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An Investigation of Methods For Analyzing Networks Containing One Nonlinear Element

Dustin J. Wilson

ANALYZING NETWORKS CONTAINING ONE NONLINEAR ELEMENT - WILSON

AN INVESTIGATION OF METHODS FOR ANALYZING NETWORKS CONTAINING ONE NONLINEAR ELEMENT

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Dustin J. Wilson

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Roy Parish
Dean

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Date

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By

Dustin J. Wilson

Thesis committee

Shlomo Karni
Chairman

E. L. Jordan

W. W. Koepsel

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CHAPTER I

INTRODUCTION

Purpose

The inclusion of a single nonlinear element in a general N-loop network greatly complicates the resulting equations and prohibits a straightforward solution. The object of this investigation is to provide a method for solving such networks. In particular, a reduction of the network coefficient matrix will be considered in order to obtain one nonlinear equation and (N-1) linear equations. It will be shown that the form of the nonlinear equations is always the same, and a procedure for approximating the solution of equations of this type will be given.

Scope

The networks to be considered, except for the nonlinear element, consist of linear, constant, lumped, passive elements and excitation sources of known time variation. The network is comprised of N loops, each branch of which contains a series combination of passive elements and excitations. The nonlinear element may be placed in series with any branch, and its excitation-response relationship is, in general,

$$v = f(i) = \sum_{q=0}^{m} a_q i^q \qquad (1\text{-}1)$$

where, in the electrical case, v is the voltage across the

nonlinear element, i is the current in the element, and f(i) is assumed to be expanded in a finite power series in i for which the constants a_q are given. Although the presentation is in terms of electrical networks, the method applies equally to other analogous networks.

Method

In the standard fashion, integro-differential equations will be written for each loop using the differential operator, p.[1*] In Chapter II, symmetry of the coefficient matrix will be considered and a standard convention will be adopted whereby this matrix is symmetric about its main diagonal and such that the system of equations contains only one which is nonlinear.

Chapter III will deal with the solution of the linear equations in terms of the variable of the argument of the nonlinear function. These solutions will be substituted in the nonlinear equation.

In Chapter IV, a transformation of variables will be used to reduce the original coefficient matrix to one which contains zeros everywhere except along the main diagonal and along the row associated with the nonlinear equation. This row will consist of unity elements except for the main diagonal term; the resulting nonlinear equation will be simpler than that derived in Chapter III.

A method for solving the nonlinear equation will be

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*Superscripts refer to references in Bibliography.

given in Chapter V. A power series in t is assumed for the unknown variable. By equating coefficients of like powers in t, the unknown coefficients are determined and an approximation to the solution is thus obtained. An example of the procedure is given.

CHAPTER II

THE MATRIX EQUATION FOR AN N-LOOP NETWORK

Comparison of Loop Equation Conventions

Two possible conventions are shown in Figure 2-1. For each, the nonlinear branch is assumed to consist of a voltage source, $V_{k,k+1}$, an impedance, $Z_{k,k+1}$, and the nonlinear element represented by f(i). The subscripts on the source and the impedance indicate that these elements are common to loop k and to loop (k+1). This is the only nonlinear branch and is part of a general N-loop network.

I_k and I_{k+1} are assumed current references. In (a), each of these currents is in the nonlinear element while in (b) only I_k is assumed in the nonlinear element.

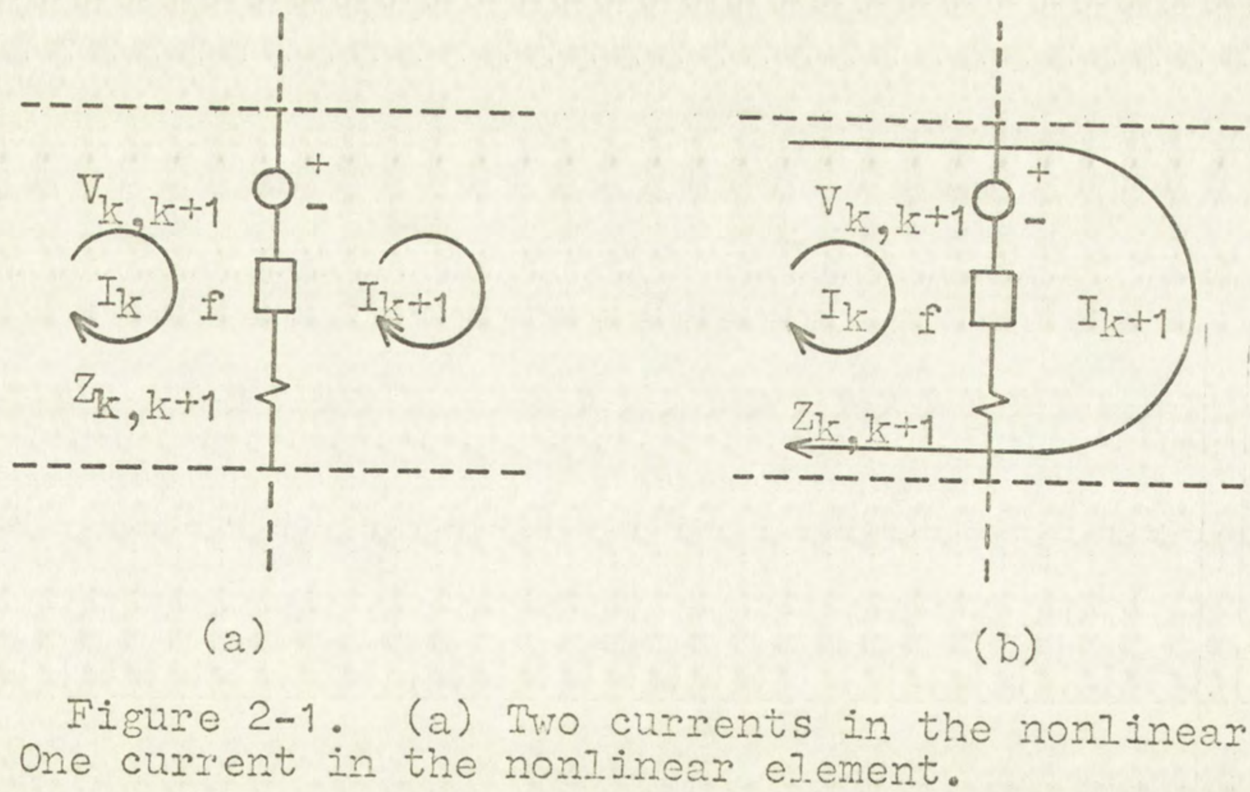

Figure 2-1. (a) Two currents in the nonlinear element. (b) One current in the nonlinear element.

In the first case, two equations are of interest. These appear as:

$$\ldots + Z_{k,k+1} I_k - Z_{k,k+1} I_{k+1} + \ldots = V_{k,k+1} + \ldots - f(I_k - I_{k+1}) \qquad (2\text{-}1a)$$

$$\ldots - Z_{k,k+1} I_k + Z_{k,k+1} I_{k+1} + \ldots = -V_{k,k+1} + \ldots + f(I_k - I_{k+1}) \qquad (2\text{-}1b)$$

In the second case, the only nonlinear equation will be

$$\ldots + Z_{k,k+1} I_k + \ldots = V_{k,k+1} + \ldots - f(I_k) \qquad (2\text{-}2a)$$

Equations (2-1a) and (2-1b) are both nonlinear and the argument of the nonlinear function, f, contains two unknowns. Certainly, it is possible to eliminate the nonlinear function from equation (2-1b) by adding to it equation (2-1a), but the argument of f will be unchanged. For this reason, the current reference shown in Figure 2-1b would seem to be preferable.

