HIGHER K-THEORY OF FORMS I.
FROM RINGS TO EXACT CATEGORIES

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Abstract. We prove the analog for the K-theory of forms of the $Q = +$ theorem in algebraic K-theory. That is, we show that the K-theory of forms defined in terms of an $S_\bullet$-construction is a group completion of the category of quadratic spaces for form categories in which all admissible exact sequences split. This applies for instance to quadratic and hermitian forms defined with respect to a form parameter.

Contents

1. Introduction

This is the first in a series of articles addressing fundamental and computational aspects of higher algebraic K-theory of quadratic, hermitian, symplectic, skew-symmetric and other kinds of forms over rings and schemes avoiding the unnecessary but common assumption of inverting the prime 2. We will study these forms in the framework of additive and exact form categories and show in Theorem 6.6 the analog for forms of the "Q=+" theorem in algebraic K-theory [Gra76, Th p. 11].

The lower K-theory of forms has been studied by several authors, notably by Bak [Bak81]. But to our knowledge, there seems to be no systematic treatment of the higher K-theory of forms, at least not in the desired generality. When 2 is a unit in a ring with involution R, the higher K-theory of forms was introduced by Karoubi [Kar73], [Kar80a] under the name of Hermitian K-theory, and a modern treatment for dg categories with uniquely 2-divisible mapping complexes was given by the author in [Sch17b]. Motivated by applications in $\mathbb{A}^1$-homotopy theory, we

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initiated the study of higher K-theory of symmetric bilinear forms in [Sch10a] and [Sch10b] taking care to avoid the assumption $1/2 \in R$. It turns out, however, that if we want to study symmetric bilinear forms through the lens of derived categories – as was done in [Sch17b] under the uniquely 2-divisibility hypothesis – we also need to consider other types of forms as the examples in [Sch17b, Proposition 2.1] show. Indeed, we will see in forthcoming work that symmetric bilinear forms on one piece of the derived category of an algebraic variety can correspond to some different type of forms on another but derived-equivalent piece. This requires us to consider all types of forms in a unified way.

We do so in this paper by considering additive and exact form categories. These are additive categories $\mathcal{A}$ equipped with a quadratic functor $Q : \mathcal{A}^{op} \to \mathrm{Ab}$ which associates to each object X of $\mathcal{A}$ an abelian group $Q(X)$ of "quadratic forms" on X such that the associated symmetric bilinear functor comes from a duality functor $\sharp : \mathcal{A}^{op} \to \mathcal{A}$, $\mathrm{can}_X : X \to X^{\sharp\sharp}$. For instance, $Q(X)$ could be the set of quadratic (hermitian, symmetric, symplectic, anti-symmetric, alternating, even hermitian) forms on a projective module X over a ring with involution. By Definition 2.1 below, the quadratic functor Q comes with a functorial C_2-equivariant diagram of functors $\mathcal{A}^{op} \to \mathrm{Ab}$,

$$(\mathcal{A}(X, X^\sharp), \sigma) \xrightarrow{\tau} Q(X) \xrightarrow{\rho} (\mathcal{A}(X, X^\sharp), \sigma),$$

where $\sigma(f) = f^\sharp\,\mathrm{can}$, $\rho\tau = 1 + \sigma$ and $Q(f+g)(\xi) = Q(f)(\xi) + Q(g)(\xi) + \tau(f^\sharp \rho(\xi) g)$. A quadratic space in $(\mathcal{A}, Q)$ is an object X of $\mathcal{A}$ equipped with a quadratic form $\xi \in Q(X)$ on X such that the associated symmetric bilinear form $\rho(\xi) : X \to X^\sharp$ is an isomorphism. The groupoid $i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ of quadratic spaces and isometries in $\mathcal{A}$ is symmetric monoidal under orthogonal sum, and we define the orthogonal sum Grothendieck-Witt space $GW^\oplus(\mathcal{A}, Q)$ as the group completion of that symmetric monoidal groupoid, a model of which is given by Quillen's $\mathcal{S}^{-1}\mathcal{S}$ construction [Gra76] for $\mathcal{S} = i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$.

If $\mathcal{A}$ is equipped with a notion of exact sequences, we require $\sharp$ to be exact and Q to be quadratic left exact (Definitions 2.22, A.13). As was done for symmetric bilinear forms in [Sch10b], we use a variant of Waldhausen's $S_\bullet$-construction to define the Grothendieck-Witt space $GW(\mathcal{A}, Q)$ of an exact form category $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ by the homotopy fibration

$$GW(\mathcal{A}, Q) \longrightarrow |i\,\mathcal{Q}\mathrm{uad}(S_{2\bullet+1}\mathcal{A})| \longrightarrow |iS_\bullet\mathcal{A}|.$$

The main theorem of this paper is the following; see Theorem 6.6 in the text.

Theorem 1.1 (Group Completion Theorem)**.** *Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an exact form category with strong duality*[1] *in which every admissible exact sequence splits. Then there is a natural homotopy equivalence of spaces*

$$GW^\oplus(\mathcal{A}, Q) \xrightarrow{\sim} GW(\mathcal{A}, Q).$$

The importance of the theorem lies in the fact that the left space $GW^\oplus(\mathcal{A}, Q)$ is related to the homology of orthogonal and symplectic groups (Remark 2.19) whereas the right space $GW(\mathcal{A}, Q)$ leads to fibration sequences [Sch10b], [Scha] and the use of derived category methods [Sch17b], [Schb].

Special cases of Theorem 1.1 can already be found in the literature. In [CL86], the authors claimed a version of the theorem for symmetric bilinear forms, but

[1] *Strong duality* means the natural map $\mathrm{can}_X : X \to X^{\sharp\sharp}$ is an isomorphism for all $X \in \mathcal{A}$.

as noted in the introduction to [Sch04], their argument has an error. In [Sch04], we gave a proof of Theorem 1.1 for symmetric bilinear forms based on Karoubi's Fundamental Theorem [Kar80a] provided $\mathcal{A}$ is $\mathbb{Z}[1/2]$-linear. In [Sch17b] we gave another proof avoiding the use of the Fundamental Theorem. In the context of Real algebraic K-theory, Hesselholt and Madsen have given a proof of Theorem 1.1 for symmetric bilinear forms [HM15]. Whereas the proof in [HM15] is an elaboration of Quillen's techniques in [Qui76], our proof uses a generalisation of the techniques in [Sch17a] which go back to [NS89].

The results of this paper are used in [Schb] to prove an integral version of Karoubi's Fundamental Theorem [Kar80a] and in [Schc] to compute the symplectic and orthogonal K-groups of the integers.

Here is a more detailed outline of the contents of the article. In Section 2, we define linear, additive and exact form categories. The motivation for the definitions is explained in Appendix A. We also define the orthogonal sum Grothendieck-Witt space of an additive form category and study products in GW-theory. In Section 3, we define form parameter rings whose categories of modules are canonically equipped with a structure of form category. They give a convenient framework for defining the transfer maps in GW-theory which we will need in the proof of Theorem 5.1. In order to compute a particular transfer map, we specialise the treatment in Section 4 to form rings. These are the form categories with strict duality which have precisely one object (Lemma 4.5). Thus, for us, a *form ring* (R, Λ) is a C_2-equivariant diagram of abelian groups

$$(R, \sigma) \xrightarrow{\tau} \Lambda \xrightarrow{\rho} (R, \sigma)$$

such that $\rho\tau = 1 + \sigma$, where (R, σ) is a ring with involution, Λ is an abelian group equipped with a multiplicative quadratic action $Q : (R, \cdot, 0, 1) \to (\mathrm{End}_{\mathbb{Z}}(\Lambda), \circ, 0, 1)$ compatible with τ and ρ; see Definitions 3.3 and 4.1. Bak requires ρ to be injective [Bak81, p. 5]. This results in a parallel but separate treatment of hermitian and quadratic forms. We are forced to abandon the injectivity of ρ as it is meaningless in a homotopical context which is essential in the further development of the theory. As a byproduct, hermitian and quadratic forms can now be treated the same.

Form rings and free modules over them were also studied by Dotto and Ogle in [DOss] under the name Hermitian Mackey functor. However, form parameter rings and form categories seem not to have been studied before though there is a related notion of stable infinity category with non-degenerate quadratic functor due to Lurie [Lur13].

In Section 5 we prove an Additivity Theorem for orthogonal sum Grothendieck-Witt theory which is the main ingredient in the proof of the Group Completion Theorem 1.1. The idea of the proof is explained in Section 5.3. In Section 6 we give the proof of the main theorem. In Appendix A we recall the definition of a quadratic functor on an additive category and motivate the definition of such a functor for linear categories. Some of this material is similar to [Bau94]. In Appendix B we recall basic facts about C_2-Mackey functors, in Appendix C we consider the tensor product of unital abelian monoids which is a convenient framework for checking that certain quadratic functors on tensor products are well-defined. Finally, in Appendix D we recall and generalise some results from [Sch17a] needed in the proof of Theorem 5.1.

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Notation. Below is a list of notation used throughout the paper. They will also be explained at the appropriate place in the text.

- For rings R and S, we denote by $R\,\mathrm{Mod}$, $\mathrm{Mod}\,S$, $R\,\mathrm{Mod}\,S$ the categories of left R-modules, right S-modules, R-S-bimodules, and we denote by ${}_R[M,N]$, $[M,N]_S$ and ${}_R[M,N]_S$ the abelian groups of homomorphisms in those categories.
- If $M \in \mathrm{Mod}\,S$ and $N \in R\,\mathrm{Mod}\,S$, the abelian group $[M,N]_S$ of right S-module homomorphisms is a left R-module with scalar multiplication $(a,f) \mapsto af$ defined by $(af)(x) = a \cdot f(x)$ for $a \in R$, $f \in [M,N]_S$ and $x \in M$.
- If R is a ring with involution $R^{op} \to R : a \mapsto \bar{a}$ and $M \in R\,\mathrm{Mod}$, we let $M^{op} \in \mathrm{Mod}\,R$ be the right R-module $(M,+,0)$ with scalar multiplication $x^{op}a = (\bar{a}\cdot x)^{op}$. If M is an R-bimodule, then M^{op} is the R-bimodule with scalar multiplication $a \cdot x^{op} \cdot b = (\bar{b}x\bar{a})^{op} \in M^{op}$ for $a, b \in R$ and $x \in M$.
- For a map of sets $f : A \to B$ between abelian groups, the *deviation* of f is the map $f(\ \ {}_\top\ \ ) : A \times A \to B$ defined by $f(a\,{}_\top\, b) = f(a+b) - f(a) - f(b)$.
- C_2 is the cyclic group of order 2 with generator usually denoted by σ.
- Ab is the category of abelian groups, $R\,\mathrm{proj}$, R-free are the categories of finitely generated projective respectively free left R-modules.
- $\mathcal{S}^\dagger = \mathcal{S}^{-1}\mathcal{S}$ is the group completion of a symmetric monoidal groupoid $\mathcal{S}$.
- $B\mathcal{C} = |\mathcal{C}|$ is the classifying space of a small category $\mathcal{C}$.
- $\underline{\mathbb{Z}} = (\mathbb{Z}, \mathbb{A}(\mathbb{Z}))$ denotes the Burnside form ring (Example 4.10). This is *not* the constant Mackey functor.

2. K-theory of form categories

Recall [Sch10a] that a *category with duality* is a triple $(\mathcal{A}, \sharp, \mathrm{can})$ where $\mathcal{A}$ is a category, $\sharp : \mathcal{A}^{op} \to \mathcal{A}$ is a functor, and $\mathrm{can} : 1 \to \sharp\sharp$ is a natural transformation, called *double dual identification*, such that $1_{A^\sharp} = \mathrm{can}_A^\sharp \circ \mathrm{can}_{A^\sharp}$ for all objects A of $\mathcal{A}$. If the double dual identification is a natural isomorphism, we say that the duality is *strong*. In case it is the identity (in which case $\sharp\sharp = id$), we call the duality *strict*. The functor $\mathcal{A}^{op} \times \mathcal{A}^{op} \to \mathrm{Sets} : (A,B) \mapsto \mathcal{A}(A, B^\sharp)$ is equipped with an automorphism σ of order two:

$$\sigma : \mathcal{A}(A, B^\sharp) \to \mathcal{A}(B, A^\sharp) : f \mapsto f^\sharp \circ \mathrm{can}_B\,. \tag{2.1}$$

In particular, for every object A of $\mathcal{A}$ the set of arrows $\mathcal{A}(A, A^\sharp)$ from A to its dual carries an action of the cyclic group $C_2 = \{1, \sigma\}$ of order 2. A *symmetric form on A* is a fixed point for that action, that is, an arrow $f : A \to A^\sharp$ such that $f = f^\sharp\,\mathrm{can}_A$.

A *linear category with duality* is a category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ where $\mathcal{A}$ is a linear category (that is, a category enriched over abelian groups) and the duality functor $\sharp$ is linear. An *additive category with duality* is a linear category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ where $\mathcal{A}$ is additive, that is, has finite direct sums. An *exact category with duality* is a category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ where $\mathcal{A}$ is an exact category [Qui73] and the duality functor $\sharp$ is exact.

A *form functor* from a category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ to another such category $(\mathcal{B}, \sharp, \mathrm{can})$ is a pair (F, φ) where $F : \mathcal{A} \to \mathcal{B}$ is a functor and $\varphi : F\sharp \to \sharp F$ is a natural transformation, called *duality compatibility map*, such that $\varphi_A^\sharp \circ \mathrm{can}_{FA} = \varphi_{A^\sharp} \circ F(\mathrm{can}_A)$ for every object A of $\mathcal{A}$. There is an evident definition of composition of form functors, see [Sch10a, 3.2]. If $\mathcal{A}$ and $\mathcal{B}$ are linear (additive, exact) categories, then a form functor (F, φ) as above is called *linear (additive, exact)* if the functor F is linear (additive, exact). In this paper, if $\mathcal{A}$ and $\mathcal{B}$ have strong dualities, a form functor is called *non-singular* if the duality compatibility map φ is a natural isomorphism.

Now we come to the definition of the main objects of study in this paper, the form categories. In a nutshell, a form category is a linear category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ together with a quadratic functor (Definition A.14) on $\mathcal{A}^{op}$ whose associated symmetric bilinear functor is (2.1). The motivation for the actual formulation of the Definition is explained in the introduction to Section A.4.

Definition 2.1. A *form category* is a linear category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ together with a functor $Q : \mathcal{A}^{op} \to \mathrm{Ab}$ and natural transformations τ and ρ of functors $\mathcal{A}^{op} \to \mathrm{Ab}$

$$\mathcal{A}(A, A^\sharp) \xrightarrow{\tau} Q(A) \xrightarrow{\rho} \mathcal{A}(A, A^\sharp), \qquad A \in \mathcal{A}, \tag{2.2}$$

called *transfer* and *restriction* such that (1) - (3) below hold. An element $\xi \in Q(A)$ is called a *quadratic form on A*. For $f \in \mathcal{A}(A, B)$ and $\xi \in Q(B)$ we may write $f^\bullet(\xi)$ for $Q(f)(\xi)$ if Q is understood and call $f^\bullet(\xi)$ the *restriction of ξ along f*. We require the following.

1. $\tau\sigma = \tau$ and $\sigma\rho = \rho$.
2. $\rho\tau = 1 + \sigma$.
3. For all $f, g \in \mathcal{A}(A, B)$ and $\xi \in Q(B)$ we have
$$(f+g)^\bullet(\xi) = f^\bullet(\xi) + g^\bullet(\xi) + \tau(g^\sharp \circ \rho(\xi) \circ f).$$

For a quadratic form $\xi \in Q(A)$ on A, the map $\rho(\xi) : A \to A^\sharp$ is called the *associated symmetric (bilinear) form*. It is indeed symmetric in view of (1). An *additive form category* is a form category as above where $\mathcal{A}$ is additive.

Remark 2.2. As natural transformations of functors with values in abelian groups, τ and ρ are abelian group homomorphisms. Definition 2.1 (1) and (2) say that diagram (2.2) is a C_2-Mackey functor (Appendix B) contravariantly functorial in $A \in \mathcal{A}$, and Definition 2.1 (3) says that for $\xi \in Q(B)$, the map of sets $\mathcal{A}(A, B) \to \mathrm{Ab} : f \mapsto f^\bullet(\xi)$ is quadratic (Appendix A.1) with deviation

$$(f \top g)^\bullet(\xi) := (f+g)^\bullet(\xi) - f^\bullet(\xi) - g^\bullet(\xi)$$

the symmetric bilinear form

$$(f \top g)^\bullet(\xi) = \tau(g^\sharp \circ \rho(\xi) \circ f).$$

Remark 2.3. From Definition 2.1 (3) we have $Q(0) = Q(0+0) = Q(0) + Q(0) + 0$ and thus, $Q(0) = 0$. Furthermore, $Q(1) = 1$ since Q is a functor. Then $0 = Q(-1+1) = Q(-1) + 1 - \tau\rho$, that is, $Q(-1) = -1 + \tau\rho$. By induction we obtain

$$Q(n) = n + \binom{n}{2} \tau\rho, \qquad n \in \mathbb{Z}.$$

Remark 2.4. In view of Lemma A.10 below, an additive form category is the same as an additive category with duality $(A, \sharp, \mathrm{can})$ together with a quadratic functor $Q : \mathcal{A}^{op} \to \mathrm{Ab}$ whose cross effect $Q(A\,|\,B)$ is naturally isomorphic to $\mathcal{A}(A, B^\sharp)$ as symmetric bilinear functors. In fact, Yoneda's Lemma implies that the functor Q determines $(\sharp, \mathrm{can})$ up to natural isomorphism.

Example 2.5 (Classical quadratic and symmetric bilinear forms)**.** Consider a linear category with duality $\mathcal{A} = (\mathcal{A}, \sharp, \mathrm{can})$. Recall from (2.1) that for every object A of $\mathcal{A}$, the abelian group $\mathcal{A}(A, A^\sharp)$ is equipped with a C_2-action. There are two standard ways to make $\mathcal{A}$ into a form category. The form category of *symmetric forms* in $(\mathcal{A}, \sharp, \mathrm{can})$ has the fixed set $Q^s(A) = \mathcal{A}(A, A^\sharp)^{C_2}$ of the C_2-action (2.1) as quadratic set of forms where transfer and restriction are

$$\mathcal{A}(A, A^\sharp) \xrightarrow{1+\sigma} \mathcal{A}(A, A^\sharp)^{C_2} \rightarrowtail^{1} \mathcal{A}(A, A^\sharp). \tag{2.3}$$

The form category of *classical quadratic forms* in $(\mathcal{A}, \sharp, \mathrm{can})$ has the orbit abelian group $Q^q(A) = A(A, A^\sharp)_{C_2}$ as set of quadratic forms where transfer and restriction are

$$\mathcal{A}(A, A^\sharp) \overset{1}{\twoheadrightarrow} \mathcal{A}(A, A^\sharp)_{C_2} \xrightarrow{1+\sigma} \mathcal{A}(A, A^\sharp).$$

Indeed, the following is a well-known and easy exercise. Suppose the category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ is the category $R\,\mathrm{proj}$ of finitely generated projective modules over a commutative ring R with duality $P^\sharp = \mathrm{Hom}_R(P, R)$ and standard double dual identification. Then the map $\mathrm{Hom}_R(P, P^\sharp)_{C_2} \to Q(P) : f \mapsto (x \mapsto f(x)(x))$ is an isomorphism for every $P \in R\,\mathrm{proj}$ where $Q(P)$ denotes the abelian group of classical quadratic forms on P in the sense of [Bou07, §3 no. 4 Définition 2]; see [Bou07, §3 no. 4 Proposition 2].

We will see many more examples in Sections 3 and 4; see (3.6) and (3.7).

Remark 2.6 (Classical form categories when 2 is invertible)**.** Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be a linear form category. By Definition 2.1 we have $\rho\tau = 1 + \sigma$ but that definition does not say anything about the other composition $\tau\rho$. However, it often happens that $\tau\rho = 2$ as in Example 2.5. When this is the case, I call such form categories *classical*. By Remark 2.3, a form category is classical if and only if $Q(-1) = 1$. If $\mathcal{A}$ is a $\mathbb{Z}[1/2]$-linear classical form category, then $\tau\rho = 2$ is an isomorphism, and thus, ρ is injective, τ is surjective, and $Q(A)$ is the image of $1 + \sigma$. In particular, a $\mathbb{Z}[1/2]$-linear category with duality $(\mathcal{A}, \sharp, \mathrm{can})$ has a unique structure of classical form category. For instance, if $\mathcal{A}$ is $\mathbb{Z}[1/2]$-linear, then symmetric and classical quadratic forms as in Example 2.5 coincide.

In general, we don't have $\tau\rho = 2$. See Example 4.10 below.

Definition 2.7 (Form functors)**.** A *form functor* or *homomorphism of form categories* from a form category $(\mathcal{A}, \sharp, \mathrm{can}, Q_{\mathcal{A}})$ to a form category $(\mathcal{B}, \sharp, \mathrm{can}, Q_{\mathcal{B}})$ is a triple (F, φ_q, φ) where $(F, \varphi) : (\mathcal{A}, \sharp, \mathrm{can}) \to (\mathcal{A}, \sharp, \mathrm{can})$ is a linear form functor between linear categories with duality, $\varphi_q : Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ F$ is a natural transformation of functors $\mathcal{A}^{op} \to \mathrm{Ab}$ such that for all objects $X \in \mathcal{A}$ the following diagram of abelian groups commutes

$$\begin{tikzcd}
\mathcal{A}(X, X^\sharp) \arrow[r, "\tau"] \arrow[d, "f \mapsto \varphi_X F(f)"'] & Q_{\mathcal{A}}(X) \arrow[r, "\rho"] \arrow[d, "\varphi_q"] & \mathcal{A}(X, X^\sharp) \arrow[d, "f \mapsto \varphi_X F(f)"] \\
\mathcal{B}(FX, (FX)^\sharp) \arrow[r, "\tau"] & Q_{\mathcal{B}}(FX) \arrow[r, "\rho"] & \mathcal{B}(FX, (FX)^\sharp).
\end{tikzcd}$$

The form functor is called *non-singular* if (F, φ) is. There is an obvious definition of composition of form functors:

$$(G, \psi_q, \psi) \circ (F, \varphi_q, \varphi) = (G \circ F, (\psi_q F) \circ \varphi_q, (\psi F) \circ G(\varphi)).$$

A *natural transformation* of form functors $\eta : (F, \varphi_q, \varphi) \to (G, \psi_q, \psi)$ is a natural transformation of functors $\eta : F \to G$ such that the following two diagrams commute

$$\begin{tikzcd}
F\sharp \arrow[r, "\eta_\sharp"] \arrow[d, "\varphi"'] & G\sharp \arrow[d, "\psi"] \\
\sharp F & \sharp G \arrow[l, "\sharp\eta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_{\mathcal{A}} \arrow[r, equal] \arrow[d, "\varphi_q"'] & Q_{\mathcal{A}} \arrow[d, "\psi_q"] \\
Q_{\mathcal{B}} \circ F & Q_{\mathcal{B}} \circ G. \arrow[l, "Q_{\mathcal{B}}(\eta)"]
\end{tikzcd}$$

We will see in Remark 2.37 below that form functors $(\mathcal{A}, Q) \to (\mathcal{B}, Q)$ are precisely the quadratic forms in the form category of linear functors from $\mathcal{A}$ to $\mathcal{B}$.

Definition 2.8 (Category of quadratic forms)**.** Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be a form category. The *category of quadratic forms in* $\mathcal{A}$ is the category

$$\mathcal{Q}\mathrm{uad}(\mathcal{A}, \sharp, \mathrm{can}, Q)$$

whose objects are pairs (X, ξ) where X is an object of $\mathcal{A}$ and $\xi \in Q(X)$ is a quadratic form on X. An arrow $f : (X, \xi) \to (Y, \zeta)$ in $\mathcal{Q}\mathrm{uad}(\mathcal{A}, \sharp, \mathrm{can}, Q)$ is a map $f : X \to Y$ in $\mathcal{A}$ such that $\xi = f^\bullet(\zeta)$. Composition is composition of maps in $\mathcal{A}$. We may write $\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ or simply $\mathcal{Q}\mathrm{uad}(\mathcal{A})$ if the remaining data are understood. Isomorphisms in $\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ are called *isometries.* A form functor $(F, \varphi_q, \varphi) : (\mathcal{A}, \sharp, \mathrm{can}, Q) \to (\mathcal{B}, \sharp, \mathrm{can}, Q)$ defines a functor of categories of quadratic forms

$$\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q) \to \mathcal{Q}\mathrm{uad}(\mathcal{B}, Q) : (X, \xi) \mapsto (FX, \varphi_q(\xi)).$$

Definition 2.9 (Orthogonal sum)**.** Let X and Y be objects of an additive form category $(\mathcal{A}, \sharp, \mathrm{can}, Q)$. Denote by p_X and p_Y the canonical projections from $X \oplus Y$ onto X and Y. The *orthogonal sum* of the quadratic forms $\xi \in Q(X)$ and $\zeta \in Q(Y)$ on X and Y is the quadratic form $\xi \perp \zeta$ on $X \oplus Y$ defined by

$$\xi \perp \zeta = p_X^\bullet(\xi) + p_Y^\bullet(\zeta) \in Q(X \oplus Y).$$

Orthogonal sum makes the category $\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ of quadratic forms in $\mathcal{A}$ into a symmetric monoidal category.

Definition 2.10 (Category of quadratic spaces)**.** Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an form category with strong duality. A quadratic form $\xi \in Q(X)$ on an object X of $\mathcal{A}$ is called *non-degenerate* if the associated symmetric bilinear form $\rho(\xi) : X \to X^\sharp$ is an isomorphism. A *quadratic space* is a pair (X, ξ) where X is an object of $\mathcal{A}$ and $\xi \in Q(X)$ is a non-degenerate quadratic form on X. We denote by

$$i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$$

the subcategory of $\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ whose objects are the quadratic spaces in $\mathcal{A}$ and whose arrows are the isometries.

Note that a non-singular form functor between form categories with strong duality preserves quadratic spaces.

Example 2.11 (Hyperbolic space)**.** Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be an additive form category with strong duality. The *hyperbolic quadratic space* of an object X of $\mathcal{A}$ is the quadratic space $H(X)=(X\oplus X^\sharp,h_X)$ where $h_X=\tau(p_X^\sharp\circ p_{X^\sharp})$ and p_X , $p_{X^\sharp}$ are the canonical projections from $X\oplus X^\sharp$ onto X and $X^\sharp$. The hyperbolic space $H(X)=(X\oplus X^\sharp,h_X)$ is indeed non-degenerate because

$$\rho(h_X)=\rho\tau(p_X^\sharp\circ p_{X^\sharp})=p_X^\sharp\circ p_{X^\sharp}+p_{X^\sharp}^\sharp\circ p_X^{\sharp\sharp}\,\mathrm{can}_{X\oplus X^\sharp}=\begin{pmatrix}0 & 1_{X^\sharp}\\ \mathrm{can}_X & 0\end{pmatrix}.$$

A quadratic space (Y,ξ) is called *hyperbolic* if it is isometric to $H(X)$ for some object $X\in\mathcal{A}$.

For an additive category $\mathcal{A}$, denote by $i\mathcal{A}\subset\mathcal{A}$ the subcategory that has the same objects as $\mathcal{A}$ and whose arrows are the isomorphisms in $\mathcal{A}$. This notation must not be confused with the notation $i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ introduced in Definition 2.10; $\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ is rarely additive.

Remark 2.12 (Hyperbolic and forgetful functors)**.** Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be an additive form category with strong duality. Then we have a *forgetful functor* $F: i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)\to i\mathcal{A}$ sending a quadratic space (X,ξ) to its underlying object X and an isometry f to the isomorphism f. The *hyperbolic functor* $H: i\mathcal{A}\to i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ sends an object X of $\mathcal{A}$ to the hyperbolic space $H(X)$ and an isomorphism $f:X\to Y$ to the isometry $H(f)=f\oplus(f^\sharp)^{-1}:H(X)\to H(Y)$.

Example 2.13 (Hyperbolic form functor)**.** See also Definition 2.36. Let $(\mathcal{A},\mathrm{can},\sharp,Q)$ and $(\mathcal{B},\mathrm{can},\sharp,Q)$ be additive form categories, and let $G:\mathcal{A}\to\mathcal{B}$ be an additive functor. The associated hyperbolic form functor

$$H(G):(\mathcal{A},\mathrm{can},\sharp,Q)\to(\mathcal{B},\mathrm{can},\sharp,Q)$$

has underlying functor $G\oplus\sharp G\sharp$, duality compatibility map

$$\begin{pmatrix}0 & \sharp G(\mathrm{can})\\ \mathrm{can}_{G\sharp} & 0\end{pmatrix}$$

and on quadratic forms it is the map

$$Q(A)\to Q(GA\oplus\sharp G(A^\sharp)):\xi\mapsto\tau\begin{pmatrix}0 & \sharp G\rho(\xi)\\ 0 & 0\end{pmatrix}.$$

If the dualities on $\mathcal{A}$ and $\mathcal{B}$ are strong, then on categories of quadratic spaces, the hyperbolic form functor is naturally isomorphic to the composition of forgetful functor, G and the hyperbolic functor $H(G)\cong H\circ G\circ F$:

$$i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)\xrightarrow{F}i\mathcal{A}\xrightarrow{G}i\mathcal{B}\xrightarrow{H}i\,\mathcal{Q}\mathrm{uad}(\mathcal{B},Q)$$

where the natural isomorphism at the quadratic space (A,ξ) in $\mathcal{A}$ is the map

$$\begin{pmatrix}1 & 0\\ 0 & \sharp G\rho(\xi)\end{pmatrix}:\left(GA\oplus\sharp G\sharp A,\tau\begin{pmatrix}0 & \sharp G\rho(\xi)\\ 0 & 0\end{pmatrix}\right)\xrightarrow{\cong}\left(GA\oplus\sharp GA,\tau\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right).$$

The following generalises the classical definitions as for example in [Bak81].

Definition 2.14 (Grothendieck-Witt group)**.** Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an additive form category with strong duality. The orthogonal sum of non-degenerate forms is non-degenerate. Hence, the groupoid $i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ becomes a unital symmetric monoidal category under orthogonal sum. In particular, the set $\mathrm{quad}(\mathcal{A}, Q) = \pi_0 i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ of isometry classes of non-degenerate quadratic forms is a unital commutative monoid, and we define the *(orthogonal sum) Grothendieck-Witt group* of $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ as the Grothendieck group of that abelian monoid

$$GW_0^{\oplus}(\mathcal{A}, Q) = K_0(\mathrm{quad}(\mathcal{A}, Q), \perp).$$

The hyperbolic functor from Remark 2.12 induces a map $H : K_0^{\oplus}(\mathcal{A}) \to GW_0^{\oplus}(\mathcal{A}, Q)$ whose cokernel $W^{\oplus}(\mathcal{A}, Q)$ is the *Witt group* of $(\mathcal{A}, Q)$. Orthogonal sum Witt and Grothendieck-Witt groups are functorial for non-singular additive form functors.

To any unital symmetric monoidal groupoid $(\mathcal{S}, \oplus, 0)$, Quillen [Gra76] associates a symmetric monoidal category $\mathcal{S}^{-1}\mathcal{S}$ together with a symmetric monoidal functor $\mathcal{S} \to \mathcal{S}^{-1}\mathcal{S}$ which on π_0 is the universal map $\pi_0\mathcal{S} \to \pi_0\mathcal{S}^{-1}\mathcal{S} = K_0(\mathcal{S})$ to the Grothendieck-group of the abelian monoid $\pi_0\mathcal{S}$ and which on integral homology groups yields an isomorphism[2]

$$[(\pi_0\mathcal{S})^{-1}]H_*(\mathcal{S}) \xrightarrow{\cong} H_*(\mathcal{S}^{-1}\mathcal{S}) \tag{2.4}$$

after inverting the multiplicative action of $\pi_0\mathcal{S}$ on $H_*(\mathcal{S})$. To simplify notation, we will write $\mathcal{S}^{\dagger}$ for $\mathcal{S}^{-1}\mathcal{S}$. The isomorphism (2.4) characterises the symmetric monoidal category $\mathcal{S}^{\dagger} = \mathcal{S}^{-1}\mathcal{S}$ up to homotopy, and one defines the K-theory space of $\mathcal{S}$ as the classifying space

$$K(\mathcal{S}) = B(\mathcal{S}^{\dagger})$$

of $\mathcal{S}^{\dagger}$ and writes $K_i(\mathcal{S}) = \pi_i K(\mathcal{S})$ for the homotopy groups of $K(\mathcal{S})$ with respect to 0 as base-point.

