Limits of the Stokes and Navier–Stokes equations in a punctured periodic domain

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Abstract

We treat three problems on a two-dimensional 'punctured periodic domain': we take $\Omega_r = (-L, L)^2 \setminus rK$, where $r > 0$ and K is the closure of an open connected set that is star-shaped with respect to 0 and has a C^1 boundary. We impose periodic boundary conditions on the boundary of $\Omega = (-L, L)^2$, and Dirichlet boundary conditions on $\partial(rK)$. In this setting we consider the Poisson equation, the Stokes equations, and the time-dependent Navier–Stokes equations, all with a fixed forcing function f, and examine the behaviour of solutions as $r \to 0$. In all three cases we show convergence of the solutions to those of the limiting problem, i.e. the problem posed on all of Ω with periodic

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boundary conditions.

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1. Introduction

The study of fluid flow around an obstacle is a challenging and interesting problem in fluid mechanics, and has been the subject of much experimental and numerical investigation (see, among others, [1, 4, 10, 11, 25, 29, 33, 34]).

The mathematical analysis of the influence of an obstacle on the behaviour of the flow when the size of the obstacle is small when compared to that of the reference spatial scale has recently received increased attention. The case of a single obstacle in a two-dimensional ideal flow was analysed by Iftimie, Lopes Filho, & Nussenzveig Lopes [13]; then Iftimie et al. [14] and Iftimie & Kelliher [12] considered the viscous case, Lopes Filho [21] treated bounded domains with several holes, Lacave [16, 17, 18] considered obstacles that shrink to a curve. For problems in exterior domains (i.e. extending to infinity) the flow is usually assumed to vanish at infinity, although the case of flows constant at infinity has been considered by Lopes Filho, Nguyen, & Nussenzveig Lopes [22]. A related 'small body' problem was considered by Robinson [27], who treated a simplified model of combustion in which physical particles were replaced by diffuse but compact regions of influence in the flow. Very recently, Lu [23] treated the Dirichlet problem in the three-dimensional unit ball with a shrinking hole. Uniform estimates, as the size of the hole goes to zero, in $W^{1,p}$ for any $3/2 < p < 3$ and counterexamples that the uniform $W^{1,p}$-estimates do not hold when $1 < p < 3/2$ or $3 < p < +\infty$ are provided. These estimates were extended by the same author [24] to the Stokes problem in a n-dimensional bounded domain, showing uniform estimates for any $n' < p < n$ and counterexamples for $1 < p < n'$ or $n < p < +\infty$. Notice that last two papers do not consider the two-dimensional case for $p = 2$.

Here we are interested in the vanishing obstacle problem in a two-dimensional periodic domain with a particularly simple geometry. More precisely, we are concerned with periodic flows on the punctured domain

$$\Omega_r = (-L, L)^2 \setminus K_r, \ L > 0,$$

where $K_r = rK$ with $r > 0$ and K is the closure of an open set that is star-shaped with respect to 0 and has a C^1 boundary, and we study the behaviour of the solutions of various models when r tends to zero. Throughout the paper we refer to the excised compact set K_r as the 'obstacle' in keeping with the ultimate application to problems of fluid flow. An illustration of the domain is provided in Figure 1.

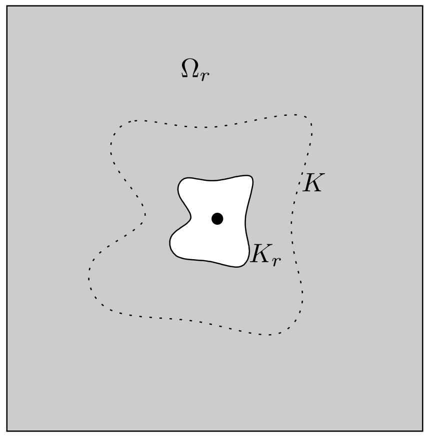

Figure 1: Domain Ω_r (greyed).