Symmetry of the Coefficient Matrix

In the linear network, the loop equations can be written in the following matrix form:

$$(a_{ij})(x_j) = (v_i) \qquad (2\text{-}3)$$

where a_{ij} is the coefficient of the j^{th} current in the i^{th} equation, x_j is the assumed current in the j^{th} loop, and v_i is the sum of the voltage sources in the i^{th} loop. If the loops are chosen properly, the matrix (a_{ij}) will be symmetric about its main diagonal. In fact, the criteria which guarantee this symmetry are:

(1) The equations must be written in the same order

as the assumed currents are numbered.

(2) Each voltage loop must coincide with its assumed current loop.[2]

If a nonlinear element is present and if the nonlinear functions are written as part of the matrix (v_i), then the coefficient matrix (a_{ij}) will be symmetric in either of the cases of the previous section. The current references shown in Figure 2-1b yield an equation of the form

$$(a_{ij})(x_j) = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k - f(x_k) \\ v_{k+1} \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix}; \qquad (2\text{-}4)$$

the current reference shown in Figure 2-1a results in

$$(a_{ij})(x_j) = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_k - f(x_k - x_{k+1}) \\ v_{k+1} + f(x_k - x_{k+1}) \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix}. \qquad (2\text{-}5)$$

In equation (2-5), the addition of equation k to equation (k+1) will yield a matrix (a'_{ij}) which is no longer symmetric. Clearly,

$$a'_{k+1,j} = a_{kj} + a_{k+1,j}\,, \qquad (2\text{-}6a)$$

$$a'_{j,k+1} = a_{j,k+1} \quad (j \neq k+1), \tag{2-6b}$$

and

$$a'_{k+1,k+1} = a_{k,k+1} + a_{k+1,k+1} \,. \tag{2-6c}$$

Thus,

$$a'_{k+1,j} \neq a'_{j,k+1} \tag{2-6d}$$

unless $j = k+1$.

The proposed method of solution depends on the evaluation of cofactors of the matrix (a_{ij}). Since in a symmetric matrix $\mathrm{cof}(a_{ij})=\mathrm{cof}(a_{ji})$, the current reference shown in Figure 2-1b will result in fewer computations and will be adhered to throughout the succeeding analysis.

General Form of the Matrix Equation

The result of using the conventions dictated in the preceeding sections is the following matrix equation:

$$\begin{bmatrix} a_{11} & a_{12}\cdots a_{1k}\cdots a_{1N} \\ a_{21} & a_{22}\cdots a_{2k}\cdots a_{2N} \\ \vdots & \vdots \\ a_{k1} & a_{k2}\cdots a_{kk}\cdots a_{kN} \\ \vdots & \vdots \\ a_{N1} & a_{N2}\cdots a_{Nk}\cdots a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k - f(x_k) \\ \vdots \\ v_N \end{bmatrix} \tag{2-7}$$

The elements a_{ij} represent functions of the operator p which correspond to the series combination of linear, passive network elements common to loops i and j. In general,

$$a_{ij} = \frac{Lp^2 + Rp + D}{p} \tag{2-8}$$

Equation (2-7) may represent any network having N loops with a nonlinear element in the k^{th} loop. Moreover, the form of the equation will be identical for other analogous systems.

CHAPTER III

REDUCTION OF SYSTEM BY DIRECT ALGEBRAIC METHODS

If the nonlinear equation is removed from equation (2-7) and the terms in x_k are transposed to the right hand side, the following system of equations results:

$$\begin{bmatrix} a_{11} & \cdots a_{1,k-1} & a_{1,k+1} & \cdots a_{1N} \\ a_{21} & \cdots a_{2,k-1} & a_{2,k+1} & \cdots a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k-1,1} & \cdots a_{k-1,k-1} & a_{k-1,k+1} & \cdots a_{k-1,N} \\ a_{k+1,1} & \cdots a_{k+1,k-1} & a_{k+1,k+1} & \cdots a_{k+1,N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & \cdots a_{N,k-1} & a_{N,k+1} & \cdots a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_{k+1} \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} v_1 & -a_{1k} & x_k \\ v_2 & -a_{2k} & x_k \\ \vdots & & \\ v_{k-1} & -a_{k-1,k} & x_k \\ v_{k+1} & -a_{k+1,k} & x_k \\ \vdots & & \\ v_N & -a_{Nk} & x_k \end{bmatrix} \qquad (3\text{-}1)$$

The nonlinear equation is

$$\sum_{j=1}^{N} a_{kj} x_j = v_k - f(x_k) \qquad (3\text{-}2)$$

Equation (3-1) will be solved for each x_j, ($j \neq k$), in terms of x_k. The results will then be substituted in equation (3-2) to yield a nonlinear equation in x_k.

By Cramer's rule, we have, for $j \neq k$,

$$x_j = \frac{\sum_{i \neq k} v_i \ \mathrm{cof}(a_{ij})_k - x_k \sum_{i \neq k} a_{ik} \ \mathrm{cof}(a_{ij})_k}{\det(a_{ij})_k} \qquad (3\text{-}3)$$

where the subscript k on the cofactor and determinant terms refers to the matrix $(a_{ij})_k$ of equation (3-1).

Substitution of equation (3-3) in equation (3-2) yields the following:

$$a_{kk}x_k + \frac{1}{\det(a_{ij})_k}\left[\sum_{j \neq k} a_{kj} \sum_{i \neq k} v_i \mathrm{cof}(a_{ij})_k \right.$$

$$\left. - x_k \sum_{j \neq k} a_{kj} \sum_{i \neq k} a_{ik} \ \mathrm{cof}(a_{ij})_k \right] = v_k - f(x_k)$$

Upon rearrangement, this equation can be written in the form

$$\left[\frac{N_1(p)}{D_1(p)} + a_{kk}\right] x_k + f(x_k) = \frac{1}{D_1(p)} \ V_1(t) + v_k(t) \qquad (3\text{-}4)$$

where

$$N_1(p) = -\sum_{j \neq k} a_{kj} \sum_{i \neq k} a_{ik} \ \mathrm{cof}(a_{ij})_k$$

$$= \frac{A_{2N}p^{2N} + A_{2N-1}p^{2N-1} + \ldots + A_0}{p^N} \qquad (3\text{-}5)$$

$$D_1(p) = \det(a_{ij})_k$$

$$= \frac{B_{2(N-1)}p^{2(N-1)} + B_{2(N-1)-1}p^{2(N-1)-1} + \ldots + B_0}{p^{N-1}} \qquad (3\text{-}6)$$

and

$$V_1(t) = \sum_{j\neq k} a_{kj} \sum_{i\neq k} v_i \; \mathrm{cof}(a_{ij})_k \qquad (3\text{-}7)$$

The evaluation of $V_1(t)$ will, in general, involve $2(N-1)$ differentiations of each v_i; this implies that the voltage sources must be well-behaved functions. In fact, the necessity for these differentiations and the integrations implied by $\frac{1}{D_1(p)}$ to be performed on the given voltage time functions is the primary disadvantage of this method.

The use of equations (3-5) and (3-6) reduces equation (3-4) to the following form:

$$\left[\frac{N_2(p)}{pD_2(p)} + a_{kk}(p)\right] x_k + f(x_k) = \left[\frac{p^{N-1}}{D_2(p)}\right] V_1(t) + v_k(t) \qquad (3\text{-}8)$$

where

$$N_2(p) = A_{2N}p^{2N} + A_{2N-1}p^{2N-1} + \ldots + A_0$$

and

$$D_2(p) = B_{2(N-1)}p^{2(N-1)} + B_{2(N-1)-1}p^{2(N-1)-1} + \ldots + B_0$$

The method to be developed in Chapter IV yields an equation of form similar to equation (3-8); the difference is that the function operating on $V_1(t)$ is absent.

