Bivariate polynomial interpolation on the square at new nodal sets ☆

Marco Caliari [a], Stefano De Marchi [a,*], Marco Vianello [b]

[a] *Department of Computer Science, University of Verona, S.da Le Grazie, 15-Ca 'Vignal 2, Verona I-37137, Italy*

[b] *Department of Pure and Applied Mathematics, University of Padova, Italy*

Abstract

As known, the problem of choosing "good" nodes is a central one in polynomial interpolation. While the problem is essentially solved in one dimension (all good nodal sequences are asymptotically equidistributed with respect to the arc-cosine metric), in several variables it still represents a substantially open question. In this work we consider new nodal sets for bivariate polynomial interpolation on the square. First, we consider *fast Leja points* for tensor-product interpolation. On the other hand, for interpolation in P_n^2 on the square we experiment four families of points which are (asymptotically) equidistributed with respect to the *Dubiner metric*, which extends to higher dimension the arc-cosine metric. One of them, nicknamed *Padua points*, gives numerically a Lebesgue constant growing like log square of the degree.

☆ Work supported by the research project CPDA028291 "Efficient approximation methods for nonlocal discrete transforms" of the University of Padova, the ex-60% funds of the University of Verona, and by the GNCS-INdAM.

* Corresponding author.

E-mail address: demarchi@sci.univr.it (S. De Marchi).

1. Optimal and near-optimal interpolation points

Let $\Omega \subset \mathbb{R}^d$ be compact. We call *optimal* polynomial interpolation points a set $X_N^* \subset \Omega$ of cardinality N, such that the *Lebesgue constant*

$$\Lambda_n(X_N) = \max_{\mathbf{x}\in\Omega} \lambda_n(\mathbf{x}; X_N), \qquad \lambda_n(\mathbf{x}; X_N) := \sum_{i=1}^{N} |\ell_i(\mathbf{x}; X_N)|, \tag{1}$$

defined for all sets $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \Omega$ which are unisolvent for polynomial interpolation of degree n, attains its minimum on $X_N = X_N^*$. Here, $\lambda_n(\mathbf{x}; X_N)$ is the *Lebesgue function* of X_N, the ℓ_i are the fundamental Lagrange polynomials of degree n, and N is the dimension of the corresponding polynomial space, i.e. $N = \binom{n+d}{d}$, or $N = (n+1)^d$ for the tensor-product case (cf. e.g. [2,4,10]). To be more precise, the fundamental Lagrange polynomials are defined as the ratio

$$\ell_i(\mathbf{x}; X_N) = \frac{\mathrm{VDM}\left(X_N^{(i)}\right)}{\mathrm{VDM}(X_N)}, \tag{2}$$

where VDM denotes the Vandermonde determinants with respect to *any given basis* of the corresponding polynomial space, and where $X_N^{(i)}$ represents the set X_N in which $\mathbf{x}$ replaces $\mathbf{x}_i$. It comes easy to see that tensor-product Lagrange polynomials are simply the product of univariate Lagrange polynomials.

As well-known optimal points are not known explicitly, therefore in applications people consider near-optimal points, i.e. roughly speaking, points whose Lebesgue constant increases asymptotically like the optimal one. Moreover, letting $E_n(X_N) = \|f - P_n\|_{\infty,\Omega}$, where P_n is the interpolating polynomial of degree $\leqslant n$ on X of a given continuous function f, and $E_n^* = \|f - P_n^*\|_{\infty,\Omega}$ the *best uniform approximation error*, then

$$E_n(X_N) \leqslant (1 + \Lambda_n(X_N))E_n^*,$$

which represents an estimate for the *interpolation error*. Thus, near-optimal nodes minimize also (asymptotically) the interpolation error.

In the *one-dimensional case*, as well-known, Chebyshev, Fekete, Leja as well as the zeros of Jacobi orthogonal polynomials are near-optimal points for polynomial interpolation, and their Lebesgue constants increase *logarithmically* in the dimension N of the corresponding polynomial space (cf. [5,13]). All these points have asymptotically the *arc-cosine distribution*, that is they are asymptotically equidistributed w.r.t. the arc-cosine metric. We now recall the definition of two important univariate nodal sets: it Fekete and *Leja* points.

