\varphi_{0,1} + \mathcal{M}_1\varphi_0 \rangle = 0\,. \tag{24}$$

Multiplying eq. (24) by $\sqrt{\delta}$ and averaging $\mathcal{M}_1$ with respect to the invariant distribution Φ, we get

$$\begin{aligned}\sqrt{\delta}\,\langle\mathcal{M}_1\rangle\,\varphi_0 &= \sqrt{\delta}\left(\sum_{i,j=1}^{d}\rho_{jz}g(z)\,\langle\Sigma_{ij}(y,z)\rangle\,\partial_{x_i}\partial_z - g(z)\,\langle\Lambda_2(y,z)\rangle\,\partial_z\right)\varphi_0 \\ &= 2\mathcal{A}^{\delta}\varphi_0\,,\end{aligned} \tag{25}$$

where, since $\partial_z\varphi_0 = \partial_\sigma\varphi_0\sigma'(z)$, we have defined

$$\mathcal{A}^{\delta} := \sum_{ij=1}^{d}\mathbf{A}_{ij}^{1,\delta}(z)\partial_{\sigma_i}\partial_{x_i} - \sum_{i=1}^{d}\mathbf{A}_{i}^{2,\delta}(z)\partial_{\sigma_i}\,, \tag{26}$$

with $\mathbf{A}_{ij}^{1,\delta}(z), \mathbf{A}_{i}^{2,\delta}$ being defined as follows

$$\mathbf{A}_{ij}^{1,\delta}(z) := \frac{\rho_{jz}\mu(z)\sqrt{\delta}}{2}\,\langle\Sigma_{ij}(y,z)\rangle\,\bar{\sigma}'(z), \quad \mathbf{A}_{i}^{2,\delta}(z) := \frac{\mu(z)\sqrt{\delta}}{2}\,\langle\Lambda_2(y,z)\rangle\,\bar{\sigma}'(z)\,.$$

Moreover since the action of $\mathcal{A}^{\delta}$ on φ_0 is given by

$$\mathcal{A}^{\delta}\varphi_0 = t\left[\sum_{i,j=1}^{d}\mathbf{A}_{ij}^{1,\delta}(z)\left(\mathbf{i}u_i^2 + u_ih_{ij}u_j\right) + \sum_{i=1}^{d}\mathbf{A}_{i}^{2,\delta}(z)\left(\mathbf{i}u_i - \sum_{j=1}^{d}u_ih_{ij}u_j\right)\right]\varphi_0$$

$$= tC_u^\delta(z,\bar\sigma)\varphi_0 \ , \quad (27)$$

where

$$\begin{aligned} C_u^\delta &= C_u^\delta(z,\bar\sigma) \\ &:= t\left[\sum_{i,j=1}^d \mathbf{A}_{ij}^{1,\delta}(z)\left(\mathbf{i}u_i^2 + u_i h_{ij} u_j\right) + \sum_{i=1}^d \mathbf{A}_i^{2,\delta}(z)\left(\mathbf{i}u_i - \sum_{j=1}^d u_i h_{ij} u_j\right)\right] \ , \end{aligned} \quad (28)$$

we have that the first order slow scale correction $\varphi_{0,1}^\delta$ is the unique classical solution of

$$\begin{cases} \langle \mathcal{L}_2 \rangle \varphi_{0,1}^\delta = -2\mathcal{A}^\delta \varphi_0 \\ \varphi_{0,1}^\delta(u;(0,x,z)) = 0 \end{cases} . \quad (29)$$

Proposition 3.3. *The first order slow scale correction* $\varphi_{0,1}^\delta$ *which solves* (29), *is given by* $\varphi_{0,1}^\delta = t^2 C_u^\delta \varphi_0$, *with* $\mathcal{A}^\delta$ *given in* (26) *and* φ_0 *given in* (17).