Recall that the (direct sum) K-theory space $K^{\oplus}(\mathcal{A})$ of an additive category $\mathcal{A}$ is the K-theory space $K^{\oplus}(\mathcal{A}) = K(i\mathcal{A}, \oplus)$ of the symmetric monoidal groupoid $(i\mathcal{A}, \oplus)$ of isomorphisms in $\mathcal{A}$ with respect to direct sum.

Definition 2.15 (Grothendieck-Witt space)**.** Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an additive form category with strong duality. Its (orthogonal sum) *Grothendieck-Witt space* is the K-theory space

$$GW^{\oplus}(\mathcal{A}, Q) = K(i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q), \perp)$$

of the symmetric monoidal groupoid of non-degenerate quadratic forms in $\mathcal{A}$. Hyperbolic and forgetful functors from Remark 2.12 induce maps on K-theory spaces

$$K^{\oplus}(\mathcal{A}) \xrightarrow{H} GW^{\oplus}(\mathcal{A}, Q) \xrightarrow{F} K^{\oplus}(\mathcal{A}). \tag{2.5}$$

The higher Grothendieck-Witt groups of $(\mathcal{A}, Q)$ are the homotopy groups of its Grothendieck-Witt space $GW_i^{\oplus}(\mathcal{A}, Q) = \pi_i GW^{\oplus}(\mathcal{A}, Q)$ taken with respect to a zero object as base-point.

By definition, the zeroth homotopy group of $GW^{\oplus}(\mathcal{A}, Q)$ is the Grothendieck-Witt group of Definition 2.14. The orthogonal sum Grothendieck-Witt space is functorial for non-singular additive form functors.

[2] In [Gra76] the translation functors $\mathcal{S} \to \mathcal{S} : A \mapsto A \oplus S$ are required to be faithful for all $S \in \mathcal{S}$. This assumption automatically holds for all symmetric monoidal groupoids of the form $i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ where $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ is an additive form category with strong duality.

Remark 2.16 (Homology of connected components)**.** Denote by $K(\mathcal{S})_0$ the connected component of 0 of the K-theory space $K(\mathcal{S})$ of a symmetric monoidal groupoid $\mathcal{S}$. Then the isomorphism (2.4) yields the isomorphism

$$\operatorname*{colim}_{[X]\in\pi_0\mathcal{S}} H_*\operatorname{Aut}(X) \xrightarrow{\cong} H_*(K(\mathcal{S})_0) \tag{2.6}$$

where the indexing category for the colimit on the left is the filtered category whose objects are the elements $[X]\in\pi_0\mathcal{S}$ and where a map $[X]\to[Y]$ is an element $A\in\pi_0\mathcal{S}$ such that $[X]+[A]=[Y]\in\pi_0\mathcal{S}$. Composition is addition of the $[A]$'s. Note that there is a (non-functorial) homotopy equivalence $K(\mathcal{S})\sim K_0(\mathcal{S})\times K(\mathcal{S})_0$ which depends on the choice of a section of $K(\mathcal{S})\to K_0(\mathcal{S})$.

Remark 2.17 (Cofinality for $K(\mathcal{S})$)**.** Let $\mathcal{S}$ be a symmetric monoidal groupoid. A full symmetric monoidal subgroupoid $\mathcal{S}'\subset\mathcal{S}$ is called *cofinal* if for every object X of $\mathcal{S}$, there is an object Y of $\mathcal{S}$ such that $X\oplus Y$ is isomorphic to an object of $\mathcal{S}'$. In this case, the indexing category $\pi_0\mathcal{S}'$ of the colimit in (2.6) is cofinal in $\pi_0\mathcal{S}$ and hence the colimits over $\pi_0\mathcal{S}'$ and $\pi_0\mathcal{S}$ agree. In view of the isomorphism (2.6), the inclusion $\mathcal{S}'\subset\mathcal{S}$ induces a homology isomorphism $H_*(K(\mathcal{S}')_0)\cong H_*(K(\mathcal{S})_0)$ between connected H-spaces and thus a homotopy equivalence of connected components $K(\mathcal{S}')_0 \xrightarrow{\sim} K(\mathcal{S})_0$. Moreover, the map $K_0(\mathcal{S}')\to K_0(\mathcal{S})$ is easily seen injective. In summary, if $\mathcal{S}'\subset\mathcal{S}$ is a fully faithful cofinal inclusion of symmetric monoidal groupoids, then

$$K_i(\mathcal{S}')\to K_i(\mathcal{S}) \text{ is } \begin{cases} \text{an isomorphism} & i>0,\\ \text{a monomorphism} & i=0.\end{cases}$$

Lemma 2.18 (Cofinality for $GW^{\oplus}$)**.** *Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be an additive form category with strong duality. Let $\mathcal{B}\subset\mathcal{A}$ be a full additive subcategory closed under the duality $\sharp$. Assume that for every object A of $\mathcal{A}$ there is an object A' of $\mathcal{A}$ such that $A\oplus A'$ is isomorphic to an object of $\mathcal{B}$. Then*

$$GW_i^{\oplus}(\mathcal{B},Q)\to GW_i^{\oplus}(\mathcal{A},Q) \textit{ is } \begin{cases} \textit{an isomorphism} & i>0,\\ \textit{a monomorphism} & i=0.\end{cases}$$

Proof. This is a special case of Remark 2.17. Indeed, $i\,\mathcal{Q}\mathrm{uad}(\mathcal{B},Q)\subset i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ is cofinal since for any $(A,\xi)\in i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ we have

$$(A,\xi)\perp(A,\xi)\perp H(A')\in i\mathcal{Q}(\mathcal{B},Q)$$

for any $A'\in\mathcal{A}$ with $A\oplus A'\in B$. □

Remark 2.19 ($GW^{\oplus}$ and the Plus Construction)**.** Remark 2.17 applies to the inclusion of free modules into projectives and gives rise to the usual homotopy equivalence $K(R)\sim K_0(R)\times BGL(R)^+$; see [Gra76, Theorem p. 7].

In general, there is no nice cofinal subcategory of $i\,\mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$ which would give a similar description of the Grothendieck-Witt space even if $\mathcal{A}$ is the category of finitely generated projective modules over a ring. However, if for all objects X of an additive form category, the transfer τ_X is surjective, then the hyperbolic spaces are cofinal in the category of all quadratic spaces[3]. In particular, we have

$$GW^{\oplus}(\mathcal{P}(R),Q)\sim GW_0^{\oplus}(\mathcal{P}(R),Q)\times BO_\infty(R,Q)^+,$$

---

[3] Split metabolic spaces are always cofinal (Example 2.26). When τ is surjective, split metabolic spaces are hyperbolic which follows from Lemma 5.3 with $a=c=1$ and $\mu=0$.

where $\mathcal{P}(R)$ is the category of finitely generated projective modules over a ring R equipped with a structure of form category with strong duality for which all transfers τ are surjective, and $O_\infty(R,Q)$ is the union of the automorphism groups $O_{2n}(R,Q) = \mathrm{Aut}(HR^n)$ of the hyperbolic spaces $H(R^n)$ of Example 2.11. This applies for example to the classical quadratic forms Q^q of Example 2.5 [Kar80b, Théorème 1.6] and to symplectic forms over a commutative ring.

For an example where τ is not surjective, consider the category of symmetric bilinear spaces over the integers $\mathbb{Z}$. It contains the cofinal subcategory of symmetric bilinear spaces $n\langle 1\rangle \perp n\langle -1\rangle$, $n \in \mathbb{N}$; see [MH73, Theorem 4.3]. In particular,

$$GW^\oplus(\mathbb{Z}, Q^s) \sim \mathbb{Z} \times \mathbb{Z} \times BO_{\infty,\infty}(\mathbb{Z})^+$$

where $O_{\infty,\infty}(\mathbb{Z})$ is the union of the groups $O_{n,n}(\mathbb{Z}) = \mathrm{Aut}(n\langle 1\rangle \perp n\langle -1\rangle)$.

Remark 2.20 (Extending $GW^\oplus$ to linear categories)**.** Sometimes it is useful to have a Grothendieck-Witt space $GW^\oplus$ associated with a form category with strong duality which may not be additive. It follows from Lemma A.16 below that the structure of form category $(\sharp, \mathrm{can}, Q)$ on a linear category $\mathcal{A}$ extends uniquely (up to natural isomorphism) to a structure of form category $(\sharp, \mathrm{can}, Q)$ on its additive hull $\mathcal{A}^\oplus$. If $\mathcal{A}$ is additive, then the inclusion $\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q) \subset \mathcal{Q}\mathrm{uad}(\mathcal{A}^\oplus, Q)$ is an equivalence and therefore induces an equivalence of K-theory spaces $GW^\oplus(\mathcal{A}, Q) \simeq GW^\oplus(\mathcal{A}^\oplus, Q)$. Thus if $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ is a form category with strong duality, I may write $GW^\oplus(\mathcal{A}, Q)$ to mean $GW^\oplus(\mathcal{A}^\oplus, Q)$ without causing confusion in case $\mathcal{A}$ was additive.

As for Quillen's K-theory, there is a definition of a Grothendieck-Witt space $GW(\mathscr{E}, Q)$ for exact categories which generalises Definition 2.15; see Remark 2.33 and Definition 6.3. But first we need to define what we mean by an exact form category. The following lemma motivates the definition.

Lemma 2.21. *Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an additive form category. Then for every split exact sequence*

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

in $\mathcal{A}$, the following is an exact sequence of abelian groups

$$0 \longrightarrow Q(Z) \xrightarrow{p^\bullet} Q(Y) \xrightarrow{(i^\sharp \circ \rho(\_),\ i^\bullet)} \mathcal{A}(Y, X^\sharp) \times Q(X). \tag{2.7}$$

Proof. This is a special case of Lemma A.12 in view of Lemma A.10. But it is also easy to prove directly. Since the sequence in $\mathcal{A}$ is split exact, there are maps $s : Z \to X$ and $r : Y \to X$ such that $ps = 1_Z$, $ri = 1_X$ and $sp + ir = 1_Y$. In particular, $1 = (ps)^\bullet = s^\bullet p^\bullet$ and $p^\bullet$ is injective. For exactness at $Q(Y)$, first note that the composition of the two maps in (2.7) is zero since for $\xi \in Q(Z)$ we have

$$(i^\sharp \circ \rho(p^\bullet\xi), i^\bullet p^\bullet \xi) = ((pi)^\sharp \circ \rho(\xi) \circ p, (pi)^\bullet \xi) = (0,0)$$

as $\rho(p^\bullet\xi) = p^\sharp \circ \rho(\xi) \circ p$ and $pi = 0$. Now, let $\xi \in Q(Y)$ such that $i^\bullet(\xi) = 0$ and $i^\sharp \circ \rho(\xi) = 0$. Then

$$\xi = (ir + sp)^\bullet(\xi) = r^\bullet i^\bullet(\xi) + p^\bullet s^\bullet(\xi) + \tau(r^\sharp i^\sharp \circ \rho(\xi) \circ sp) = p^\bullet s^\bullet(\xi)$$

is in the image of $p^\bullet$. $\square$

Definition 2.22 (Exact form category)**.** An *exact form category* is an additive form category $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ where $\mathscr{E}$ is an exact category such that for every admissible short exact sequence

$$X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z \tag{2.8}$$

in $\mathscr{E}$, the sequence (2.7) is exact.

Example 2.23. An additive category $\mathcal{A}$ can be considered as an exact category where a sequence (2.8) is admissible exact if it splits, that is, if p has a section, or equivalently, if i has a retraction. Such exact categories are called *split exact.* By Lemma 2.21, any additive form category is canonically a split exact form category.

Example 2.24. Any exact category with duality $(\mathscr{E}, \sharp, \mathrm{can})$ canonically defines an exact form category $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ of symmetric forms in $\mathscr{E}$. So

$$Q(E) = \mathscr{E}(E, E^\sharp)^{C_2}$$

with restriction and transfer as in Example 2.5 (2.3). The exact sequence (2.7) is an easy diagram chase.

Definition 2.25 (Sublagrangian, Lagrangian, metabolic space)**.** Let $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ be an exact form category with strong duality, and let (X, ξ) be a quadratic space in $(\mathscr{E}, Q)$. Let $i : Y \rightarrowtail X$ be an admissible subobject. The *orthogonal complement* of Y in X is the admissible subobject $Y^\perp = \ker(i^\sharp \circ \rho(\xi)) \rightarrowtail X$ of X. The subobject $Y \subset X$ is called *sublagrangian*, or *totally isotropic subspace*, if $i^\bullet(\xi) = 0$ and the inclusion $Y \subset Y^\perp$ is an admissible monomorphism.[4] A *Lagrangian* of (X, ξ) is a sublagrangian $Y \subset X$ such that $Y = Y^\perp$. A Lagrangian is called *split* if it is a direct summand. A quadratic space (X, ξ) is called *metabolic* if it has a Lagrangian. It is called *split metabolic* if it has a split Lagrangian. Note that a nonsingular exact form functor between exact form categories with strong dualities preserves metabolic and split metabolic objects.

Example 2.26. For every quadratic space (X, ξ), the quadratic space $(X, \xi) \perp (X, -\xi)$ is split metabolic with Lagrangian the diagonal embedding $X \subset X \oplus X$. The hyperbolic space $H(X)$ of an object X is split metabolic with Lagrangian the canonical inclusion $X \subset X \oplus X^\sharp : x \mapsto (x, 0)$.

Definition 2.27. The *Grothendieck-Witt group* of an exact form category with strong duality $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ is the abelian group

$$GW_0(\mathscr{E}, Q)$$

generated by isometry classes of quadratic spaces $[X, \xi]$ in $(\mathscr{E}, Q)$ subject to the relations

(1) $[X \perp Y] = [X] + [Y]$
(2) $[M] = [H(L)]$ if M is metabolic with Lagrangian L.

The Witt group $W(\mathscr{E}, Q)$ of $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ is the abelian monoid of isometry classes $[X, \xi]$ of quadratic spaces (X, ξ) in $(\mathscr{E}, Q)$ modulo the submonoid of metabolic spaces. The quotient monoid is indeed a group since $(X, \xi) \perp (X, -\xi)$ is metabolic for every quadratic space (X, ξ) and thus $-[X, \xi] = [X, -\xi] \in W(\mathscr{E}, Q)$.

[4]Admissibility comes for free when $\mathscr{E}$ is semi-idempotent complete, that is, when every arrow in $\mathscr{E}$ which has a section also has a kernel.

Remark 2.28. For any exact form category with strong duality $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ we have an exact sequence of abelian groups

$$K_0(\mathscr{E}) \xrightarrow{H} GW_0(\mathscr{E}, Q) \longrightarrow W(\mathscr{E}, Q) \to 0$$

where the first map sends $[X]$ to $[HX]$ and the second map sends $[X, \xi]$ to $[X, \xi]$.

Indeed, the map $W(\mathscr{E}, Q) \to \mathrm{coker}(H) : [X, \xi] \mapsto [X, \xi]$ is well-defined, surjective, and the composition with $\mathrm{coker}(H) \to W(\mathscr{E}, Q) : [X, \xi] \mapsto [X, \xi]$ is the identity on $W(\mathscr{E}, Q)$.

Many constructions and properties of classical quadratic forms carry over to the context of form categories. Here are some examples.

Lemma 2.29 (Existence of orthogonal decompositions)**.** *Let $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ be an exact form category with strong duality. Let $\xi \in Q(X)$ be a non-degenerate quadratic form on an object X of $\mathscr{E}$, and let $i : Y \rightarrowtail X$ be an admissible subobject such that $i^\bullet(\xi) \in Q(Y)$ is non-degenerate. Let $j : Y^\perp \subset X$ be the inclusion of the orthogonal complement of Y in X with respect to $\rho(\xi)$, that is, $Y^\perp = \ker(i^\sharp \circ \rho(\xi))$. Then we have the following orthogonal sum decomposition*

$$(X, \xi) = (Y, i^\bullet(\xi)) \perp (Y^\perp, j^\bullet(\xi)).$$

Proof. Since $\rho(\xi)$ is an isomorphism, the usual argument for symmetric forms implies that $(i, j) : Y \oplus Y^\perp \to X$ is an isomorphism. Denote by $p : X \to Y$ and $q : X \to Y^\perp$ the corresponding projections under this isomorphism. Then $ip + jq = 1_X$, and since $i^\sharp \circ \rho(\xi) \circ j = 0$ we have

$$\xi = (ip + jq)^\bullet(\xi) = (ip)^\bullet(\xi) + (jq)^\bullet(\xi) + \tau(p^\sharp i^\sharp \circ \rho(\xi) \circ jq) = (ip)^\bullet(\xi) + (jq)^\bullet(\xi).$$

□

Lemma 2.30 (Sublagrangian construction)**.** *Let $Y \rightarrowtail X$ be a sublagrangian of a quadratic space (X, ξ) in an exact form category with strong duality $(\mathscr{E}, \sharp, \mathrm{can}, Q)$. Denote by $j : Y^\perp \rightarrowtail X$ its orthogonal complement. Then there is a unique non-degenerate quadratic form $\zeta \in Q(Y^\perp/Y)$ on the quotient $Y^\perp/Y$ such that $j^\bullet(\xi) = p^\bullet(\zeta)$ where $p : Y^\perp \twoheadrightarrow Y^\perp/Y$ is the quotient map. Moreover, in the Witt and Grothendieck-Witt groups we have the following equalities*

$$[X, \xi] = [Y^\perp/Y, \zeta] \in W(\mathscr{E}, Q), \qquad [X, \xi] = [Y^\perp/Y, \zeta] + [H(Y)] \in GW_0(\mathscr{E}, Q).$$

Proof. The existence of the non-degenerate form ζ follows from the exact sequence (2.7) associated with the admissible short exact sequence $Y \rightarrowtail Y^\perp \twoheadrightarrow Y^\perp/Y$. For the relation in the Grothendieck-Witt group, we note that $(X, \xi) \perp (Y^\perp/Y, -\zeta)$ is metabolic with Lagrangian $Y^\perp$ and thus we have in $GW_0(\mathscr{E}, Q)$

$$\begin{aligned} [X, \xi] &= -[Y^\perp/Y, -\zeta] + [H(Y^\perp)] \\ &= [Y^\perp/Y, \zeta] - [H(Y^\perp/Y)] + [H(Y^\perp)] \\ &= [Y^\perp/Y, \zeta] + [H(Y)] \end{aligned}$$

since $H(Y^\perp)$ has Lagrangian $Y \oplus (Y^\perp/Y)^\sharp$ and $\rho(\zeta) : Y^\perp/Y \cong (Y^\perp/Y)^\sharp$. □

Lemma 2.31 (Split metabolic forms are stably hyperbolic)**.** *Let M be a split metabolic space with split Lagrangian $L \subset M$ in an exact form category with strong duality $(\mathscr{E}, \sharp, \mathrm{can}, Q)$. Then there is a metabolic space N and an isometry $M \perp N \cong H(L) \perp N$.*

Proof. The usual proof [Sch10a, Lemma 2.9] easily generalises. It also follows from Lemma 5.5 below in view of Lemma 5.3. □

Corollary 2.32. *Let* $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ *be a split exact form category with strong duality. Then the following surjective map is an isomorphism of abelian groups*

$$GW_0^{\oplus}(\mathcal{A}, Q) \xrightarrow{\cong} GW_0(\mathcal{A}, Q) : [X, \xi] \mapsto [X, \xi].$$

Proof. The inverse map $GW_0(\mathcal{A}, Q) \to GW_0^{\oplus}(\mathcal{A}, Q) : [X, \xi] \mapsto [X, \xi]$ is well-defined in view of Lemma 2.31. □

Remark 2.33 (Q-construction)**.** As was done for symmetric bilinear forms in [Sch10a], Quillen's Q-construction can be generalised to exact form categories $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ in order to define its Grothendieck-Witt space $GW(\mathscr{E}, \sharp, \mathrm{can}, Q)$. We will pursue this in [Scha]. In Section 6 we will give a model for $GW(\mathscr{E}, \sharp, \mathrm{can}, Q)$ in terms of Waldhausen's $S_\bullet$-construction. Our Group Completion Theorem 6.6 below generalises Corollary 2.32 to higher homotopy groups.

We finish the section with a quick overview of products for orthogonal sum GW-theory. This is needed in the proof of Theorem 5.1 below. We will define tensor product and internal homomorphism objects of form categories.

Definition 2.34 (Tensor product of form categories)**.** Let $(\mathcal{A}, \sharp_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}})$ and $(\mathcal{B}, \sharp_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}})$ be two form categories. Their *tensor product*

$$(\mathcal{A}, \sharp_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}}) \otimes (\mathcal{B}, \sharp_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{A}}) = (\mathcal{A} \otimes \mathcal{B}, \sharp, \mathrm{can}, Q)$$

is the linear category $\mathcal{A} \otimes \mathcal{B}$ whose objects are pairs (A, B) of objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and whose abelian group of homomorphisms from (A, B) to (X, Y) is $\mathcal{A}(A, X) \otimes \mathcal{B}(B, Y)$ with composition given by $(f_2 \otimes g_2) \otimes (f_1 \otimes g_1) = (f_2 f_1) \otimes (g_2 g_1)$. The duality functor is

$$\sharp = \sharp_{\mathcal{A}} \otimes \sharp_{\mathcal{B}} : (\mathcal{A} \otimes \mathcal{B})^{op} = \mathcal{A}^{op} \otimes \mathcal{B}^{op} \to \mathcal{A} \otimes \mathcal{B} : (A, B) \mapsto (A^\sharp, B^\sharp)$$

with double dual identification $\mathrm{can}_{(A,B)} = \mathrm{can}_{\mathcal{A},A} \otimes \mathrm{can}_{\mathcal{B},B} : (A, B) \to (A^{\sharp\sharp}, B^{\sharp\sharp})$. The C_2-Mackey functor

$$\mathcal{A}(A, A^\sharp) \otimes \mathcal{B}(B, B^\sharp) \xrightarrow{\tau} Q_{\mathcal{A}}(A) \hat{\otimes} Q_{\mathcal{B}}(B) \xrightarrow{\rho} \mathcal{A}(A, A^\sharp) \otimes \mathcal{B}(B, B^\sharp)$$

of quadratic forms at $(A, B) \in \mathcal{A} \otimes \mathcal{B}$ is the tensor product of the Mackey functors (Appendix B) of quadratic forms $(\mathcal{A}(A, A^\sharp), Q_{\mathcal{A}}(A))$ and $(\mathcal{B}(B, B^\sharp), Q_{\mathcal{B}}(B))$. That is, $Q(A, B) = Q_{\mathcal{A}}(A) \hat{\otimes} Q_{\mathcal{B}}(B)$ is the quotient abelian group of

$$Q_{\mathcal{A}}(A) \otimes Q_{\mathcal{B}}(B) \oplus \mathcal{A}(A, A^\sharp) \otimes \mathcal{B}(B, B^\sharp)$$

modulo the three relations

$$\xi \otimes \tau(b) = \rho(\xi) \otimes b, \qquad \tau(a) \otimes \zeta = a \otimes \rho(\zeta), \qquad a \otimes b = \sigma(a) \otimes \sigma(b) \tag{2.9}$$

where $a \in \mathcal{A}(A, A^\sharp)$, $b \in \mathcal{B}(B, B^\sharp)$, $\xi \in Q_{\mathcal{A}}(A)$ and $\zeta \in Q_{\mathcal{B}}(B)$. Restriction and transfer are the linear maps

$$\rho(\xi \otimes \zeta + a \otimes b) = \rho(\xi) \otimes \rho(\zeta) + a \otimes b + \sigma(a) \otimes \sigma(b), \qquad \tau(a \otimes b) = a \otimes b.$$

We need to make $Q = Q_{\mathcal{A}} \hat{\otimes} Q_{\mathcal{B}}$ into a functor $(\mathcal{A} \otimes \mathcal{B})^{op} \to \mathrm{Ab}$ satisfying the requirement of Definition 2.1 (3). For simple tensors $f \otimes g$, this is given by the bifunctoriality of the tensor product of Mackey functors, and the formula for general tensors is then forced by Definition 2.1 (3). For the details we will use the description of the tensor product of unital abelian monoids in Appendix C. Let

$$\eta = \sum_{i=1}^{r} f_i \otimes g_i$$

be an element of the free unital abelian monoid generated by symbols $f \otimes g$ with $f \in \mathcal{A}(X, A)$ and $g \in \mathcal{B}(Y, B)$. We set

$$(\eta)^\bullet (\xi \otimes \zeta) = \sum_{i=1}^{r} f_i^\bullet(\xi) \otimes g_i^\bullet(\zeta) + \sum_{1 \le i < j \le r} (f_i^\sharp \rho(\xi) f_j) \otimes (g_i^\sharp \rho(\zeta) g_j)$$

and

$$(\eta)^\bullet (a \otimes b) = \sum_{i,j=1}^{r} (f_i^\sharp \circ a \circ f_j) \otimes (g_i^\sharp \circ b \circ g_j) \tag{2.10}$$

where ξ, ζ, a and b are as above. One checks that for fixed f_i, g_i, the three relations (2.9) are preserved so that the above defines a homomorphism of abelian groups $Q(\eta) : Q_{\mathcal{A}}(A) \hat{\otimes} Q_{\mathcal{B}}(B) \to Q_{\mathcal{A}}(X) \hat{\otimes} Q_{\mathcal{B}}(Y)$. For elements η, ε of the free unital abelian monoid on symbols $f \otimes g$, we have

$$(\eta + \varepsilon)^\bullet = \eta^\bullet + \varepsilon^\bullet + \tau(\eta^\sharp \cdot \rho(\ \_\ ) \cdot \varepsilon).$$

It follows that if $\eta_1^\bullet = \eta_2^\bullet$ and $\tau(\eta_1^\sharp \cdot \rho(\ \_\ ) \cdot \varepsilon) = \tau(\eta_2^\sharp \cdot \rho(\ \_\ ) \cdot \varepsilon)$ then $(\eta_1 + \varepsilon)^\bullet = (\eta_2 + \varepsilon)^\bullet$. In other words, in order to check that $\eta^\bullet$ is well-defined for $\eta \in \mathcal{A}(X, A) \otimes \mathcal{B}(Y, B)$, we only need to check that the relations $(f \otimes (g_1 + g_2))^\bullet = (f \otimes g_1)^\bullet + (f \otimes g_2)^\bullet$, $((f_1 + f_2) \otimes g)^\bullet = (f_1 \otimes g)^\bullet + (f_2 \otimes g)^\bullet$ and $(0 \otimes g)^\bullet = (f \otimes 0)^\bullet = 0$ hold; see Section C. These relations hold when applied to $a \otimes b$ with $a : A \to A^\sharp$ and $b : B \to B^\sharp$ on account of bilinearity of (2.10). They also hold when applied to $\xi \otimes \zeta$ with $\xi \in Q(A)$ and $\zeta \in Q(B)$. For instance,

$$\begin{aligned}
(f \otimes (g_1 + g_2))^\bullet(\xi \otimes \zeta) &= f^\bullet(\xi) \otimes (g_1 + g_2)^\bullet(\zeta) \\
&= f^\bullet(\xi) \otimes g_1^\bullet(\zeta) + f^\bullet(\xi) \otimes g_2^\bullet(\zeta) + f^\bullet(\xi) \otimes \tau(g_1^\sharp \rho(\zeta) g_2) \\
&= f^\bullet(\xi) \otimes g_1^\bullet(\zeta) + f^\bullet(\xi) \otimes g_2^\bullet(\zeta) + \rho(f^\bullet(\xi)) \otimes (g_1^\sharp \rho(\zeta) g_2) \\
&= f^\bullet(\xi) \otimes g_1^\bullet(\zeta) + f^\bullet(\xi) \otimes g_2^\bullet(\zeta) + (f^\sharp \rho(\xi) f) \otimes (g_1^\sharp \rho(\zeta) g_2) \\
&= (f \otimes g_1 + f \otimes g_2)^\bullet(\xi \otimes \zeta).
\end{aligned}$$

Remark 2.35. Even if $\mathcal{A}$ and $\mathcal{B}$ are additive categories, $\mathcal{A} \otimes \mathcal{B}$ is linear but not additive, in general.

For any two form categories as in Definition 2.34 we have a canonical functor of categories of quadratic forms

$$\otimes : \mathcal{Q}\mathrm{uad}(\mathcal{A}, Q) \times \mathcal{Q}\mathrm{uad}(\mathcal{B}, Q) \longrightarrow \mathcal{Q}\mathrm{uad}(\mathcal{A} \otimes \mathcal{B}) : (\xi, \zeta) \mapsto \xi \otimes \zeta$$

If the dualities are strong, this induces homomorphisms of abelian groups

$$GW_i^\oplus(\mathcal{A}, Q_{\mathcal{A}}) \otimes GW_j^\oplus(\mathcal{B}, Q_{\mathcal{B}}) \xrightarrow{\cup} GW_{i+j}^\oplus(\mathcal{A} \otimes \mathcal{B}, Q_{\mathcal{A}} \hat{\otimes} Q_{\mathcal{B}})$$

using the machinery of [EM06]. The map is easy to define for $i = 0$. This is all we need below, so we will only give details in this case. Let $(\mathcal{A}, \sharp_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}})$ and $(\mathcal{B}, \sharp_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}})$ be two form categories with strong duality, and let (X, ξ) be a quadratic space in $(\mathcal{A}, Q)$. We define the linear form functor

$$\xi \otimes \_ : (\mathcal{B}, \sharp_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}}) \longrightarrow (\mathcal{A} \otimes \mathcal{B}, \sharp, \mathrm{can}, Q_{\mathcal{A}} \hat{\otimes} Q_{\mathcal{B}}) \tag{2.11}$$

where the right hand side was defined in Definition 2.34. On objects it sends $B \in \mathcal{B}$ to $(X, B) \in \mathcal{A} \otimes \mathcal{B}$, on morphisms it sends $g \in \mathcal{B}(Y, B)$ to $1_X \otimes g$, the duality compatibility map is the isomorphism $\rho(\xi) \otimes 1 : (X, B^{\sharp}) \to (X, B)^{\sharp} = (X^{\sharp}, B^{\sharp})$, and on quadratic forms it is the map $Q_{\mathcal{B}}(B) \to Q_{\mathcal{A}}(X) \hat{\otimes} Q_{\mathcal{B}}(B) : \zeta \mapsto \xi \otimes \zeta$. Since the duality compatibility map is an isomorphism the linear form functor (2.11) is non-singular and induces a map of Grothendieck-Witt spaces

$$\xi \cup \_ : GW^{\oplus}(\mathcal{B}, \sharp, \mathrm{can}, Q_{\mathcal{B}}) \longrightarrow GW^{\oplus}(\mathcal{A} \otimes \mathcal{B}, \sharp, \mathrm{can}, Q_{\mathcal{A}} \hat{\otimes} Q_{\mathcal{B}}) \tag{2.12}$$

with the understanding that $GW^{\oplus}$ of a linear category means $GW^{\oplus}$ of its additive hull; see Remark 2.20. An isometry $\xi \cong \xi'$ induces a natural isomorphism of form functors $(\xi \otimes \_\,) \cong (\xi' \otimes \_\,)$. Moreover, one has a natural isomorphism of form functors $(\xi \perp \xi') \otimes \_ \cong (\xi \otimes \_\,) \perp (\xi' \otimes \_\,)$ so that we obtain a homomorphism of abelian groups

$$GW_0^{\oplus}(\mathcal{A}, Q_{\mathcal{A}}) \otimes GW_j^{\oplus}(\mathcal{B}, Q_{\mathcal{B}}) \xrightarrow{\cup} GW_j^{\oplus}(\mathcal{A} \otimes \mathcal{B}, Q_{\mathcal{A}} \hat{\otimes} Q_{\mathcal{B}}). \tag{2.13}$$

which sends $[\xi] \otimes y$ to the image of y under the map (2.12). The cup-product (2.13) is associative in the sense that $x \cup (y \cup z) = (x \cup y) \cup z$ in $GW_j^{\oplus}(\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}, Q)$ for $x \in GW_0^{\oplus}(\mathcal{A}, Q)$, $y \in GW_0^{\oplus}(\mathcal{B}, Q)$ and $z \in GW_j^{\oplus}(\mathcal{C}, Q)$.