Our primary motivation for this geometry was the moving 'tracer particle' problem considered in two dimensions by Dashti & Robinson [3] and in three dimensions by Silvestre & Takahashi [28]: given a solid disc/sphere of radius r moving in the fluid, does the motion of the particle follow that of the fluid in the limit $r \to 0$? Our aim was to include rotation of the tracer in the 2D case, which was excluded in [3]. However, in the course of the analysis that follows we observed the failure of certain uniform elliptic regularity estimates that are required in both these papers (see Section 2.1). The two-dimensional case has now been resolved by Lacave & Takahashi [19] for small initial data and when the density of the solid is independent of r (using maximal regularity estimates for the Stokes equation). Moreover, assuming that the density of the rigid body goes to infinity, He & Iftimie [7, 8] were able to tackle the problem in both dimensions (using a truncation procedure.) The general case remains open. (We choose a particularly simple geometry and a somewhat simpler problem in which these uniform estimates fail, but there is no reason to believe that this has any significant effect of the nature of this phenomenon.)

In order to clarify the setting and provide some background to these uniform elliptic estimates, as well as allowing us to outline the main ideas that will then be applied in the more complicated Stokes and time-dependent Navier–Stokes problems (which have the added component of incompressibility) we first consider the Poisson equation as a model problem. Thus our initial aim (in Section 2) will be to determine the asymptotic behaviour of the solution of the following problem when $r \to 0$:

$$-\Delta u_r = f \ \text{ in } \Omega_r, \qquad u_r \text{ periodic}, \qquad u_r = 0 \text{ on } \partial K_r. \tag{1.1}$$

While this problem has a solution for any $f \in L^2(\Omega_r)$, the limiting problem,

$$-\Delta u = f \ \text{ in } \Omega = (-L, L)^2, \qquad u \text{ periodic},$$

only has a solution when

$$\int_\Omega f = 0. \tag{1.2}$$

We will show that when (1.2) holds then the solutions of (1.1) are uniformly bounded in r in the sense that

$$\int_{\Omega_r} |\nabla u_r|^2 + \int_{\Omega_r} \left| u_r - \fint_\Omega u_r \right|^2$$

is uniformly bounded, where $\fint_\Omega u = |\Omega|^{-1} \int_\Omega u$ denotes the average of u over Ω (note that this is the whole domain and not just Ω_r). This is enough to show that

$$u_r - \fint_\Omega u_r \to u$$

in $H^1(\Omega)$ and that u satisfies the limiting equation. If (1.2) does not hold then the limiting problem has no solution, and in this case it follows that $\|u_r\|_{H^1}$ is unbounded as $r \to 0$.

We remark here, and will return to this later, that we have been unable to obtain a uniform bound on $\fint_\Omega u_r$, since the constant in the Poincaré inequality available on Ω_r degrades as $r \to 0$ (see Lemma 2.2).

In Section 3 we obtain similar results for the Stokes problem

$$\begin{cases} -\Delta \mathbf{u}_r + \nabla p_r = \mathbf{f} \ \text{ in } \Omega_r, \\ \operatorname{div} \mathbf{u}_r = 0, \\ \mathbf{u}_r \text{ periodic}, \\ \mathbf{u}_r = 0 \text{ on } \partial K_r. \end{cases}$$

The main change from the case of the pure Laplacian is that we now have to deal with divergence-free vector-valued functions. The key technical result that allows us to do this is a method for approximating divergence-free periodic functions defined on the whole of Ω by a sequence of divergence-free functions that satisfy the zero boundary condition on ∂K_r (Lemma 3.4). Once again, we require that $\int_\Omega \mathbf{f} = 0$. As before, we can find uniform estimates sufficient to show that $\mathbf{u}_r - \fint_\Omega \mathbf{u}_r$ converges to a solution of the limiting problem, but we are unable to bound the average of $\mathbf{u}_r$ over Ω.