Upon obtaining a solution for $x_k(t)$, this result can be substituted in equation (3-3) to evaluate each $x_j(t)$. The differential equations which result are of the form

$$pD_2(p)x_j = V_j(t) - N_j(p)x_k(t) \;, \; (j\neq k) \qquad (3\text{-}9)$$

where

$$V_j(t) = p^N \sum_{i \neq k} v_i \operatorname{cof}(a_{ij})_k,$$

and

$$N_j(p) = p^N \sum_{i \neq k} a_{ik} \operatorname{cof}(a_{ij})_k .$$

Equation (3-9) is a differential equation of order (N-1).

CHAPTER IV

REDUCTION OF COEFFICIENT MATRIX USING TRANSFORMATION OF VARIABLES

Another technique for solving the system of equations under consideration will be described. Although more complicated than the method of Chapter III, the present one will yield a more elegant representation of the network. A transformation of variables will be used to reduce the coefficient matrix to one in which only the diagonal and k^{th} row elements are non-zero.[3]

Transformation of Variables

Consider the system of equations given in Chapter II:

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1k} \cdots & a_{1N} \\ a_{21} & a_{22} \cdots & a_{2k} \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & a_{kk} & a_{kN} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{Nk} & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k - f(x_k) \\ \vdots \\ v_N \end{bmatrix} \qquad (4\text{-}1)$$

Equations cannot be added or subtracted to yield a triangular matrix since the result would include more than one non-linear equation.

Define a new set of variables (y_i) as

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_k \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \cdots b_{1k} \cdots b_{1N} \\ b_{21} & b_{22} \cdots b_{2k} \cdots b_{2N} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ b_{k1} & b_{k2} \cdots b_{kk} \cdots b_{kN} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ b_{N1} & b_{N2} \cdots b_{Nk} \cdots b_{NN} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix} \tag{4-2}$$

and substitute this relationship in equation (4-1). The resulting system of equations is:

$$\begin{bmatrix} c_{11} & c_{12} \cdots c_{1k} \cdots c_{1N} \\ c_{21} & c_{22} \cdots c_{2k} \cdots c_{2N} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ c_{k1} & c_{k2} \cdots c_{kk} \cdots c_{kN} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ c_{N1} & c_{N2} \cdots c_{Nk} \cdots c_{NN} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_k - f(x_k) \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} \tag{4-3}$$

in which

$$c_{ij} = \sum_{q=1}^{N} a_{iq} b_{qj} \tag{4-4}$$

Observe that, from equation (4-2),

$$x_k = b_{k1} y_1 + b_{k2} y_2 + \cdots + b_{kk} y_k + \cdots + b_{kN} y_N \; ;$$

therefore,

$$f(x_k) = f(b_{k1} y_1 + b_{k2} y_2 + \cdots + b_{kk} y_k + \cdots + b_{kN} y_N)$$

So that the argument of f will contain only one unknown,

y_k, the restrictions are imposed that

$$b_{kj} = 0 \ , \ (j \neq k) \tag{4-5}$$

and

$$b_{kk} = 1 \ . \tag{4-6}$$

Thus,

$$f(x_k) = f(y_k) \ .$$

To investigate the effect of these restrictions, equation (4-4) can be expanded for a specific column, J, of (c_{ij}), as follows:

$$\begin{array}{l}
c_{1J} = a_{11}b_{1J} + a_{12}b_{2J} + \dots + a_{1k}b_{kJ} + \dots + a_{1N}b_{NJ} \\
c_{2J} = a_{21}b_{1J} + a_{22}b_{2J} + \dots + a_{2k}b_{kJ} + \dots + a_{2N}b_{NJ} \\
\vdots \\
c_{JJ} = a_{J1}b_{1J} + a_{J2}b_{2J} + \dots + a_{Jk}b_{kJ} + \dots + a_{JN}b_{NJ} \\
\vdots \\
c_{kJ} = a_{k1}b_{1J} + a_{k2}b_{2J} + \dots + a_{kk}b_{kJ} + \dots + a_{kN}b_{NJ} \\
\vdots \\
c_{NJ} = a_{N1}b_{1J} + a_{N2}b_{2J} + \dots + a_{Nk}b_{kJ} + \dots + a_{NN}b_{NJ}
\end{array} \tag{4-7}$$

<u>Solution for b_{qJ}, $(J \neq k)$</u>

In case $J \neq k$, substitution of equation (4-5) in equation (4-7) yields the following relations for c_{iJ}:

$$\begin{array}{l}
c_{1J} = a_{11}b_{1J} + \dots + a_{1,k-1}b_{k-1,J} + a_{1,k+1}b_{k+1,J} + \dots + a_{1N}b_{NJ} \\
\vdots \\
c_{JJ} = a_{J1}b_{1J} + \dots + a_{J,k-1}b_{k-1,J} + a_{J,k+1}b_{k+1,J} + \dots + a_{JN}b_{NJ} \\
\vdots \\
c_{kJ} = a_{k1}b_{1J} + \dots + a_{k,k-1}b_{k-1,J} + a_{k,k+1}b_{k+1,J} + \dots + a_{kN}b_{NJ} \\
\vdots \\
c_{NJ} = a_{N1}b_{1J} + \dots + a_{N,k-1}b_{k-1,J} + a_{N,k+1}b_{k+1,J} + \dots + a_{NN}b_{NJ}
\end{array} \tag{4-8}$$

This set contains N equations in (N-1) unknowns. Therefore, one of the terms, c_{iJ}, is not independent and, since a diagonal matrix is desired, this dependent term is chosen to be c_{JJ}. Obviously, when the J^{th} equation is eliminated, the resulting system of equations cannot be homogeneous; i.e., the remaining terms, c_{iJ}, $(i \neq J)$, cannot all be made zero. But, since the nonlinear equation, of which c_{kJ} is a part, will be the last to be solved in equation (4-3), it is valid to let

$$c_{kJ} = 1 \qquad (4\text{-}9)$$

and

$$c_{iJ} = 0 \;, \; (i \neq k, i \neq J) \;. \qquad (4\text{-}10)$$

The result of these restrictions is the following set of equations in b_{qJ}:

$$\begin{array}{l}
a_{11} b_{1J} + \ldots + a_{1,k-1} b_{k-1,J} + a_{1,k+1} b_{k+1,J} + \ldots + a_{1N} b_{NJ} = 0 \\
\vdots \\
a_{J-1,1} b_{1J} + \ldots + a_{J-1,k-1} b_{k-1,J} + a_{J-1,k+1} b_{k+1,J} + \ldots + a_{J-1,N} b_{NJ} = 0 \\
a_{J+1,1} b_{1J} + \ldots + a_{J+1,k-1} b_{k-1,J} + a_{J+1,k+1} b_{k+1,J} + \ldots + a_{J+1,N} b_{NJ} = 0 \\
\vdots \\
a_{k-1,1} b_{1J} + \ldots + a_{k-1,k-1} b_{k-1,J} + a_{k-1,k+1} b_{k+1,J} + \ldots + a_{k-1,N} b_{NJ} = 0 \\
a_{k1} b_{1J} + \ldots + a_{k,k-1} b_{k-1,J} + a_{k,k+1} b_{k+1,J} + \ldots + a_{kN} b_{NJ} = 1 \\
a_{k+1,1} b_{1J} + \ldots + a_{k+1,k-1} b_{k-1,J} + a_{k+1,k+1} b_{k+1,J} + \ldots + a_{k+1,N} b_{NJ} = 0 \\
\vdots \\
a_{N1} b_{1J} + \ldots + a_{N,k-1} b_{k-1,J} + a_{N,k+1} b_{k+1,J} + \ldots + a_{NN} b_{NJ} = 0
\end{array} \qquad (4\text{-}11)$$

Also, from equation (4-8),

$$c_{JJ} = \sum_{q=1}^{N} a_{Jq} b_{qJ} \qquad (4\text{-}12)$$

Symbolically, equations (4-11) appear as

$$(a_{ij})_{Jk} \cdot (b_{qJ}) = (1)_k \qquad (4\text{-}13)$$

Observe that the matrix, $(a_{ij})_{Jk}$, is the original a_{ij} matrix of equation (4-1) with row J and column k deleted, and that the column matrix, $(1)_k$, consists of zeros except that the k^{th} element is unity. It is assumed that the numbering scheme used for (a_{ij}) will be retained for matrices of smaller order which have certain rows and columns deleted.