Definition 2.36 (The form category of functors)**.** Given two small form categories $(\mathcal{A}, *, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}})$ and $(\mathcal{B}, *, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}})$, we will make the category $\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B})$ of linear functors from $\mathcal{A}$ to $\mathcal{B}$ into a form category

$$(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), \sharp, \mathrm{can}, Q)$$

as follows. For linear functors $F, G : \mathcal{A} \to \mathcal{B}$ denote by $[F, G]$ the abelian group of natural transformations $F \to G$.

(1) The dual of a functor $F : \mathcal{A} \to \mathcal{B}$ is $F^{\sharp} = *F*$. The dual of a natural transformation $\eta : F \to G$ is the natural transformation $\eta^{\sharp} : G^{\sharp} \to F^{\sharp}$ which at $A \in \mathcal{A}$ is the map $(\eta^{\sharp})_A = (\eta_{A^*})^*$. The double dual identification $F \to F^{\sharp\sharp}$ at $A \in \mathcal{A}$ is $\mathrm{can}_{F(A^{**})} \circ F(\mathrm{can}_A) : FA \to F^{\sharp\sharp}A = F(A^{**})^{**}$.

(2) There is an isomorphism of symmetric bilinear functors

$$[F, G^{\sharp}] \xrightarrow{\cong} [F*, *G] : \eta \mapsto (F\,\mathrm{can})^* \circ \eta_*$$

where the symmetry for the functor on the right is

$$[F*, *G] \to [G*, *F] : \varphi \mapsto \tilde{\varphi} = (F\,\mathrm{can})^* \circ \varphi_*^* \circ \mathrm{can}_{G*}\,.$$

We have $\tilde{\tilde{\varphi}} = \varphi$ and $\varphi_A^* \circ \mathrm{can}_{GA} = \tilde{\varphi}_{A^*} \circ G(\mathrm{can}_A)$ for all $A \in \mathcal{A}$.

(3) For any linear functor $F : \mathcal{A} \to \mathcal{B}$, any natural transformation $\varphi : F* \to *F$ and any arrow $a : A \to A^*$ in $\mathcal{A}$ we have

$$\overline{\varphi_A \circ F(a)} = \tilde{\varphi}_A \circ F(\bar{a}).$$

(4) The set $Q(F)$ of quadratic forms $Q(F)$ on a functor F is the set of all pairs (φ_q, φ) of natural transformations $\varphi_q : Q_{\mathcal{A}} \to Q_{\mathcal{B}} F$ and $\varphi : F* \to *F$ such that (F, φ_q, φ) is a form functor (Definition 2.7) from $(\mathcal{A}, *, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}})$ to $(\mathcal{B}, *, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}})$. The set $Q(F)$ is an abelian group under addition of natural transformations.

(5) The functorial Mackey-functor of quadratic forms (under the C_2-equivariant isomorphism $[F, F^\sharp] \cong [F*, *F]$ of (2))
$$[F*, *F] \xrightarrow{\tau} Q(F) \xrightarrow{\rho} [F*, *F]$$
has restriction $\rho(\varphi_q, \varphi) = \varphi$ and transfer $\tau(\varphi) = (\varphi_q, \varphi + \tilde{\varphi})$ where for $A \in \mathcal{A}$ we set
$$\varphi_q : Q(A) \to Q(FA) : \xi \mapsto \tau(\varphi_A \circ F\rho(\xi)).$$
Using (3) one checks that the diagram in Definition 2.7 commutes for $(F, \varphi_q, \varphi + \tilde{\varphi})$.

Remark 2.37. By definition, a quadratic form in $(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), Q)$ is the same as a form functor $(\mathcal{A}, Q) \to (\mathcal{B}, Q)$, and a map of quadratic forms in $(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), Q)$ is the same as a natural transformation of form functors.

If $(\mathcal{A}, Q)$ and $(\mathcal{B}, Q)$ have strong dualities, then so has $(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), Q)$. In this case, a quadratic space in $(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), Q)$ is the same as a non-singular form functor $(\mathcal{A}, Q) \to (\mathcal{B}, Q)$.

Remark 2.38. Let $(\mathcal{A}, *, \mathrm{can}_{\mathcal{A}}, Q_{\mathcal{A}})$ and $(\mathcal{B}, *, \mathrm{can}_{\mathcal{B}}, Q_{\mathcal{B}})$ be two additive from categories, and let $\mathcal{A}_0 \subset \mathcal{A}$ be a full subcategory closed under the duality but not necessarily additive. If $\mathcal{A}_0$ generates $\mathcal{A}$ as an additive category, that is, if every object of $\mathcal{A}$ is a finite direct sum of objects in $\mathcal{A}_0$, then restriction induces an equivalence of categories $\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}_{add}(\mathcal{A}_0, \mathcal{B})$. In particular, if the dualities on $\mathcal{A}$ and $\mathcal{B}$ are strong, restriction yields an equivalence of categories of quadratic spaces
$$i\,\mathcal{Q}\mathrm{uad}(\mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}), Q) \xrightarrow{\sim} i\,\mathcal{Q}\mathrm{uad}(\mathrm{Fun}_{add}(\mathcal{A}_0, \mathcal{B}), Q).$$
In particular, any non-singular form functor $\mathcal{A}_0 \to \mathcal{B}$ extends uniquely (up to isomorphism of form functors) to a non-singular form functor $\mathcal{A} \to \mathcal{B}$.

3. Form parameter rings and their modules

In classical (linear) ring theory, a ring homomorphism $R \to S$ induces an adjoint pair of functors between categories of left modules $R\,\mathrm{Mod} \to S\,\mathrm{Mod} : M \mapsto S \otimes_R M$ and $S\,\mathrm{Mod} \to R\,\mathrm{Mod} : M \mapsto M$. In this section we will spell out the analog for form parameter rings. Their restrictions to projective modules govern the covariant and contravariant functoriality behaviour of Grothendieck-Witt groups.

We start by providing a large supply of duality functors for the category of left R-modules. We emphasize however, that it is not our aim here to give a complete classification of such functors but rather to provide a context for the transfer computations in Sections 4 and 5 needed in the proof of our main theorems 5.1 and 6.6.

Let R be a ring with involution. For a left R-module M, denote by M^{op} the right R-module with underlying abelian group M and scalar multiplication $M^{op} \times R \to M^{op} : (x^{op}, a) \mapsto (\bar{a}x)^{op}$ where x^{op} is the element in M^{op} corresponding to $x \in M$. We sometimes drop the superscript and simply write x for x^{op} if no confusion may arise. If M is a right R-module or an R-bimodule, then we similarly define the left R-module or R-bimodule M^{op}. For instance, $a \cdot x^{op} \cdot b = (\bar{b}x\bar{a})^{op}$ for $a, b \in R$ and x an element of the R-bimodule M. If R is a ring, then R^{op} denotes the opposite ring.

Definition 3.1. Let R be a ring with involution $R^{op} \to R : a \mapsto \bar{a}$. An *$R$-bimodule with involution*, also called *duality coefficient for R*, is an R-bimodule I together with a bimodule homomorphism $\sigma : I^{op} \to I$ such that $\sigma^{op} \circ \sigma = 1$. Recall that $\sigma : I^{op} \to I$ is an R-bimodule homomorphism if $\sigma(axb) = \bar{b}\sigma(x)\bar{a}$ for all $x \in I$, $a, b \in R$ since $(axb)^{op} = \bar{b}x^{op}\bar{a}$.

Example 3.2. If R is a ring with involution, then $(R, a \mapsto \varepsilon \cdot \bar{a})$ is a duality coefficient for any element $\varepsilon \in R$ in the centre $Z(R)$ of R such that $\varepsilon \cdot \bar{\varepsilon} = 1$.

If $A \to B$ is a homomorphism of rings with involution, then B is an A-bimodule with involution.

A duality coefficient (I, σ) makes the category $R\,\mathrm{Mod}$ of left R-modules into a category with duality

$$(R\,\mathrm{Mod}, \sharp_I, \mathrm{can}^I)$$

where the duality functor is

$$\sharp = \sharp_I : (R\,\mathrm{Mod})^{op} \to R\,\mathrm{Mod} : M \mapsto M^\sharp = [M^{op}, I]_R$$

with double dual identification $\mathrm{can}_M = \mathrm{can}^I_M : M \to M^{\sharp\sharp}$ defined by

$$\mathrm{can}_M(x)(f) = \sigma(f(x^{op})), \quad x \in M,\ f \in M^\sharp = [M^{op}, I]_R.$$

For a left R-module M we frequently identify the set ${}_R[M, M^\sharp]$ of left R-module homomorphisms from M to $M^\sharp = [M^{op}, I]_R$ with the set ${}_R[M \otimes_{\mathbb{Z}} M^{op}, I]_R$ of R-bimodule maps $M \otimes_{\mathbb{Z}} M^{op} \to I$, (which we may call R-bilinear maps with values in I) under the adjunction

$$(3.1) \qquad {}_R[M, M^\sharp] \xrightarrow{\cong} {}_R[M \otimes_{\mathbb{Z}} M^{op}, I]_R : f \mapsto (x \otimes y^{op} \mapsto f(x)(y^{op})).$$

The abelian group I of a duality coefficient (I, σ) is canonically equipped with

- the C_2-action $x \mapsto \sigma(x^{op})$, and
- the quadratic multiplicative left action $Q : (R, \cdot, 0, 1) \to (\mathrm{End}_{\mathbb{Z}}(I), \circ, 0, 1)$ defined by $Q(a)(x) = a \cdot x \cdot \bar{a}$.

The map Q is indeed quadratic since its deviation is the $\mathbb{Z}$-bilinear map $Q(a \top b)(x) = a \cdot x \cdot \bar{b} + b \cdot x \cdot \bar{a}$; see Section A.1 for terminology.

The following generalises the notion of form parameter due to Bak [Bak81].

Definition 3.3. Let R be a ring with involution. A *form parameter* for R is a pair (I, Λ) where $I = (I, \sigma)$ is an R-bimodule with involution and Λ is an abelian group equipped with the trivial C_2-action together with C_2-equivariant abelian group homomorphisms τ and ρ,

$$(3.2) \qquad (I, \sigma) \xrightarrow{\tau} \Lambda \xrightarrow{\rho} (I, \sigma)$$

called transfer and restriction, and a multiplicative $\mathbb{Z}$-linear left action

$$(3.3) \qquad Q : (R, \cdot, 0, 1) \to (\mathrm{End}_{\mathbb{Z}}(\Lambda), \circ, 0, 1)$$

of R on Λ preserving 0 and 1 such that the following holds.

(1) Diagram (3.2) defines a C_2-Mackey functor (Section B).

(2) The deviation (Section A.1) of Q is given by the formula
$$Q(a \top b)(x) = \tau(a \cdot \rho(x) \cdot \bar{b})$$
for $a, b \in R$ and $x \in \Lambda$. In particular, the map Q in (3.3) defines a *quadratic action* of R on Λ.

(3) The maps ρ and τ commute with the quadratic actions.

A *form parameter ring* (R, I, Λ) is a ring with involution R equipped with a form parameter (I, Λ).

We will see below that the category of left modules over a form parameter ring is canonically endowed with the structure of a form category.

Remark 3.4. Definition 3.3 (1) means that $\rho \circ \tau = 1 + \sigma$, that is, $\rho(\tau(x)) = x + \sigma(x^{op})$ for all $x \in I$. Definition 3.3 (2) means that $Q(a+b)(x) = Q(a)(x) + Q(b)(x) + \tau(a \cdot \rho(x) \cdot \bar{b})$ for $a, b \in R$ and $x \in \Lambda$. Definition 3.3 (3) means that $\tau(a \cdot x \cdot \bar{a}) = Q(a)\tau(x)$ and $\rho(Q(a)\xi) = a \cdot \rho(\xi) \cdot \bar{a}$ for all $a \in R$, $x \in I$ and $\xi \in \Lambda$.

Definition 3.5. Let R be a ring with involution. A homomorphism of form parameters $(f_1, f_0) : (I, \Lambda) \to (J, \Gamma)$ for R is a pair of abelian group homomorphisms $f_1 : I \to J$, $f_0 : \Lambda \to \Gamma$ such that the diagram

$$\begin{tikzcd} I \arrow[r, "\tau"] \arrow[d, "f_1"'] & \Lambda \arrow[r, "\rho"] \arrow[d, "f_0"] & I \arrow[d, "f_1"] \\ J \arrow[r, "\tau"'] & \Gamma \arrow[r, "\rho"'] & J \end{tikzcd}$$

commutes, the map $f_1 : I \to J$ is a homomorphism of R-bimodules commuting with the involutions on I and J, and the map f_0 commutes with the quadratic actions of R on Λ and Γ, that is, $f_0(Q(a)(x)) = Q(a)(f_0(x))$.

Example 3.6. Our definition of form parameter generalises that of Bak [Bak81]. If (R, σ) is a ring with involution and $\varepsilon \in R$ a central element with $\varepsilon \cdot \sigma(\varepsilon) = 1$, we will denote by ${}_{\varepsilon}R$ the duality coefficient $(R, \varepsilon\sigma)$. A form parameter ${}_{\varepsilon}(R, \Lambda)$ in the sense of Bak [Bak81] gives rise to a form parameter $({}_{\varepsilon}R, \Lambda, \tau, \rho)$ in the sense of Definition 3.3 where $\rho : \Lambda \to R$ is the inclusion $\Lambda \subset R$ and $\tau = 1 + \varepsilon\sigma$.

Example 3.7. Let $f : R \to S$ be a homomorphism of rings with involutions. Then a form parameter (J, Γ, τ, ρ) for S defines a form parameter (J, Γ, τ, ρ) for R via restriction of scalars along f.

Definition 3.8. Let (R, I, Λ) be a form parameter ring. A Λ-*quadratic form* on a left R-module M is a pair (q, β) where

(1) $q : M \to \Lambda$ is a map of sets such that $q(ax) = Q(a)q(x)$ for all $a \in R$ and $x \in M$,

(2) $\beta : M \otimes_{\mathbb{Z}} M^{op} \to I$ is a symmetric bilinear map such that $\beta(x, x^{op}) = \rho(q(x))$ for all $x \in M$, and

(3) the deviation of q satisfies $q(x \top y) = \tau(\beta(x, y^{op}))$ for all $x, y \in M$.

Recall that $\beta : M \otimes_{\mathbb{Z}} M^{op} \to I$ is bilinear if $\beta(ax, (by)^{op}) = a\beta(x, y^{op})\bar{b}$ and symmetric if $\beta(x, y^{op}) = \sigma(\beta(y, x^{op}))$ for all $a, b \in R$ and $x, y \in M$. Part (3) in Definition 3.8 says that $q : M \to \Lambda$ is quadratic with associated $\mathbb{Z}$-bilinear form $(x, y) \mapsto \tau(\beta(x, y^{op}))$. We denote by

$$Q(M)$$

the set of all Λ-quadratic forms on M. This is an abelian group under addition of functions $M \to \Lambda$, $M \otimes M^{op} \to I$ using the abelian group structures on Λ and I. To emphasise dependence on (R, I, Λ) we may sometimes write $Q_{I,\Lambda}(M)$ or versions thereof for $Q(M)$.

Example 3.9 (Hermitian and quadratic modules in the sense of Bak)**.** Let R be a ring with involution and form parameter where ρ is injective:

$$(I, \sigma) \xrightarrow{\tau} \Lambda \rightarrowtail^{\rho} (I, \sigma). \tag{3.4}$$

Then one checks that

$$(I, -\sigma) \xrightarrow{x \mapsto [x]} I/\rho(\Lambda) \xrightarrow{[x] \mapsto x - \sigma(x)} (I, -\sigma). \tag{3.5}$$

is another form parameter for R. For instance, the deviation of the quadratic action $Q(a)[x] = [ax\bar{a}]$ of the middle term of (3.5) satisfies Definition 3.3 (2) since

$$Q(a \top b)[x] = [ax\bar{b} + bx\bar{a}] = [a(x - \sigma(x))\bar{b}] + [bx\bar{a} + a\sigma(x)\bar{b}] = [a(x - \sigma(x))\bar{b}]$$

as $bx\bar{a} + a\sigma(x)\bar{b} = bx\bar{a} + \sigma(bx\bar{a}) = \rho\tau(bx\bar{a}) \in \rho(\Lambda)$.

Assume that $(I, \sigma) = {}_{\varepsilon}R$ as in Example 3.6. Then a Λ-hermitian module in the sense of Bak is a quadratic module in our sense for the form parameter (3.4). A Λ-quadratic module in the sense of Bak is a quadratic module in our sense for the form parameter (3.5). In this paper and its sequels, it is important to dispense with the injectivity and surjectivity requirements in (3.4) and (3.5). For instance, these properties are not preserved under basic constructions such as those in Examples 4.12 and 4.13.

$R\,\mathrm{Mod}$ **as a form category**. Let (R, I, Λ) be a form parameter ring. For a homomorphism $f : N \to M$ of left R-modules, we define the homomorphism of abelian groups

$$f^{\bullet} : Q(M) \to Q(N) : (q, \beta) \mapsto (q \circ f, \beta \circ (f \otimes f^{op}))$$

by restricting quadratic maps and symmetric bilinear forms along f. This defines the functor

$$Q : (R\,\mathrm{Mod})^{op} \to \mathrm{Ab}$$

and makes the quadruple

$$R\,\mathrm{Mod}_{I,\Lambda} = (R\,\mathrm{Mod}, \sharp_I, \mathrm{can}, Q_{I,\Lambda}) \tag{3.6}$$

into a form category. The structure maps

$${}_R[M, M^{\sharp}] \xrightarrow{\tau} Q(M) \xrightarrow{\rho} {}_R[M, M^{\sharp}]$$

under the identification (3.1) are defined by the formulas $\tau(\alpha) = (q_\alpha, \alpha + \bar{\alpha})$ and $\rho(q, \beta) = \beta$ where

$$q_\alpha(x) = \tau(\alpha(x, x^{op})) \quad \text{and} \quad \bar{\alpha}(x, y^{op}) = \sigma(\alpha(y, x^{op})).$$

Definition 3.10. Let (R, I, Λ) be a form parameter ring. Assume that the bimodule I is finitely generated projective as left R-module and that the canonical double dual identification $\mathrm{can} : R \to R^{\sharp\sharp}$ is an isomorphism. Then for any finitely generated projective left R-module P, the dual $P^{\sharp} = [P^{op}, I]_R$ is again finitely generated projective and $\mathrm{can}_P : P \to P^{\sharp\sharp}$ is an isomorphism since this is true for $P = R$, and those properties are preserved under taking finite direct sums and direct factors.

Restricting the quadratic functor $Q_{I,\Lambda}$ of $R\,\mathrm{Mod}_{I,\Lambda}$ to such modules then defines an additive form category with strong duality

$$R\,\mathrm{proj}_{I,\Lambda} = (R\,\mathrm{proj}, \sharp, \mathrm{can}, Q_{I,\Lambda}) \tag{3.7}$$

with underlying additive category the category of finitely generated projective left R-modules. The (orthogonal sum) Grothendieck-Witt space

$$GW^{\oplus}(R, I, \Lambda) = GW^{\oplus}(R\,\mathrm{proj}_{I,\Lambda})$$

of such a form parameter ring (R, I, Λ) is the Grothendieck-Witt space of the form category $R\,\mathrm{proj}_{I,\Lambda}$. As usual, the higher Grothendieck-Witt groups $GW^{\oplus}_i(R, I, \Lambda)$ of (R, I, Λ) are the homotopy groups of $GW^{\oplus}(R, I, \Lambda)$.

Example 3.11. In the notation of Example 3.6, for a commutative ring R with trivial involution quadratic modules over the form ring $(R, {}_{-1}R, 0)$ are the usual symplectic modules, and thus the *symplectic K-theory* of R is defined as

$$K\,\mathrm{Sp}(R) = GW^{\oplus}(R, {}_{-1}R, 0).$$

For a ring R with involution quadratic forms over the form ring $(R, {}_{\varepsilon}R, R_{\varepsilon\sigma})$ from Example 3.9 (3.5) are the usual classical ε-quadratic forms over R and thus the *ε-quadratic K-theory* of R is defined as

$${}_{\varepsilon}KQ(R) = GW^{\oplus}(R, {}_{\varepsilon}R, R_{\varepsilon\sigma}).$$

Again, for a ring R with involution quadratic forms over the form ring $(R, {}_{\varepsilon}R, R^{\varepsilon\sigma})$ from Example 3.9 (3.4) are the usual classical ε-hermitian forms over R and thus the *ε-hermitian K-theory* of R is defined as

$${}_{\varepsilon}GW(R) = GW^{\oplus}(R, {}_{\varepsilon}R, R^{\varepsilon\sigma}).$$

When $\varepsilon = 1$, we usually drop the index ε from the notation.

Definition 3.12. A *homomorphism of form parameter rings*

$$f : (R, I, \Lambda) \to (S, J, \Gamma)$$

is a homomorphism of rings with involution $f : R \to S$ together with a homomorphism of form parameters $(f_1, f_0) : (I, \Lambda) \to (J, \Gamma)$ for R where the latter is considered a form parameter for R via restriction of scalars along the map $f : R \to S$. Composition is composition of underlying maps of sets. To simplify notation, we may denote the maps $I \to J$ and $\Lambda \to \Gamma$ by the same letter as the homomorphism $R \to S$.

Covariant functoriality. A homomorphism $f : (R, I, \Lambda) \to (S, J, \Gamma)$ of form parameter rings defines a homomorphism of associated form categories

$$R\,\mathrm{Mod}_{I,\Lambda} \xrightarrow{f^*} S\,\mathrm{Mod}_{J,\Gamma}\,.$$

On objects, the functor f^* is the usual extension of scalars $M \mapsto S \otimes_R M$. The duality compatibility map

$$S \otimes_R [M^{op}, I]_R \longrightarrow [(S \otimes_R M)^{op}, J]_S : a \otimes \varphi \mapsto a \otimes f\varphi$$

is given by the formula

$$(a \otimes f\varphi)\,((b \otimes x)^{op}) = a \cdot f\varphi(x^{op}) \cdot \bar{b},$$

and on quadratic forms f^* is the map

$$Q_{I,\Lambda}(M) \longrightarrow Q_{J,\Gamma}(S \otimes_R M) : (q, \beta) \mapsto (\tilde{q}, \tilde{\beta})$$

where

$$\tilde{q}\left(\sum_{i=1}^{r} m_i(s_i \otimes x_i)\right) = \sum_{i=1}^{r} Q(m_i s_i)\,(fq(x_i)) + \sum_{1\le i<j\le r} m_i m_j \cdot \tau\,(s_i \cdot f\beta(x_i,x_j)\cdot \bar{s}_j)$$

and

$$\tilde{\beta}\left(\sum_i m_i(s_i\otimes x_i), \sum_j m_j(t_j\otimes y_j)\right) = \sum_{i,j} m_i n_j \cdot s_i \cdot f\beta(x_i,y_j)\cdot \bar{t}_j$$

for $m_i, n_j \in \mathbb{Z}$, $s_i, t_j \in S$ and $x_i, y_j \in M$. It is standard that $\tilde{\beta}$ is well-defined and symmetric bilinear. Moreover, $\tilde{q}$ is also well-defined by an argument similar to that in Definition 2.34. First, $\tilde{q}$ is well-defined as a map from the free abelian group on the symbols $a\otimes x$, where $s \in S$, $x \in M$. Then for elements ξ and ζ of that free abelian group we check that $\tilde{q}(\xi+\zeta) = \tilde{q}(\xi)+\tilde{q}(\zeta)+\tau\tilde{\beta}(\xi,\zeta)$ and $\tilde{\beta}(\xi,\xi) = \rho\tilde{q}(\xi)$. In particular, for all ε, ξ, ζ in the free abelian group of symbols, if $\tilde{q}(\xi) = \tilde{q}(\zeta)$ and $\tilde{\beta}(\xi,\varepsilon) = \tilde{\beta}(\zeta,\varepsilon)$ then $\tilde{q}(\xi+\varepsilon) = \tilde{q}(\zeta+\varepsilon)$. Thus, in order to verify that $\tilde{q}$ is well-defined on $S\otimes_R M$ it suffices to check that $\tilde{q}(ar\otimes x) = \tilde{q}(a\otimes rx)$, $\tilde{q}((a+b)\otimes x) = \tilde{q}(a\otimes x + b\otimes x)$ and $\tilde{q}(a\otimes(x+y)) = \tilde{q}(a\otimes x + a\otimes y)$ for all $r\in R$, $a,b\in S$ and $x,y\in M$. But this is clear since

$$\tilde{q}(ar\otimes x) = Q(ar)fq(x) = Q(a)Q(r)fq(x) = Q(a)fq(rx) = \tilde{q}(a\otimes rx),$$

$$\begin{aligned}
\tilde{q}((a+b)\otimes x) &= Q(a+b)fq(x)\\
&= (Q(a)+Q(b))\,fq(x) + \tau(a\cdot\rho(fq(x))\cdot\bar{b})\\
&= \tilde{q}(a\otimes x) + \tilde{q}(b\otimes x) + \tau(\tilde{\beta}(a\otimes x, b\otimes y))\\
&= \tilde{q}(a\otimes x + b\otimes x),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{q}\,(a\otimes(x+y)) &= Q(a)fq(x+y)\\
&= Q(a)f(q(x)+q(y)+\tau(\beta(x,y)))\\
&= Q(a)f(q(x)+q(y)) + \tau(\tilde{\beta}(a\otimes x, a\otimes y))\\
&= \tilde{q}(a\otimes x + a\otimes y).
\end{aligned}$$

Contravariant functoriality. Let (S,J,Γ) be a form parameter ring, and let $f: R\to S$ be a homomorphism of rings with involutions. Restriction of scalars along f defines a form parameter (J,Γ) for R, and we obtain a homomorphism of form categories

$$S\,\mathrm{Mod}_{J,\Gamma} \xrightarrow{f_*} R\,\mathrm{Mod}_{J,\Gamma}$$

whose underlying functor is the functor $M\mapsto M$ restricting scalars along f. The duality compatibility map

$$[M^{op},J]_S \to [M^{op},J]_R : g\mapsto g$$

is the canonical inclusion. For quadratic forms on $M\in S\,\mathrm{Mod}$, the functor is the map $Q_{S,J,\Gamma}(M)\to Q_{R,J,\Gamma}(M)$ sending (q,β) to (q,β).

If moreover, we have a homomorphisms $\iota : (J,\Gamma) \to (I,\Lambda)$ of form parameters for R, then we obtain an induced functor of form categories as the composition

$$(f,\iota)_* : S\,\mathrm{Mod}_{J,\Gamma} \xrightarrow{f_*} R\,\mathrm{Mod}_{J,\Gamma} \xrightarrow{\iota^*} R\,\mathrm{Mod}_{I,\Lambda}\,. \tag{3.8}$$

4. Form rings and a transfer map

In many cases of interest, the duality coefficient (I,σ) of a form parameter ring (R,I,Λ) equals the ring with involution (R,σ). When this happens, many formulas simplify. It is therefore convenient to introduce notation adapted to those kinds of form parameter rings.

Definition 4.1. A *form ring* is a form parameter ring (R,I,Λ) where the duality coefficient (I,σ) satisfies $I = R$, and $\sigma : I^{op} \to I$ is $a \mapsto \bar{a}$, $a \in R$.

When talking about form rings, we usually omit the mention of the duality coefficient $I = R$ and simply write (R,Λ) for the form parameter ring (R,R,Λ). For instance, $GW^{\oplus}(R,\Lambda)$ will mean $GW^{\oplus}(R,R,\Lambda)$ from Definition 3.10[5]. Note that the Grothendieck-Witt space $GW^{\oplus}(R,\Lambda)$ is defined for all form rings (R,Λ).

A homomorphism of form rings $(f,f_0) : (R,\Lambda) \to (S,\Gamma)$ is a homomorphism of form parameters rings $(f,f,f_0) : (R,R,\Lambda) \to (S,S,\Gamma)$ where the map of duality coefficients equals the homomorphism of rings with involutions.

Remark 4.2. Bak has coined the name *form ring* in [Bak81] to mean a form parameter ring $(R,{}_{\varepsilon}R,\Lambda)$ with injective restriction as in Example 3.6. See also Examples 3.9 and 3.11. We will see in Lemma 4.5 below, that a form ring in the sense of Definition 4.1 is the same as a form category with strict duality that has precisely one object. From that point of view, our definition is very natural.

Form rings were called *Hermitian Mackey functors* in [DOss].

Example 4.3 (The symmetric form ring)**.** Any ring with involution (B,σ) defines a form ring

$$B \xrightarrow{1+\sigma} B^{\sigma} \rightarrowtail^{1} B$$

whose quadratic modules are the usual hermitian modules over (B,σ).

Example 4.4 (The endomorphism form ring)**.** Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be a form category with strong duality. Let (P,φ) be a symmetric space in $(\mathcal{A},\sharp,\mathrm{can})$, that is, $\varphi : P \to P^{\sharp}$ is an isomorphism such that $\varphi^{\sharp}\,\mathrm{can} = \varphi$. Then $\mathrm{End}_{\mathcal{A}}(P)$ is a ring with involution $f \mapsto \varphi^{-1} f^{\sharp}\varphi$. In particular, $\mathrm{End}_{\mathcal{A}}(P)^{op}$ is a ring with involution and we have a C_2-equivariant isomorphism of abelian groups commuting with the quadratic action of $\mathrm{End}_{\mathcal{A}}(P)^{op}$

$$\Phi : \mathrm{End}_{\mathcal{A}}(P)^{op} \xrightarrow{\cong} \mathcal{A}(P,P^{\sharp}) : f \mapsto \varphi \circ f.$$

Under this isomorphism, the Mackey functor

$$\mathcal{A}(P,P^{\sharp}) \xrightarrow{\tau} Q(P) \xrightarrow{\rho} \mathcal{A}(P,P^{\sharp})$$

becomes the form ring

$$\mathrm{End}_{\mathcal{A}}(P)^{op} \xrightarrow{\tau\circ\Phi} Q(P) \xrightarrow{\Phi^{-1}\circ\rho} \mathrm{End}_{\mathcal{A}}(P)^{op}\,.$$

---

[5] In view of Lemma 4.5, there is a potential conflict with the convention in Remark 2.20. Nevertheless, $GW^{\oplus}(R,\Lambda)$ will always be based on projective modules.

Recall that a ring is the same as a linear category with precisely one object. The analog for form rings is true as well.

Lemma 4.5. *To give a form ring is the same as to give a form category with strict duality that has precisely one object*[6].