It would seem that the next natural step would be to consider the stationary Navier–Stokes equations in Ω_r,

$$-\Delta \mathbf{u}_r + (\mathbf{u}_r \cdot \nabla)\mathbf{u}_r + \nabla p_r = \mathbf{f}, \qquad \nabla \cdot \mathbf{u}_r = 0. \tag{1.3}$$

However, while in the linear problems considered so far bounds on $\mathbf{u}_r - \fint_\Omega \mathbf{u}_r$ were sufficient to pass to the limit, this is not the case here. Informally, if we set $\langle \mathbf{u}_r \rangle = \fint_\Omega \mathbf{u}_r$ and consider the equation for $\tilde{\mathbf{u}}_r = \mathbf{u}_r - \langle \mathbf{u}_r \rangle$ then we obtain

$$-\Delta \tilde{\mathbf{u}}_r + (\tilde{\mathbf{u}}_r \cdot \nabla)\tilde{\mathbf{u}}_r + (\langle \mathbf{u}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r + \nabla p_r = \mathbf{f},$$

which contains the additional term $(\langle \mathbf{u}_r \rangle \cdot \nabla)\tilde{\mathbf{u}}_r$. A uniform bound on $\langle \mathbf{u}_r \rangle$ would enable us to pass to the limit in this term, but we do not currently have such a bound.

An additional factor that makes this problem different in character from the others we consider here is that there is no known general uniqueness result for solutions of (1.3), even on the entire periodic domain. As such, it is perhaps more natural to consider a perturbation problem (given a solution of the equation on Ω, investigate the existence of nearby solutions for r small) than as a limiting problem; or to treat a restricted setting in which uniqueness results are available (when $\mathbf{f}$ is small in an appropriate sense). For more discussion of this stationary problem we refer to the classical work of Ladyzhenskaya [20] and Temam [31, 32].

We therefore instead turn in Section 4 to the time-dependent Navier–Stokes problem, which turns out to be more straightforward and for which we do not require the use of the Poincaré inequality, since a bound on the $\mathbb{L}^2$ norm follows immediately from the energy inequality (here and in the following, given any space X of scalar functions, we set $\mathbb{X} = X^2$). In this case we obtain convergence of $\mathbf{u}_r$ to the solution $\mathbf{u}$ of the periodic Navier–Stokes equations,

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0,$$

where the convergence is strong in $L^2(0, T; \mathbb{L}^2(\Omega))$ and weak in $L^2(0, T; \mathbb{H}^1(\Omega))$. We note that this falls short of $\mathbb{L}^\infty$ convergence of the velocity field; this is unsurprising since uniform convergence coupled with the fact that $\mathbf{u}_r = 0$ on ∂K_r would imply that the limiting flow was stationary at the origin.

Let us conclude this introduction by noticing that our equations are set and analysed in dimension 2 for ease of presentation. All the results in this paper are valid in dimension 3, modulo small modifications due to the absence of uniqueness result for Navier–Stokes in 3D (see Remark 4.2).

2. Poisson equation

In this section we discuss the asymptotic behaviour of weak solutions for the Poisson problem

$$\begin{cases} -\Delta u_r = f \ \text{ in } \Omega_r, \\ u_r \text{ periodic}, \\ u_r = 0 \text{ on } \partial K_r. \end{cases}$$

Let us introduce some notation. Set $\Omega_0 = (-L, L)^2 = \Omega$ and $\Omega_r = (-L, L)^2 \setminus K_r$, where $K_r = rK$ with $r > 0$ and K is the closure of an open set containing 0 and with a C^1 boundary. The disc of centre 0 and radius ζ is denoted by D_ζ. We use the subscript 'per' on a space X to denote the restriction to Ω (or to Ω_r) of a function that is $2L$-periodic on $\mathbb{R}^2$ in both directions and is in $X_{\text{loc}}(\mathbb{R}^2)$. In this way we define the function spaces $H^1_{\text{per}}(\Omega)$ and, for $r > 0$,

$$H^1_{\text{per}}(\Omega_r) = \text{ the closure of } C^1_{\text{per}}(\overline{\Omega}_r) \text{ in } H^1(\Omega_r)$$

and

$$V_{0,r} = \{v \in H^1_{\text{per}}(\Omega_r) : \ v = 0 \text{ on } \partial K_r\}.$$

Note that any function in $V_{0,r}$ can be extended by zero inside K_r to give a function in $H^1_{\text{per}}(\Omega)$; this observation is fundamental to our analysis, and we will implicitly perform such extension when comparing different $V_{0,r}$ spaces.