Equations (4-11) are linear and can be solved for each b_{iJ}, $(i \neq k)$, by Cramer's rule. Thus,

$$b_{qJ} = \frac{D_q}{\Delta_J} \qquad (4\text{-}14)$$

Here, D_q is the determinant of $(a_{ij})_{Jk}$ with column q replaced by $(1)_k$; it can be evaluated by expanding the cofactor of element a_{kq} in $(a_{ij})_{Jk}$. The following notation will be used:

$$D_q = \text{cofactor } (a_{kq,Jk}) \qquad (4\text{-}15)$$

Thus, D_q is the determinant of (a_{ij}) with rows k and J and columns q and k deleted.

Also, Δ_J represents the determinant of $(a_{ij})_{Jk}$ and can be written as

$$\Delta_J = (-1)^{J+k} \text{cof}(a_{Jk}) \qquad (4\text{-}16)$$

Equations (4-15) and (4-16) can be combined with equation (4-14) to yield:

$$b_{qJ} = (-1)^{q-j+1} \frac{\text{cof}(a_{kq,Jk})}{\text{cof}(a_{Jk})} , \begin{bmatrix} q<k, \ J<k \\ \text{or} \\ q>k, \ J>k \end{bmatrix} ; \quad (4\text{-}17a)$$

and

$$b_{qJ} = (-1)^{q-J} \frac{\text{cof}(a_{kq,Jk})}{\text{cof}(a_{Jk})} , \begin{bmatrix} q>k, \ J<k \\ \text{or} \\ q<k, \ J>k \end{bmatrix} . \quad (4\text{-}17b)$$

Solution for b_{qk}

To evaluate b_{qk}, $(q\neq k)$, equation (4-6) is substituted in the set of equations (4-7) with the additional requirement that J=k. The result is:

$$\begin{array}{l} c_{1k}=a_{11}b_{1k}+\cdots+a_{1,k-1}b_{k-1,k}+a_{1k}+a_{1,k+1}b_{k+1,k}+\cdots+a_{1N}b_{Nk} \\ c_{2k}=a_{21}b_{1k}+\cdots+a_{2,k-1}b_{k-1,k}+a_{2k}+a_{2,k+1}b_{k+1,k}+\cdots+a_{2N}b_{Nk} \\ \vdots \\ c_{kk}=a_{k1}b_{1k}+\cdots+a_{k,k-1}b_{k-1,k}+a_{kk}+a_{k,k+1}b_{k+1,k}+\cdots+a_{kN}b_{Nk} \\ \vdots \\ c_{Nk}=a_{N1}b_{1k}+\cdots+a_{N,k-1}b_{k-1,k}+a_{Nk}+a_{N,k+1}b_{k+1,k}+\cdots+a_{NN}b_{Nk} \end{array} \quad (4\text{-}18)$$

Again, there are N equations and (N-1) unknowns. Therefore, the diagonal element c_{kk} is selected as the dependent term. However, the column of constants (a_{ik}) insures that equations (4-18) will not all be homogeneous and it is possible to specify that

$$c_{ik} = 0 , \ (i\neq k) . \quad (4\text{-}19)$$

Substitution of this requirement in equations (4-18) and

rearrangement yields the desired relations for b_{qk}. Thus,

$$a_{11}b_{1k} + \dots + a_{1,k-1}b_{k-1,k} + a_{1,k+1}b_{k+1,k} + \dots + a_{1N}b_{Nk} = -a_{1k}$$

$$a_{21}b_{1k} + \dots + a_{2,k-1}b_{k-1,k} + a_{2,k+1}b_{k+1,k} + \dots + a_{2N}b_{Nk} = -a_{2k}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{k-1,1}b_{1k} + \dots + a_{k-1,k-1}b_{k-1,k} + a_{k-1,k+1}b_{k+1,k} + \dots + a_{k-1,N}b_{Nk} = -a_{k-1,k}$$

$$a_{k+1,1}b_{1k} + \dots + a_{k+1,k-1}b_{k-1,k} + a_{k+1,k+1}b_{k+1,k} + \dots + a_{k+1,N}b_{Nk} = -a_{k+1,k}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{N1}b_{1k} + \dots + a_{N,k-1}b_{k-1,k} + a_{N,k+1}b_{k+1,k} + \dots + a_{NN}b_{Nk} = -a_{Nk} \qquad (4\text{-}20)$$

These equations can be rewritten as

$$(a_{ij})_{kk}(b_{qk}) = (-a_{ik}) \, , \; (q \neq k, \; i \neq k) \; . \qquad (4\text{-}21)$$

Also,

$$c_{kk} = \sum_{q=1}^{N} a_{kq} b_{qk} \; . \qquad (4\text{-}22)$$

As before,

$$b_{qk} = \frac{D_{qk}}{\Delta_k} \qquad (4\text{-}23)$$

D_{qk} is the determinant of $(a_{ij})_{kk}$ with column q replaced by $(-a_{ik})$, $(i \neq k)$. Δ_k is the determinant of $(a_{ij})_{kk}$. If D_{qk} is expanded by column q, then

$$D_{qk} = \sum_{i=1}^{k-1} (-1)^{q+k+i} \; a_{ik} \mathrm{cof}(a_{iq,kk})$$

$$+ \sum_{i=k+1}^{N} (-1)^{q+k+i+1} \; a_{ik} \mathrm{cof}(a_{iq,kk}) \qquad (4\text{-}24)$$

Obviously,

$$\Delta_k = \det(a_{ij})_{kk} = +\mathrm{cof}(a_{kk}) \qquad (4\text{-}25)$$

Combining equations (4-24) and (4-25) in equation (4-23) results in

$$b_{qk} = \sum_{i=1}^{k-1} (-1)^{q+k+i} \; a_{ik} \frac{\mathrm{cof}(a_{iq,kk})}{\mathrm{cof}(a_{kk})}$$

$$+ \sum_{i=k+1}^{N} (-1)^{q+k+i+1} \; a_{ik} \frac{\mathrm{cof}(a_{iq,kk})}{\mathrm{cof}(a_{kk})} \qquad (4\text{-}26)$$

Result of Transformation

Having defined the parameters of the transformation in terms of the known a_{ij}, the system of equations becomes:

$$\begin{bmatrix} c_{11} & 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & 0 \\ 0 & c_{22} & 0 \ldots 0 & 0 & 0 \ldots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 \ldots 1 & c_{kk} & 1 \ldots 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & c_{NN} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_k \\ \cdots \\ y_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_k - f(y_k) \\ \cdots \\ v_N \end{bmatrix} \qquad (4\text{-}27)$$

In the c_{ij} matrix, the diagonal elements, c_{jj}, are functions of p; the elements of row k, c_{kj}, $(j \neq k)$, are unity; the remainder of the elements are zero.