Definition 1. Given $X_N = \{x_1, \ldots, x_N\} \subset [a,b]$ let $\mathrm{VDM}(X_N) = \det(x_i^{N-j})_{1\leqslant i,j\leqslant N}$ be the classical Vandermonde determinant. The *Fekete* points are the set $F = \{f_1, \ldots, f_N\}$ such that

$$|\mathrm{VDM}(F_N)| = \max_{X_N \subset [a,b]} |\mathrm{VDM}(X_N)|.$$

Definition 2. Let λ_1 arbitrarily chosen in $[a,b]$. The points $\lambda_s \in [a,b]$, $s = 2, \ldots, N$, such that

$$\prod_{k=1}^{s-1} |\lambda_s - \lambda_k| = \max_{x \in [a,b]} \prod_{k=1}^{s-1} |x - \lambda_k| \tag{3}$$

are called a *Leja sequence* for the interval $[a,b]$ (cf. [12]).

The relation that makes the connection between Fekete and Leja points is the maximization of the Vandermonde determinant $\mathrm{VDM}(X_N)$ on a set $X_N = \{x_1, \ldots, x_N\} \subset [a,b]$. The set $F_N = \{f_1, \ldots, f_N\}$ of Fekete points is the one that *globally* solve the multi-dimensional optimization problem $\max_{X_N} \subset [a,b]|\mathrm{VDM}(X_N)|$. On the other hand, as $|\mathrm{VDM}(X_N)| = |\mathrm{VDM}(X_{N-1})| \cdot \prod_{i=1}^{N-1} |x_N - x_i|$, to determine the kth point of the set $L_N = \{x_1, \ldots, x_N\}$ of Leja points, once we have computed the $x_1, \ldots, x_{k-1}$, we simply solve the one-dimensional problem $\max_{x \in [a,b]} \prod_{i=1}^{k-1} |x - x_i|$. Both sets of points F_N or L_N tend to minimize the associated Lebesgue constant, since from their definition they reduce the size of the fundamental Lagrange polynomials. We recall that Leja points are computationally effective for polynomial interpolation in Newton form, since they give an increasing sequence $L_{N-1} \subset L_N$, and they stabilize the computation of divided differences [15].

Differently from Leja points [7], the definition of Fekete points can be immediately extended to the multi-variate setting, and observing that by construction $\max_{\mathbf{x} \in \Omega} |\ell_i(\boldsymbol{x}, F_N)| \leqslant 1$ we obtain the rough overestimate

$$\Lambda_n(F_N) \leqslant N, \tag{4}$$

which is valid in any dimension d. However, for $d > 1$ the Fekete points are not known explicitly except for the tensor-product case (see the next section), and their computation for a given compact set is a difficult task, as discussed for example in the case of the triangle in [17].

2. Tensor-product Chebyshev–Lobatto and Leja points

Tensor-product interpolation is well studied and used in many applications (cf. e.g. [6,11]). Tensor-product Fekete points have been recently studied by Bos et al. in [4], where it has been proved that the n-dimensional tensor-products of Gauss–Lobatto quadrature points are also Fekete points for the cube.

Here we consider two sets of tensor-product nodes in the square $[a,b] \times [a,b]$, i.e. the *tensor-product Chebyshev–Lobatto* and *tensor-product Leja points*, which have the same asymptotic distribution of the tensor-product Fekete points. Tensor-product Leja points are generated by using the so-called *Fast Leja Points*, introduced by Baglama et al. in [1]. Fast Leja points are obtained by maximization over *adaptive* discretization of the interval $[a,b]$. This method allows to compute m Leja points with a *complexity* of roughly $\frac{1}{2}m^2$ flops.

In Fig. 1 we compare the growth of Lebesgue constants for tensor-product Chebyshev–Lobatto points (shortly TPC) and tensor-product fast Leja points (shortly TPL) with the theoretical bound $(1 + 2/\pi \log(n+1))^2$ for near-optimal points (tensor-product Chebyshev points) (cf. [5]). In fact, it is immediate to see that the Lebesgue constant for tensor-product interpolation points is the square of the univariate constant. In practice, we have estimated the Lebesgue constants by maximizing the Lebesgue function (cf. (1)) on a grid of 101×101 equally spaced points on the reference square. In Tables 1–3 we then show the errors of tensor-product interpolation with degrees $n = 24, 34, 44, 54$, corresponding to three test functions with different degree of regularity: the well-known *Franke* function

$$f_1(x_1,x_2) = \frac{3}{4}\mathrm{e}^{-\frac{1}{4}((9x_1-2)^2+(9x_2-2)^2)} + \frac{3}{4}\mathrm{e}^{-\frac{1}{49}((9x_1-2)^2-\frac{1}{10}(9x_2-2)^2)} + \frac{1}{2}\mathrm{e}^{-\frac{1}{4}((9x_1-7)^2+(9x_2-3)^2)} - \frac{1}{5}\mathrm{e}^{-((9x_1-4)^2+(9x_2-7)^2)}$$