Proof. Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be a form category with strict duality that has precisely one object. Call that object A. Then the identity $1 : A \to A^\sharp = A$ is a symmetric isomorphism since $\mathrm{can}_A = 1_A$, and we obtain a form ring as in Example 4.4

$$\mathrm{End}_{\mathcal{A}}(A)^{op} \xrightarrow{\tau} Q(A) \xrightarrow{\rho} \mathrm{End}_{\mathcal{A}}(A)^{op}.$$

Conversely, a form ring

$$R \xrightarrow{\tau} \Lambda \xrightarrow{\rho} R \tag{4.1}$$

defines a form category with strict duality $\mathcal{A}$ which has one object, say A, and arrows the left R-module homomorphisms $R \to R$. The duality functor $\sharp : \mathcal{A}^{op} \to \mathcal{A}$ sends A to A and the left R-module homomorphism $f_a : R \to R : x \mapsto xa$ given by right multiplication with a to right multiplication $f_{\bar{a}} : x \mapsto x\bar{a}$ with $\bar{a}$. Double dual identification $\mathrm{can} : A \to A^{\sharp\sharp} = A$ is the identity. So, $(\mathcal{A}, \sharp, \mathrm{can})$ is a linear category with strict duality. Posing $Q(A) = \Lambda$ and $Q(f_a) = Q_\Lambda(a)$ defines a functor $Q : \mathcal{A}^{op} \to \mathrm{Ab}$ since

$$Q(f_a \circ f_b) = Q(f_{ba}) = Q_\Lambda(ba) = Q_\Lambda(b) \circ Q_\Lambda(a) = Q(f_b) \circ Q(f_a).$$

Under the identification $R^{op} = \mathrm{End}_{\mathcal{A}}(A) = \mathcal{A}(A, A^\sharp) : a \mapsto f_a$ the C_2-Mackey functor 4.1 becomes the Mackey functor

$$\mathcal{A}(A, A^\sharp) \xrightarrow{\tau} Q(A) \xrightarrow{\rho} \mathcal{A}(A, A^\sharp)$$

giving $\mathcal{A}$ the structure of a form category with strict duality $(\mathcal{A}, \sharp, \mathrm{can}, Q)$.

It is clear that the two constructions above are inverse to each other. □

Definition 4.6. A form ring (R, Λ, τ, ρ) is called *commutative* if R and Λ are commutative rings, $\rho : \Lambda \to R$ is a ring homomorphism, $Q(x) : \Lambda \to \Lambda$ is Λ-linear for all $x \in R$, and $\tau : R \to \Lambda$ is Λ-linear where R is considered a Λ-module via ρ. A homomorphism $(A, \Lambda_A) \to (B, \Lambda_B)$ of commutative form rings is a homomorphism of form rings such that $\Lambda_A \to \Lambda_B$ is a ring homomorphism.

Remark 4.7. We will note below that the commutative form rings are precisely the commutative monoids in the unital symmetric monoidal category of form rings under the tensor product.

Remark 4.8. Tambara functors give rise to commutative form rings. Recall that a C_2-Tambara functor is a diagram

$$\begin{tikzcd} R \arrow[r, shift left, "\eta"] \arrow[r, shift right, "\tau"'] & \Lambda \arrow[r, "\rho"] & R \end{tikzcd}$$

[6]It is a little confusing that the quadratic action in a form category is contravariant whereas the quadratic action in a form ring is covariant. This is an amplification of the ring isomorphism $R \cong \mathrm{End}_{R\,\mathrm{Mod}}(R)^{op}$.

where (R,Λ,τ,ρ) is a C_2-Mackey functor, R, Λ are commutative rings, ρ is a ring homomorphism, the C_2-action on R is a homomorphism of rings, and $\eta : (R,\cdot,0,1) \to (\Lambda,\cdot,0,1)$ is a C_2-equivariant multiplicative map preserving 0 and 1 such that

$$\tau(a)\cdot\lambda = \tau(a\cdot\rho(\lambda)), \qquad \rho(\eta(a)) = a\cdot\bar{a}, \qquad \eta(a \top b) = \tau(a\cdot\bar{b})$$

for all $a, b \in R$ and $\lambda \in \Lambda$. It was noted in [DOss, Example 1.4] that a C_2-Tambara functor defines a form ring (R,Λ,τ,ρ) with quadratic action Q on Λ given by $Q(a)\lambda = \eta(a)\cdot\lambda$, $a \in R$, $\lambda \in \Lambda$. This form ring is clearly commutative.

Note that a Tambara functor has *equivariant* multiplicative transfer, that is, $\eta(a) = \eta(\bar{a})$. It is easy to come up with commutative form rings for which $Q(a) \neq Q(\bar{a})$. For instance, the extension of scalars (Definition 4.13, Lemma 4.14) from the symmetric form ring $\mathbb{Z}$ of Example 4.3 along the homomorphisms of rings with involution $\mathbb{Z} \to \mathbb{Z}[X,Y]$ where $\overline{X} = Y$, $\overline{Y} = X$. Thus, not every commutative form ring comes from a Tambara functor. However, if a commutative form ring (R,Λ) satisfies $Q(a) = Q(\bar{a})$ for all $a \in R$ then it defines a C_2-Tambara functor with multiplicative transfer $\eta(a) = Q(a)1_\Lambda$.

Example 4.9. Examples of C_2-Tambara functors and hence of commutative form rings come from C_2-equivariant topological K-theory. Let X be a compact topological space. Then we have the Tambara functor

$$K_0\,\mathrm{Vect}_{\mathbb{C}}(X) \underset{\tau}{\overset{\eta}{\rightrightarrows}} K_0\,\mathrm{Vect}_{\mathbb{C}}^{C_2}(X) \xrightarrow{\rho} K_0\,\mathrm{Vect}_{\mathbb{C}}(X) \tag{4.2}$$

where $\mathrm{Vect}_{\mathbb{C}}(X)$ and $\mathrm{Vect}_{\mathbb{C}}^{C_2}(X)$ are the categories of complex vector bundles and C_2-equivariant complex vector bundles on X, respectively. The structure maps are given by

$$\tau(V) = (V\oplus V,\ (x,y)\mapsto(y,x));\ \eta(V) = (V\otimes V : x\otimes y \mapsto y\otimes x);\ \rho(V) = V.$$

Example 4.10 (The Burnside form ring $\underline{\mathbb{Z}}$)**.** The Burnside form ring $\underline{\mathbb{Z}} = (\mathbb{Z}, \mathbb{A}(\mathbb{Z}))$ of the integers[7] is the commutative form ring associated with the Tambara functor (4.2) where X is the one-point space. So, it is given by the diagram

$$\mathbb{Z} \underset{\tau}{\overset{\eta}{\rightrightarrows}} \mathbb{A}(\mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}$$

where $\mathbb{A}(\mathbb{Z}) = \mathbb{Z}C_2 = \mathbb{Z}[t]/(t^2-1)$ is the integral group ring over C_2 and

$$\tau(n) = n(1+t), \qquad \rho(a+bt) = a+b, \qquad \eta(n) = \frac{(n+1)n}{2} + \frac{n(n-1)}{2}t.$$

The Burnside form ring $\underline{\mathbb{Z}}$ is the initial object in the category of commutative form rings.

Remark 4.11. Consider the Burnside form ring $\underline{\mathbb{Z}}$ as a form category with one object (Lemma 4.5). Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be a form category. To give a form functor $\underline{\mathbb{Z}} \to (\mathcal{A},\sharp,\mathrm{can},Q)$ is the same as to give an object $A \in \mathcal{A}$ together with a quadratic form $q \in Q(A)$. Indeed, given $(A,q) \in \mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$, the corresponding form functor sends $\mathbb{Z}$ to A, has duality compatibility $\rho(q)$ and on quadratic forms it is the $\mathbb{Z}$-linear map $\mathbb{A}(\mathbb{Z}) \to Q(P)$ sending 1 to q and t to $Q(-1)q$. Conversely, a form functor $(F,\varphi_q,\varphi) : \mathbb{Z} \to (\mathcal{A},Q)$ defines the object $(F(\mathbb{Z}),\varphi_q(1_{\mathbb{A}})) \in \mathcal{Q}\mathrm{uad}(\mathcal{A},Q)$.

[7] Our notation here deviates from [DOss]. Below, we will define a Burnside form ring $\underline{R}$ for any commutative ring R.

Example 4.12 (Restriction of scalars)**.** Let $A \overset{\tau}{\to} \Lambda \overset{\rho}{\to} A$ be a form ring and let $\iota : B \to A$ be a homomorphism of rings with involution. Then we obtain an induced form ring $B \to \Lambda_B = \Lambda \times_A B^\sigma \to B$ and a homomorphism of form rings as in the diagram

$$\begin{tikzcd} B \arrow[r, "(\tau\circ\iota)\times(1+\sigma)"] \arrow[d, "\iota"'] & \Lambda \times_A B^\sigma \arrow[r, "(q,x)\mapsto x"] \arrow[d, "(q,x)\mapsto q"] & B \arrow[d, "\iota"] \\ A \arrow[r, "\tau"'] & \Lambda \arrow[r, "\rho"'] & A \end{tikzcd}$$

where the quadratic action of B on Λ_B is

$$Q(a)(q,x) = (Q(\iota a)q, ax\bar{a})$$

for $a \in B$, $q \in \Lambda$ and $x \in B^\sigma$.

The following definition and lemma are crucial for the transfer arguments in the proof of Theorem 5.1.

Definition 4.13 (Extension of scalars)**.** Let (A, Λ, τ, ρ) be a form ring and let $f : A \to B$ be a homomorphism of rings with involution. We define a new form ring $(B, \Lambda_B, \tau_B, \rho_B)$ called *extension of scalars* of (A, Λ) along f. Here, the abelian group Λ_B is generated by symbols

$$[x], [y, \lambda], \quad x, y \in B,\ \lambda \in \Lambda,$$

subject to the relations

(1) $[x \cdot f(a) \cdot \bar{x}] = [x, \tau_A(a)]$,
(2) $[x + y] = [x] + [y]$,
(3) $[x] = [\bar{x}]$,
(4) $[x, \lambda_1 + \lambda_2] = [x, \lambda_1] + [x, \lambda_2]$,
(5) $[x + y, \lambda] = [x, \lambda] + [y, \lambda] + [x \cdot f(\rho\lambda) \cdot \bar{y}]$,
(6) $[x, Q(a)\lambda] = [x \cdot f(a), \lambda]$,

where $a \in A$, $x, y \in B$, $\lambda, \lambda_1, \lambda_2 \in \Lambda$. We define the linear maps $\rho_B : \Lambda_B \to B$ and $Q(x) : \Lambda_B \to \Lambda_B$ for $x \in B$ on symbols by

$$\begin{array}{lcl} \rho_B([y]) & = & y + \bar{y}, \\ \rho_B([y, \lambda]) & = & y \cdot f(\rho\lambda) \cdot \bar{y}, \\ Q(x)([y]) & = & [x \cdot y \cdot \bar{x}], \\ Q(x)([y, \lambda]) & = & [xy, \lambda], \end{array}$$

where $y \in B$ and $\lambda \in \Lambda$. One checks that ρ_B and $Q(x)$ are well-defined. Setting $\tau_B(x) = [x]$ for $x \in B$ we obtain the form ring $(B, \Lambda_B, \tau_B, \rho_B)$. Setting $f_0(\lambda) = [1, \lambda]$ for $\lambda \in \Lambda$ we obtain the homomorphism of form rings $(f, f_0) : (A, \Lambda) \to (B, \Lambda_B)$.

Lemma 4.14. *Let (A, Λ) be a commutative form ring and $A \to B$ a homomorphism of commutative rings with involution. Let (B, Λ_B) be the extension of scalars of (A, Λ) along $A \to B$ as in Definition 4.13. Then (B, Λ_B) is a commutative form ring with unit $1_{\Lambda_B} = [1_B, 1_\Lambda]$ and multiplication $\Lambda_B \otimes \Lambda_B \to \Lambda_B$ defined on symbols* *by*

$$\begin{array}{lcll} [x]\cdot[y] & = & [x\cdot y]+[x\cdot\bar{y}], & x,y\in B \\ [x]\cdot[y,\lambda] & = & [x\cdot y\cdot f(\rho\lambda)\cdot\bar{y}], & x,y\in B,\ \lambda\in\Lambda \\ [x,\lambda]\cdot[y,\xi] & = & [x\cdot y,\lambda\cdot\xi], & x,y\in B,\ \lambda,\xi\in\Lambda. \end{array}$$

The homomorphism of form rings $(A,\Lambda)\to(B,\Lambda_B)$ is a homomorphism of commutative form rings.

Proof. Direct verification. □

Notation 4.15 (The form rings R and $\underline{R}$)**.** Let R be a commutative ring with involution σ. I will denote by R and $\underline{R}$ the form rings

$$R=(R,R^{\sigma})\qquad\text{and}\qquad\underline{R}=(R,\mathbb{A}(R)),$$

where R is the symmetric form ring of Example 4.3, and $\underline{R}$ is the extension of scalars (Definition 4.13) of the Burnside form ring $\underline{\mathbb{Z}}$ from Example 4.10 along the unique ring homomorphism $\mathbb{Z}\to R$. The form ring $\underline{R}$ is called the *Burnside form ring* of R. By Lemma 4.14, it is a commutative form ring.

Tensor product of form rings. The tensor product of form rings is given by the tensor product of rings with involution and the tensor product of Mackey functors of form parameters. In more detail, recall from Lemma 4.5 that form rings are the same as form categories with strict duality that have one object. The tensor product of the latter was defined in Definition 2.34 and yields another such form category with strict duality that has one object. In that sense, we have already defined the tensor product of form rings. As we will need the details below, we spell out what this means. The tensor product of form rings

$$(A\otimes B,\Lambda_A\hat{\otimes}\Lambda_B)=(A,\Lambda_A)\otimes(B,\Lambda_B)$$

has underlying ring with involution $\sigma:(A\otimes B)^{op}\to A\otimes B:(a\otimes b)\mapsto\bar{a}\otimes\bar{b}$ and abelian group of quadratic forms $\Lambda_A\hat{\otimes}\Lambda_B$ the quotient of

$$(A\otimes B)/\sigma\oplus\Lambda_A\otimes\Lambda_B$$

by the two relations

$$[\rho(\xi)\otimes y]=\xi\otimes\tau(y),\qquad[x\otimes\rho(\zeta)]=\tau(x)\otimes\zeta$$

for $x\in A$, $y\in B$, $\xi\in\Lambda_A$ and $\zeta\in\Lambda_B$. The structure maps ρ and τ are defined as

$$A\otimes B\xrightarrow{\tau}\Lambda_{A\otimes B}:x\otimes y\mapsto[x\otimes y],\quad\text{and}$$

$$\Lambda_A\hat{\otimes}\Lambda_B\xrightarrow{\rho}A\otimes B:[x\otimes y]+\xi\otimes\zeta\mapsto x\otimes y+\bar{x}\otimes\bar{y}+\rho(\xi)\otimes\rho(\zeta).$$

The quadratic action of $A\otimes B$ on $\Lambda_A\hat{\otimes}\Lambda_B$ is defined by

$$Q(a\otimes b)([x\otimes y])=[ax\bar{a}\otimes by\bar{b}],\quad Q(a\otimes b)(\xi\otimes\zeta)=Q(a)(\xi)\otimes Q(b)(\zeta)$$

extended to all of $A\otimes B$ by the requirement $Q(x\top y)=\tau(x\rho(\_)\bar{y})$. The unit of the tensor product is the Burnside form ring $\underline{\mathbb{Z}}$ of the integers from Example 4.10. The isomorphism $(\mathbb{Z},\mathbb{A}(\mathbb{Z}))\otimes(A,\Lambda_A)\cong(A,\Lambda_A)$ on quadratic forms is the map $\mathbb{A}(\mathbb{Z})\hat{\otimes}\Lambda_A\to\Lambda_A$ from Section B.4 given by

$$A/\sigma\oplus\mathbb{Z}C_2\otimes\Lambda_A\to\Lambda_A:[a]\mapsto\tau(a),\ (m+nt)\otimes\xi\mapsto n\tau\rho(\xi)+(m-n)\xi.$$

The same formula shows that $\underline{\mathbb{Z}}$ considered as a form category (Lemma 4.5) is the unit for the tensor product of form categories (Definition 2.34). Tensor product makes the category of form rings into a unital symmetric monoidal category

$$(\mathrm{FormRings}, \otimes, \underline{\mathbb{Z}}) \tag{4.3}$$

Example 4.16. A commutative form ring (A, Λ) is a unital commutative monoid in $(\mathrm{FormRings}, \otimes, \underline{\mathbb{Z}})$ with multiplication

$$\mu_A = (\mu, \mu_0) : (A, \Lambda) \otimes (A, \Lambda) \to (A, \Lambda)$$

and unit map the unique homomorphism of commutative form rings

$$\underline{\mathbb{Z}} \to (A, \Lambda)$$

where μ is multiplication in A and μ_0 is defined as

$$\mu_0 : \Lambda \hat{\otimes} \Lambda \to \Lambda : [a \otimes b] + \xi \otimes \zeta \mapsto \tau(a \cdot b) + \xi \cdot \zeta.$$

The converse also holds. Since we do not need this here, we will formulate it as an exercise.

Exercise 4.17. The unital commutative monoids in the symmetric monoidal category of form rings (4.3) are precisely the commutative form rings.

Tensor product of form rings induces a form functor of form categories

$$(\otimes, q, \varphi) : A\,\mathrm{Mod}_\Lambda \otimes B\,\mathrm{Mod}_\Gamma \to (A \otimes B)\,\mathrm{Mod}_{\Lambda \hat{\otimes} \Gamma} \tag{4.4}$$

extending the usual tensor product $(M, N) \mapsto M \otimes N$. On quadratic forms the functor is

$$\begin{array}{ccc} Q_\Lambda(M) \hat{\otimes} Q_\Gamma(N) & \xrightarrow{q} & Q_{\Lambda \hat{\otimes} \Gamma}(M \otimes N) \\ (\xi, b_\xi) \otimes (\zeta, b_\zeta) + [f \otimes g] & \mapsto & (\xi \hat{\otimes} \zeta, b_\xi \cdot b_\zeta) + \tau(f \otimes g) \end{array}$$

where

$$(\xi \hat{\otimes} \zeta) \left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} \xi(x_i) \otimes \xi(y_i) + \sum_{1 \le i < j \le n} \tau(b_\xi(x_i, x_j) \otimes b_\zeta(y_i, y_j))$$

and

$$(b_\xi \cdot b_\zeta)(x \otimes y, x' \otimes y') = b_\xi(x, x') \otimes b_\zeta(y, y').$$

The duality compatibility map is

$$\begin{array}{ccccc} [M^{op}, A]_A & \otimes & [N^{op}, B]_B & \xrightarrow{\varphi} & [(M \otimes N)^{op}, A \otimes B]_{A \otimes B} \\ f & \otimes & g & \mapsto & f \otimes g \end{array}$$

where $(f \otimes g)((x \otimes y)^{op}) = f(x^{op}) \otimes g(y^{op})$.

Restricting (4.4) to finitely generated projective modules, (2.13) defines the product

$$GW_0^{\oplus}(A, \Lambda) \otimes GW_j^{\oplus}(B, \Gamma) \xrightarrow{\cup} GW_j^{\oplus}(A \otimes B, \Lambda \hat{\otimes} \Gamma)$$

which is associative and unital in the first variable.

Transfer for Grothendieck-Witt groups. Recall from Example 4.16 that a commutative form ring (A, Λ) is a unital commutative monoid for the tensor product of form rings. In particular, the Grothendieck-Witt group $GW_0(A,\Lambda)$ is a unital commutative ring under the multiplication map

$$GW_0(A,\Lambda)\otimes GW_0(A,\Lambda) \xrightarrow{\cup} GW_0(A\otimes A,\Lambda\hat{\otimes}\Lambda) \xrightarrow{\mu^*} GW_0(A,\Lambda)$$

defined in (2.13) with unit the quadratic space $\langle 1_\Lambda\rangle$ given by the multiplicative unit 1_Λ in Λ. A homomorphism of commutative form rings $(f,f_0):(A,\Lambda)\to(B,\Gamma)$ induces a ring homomorphism

$$f^*: GW_0(A,\Lambda)\to GW_0(B,\Lambda)$$

in view of the following commutative diagram

$$\begin{tikzcd}
(A,\Lambda)\otimes(A,\Lambda) \arrow[r, "\mu_A"] \arrow[d, "f^*\otimes f^*"'] & (A,\Lambda) \arrow[d, "f^*"] \\
(B,\Gamma)\otimes(B,\Gamma) \arrow[r, "\mu_B"'] & (B,\Gamma).
\end{tikzcd}$$

Recall that (B,Γ) is a form parameter for A by restriction of scalars along f. Assume we are given a homomorphism $(t,t_0):(B,\Gamma)\to(A,\Lambda)$ of form parameters for A. In particular, t is A-linear and t_0 commutes with the quadratic actions of A on Γ and Λ. If, moreover, t_0 is Λ-linear, then we have a commutative diagram of Mackey functors

$$\begin{tikzcd}
(A,\Lambda)\otimes(B,\Gamma) \arrow[r, "\mu_B\circ(f\otimes 1)"] \arrow[d, "1\otimes(t,t_0)"'] & (B,\Gamma) \arrow[d, "(t,t_0)"] \\
(A,\Lambda)\otimes(A,\Lambda) \arrow[r, "\mu_A"'] & (A,\Lambda).
\end{tikzcd} \tag{4.5}$$

We obtain from (3.8) the form functor

$$(f,t,t_0)_*: B\,\mathrm{Mod}_\Gamma \longrightarrow A\,\mathrm{Mod}_\Lambda : M\mapsto M$$

which on quadratic forms on $M\in B\,\mathrm{Mod}$ is

$$Q_{B,\Gamma}(M)\to Q_{A,\Lambda}(M):(q,b)\mapsto(t_0\circ q, t\circ b).$$

If B is finitely generated projective over A then the form functor restricts to finitely generated projective modules

$$(f,t,t_0)_*: B\,\mathrm{proj}_\Gamma \longrightarrow A\,\mathrm{proj}_\Lambda : M\mapsto M.$$

If, moreover, the symmetric form $B\otimes B^{op}\to A: x\otimes y\mapsto t(x\cdot\bar{y})$ is non-degenerate over A, then the last form functor is non-singular and it induces the transfer map

$$(f,t,t_0)_*: GW_i(B,\Gamma)\to GW_i(A,\Lambda).$$

The transfer map is a homomorphism of $GW_0(A,\Lambda)$-modules in view of (4.5).

Example 4.18. We apply the observation above as follows. Let (A,Λ) be a commutative form ring, and let $f:A\to B$ be a homomorphism of commutative rings with involution such that B is finitely generated projective over A. Let $(f,f_0):(A,\Lambda)\to(B,\Gamma)$ be the extension of scalars (Definition 4.13) of (A,Λ) along f. By Lemma 4.14, (B,Γ) is a commutative form ring. Let $t:B\to A$ be an A-linear C_2-equivariant map such that the symmetric bilinear form $b:B\otimes B^{op}\to A: x\otimes y\mapsto t(x\cdot\bar{y})$ is non-degenerate over A. Let $(q,b)\in Q_\Lambda(B)$ be a quadratic form in (A, Λ) with associated symmetric form b. That is, $q : B \to \Lambda$ is a map satisfying $q(ax) = Q_\Lambda(a)q(x)$, $q(x \top y) = \tau_A(t(x \cdot \bar{y}))$ and $t(x \cdot \bar{x}) = \rho_A(q(x))$ for $a \in A$ and $x, y \in B$. We define the $\mathbb{Z}$-linear map $t_0 : \Gamma \to \Lambda$ on symbols by

$$t_0([x]) = \tau_A(t(x)), \quad t_0([x, \lambda]) = q(x) \cdot \lambda$$

and obtain the homomorphism $(t, t_0) : (B, \Gamma) \to (A, \Lambda)$ of form parameters for A. Note that t_0 is Λ-linear and thus makes diagram (4.5) commutative. The non-singular form functor

$$q_* := (f, t, t_0)_* : B\,\mathrm{proj}_\Gamma \to A\,\mathrm{proj}_\Lambda$$

induces the $GW_0(A, \Lambda)$-linear transfer map

$$q_* : GW_i(B, \Gamma) \to GW_i(A, \Lambda)$$

generalising the usual transfer for symmetric bilinear forms as in [Sch85, Chapter 2 §§5, 8].

Lemma 4.19. *In the situation of Example 4.18, the composition*

$$GW_0(A, \Lambda) \xrightarrow{f^*} GW_0(B, \Gamma) \xrightarrow{q_*} GW_0(A, \Lambda)$$

is multiplication with the quadratic space $[B, q, b]$.

Proof. The maps in the lemma are module homomorphisms over the commutative unital ring $GW_0(A, \Lambda)$. It follows that the composition is given by multiplication with the image of $1 \in GW_0(A, \Lambda)$. But that image is $[B, q, b]$. □

5. Additivity for orthogonal sum K-theory

5.1. **The form category of extensions.** Let $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ be an exact form category (Definition 2.22). Let $S_2\mathscr{E}$ be the category of admissible exact sequences

$$0 \to M_{-1} \xrightarrow{i} M_0 \xrightarrow{p} M_1 \to 0 \tag{5.1}$$

in $\mathscr{E}$. Defining duality and double dual identification object-wise makes $S_2\mathscr{E}$ into an exact category with duality $(S_2\mathscr{E}, \sharp, \mathrm{can})$. The dual $(M_\bullet)^\sharp$ of the admissible exact sequence $M_\bullet$ in (5.1) is the admissible exact sequence

$$0 \to M_1^\sharp \xrightarrow{p^\sharp} M_0^\sharp \xrightarrow{i^\sharp} M_{-1}^\sharp \to 0.$$

The double dual identification $M_\bullet \to (M_\bullet)^{\sharp\sharp}$ is the map $(\mathrm{can}_{M_{-1}}, \mathrm{can}_{M_0}, \mathrm{can}_{M_1})$. In view of the universal property of kernel and cokernel in an exact sequence, a map $(\varphi_{-1}, \varphi_0, \varphi_1) : M_\bullet \to M_\bullet^\sharp$ in $S_2\mathscr{E}$ is determined by φ_0. More precisely, the map $(\varphi_{-1}, \varphi_0, \varphi_1) \mapsto \varphi_0$ defines an isomorphism of abelian groups

$$S_2\mathscr{E}(\mathcal{M}_\bullet, M_\bullet^\sharp) \cong \{\varphi_0 \in \mathscr{E}(M_0, M_0^\sharp)|\ i^\sharp \circ \varphi_0 \circ i = 0\}. \tag{5.2}$$

Note that the map $(\varphi_{-1}, \varphi_0, \varphi_1) : M_\bullet \to (M_\bullet)^\sharp$ is symmetric if and only if $\varphi_0 : M_0 \to (M_0)^\sharp$ is.

We make $S_2\mathscr{E}$ into an exact form category $(S_2\mathscr{E}, \sharp, \mathrm{can}, Q)$ as follows. For a generalisation, see Section 6, after Lemma 6.2. For an exact sequence $M_\bullet$ as in (5.1), the abelian group of quadratic forms $Q(M_\bullet)$ on $M_\bullet$ is the abelian group

$$Q(M_\bullet) = \{\zeta \in Q(M_0)|\ i^\bullet(\zeta) = 0\}$$

of quadratic forms ζ on M_0 whose restriction $\zeta_{|M_{-1}} = Q(i)(\zeta)$ to M_{-1} is zero. In view of (5.2), the structure maps

$$S_2\mathscr{E}(M_\bullet, M_\bullet^\sharp) \xrightarrow{\tau} Q(M_\bullet) \xrightarrow{\rho} S_2\mathscr{E}(M_\bullet, M_\bullet^\sharp)$$

are defined by

$$\tau_{S_2\mathscr{E}}(\varphi_0) = \tau_{\mathscr{E}}(\varphi_0), \quad \rho_{S_2\mathscr{E}}(\zeta) = \rho_{\mathscr{E}}(\zeta).$$

Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an additive form category with strong duality. Consider $\mathcal{A}$ as an exact category in which every admissible exact sequence splits. Then $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ is a split exact form category with strong duality. Let $i\mathcal{A} \subset \mathcal{A}$ be the subcategory of isomorphisms in $\mathcal{A}$. There is an evident functor

$$F : i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q) \to i\mathcal{A} : (M_\bullet, \zeta) \mapsto M_{-1} \tag{5.3}$$

The proof of the following theorem will occupy the rest of this section.

Theorem 5.1 (Additivity for $GW^\oplus$)**.** *For any additive form category with strong duality $(\mathcal{A}, \sharp, \mathrm{can}, Q)$, the symmetric monoidal functor (5.3) induces a homotopy equivalence after group completion:*

$$GW^\oplus(S_2\mathcal{A}, Q) \xrightarrow{\simeq} K^\oplus(\mathcal{A}). \tag{5.4}$$

Remark 5.2. The Additivity Theorem for symmetric bilinear forms proved in [Sch10a, Theorem 7.2] uses the hermitian Q-construction rather than the group completion definition of Hermitian K-theory. To relate the two one has to show that the two definitions agree (for split exact categories). This is the content of the Group Completion Theorem 6.6 which we will prove below. The point is that we need the $GW^\oplus$-version of Additivity in order to prove our Group Completion Theorem. Also, the Additivity Theorem in [Sch10a, Theorem 7.2] is actually an S_3-version of Additivity. As we will see in Proposition 6.8 below, the S_n-version formally follows from the S_2-version. In [Scha] we generalise the Additivity Theorem to exact form categories using the appropriate version of the Q-construction.

For the rest of this section, let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be a split exact form category with strong duality as in Theorem 5.1, and let $(S_2\mathcal{A}, \sharp, \mathrm{can}, Q)$ be the form category of extensions in $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ as defined above.

5.2. **Quadratic spaces in $(S_2\mathcal{A}, Q)$.** We start by giving a more explicit description of the category of quadratic spaces in $(S_2\mathcal{A}, Q)$. From the definition of the form category $(S_2\mathcal{A}, , \sharp, \mathrm{can}, Q)$, it follows that its category $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ of quadratic spaces is equivalent to the category whose objects are triples (M, L, ξ) where $(M, \xi) \in i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q)$ is a metabolic quadratic space in $(\mathcal{A}, Q)$ and $L \subset M$ is an Lagrangian of (M, ξ) such that $\xi_{|L} = 0$. Maps $f : (M, L, \xi) \to (N, L', \zeta)$ in $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ are isometries $f : (M, \xi) \to (N, \zeta)$ in $(\mathcal{A}, Q)$ mapping L isomorphically onto L'.

Generalising the hyperbolic spaces of Example 2.11, for any object $L \in \mathcal{A}$ and quadratic form $\lambda \in Q(L^\sharp)$ on its dual, we define the quadratic space

$$M_\lambda(L) = (L \oplus L^\sharp, L, h_L + p_{L^\sharp}^\bullet(\lambda))$$

in $(S_2\mathcal{A}, Q)$ where $i_L : L \subset L \oplus L^\sharp : x \mapsto (x, 0)$ and $p_{L^\sharp} : L \oplus L^\sharp \to L^\sharp : (x, y) \mapsto y$ are the canonical inclusion and projection, respectively, and $h_L = \tau(p_L^\sharp p_{L^\sharp})$ is the standard hyperbolic form. The object $M_\lambda(L)$ is indeed a quadratic space in $(S_2\mathcal{A}, Q)$ since

$$\rho(h_L + p^\bullet_{L^\sharp}(\lambda)) = \rho(h_L) + \rho(p^\bullet_{L^\sharp}(\lambda)) = \rho(h_L) + p^\sharp_{L^\sharp}\rho(\lambda)p_{L^\sharp} = \left(\begin{smallmatrix} 0 & 1 \\ \mathrm{can} & \rho(\lambda)\end{smallmatrix}\right).$$

is a metabolic symmetric bilinear space in $\mathcal{A}$ with Lagrangian L and

$$i^\bullet_L(h_L + p^\bullet_{L^\sharp}(\lambda)) = i^\bullet_L(h_L) + i^\bullet_L p^\bullet_{L^\sharp}(\lambda) = 0.$$

Note that we have a natural isometry in $(S_2\mathcal{A}, Q)$

$$1_A \oplus \left(\begin{smallmatrix} 0 & 1_{A^\sharp} \\ 1_B & 0\end{smallmatrix}\right) \oplus 1_{B^\sharp} : M_{\lambda\perp\mu}(A \oplus B) \xrightarrow{\cong} M_\lambda(A) \perp M_\mu(B). \tag{5.5}$$

Lemma 5.3. *Every quadratic space in $(S_2\mathcal{A}, Q)$ is isometric to a quadratic space $M_\lambda(L)$ for some $\lambda \in Q(L^\sharp)$. The set of maps $M_\lambda(A) \to M_\mu(B)$ in $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ is the set of matrices*

$$\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} : A \oplus A^\sharp \to B \oplus B^\sharp,$$

where $a : A \to B$, $b : A^\sharp \to B$ and $c : A^\sharp \to B^\sharp$ are maps in $\mathcal{A}$, and a and c are isomorphisms satisfying

$$a^\sharp \circ c = 1_{A^\sharp}, \qquad \lambda = \tau(b^\sharp \circ c) + c^\bullet(\mu). \tag{5.6}$$

Composition is matrix multiplication.