Equation (4-27) provides a representation of the network which lends itself to a systematic solution. A solution for the transformed variables, y_i, is a solution for the network except that the original unknowns, x_i, can be obtained only by use of the inverse transformation as defined in equation (4-2).

Equivalent Circuit for the Transformed Network

It is possible to develop several equivalent circuits which represent matrix equation (4-27); one choice is shown in Figure 4-1.

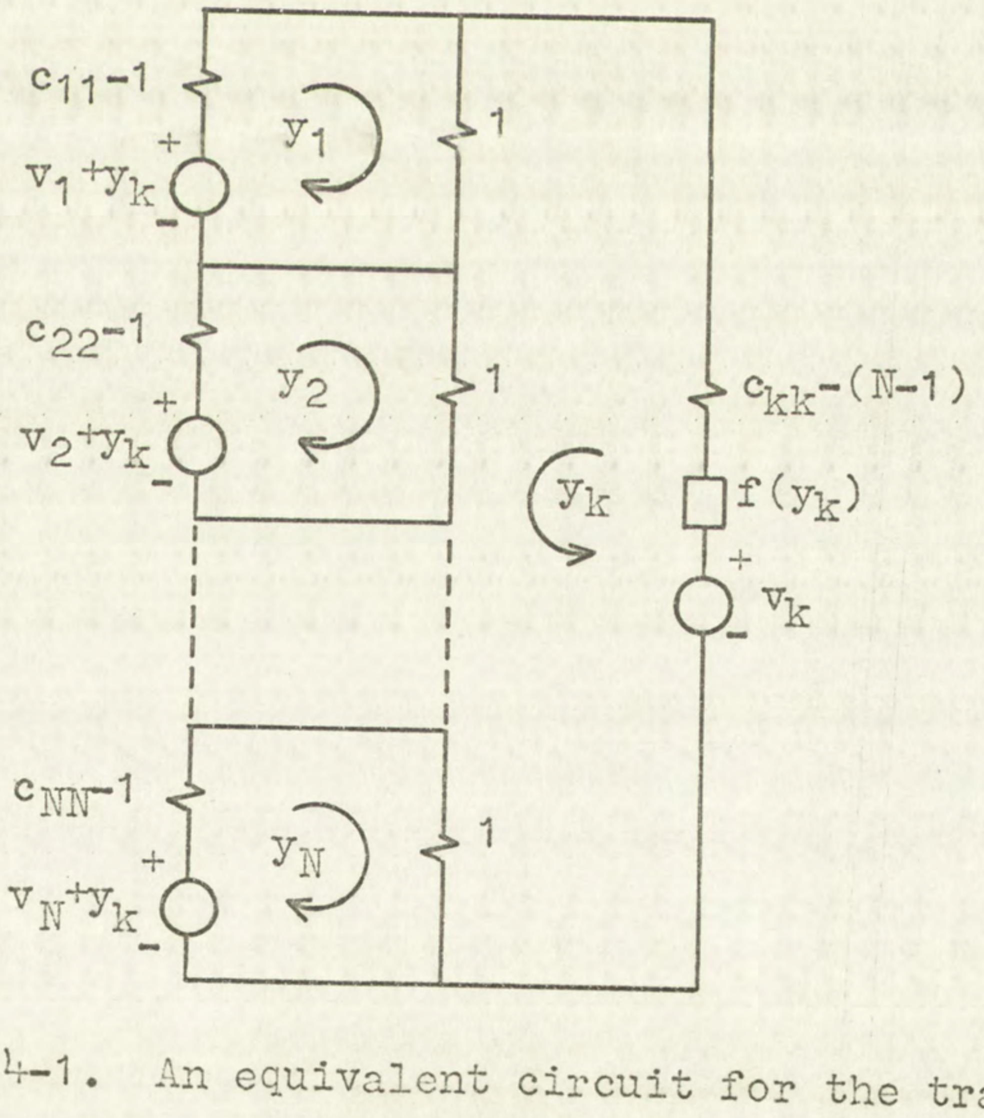

Figure 4-1. An equivalent circuit for the transformed network.

In this circuit, the terms $(c_{ii}-1)$ are rational functions of p and represent equivalent impedances; each voltage generator $(v_i + y_k)$ is dependent on the loop current y_k; a one ohm resistor appears in each linear loop.

A more interesting equivalent circuit is shown in Figure 4-2.

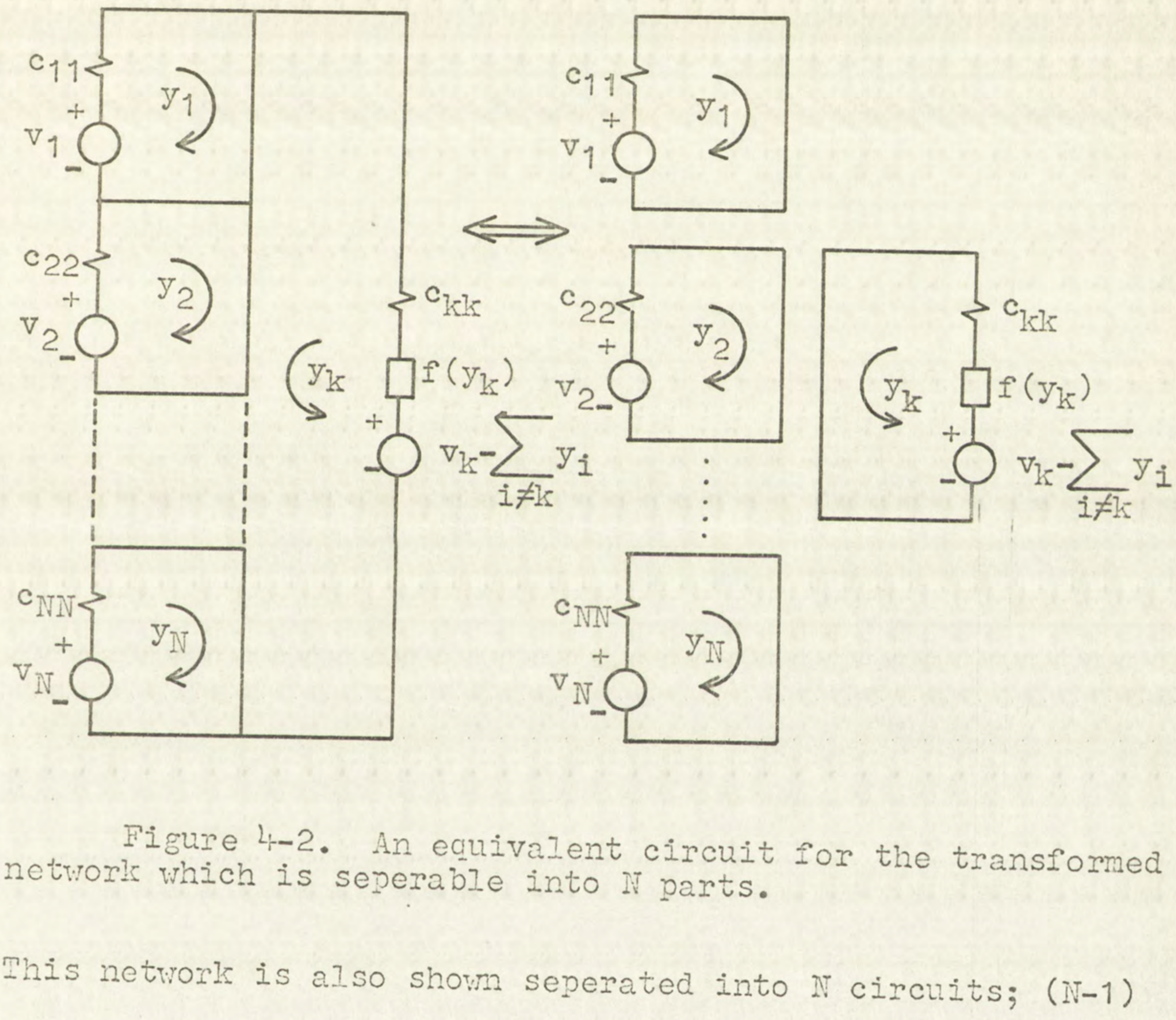

Figure 4-2. An equivalent circuit for the transformed network which is seperable into N parts.