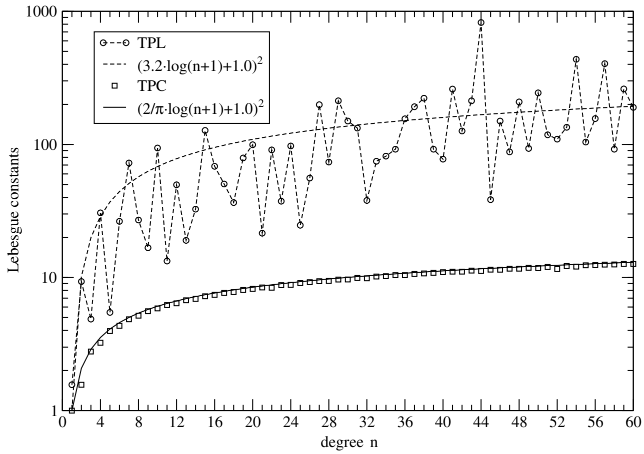

Fig. 1. Lebesgue constants for tensor-product Chebyshev–Lobatto (TPC) and Leja (TPL) points up to degree 60, compared with the theoretical bound for TPC and with a least-square fitting for TPL.

Table 1
Tensor-product interpolation errors for the Franke function

	N			
	25^2	35^2	45^2	55^2
TPC	1.2×10^{-3}	2.3×10^{-6}	1.5×10^{-9}	1.9×10^{-13}
TPL	2.5×10^{-3}	6.4×10^{-6}	8.9×10^{-9}	1.4×10^{-12}

Table 2
Tensor-product interpolation errors for the function $f_2(x_1, x_2) = (x_1^2 + x_2^2)^{5/2}$

	N			
	25^2	35^2	45^2	55^2
TPC on $[-1,1]^2$	6.0×10^{-5}	8.2×10^{-6}	1.8×10^{-6}	5.4×10^{-7}
TPC on $[0,2]^2$	8.5×10^{-9}	1.7×10^{-10}	1.4×10^{-11}	1.1×10^{-11}
TPL on $[-1,1]^2$	8.4×10^{-5}	1.6×10^{-5}	9.4×10^{-6}	8.3×10^{-7}
TPL on $[0,2]^2$	2.3×10^{-8}	6.3×10^{-10}	1.4×10^{-11}	1.8×10^{-11}

Table 3
Tensor-product interpolation errors for the function $f_3(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$

	N			
	25^2	35^2	45^2	55^2
TPC on $[-1,1]^2$	2.1×10^{-1}	1.1×10^{-1}	6.8×10^{-2}	4.6×10^{-2}
TPC on $[0,2]^2$	2.8×10^{-3}	5.8×10^{-4}	1.1×10^{-4}	8.9×10^{-5}
TPL on $[-1,1]^2$	5.7×10^{-1}	5.6×10^{-1}	6.2×10^{-1}	1.1×10^{-1}
TPL on $[0,2]^2$	3.9×10^{-3}	1.2×10^{-3}	5.8×10^{-5}	2.8×10^{-5}

considered as usual on $[0,1]^2$, $f_2(x_1, x_2) = (x_1^2 + x_2^2)^{5/2}$ and $f_3(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$. Observe that f_2 and f_3 are not regular at the origin, in particular f_2 is $\mathscr{C}^4$ with lipschitzian fourth partial derivatives and fifth partial derivatives discontinuous at the origin, while f_3 is lipschitzian with first partial derivatives discontinuous at the origin.

Even if the behavior of TPL Lebesgue constant is worse than that of TPC (see again Fig. 1), in the numerical tests the TPL approximation errors turn out to be closer to TPC errors than predicted by the ratio of Lebesgue constants (the errors have been computed on the same uniform control grid used to estimate the Lebesgue constant). Moreover, one can notice that the approximation performs better when the singularity is located at a corner of the square, since both TPC and TPL cluster by construction at the sides and especially at the corners.

3. Bivariate interpolation and Dubiner metric

3.1. Generalized arc-cosine metric

In [9], Dubiner proposed a metric which encapsulates the local properties of polynomial spaces on a given multi-variate compact set, and in one dimension coincides with the *arc-cosine metric*:

$$\mu_{[-1,1]}(x,y) := |\cos^{-1}(x) - \cos^{-1}(y)| \quad \forall x,y \in [-1,1].$$

Following [9], it can be proven by means of the *van der Corput-Schaake inequality* [20] (cf. [3,7] for details) that

$$\mu_{[-1,1]}(x,y) = \sup_{\|P\|_{\infty,[-1,1]} \leqslant 1} (\deg P)^{-1} |\cos^{-1}(P(x)) - \cos^{-1}(P(y))|, \tag{5}$$

where P varies in $\mathbb{P}([-1,1])$. By generalizing, define

$$\mu_{\Omega}(\mathbf{x},\mathbf{y}) = \sup_{\|P\|_{\infty,\Omega} \leqslant 1} (\deg P)^{-1} |\cos^{-1}(P(\mathbf{x})) - \cos^{-1}(P(\mathbf{y}))|, \quad \mathbf{x},\mathbf{y} \in \Omega \subset \mathbb{R}^d, \tag{6}$$

where P varies in $\mathbb{P}(\Omega)$, which is the *Dubiner metric* on the compact Ω.