Proof. Let (M, L, ξ) be an object of $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ and denote by $i : L \subset M$ the inclusion. Then $(i, i^\sharp\rho(\xi))$ defines a split exact sequence and we can choose a map $s : L^\sharp \to M$ such that $i^\sharp\rho(\xi)s = 1_{L^\sharp}$. Set $\lambda = s^\bullet(\xi)$, and let $p_L : L \oplus L^\sharp \to L$ and $p_{L^\sharp} : L \oplus L^\sharp \to L^\sharp$ be the canonical projections. Then $f = ip_L + sp_{L^\sharp} : L \oplus L^\sharp \to M$ is an isomorphism in $\mathcal{A}$ which is the identity on L and which satisfies

$$f^\bullet(\xi) = (ip_L)^\bullet(\xi) + (sp_{L^\sharp})^\bullet(\xi) + \tau(p^\sharp_L i^\sharp\rho(\xi)sp_{L^\sharp}) = 0 + p^\bullet_{L^\sharp}(\lambda) + h_L$$

since $i^\sharp\rho(\xi)s = 1$ and $h_L = \tau(p^\sharp_L p_{L^\sharp})$. Hence, we have an isometry

$$f : M_\lambda(L) \xrightarrow{\cong} (M, L, \xi)$$

in $(S_2\mathcal{A}, Q)$ as required.

A map $M_\lambda(A) \to M_\mu(B)$ is given by an invertible matrix

$$\left(\begin{smallmatrix} a & b \\ d & c\end{smallmatrix}\right) : A \oplus A^\sharp \to B \oplus B^\sharp$$

sending A isomorphically onto B and preserving quadratic forms. So, $d = 0$, a is an isomorphism, and

$$h_A + (0,1)^\bullet\lambda = \left(\begin{smallmatrix} a & b \\ 0 & c\end{smallmatrix}\right)^\bullet(h_B + (0,1)^\bullet\mu). \tag{5.7}$$

Applying the restriction ρ to (5.7) yields the equation

$$\left(\begin{smallmatrix} 0 & 1 \\ \mathrm{can} & \rho(\lambda)\end{smallmatrix}\right) = \begin{pmatrix} a^\sharp & 0 \\ b^\sharp & c^\sharp\end{pmatrix}\left(\begin{smallmatrix} 0 & 1 \\ \mathrm{can} & \rho(\mu)\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & c\end{smallmatrix}\right)$$

from which we deduce $a^\sharp c = 1$. Applying $\left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)^\bullet$ to (5.7) yields

$$\lambda = \left(\begin{smallmatrix} b \\ c\end{smallmatrix}\right)^\bullet(h_B + (0,1)^\bullet\mu)$$

from which we deduce the second equation in (5.6) since

$$\left(\begin{smallmatrix} b \\ c\end{smallmatrix}\right)^\bullet h_B = \left(\begin{smallmatrix} b \\ c\end{smallmatrix}\right)^\bullet\tau\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right) = \tau\left((b^\sharp, c^\sharp)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} b \\ c\end{smallmatrix}\right)\right) = \tau(b^\sharp c).$$

The conditions (5.6) are also sufficient as they imply (5.7):

$$\begin{aligned}
\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)^{\bullet}(h_B + (0,1)^{\bullet}\mu) &= \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)^{\bullet} h_B + (0,1)^{\bullet} c^{\bullet}(\mu) \\
&= \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)^{\bullet} \tau\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) - (0,1)^{\bullet}\tau(b^{\sharp}c) + (0,1)^{\bullet}\lambda \\
&= (0,1)^{\bullet}\lambda + \tau\left(\left(\begin{smallmatrix} a^{\sharp} & 0 \\ b^{\sharp} & c^{\sharp} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) b^{\sharp}c(0,1)\right) \\
&= (0,1)^{\bullet}\lambda + \tau\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).
\end{aligned}$$

It is clear that composition is matrix multiplication. □

Lemma 5.4. *The functor F in (5.3) has a section.*

Proof. The section is given by the functor

$$L \mapsto M_0(L) : a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^{\sharp})^{-1} \end{pmatrix}.$$

□

Lemma 5.5. *For any $\lambda \in Q(L^{\sharp})$, the map*

$$\sigma : M_{\lambda\perp\lambda}(L \oplus L) \to M_{\lambda\perp 0}(L \oplus L) \tag{5.8}$$

is an isomorphism in $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ where σ and σ^{-1} are given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & \beta & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -\beta & -\beta \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\beta = \mathrm{can}_L^{-1} \circ \rho(\lambda)$.

Proof. It is clear that σ^{-1} is the inverse of σ, and that σ has the prescribed form as in Lemma 5.3 with

$$a = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \quad b = \left(\begin{smallmatrix} 0 & 0 \\ \beta & 0 \end{smallmatrix}\right), \quad c = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right).$$

Note that $\beta^{\sharp} = \rho(\lambda)^{\sharp}(\mathrm{can}_L^{\sharp})^{-1} = \rho(\lambda)^{\sharp}\,\mathrm{can}_{L^{\sharp}} = \rho(\lambda)$. Now, the second equation in (5.6) is also immediate:

$$\begin{aligned}
c^{\bullet}(\lambda \perp 0) + \tau(b^{\sharp}c) &= \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)^{\bullet}(1\ 0)^{\bullet}\lambda + \tau\left(\left(\begin{smallmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\right) \\
&= (1\ \ -1)^{\bullet}\lambda + \tau\left(\begin{smallmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{smallmatrix}\right) \\
&= (1\ 0)^{\bullet}\lambda + (0\ \ -1)^{\bullet}\lambda + \tau\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\rho(\lambda)(0\ \ -1)\right) + \tau\left(\begin{smallmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{smallmatrix}\right) \\
&= (1\ 0)^{\bullet}\lambda + (0\ 1)^{\bullet}\lambda.
\end{aligned}$$

□

Corollary 5.6. *The functor (5.3) induces an isomorphism of abelian groups*

$$GW_0^{\oplus}(S_2\mathcal{A}, Q) \xrightarrow{\cong} K_0^{\oplus}(\mathcal{A}).$$

Proof. The map $[L] \mapsto [M_0(L)]$ is a section to the map in the corollary. It suffices to show that the section is an isomorphism. It is clearly injective. By Lemmas 5.3, 5.5 and the isomorphism (5.5), every object in $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A}, Q)$ is stably isomorphic to $M_0(L)$ for some $L \in \mathcal{A}$. Hence the section is also surjective. □

5.3. **Strategy of proof of Theorem 5.1.** The map (5.4) in Theorem 5.1 is a map of group complete H-spaces which is an isomorphism on π_0, by Corollary 5.6. All components of a group complete H-space have the same homotopy type. Therefore, it suffices to show that the map is a homotopy equivalence between the connected components of 0. In fact, it suffices to show that the last map is an isomorphism on integral homology groups as a homology isomorphism between simple spaces is a homotopy equivalence. By (2.6), this is the map

$$\operatorname*{colim}_{[M_\lambda(L)]\in\pi_0 i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A},Q)} H_*\operatorname{Aut}(M_\lambda(L)) \xrightarrow{F} \operatorname*{colim}_{[A]\in\pi_0 i\mathcal{A}} H_*\operatorname{Aut}(A). \tag{5.9}$$

By Lemma 5.4, this map is surjective in all degrees. We therefore need to show that it is also injective. For $L\in\mathcal{A}$ and $\lambda\in Q(L^\sharp)$ we will construct in Lemma 5.7 below a commutative diagram of groups

$$\begin{tikzcd}
 & \operatorname{Aut}(M_\lambda(L)) \arrow[dl, "F"'] \arrow[d] \arrow[dr, "M_\lambda(L)\perp"] & \\
\operatorname{Aut}(L) & G_\lambda(L) \arrow[l, "(5.14)"] \arrow[r] & \operatorname{Aut}(M_\lambda(L)\perp M_\lambda(L))
\end{tikzcd} \tag{5.10}$$

in which the lower left horizontal arrow (5.14) is an isomorphism on integral homology groups in a range of degrees depending on the global units in $\mathcal{A}$; see Proposition 5.9 below. In particular, a homology class in $H_*\operatorname{Aut}(M_\lambda(L))$ which is sent to zero under F is sent to zero under $M_\lambda(L)\perp(\_)$. This implies that the map (5.9) is an isomorphism in case $\mathcal{A}$ has enough global units. The general case will follow from a transfer argument at the end of this section.

5.4. **The group $G_\lambda(L)$.** We define the subgroup

$$G_\lambda(L)\subset\operatorname{Aut}(M_{\lambda\perp 0}(L\oplus L))$$

of the group of automorphisms of $M_{\lambda\perp 0}(L\oplus L)$ as the set of tuples (a,b,c,f) corresponding to the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ -a & f^{-1} & b & c\\ 0 & 0 & 1 & a^\sharp f^\sharp\\ 0 & 0 & 0 & f^\sharp\end{pmatrix}\in\operatorname{Aut}(M_{\lambda\perp 0}(L\oplus L)).$$

By Lemma 5.3, such a matrix defines an automorphism of $M_{\lambda\perp 0}(L\oplus L)$ in $i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{A},Q)$ if and only if the equation

$$(1\ \ 0)^\bullet\lambda = \begin{pmatrix} 1 & a^\sharp f^\sharp\\ 0 & f^\sharp\end{pmatrix}^\bullet (1\ \ 0)^\bullet\lambda + \tau\left(\begin{pmatrix} 0 & b^\sharp\\ 0 & c^\sharp\end{pmatrix}\circ\begin{pmatrix} 1 & a^\sharp f^\sharp\\ 0 & f^\sharp\end{pmatrix}\right) \tag{5.11}$$

holds. Since

$$\begin{pmatrix} 1 & a^\sharp f^\sharp\\ 0 & f^\sharp\end{pmatrix}^\bullet (1\ \ 0)^\bullet\lambda = (1\ \ a^\sharp f^\sharp)^\bullet\lambda = (1\ \ 0)^\bullet\lambda + (0\ \ a^\sharp f^\sharp)^\bullet\lambda + \tau\begin{pmatrix} 0 & \rho(\lambda)a^\sharp f^\sharp\\ 0 & 0\end{pmatrix},$$

equation (5.11) is equivalent to

$$(1\ \ 0)^\bullet\lambda = (1\ \ a^\sharp f^\sharp)^\bullet\lambda + \tau\begin{pmatrix} 0 & b^\sharp f^\sharp\\ 0 & c^\sharp f^\sharp\end{pmatrix} \tag{5.12}$$

and equivalent to

$$0 = (0\ \ a^\sharp f^\sharp)^\bullet\lambda + \tau\begin{pmatrix} 0 & b^\sharp f^\sharp + \rho(\lambda)a^\sharp f^\sharp\\ 0 & c^\sharp f^\sharp\end{pmatrix}$$

and to

$$0 = (0\ \ a^\sharp)^\bullet\lambda + \tau\left(\begin{pmatrix} (f^{\sharp\sharp})^{-1} & 0\\ 0 & (f^{\sharp\sharp})^{-1}\end{pmatrix}\circ\begin{pmatrix} 0 & b^\sharp+\rho(\lambda)a^\sharp\\ 0 & c^\sharp\end{pmatrix}\right). \tag{5.13}$$

From this equation, we see that the group homomorphism

$$G_\lambda(L) \to \mathrm{Aut}(L) : (a,b,c,f) \mapsto (f^{-1}) \tag{5.14}$$

has a section $g \mapsto (0,0,0,g^{-1})$ and is thus surjective.

Lemma 5.7. *The diagram of groups (5.10) commutes where the two non-labelled group homomorphisms are defined as follows (suppressing L in the notation)*

$$\mathrm{Aut}(M_\lambda) \xrightarrow{g\mapsto 1_{M_\lambda}\perp g} \mathrm{Aut}\left(M_\lambda \perp M_\lambda\right) \underset{\cong}{\xrightarrow{(5.5)}} \mathrm{Aut}\left(M_{\lambda\perp\lambda}\right) \underset{\cong}{\xrightarrow{(5.8)}} \mathrm{Aut}\left(M_{\lambda\perp 0}\right) \supset G_\lambda$$

$$G_\lambda \subset \mathrm{Aut}\left(M_{\lambda\perp 0}\right) \underset{\cong}{\xrightarrow{(5.8)}} \mathrm{Aut}\left(M_{\lambda\perp\lambda}\right) \underset{\cong}{\xrightarrow{(5.5)}} \mathrm{Aut}\left(M_\lambda \perp M_\lambda\right).$$

Proof. The main point is to check that the image of the first map is in G_λ. In the notation of Lemma 5.3, the first map is

$$\begin{pmatrix} f^{-1} & \alpha \\ 0 & f^\sharp \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-f^{-1} & f^{-1} & (1-f^{-1})\beta & \alpha + (1-f^{-1})\beta \\ 0 & 0 & 1 & 1-f^\sharp \\ 0 & 0 & 0 & f^\sharp \end{pmatrix} \in \mathrm{Aut}\left(M_{\lambda\perp 0}\right)$$

which lands indeed in G_λ. Commutativity of diagram (5.10) is now clear. $\square$

5.5. **The ring ΣR^2.** For a commutative ring R, let $\Sigma R^2 \subset R$ be the abelian subgroup generated by the squares $a^2 \in R$ of elements $a \in R$. The abelian group ΣR^2 is clearly a subring of R. As examples, we have $\Sigma\mathbb{Z}^2 = \mathbb{Z}$ and $\Sigma R^2 = R$ for every commutative ring R with $2 \in R^*$ a unit: in this case every $a \in R$ can be written as $a = ((1/2)^2 + (1/2)^2)\cdot((a+1)^2 - a^2 - 1^2) \in \Sigma R^2$. However, $\Sigma F^2 \neq F$ for an imperfect field of characteristic 2. Note that the inclusion $\Sigma R^2 \subset R$ detects units: if an element $a \in \Sigma R^2$ is a unit in R with inverse $b \in R$ then $b = ab^2 \in \Sigma R^2$ and a is a unit in ΣR^2.

Definition 5.8. Let R be a commutative ring. An *R-linear form category* is a form category $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ such that $\mathcal{A}$ is an R-linear category and the duality functor $\sharp : \mathcal{A}^{op} \to \mathcal{A}$ is R-linear. This means that homomorphism sets in $\mathcal{A}$ are left R-modules, composition is R-bilinear, and we have $(ta)^\sharp = t(a^\sharp)$ for all $t \in R$ and $a \in \mathcal{A}(A,B)$.

For an R-linear form category (A,Q) and $A \in \mathcal{A}$, the kernel of the transfer $\tau : \mathcal{A}(A,A^\sharp) \to Q(A)$ is a ΣR^2-submodule of the R-module $\mathcal{A}(A,A^\sharp)$. Indeed, this kernel is an abelian subgroup and if $\tau(f) = 0$ then $\tau(t^2 f) = Q(t\cdot 1_A)\tau(f) = 0$ for all $t \in R$ and $f : A \to A^\sharp$. Similarly, the image of the restriction $\rho : Q(A) \to \mathcal{A}(A,A^\sharp)$ is also a ΣR^2-submodule.

For the next proposition, recall from Definition D.1 the notion of $S(m)$-algebra and the element $s_m \in \mathbb{Z}[S(m)^*]$.

Proposition 5.9. *Let $m \geq 1$ be an integer. Let R be a commutative ring such that ΣR^2 is an $S(m)$-algebra. Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an R-linear additive form category with strong duality. Then for every object $L \in \mathcal{A}$ and every quadratic form $\lambda \in Q(L^\sharp)$ the group homomorphism (5.14) induces an isomorphism on integral homology groups*

$$H_i\left(G_\lambda(L)\right) \xrightarrow{\cong} H_i\left(\mathrm{Aut}(L)\right) \quad \text{for } i < m/2.$$

Proof. To simplify, write S for the subring ΣR^2 of R. Let $G^0_\lambda(L)$ be the kernel of the surjective group homomorphism (5.14). Thus, $G^0_\lambda(L)$ is the group of matrices under multiplication

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & b & c \\ 0 & 0 & 1 & a^\sharp \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfying

$$0 = (0 \ \ a^\sharp)^\bullet\lambda + \tau \begin{pmatrix} 0 & b^\sharp+\rho(\lambda)a^\sharp \\ 0 & c^\sharp \end{pmatrix} \in Q(L^\sharp \oplus L^\sharp). \tag{5.15}$$

Under the isomorphism

$$\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^\bullet,\ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)^\bullet,\ (1 \ \ 0) \circ \rho(\_) \circ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right) : Q(A \oplus B) \xrightarrow{\cong} Q(A) \oplus Q(B) \oplus \mathcal{A}(B, A^\sharp)$$

of (A.10) in Remark A.11, the element on the right hand side of the equation corresponds to $(0, (a^\sharp)^\bullet\lambda + \tau(c^\sharp), b^\sharp + \rho(\lambda)a^\sharp)$. Therefore, equation (5.15) is equivalent to the set of equations

$$0 = (a^\sharp)^\bullet\lambda + \tau(c^\sharp) \in Q(L^\sharp), \qquad 0 = b^\sharp + \rho(\lambda)a^\sharp \in \mathcal{A}(L^\sharp, L^{\sharp\sharp}). \tag{5.16}$$

Since

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_0 & 1 & b_0 & c_0 \\ 0 & 0 & 1 & a_0^\sharp \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_1 & 1 & b_1 & c_1 \\ 0 & 0 & 1 & a_1^\sharp \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_0-a_1 & 1 & b_0+b_1 & c_0+c_1+b_0a_1^\sharp \\ 0 & 0 & 1 & a_0^\sharp+a_1^\sharp \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

multiplication in the group $G^0_\lambda(L)$ is given by

$$(a_0, b_0, c_0) \cdot (a_1, b_1, c_1) = (a_0 + a_1, b_0 + b_1, c_0 + c_1 + b_0a_1^\sharp).$$

Unit and inverse are

$$1 = (0, 0, 0), \qquad (a, b, c)^{-1} = (-a, -b, -c + ba^\sharp).$$

By assumption, $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ is an R-linear additive form category. For $t \in R$ and $(a, b, c) \in G^0_\lambda(L)$, we set

$$t.(a, b, c) = (ta, tb, t^2c). \tag{5.17}$$

Applying $(t \cdot 1_{L^\sharp})^\bullet$ to the first equation of (5.16) and composing from the left with $t \cdot 1_{L^{\sharp\sharp}}$ in the second equation we see that $t.(a, b, c) \in G^0_\lambda(L)$ for all $(a, b, c) \in G^0_\lambda$. It follows that the group homomorphism

$$G^0_\lambda(L) \to \mathcal{A}(L, L) \times \mathcal{A}(L^\sharp, L) : (a, b, c) \mapsto (a, b)$$

has image $G^1_\lambda(L)$ an R-submodule of $\mathcal{A}(L, L) \times \mathcal{A}(L^\sharp, L)$ and kernel $N_\lambda(L)$ an $S = \Sigma R^2$-submodule of $\mathcal{A}(L^\sharp, L)$. Summarizing, we have an exact sequence of groups

$$0 \to N_\lambda(L) \to G^0_\lambda(L) \to G^1_\lambda(L) \to 0 \tag{5.18}$$

in which subgroup and quotient group are the underlying abelian groups of S-modules. There is an action of R^* on the exact sequence (5.18). For $t \in R$, it is given by (5.17) on the middle term and by multiplication with t and t^2 on quotient and kernel. Via the inclusion $S \subset R$, this action restricts to an action of S^* on (5.18). On $G^1_\lambda(L)$ this action is the linear S-module action, and on $N_\lambda(L)$ it is the square of the linear S-module action. The conjugation action of $G^0_\lambda(L)$ on $N_\lambda(L)$ is trivial because for all $(a, b, c) \in G^0_\lambda(L)$ and $(0, 0, c_0) \in N_\lambda(L)$, we have

$$(a, b, c) \cdot (0, 0, c_0) \cdot (a, b, c)^{-1} = (a, b, c + c_0) \cdot (-a, -b, -c + ba^\sharp) = (0, 0, c_0).$$

In other words, the exact sequences (5.18) is a central extension. Since S is an $S(m)$-algebra we have

$$s_m^{-1}H_n(G_\lambda^0(L)) = 0 \quad \text{for} \quad 1 \leq n < m/2,$$

by Proposition D.4.

For $t \in S^* \subset R^*$, conjugation with the element $(0,0,0,t^{-1}) \in G_\lambda(L)$ induces an action of S^* on the exact sequence

$$1 \to G_\lambda^0(L) \to G_\lambda(L) \to \mathrm{Aut}(L) \to 1$$

which is trivial on the base $\mathrm{Aut}(L)$ and which is the action on $G_\lambda^0(L)$ considered above. The integral homology Hochschild-Serre spectral sequence of that exact sequence

$$E^2_{p,q} = H_p(\mathrm{Aut}(L), H_q(G_\lambda^0(L))) \Rightarrow H_{p+q}(G_\lambda(L))$$

localised at s_m has E^2-terms

$$s_m^{-1}H_p(\mathrm{Aut}(L), H_q(G_\lambda^0(L))) = H_p(\mathrm{Aut}(L), s_m^{-1}H_q(G_\lambda^0(L)))$$

since the action on the base is trivial. As shown above, this group is trivial for $1 \leq q < m/2$. Hence, the edge map in the localised spectral sequence is an isomorphism for $n < m/2$

$$s_m^{-1}H_n(G_\lambda(L)) \xrightarrow{\cong} H_n(\mathrm{Aut}(L), s_m^{-1}H_0(G_\lambda^0(L))).$$

But this map is the map $H_n(G_\lambda(L)) \to H_n(\mathrm{Aut}(L))$ in the lemma since S^* acts on $G_\lambda(L)$ via conjugation, hence trivially on its homology groups, and the augmentation $\mathbb{Z}[S(m)^*] \to \mathbb{Z} : \langle r\rangle \mapsto 1$ sends s_m to 1. □

Corollary 5.10. *Let $m \geq 2$ be an integer. Let R be a commutative ring such that ΣR^2 is an $S(m)$-algebra. Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an R-linear split exact form category with strong duality. Then the functor (5.3) induces an isomorphism of K-groups for $0 \leq i < m/2$*

$$GW_i^\oplus(S_2\mathcal{A}, Q) \xrightarrow{\cong} K_i^\oplus(\mathcal{A}).$$

Proof. For $n = 0$ this is Lemma 5.6. In view of the commutativity of diagram (5.10) proved in Lemma 5.7, Proposition 5.9 implies that the map (5.9) is an isomorphism in degrees $< m/2$; see Subsection 5.3. Thus, the map (5.4) is an isomorphism on integral homology in degrees $< m/2$. This implies the claim. □

If $\mathcal{A}$ is linear over a ring R such that ΣR^2 has many units, Corollary 5.10 holds for all n, that is, Theorem 5.1 is proved in this case. This holds for instance when R is an infinite field. In order to reduce the general case of Theorem 5.1 to Corollary 5.10, we will employ a transfer argument. To set up the argument, we need the next few lemmas.

Lemma 5.11. *Let $m \geq 1$ be an integer. Let R be a commutative ring such that ΣR^2 is an $S(m)$-algebra. Then for any integer $d \geq 2^{m+1}$ there is a monic polynomial $f = T^d + a_{d-1}T^{d-1} + \cdots + a_1T + a_0 \in R[T]$ of degree d with $a_{d-1} = a_0 = 1$ such that ΣS^2 is an $S(m+1)$-algebra where $S = R[T]/f$.*

Proof. Since ΣR^2 is an $S(m)$-algebra (Definition 5.4) there are units $(u_1, ..., u_m)$ in ΣR^2 such that $u_J \in (\Sigma R^2)^*$ for $\emptyset \neq J \subset \{1, ..., m\}$. Set

$$\begin{array}{rclcl} h & = & T \cdot \prod_{\emptyset \neq J \subset \{1,\dots,m\}} (T^2 + u_J) & = & T^n + c_{n-1}T^{n-1} + \cdots + c_1 T, \\ f & = & 1 + gh & = & T^d + a_{d-1}T^{d-1} + \cdots + a_1 T + a_0 \quad \in R[T] \end{array}$$

where $n = 2^{m+1} - 1$ and $g = T^e + b_{e-1}T^{e-1} + \cdots + b_1 T + b_0$ is a monic polynomial of degree $e = d - n \geq 1$. Set $S = R[T]/f$. Recall that the inclusion of rings $\Sigma S^2 \subset S$ detects units. In particular, T^2 and $T^2 + u_J$ are units in ΣS^2 since they are units in S. Hence, ΣS^2 is an $S(m+1)$-algebra with $S(m+1)$-sequence $(u_1, ..., u_m, T^2)$, and f is monic of degree d. We have $a_0 = 1$ as $c_0 = 0$. Finally, since h is monic we have $a_{d-1} = b_{e-1} + c_{n-1}$ and we can choose b_{e-1} such that $a_{d-1} = 1$. □

Lemma 5.12. *Let $d \geq 2$ be an integer, let R be a commutative ring and let $f = T^d + a_{d-1}T^{d-1} + \cdots + a_1 T + a_0 \in R[T]$ be a monic polynomial of degree d with $a_{d-1} = a_0 = 1$. Let $t : S = R[T]/f \to R$ be the R-linear map $1 \mapsto 1$, $T^j \mapsto 0$ for $1 \leq j < d$, and let $\mu : S \otimes S \to S$ the multiplication map. Then there are a metabolic symmetric bilinear space M over R and an isometry of symmetric bilinear spaces over R*

$$(S, t \circ \mu) \cong \begin{cases} \langle 1 \rangle \perp \langle -1 \rangle \perp M & \textit{if } d = 2r \\ 2\langle 1 \rangle \perp \langle -1 \rangle \perp M & \textit{if } d = 2r + 1. \end{cases}$$

Proof. The symmetric bilinear form $(S, t\mu)$ over R is non-degenerate, by inspection of its Gram matrix with respect to the basis $\{1, T, ..., T^{d-1}\}$. If $d = 2r$ then the symmetric bilinear space of the lemma $S \otimes S \to R : x \otimes y \mapsto t(xy)$ has a non-degenerate subspace $R \cdot 1 \perp R \cdot T^r = \langle 1 \rangle \perp \langle -1 \rangle$ whose orthogonal complement M is metabolic with Lagrangian $RT \oplus RT^2 \oplus \cdots RT^{r-1}$. If $d = 2r + 1$ then S has a non-degenerate subspace

$$R \cdot 1 \perp (R \cdot T^r \oplus R \cdot T^{r+1}) = \langle 1 \rangle \perp \langle \left(\begin{smallmatrix} 0 & -1 \\ -1 & 1 \end{smallmatrix} \right) \rangle \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \langle -1 \rangle$$

whose orthogonal complement M is metabolic with Lagrangian $RT \oplus RT^2 \oplus \cdots RT^{r-1}$. □

Lemma 5.13. *For every integer $m \geq 2$ there are an integer $n \geq 0$, commutative rings B_1 and B_2 with multiplication μ_1 and μ_2 together with abelian group homomorphisms $t_1 : B_1 \to \mathbb{Z}$, $t_2 : B_2 \to \mathbb{Z}$ such that*

1. *$\Sigma(B_1)^2$ and $\Sigma(B_2)^2$ are $S(m)$-algebras,*
2. *B_1 and B_2 are finitely generated free abelian groups of rank $2n+1$ and $2n$, and*
3. *there exist isometries of symmetric bilinear forms over $\mathbb{Z}$*

$$g_1 : (B_1, t_1 \circ \mu_1) \cong n\langle 1 \rangle \perp n\langle -1 \rangle, \quad g_2 : (B_2, t_2 \circ \mu_2) \cong (n+1)\langle 1 \rangle + n\langle -1 \rangle.$$

Proof. We start with a preliminary remark based on Lemma 5.12 and the fact that the transfer of symmetric bilinear forms associated with an R-linear section $t : S \to R$ of a commutative R-algebra S preserves metabolic spaces and sends $\langle 1 \rangle$ and $\langle -1 \rangle$ to $(S, t\mu)$ and $(S, -t\mu)$. Consider a sequence of commutative rings $R_0 \to R_1 \to \cdots \to R_{m-1}$ where $R_{i+1} = R_i[T_i]/f_i$ and f_i is monic and has lowest and second highest coefficient equal 1 as in Lemma 5.12. Let $t_{i+1} : R_{i+1} \to R_i$ be the R_i-linear map defined in Lemma 5.12. Write $t : R_{m-1} \to R_0$ for the R_0-linear map which is the composition $t = t_1 \circ \cdots \circ t_{m-1}$. Then the symmetric bilinear space $(R_{m-1}, t \circ \mu_{m-1})$ over R_0 is isometric to $\varepsilon\langle 1\rangle \perp \langle -1\rangle \perp M$ where M is metabolic and $\varepsilon = 1, 2$ depending on wether the rank of R_{m-1} over R_0 is even or odd, by Lemma 5.12. If $R_0 = \mathbb{Z}$ then $\langle 1\rangle \perp \langle -1\rangle \perp M \cong (r+1)(\langle 1\rangle + \langle -1\rangle)$ where $\mathrm{rk}_{\mathbb{Z}} M = 2r$ since any indefinite inner product space of type I is determined by its rank and signature [MH73, Theorem 4.3].

Now we prove Lemma 5.13. We will use the identities

$$\begin{aligned} X^{(2^m)} - 1 &= (X-1)\textstyle\prod_{i=0}^{m-1}\left(X^{(2^i)} + 1\right) \\ X^{(2^m)} &= X \textstyle\prod_{i=0}^{m-1} X^{(2^i)} \end{aligned}$$

in $\mathbb{Z}[X]$ for $X = 3$ and we set

$$\begin{aligned} &n = \textstyle\prod_{i=0}^{m-1}\left(X^{(2^i)} + 1\right), \\ &d_0 = (X-1)(X+1), \qquad d_0' = X^2, \\ &d_i = X^{(2^i)} + 1, \qquad\qquad d_i' = X^{(2^i)} \quad \text{for } i = 1, ..., m-1. \end{aligned}$$

Then we have $2n = d_0 d_1 \cdots d_{m-1}$ and $2n+1 = d_0' d_1' \cdots d_{m-1}'$ with $d_i, d_i' \geq 2^{i+2}$ for $i = 0, ..., m-1$. By Lemma 5.11, there are sequences of commutative rings $\mathbb{Z} = A_0 \to A_1 \to \cdots \to A_{m-1} = B_1$ and $\mathbb{Z} = A_0' \to A_1' \to \cdots \to A_{m-1}' = B_2$ where $\Sigma(A_r)^2$ and $\Sigma(A_r')^2$ are $S(r+1)$-rings, $A_{r+1} \cong A_r[T]/f_r$ and $A_{r+1}' \cong A_r'[T]/f_r'$ are free A_r- and A_r'-modules of rank d_r and d_r'. Moreover, f_r and f_r' are monic and have lowest and second highest coefficient equal to 1. Now we apply the remark at the beginning of the proof to the two sequences of rings and conclude the lemma. □

End of proof of Theorem 5.1. We will use the multiplicative structure on Grothendieck-Witt groups (2.13) and the transfer computation in Lemma 4.19. Recall that the Burnside form ring $\underline{\mathbb{Z}} = (\mathbb{Z}, \mathbb{A}(\mathbb{Z}))$ is the tensor unit for form categories. In particular, the higher Grothendieck-Witt groups of any form category with strong duality are modules over the ring $GW_0^{\oplus}(\underline{\mathbb{Z}})$. Any quadratic space (P, q) in $\mathbb{Z}\,\mathrm{proj}_{\mathbb{A}(\mathbb{Z})}$ defines a non-singular form functor $(P, q) : \underline{\mathbb{Z}} \to \mathbb{Z}\,\mathrm{proj}_{\mathbb{A}(\mathbb{Z})}$ (Remark 4.11). For any additive form category with strong duality $\mathcal{A} = (\mathcal{A}, \sharp, \mathrm{can}, Q)$ the quadratic space $(P, q) \in i\,\mathcal{Q}\mathrm{uad}(\underline{\mathbb{Z}})$ defines a non-singular form functor $\mathcal{A} \to \mathcal{A}$, called *tensor product with* (P, q), given by

$$\mathcal{A} \cong \underline{\mathbb{Z}} \otimes \mathcal{A} \xrightarrow{(P,q)\otimes 1} \mathbb{Z}\,\mathrm{proj}_{\mathbb{A}(\mathbb{Z})} \otimes \mathcal{A} \xleftarrow[\simeq]{(\mathbb{Z}, 1_{\mathbb{A}})\otimes 1} \underline{\mathbb{Z}} \otimes \mathcal{A} \cong \mathcal{A}$$

where the second arrow is an equivalence since $\mathcal{A}$ is additive.