This network is also shown seperated into N circuits; (N-1) of these involve only linear functions and each is independent of all the other circuits. However, the k^{th} circuit contains the nonlinear element and a voltage generator which is dependent on each of the other currents.

All of the equivalent circuits shown demonstrate the impossibility of achieving N completely independent equations through use of a transformation of variables if a nonlinear function is present. This is to be compared with the completely linear case in which a transformation of variables will yield a coefficient matrix which is strictly diagonal.[3]

The question of the realizability of the elements c_{ii} and c_{ii}^{-1} would bear further investigation. Equations (4-12) and (4-22) would have to be expanded as functions of p and the resulting coefficients examined for sign.

Parameters of the Transformation in Terms of p

From Chapter II,

$$a_{ij} = \frac{Lp^2 + Rp + D}{p}$$

Therefore,

$$\text{cof}(a_{iq,Jk}) = \frac{A_1 p^{2(N-2)} + B_1 p^{2(N-2)-1} + \ldots + K_1}{p^{N-2}}$$

and

$$\text{cof}(a_{Jk}) = \frac{A_2 p^{2(N-1)} + B_2 p^{2(N-1)-1} + \ldots + K_2}{p^{N-1}}$$

From equations (4-17),

$$b_{qJ} = \frac{A_1 p^{2N-3} + B_1 p^{2N-4} + \ldots + K_1 p}{A_2 p^{2N-2} + B_2 p^{2N-3} + \ldots + K} \; ; \qquad (4\text{-}28)$$

and from equation (4-26),

$$b_{qk} = \frac{A_3p^{2N} + B_3p^{2N-1} + \ldots + K_3}{A_4p^{2N} + B_4p^{2N-1} + \ldots + K_4} \qquad (4\text{-}29)$$

Using equations (4-12) and (4-22) it is seen that

$$c_{JJ} = \frac{A_5p^{2N} + B_5p^{2N-1} + \ldots + K_5}{A_6p^{2N} + B_6p^{2N-1} + \ldots + K_6} \qquad (4\text{-}30)$$

and

$$c_{kk} = \frac{A_7p^{2N+2} + B_7p^{2N+1} + \ldots + K_7}{p(A_8p^{2N} + B_8p^{2N-1} + \ldots + K_8)} \qquad (4\text{-}31)$$

Solution of the Transformed Linear Equations

It is proposed to solve each of the linear equations in system (4-27) and substitute the results in the nonlinear equation. Each linear equation will be of the form

$$c_{JJ}y_J = v_J$$

By equation (4-31), this becomes, upon rearrangement,

$$(A_5p^{2N} + B_5p^{2N-1} + \ldots + K_5)y_J(t)$$
$$= (A_6p^{2N} + B_6p^{2N-1} + \ldots + K_6)v_J(t) \qquad (4\text{-}32)$$

If initial conditions are unknown, the solution for $y_J(t)$ will involve 2N undetermined constants. Thus, the nonlinear equation will ultimately include 2N(N-1) of these constants. Only by inverse transformation could the constants

be evaluated. For large N, this would be cumbersome. However, it should be possible to determine the initial conditions for y_i from the initial conditions of x_i by means of the transformation, equation (4-2). In this case, the solution of equation (4-32) is straightforward.

<u>General Form of the Nonlinear Equation</u>

In equation (4-27), the nonlinear equation is

$$c_{kk}y_k + \sum_{i=1, i\neq k}^{N} y_i = v_k - f(y_k) \qquad (4\text{-}33)$$

Using equation (4-31), this becomes

$$\left[\frac{A_7p^{2N+2} + B_7p^{2N+1} + \ldots + K_7}{p(A_8p^{2N} + B_8p^{2N-1} + \ldots + K_8)}\right]y_k + f(y_k)$$

$$= v_k - \sum_{i=1, i\neq k}^{N} y_i = V(t) \; ; \qquad (4\text{-}34)$$

or

$$\left[\frac{N(p)}{pD(p)}\right]y_k + f(y_k) = V(t) \qquad (4\text{-}35)$$

where

$$N(p) = A_7p^{2N+2} + B_7p^{2N+1} + \ldots + K_7 \qquad (4\text{-}36)$$

and

$$D(p) = A_8p^{2N} + B_8p^{2N-1} + \ldots + K_8 \qquad (4\text{-}37)$$

A Concise Form for the Matrix Equation

If both sides of equation (4-35) are multiplied by p, the result is

$$\left[\frac{N(p)}{D(p)}\right]y_k + pf(y_k) = pV(t) = g'(t)$$

But,

$$pf(y_k) = (a_1 + 2a_2y_k + \ldots + na_ny_k^{n-1})py_k = c(y_k)py_k$$

and

$$\frac{N(p)}{D(p)} = pc_{kk}(p)$$

so that equation (4-35) becomes

$$\left[pc_{kk}(p) + c(y_k)p\right]y_k = g'(t) \qquad (4\text{-}38)$$

Then, in case k=1, equation (4-27) can be written

$$\begin{bmatrix} pc_{11} + cp & 1 \\ 0 & C \end{bmatrix}\begin{bmatrix} y \\ Y \end{bmatrix} = \begin{bmatrix} g' \\ V \end{bmatrix} \qquad (4\text{-}39)$$

in which C represents a diagonal submatrix having linear elements only, 1 is a row submatrix of unity elements, Y is a column submatrix of the unknown variables, and V is a column submatrix of the known excitations.[2]

While equation (4-39) is a concise representation of the network, the form of equation (4-27) is preferable for computational purposes.

CHAPTER V

SOLVING THE NONLINEAR EQUATION BY POWER SERIES SUBSTITUTION

The procedure described in Chapter III results in a nonlinear integro-differential equation of the type:

$$\left[\frac{N(p)}{pD(p)} + a_{kk}\right]x(t) + f(x) = \frac{1}{D(p)}.V(t) + v_k(t) \;; \qquad (5\text{-}1)$$

and the result in Chapter IV was:

$$\left[\frac{N(p)}{pD(p)}\right]y(t) + f(y) = V(t) \qquad (5\text{-}2)$$

Obviously, similar methods can be used to solve either equation. It is proposed to expand the term $\frac{N(p)}{pD(p)}$ as a power series in t. By equating coefficients of like terms, the coefficients of the series for y(t) can be determined. Thus, an approximation for y(t) can be obtained. The method will be described for equation (5-2); however, in the series expansions, the resulting form of equation (5-1) will be identical to that of equation (5-2).