In view of the properties of such a metric (cf. [9]), one may state [3] the following

- *conjecture*: *nearly optimal interpolation points on a compact Ω are asymptotically equidistributed with respect to the Dubiner metric on Ω.*

This suggests a general way to produce candidates to be good interpolation points, *once we know the Dubiner metric* for the compact set Ω. Unfortunately the Dubiner metric is explicitly known only in very few cases, for $d = 2$ namely the square and the circle

- $\Omega = [-1,1]^2$, $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$:

$$\mu_{\Omega}(\mathbf{x},\mathbf{y}) = \max\{|\cos^{-1}(x_1) - \cos^{-1}(y_1)|, |\cos^{-1}(x_2) - \cos^{-1}(y_2)|\}.$$

- $\Omega = \{\boldsymbol{x} : |\mathbf{x}| \leqslant 1\}$, $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$:

$$\mu_{\Omega}(\mathbf{x},\mathbf{y}) = \left|\cos^{-1}\left(x_1x_2 + y_1y_2 + \sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}\right)\right|.$$

In the following subsections we test the conjecture above on four sets of points on the square which are (asymptotically) equidistributed with respect to the Dubiner metric. The first one is obtained numerically using a reasonable definition of asymptotic equidistribution in a given metric. The other three are given by explicit formulas and are exactly equidistributed in the Dubiner metric.

3.2. Quasi-uniform points in the Dubiner metric

Following [8] we can construct a sequence of points which are asymptotically equidistributed in a compact Ω with respect to a given metric v, by means of the following *geometric greedy algorithm*:

- Let Ω be a compact set in $\mathbb{R}^d$, and consider $X_0 = \{\boldsymbol{x}_0\}$ where $\mathbf{x}_0 \in \partial\Omega$.
- If $X_j \subset \Omega$ is finite and consisting of j points, choose $\mathbf{x}_{j+1} \in \Omega \setminus X_j$ so that its distance to X_j is maximal and form $X_{j+1} := X_j \cup \{\mathbf{x}_{j+1}\}$.

Remarks

- For numerical purposes Ω must be finite with *cardinality* C_Ω (i.e. a discretization of Ω). Then, each step of the algorithm can be carried out in $\mathcal{O}(C_\Omega)$ operations, since for each $x \in \Omega \setminus X_j$ we should compute the distance to its nearest neighbor within X_j. To update this array of length C_Ω requires firstly to calculate the $C_\Omega - j$ values $v(\mathbf{x}, \mathbf{x}_i)$, $i = 1, \ldots, j$ and then taking the componentwise minimum within the ith array of distances. The next point $\mathbf{x}_{j+1}$ is then easily found by picking the maximum of the array of minima.
- It is worth noticing that the construction technique in the geometric greedy algorithm, is conceptually similar to that used in generating univariate *Leja sequences*. Indeed, in both approaches we maximize a function of distances from already computed points (in practice, on a suitable discretization of Ω).

Defining the *separation distance*

$$q_j := \frac{1}{2} \min_{\substack{\mathbf{x},\mathbf{y} \in X_j \\ \mathbf{x} \neq \mathbf{y}}} v(\mathbf{x}, \mathbf{y}),$$

and the *fill distance*

$$h_j := \max_{\mathbf{x} \in \Omega} \min_{\mathbf{y} \in X_j} v(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{y} \in X_j} v(\mathbf{x}_{j+1}, \mathbf{y}),$$

by a generalization to an arbitrary metric v of the proof in [8], it can be shown that:

Proposition 3. *The geometric greedy algorithm produces sequences which are quasi-uniform in the metric* v, *that is*

$$h_j \geqslant q_j \geqslant \frac{1}{2}h_{j-1} \geqslant \frac{1}{2}h_j \quad \forall j \geqslant 2.$$

In Fig. 2 we show the distribution of $N = 496$ (which correspond to polynomial degree 30) quasi-uniform Dubiner (shortly quD) points on the square, computed by the geometric greedy algorithm starting from a sufficiently dense random discretization of the square. We chose a discretization with N^3 random points, in analogy with the considerations in [16] for the extraction of Leja points from compact sets in the complex plane (see also our Remarks above). We also show the behavior of Lebesgue constants up to degree 28 for quasi-uniform Dubiner points, compared with quasi-uniform points in the Euclidean metric and random points on the square $[-1, 1]^2$. Here and below, the fundamental Lagrange polynomials are computed by inverting the $N \times N$ Vandermonde matrix built, for stability reasons, by using the Chebyshev basis, $\{T_i(x_1)T_j(x_2),\ i + j \leqslant n\}$. As in the tensor-product case, we have estimated numerically the Lebesgue constants by maximizing the Lebesgue function on a suitable grid.