Let R be a commutative ring with multiplication μ and trivial involution. Recall from Notation 4.15 the Burnside form ring $\underline{R} = (R, \mathbb{A}(R))$ of R. Assume that R is finitely generated free as abelian group, that $t : R \to \mathbb{Z}$ is a homomorphism of abelian groups such that $(R, t \circ \mu)$ is a non-degenerate symmetric bilinear form over $\mathbb{Z}$, and that $q \in Q_{\mathbb{A}(\mathbb{Z})}(R)$ is a quadratic form on R over $\underline{\mathbb{Z}}$ with associated symmetric bilinear form $\rho(q) = t \circ \mu$. As explained in Example 4.18, we have induced form functors of additive form categories with strong duality

$$\mathbb{Z}\,\mathrm{proj}_{\mathbb{A}(\mathbb{Z})} \longrightarrow R\,\mathrm{proj}_{\mathbb{A}(R)} \xrightarrow{q_*} \mathbb{Z}\,\mathrm{proj}_{\mathbb{A}(\mathbb{Z})}. \tag{5.19}$$

By Remark 2.38, the composition is tensor product with the quadratic space $(R,q) \in i\,\mathcal{Q}\mathrm{uad}(\underline{\mathbb{Z}})$ since it sends $(\mathbb{Z}, 1_{\mathbb{A}})$ to (R,q). For an additive form category $\mathcal{A}$, write $\mathcal{A}_R$ for the additive hull of the form category $\underline{R} \otimes \mathcal{A}$; see Definition 2.34. Note that $\mathcal{A}_R$ and its category of extensions $S_2(\mathcal{A}_R)$ are R-linear form categories with strong duality. Tensoring diagram (5.19) with $\mathcal{A}$, taking additive hulls and applying the functor S_2 of extensions induces diagrams of additive form categories

$$\mathcal{A} \longrightarrow \mathcal{A}_R \xrightarrow{q_*} \mathcal{A} \quad \text{and} \quad S_2\mathcal{A} \longrightarrow S_2(\mathcal{A}_R) \xrightarrow{q_*} S_2\mathcal{A}$$

whose compositions are given by the tensor product with the quadratic space $(R,q) \in i\,\mathcal{Q}\mathrm{uad}(\underline{\mathbb{Z}})$. This induces the string of maps of Grothendieck-Witt groups

$$GW_i^{\oplus}(S_2\mathcal{A}) \longrightarrow GW_i^{\oplus}(S_2(\mathcal{A}_R)) \xrightarrow{q_*} GW_i^{\oplus}(S_2\mathcal{A})$$

whose composition is multiplication with $[q] \in GW_0^{\oplus}(\underline{\mathbb{Z}})$.

Now we are ready to prove Theorem 5.1. Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be any small split exact form category with strong duality. Let m and i be integers such that $m \geq 2$ and $0 \leq i < m/2$. As noted in Lemma 5.4, the functor (5.3) has a section $K^{\oplus}(\mathcal{A}) \to GW^{\oplus}(S_2\mathcal{A})$. It suffices to show that this section is an isomorphism in degree i. Choose an integer $n \geq 0$ such that the conclusion of Lemma 5.13 holds and keep the notation of that lemma. The quadratic space $\langle 1_{\mathbb{A}} \rangle = (\mathbb{Z}, 1_{\mathbb{A}})$ over $\underline{\mathbb{Z}}$ has associated symmetric bilinear space $\langle 1_{\mathbb{Z}} \rangle$ where $1_{\mathbb{A}} \in \mathbb{A}(\mathbb{Z})$ is the multiplicative identity. Define the quadratic form $q_j \in Q_{\mathbb{A}(\mathbb{Z})}(B_j)$ on B_j by the formulas

$$q_1 = g_1^{\bullet}\left(n\langle 1_{\mathbb{A}} \rangle \perp n\langle -1_{\mathbb{A}} \rangle\right), \qquad q_2 = g_2^{\bullet}\left((n+1)\langle 1_{\mathbb{A}} \rangle \perp n\langle -1_{\mathbb{A}} \rangle\right)$$

and note that $\rho(q_j) = t_j \mu_j$. Then we have isometries of quadratic spaces

$$g_1 : (B_1, q_1) \cong n\langle 1_{\mathbb{A}} \rangle \perp n\langle -1_{\mathbb{A}} \rangle, \qquad g_2 : (B_2, q_2) \cong (n+1)\langle 1_{\mathbb{A}} \rangle \perp n\langle -1_{\mathbb{A}} \rangle$$

in $i\,\mathcal{Q}\mathrm{uad}(\underline{\mathbb{Z}})$. For $j = 1,2$ we have a commutative diagram of abelian groups

$$\begin{tikzcd}
K_i^{\oplus}(\mathcal{A}) \arrow[r] \arrow[d] & K_i^{\oplus}(\mathcal{A}_{B_j}) \arrow[r, "(q_j)_*"] \arrow[d, "\cong"] & K_i^{\oplus}(\mathcal{A}) \arrow[d] \\
GW_i^{\oplus}(S_2\mathcal{A}) \arrow[r] & GW_i^{\oplus}(S_2(\mathcal{A}_{B_j})) \arrow[r, "(q_j)_*"'] & GW_i^{\oplus}(S_2\mathcal{A})
\end{tikzcd}$$

in which the middle vertical arrow is an isomorphism, by Corollary 5.10. The composition of the two lower horizontal maps is multiplication with $[q_j] \in GW_0(\underline{\mathbb{Z}})$. It follows that for any $x \in GW_i^{\oplus}(S_2\mathcal{A})$, the element $[q_j] \cdot x$ is in the image of the right vertical map. Since $[q_2] - [q_1] = 1 \in GW_0(\underline{\mathbb{Z}})$ we have $x = [q_2] \cdot x - [q_1] \cdot x$. Both summands are in the image of the right vertical map, hence the element x is, too. $\square$

6. The Group Completion Theorem

In order to state and prove the Group Completion Theorem 6.6 below it is convenient to give a model of the Grothendieck-Witt space in terms of Waldhausen's $S_\bullet$-construction adapted to form categories as was done for symmetric forms in [Sch10b, Definition 3].

Recall [Sch10a, §3.2] that the category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ between categories with dualities $(\mathcal{A}, \sharp_{\mathcal{A}}, \mathrm{can}_{\mathcal{A}})$, $(\mathcal{B}, \sharp_{\mathcal{B}}, \mathrm{can}_{\mathcal{B}})$ is canonically a category with duality where the dual of a functor F is $\sharp \circ F \circ \sharp$ and the double dual identification is $\mathrm{can}_{\mathcal{B}} \circ F(\mathrm{can}_{\mathcal{A}})$. If $\mathcal{B}$ is linear (additive, exact) then so is $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. For instance, if $\mathcal{B}$ is exact, then a sequence in $\mathrm{Fun}(\mathcal{A}, B)$ is exact if its value at A is exact in $\mathcal{B}$ for all objects $A \in \mathcal{A}$. There is a bijection between symmetric forms (F, φ) in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ and form functors $(F, \hat{\varphi}) : \mathcal{A} \to \mathcal{B}$ between categories with dualities which sends $\varphi : F \to \sharp F \sharp$ to $\hat{\varphi} = \varphi^{\sharp_{\mathcal{B}}} \circ \mathrm{can}_{\mathcal{B}} : F\sharp \to \sharp F$. In this sense we may identify these two notions.

Definition 6.1. Let $(\mathcal{P}, \leq)$ be a poset with strong duality $\mathcal{P}^{op} \to \mathcal{P} : x \mapsto x'$. We assume that $x \leq y$ and $y \leq x$ implies $x = y$ so that the duality is strict and $x'' = x$. Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be a form category. A *quadratic form on a functor* $A : \mathcal{P} \to \mathcal{A} : i \mapsto A_i$ is a pair (ξ, φ) where $\xi = (\xi_i)_{i \leq i'}$ is a family of quadratic forms $\xi_i \in Q(A_i)$ indexed over those $i \in \mathcal{P}$ satisfying $i \leq i'$ and $\varphi : A \to A^\sharp$ is a symmetric form in $\mathrm{Fun}(\mathcal{P}, \mathcal{A})$. We require compatibility among the forms ξ_i and φ in the sense that $A^\bullet_{i \leq j}(\xi_j) = \xi_i$ whenever $i \leq j \leq j' \leq i'$ and $\rho(\xi_i) = \varphi_{i'} \circ A_{i \leq i'}$ is the diagonal map in the commutative diagram

$$\begin{tikzcd}
A_i \arrow[r, "\varphi_i"'] \arrow[d, "A_{i\leq i'}"'] & (A^\sharp)_i \arrow[r, equal] \arrow[d, "(A^\sharp)_{i\leq i'}"] & (A_{i'})^\sharp \arrow[d, "(A_{i\leq i'})^\sharp"] \\
A_{i'} \arrow[r, "\varphi_{i'}"'] & (A^\sharp)_{i'} \arrow[r, equal] & (A_i)^\sharp .
\end{tikzcd} \tag{6.1}$$

In other words, the set of quadratic forms $Q(A)$ on a functor A is defined by the equalizer diagram

$$Q(A) \rightarrowtail \lim_{i \leq i'} Q(A_i) \times [A, A^\sharp] \overset{\rho}{\underset{\varepsilon}{\rightrightarrows}} \prod_{i \leq i'} \mathcal{A}(A_i, (A_i)^\sharp)$$

where $[A, A^\sharp] = \mathrm{Fun}(\mathcal{P}, \mathcal{A})(A, A^\sharp)$ is the set of natural transformations $A \to A^\sharp$ and where the i-th component of ρ and ε are $\rho_i(\xi, \varphi) = \rho_{A_i}(\xi_i)$ and $\varepsilon(\xi, \varphi) = \varphi_{i'} \circ A_{i \leq i'}$. The set $Q(A)$ is an abelian group with component-wise addition $(\xi, \varphi) + (\zeta, \psi) = (\xi + \zeta, \varphi + \psi)$ where $(\xi_i)_{i \leq i'} + (\zeta_i)_{i \leq i'} = (\xi_i + \zeta_i)_{i \leq i'}$. The abelian group $Q(A)$ is part of the following Mackey functor structure

$$[A, A^\sharp] \xrightarrow{\tau} Q(A) \xrightarrow{\rho} [A, A^\sharp].$$

The restriction is $\rho : Q(A) \to [A, A^\sharp] : (\xi, \varphi) \mapsto \varphi$. The transfer $\tau : [A, A^\sharp] \to Q(A)$ for $\varphi \in [A, A^\sharp]$ at $i \leq i'$ is the pair $(\tau(\varphi), \varphi + \varphi^\sharp \,\mathrm{can})$ where

$$\tau(\varphi)_i = \tau(\varphi_{i'} \circ A_{i \leq i'}) = \tau((A_{i \leq i'})^\sharp \circ \varphi_i)$$

is the transfer in $\mathcal{A}$ of the diagonal arrow in the commutative diagram (6.1). The family of quadratic forms $(\tau(\varphi)_i)_{i \leq i'}$ is indeed compatible: for $j \leq i$ we have

$$Q(A_{j \leq i})\tau(\varphi_{i'} \circ A_{i \leq i'}) = \tau((A_{j \leq i})^\sharp \circ \varphi_{i'} \circ A_{i \leq i'} A_{j \leq i}) = \tau(\varphi_{j'} \circ A_{j \leq j'})$$

since $(A_{j \leq i})^\sharp \circ \varphi_{i'} = \varphi_{j'} \circ A_{i' \leq j'}$ and

$$\rho(\tau(\varphi)_i) = \varphi_{i'} A_{i \leq i'} + (A_{i \leq i'})^\sharp (\varphi_{i'})^\sharp \,\mathrm{can} = (\varphi + \varphi^\sharp \,\mathrm{can})_{i'} \circ A_{i \leq i'}.$$

It is clear that ρ is equivariant and that $\rho\tau = 1 + \sigma$. The transfer τ is also equivariant because of the commutativity of (6.1) and equivariance of τ in $\mathcal{A}$. Finally,

$$Q : \mathrm{Fun}(\mathcal{P}, \mathcal{A})^{op} \to \mathrm{Ab} \tag{6.2}$$

is a functor sending a natural transformation $f : B \to A$ to the map $f^\bullet : Q(A) \to Q(B)$ defined by $f^\bullet(\xi, \varphi) = (f^\bullet(\xi), f^\bullet(\varphi))$ where $f^\bullet(\xi)_i = f_i^\bullet(\xi_i)$ and $f^\bullet(\varphi) = f^\sharp \varphi f$.

For natural transformations $f, g : B \to A$ and elements $i \leq i'$ in $\mathcal{P}$ we have

$$\begin{aligned} ((f \top g)^\bullet(\xi))_i &= (f_i \top g_i)^\bullet(\xi_i) \\ &= \tau_{B_i}\left((f_i)^\sharp \circ \rho_{A_i}(\xi_i) \circ g_i\right) \\ &= \tau_{B_i}\left((f_i)^\sharp \circ (A_{i\leq i'})^\sharp \circ \varphi_i \circ g_i\right) \\ &= \tau_{B_i}\left((B_{i\leq i'})^\sharp \circ (f^\sharp)_i \circ \varphi_i \circ g_i\right) \\ &= \left(\tau_B\left(f^\sharp \circ \rho_A(\xi, \varphi) \circ g\right)\right)_i \end{aligned}$$

and condition (3) in Definition 2.1 holds so that we have a form category

$$(\mathrm{Fun}(\mathcal{P}, \mathcal{A}), \sharp, \mathrm{can}, Q). \tag{6.3}$$

Lemma 6.2. *Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an exact form category. Then for every poset with strict duality $\mathcal{P}$, the category of functors (6.3) is an exact form category where a sequence of functors is admissible exact if its evaluation at all $i \in \mathcal{P}$ is admissible exact in $\mathcal{A}$.*

Proof. We have to show that for every admissible short exact sequence $X \overset{s}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$ of functors $\mathcal{P} \to \mathcal{A}$ the induced sequence of abelian groups

$$0 \longrightarrow Q(Z) \xrightarrow{p^\bullet} Q(Y) \xrightarrow{(s^\sharp \circ \rho(\_),\ s^\bullet)} [Y, X^\sharp] \times Q(X)$$

is exact. Injectivity of $p^\bullet$ is clear since the component maps $Q(Z_i) \to Q(Y_i)$ and $[Z, Z^\sharp] \to [Y, Y^\sharp]$ are injective. For exactness at $Q(Y)$, let $(\xi, \varphi) \in Q(Y)$ be such that $s^\bullet(\xi, \varphi) = 0$ and $s^\sharp \varphi = 0$. Since $(\mathrm{Fun}(\mathcal{P}, \mathcal{A}), \sharp, \mathrm{can})$ is an exact category with duality, there is a unique symmetric $\bar{\varphi} \in [Z, Z^\sharp]$ such that $\varphi = p^\sharp \bar{\varphi} p$; see Example 2.24. Moreover, in view of the functorial exact sequence

$$0 \longrightarrow Q(Z_i) \xrightarrow{p_i^\bullet} Q(Y_i) \xrightarrow{(s_i^\sharp \circ \rho_i(\_),\ s_i^\bullet)} \mathcal{A}(Y_i, (X_i)^\sharp) \times Q(X_i)$$

there is a unique $\bar{\xi} \in \lim_{i \leq i'} Q(Z_i)$ such that $p_i^\bullet(\bar{\xi}_i) = \xi_i$ since $s_i^\bullet(\xi_i) = 0$ and $s_i^\sharp \rho_i(\xi_i) = s_i^\sharp (A_{i\leq i'})^\sharp \varphi_i = (s^\sharp \varphi)_i = 0$. Then $(\bar{\xi}, \bar{\varphi}) \in Q(Z)$ satisfies $p^\bullet(\bar{\xi}, \bar{\varphi}) = (\xi, \varphi)$. □

Let $\mathrm{Ar}[n]$ denote the poset whose objects are the arrows of the poset $[n] = \{0 < 1 < \ldots < n\}$ and whose morphisms are the commutative squares in $[n]$. For an exact category $\mathscr{E}$, we let as usual $S_n\mathscr{E} \subset \mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$ be the full subcategory of those functors

$$A : \mathrm{Ar}[n] \to \mathscr{E} : (p \leq q) \mapsto A_{p,q}$$

for which $A_{p,p} = 0$ and $A_{p,q} \rightarrowtail A_{p,r} \twoheadrightarrow A_{q,r}$ is an admissible short exact sequence whenever $p \leq q \leq r$, $p, q, r \in [n]$. The category $S_n\mathscr{E}$ is closed under extensions in $\mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$. This makes it into an exact category by declaring a sequence in $S_n\mathscr{E}$ to be admissible exact if it is in $\mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$. The cosimplicial category $n \mapsto \mathrm{Ar}[n]$ makes the assignment $n \mapsto S_n\mathscr{E}$ into a simplicial exact category.

Recall [Wal85, §1.9] that the K-theory space $K(\mathscr{E})$ of an exact category $\mathscr{E}$ can be defined as the space

$$K(\mathscr{E}) = \Omega|iS_\bullet\mathscr{E}|$$

where $iS_n\mathscr{E} \subset S_n\mathscr{E}$ is the subcategory of isomorphisms.

The category $[n]$ has a unique structure of a poset with strict duality $[n]^{op} \to [n] : i \mapsto n - i$. This induces a strict duality on the category $\mathrm{Ar}[n]$ of arrows in $[n]$.

For an exact form category $(\mathscr{E}, \sharp, \mathrm{can}, Q)$, the category $\mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$ is therefore an exact form category (see Lemma 6.2). This duality preserves the subcategory $S_n\mathscr{E} \subset \mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$, and we obtain an exact form category

$$(S_n\mathscr{E}, \sharp, \mathrm{can}, Q)$$

by restricting the quadratic functor Q on $\mathrm{Fun}(\mathrm{Ar}[n], \mathscr{E})$ to $S_n\mathscr{E}$. The simplicial structure maps of $n \mapsto S_n\mathscr{E}$ are not compatible with dualities. However, its edgewise subdivision [Wal85, §1.9]

$$S^e_\bullet\mathscr{E} : n \mapsto S^e_n\mathscr{E} = S_{2n+1}\mathscr{E}$$

does defines a simplicial exact form category where for a simplicial object $n \mapsto X[n]$ its edgewise subdivision X^e is the simplicial object $n \mapsto X([n]^{op}[n])$ and $[n]^{op}[n] \cong \{n' < \cdots < 0' < 0 < \cdots n\} \cong [2n+1]$ is the concatenation (or join) of $[n]^{op}$ and $[n]$; see [Wal85, §1.9], [Sch10b, §2.4-2.6].

The following generalises the definition of the Grothendieck-Witt space for symmetric bilinear forms given in [Sch10b], [Sch17b].

Definition 6.3 (Grothendieck-Witt space)**.** Let $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ be an exact form category with strong duality. The assignment $n \mapsto (S^e_n\mathscr{E}, \sharp, \mathrm{can}, Q)$ defines a simplicial exact form category with strong duality. The associated categories of quadratic spaces define a simplicial category $n \mapsto i\,\mathcal{Q}\mathrm{uad}(S^e_n\mathscr{E}, Q)$.

The composition $i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E}, Q) \to iS^e_\bullet\mathscr{E} \to iS_\bullet\mathscr{E}$ of simplicial categories, in which the first arrow is the forgetful functor $(X, \xi) \mapsto X$, and the second is the canonical map $X^e_\bullet \to X_\bullet$ of simplicial objects induced by $[n] \subset [n]^{op}[n] : i \mapsto i$ yields a map of classifying spaces

$$|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E}, Q)| \to |iS_\bullet\mathscr{E}| \tag{6.4}$$

whose homotopy fibre (with respect to a zero object of $\mathscr{E}$ as base point of $iS_\bullet\mathscr{E}$) is defined to be the *Grothendieck-Witt space*

$$GW(\mathscr{E}, Q)$$

of $(\mathscr{E}, \sharp, \mathrm{can}, Q)$. Thus, we have a homotopy fibration

$$GW(\mathscr{E}, Q) \to |i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E}, Q)| \to |iS_\bullet\mathscr{E}|.$$

We define the *higher Grothendieck-Witt groups* of $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ as the homotopy groups

$$GW_i(\mathscr{E}, Q) = \pi_i GW(\mathscr{E}, Q), \qquad i \geq 1,$$

and show in Theorem 6.5 below that $\pi_0 GW(\mathscr{E}, Q) \cong GW_0(\mathscr{E}, Q)$, so that our definition here extends that in Definition 2.27.

The topological realisation $|\mathcal{C}|$ of a symmetric monoidal category $(\mathcal{C}, \oplus, e, \alpha, \lambda, \rho, \gamma)$ as defined in [Mac71, p. 180] is a homotopy commutative, associative and unital H-space $\oplus : |\mathcal{C} \times \mathcal{C}| = |\mathcal{C}| \times |\mathcal{C}| \to |\mathcal{C}|$ with specified homotopies for the unit, associativity and commutativity diagrams given by the structure maps $e, \alpha, \lambda, \rho, \gamma$. A choice of direct sum in $\mathscr{E}$ makes $\mathscr{E}$ and all functor categories $\mathrm{Fun}(\mathcal{P}, \mathscr{E})$ into a symmetric monoidal category [Mac71, p. 180] where the structure maps on $\mathrm{Fun}(\mathcal{P}, \mathscr{E})$ are defined object-wise. For instance, for $F, G \in \mathrm{Fun}(\mathcal{P}, \mathscr{E})$ and $P \in \mathcal{P}$ we set $(F \oplus G)(P) = F(P) \oplus G(P)$. This makes $i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E}, Q)$ into a simplicial symmetric monoidal category where all simplicial structure maps strictly commute with the symmetric monoidal structures maps. In particular, its geometric realisation $|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)|$ becomes a homotopy commutative, associative and unital H-space. Its set of connected components is therefore a unital abelian monoid. The terminal object $[0]$ in Δ induces the inclusion

$$i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q) = i\,\mathcal{Q}\mathrm{uad}(S^e_0\mathscr{E},Q) \to i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q) \tag{6.5}$$

of simplicial categories in which the first is considered simplicially constant. Taking π_0, we obtain a homomorphism of abelian monoids

$$\pi_0\ i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q) \to \pi_0|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)|. \tag{6.6}$$

Recall that the Witt group $W(\mathscr{E},Q)$ of $(\mathscr{E},Q)$ is the quotient of $\pi_0\ i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q)$ modulo the submonoid of metabolic spaces.

Lemma 6.4. *Let $(\mathscr{E},\sharp,\mathrm{can},Q)$ be an exact form category with strong duality. Then the map (6.6) sends metabolic spaces to zero and the induced map of abelian monoids is an isomorphism:*

$$W(\mathscr{E},Q) \xrightarrow{\cong} \pi_0|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)|.$$

Proof. We have a coequalizer diagram

$$\pi_0\ i\,\mathcal{Q}\mathrm{uad}(S^e_1\mathscr{E},Q) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \pi_0\ i\,\mathcal{Q}\mathrm{uad}(S^e_0\mathscr{E},Q) \longrightarrow \pi_0|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)|$$

which is precisely the presentation of $W(\mathscr{E},Q)$ in view of Lemma 2.30. $\square$

The H-spaces $|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)|$ and $|iS_\bullet\mathscr{E}|$ are group complete since their π_0 are $W(\mathscr{E},Q)$ and the trivial group; see Lemma 6.4. Therefore, the following degree-wise group completion maps are weak equivalences (see for instance [Sch04, Lemma 2.6])

$$|i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)| \xrightarrow{\sim} |i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)^\dagger|, \qquad |iS_\bullet\mathscr{E}| \xrightarrow{\sim} |(iS_\bullet\mathscr{E})^\dagger|.$$

In particular, from the definition of the Grothendieck-Witt space we have the homotopy fibration

$$GW(\mathscr{E},Q) \to |i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)^\dagger| \to |(iS_\bullet\mathscr{E})^\dagger|.$$

The maps (6.5) and (6.4) define the sequence of simplicial categories

$$i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q) \to i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q) \to iS_\bullet\mathscr{E} \tag{6.7}$$

whose composition is trivial since $iS_0\mathscr{E}$ is trivial. This defines the natural map of topological spaces

$$|i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q)| \to GW(\mathscr{E},Q). \tag{6.8}$$

We will see in Theorem 6.6 below that this map is a group completion provided all admissible exact sequences in $\mathscr{E}$ split. But first we study the effect of this map on π_0.

Theorem 6.5. *Let $(\mathscr{E},\sharp,\mathrm{can},Q)$ be an exact form category with strong duality. Then the map (6.8) on π_0 induces an isomorphism of abelian groups*

$$GW_0(\mathscr{E},Q) \xrightarrow{\cong} \pi_0 GW(\mathscr{E},Q)$$

where the group on the left was defined in Definition 2.27.

Proof. Let V_n be the homotopy fibre of $|i\,\mathcal{Q}\mathrm{uad}(S^e_n\mathscr{E},Q)^\dagger| \to |(iS_n\mathscr{E})^\dagger|$. Since the last map is a map of group complete H-spaces which is surjective on π_0, the Bousfield-Friedlander Theorem [BF78, Theorem B.4] implies that the induced map $GW(\mathscr{E},Q) \to |V_\bullet|$ to the realisation of the simplicial space $n \mapsto V_n$ is a weak equivalence. Hence, we have a coequalizer diagram

$$\pi_0 V_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \pi_0\ V_0 \longrightarrow \pi_0|V_\bullet| = \pi_0 GW(\mathscr{E},Q).$$

For any monoidal functor $F:\mathcal{A}\to\mathcal{B}$ between symmetric monoidal groupoids which is essentially surjective on objects, Bass [Bas68], [Kar08, Ch2 §2.13] computes the π_0 of the homotopy fibre of $K(\mathcal{A})\to K(\mathcal{B})$ as the abelian group $K(F)$ generated by triples (A,B,f) where A,B are objects of $\mathcal{A}$ and $f:FA\cong FB$ is an isomorphism in $\mathcal{B}$, modulo some relations which are irrelevant for our purpose. So, π_0V_1 is generated by pairs (X,Y) of quadratic spaces $X,Y\in i\,\mathcal{Q}\mathrm{uad}(S_3\mathscr{E},Q)$ such that $X_{01}\cong Y_{01}$. Since $V_0 = GW^\oplus(\mathscr{E},Q)$, we have $\pi_0V_0 = GW_0^\oplus(\mathscr{E},Q)$ and d_1-d_0 is the map

$$\pi_0V_1 \xrightarrow{d_1-d_0} GW_0^\oplus(\mathscr{E},Q) : (X,Y)\mapsto X_{12}-Y_{12}-X_{03}+Y_{03}\ .$$

By Lemma 2.30, we have $X_{12}-Y_{12}-X_{03}+Y_{03} = H(X_{01})-H(Y_{01}) = 0 \in GW_0(\mathscr{E},Q)$ since $X_{01}\cong Y_{01}$. Hence, the map $GW_0^\oplus(\mathscr{E},Q)/(d_1-d_0)\to GW_0(\mathscr{E},Q):[X]\mapsto[X]$ is well-defined. The inverse map $GW_0(\mathscr{E})\to GW_0^\oplus(\mathscr{E},Q)/(d_1-d_0):[X]\mapsto[X]$ is also well-defined since a metabolic quadratic space X with Lagrangian L defines a pair of quadratic spaces in $S_3\mathscr{E}$, namely $L\subset L\subset X$ and $L\subset L\subset H(L)$, which defines an element of π_0V_1. Hence, the result follows. □

The following is the main theorem of the article. Its proof will occupy the rest of the section.

Theorem 6.6 (Group Completion Theorem)**.** *Let $(\mathscr{E},\sharp,\mathrm{can},Q)$ be an exact form category with strong duality in which every admissible exact sequence splits. Then the sequence (6.7) induces a homotopy fibration after degree-wise group completion*

$$|i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q)^\dagger| \to |i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)^\dagger| \to |(iS_\bullet\mathscr{E})^\dagger|. \tag{6.9}$$

In particular, the induced map into the homotopy fibre of the right map is a weak equivalence

$$GW^\oplus(\mathscr{E},Q) \xrightarrow{\sim} GW(\mathscr{E},Q).$$

Let $(\mathcal{A},\sharp,\mathrm{can},Q)$ be an exact form category with strong duality. Recall that the poset $[n]$ has a unique strict duality $i\mapsto n-i$ which gives the category $S_n\mathcal{A}$ the structure of an exact form category $(S_n\mathcal{A},\sharp,\mathrm{can},Q)$ as explained at the beginning of Section 6. For any $i\le j$, the map $[0]\to\mathrm{Ar}[n]:0\mapsto(i\le j)$ induces by restriction the exact evaluation functor $S_n\mathcal{A}\to\mathcal{A}:A\mapsto A_{ij}$ evaluating at $i\le j$. The map $[0]\to\mathrm{Ar}[n]:0\mapsto(i\le j)$ preserves dualities if $j=n-i$ in which case restriction along $[0]\to\mathrm{Ar}[n]$ defines an exact form functor $(S_n\mathcal{A},Q)\to(\mathcal{A},Q):A\mapsto A_{ij}$. For a form functor $G:(\mathcal{A},Q)\to S_n(\mathcal{B},Q)$, I will denote by $G_{ij}:(\mathcal{A},Q)\to(\mathcal{B},Q)$ the composition of G and evaluation at $i\le j$ if $j=n-i$. Recall from Example 2.13 the hyperbolic form functor $H(G):(\mathcal{A},Q)\to(\mathcal{B},Q)$ of an exact functor $G:\mathcal{A}\to\mathcal{B}$.

The following generalises [Sch17b, Corollary A.9].

Proposition 6.7. *Let $(A, \sharp, \mathrm{can}, Q)$ and $(\mathcal{B}, \sharp, \mathrm{can}, Q)$ be split exact form categories with strong duality.*

(1) *Let $G = (G, \varphi_q, \varphi) : (\mathcal{A}, Q) \to (S_2\mathcal{B}, Q)$ be a non-singular exact form functor. Then the two non-singular exact form functors G_{02} and $H(G_{01})$ induce homotopic maps of orthogonal sum Grothendieck-Witt spaces*
$$G_{02} \sim H(G_{01}) : GW^{\oplus}(\mathcal{A}, Q) \to GW^{\oplus}(\mathcal{B}, Q).$$

(2) *Let $G = (G, \varphi_q, \varphi) : (\mathcal{A}, Q) \to (S_3\mathcal{B}, Q)$ be a non-singular exact form functor. Then the two non-singular exact form functors G_{03} and $G_{12} \perp H(G_{01})$ induce homotopic maps of Grothendieck-Witt spaces*
$$G_{03} \sim (G_{12} \perp H(G_{01})) : GW^{\oplus}(\mathcal{A}, Q) \to GW^{\oplus}(\mathcal{B}, Q).$$

Proof. Consider the commutative diagram of categories
$$\begin{tikzcd}
i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{B}, Q) \arrow[r, "F"] & iS_2\mathcal{B} \arrow[r, "\delta_2"] & i\mathcal{B} \arrow[r, "\sigma_1"] \arrow[dr, "1"] & iS_2\mathcal{B} \arrow[r, "H"] \arrow[d, "\delta_1"] & i\,\mathcal{Q}\mathrm{uad}(S_2\mathcal{B}, Q) \arrow[d, "\delta_1"] \\
i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q) \arrow[r, "F"'] \arrow[u, "G"] & i\mathcal{A} \arrow[u, "G"] \arrow[ur, "G_{01}"] \arrow[rr, "G_{01}"'] & & i\mathcal{B} \arrow[r, "H"'] & i\,\mathcal{Q}\mathrm{uad}(\mathcal{B}, Q)
\end{tikzcd}$$
where F and H denote forgetful and hyperbolic functors, and $\delta_i : [1] \to [2]$ and $\sigma_i : [2] \to [1]$ are the standard i-th face and degeneracy maps. By Theorem 5.1, the composition $\delta_2 F$ of the top two left arrows induces an equivalence after group completion. The composition $(\delta_2 F) \circ (H\sigma_1)$ of the top two right maps followed by the top two left maps is the identity. It follows that $H\sigma_1$ is the homotopy inverse of $\delta_2 F$ after group completion. In particular, the composition of the top four horizontal maps is homotopic to the identity after group completion. It follows that $\delta_1 \circ G$ and $H \circ G_{01} \circ F$ induce homotopic maps after group completion. This proves the first claim.