The vanishing obstacle problem for the Poisson equation

$$-\Delta u_r = f \ \text{ in } \Omega_r, \qquad u_r \in V_{0,r}, \tag{2.1}$$

consists in determining the asymptotic behaviour of the solution u_r when r tends to 0.

The precise statement of our first convergence result is as follows.

Theorem 2.1. *Let $f \in L^2(\Omega)$. For every $r > 0$ there exists a unique solution $u_r \in V_{0,r}$ of the problem*

$$\int_{\Omega_r} \nabla u_r \cdot \nabla v = \int_{\Omega_r} fv \qquad \text{for all } v \in V_{0,r}. \tag{2.2}$$

Moreover

a) if $\int_\Omega f = 0$ then as $r \to 0$

$$u_r - \frac{1}{|\Omega|}\int_\Omega u_r \to u_0 \qquad \text{and} \qquad \nabla u_r \to \nabla u_0,$$

where the limits are taken in $L^2(\Omega)$ and $u_0 \in H^1_{\mathrm{per}}(\Omega)$ is the unique solution of the problem

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega fv \qquad \text{for all } v \in H^1_{\mathrm{per}}(\Omega) \tag{2.3}$$

that satisfies $\int_\Omega u_0 = 0$.

b) If $\int_\Omega f \neq 0$ then $\|\nabla u_r\|_{L^2}$ is unbounded as $r \to 0$.

A few comments are in order.

Note that one can use $v = 1$ as a test function in (2.3), from which it follows immediately that there can be no solution of the limiting problem unless

$$\int_\Omega f = 0.$$

Observe that we do not have convergence of u_r itself in $L^2(\Omega)$. The main reason for this is that the constant in the Poincaré inequality for the punctured domain Ω_r tends to degrade as $r \to 0$. We first recall the classical Poincaré–Wirtinger inequality: there exists a constant $C > 0$ such that for any $v \in H^1_{\mathrm{per}}(\Omega)$

$$\left\| v - \fint_\Omega v \right\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}. \tag{2.4}$$

Notice that inequality (2.4) is still valid for functions in $v \in V_{0,r}$, and in particular the constant does not depend on r. However, without subtraction of the average we have only the following estimate.

Lemma 2.2. *Let $\zeta > 0$ be such that $D_\zeta \subset K$. Take $r > 0$ such that $r\zeta < (2 - \sqrt{2})L$. Then for all $v \in V_{0,r}$*

$$\|v\|_{L^2(\Omega_r)} \leq c|\log(r\zeta)|\|\nabla v\|_{L^2(\Omega_r)}. \tag{2.5}$$

Proof. We first notice that we can assume that $K_r = D_{r\zeta}$. Indeed, assume that (2.5) holds for $V_{0,r}$ and Ω_r defined using $D_{r\zeta}$ instead of K_r; then $v \in V_{0,r}$ can be extended by 0 to $K_r \backslash D_{r\zeta}$, and this extension is periodic and vanishes on $\partial D_{r\zeta}$. Applying (2.5) to this extension shows that this estimate is also satisfied by v itself. From hereon in this proof, we therefore assume that $K_r = D_{\tilde{r}}$ with $\tilde{r} = r\zeta$.

We assume that $v \in C^1_{\mathrm{per}}(\overline{\Omega}_r)$ with $v = 0$ on $\partial D_{\tilde{r}}$, with the result for $v \in V_{0,r}$ obtained by a density argument. We extend v periodically outside Ω_r, the assumption that $\tilde{r} < (2 - \sqrt{2})L$ meaning that any x with $|x| \leq \sqrt{2}L$ in the extended domain does not lie within one of the additional 'holes', see Figure 1.