Modification of equations (4-36) and (4-37), with n=2N+2, yields

$$N(p) = A_n p^n + A_{n-1}p^{n-1} + \ldots + A_0 \qquad (5\text{-}3)$$

and

$$pD(p) = B_{n-1}p^{n-1} + B_{n-2}p^{n-2} + \ldots + B_1 p \qquad (5\text{-}4)$$

By long division,

$$\frac{N(p)}{pD(p)} = F_1 p + F_0 + F_{-1}p^{-1} + F_{-2}p^{-2} + \ldots \qquad (5\text{-}5)$$

Let

$$y(t) = \sum_{i=0}^{\infty} b_i t^i \qquad (5\text{-}6)$$

and from Chapter I,

$$f(y) = a_0 + a_1 y + a_2 y^2 + \ldots + a_m y^m \qquad (5\text{-}7)$$

Since y(t) is given as a power series, higher powers of y will also be power series. Thus, let

$$y^2(t) = \sum_{i=0}^{\infty} c_i t^i , \qquad (5\text{-}8)$$

$$y^3(t) = \sum_{i=0}^{\infty} d_i t^i , \qquad (5\text{-}9)$$

$$y^{m-1}(t) = \sum_{i=0}^{\infty} m_i' t^i , \qquad (5\text{-}10)$$

and

$$y^m(t) = \sum_{i=0}^{\infty} m_i t^i . \qquad (5\text{-}11)$$

It can be shown that

$$c_i = 2\sum_{j=0}^{\frac{i-1}{2}} b_j b_{i-j} \text{ (for i odd)}, \qquad (5\text{-}12)$$

and

$$c_i = b_{\frac{i}{2}}^2 + 2\sum_{j=0}^{\frac{i-1}{2}} b_j b_{i-j} \text{ (for i even)} . \quad (5\text{-}13)$$

By induction,

$$m_i = \sum_{j=0}^{i} b_j m'_{i-j} . \quad (5\text{-}14)$$

Thus, each of the coefficients in the series for higher order powers of y(t) is expressible in terms of the coefficients in the series for y(t).

In addition, let

$$V(t) = \sum_{i=0}^{\infty} K_i t^i \quad (5\text{-}15)$$

where each K_i can be determined since V(t) is a known time function.

By use of equations (5-5) through (5-15), equation (5-2) becomes

$$\left[F_1 p + F_0 + F_{-1}p^{-1} + F_{-2}p^{-2} + \ldots\right]\sum_{i=0}^{\infty} b_i t^i$$

$$+ a_0 + a_1\sum_{i=0}^{\infty} b_i t^i + a_2\sum_{i=0}^{\infty} c_i t^i + \ldots + \sum_{i=0}^{\infty} m'_i t^i + \sum_{i=0}^{\infty} m_i t^i$$

$$= \sum_{i=0}^{\infty} K_i t^i \quad (5\text{-}16)$$

Performing the indicated operations, we obtain

$$\sum_{i=0}^{\infty}\left(F_1 i b_i t^{i-1} + F_0 b_i t^i + \frac{F_{-1} b_i t^{i+1}}{i+1} + \ldots + \frac{F_{-k} b_i t^{i+k}}{(i+k)\ldots(i+1)} + \ldots\right)$$

$$+ \sum_{i=0}^{\infty}(a_2 c_i t^i + a_3 d_i t^i + \ldots + a_m m_i t^i) = \sum_{i=0}^{\infty} K_i t^i \quad (5\text{-}17)$$

where K_0 now includes a_0 and a constant of integration and F_0 includes a_1.

Equating coefficients of like powers in t yields the following set of equations:

$$F_1 b_1 + F_0 b_0 + a_2 c_0 + \ldots + a_m m_0 = K_0 \ ,$$

$$2F_1 b_2 + F_0 b_1 + F_{-1} b_0 + a_2 c_1 + \ldots + a_m m_1 = K_1 \ ,$$

$$3F_1 b_3 + F_0 b_2 + \frac{F_{-1}}{2} b_1 + \frac{F_{-2}}{2} b_0 + a_2 c_2 + \ldots + a_m m_2 = K_2 \ ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(q+1)F_1 b_{q+1} + F_0 b_q + \frac{F_{-1}}{q} b_{q-1} + \ldots + \frac{F_{-(q-1)}}{q!} b_1 + \frac{F_{-q}}{q!} b_0$$

$$+ a_2 c_q + \ldots + a_m m_q = K_q \ .$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5\text{-}18)$$

Since the series for y(t) is a Maclaurin expansion about the origin, it is well known that

$$b_i = \left.\frac{d^{(i)}y}{dt^i}\right|_{t=0^+} \quad (5\text{-}19)$$

Specifically,

$$b_0 = y(0^+)$$

and

$$b_1 = \left.\frac{dy}{dt}\right|_{t=0^+} .$$

In chapter IV, it was specified that $y_k(t) = x_k(t)$; therefore, b_0 and b_1 are readily obtainable as network initial conditions on $x_k(t)$.

If these initial conditions are substituted into the first of equations (5-18), K_0, which contains an arbitrary constant of integration, may be evaluated. However, this is not necessary since the same substitution in the second equation yields an equation solvable for b_2. Obviously, this procedure can be continued to find as many of the b's as desired. The resulting set of b's, when substituted in equation (5-6), will constitute an approximation to the solution of equation (5-2). The inverse transformation, equation (4-2), will then yield solutions for the remainder of the network unknowns.

Example

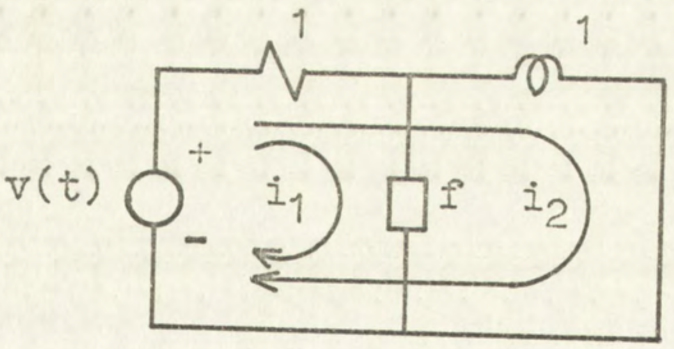

Figure 5-1

In the network of Figure 5-1, it is assumed that

$$f(i) = i + i^2, \tag{5-20}$$

$$v(t) = e^{-t} + \frac{1}{2}e^{-2t}, \tag{5-21}$$

and

$$i_1(0^+) = 1 . \tag{5-22}$$

The loop equations are as follows:

$$i_1 + \qquad i_2 = v(t) - f(i_1) \tag{5-23}$$

$$i_1 + (p+1)i_2 = v(t) \tag{5-24}$$

The transformation of variables is defined as

$$i_1 = b_{11}y_1 + b_{12}y_2 \tag{5-25}$$

$$i_2 = b_{21}y_1 + b_{22}y_2 \tag{5-26}$$

According to equations (4-5) and (4-6), let

$$b_{11} = 1 \tag{5-27}$$

and

$$b_{12} = 0 . \tag{5-28}$$

Substitution of equations (5-25) through (5-28) in equations (5-23) and (5-24) yields

$$(1 + b_{21})y_1 + \qquad b_{22}y_2 = v(t) - f(y_1)$$

$$1 + b_{21}(p+1)\ y_1 + (p+1)b_{22}y_2 = v(t)$$

According to equations (4-9), let

$$b_{22} = 1 \tag{5-29}$$

and

$$1 + b_{21}(p+1) = 0 .$$

Therefore,

$$b_{21} = \frac{-1}{p+1} . \tag{5-30}$$

The resulting system of equations is

$$\left(\frac{p}{p+1}\right)y_1 + \qquad y_2 = v(t) - f(y_1)$$

$$(p+1)y_2 = v(t)$$

When equations (5-20) and (5-21) are inserted, this becomes

$$\left(\frac{2p+1}{p+1}\right)y_1 + y_1^2 = e^{-t} + \tfrac{1}{2}e^{-2t} - y_2 \qquad (5\text{-}31)$$

$$(p+1)y_2 = e^{-t} + \tfrac{1}{2}e^{-2t} \qquad (5\text{-}32)$$

The solution to equation (5-32) is

$$y_2 = K_1 e^{-t} + te^{-t} - \tfrac{1}{2}e^{-2t} \ . \qquad (5\text{-}33)$$

Thus, equation (5-31) becomes

$$\left(\frac{2p+1}{p+1}\right)y_1 + y_1^2 = (K-t)e^{-t} + e^{-2t} \ . \qquad (5\text{-}34)$$