The second claim follows formally from part (1) and Lemma 2.30. Here are the details. Write $\sim$ for "homotopic after applying $GW^{\oplus}$". For any non-singular exact form functor $F = (F, \varphi_q, \varphi) : (\mathcal{A}, \sharp, \mathrm{can}, Q) \to (\mathcal{B}, \sharp, \mathrm{can}, Q)$, we have a non-singular exact form functor $(\mathcal{A}, \sharp, \mathrm{can}, Q) \to (S_2\mathcal{B}, \sharp, \mathrm{can}, Q)$ with underlying functor and duality compatibility map
$$F \overset{\binom{1}{1}}{\rightarrowtail} F \oplus F \overset{(1\ -1)}{\twoheadrightarrow} F, \qquad (\varphi, \left(\begin{smallmatrix} \varphi & 0 \\ 0 & -\varphi \end{smallmatrix}\right), \varphi),$$
which on quadratic forms is
$$Q_{\mathcal{A}}(A) \to Q_{S_2\mathcal{B}}(FA \to FA \oplus FA \to FA) \subset Q_{\mathcal{B}}(FA \oplus FA) : \xi \mapsto \xi \perp -\xi.$$
By the first part we have
$$H(F) \sim F \perp (-F). \tag{6.10}$$
where $(-F) = (F, -\varphi_q, -\varphi)$.

For any exact form category $(\mathscr{E}, \sharp, \mathrm{can}, Q)$, we have an exact form functor $\Phi : (S_3\mathscr{E}, Q) \to (S_2\mathscr{E}, Q)$ which sends $X \in S_3\mathscr{E}$ to the admissible exact sequence
$$X_{02} \xrightarrow{\binom{1}{1}} X_{12} \oplus X_{03} \xrightarrow{(-1\ 1)} X_{23}$$
with duality compatibility $(1, -1 \oplus 1, 1)$ and map on quadratic forms
$$Q_{S_3\mathscr{E}}(X) \to Q_{S_2\mathscr{E}}(\Phi X) \subset Q_{\mathscr{E}}(X_{12} \oplus X_{03}) : (\xi)_{i \le i'} \mapsto -\xi_{12} \perp \xi_{03}.$$

For the non-singular exact form functor $G = (G, \varphi_q, \varphi) : (\mathcal{A}, Q) \to (S_3\mathcal{B}, Q)$ in the proposition, we have

$$\begin{aligned} G_{12} \perp H(G_{01}) \perp H(G_{12}) &\sim G_{12} \perp H(G_{02}) \\ &\sim G_{12} \perp (-G_{12}) \perp G_{03} \\ &\sim H(G_{12}) \perp G_{03} \end{aligned}$$

where the first homotopy follows from Additivity in K-theory, the second is part (1) applied to $\Phi \circ G$, and the third is (6.10). After applying $GW^{\oplus}$, these are maps between commutative H-spaces. Thus, we can cancel $H(G_{12})$ after applying $GW^{\oplus}$ and obtain the result. □

Proposition 6.8. *Let $(\mathcal{A}, \sharp, \mathrm{can}, Q)$ be an exact form category with strong duality in which every admissible exact sequence splits. Then the symmetric monoidal functors*

$$i\,\mathcal{Q}\mathrm{uad}(S_{2n}\mathcal{A}, Q) \to (i\mathcal{A})^n : \qquad (A, \xi) \mapsto (A_{01}, A_{12}, ..., A_{n-1,n})$$

$$i\,\mathcal{Q}\mathrm{uad}(S_{2n+1}\mathcal{A}, Q) \to (i\mathcal{A})^n \times i\,\mathcal{Q}\mathrm{uad}(\mathcal{A}, Q) : \quad (A, \xi) \mapsto (A_{01}, A_{12}, ..., A_{n-1,n}), (A_{n,n+1}, \xi_{n,n+1})$$

are homotopy equivalences after group completion.

Proof. This immediately follows from Proposition 6.7 and is *mutatis mutandis* the same as [Sch17b, Proposition A.8.]. □

Proof of Theorem 6.6. Using Proposition 6.8, the proof now is the same as [Sch04, Theorem 4.2]. Here are the details. Let $(S, \oplus, 0)$ be a symmetric strict monoidal category [Mac71, p. 157] acting on a category T from the right. Then there is a simplicial category $\mathrm{Bar}(T, S)$ which in simplicial degree n is

$$\mathrm{Bar}_n(T, S) = T \times S^n$$

with face maps induced by $\oplus$ and the action, and the degeneracy maps insert 0's. If S acts invertibly on T, that is, if for every object A of S, the map $\oplus A : T \to T$ induced by the action is a homotopy equivalence, then this space fits into a homotopy fibration

$$T \longrightarrow \mathrm{Bar}(T, S) \longrightarrow \mathrm{Bar}(0, S) \tag{6.11}$$

where the first map is inclusion of degree 0 and the second map is induced by $T \to 0$, see for instance [Sch04, Lemma 2.2] which is a special case of [Moe89, Theorem 2.1]. We will identify the sequence (6.9) in Theorem 6.6 with the sequence (6.11) up to homotopy. To that end, we replace $(\mathscr{E}, \sharp, \mathrm{can}, Q)$ with an equivalent form category that has a strictly monoidal direct sum, a strict duality and satisfies $(A \oplus B)^\sharp = B^\sharp \oplus A^\sharp$ for all $A, B \in \mathscr{E}$; see [Sch04, Lemma A.8.]. We define an action

$$i\,\mathcal{Q}\mathrm{uad}(\mathscr{E}, Q) \times i\mathscr{E} \to i\,\mathcal{Q}\mathrm{uad}(\mathscr{E}, Q)$$

of $(i\mathscr{E}, \oplus, 0)$ on $i\,\mathcal{Q}\mathrm{uad}(\mathscr{E}, Q)$ by sending $((B, \xi), A)$ to $(A^\sharp \oplus B \oplus A, \alpha^\bullet(h_A \perp \xi))$ where $\alpha : A^\sharp \oplus B \oplus A \cong A \oplus A^\sharp \oplus B$ is the canonical isomorphism switching A and $A^\sharp \oplus B$. The action sends the arrow (a, b) to the arrow $(a^\sharp)^{-1} \oplus b \oplus a$. This defines the simplicial category $\mathrm{Bar}_\bullet(i\,\mathcal{Q}\mathrm{uad}(\mathscr{E}, Q), i\mathscr{E})$. We define a map of simplicial categories which in simplicial degree n is

$$\beta_n : \mathrm{Bar}_n(i\,\mathcal{Q}\mathrm{uad}(\mathscr{E}, Q), i\mathscr{E}) \to i\,\mathcal{Q}\mathrm{uad}(S^e_n(\mathscr{E}, Q))$$

sending the object $((B,\xi),A_1,...,A_n)$ to the object $X \in S^e_n\mathscr{E}$ with

$$X_{i,j} = X_{i,i+1} \oplus \cdots \oplus X_{j-1,j} = \bigoplus_{i \le r < j} X_{r,r+1}$$

for $i \le j$ and $i,j \in \{n' < \cdots < 0' < 0 < \cdots < n\}$ where

$$(X_{n',(n-1)'},\dots,X_{1',0'},X_{0'0},X_{0,1},\dots,X_{n-1,n}) = (A_n^\sharp,\dots,A_1^\sharp,B,A_1,\dots,A_n).$$

Note that for $i \neq 0'$ we have $X^\sharp_{i,i+1} = X_{(i+1)',i'} = X_{i'-1,i'}$ since the duality on $\mathscr{E}$ is strict. The maps $X_{i,j} \to X_{r,s}$ for $(i,j) \le (r,s)$ are induced by the canonical partial inclusions and projections. The object X is equipped with the form $(\xi_\bullet, \varphi)$ where $\varphi : X \to X^\sharp$ has components

$$\begin{array}{lcr} X_{i,j} & = & X_{i,i+1} \oplus \cdots \oplus X_{j-1,j} \\ \varphi_{i,j} \downarrow & & \\ (X^\sharp)_{i,j} & = & (X_{j',i'})^\sharp = (X_{j',j'+1} \oplus \cdots \oplus X_{i'-1,i'})^\sharp = X^\sharp_{i'-1,i'} \oplus \cdots \oplus X^\sharp_{j',j'+1} \end{array}$$

which are 1 on the summands $X_{i,i+1}$ for $i \neq 0'$ and $\rho(\xi) : X_{0'0} \to X^\sharp_{0'0}$ for $i = 0'$. The compatible collection $\xi_\bullet$ of forms $\xi_{i,j} \in Q(X_{i,j})$ for $(i,j) \le (j',i')$ is

$$\alpha^\bullet(h_{X_{0,j}} \perp \xi) \in Q(X_{j',j}) = Q(X^\sharp_{0,j} \oplus X_{0',0} \oplus X_{0,j})$$

when $i = j'$, $j = 0,...,n$ where $\alpha : X^\sharp_{0,j} \oplus X_{0',0} \oplus X_{0,j} \cong X_{0,j} \oplus X^\sharp_{0,j} \oplus X_{0',0}$ is the canonical isomorphism switching $X_{0,j}$ and $X^\sharp_{0,j} \oplus X_{0',0}$. For $(r,s) \le (j',j)$, we let $\xi_{r,s}$ be the restriction of $\xi_{j',j}$ along the map $X_{r,s} \to X_{j',j}$.

Similarly, we have a map of simplicial categories $\gamma : \mathrm{Bar}_\bullet(0, i\mathscr{E}) \to iS_\bullet\mathscr{E}$ which in degree n sends $(A_1,...,A_n)$ to $X \in iS_n\mathscr{E}$ with

$$X_{i,j} = X_{i,i+1} \oplus \cdots \oplus X_{j-1,j} = \bigoplus_{i \le r < j} X_{r,r+1}$$

for $i \le j$ and $i,j \in \{0 < \cdots < n\}$ where $(X_{0,1},\dots X_{n-1,n}) = (A_1,\dots,A_n)$.

For $S = i\mathscr{E}$ and $T = i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q)$, β and γ induce a commutative diagram of simplicial categories

$$\begin{tikzcd} T^\dagger \arrow[r] \arrow[d, equal] & \mathrm{Bar}_\bullet(S,T)^\dagger \arrow[r] \arrow[d, "\beta^\dagger"] & \mathrm{Bar}_\bullet(S,0)^\dagger \arrow[d, "\gamma^\dagger"] \\ i\,\mathcal{Q}\mathrm{uad}(\mathscr{E},Q)^\dagger \arrow[r] & i\,\mathcal{Q}\mathrm{uad}(S^e_\bullet\mathscr{E},Q)^\dagger \arrow[r] & (iS_\bullet\mathscr{E})^\dagger. \end{tikzcd}$$

Note that $\mathrm{Bar}_n(S,T)^\dagger = \mathrm{Bar}_n(S^\dagger,T^\dagger)$. Since $T^\dagger$ is group complete, $S^\dagger$ acts invertibly on $T^\dagger$, and thus, the top row is a homotopy fibration. Using the second part of Proposition 6.8, the middle vertical arrow is degree-wise a homotopy equivalence. By Additivity in K-theory [Wal85], [Qui73], the right vertical map is also degree-wise a homotopy equivalence. Since the top sequence is a homotopy fibration (after realisation), so is the bottom sequence. $\square$

Appendix A. Quadratic functors on linear categories

A.1. **Quadratic maps and deviation.** Let M and N be abelian groups. Recall that a map of sets $g : M \to N$ is called *linear* if $g(x + y) = g(x) + g(y)$ for all $x, y \in M$. A map of sets $b : M \times M \to N$ is called *bilinear* if it is linear in both variables, that is the maps $M \to N$ defined by $z \mapsto b(x, z)$ and $z \mapsto b(z, y)$ are linear for all $x, y \in M$. A map of sets $g : M \to N$ is called *quadratic* if its *deviation*

$$M \times M \to N : (x, y) \mapsto g(x \top y) = g(x + y) - g(x) - g(y)$$

is bilinear. Note that a quadratic map g satisfies $g(0) = 0$ since $0 = g(0 \top 0) = g(0 + 0) - g(0) - g(0) = -g(0)$. The notion of linear and quadratic maps between abelian groups categorify to notions about functors between linear categories. The purpose of this section is to review these concepts and to establish notation and simple facts used throughout the paper. Some of the material is similar to [Bau94].

A.2. **Reduced functors, deviation and cross effect.** Let $\mathcal{A}$ be an additive category. A functor $F : \mathcal{A} \to \mathrm{Ab}$ with values in the category Ab of abelian groups is called *reduced* if $F(0) = 0$. To any reduced functor F is associated a functor $\mathcal{A} \times \mathcal{A} \to \mathrm{Ab} : (X, Y) \mapsto F(X \,|\, Y)$, called cross effect, equipped with a natural automorphism of order 2

$$\sigma_{X\,|\,Y} : F(X \,|\, Y) \xrightarrow{\cong} F(Y \,|\, X), \quad \sigma_{Y\,|\,X} \circ \sigma_{X\,|\,Y} = 1$$

and a C_2-equivariant natural diagram

$$F(X \,|\, X) \xrightarrow{\tau_X} F(X) \xrightarrow{\rho_X} F(X \,|\, X)$$

where C_2 acts via σ on $F(X \,|\, X)$ and trivially on $F(X)$; see Lemma A.2. The purpose of this subsection is to explain these extra data associated with F.

Let $\mathcal{A}$ be an additive category, $F : \mathcal{A} \to \mathrm{Ab}$ a reduced functor, and let X, Y be objects of $\mathcal{A}$. The *deviation* of F at (X, Y) is the function

$$\mathcal{A}(X, Y) \times \mathcal{A}(X, Y) \to \mathrm{Ab}(FX, FY) : (f_1, f_2) \mapsto F(f_1 \top f_2)$$

defined by

$$F(f_1 \top f_2) = F(f_1 + f_2) - F(f_1) - F(f_2).$$

Note that $F(f_1 \top 0) = F(0 \top f_2) = 0$ because F is reduced. Since F is a functor, we have

$$F(h) \circ F(f_1 \top f_2) = F(hf_1 \top hf_2), \qquad F(f_1 \top f_2) \circ F(g) = F(f_1 g \top f_2 g).$$

The biproduct $X \oplus Y$ in $\mathcal{A}$ is equipped with natural injection and projection maps $i_X : X \to X \oplus Y$, $i_Y : Y \to X \oplus Y$, $p_X : X \oplus Y \to X$ and $p_Y : X \oplus Y \to Y$ satisfying $1_X = p_X i_X$, $1_Y = p_Y i_Y$ and $1_{X \oplus Y} = e_X + e_Y$ where $e_X = i_X p_X$, $e_Y = i_Y p_Y$. By functoriality we have $F(e_X)F(e_Y) = F(e_Y)F(e_X) = F(0) = 0$ since F is reduced. The maps $F(e_X)$, $F(e_Y)$, $F(e_X \top e_Y) = 1 - F(e_X) - F(e_Y)$ are idempotents of $F(X \oplus Y)$ and define the decomposition of abelian groups

$$F(X \oplus Y) = \operatorname{Im} F(e_X) \oplus \operatorname{Im} F(e_Y) \oplus \operatorname{Im} F(e_X \top e_Y). \tag{A.1}$$

Note that we have isomorphisms $F(i_X) : F(X) \cong \operatorname{Im} F(e_X)$ and $F(i_Y) : F(Y) \cong \operatorname{Im} F(e_Y)$ with inverses $F(p_X)$ and $F(p_Y)$. The *cross effect* of F at (X, Y) is by definition the third term in the decomposition, that is, the abelian group

$$F(X \,|\, Y) = \operatorname{Im} F(e_X \top e_Y). \tag{A.2}$$

Denote by

$$\rho_{X,Y} : F(X \oplus Y) \to F(X \,|\, Y), \quad \tau_{X,Y} : F(X \,|\, Y) \to F(X \oplus Y) \tag{A.3}$$

the projection and inclusion maps induced by the decomposition (A.1) of $F(X \oplus Y)$. They are determined by the equations

$$\rho_{X,Y} \circ \tau_{X,Y} = 1, \quad \tau_{X,Y} \circ \rho_{X,Y} = F(e_X \top e_Y).$$

With this notation we have the natural isomorphism

$$F(X \oplus Y) \cong F(X) \oplus F(Y) \oplus F(X \,|\, Y) \tag{A.4}$$

given by the map $(F(p_X), F(p_Y), \rho_{X,Y})$ with inverse $(F(i_X), F(i_Y), \tau_{X,Y})$. For maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ in $\mathcal{A}$, the homomorphism $F(f_1 \oplus f_2)$ corresponds under this isomorphism to the map $F(f_1) \oplus F(f_2) \oplus F(f_1 \,|\, f_2)$ where

$$\begin{aligned} F(f_1 \,|\, f_2) &= \rho_{Y_1,Y_2} \circ F(f_1 \oplus f_2) \circ \tau_{X_1,X_2} \\ &= \rho_{Y_1,Y_2} \circ F(i_{Y_1} \cdot f_1 \cdot p_{X_1} \top i_{Y_2} \cdot f_2 \cdot p_{X_2}) \circ \tau_{X_1,X_2}. \end{aligned}$$

Note that for $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$ and $f_1, f_2 : X \to Y$ we have

$$F(f_1 \top f_2) = F(\nabla_Y) \circ F(i_{Y_1} f_1 p_{X_1} \top i_{Y_2} f_2 p_{X_2}) \circ F(\Delta_X) \tag{A.5}$$

where $\Delta_X : X \to X \oplus X : x \mapsto (x, x)$ and $\nabla_X : X \oplus X \to X : (x, y) \mapsto x + y$ are diagonal and codiagonal of X. If we set

$$\begin{aligned} \rho_X &= \rho_{X,X} \circ F(\Delta_X) : F(X) \xrightarrow{\Delta_X} F(X \oplus X) \xrightarrow{\rho_{X,X}} F(X|X) \\ \tau_X &= F(\nabla_X) \circ \tau_{X,X} : F(X|X) \xrightarrow{\tau_{X,X}} F(X \oplus X) \xrightarrow{\nabla_X} F(X) \end{aligned} \tag{A.6}$$

then we have the following.

Lemma A.1. *Let $\mathcal{A}$ be an additive category and let $F : \mathcal{A} \to \mathrm{Ab}$ be a reduced functor. Then the following holds.*

(1) *For all $f_1 \in \mathcal{A}(X_1, Y_1)$ and $f_2 \in \mathcal{A}(X_2, Y_2)$ we have*
$$F(f_1 \,|\, f_2) = \rho_{Y_1,Y_2} \circ F(i_{Y_1} \cdot f_1 \cdot p_{X_1} \top i_{Y_2} \cdot f_2 \cdot p_{X_1}) \circ \tau_{X_1,X_2}.$$

(2) *For all $f, g \in \mathcal{A}(X, Y)$ we have*
$$F(f \top g) = \tau_Y \circ F(f \,|\, g) \circ \rho_X.$$

Proof. The first holds by definition, and the second is a restatement of equation (A.5). $\square$

Let $\mathcal{A}$ be an additive category and $F : \mathcal{A} \to \mathrm{Ab}$ a reduced functor. Denote by σ the natural switch isomorphism $\sigma_{X,Y} : X \oplus Y \to Y \oplus X : (x, y) \mapsto (y, x)$ in $\mathcal{A}$. Since $F(\sigma_{X,Y}) \circ F(e_X \top e_Y) = F(e_Y \top e_X) \circ F(\sigma_{Y,X})$, there is a unique homomorphism of abelian groups

$$\sigma_{X\,|\,Y} : F(X \,|\, Y) \to F(Y \,|\, X) \tag{A.7}$$

characterized by the equations

$$\rho_{Y,X} \circ F(\sigma_{X,Y}) = \sigma_{X\,|\,Y} \circ \rho_{X,Y}, \quad F(\sigma_{X,Y}) \circ \tau_{X,Y} = \tau_{Y,X} \circ \sigma_{X\,|\,Y}.$$

Moreover, we have $\sigma_{X\,|\,Y} \circ \sigma_{Y\,|\,X} = 1$. The map

$$\sigma_X = \sigma_{X\,|\,X} : F(X \,|\, X) \to F(X \,|\, X).$$

defines a C_2-action on $F(X \,|\, X)$. Since Δ_X and ∇_X are C_2-equivariant where the action on X is trivial and on $X \oplus X$ it is the switch, we have

$$\tau_X \sigma_X = \tau_X, \quad \sigma_X \rho_X = \rho_X.$$

We equip $F(X)$ with the trivial involution. Summarising we have the following.

Lemma A.2. *Let $\mathcal{A}$ be an additive category and $F : \mathcal{A} \to \mathrm{Ab}$ a reduced functor. Then the cross effect $(X, Y) \mapsto F(X \,|\, Y)$ is equipped with a natural automorphism (A.7) of order 2 and we have a natural C_2-equivariant diagram*

$$F(X \,|\, X) \xrightarrow{\tau_X} F(X) \xrightarrow{\rho_X} F(X \,|\, X) \tag{A.8}$$

defined in (A.6). Moreover for any two maps $f, g \in \mathcal{A}(X, Y)$ we have

$$F(f + g) = F(f) + F(g) + \tau_Y F(f \,|\, g)\rho_X.$$

Proof. This follows from Lemma A.1 (2) and the definitions as discussed above. □

We will refer to the natural maps τ, ρ, σ associated with a reduced functor F as its *structure maps*.

A.3. **Quadratic functors on additive categories.** Let $\mathcal{A}$ be a linear category. Recall that a functor $F : \mathcal{A} \to \mathrm{Ab}$ is called *linear* if for all objects $X, Y \in \mathcal{A}$ the map

$$\mathcal{A}(X, Y) \to \mathrm{Ab}(FX, FY) : f \mapsto F(f) \tag{A.9}$$

is a linear map of abelian groups.

Definition A.3. Let $\mathcal{A}$ be an additive category. A functor $F : \mathcal{A} \to \mathrm{Ab}$ is called *quadratic* if for all objects X, Y of $\mathcal{A}$ the map (A.9) is a quadratic map of abelian groups, that is, $(f, g) \mapsto F(f \top g)$ is bilinear.

There are many equivalent definitions for a functor on an additive category to be quadratic. We will review some of them in this subsection. Recall that for a linear category $\mathcal{A}$, a functor $B : \mathcal{A} \times \mathcal{A} \to \mathrm{Ab}$ is called *bilinear* if it is linear in each variable, that is, for all objects $X, Y \in \mathcal{A}$ the functors $\mathcal{A} \to \mathrm{Ab}$ given by $Z \mapsto B(X, Z)$ and $Z \mapsto B(Z, Y)$ are linear.

Lemma A.4. *Let $\mathcal{A}$ be an additive category and let $F : \mathcal{A} \to \mathrm{Ab}$ be a functor. Then the following are equivalent.*

1. *F is quadratic.*
2. *F is reduced and its cross effect $F(\ \,|\ \,) : \mathcal{A} \times \mathcal{A} \to \mathrm{Ab}$ is bilinear.*

Proof. Assume that F is quadratic. Then F is reduced. Indeed, any quadratic map between abelian groups sends 0 to 0, and any functor sends identity morphisms to identity morphisms. For the zero-object 0 of $\mathcal{A}$ we have $1_0 = 0_0 \in \mathcal{A}(0, 0)$. Therefore $1_{F(0)} = F(1_0) = F(0_0) = 0_{F(0)}$ and $F(0)$ is the zero abelian group. The rest is immediate from Lemma A.1. □

In particular, any quadratic functor is reduced and thus is equipped with extra structure maps as in Lemma A.2.

Example A.5. Let $B : \mathcal{A} \times \mathcal{A} \to \mathrm{Ab}$ be a bilinear functor. Then the functor $X \mapsto F(X) = B(X, X)$ is quadratic with cross effect

$$F(X|Y) = B(X, Y) \oplus B(Y, X)$$

and structure maps

$$
\begin{array}{lll}
\rho: & B(X,X) \to B(X,X) \oplus B(X,X): & q \mapsto (q,q) \\
\tau: & B(X,X) \oplus B(X,X) \to B(X,X): & (f,g) \mapsto f+g \\
\sigma: & B(X,Y) \oplus B(Y,X) \to B(Y,X) \oplus B(X,Y): & (f,g) \mapsto (g,f).
\end{array}
$$

Recall that the cross effect of a reduced functor F is equipped with an automorphism $\sigma : F(X\,|\,Y) \to F(Y\,|\,X)$ of order 2.

Lemma A.6. *Let $\mathcal{A}$ be an additive category, and let $F : \mathcal{A} \to \mathrm{Ab}$ be a quadratic functor. Then for every object $X \in \mathcal{A}$ we have*

$$\rho_X \circ \tau_X = 1 + \sigma_X : F(X\,|\,X) \to F(X\,|\,X).$$

Proof. By definition (A.3), the map $\rho_{X,X}$ is surjective and $\tau_{X,X}$ is injective. Thus, it suffices to check

$$\tau_{X,X} \circ \rho_X \tau_X \circ \rho_{X,X} = \tau_{X,X} \circ (1+\sigma_X) \circ \rho_{X,X}.$$

Note that

$$
\begin{aligned}
F(1{+}\sigma)F(e_{X_1} \top e_{X_2}) &= F(e_{X_1}{+}\sigma e_{X_1} \top e_{X_2}{+}\sigma e_{X_2}) = F(e_{X_1}{+}e_{X_2}\sigma \top e_{X_2}{+}e_{X_2}\sigma) \\
&= F(e_{X_1} \top e_{X_2})(1+F(\sigma)) + (F(e_{X_1})+F(e_{X_2}))F(1 \top \sigma)
\end{aligned}
$$

using bilinearity of the deviation. Therefore,

$$
\begin{aligned}
\tau_{X,X} \circ \rho_X \tau_X \circ \rho_{X,X} &= F(e_{X_1} \top e_{X_2})F(\Delta_X)F(\nabla_X)F(e_{X_1} \top e_{X_2}) \\
&= F(e_{X_1} \top e_{X_2})F(1+\sigma)F(e_{X_1} \top e_{X_2}) \\
&= F(e_{X_1} \top e_{X_2})(1+F(\sigma)) \\
&= \tau_{X,X} \circ (1+\sigma_X) \circ \rho_{X,X}
\end{aligned}
$$

because $F(e_{X_1} \top e_{X_2})F(e_{X_1}) = F(e_{X_1} \top e_{X_2})F(e_{X_2}) = 0$ as product of orthogonal idempotents. $\square$

Definition A.7. Let $\mathcal{A}$ be a linear category. A *symmetric bilinear functor* on $\mathcal{A}$ (with values in Ab) is a pair (B,ι) where $B : \mathcal{A} \times \mathcal{A} \to \mathrm{Ab}$ is a bilinear functor and $\iota : B(X,Y) \to B(Y,X)$ is a natural automorphism of order 2. A homomorphism of symmetric bilinear functors $(B_1,\iota) \to (B_2,\iota)$ is a natural transformation $B_1 \to B_2$ commuting with ι.

Example A.8. The cross effect $(F(\ \ |\ \ ),\sigma)$ of a quadratic functor F on an additive category is symmetric bilinear; see Lemma A.4.

Let (B,ι) be a symmetric bilinear functor on an additive category $\mathcal{A}$. Then the automorphism ι defines a C_2-action on $B(X,X)$ and we denote by $B_{C_2}, B^{C_2} : \mathcal{A} \to \mathrm{Ab}$ the orbit and fixed point functors defined by $B_{C_2}(X) = B(X,X)_{C_2}$ and $B^{C_2}(X) = B(X,X)^{C_2}$. The map $1+\iota : B(X,X) \to B(X,X)$ naturally factors as

$$B \xrightarrow{\tau_B} B_{C_2} \xrightarrow{N} B^{C_2} \xrightarrow{\rho_B} B$$

where τ_B and ρ_B are the natural quotient and inclusion maps. The map N is called *norm.*

Lemma A.9. *Let $\mathcal{A}$ be an additive category and $(B,\iota):\mathcal{A}\times\mathcal{A}\to \mathrm{Ab}$ a symmetric bilinear functor on $\mathcal{A}$. Then the orbit and fixed point functors B_{C_2} and B^{C_2} are quadratic with associated symmetric bilinear cross effects isomorphic to $(B(X,Y),\iota)$ and structure maps $(\tau_B,\rho_B N)$ and $(N\tau_B,\rho_B)$. In particular, the norm map $N: B_{C_2}\to B^{C_2}$ induces an isomorphism of associated symmetric bilinear cross effects $N: B_{C_2}(X\,|\,Y)\cong B^{C_2}(X\,|\,Y)$.*

Proof. The functor $X\mapsto F(X)=B(X,X)$ carries a C_2-action defined by ι which induces the following C_2-action on the cross effect $F(X\,|\,Y)$, by Example A.5:

$$\iota: B(X,Y)\oplus B(Y,X)\to B(X,Y)\oplus B(Y,X):(a,b)\mapsto(\iota b,\iota a).$$

The structure maps ρ, τ and σ in Example A.5 commute with the action of ι. Since the construction of the cross effect commutes with taking fixed points and orbits we obtain the following isomorphisms of symmetric bilinear functors

$$B(X,Y)\cong F^{C_2}(X\,|\,Y)=F(X\,|\,Y)^{C_2}:a\mapsto(a,\iota a)$$
$$F_{C_2}(X\,|\,Y)=F(X\,|\,Y)_{C_2}\cong B(X,Y):(a,b)\mapsto a+\iota(b)$$

Under these isomorphisms the left hand diagram below maps isomorphically to the right diagram

$$\begin{tikzcd} F(X\,|\,Y) \arrow[r, "1+\iota"] \arrow[d, two heads] & F(X\,|\,Y) \\ F_{C_2}(X\,|\,Y) \arrow[r, "N"'] & F^{C_2}(X\,|\,Y) \arrow[u, tail] \end{tikzcd} \qquad \begin{tikzcd} B(X,Y)\oplus B(Y,X) \arrow[r, "1+\iota"] \arrow[d, two heads] & B(X,Y)\oplus B(Y,X) \\ B(X,Y) \arrow[r, "1"'] & B(X,Y). \arrow[u, tail] \end{tikzcd}$$

In particular, N is an isomorphism on cross effects. □

Here comes our final characterisation of quadratic functors on additive categories.

Lemma A.10. *Let $\mathcal{A}$ be an additive category and let $F:\mathcal{A}\to\mathrm{Ab}$ be a functor. Then the following are equivalent.*

(1) *F is quadratic.*
(2) *There is a symmetric bilinear functor $(B,\sigma):\mathcal{A}\times\mathcal{A}\to\mathrm{Ab}$ and a natural C_2-equivariant diagram of functors $\mathcal{A}\to\mathrm{Ab}$*

$$B(X,X)\xrightarrow{\tau}F(X)\xrightarrow{\rho}B(X,X)$$

such that $\rho\tau=1+\sigma$ and $F(f\top g)=\tau B(f,g)\rho$.