At $x = \rho\hat{x}$ (where $\hat{x} = x/|x|$), we can write

$$|v(x)| = |v(\rho\hat{x}) - v(\tilde{r}\hat{x})| = \left| \int_{\tilde{r}}^{\rho} \frac{\mathrm{d}}{\mathrm{d}s} v(s\hat{x}) \,\mathrm{d}s \right| \leq \int_{\tilde{r}}^{\rho} |\nabla v(s\hat{x})| \,\mathrm{d}s.$$

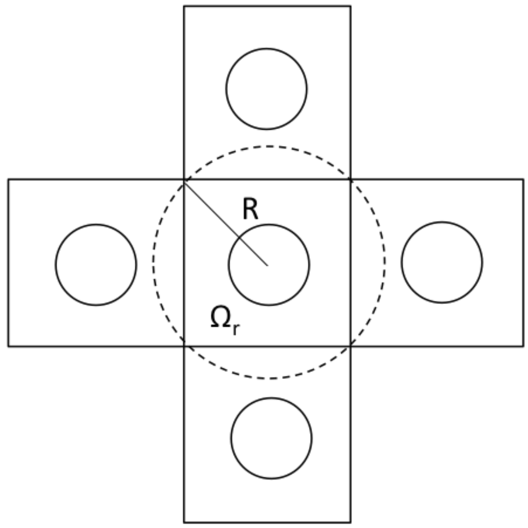

Figure 2: Periodic extension of the domain Ω_r used in the proof of Lemma 2.2

Then, since $D_{\sqrt{2}L} \supset \Omega_r$, setting $R = \sqrt{2}L$ we have

$$
\begin{aligned}
\int_{\Omega_r} |v(x)|^2 &\le \int_0^{2\pi} \int_{\tilde r}^{R} \rho |v(\rho \hat x)|^2 \,\mathrm{d}\rho\,\mathrm{d}\theta \\
&\le \int_0^{2\pi} \int_{\tilde r}^{R} \rho \left(\int_{\tilde r}^{\rho} |\nabla v(s\hat x)| \,\mathrm{d}s \right)^2 \mathrm{d}\rho\,\mathrm{d}\theta \\
&\le \int_0^{2\pi} \int_{\tilde r}^{R} \rho \left(\int_{\tilde r}^{\rho} s^{-1} \,\mathrm{d}s \right) \left(\int_{r}^{\rho} s |\nabla v(s\hat x)|^2 \,\mathrm{d}s \right) \mathrm{d}\rho\,\mathrm{d}\theta \\
&\le \int_0^{2\pi} \int_{\tilde r}^{R} \rho \log(\rho/\tilde r) \left(\int_{r}^{\rho} s |\nabla v(s\hat x)|^2 \,\mathrm{d}s \right) \mathrm{d}\rho\,\mathrm{d}\theta \\
&\le \left(\int_{\tilde r}^{R} \rho \log(\rho/\tilde r) \,\mathrm{d}\rho \right) \left(\int_{D_{\sqrt{2}L}} |\nabla v|^2 \,\mathrm{d}x \right), \\
&\le c |\log \tilde r| \|\nabla v\|^2_{L^2(\Omega_r)},
\end{aligned}
$$

using the fact that $\int_{D_R} |\nabla v|^2 \le 2 \int_{\Omega_r} |\nabla v|^2$ since we have extended v periodically outside Ω_r. □