The comparison of the Lebesgue constants in Fig. 2 shows that the quasi-uniform Dubiner points are much better for polynomial interpolation than the quasi-uniform Euclidean (shortly EUC) and the random ones (shortly RND), since the growth of their Lebesgue constant is polynomial instead of exponential in the degree. However, they are not still satisfactory since the growth is of order $N^{3/2}$, which is bigger than the theoretical bound of the Fekete points (cf. (4)). This suggests that quasi-uniformity in the Dubiner metric is not sufficient for

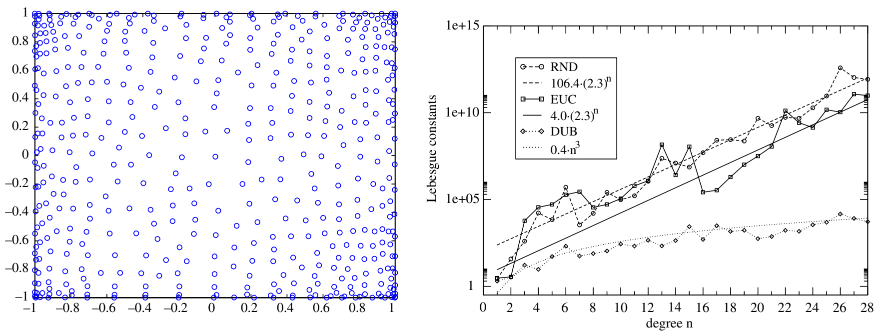

Fig. 2. Left: 496 (i.e. degree 30) quasi-uniform Dubiner (DUB) points for the square; right: Lebesgue constants for DUB points, quasi-uniform Euclidean (EUC) points and random (RND) points.

near-optimality of the interpolation points. This it will be confirmed by the three sets of points that we present in the next section.

3.3. Morrow–Patterson (MP) points

Morrow and Patterson (cf. [14]), proposed for cubature purposes the following set of nodes on the square. For n, a positive *even* integer, consider the points $X_N^{\mathrm{MP}} = \{(x_m, y_k)\} \subset [-1,1]^2$ given by

$$x_m = \cos\left(\frac{m\pi}{n+2}\right), \qquad y_k = \begin{cases} \cos\left(\dfrac{2k\pi}{n+3}\right) & m \text{ odd} \\ \cos\left(\dfrac{(2k-1)\pi}{n+3}\right) & m \text{ even} \end{cases} \tag{7}$$

$1 \leqslant m \leqslant n+1$, $1 \leqslant k \leqslant \frac{n}{2}+1$. It is easily seen that these points are exactly equally spaced w.r.t. the Dubiner metric, i.e. they have a constant pointwise separation distance, cf. Section 3.1. This set consists of $N = \binom{n+2}{2}$ points, which is equal to $\dim(\mathbb{P}_n(\mathbb{R}^2))$, and is unisolvent for polynomial interpolation on the square. In fact, in view of the Christoffel–Darboux formulas of Xu [20,21], the fundamental Lagrange polynomials at the Morrow–Patterson points have an explicit expression in terms of second kind Chebyshev polynomials. Hence, the interpolation problem has a *constructive* solution, which implies that the nodes give a unisolvent set and $\mathrm{VDM}(X_N^{\mathrm{MP}}) \neq 0$.

As for the growth of the Lebesgue constant, Bos [3] proved that $\Lambda_n^{\mathrm{MP}} = \mathcal{O}(n^6)$. From our experiments we showed that this bound can be strongly improved, since $\Lambda_n^{\mathrm{MP}} = \mathcal{O}(n^2)$ as can be seen in Fig. 4. In particular we found that Λ_n^{MP} can be least-square fitted with the quadratic polynomial $(0.7n + 1)^2$, which is smaller than N, i.e. than the theoretical bound for Fekete points.