Moreover, if F is quadratic, the diagram in (2) defines a factorisation of the norm $B_{C_2}\xrightarrow{\tilde\tau}F\xrightarrow{\tilde\rho}B^{C_2}$ which induces isomorphisms of associated symmetric bilinear cross effects which are inverse to each other

$$B(X,Y)\underset{\cong}{\xrightarrow{\tilde\tau}}F(X\,|\,Y)\underset{\cong}{\xrightarrow{\tilde\rho}}B(X,Y).$$

Proof. We have already proved that (1) implies (2) with $B(X,Y)=F(X\,|\,Y)$; see Lemmas A.4, A.2 and A.6. Moreover, (2) implies (1) since the cross effect of F is bilinear in view of the equation $F(f\top g)=\tau B(f,g)\rho$ and Lemma A.1.

We are left with proving the two isomorphisms of cross effects. In view of Lemma A.9, the equation $\rho\tau=1+\sigma$ implies that on cross effects the composition $\tilde\rho\tilde\tau$ is an isomorphism. In particular, the map $\tilde{\tau}$ is injective on cross effects. The equation $F(f \top g) = \tau B(f,g)\rho$ shows that the cokernel of $\tilde{\tau} : B_{C_2}(X) \to F(X)$ is a linear functor. In particular, its cross effect $\mathrm{coker}(\tilde{\tau} : B_{C_2}(X,Y) \to F(X\,|\,Y))$ is zero, that is, $\tilde{\tau} : B(X,Y) = B_{C_2}(X\,|\,Y) \to F(X\,|\,Y)$ is also surjective. □

Remark A.11. In the situation of Lemma A.10 (2), the isomorphism of symmetric bilinear functors $\tilde{\tau} : B \to F(\ \ |\ \ )$ with inverse $\tilde{\rho}$ turns the natural isomorphism (A.4) into the isomorphism

$$F(X \oplus Y) \cong F(X) \oplus F(Y) \oplus B(X,Y) \tag{A.10}$$

given by $(F(p_X), F(p_Y), B(p_X,p_Y)\rho_{X\oplus Y})$ and inverse $(F(i_X), F(i_Y), \tau_{X\oplus Y}B(i_X,i_Y))$. Since B is bilinear, the map $(B(p_X,p_X), B(p_Y,p_Y), B(p_X,p_Y), B(p_Y,p_X))$

$$B(X\oplus Y, X\oplus Y) \cong B(X) \oplus B(Y) \oplus B(X,Y) \oplus B(Y,X) \tag{A.11}$$

is an isomorphism with inverse $(B(i_X,i_X), B(i_Y,i_Y), B(i_X,i_Y), B(i_Y,i_X))$. Under the isomorphisms (A.10) and (A.11), the structure maps (τ,ρ) at $X\oplus Y$ in Lemma A.10 (2) become

$$\begin{tikzcd}
B(X,X)\oplus B(Y,Y)\oplus B(X,Y)\oplus B(Y,X) \arrow[d, "\tau_X\oplus\tau_Y\oplus(1,\sigma)"] \\
F(X)\oplus F(Y)\oplus B(X,Y) \arrow[d, "\rho_X\oplus\rho_Y\oplus\binom{1}{\sigma}"] \\
B(X,X)\oplus B(Y,Y)\oplus B(X,Y)\oplus B(Y,X).
\end{tikzcd} \tag{A.12}$$

In particular, the value of the quadratic functor (F,B,ρ,τ) at $X\oplus Y$ is determined (up to natural isomorphism) by its values at X and Y.

Lemma A.12. *Let $\mathcal{A}$ be an additive category, and let $F : \mathcal{A} \to \mathrm{Ab}$ be a quadratic functor. Then for every split exact sequence*

$$0 \to X \xrightarrow{s} Y \xrightarrow{r} Z \to 0$$

in $\mathcal{A}$, the following is an exact sequence of abelian groups

$$0 \to F(X) \xrightarrow{F(s)} F(Y) \xrightarrow{\begin{pmatrix} F(1\,|\,r)\rho_Y \\ -F(r)\end{pmatrix}} F(Y\,|\,Z)\oplus F(Z) \xrightarrow{(F(r\,|\,1),\rho_Z)} F(Z\,|\,Z) \to 0.$$

Proof. The choice $t : Z \to Y$ of a splitting of $r : Y \to Z$ defines an isomorphism $u = (s,t) : X\oplus Z \to Y$. Consider the following diagram of abelian groups and linear maps

$$\begin{tikzcd}
F(X\,|\,Z) \arrow[r, tail, "F(u)\circ\tau_{X,Z}"] \arrow[d, equal] & F(Y)/F(X) \arrow[r, two heads, "F(r)"] \arrow[d, "F(1\,|\,r)\rho_Y"] & F(Z) \arrow[d, "\rho_Z"] \\
F(X\,|\,Z) \arrow[r, tail, "F(s\,|\,1)"'] & F(Y\,|\,Z) \arrow[r, two heads, "F(r\,|\,1)"'] & F(Z\,|\,Z)
\end{tikzcd} \tag{A.13}$$

where $F(Y)/F(X)$ is the cokernel of the injective map $F(s)$. The right square commutes by functoriality of ρ. We check commutativity of the left square. Since $\tau_{Y,Z}$ is injective and $\rho_{X,Z}$ is surjective it suffices to check

$$\tau_{Y,Z}\circ F(s\,|\,1)\circ\rho_{X,Z} = \tau_{Y,Z}\circ F(1\,|\,r)\circ\rho_Y\circ F(u)\circ\tau_{X,Z}\circ\rho_{X,Z}.$$

This equality holds since the right hand side of that equation is

$$\begin{aligned}
& \tau_{Y,Z} \circ \rho_{Y,Z} \circ F(1_Y \oplus r) \circ F(\Delta_Y) \circ F(u) \circ \tau_{X,Z} \circ \rho_{X,Z} \\
= \;& F\left(\left(\begin{smallmatrix} 1_Y & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right) \circ F\left(\begin{smallmatrix} s & t \\ 0 & 1_Z \end{smallmatrix}\right) \circ F\left(\left(\begin{smallmatrix} 1_X & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right) \\
= \;& \left(1 - F\left(\begin{smallmatrix} 1_Y & 0 \\ 0 & 0 \end{smallmatrix}\right) - F\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right) \circ F\left(\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & t \\ 0 & 1_Z \end{smallmatrix}\right)\right) \\
= \;& F\left(\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & t \\ 0 & 1_Z \end{smallmatrix}\right)\right) - F\left(\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)\right) \\
= \;& F\left(\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right)
\end{aligned}$$

whereas the left hand side of that equation is

$$\begin{aligned}
\tau_{Y,Z} \circ \rho_{Y,Z} \circ F(s \oplus 1_Z) &= F\left(\left(\begin{smallmatrix} 1_Y & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right) \circ F(s \oplus 1_Z) \\
&= F\left(\left(\begin{smallmatrix} s & 0 \\ 0 & 0 \end{smallmatrix}\right) \top \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_Z \end{smallmatrix}\right)\right).
\end{aligned}$$

To prove the lemma, note that the right horizontal maps in diagram (A.13) are split surjective and the left horizontal maps are split injective. The top row is exact, by definition of the cross effect. The bottom row is exact since F is quadratic; see Lemma A.4. It follows that the total complex of the right square is exact, that is, the sequence of abelian groups in the lemma is exact. □

Definition A.13. Let $\mathscr{E}$ be an exact category. A functor $F : \mathscr{E} \to \mathrm{Ab}$ is called *quadratic left exact* if it is quadratic on the underlying additive category and if for every admissible exact sequence in $\mathscr{E}$

$$0 \to X \xrightarrow{s} Y \xrightarrow{r} Z \to 0$$

the following sequence of abelian groups and linear maps is exact

$$0 \to F(Z) \xrightarrow{F(r)} F(Y) \xrightarrow{\left(\begin{smallmatrix} F(1|s)\rho_Y \\ -F(s) \end{smallmatrix}\right)} F(Y \mid X) \oplus F(X).$$

By Lemma A.12, any quadratic functor on a split exact category is quadratic left exact.

A.4. **Quadratic functors on linear categories.** In Section A.3, we have given three equivalent characterisations of quadratic functors on additive categories. But what should be a quadratic functor on a linear category which is not additive such as a ring R? The characterisation in Lemma A.4 doesn't make sense since the cross effect is only defined in the presence of finite direct sums. It turns out that the characterisation in Definition A.3 is not appropriate either since non-isomorphic quadratic functors on finitely generated free R-modules may be isomorphic when restricted to R. For instance, the quadratic functors $P \mapsto \mathrm{Hom}_{\mathbb{Z}}(P, P^\sharp)_\sigma$ and $P \mapsto \mathrm{Hom}_{\mathbb{Z}}(P, P^\sharp)^\sigma$ are isomorphic when restricted to $P = \mathbb{Z}$, but they are not isomorphic on finitely generated free $\mathbb{Z}$-modules, where $\sigma(f) = f^\sharp\,$can and $P^\sharp = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. The first defines quadratic forms over $\mathbb{Z}$ whereas the second functor defines symmetric bilinear forms over $\mathbb{Z}$. Therefore, we are forced to use the characterisation given in Lemma A.10. This is the reason for the formulation of Definition 2.1 on form categories.

Recall from Definition A.7 the notion of a symmetric bilinear functor.

Definition A.14. Let $\mathcal{A}$ be a linear category. A quadratic functor on $\mathcal{A}$ is the datum of a functor $F : \mathcal{A} \to \mathrm{Ab}$, a symmetric bilinear functor $(B, \sigma) : \mathcal{A} \times \mathcal{A} \to \mathrm{Ab}$ and a C_2-equivariant diagram of functors $\mathcal{A} \to \mathrm{Ab}$

$$B(X, X) \xrightarrow{\tau} F(X) \xrightarrow{\rho} B(X, X)$$

such that $\rho\tau = 1+\sigma$ and $F(f \top g) = \tau B(f, g)\rho$ for all $f, g \in \mathcal{A}(X, Y)$ and $X, Y \in \mathcal{A}$. Here C_2 acts trivially on $F(X)$ and via σ on $B(X, X)$.

A map of quadratic functors $(F_1, B_1, \sigma, \tau, \rho) \to (F_2, B_2, \sigma, \tau, \rho)$ on $\mathcal{A}$ is a pair of natural transformations of functors $F_1 \to F_2$ and $B_1 \to B_2$ commuting with the structure maps σ, τ and ρ. Composition is composition of natural transformations. This defines the category $\mathrm{Fun}_{quad}(\mathcal{A}, \mathrm{Ab})$ of quadratic functors on $\mathcal{A}$.

Remark A.15. In view of Definition B.1, a quadratic functor on a linear category $\mathcal{A}$ is a functor

$$(F, B, \sigma, \tau, \rho) : \mathcal{A} \to C_2\,\mathrm{Mac}$$

into C_2-Mackey functors together with an extension of (B, σ) to a symmetric bilinear functor on $\mathcal{A}$ such that $F(f \top g) = \tau B(f, g)\rho$ for all $f, g \in \mathcal{A}(X, Y)$.

Recall that to any linear category $\mathcal{A}$ is associated an additive category $\mathcal{A}^{\oplus}$, its *additive hull*, together with a fully faithful embedding $\mathcal{A} \subset \mathcal{A}^{\oplus}$ such that for every additive category $\mathcal{B}$, the restriction along the embedding yields an equivalence of categories of additive functors

$$\mathrm{Fun}_{add}(\mathcal{A}^{\oplus}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Fun}_{add}(\mathcal{A}, \mathcal{B}).$$

Explicitly, an object of $\mathcal{A}^{\oplus}$ is a pair (A, n) where $n \geq 0$ is an integer and $A = (A_1, ..., A_n)$ is a sequence of objects in $\mathcal{A}$ of length n formally written as $\bigoplus_{i=1}^{n} A_i$. Maps from (A, n) to (B, m) are matrices (f_{ij}) of maps $f_{ij} : A_j \to B_i$ in $\mathcal{A}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Composition is matrix multiplication. The category $\mathcal{A}^{\oplus}$ is an additive category with biproduct

$$(A_1, ..., A_n) \oplus (B_1, ..., B_m) = (A_1, ..., A_n, B_1, ..., B_m)$$

and zero object (0) the unique sequence of length 0. Moreover, $\mathcal{A}$ is embedded in $\mathcal{A}^{\oplus}$ via the functor $\mathcal{A} \to \mathcal{A}^{\oplus} : A \mapsto (A, 1)$.

Lemma A.16. *A quadratic functor $(B, \sigma) \xrightarrow{\tau} F \xrightarrow{\rho} (B, \sigma)$ on a linear category $\mathcal{A}$ extends essentially uniquely to a quadratic functor on the additive hull $\mathcal{A}^{\oplus}$ of $\mathcal{A}$. More precisely, the embedding $\mathcal{A} \subset \mathcal{A}^{\oplus}$ induces an equivalence of categories of quadratic functors*

$$\mathrm{Fun}_{quad}(\mathcal{A}^{\oplus}, \mathrm{Ab}) \xrightarrow{\sim} \mathrm{Fun}_{quad}(\mathcal{A}, \mathrm{Ab}). \tag{A.14}$$

Proof. It follows from Remark A.11 that the functor (A.14) is fully faithful with quasi-inverse given by

$$\begin{array}{ccc}
B(\bigoplus_{i=1}^n A_i, \bigoplus_{i=1}^n A_i) & = & \bigoplus_{i=1}^n B(A_i,A_i) \oplus \bigoplus_{1\le i<j\le n} (B(A_i,A_j)\oplus B(A_j,A_i)) \\
\tau\downarrow & & \downarrow \tau \oplus (1,\sigma) \\
Q(\bigoplus_{i=1}^n A_i) & = & \bigoplus_{i=1}^n Q(A_i) \oplus \bigoplus_{1\le i<j\le n} B(A_i,A_j) \\
\rho\downarrow & & \downarrow \rho \oplus \binom{1}{\sigma} \\
B(\bigoplus_{i=1}^n A_i, \bigoplus_{i=1}^n A_i) & = & \bigoplus_{i=1}^n B(A_i,A_i) \oplus \bigoplus_{1\le i<j\le n} (B(A_i,A_j)\oplus B(A_j,A_i))
\end{array}$$

□

Appendix B. C_2-Mackey functors

The tensor product and internal hom of form categories is based on the tensor product and internal hom of C_2-Mackey functors which we will review in this section. The material here can be found in much greater generality in [Bou97]. We denote by $\otimes$ the tensor product of abelian groups.

Definition B.1 (C_2-Mackey functor)**.** Let $C_2 = \{1,\sigma\}$ be the cyclic group of order 2 with generator σ. A C_2-*Mackey functor* is a diagram $M = (M(e), M(C_2), \tau, \rho)$ of C_2-abelian groups and C_2-equivariant linear maps

$$M(e) \xrightarrow{\tau} M(C_2) \xrightarrow{\rho} M(e)$$

where the action on $M(C_2)$ is trivial and $\rho\circ\tau = 1+\sigma$. It helps to think of $M(G)$ as the "G-fixed points" of M for $G \subset C_2$ a subgroup. The maps τ and ρ are called *transfer* and *restriction*.

A homomorphism $f : M \to N$ of Mackey functors is a pair $f = (f^e, f^{C_2})$ of C_2-equivariant maps $f^e : M(e) \to N(e)$ and $f^{C_2} : M(C_2) \to N(C_2)$ commuting with transfer and restriction. The category of C_2-Mackey functors is denoted by $C_2\,\mathrm{Mac}$ or simply Mac.

B.1. **The internal Homomorphism Mackey functor.** Let M, N be C_2-Mackey functors. The internal homomorphism Mackey functor $\underline{\mathrm{Mac}}(M,N)$ is

$$\underline{\mathrm{Mac}}(M,N)(e) = \mathrm{Hom}(M(e),N(e)), \text{ with action } f\mapsto \bar f = \sigma\circ f\circ\sigma,$$

$$\underline{\mathrm{Mac}}(M,N)(C_2) = \mathrm{Mac}(M,N)$$

and structure maps

$$\tau : \underline{\mathrm{Mac}}(M,N)(e) \to \underline{\mathrm{Mac}}(M,N)(C_2) : f \mapsto (f+\bar f, \tau\circ f\circ\rho)$$

$$\rho : \underline{\mathrm{Mac}}(M,N)(C_2) \to \underline{\mathrm{Mac}}(M,N)(e) : (f^e, f^{C_2}) \mapsto f^e.$$

B.2. **The tensor product of Mackey functors.** The tensor product Mackey functor $M\hat\otimes N$ of two C_2-Mackey functors M and N has

$$(M\hat\otimes N)(e) = M(e)\otimes N(e), \text{ with action } \sigma\otimes\sigma,$$

and $(M\hat\otimes N)(C_2)$ is the quotient of

$$M(C_2)\otimes N(C_2) \oplus (M(e)\otimes N(e))/(1-\sigma\otimes\sigma)$$

by the two relations

$$\rho(\xi)\otimes y = \xi\otimes\tau(y),\qquad x\otimes\rho(\zeta)=\tau(x)\otimes\zeta$$

for $x\in M(e)$, $y\in N(e)$, $\xi\in M(C_2)$ and $\zeta\in N(C_2)$. Transfer and restriction are defined by

$$(M\hat{\otimes}N)(e)\xrightarrow{\tau}(M\hat{\otimes}N)(C_2): x\otimes y\mapsto [x\otimes y]$$

$$(M\hat{\otimes}N)(C_2)\xrightarrow{\rho}(M\hat{\otimes}N)(e): \xi\otimes\zeta+[x\otimes y]\mapsto \rho(\xi)\otimes\rho(\zeta)+x\otimes y+\sigma(x)\otimes\sigma(y).$$

B.3. **Adjointness of $\hat{\otimes}$ and $\underline{\mathrm{Mac}}$.** As usual, there is a natural isomorphism

$$\underline{\mathrm{Mac}}(M\hat{\otimes}N,P)\cong\underline{\mathrm{Mac}}(M,\underline{\mathrm{Mac}}(N,P))$$

given by unit and counit of an adjunction defined as follows. Evaluation is the map

$$e=(e,e^{C_2}):\underline{\mathrm{Mac}}(M,N)\otimes M\to N$$

defined by the usual evaluation map at e, and at C_2 it is

$$(\underline{\mathrm{Mac}}(M,N)\hat{\otimes}M)(C_2)\xrightarrow{e^{C_2}}N(C_2):(f^e,f^{C_2})\otimes\xi+[g\otimes x]\mapsto f^{C_2}(\xi)+\tau(g(x)).$$

The coevaluation map

$$\nabla: M\to\underline{\mathrm{Mac}}(N,M\hat{\otimes}N)$$

is the usual coevaluation map at e

$$M(e)\to\underline{\mathrm{Mac}}(N,M\hat{\otimes}N)=\mathrm{Hom}(N(e),M(e)\otimes N(e)):x\mapsto(y\mapsto x\otimes y)$$

and at C_2 it is the map

$$\nabla: M(C_2)\to\underline{\mathrm{Mac}}(N,M\hat{\otimes}N)(C_2)=\mathrm{Mac}(N,M\hat{\otimes}N):\xi\mapsto(\nabla^e_\xi,\nabla^{C_2}_\xi)$$

where

$$\nabla^e_\xi: N(e)\to M(e)\otimes N(e): y\mapsto\rho(\xi)\otimes y$$

and

$$\nabla^{C_2}_\xi: N(C_2)\to(M\otimes N)(C_2):\zeta\mapsto\xi\otimes\zeta.$$

B.4. **The unit of the tensor product.** The unit of the tensor product is the Burnside Mackey functor $\underline{\mathbb{Z}}$ where $\underline{\mathbb{Z}}(e)=\mathbb{Z}$ with trivial action, $\underline{\mathbb{Z}}(C_2)=\mathbb{Z}[C_2]=\mathbb{Z}\oplus\mathbb{Z}\sigma$, $\tau(m)=m+m\sigma$, $\rho(m+n\sigma)=m+n$. The unit isomorphism $u:\underline{\mathbb{Z}}\hat{\otimes}M\to M$ is the following isomorphism of Mackey functors

$$\begin{tikzcd}
(\underline{\mathbb{Z}}\hat{\otimes}M)(e) \arrow[r, "\cong"] \arrow[d, shift left, "\tau"] & M(e): \arrow[d, shift left, "\tau"] & m\otimes x\mapsto m\cdot x \\
(\underline{\mathbb{Z}}\hat{\otimes}M)(C_2) \arrow[r, "\cong"] \arrow[u, shift left, "\rho"] & M(C_2): \arrow[u, shift left, "\rho"] & (m+n\sigma)\otimes\xi+[r\otimes y] \\
 & & \mapsto\ (m-n)\cdot\xi+n\cdot\tau\rho(\xi)+r\cdot\tau(y)
\end{tikzcd}$$

Associativity $a:(M\hat{\otimes}N)\hat{\otimes}P\cong M\hat{\otimes}(N\hat{\otimes}P)$ and commutativity isomorphisms $c: M\hat{\otimes}N\to N\hat{\otimes}M$ are the maps induced by associativity and commutativity of the usual tensor product of abelian groups. In consequence, we have the following.

Proposition B.2. *The data $(\hat{\otimes},\underline{\mathrm{Mac}},\nabla,e,a,c,\underline{\mathbb{Z}},u)$ defined above make the category $C_2\,\mathrm{Mac}$ of C_2-Mackey functors into a closed symmetric monoidal category.*

Exercise B.3. Use Proposition B.2 to make the category of small form categories into a closed symmetric monoidal category with tensor product and internal homomorphism objects as in Definitions 2.34 and 2.36.

Appendix C. Tensor product of unital abelian monoids

Let M, P be unital abelian monoids. A map of sets $f : M \to P$ is called *linear* if $f(0) = 0$ and $f(a+b) = f(a) + f(b)$ for all $a, b \in M$. Let M, N, P be unital abelian monoids. A map of sets $M \times N \to P : (x, y) \mapsto \langle x, y\rangle$ is called *bilinear* if the maps $M \to P : m \mapsto \langle m, y\rangle$ and $N \mapsto P : n \mapsto \langle x, n\rangle$ are linear for all $x \in M$ and $y \in N$.

The *tensor product* $M \otimes N$ of unital abelian monoids M and N is the quotient unital abelian monoid of the free unital abelian monoid on symbols $m \otimes n$ with $m \in M$ and $n \in N$ modulo the relations $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$ and $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, and $0 \otimes n = m \otimes 0 = 0$. It satisfies the usual universal property where restriction along $M \times N \to M \otimes N : (m, n) \mapsto m \otimes n$ defines a bijection between linear maps from $M \otimes N$ with bilinear maps from $M \times N$.

Recall that an abelian group is a unital abelian monoid in which every element has an inverse. If M and N are abelian groups, then their tensor product $M \otimes N$ as abelian monoids is the usual tensor product of abelian groups with inverses give by $-(m \otimes n) = (-m) \otimes n = m \otimes (-n)$.

Appendix D. Homology of $S(m)$-modules

We recall from [Sch17a] the notion of $S(m)$-sequence and $S(m)$-algebra.

Definition D.1. Let R be a ring with group of units R^*, and let $m \geq 0$ be an integer. An $S(m)$-*sequence* in R is a sequence $(u_1, ..., u_m)$ of m central elements $u_1, ..., u_m$ in R all of whose non-empty partial sums are units, that is, for every index set $\emptyset \neq J \subset \{1, ..., m\}$, we require

$$u_J = \sum_{j \in J} u_j \quad \in R^*.$$

An $S(m)$-algebra is a ring together with a choice of an $S(m)$-sequence.

A ring has *many units* if it has an $S(m)$-algebra structure for all integers $m \geq 0$. For instance, $\mathbb{Z}$ is an $S(1)$-algebra but not an $S(2)$-algebra. Any infinite field and any local ring with infinite residue field has many units.

$S(m)$-algebras in the sense above are algebras over the commutative ring

$$S(m) = \mathbb{Z}[X_1, ..., X_m][\Sigma^{-1}]$$

obtained by localising the polynomial ring $\mathbb{Z}[X_1, ..., X_m]$ in the m variables $X_1, ..., X_m$ at the set of all non-empty partial sums of the variables

$$\Sigma = \{X_J |\ \emptyset \neq J \subset \{1, ..., m\}\}, \quad \text{where } X_J = \sum_{j \in J} X_j.$$

Indeed, an $S(m)$-sequence $(u_1, ..., u_m)$ in R determines a ring homomorphism $S(m) \to R : X_i \mapsto u_i$ with image in the centre of R.

For a ring R, we denote by $\mathbb{Z}[R^*]$ the integral group ring of the group of units R^* in R. It has $\mathbb{Z}$-basis the elements $\langle a\rangle$ corresponding to the units $a \in R^*$. Suppose R has an $S(m)$-sequence $u = (u_1, ..., u_m)$. For an integer $m \geq 1$, write $[1, m]$ for the set $\{1, ..., m\}$ of integers between 1 and m. As in [Sch17a], we will denote by $s(u)$ the following element in $\mathbb{Z}[R^*]$

$$s(u) = -\sum_{\emptyset \neq J \subset [1,m]} (-1)^{|J|} \langle u_J \rangle \quad \in \mathbb{Z}[R^*].$$

Note that the augmentation homomorphism $\varepsilon : \mathbb{Z}[R^*] \to \mathbb{Z} : \langle a \rangle \mapsto 1$ sends $s(u)$ to 1; see [Sch17a, §2]. For the universal $S(m)$-sequence $(X_1, ..., X_m)$ in $S(m)$, I will write s_m instead of $s(X_1, ..., X_m)$

Lemma D.2. *Let R be a commutative ring with $S(m)$-sequence $u = (u_1, ..., u_m)$. Let A, N be R-modules. Consider A and N as $\mathbb{Z}[R^*]$-modules with multiplication $\langle r \rangle (a) = ra$ and $\langle r \rangle (y) = r^2 y$ for $r \in R^*$, $a \in A$ and $y \in N$. By functoriality of exterior products, this makes $\Lambda^p_{\mathbb{Z}} A$ and $\Lambda^q_{\mathbb{Z}} N$ into R^*-modules. Consider the tensor product $\Lambda^p_{\mathbb{Z}} A \otimes \Lambda^q_{\mathbb{Z}} N$ equipped with the diagonal R^*-action. Then for all $1 \leq p + 2q < m$ we have*

$$s(u) \cdot (\Lambda^p_{\mathbb{Z}}(A) \otimes \Lambda^q_{\mathbb{Z}}(N)) = 0.$$

Proof. By definition of the functor $\Lambda^n_{\mathbb{Z}}$, we have a surjection of R^*-modules

$$\bigotimes^p A \otimes \bigotimes^q N \twoheadrightarrow \Lambda^p(A) \otimes \Lambda^q(N). \tag{D.1}$$

Thus, it suffices to show that $s(u)$ annihilates the source of that map. Consider each copy of A in the source of (D.1) as an R-module with its given scalar product and each copy of N as an $R^{\otimes 2}$-module with scalar multiplication $(a \otimes b) \cdot y = aby$ for $a, b \in R$ and $y \in N$. Together this defines a natural $R^{\otimes(p+2q)} = \bigotimes^p R \otimes \bigotimes^q (R^{\otimes 2})$-module structure on the source of (D.1). The $\mathbb{Z}[R^*]$-module structure on the source of (D.1) is the restriction of the $R^{\otimes(p+2q)}$-module structure along the ring homomorphism $\mathbb{Z}[R^*] \to R^{\otimes(p+2q)} : \langle r \rangle \mapsto r \otimes \cdots \otimes r$. Since that ring homomorphism sends $s(u)$ to zero [Sch17a, Lemma 2.2] provided $1 \leq p + 2q < m$, we are done. □

Lemma D.3. *Keep notation and hypothesis of Lemma D.2. If moreover A is torsion free as abelian group then for all $1 \leq p + 2q < m$ we have*

$$s(u)^{-1} (H_p(A) \otimes H_q(N)) = 0.$$

Proof. Assume first that N is torsion free. Then $s(u)$ annihilates

$$H_p(A) \otimes H_q(N) \cong \Lambda^p_{\mathbb{Z}} A \otimes \Lambda^q_{\mathbb{Z}} N$$

by Lemma D.2

For the general case, choose a surjective weak equivalence of simplicial $S(m)$-modules $N_* \to N$ with N_i a projective $S(m)$-module for all $i \in \mathbb{N}$. For instance, the simplicial $S(m)$-module corresponding to an $S(m)$-projective resolution of N under the Dold-Kan correspondence will do. Each N_i is a torsion free abelian group since $S(m)$ is. The classifying space functor induces an $S(m)^*$-equivariant weak equivalence of simplicial sets $BN_* \to BN$ where $S(m)^*$ acts on each N_i as the square of the natural $S(m)$ scalar multiplication. Tensoring the spectral sequence of the simplicial space $n \mapsto BN_n$,

$$E^1_{s,t} = H_t(BN_s) \Rightarrow H_{s+t}(BN_*) = H_{s+t}(BN) = H_{s+t}(N),$$

with the flat $\mathbb{Z}$-module $H_p(A)$ yields the spectral sequence of $S(m)^*$-modules

$$H_p(A) \otimes E^1_{s,t} = H_p(A) \otimes H_t(BN_s) \Rightarrow H_p(A) \otimes H_{s+t}(N)$$

Localising at $s_m \in \mathbb{Z}[S(m)^*]$, this yields a spectral sequence with trivial $E^1_{s,t}$-term for $1 \leq p + 2t < m$. This implies the claim. $\square$

The following is a version of [Sch17a, Proposition 2.4].

Proposition D.4. *Let R be a commutative ring with $S(m)$-sequence $u = (u_1, ..., u_m)$. Let*

$$N \to G \to A$$

be a central extension of groups. Assume that the group of units R^* acts on the exact sequence, that the groups A and N are the underlying abelian groups $(A, +, 0)$ and $(N, +, 0)$ of R-modules $(A, +, 0, \cdot)$ and $(N, +, 0, \cdot)$ such that the R^*-actions on A and N in the exact sequence are the scalar multiplication and its square $R^* \times A \to A : (t, a) \mapsto t \cdot a$ and $R^* \times N \to N : (t, y) \mapsto t^2 \cdot y$. Then for all $1 \leq n < m/2$ we have

$$s_m^{-1} H_n(G) = 0.$$

Proof. Assume first that A is torsion free as abelian group. Then its integral homology groups $H_*(A)$ are torsion free and the natural map $H_p(A) \otimes F \to H_p(A, F)$ is an isomorphism for any abelian group F, by the Universal Coefficient Theorem. Since the extension is central, the group A acts trivially on $H_*(N)$ and the Hochschild-Serre spectral sequence of the group extension has the form

$$E^2_{p,q} = H_p(A, H_q(N)) \cong H_p(A) \otimes H_q(N) \Rightarrow H_{p+q}(G).$$

Localising the spectral sequence at s_m yields a spectral sequence with E^2-term $s_m^{-1} E^2_{p,q} = 0$ for $1 \leq p + 2q < m$, by Lemma D.3. This implies the claim in case A is torsion free.

For general A, choose a surjective weak equivalence $A_* \to A$ of simplicial $S(m)$-modules with A_n a projective $S(m)$-module for all n. For instance, the simplicial $S(m)$-module corresponding to an $S(m)$-projective resolution of A under the Dold-Kan correspondence will do. Then each A_n is flat as abelian group since $S(m)$ is. Let $G_n = G \times_A A_n$. Then $G_* \to G$ is a surjection of simplicial groups with contractible kernel. In particular, the map on classifying spaces $B|s \mapsto G_s| = |s \mapsto BG_s| \to BG$ is an $S(m)^*$-equivariant weak equivalence. For each n we have an $S(m)^*$-equivariant central extension $N \to G_n \to A_n$ with torsion free base. By the torsion free case treated above, we have $s_m^{-1} H_r(BG_n) = 0$ for $1 \leq r < m/2$ and for all n. Therefore, the spectral sequence of the simplicial space $s \mapsto BG_s$,

$$E^2_{p,q} = \pi_p |s \mapsto H_q(BG_s)| \Rightarrow H_{p+q}(BG_*) = H_{p+q}(BG),$$

localised at s_m has trivial $E^2_{p,q}$-term for $1 \leq p + q < m/2$. The claim follows. $\square$

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