We note that the fact that the constant in Lemma 2.2 is not independent of r is not merely an artefact of our method of proof: while it may be possible to improve the dependence on r, one cannot remove it. Indeed, consider the case $K_r = D_r$ and the family of functions u_r defined on Ω_r by

$$u_r(x) = \log(1 + \log(\rho/r))$$

where ρ is distance of x from the origin. This defines a function in $V_{0,r}$, since its values on the boundary of Ω agree on opposite faces. Now, certainly

$$\begin{aligned}
\|u_r\|^2_{L^2(\Omega_r)} \geq \int_{r\leq|x|\leq L} |u_r(x)|^2 \, \mathrm{d}x = 2\pi \int_r^L \rho(\log(1+\log(\rho/r)))^2 \, \mathrm{d}\rho \\
= 2\pi r^2 \int_1^{L/r} s(\log(1+\log s))^2 \, \mathrm{d}s \\
\geq 2\pi r^2 \int_{L/2r}^{L/r} s(\log(1+\log s))^2 \, \mathrm{d}s \\
\geq 2\pi r^2 (L/2r)^2 \log(1+\log(L/2r))^2 \\
= \frac{\pi L^2}{2} \log(1+\log(L/2r))^2,
\end{aligned}$$

which is unbounded as $r \to 0$. However,

$$\partial_\rho u_r = \frac{1}{1+\log(\rho/r)} \frac{1}{\rho}$$

and so

$$\begin{aligned}
\|\nabla u_r\|^2_{L^2(\Omega_r)} \leq \int_{r\leq|x|\leq\sqrt{2}L} |\partial_\rho u_r|^2 \, \mathrm{d}x = 2\pi \int_r^{\sqrt{2}L} \frac{1}{(1+\log(\rho/r))^2} \frac{1}{\rho} \, \mathrm{d}\rho \\
\leq 2\pi \int_1^\infty \frac{1}{s(1+\log s)^2} \, \mathrm{d}s < \infty.
\end{aligned}$$

We now state a preliminary lemma on approximation of functions in $H^1_{\mathrm{per}}(\Omega)$ by functions in $V_{0,r}$, which will be used to pass to the limit.

Lemma 2.3. *Given $v \in H^1_{\mathrm{per}}(\Omega)$ there exists a sequence $v_\varepsilon \in V_{0,\varepsilon}$ such that*

$$v_\varepsilon \to v \quad \textit{in} \quad H^1(\Omega) \qquad \textit{as} \qquad \varepsilon \to 0.$$

Proof. (The proof consists essentially of showing that $\{0\}$ has zero 2-capacity in $\mathbb{R}^2$, see Heinonen, Kilpeläinen, & Martio [9].)

Let $\gamma > 0$ be such that $K \subset D_\gamma$. Without loss of generality, we can assume that $0 < \varepsilon\gamma < 1$. Let

$$\phi_\varepsilon(x) = \min(1, (-\log(\varepsilon\gamma))^\nu - (-\log|x|)^\nu)\,,\ x \in \Omega \backslash D_{\varepsilon\gamma},$$

for $\nu \in (0, 1/2)$, and ϕ_ε is extended by 0 in $D_{\varepsilon\gamma}$ and by 1 outside of D_1. It is clear that $\phi_\varepsilon(x) = 1$ where

$$(-\log|x|)^\nu \leq (-\log(\varepsilon\gamma))^\nu - 1 \Leftrightarrow |x| \geq \exp\left(-((-\log(\varepsilon\gamma))^\nu - 1)^{1/\nu}\right) =: r(\varepsilon).$$

Notice that $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, using polar coordinates, we have

$$\begin{aligned}\int_\Omega |\nabla\phi_\varepsilon|^2 &= 2\pi \int_{\varepsilon\gamma}^{r(\varepsilon)} \left(\nu(-\log\rho)^{\nu-1} \times \frac{-1}{\rho}\right)^2 \rho \mathrm{d}\rho \\ &= 2\pi \int_{\varepsilon\gamma}^{r(\varepsilon)} \nu^2(-\log\rho)^{2\nu-2}\frac{\mathrm{d}\rho}{\rho} = -\frac{2\pi\nu^2}{2\nu-1}(-\log\rho)^{2\nu-1}\Big|_{\varepsilon\gamma}^{r(\varepsilon)} \to 0 \end{aligned} \tag{2.6}$$