3.4. Extended Morrow–Patterson (EMP) points

In analogy with the one-dimensional setting [5], we tried to improve the Lebesgue constant by considering the extended Morrow–Patterson points, which correspond to using extended Chebyshev nodes in (7), i.e. $X_N^{\mathrm{EMP}} = \{(x_m, y_k)\} \subset [-1,1]^2$ given by

$$x_m = \frac{1}{\alpha_n}\cos\left(\frac{m\pi}{n+2}\right), \qquad y_k = \begin{cases} \dfrac{1}{\beta_n}\cos\left(\dfrac{2k\pi}{n+3}\right) & m \text{ odd} \\ \dfrac{1}{\beta_n}\cos\left(\dfrac{(2k-1)\pi}{n+3}\right) & m \text{ even} \end{cases} \tag{8}$$

$1 \leqslant m \leqslant n+1$, $1 \leqslant k \leqslant \frac{n}{2}+1$, where the *dilation coefficients* $1/\alpha_n$ and $1/\beta_n$ correspond to

$$\alpha_n = \cos(\pi/(n+2)), \qquad \beta_n = \cos(\pi/(n+3)).$$

As the Morrow–Patterson points, the EMP points are exactly equally spaced w.r.t. the Dubiner metric. Moreover, it is not difficult to show that these points are again insolvent for polynomial interpolation of degree n. Indeed, the Vandermonde matrix of X_N^{EMP} w.r.t. the canonical basis of $\mathbb{P}_n(\mathbb{R}^2)$, is given by the Vandermonde matrix of the Morrow–Patterson points, where each column is scaled by a suitable constant. In particular, the column corresponding to the monomial $x^i y^j$ is multiplied by $\alpha_n^{-i}\beta_n^{-j}$: hence, $|\mathrm{VDM}(X_N^{\mathrm{EMP}})|$ is strictly greater than $|\mathrm{VDM}(X_N^{\mathrm{MP}})|$, i.e. it cannot vanish.

In Fig. 4 we reported a least-square fitting of the Lebesgue constant Λ_n^{EMP} for n up to 60. The growth is again quadratic in the degree, that is linear in the dimension of the polynomial space, but slower than that of the basic Morrow–Patterson points. However, concerning the Lebesgue function, we have numerical evidence that it is not true that $\lambda_n(x_1, x_2; X_N^{\mathrm{MP}}) < \lambda_n(x_1, x_2; X_N^{\mathrm{EMP}})$ for every x_1, x_2, while $\Lambda_n(X_N^{\mathrm{MP}}) < \Lambda_n(X_N^{\mathrm{EMP}})$.

3.5. Modified Morrow–Patterson points or Padua (PD) points

For n a positive *even* integer consider the points $(x_m, y_k) \in [-1,1]^2$ given by

$$x_m = \cos\left(\frac{(m-1)\pi}{n}\right), \qquad y_k = \begin{cases} \cos\left(\dfrac{(2k-1)\pi}{n-1}\right) & m \text{ odd} \\ \cos\left(\dfrac{(2k-2)\pi}{n-1}\right) & m \text{ even} \end{cases} \tag{9}$$

$1 \leqslant m \leqslant n+1$, $1 \leqslant k \leqslant \frac{n}{2}+1$. These are *modified* Morrow–Patterson points that were firstly discussed in Padua by the authors with L. Bos and S. Waldron [19], and so we have decided to call them **Padua points** (shortly PD points); again, they are exactly equispaced w.r.t. the Dubiner metric on the square. For a sketch of the distribution of PD points and for a comparison with MP and EMP at small degree, see Fig. 3.

The Padua points are, to our knowledge, the best known nodes for polynomial interpolation on the square. In fact, from our experiments, $\Lambda_n^{\mathrm{PD}} = \mathcal{O}(\log^2 n)$ (see Fig. 4). Note that, the asymptotic growth of their Lebesgue constant turns out to be *exactly* that of the TPC nodes, cf. Fig. 2. Unfortunately, so far we have not even been able to prove that they are unisolvent, whereas we have numerical evidence of this property.

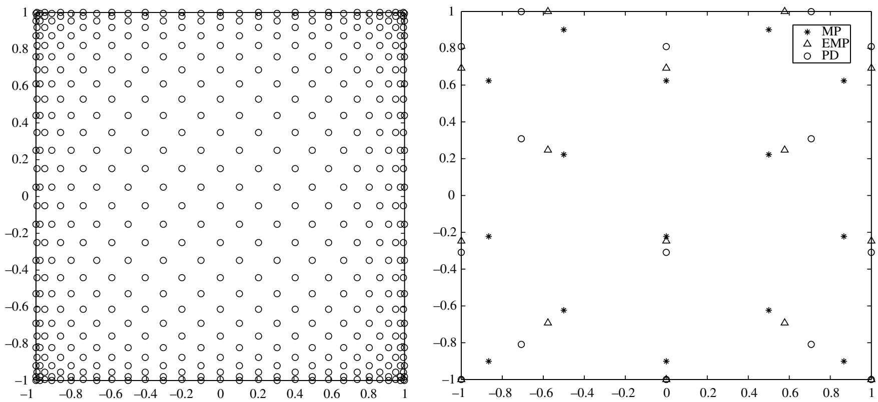

Fig. 3. Right: Morrow–Patterson (MP), extended Morrow–Patterson (EMP) and Padua (PD) points, for degree $n = 4$. Left: Padua points for degree $n = 30$.