Proof. Since $\langle \mathcal{L}_2 \rangle \varphi_0 = 0$, by direct computation we have

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \varphi_{0,1}^\delta &= \langle \mathcal{L}_2 \rangle \left(t^2 C_u^\delta(z,\bar\sigma)\varphi_0\right) = C_u^\delta(z,\bar\sigma)\langle \mathcal{L}_2 \rangle \left(t^2\varphi_0\right) = \\ &= C_u^\delta(z,\bar\sigma)\left[-2t\varphi_0 + t^2\langle \mathcal{L}_2 \rangle \varphi_0\right] = -2C_u^\delta(z,\bar\sigma)t\varphi_0 = -2\mathcal{A}^\delta\varphi_0 \ , \end{aligned}$$

and the initial condition $\varphi_{0,1}^\delta(u;(0,x,z)) = 0$ clearly holds. □

Exploiting prop. 3.3, we evaluate $\varphi_{0,1}^\delta$ at time $t = T$ to be $\varphi_{0,1}^\delta = T^2 C_u^\delta \varphi_0$. Gathering previous results, namely eq. (17), prop. (3.3) and prop. (3.2), we have that the first order approximation for the characteristic function of the process X^1 at maturity, see eq. (8), reads as follows

$$\tilde\varphi = \varphi_0\left[1 + TC_u^\epsilon + T^2 C_u^\delta\right] \ . \quad (30)$$

4. Conclusion

We derive an explicit expression, see eq. (30), for the first order correction of the characteristic function of the random variable $\mathbf{X}_T$, $\{\mathbf{X}_t\}_{t\in[0,T]}$ being the log-normal process associated to a $d-$dimensional underlying asset $\{\mathbf{S}_t\}_{t\in[0,T]}$ price process, see eq. (1). Such a result can be used to numerically retrieve the probability to end up in the money hence finding the fair price of an option written on the underlying $\mathbf{X}$, see, e.g., [2, 4, 9, 12].The result stated in eq. (30) can be also used, see the result in [5, th.4.10], to obtain a bound for the difference between the exact solution φ and the approximate solution $\tilde{\varphi}$, since for fixed $(t, \mathbf{x}, y, z)$, there exists $C > 0$ such that for the error of the approximation given in (30), the following holds

$$|\varphi - \tilde{\varphi}| \leq C(\epsilon + \delta) \, , \, \forall \epsilon \leq 1 \, , \, \delta \leq 1 \, .$$

We would like to stress that, although assumptions made in [5, th.4.10] look particularly strong, in fact they are not. Indeed, in many financial applications stochastic processes used to model volatility movements, e.g. Ornstein-Uhlenbeck process, Cox-Ingersoll-Ross process, etc., satisfy our requirements. The assumptions on the volatility function $f(y, z)$ can be relaxed with particular choices of volatility factors such as f being an exponential function or a square root. Furthermore if we freeze the slow scale process Z and we assume $f = \sqrt{Y}$ with Y to be a CIR process, we retrieve the well known Heston model which turns to satisfy our requirements.

References

[1] D. Brigo and F. Mercurio, *Interest Rates Model: Theory and Practice*, Springer-Verlag, Berlin (2006).

[2] P. Carr and D. Madan, Option valuation using the fast Fourier transform, *Journal of Computational Finance*, **2**, 4 (1999), 61-73.

[3] F. Cordoni and L. Di Persio , Asymptotic expansion for the characteristic function of a multiscale stochastic volatility model, *Intern. J. Appl. Math.*, **27**, 3 (2014).

[4] J.C. Duan, W.K. Härdle and J. E. Gentle (editors), *Handbook of computational finance* , Springer Handbooks of Computational Statistics, (2012).

[5] J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, *Multiscale stochastic volatility for equity, interest rate, and credit derivatives*, Cambridge University Press, Cambridge, (2011).

[6] J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, Singular perturbations in option pricing, *SIAM J. on App. Math.* **63**, 5 (2003), 1648-1665.

[7] J.P. Fouque, G. Papanicolaou, and R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press, Cambridge (2000).

[8] J.P. Fouque, and M.J. Lorig, A fast mean-reverting correction to Heston's stochastic volatility model, *SIAM Journal on Fin. Math.* **2**, 1 (2011), 221-254.

[9] A. Hirsa, *Computational Methods in Finance*, Chapman & Hall/CRC Financial Mathematics Series, CRC Press, Boca Raton, FL (2013).

[10] Y. Kabanov and S. Pergamenshchikov, *Two-scale Stochastic Systems: Asymptotic Analysis and Control*, Springer-Verlag, Berlin (2003).

[11] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York (1991).

[12] M. Schmelzle, Option pricing formulae using Fourier transform: Theory and Applications, http://pfadintegral.com/articles/option-pricing-formulae-using-fourier-transform/.

[13] S. E. Shreve, *Stochastic Calculus for Finance II.* Springer Finance, New York (2004).