By division, we obtain

$$\left(\frac{2p+1}{p+1}\right) = 2 - p^{-1} + p^{-2} - p^{-3} + p^{-4} - p^{-5} + \ldots \qquad (5\text{-}35)$$

Assume that

$$y_1 = \sum_{i=0}^{\infty} b_i t^i \qquad (5\text{-}36)$$

and

$$y_1^2 = \sum_{i=0}^{\infty} c_i t^i \ . \qquad (5\text{-}37)$$

where c_i is given by equation (5-12) or (5-13).

Also,

$$(K-t)e^{-t} + e^{-2t} = (K+1)t^0 + (-K-3)t^1 + (\frac{K}{2}+3)t^2 + (\frac{-K}{3!}-\frac{1}{2}-\frac{8}{3!})t^3$$
$$+ (\frac{K}{4!}+\frac{1}{3!}+\frac{16}{4!})t^4 + (-\frac{K}{5!}-\frac{1}{4!}-\frac{32}{5!})t^5 + (\frac{K}{6!}+\frac{1}{5!}+\frac{64}{6!})t^6$$
$$+ (-\frac{K}{7!}-\frac{1}{6!}-\frac{128}{7!})t^7 + \ldots \quad (5\text{-}38)$$

By substituting relations (5-35) through (5-38) in equation (5-34) and equating coefficients of like powers in t, the following set of equations is obtained:

$$2b_0 + c_0 = C_I + K + 1 \quad (5\text{-}39)$$
$$2b_1 - b_0 + c_1 = -K - 3 \quad (5\text{-}40)$$
$$2b_2 - \frac{b_1}{2} + \frac{b_0}{2} + c_2 = \frac{K}{2} + 3 \quad (5\text{-}41)$$
$$2b_3 - \frac{b_2}{3} + \frac{b_1}{6} - \frac{b_0}{6} + c_3 = -\frac{K}{3!} - \frac{1}{2} - \frac{8}{3!} \quad (5\text{-}42)$$
$$2b_4 - \frac{b_3}{4} + \frac{b_2}{12} - \frac{b_1}{24} + \frac{b_0}{24} + c_4 = \frac{K}{4!} + \frac{1}{3!} + \frac{16}{4!} \quad (5\text{-}43)$$

. . . .

In equation (5-39), C_I is a constant of integration.

Since, from equation (5-22),

$$i_1(0^+) = b_0 = 1 ,$$

it can be shown, in equations (5-23) and (5-24), that

$$i_1'(0^+) = b_1 = -1 .$$

Since b_0 and b_1 are known, K can be evaluated in equation (5-40). Thus,

$$K = -3 - 2b_1 + b_0 - 2b_0 b_1 = 2$$

The remainder of the equations can be solved successively and the results are:

$$C_I = 0$$

$$b_2 = \frac{1}{2} = \frac{1}{2!}$$

$$b_3 = \frac{-1}{6} = \frac{-1}{3!}$$

$$b_4 = \frac{1}{24} = \frac{1}{4!}$$

Therefore,

$$y_1(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \ldots = e^{-t}$$

From equations (5-25) through (5-30), it is seen that

$$i_1(t) = y_1(t) = e^{-t} \qquad (5\text{-}44)$$

and

$$i_2(t) = \frac{-1}{p+1} \cdot y_1(t) + y_2(t) \; .$$

Then,

$$(p+1)i_2 = -e^{-t} + (p+1)(2e^{-t} + te^{-t} - \tfrac{1}{2}e^{-2t})$$

$$= \tfrac{1}{2}e^{-2t} \; .$$

It can be shown that

$$i_2(0^+) = -\frac{3}{2} \; ;$$

thus

$$i_2(t) = -e^{-t} - \tfrac{1}{2}e^{-2t} \; . \qquad (5\text{-}45)$$

On the other hand, if both sides of equation (5-34)

are multiplied by (p+1), then equation (5-44) is a solution. This verifies the validity of the method in this example.

An exact solution was obtained because the series solution was recognizable as that for an exponential function. Although this will not always be the case, the terms which are calculated by this method will be identical to the corresponding terms of the actual solution. Thus, the solution is an approximation only because the higher ordered terms are omitted.

The steady-state solution for networks of this type has been treated by Lytal using an algorithm which yields uniformly convergent solutions.[4]

CHAPTER VI

CONCLUSION

Disadvantages of the Methods

The method used in Chapter III requires the operation of a rational function of p on some known voltage source time function. Since the method using a transformation of variables eliminates this requirement and also results in a similar nonlinear equation, it would seem to be preferable.

Either of the methods requires the evaluation of several determinants of order (N-1) and one determinant of order N. For large N, this is a laborious process; however, since the matrix (a_{ij}) is symmetric, only $N(N-1)/2 + N$ cofactors need be computed. In the analysis, expansion by cofactors was used only as a means to determine the form of the resultant equations; certainly, other means are available for evaluating large determinants.

The problem of initial conditions was not considered in detail. However, the method using the transformation requires that initial conditions for the transformed variables be obtainable from the original network equations. If this is impossible, then undetermined constants must be carried along in the solution of the nonlinear equation. Though more complicated, the procedure remains valid for this case.

Several implicit assumptions have been made in the analysis. The most important was that the parameters of the transformation, b_{ij}, could be treated as ordinary algebraic quantities although they are actually functions of the differential operator.[5] The validity of this assumption has not been proven here; but its use seems to yield a correct result.

In equations (5-16) and (5-17), it was assumed that differentiation and integration of a power series was possible. This is, in fact, valid only if the power series and the resulting series are convergent;[6] otherwise, another approach must be used to solve the nonlinear equation.

Minorsky has given an extensive analysis of nonlinear equations using perturbation techniques to derive statements concerning the stability of the response.[7]

Extension to Networks Having More Than One Nonlinear Element

If there is more than one nonlinear element in the network, there are two possibilities:

(1) A current in one nonlinear element is not common to a loop containing another nonlinear element.

(2) A current in one nonlinear element is common to a loop containing another nonlinear element.

In the first case, the equations can be written so that one nonlinear equation corresponds to one nonlinear element. These equations will be nonlinear in different, single unknowns. The second case would bear further investigation

[illegible]

[illegible] seems to yield a result

[illegible] second [illegible]

[illegible] power [illegible]

[illegible]

there are two possibilities:

(1) A current-mode nonlinear element is not common [illegible]

to a loop containing another nonlinear element.

(2) A current-mode nonlinear element is common to

a loop containing another nonlinear element.

In the first case, the equations can be written so that one

[illegible] contains only one nonlinear element.

[illegible]

The second case [illegible] further investigation in that the nonlinear equations may be nonlinear in more than one variable. However, it seems feasible that a method could be established which would cover each case and, therefore, the case of M nonlinear elements in an N-loop network.

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