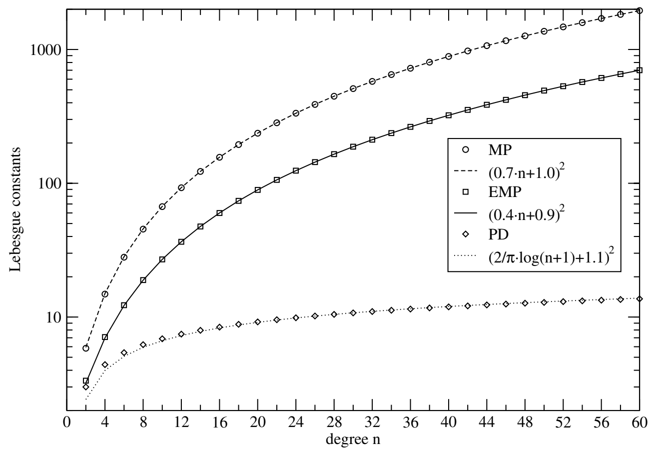

Fig. 4. The behavior of the Lebesgue constants for Morrow–Patterson (MP), extended Morrow–Patterson (EMP), Padua (PD) points up to degree 60, and their least-squares fitting curves.

Table 4
Interpolation errors for the Franke function

	$n = 34$	Λ_{34}	$n = 48$	Λ_{48}	$n = 62$	Λ_{62}	$n = 76$	Λ_{76}
MP	1.3×10^{-3}	649	2.6×10^{-6}	1264	1.1×10^{-9}	2082	2.0×10^{-13}	3102
EMP	6.3×10^{-4}	237	1.3×10^{-6}	456	5.0×10^{-10}	746	5.4×10^{-14}	1106
PD	4.3×10^{-5}	11	3.3×10^{-8}	13	5.4×10^{-12}	14	1.9×10^{-14}	15

Table 5
Interpolation errors for the function $f_2(x_1, x_2) = (x_1^2 + x_2^2)^{5/2}$

	n			
	34	48	62	76
MP on $[-1,1]^2$	1.8×10^{-4}	5.1×10^{-5}	1.9×10^{-5}	8.8×10^{-6}
MP on $[0,2]^2$	1.0×10^{-8}	3.8×10^{-10}	3.7×10^{-11}	2.3×10^{-11}
EMP on $[-1,1]^2$	6.5×10^{-5}	1.8×10^{-5}	6.7×10^{-6}	3.0×10^{-6}
EMP on $[0,2]^2$	7.2×10^{-9}	2.6×10^{-10}	2.4×10^{-11}	8.6×10^{-12}
PD on $[-1,1]^2$	3.6×10^{-6}	6.5×10^{-7}	1.8×10^{-7}	6.5×10^{-8}
PD on $[0,2]^2$	2.8×10^{-9}	9.3×10^{-11}	9.4×10^{-12}	6.4×10^{-12}

Table 6
Interpolation errors for the function $f_3(x_1, x_2) = (x_1^2 + x_2^2)^{1/2}$

	n			
	34	48	62	76
MP on $[-1,1]^2$	4.4×10^{-1}	4.4×10^{-1}	4.4×10^{-1}	4.4×10^{-1}
MP on $[0,2]^2$	8.8×10^{-4}	2.8×10^{-4}	2.6×10^{-4}	1.7×10^{-5}
EMP on $[-1,1]^2$	1.4×10^{-1}	1.4×10^{-1}	1.4×10^{-1}	1.4×10^{-1}
EMP on $[0,2]^2$	8.3×10^{-4}	2.6×10^{-4}	2.1×10^{-4}	2.1×10^{-5}
PD on $[-1,1]^2$	3.7×10^{-2}	2.7×10^{-2}	2.1×10^{-2}	1.7×10^{-2}
PD on $[0,2]^2$	7.3×10^{-4}	3.7×10^{-4}	7.0×10^{-6}	4.6×10^{-6}

3.6. Numerical tests with MP, EMP and PD points

In this section we apply interpolation at MP, EMP and PD points to the three test functions already considered in Section 2. The interpolation errors are displayed in Tables 4–6 below (the errors have been computed on the same uniform control grid used to estimate the Lebesgue constants). First, we observe that the interpolation degrees have been chosen in such a way that the dimension of polynomial spaces, and thus the number of function evaluations, is as close as possible to the dimension of the tensor-product polynomial spaces in Tables 1–3. For example, when $n = 34$ we have $N = \binom{n+2}{2} = 630$, to be compared with $25^2 = 625$ in the tensor-product case.

At a first glance, Tables 4–6 show that the errors of MP, EMP and PD are in decreasing order, with ratios of the size of the ratios between the corresponding Lebesgue constants (whose values have been rounded to the nearest integer). Moreover, by comparison with TPC we can appreciate that MP and EMP errors are comparable with TPC errors, while PD errors are one or two orders below. Concerning the functions f_2 and f_3, which have a singularity at the origin, again we see that interpolation performs better when the singularity is located at a corner of the square, where all the three families of nodes cluster by construction.

4. Conclusion

The above comparisons, together with the behavior of the Lebesgue constants, suggest that in principle the PD points should be adopted for polynomial interpolation on the square, whenever the underlying function can be evaluated everywhere. However, there is still a lot of work to do, from both the theoretical point of view, i.e. concerning unisolvence of the PD points and asymptotic analysis of Lebesgue constants for MP, EMP and PD points, and from the practical point of view, concerning efficient implementation of the interpolant. As for the last issue, it is worth recalling that an efficient construction method of the fundamental Lagrange polynomials is known only for the MP points, which is based on the Christoffel–Darboux formulas of Xu [20,21].

Acknowledgement

We are grateful to Len Bos and Shayne Waldron, for valuable discussions and suggestions during their recent visits at the Universities of Padua and Verona.

References

[1] J. Baglama, D. Calvetti, L. Reichel, Fast Leja points, Electron. Trans. Numer. Anal. 7 (1998) 124–140.

[2] L. Bos, On certain configurations of points in $\mathbb{R}^n$ which are unisolvent for polynomial interpolation, J. Approx. Theory 64 (3) (1991) 271–280.

[3] L. Bos, Multivariate interpolation and polynomial inequalities, Ph.D. course held at the University of Padua, unpublished.

[4] L. Bos, M.A. Taylor, B.A. Wingate, Tensor product Gauss–Lobatto points are Fekete points for the cube, Math. Comp. 70 (2001) 1543–1547.

[5] L. Brutman, Lebesgue functions for polynomial interpolation—a survey, Ann. Numer. Math. 4 (1997) 111–127.

[6] C. de Boor, The error in polynomial tensor-product, and Chung-Yao, interpolation, in: Surface Fitting and Multiresolution Methods, Vanderbilt University Press, Nashville, TN, 1997, pp. 35–50.

[7] S. De Marchi, On Leja sequences: some results and applications, Appl. Math. Comput. 152 (2004) 621–647.

[8] S. De Marchi, R. Schaback, H. Wendland, Near-optimal data-independent point locations for radial basis function interpolation, Adv. Comput. Math., in press.

[9] M. Dubiner, The theory of multi-dimensional polynomial approximation, J. Anal. Math. 67 (1995) 39–116.

[10] M. Gasca, T. Sauer, Polynomial interpolation in several variables, Adv. Comput. Math. 12 (2000) 377–410.
[11] P. Lancaster, K. Salkauskas, Curve and surface fitting. An introduction, Academic Press, London, 1986.
[12] F. Leja, Sur certaines suites liées aux ensembles plans er leur application à la représentation conforme, Ann. Polon. Math. 4 (1957) 8–13.
[13] G. Mastroianni, D. Occorsio, Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey, J. Comput. Appl. Math. 134 (1–2) (2001) 325–341.
[14] C.R. Morrow, T.N.L. Patterson, Construction of algebraic cubature rules using polynomial ideal theory, SIAM J. Numer. Anal. 15 (5) (1978) 953–976.
[15] L. Reichel, Newton interpolation at Leja points, BIT 30 (1990) 332–346.
[16] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Springer, Berlin, 1997.
[17] M.A. Taylor, B.A. Wingate, R.E. Vincent, An algorithm for computing Fekete points in the triangle, SIAM J. Numer. Anal. 38 (5) (2000) 1707–1720.
[18] J.G. van der Corput, G. Schaake, Ungleichungen für polynome und trigonometriske polynome, Compos. Math. 2 (1935) 321–361.
[19] S. Waldron, personal communication.
[20] Y. Xu, Gaussian cubature and bivariate polynomial interpolation, Math. Comput. 59 (1992) 547–555.
[21] Y. Xu, On multivariate orthogonal polynomials, SIAM J. Math. Anal. 24 (3) (1993) 783–794.