when $\varepsilon \to 0$. Moreover $\phi_\varepsilon \to 1$ a.e. on Ω while remaining bounded by 1. Assume that $v \in H^1_{\text{per}}(\Omega) \cap L^\infty(\Omega)$. Then by dominated convergence $\phi_\varepsilon v \to v$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Moreover, $\nabla(\phi_\varepsilon v) = (\nabla\phi_\varepsilon)v + \phi_\varepsilon\nabla v$ so that, using $v \in L^\infty(\Omega)$ and (2.6) for the first term and the dominated convergence for the second term, $\nabla(\phi_\varepsilon v) \to \nabla v$ in $L^2(\Omega)$. Hence,

$$\phi_\varepsilon v \to v \text{ in } H^1_{\text{per}}(\Omega) \text{ as } \varepsilon \to 0.$$

Let $v \in H^1_{\text{per}}(\Omega)$ supposed to be extended by periodicity to $\mathbb{R}^2$. Let ϱ_n be a standard mollifier, i.e. $\varrho_n(x) = n^2\varrho(nx)$ where ϱ is a C^∞ function with support in the unit disc and such that $\varrho \geq 0$ and $\int_{\mathbb{R}^2} \varrho = 1$. Then set

$$v_n(x) = \varrho_n * v(x) = \int_{\mathbb{R}^2} \varrho_n(y)v(x-y)\mathrm{d}y.$$

It is clear that v_n is periodic in x – with the same period as v, smooth (and thus in $L^\infty(\Omega)$) and, as $n \to \infty$,

$$v_n,\, \nabla v_n \to v,\, \nabla v \text{ in } L^2(\Omega) \text{ and } L^2(\Omega)^2, \text{ respectively.}$$

This allows us to deduce the existence of the required sequence using a diagonal argument. $\square$

We remark that we have shown that $\cup_{\varepsilon>0} V_{0,\varepsilon}$ is dense in $H^1_{\text{per}}(\Omega)$ in the strong topology. We are now in a position to prove our first convergence result.

Proof (Theorem 2.1). For fixed $r > 0$, the existence and uniqueness of u_r follow from the Lax–Milgram Lemma and Lemma 2.2.

We consider the cases when $\int_\Omega f = 0$ and $\int_\Omega f \neq 0$ separately.

a) Assume that $\int_\Omega f = 0$. We first obtain an estimate for the solution u_r. By taking $v = u_r$ in (2.2) and using the Poincaré–Wirtinger inequality (2.4) one has

$$
\begin{aligned}
\|\nabla u_r\|^2_{L^2} = \int_\Omega |\nabla u_r|^2 &= \int_\Omega f u_r \\
&= \int_\Omega f \left(u_r - \fint_\Omega u_r \right) \\
&\leq \|f\|_{L^2} \left\| u_r - \fint_\Omega u_r \right\|_{L^2} \leq C \|f\|_{L^2} \|\nabla u_r\|_{L^2},
\end{aligned}
$$

from which it follows that

$$
\|\nabla u_r\|_{L^2} \leq C \|f\|_{L^2}, \tag{2.7}
$$

with a constant $C > 0$ independent on r.

Next, define

$$
\tilde{u}_r = u_r - \fint_\Omega u_r.
$$

Then from the bound (2.7) and the Poincaré–Wirtinger inequality (2.4), $\|\tilde{u}_r\|_{H^1(\Omega_r)}$ is uniformly bounded.

It follows that, up to the extraction of a subsequence, $\nabla u_r = \nabla \tilde{u}_r \rightharpoonup \nabla u_0$ and $\tilde{u}_r \to u_0$ in $L^2(\Omega)$. Note that

$$
\int_\Omega u_0 = \lim_{r\to 0} \int_\Omega \tilde{u}_r = \lim_{r\to 0} \int_\Omega \left(u_r - \fint_\Omega u_r \right) = 0. \tag{2.8}
$$

Now, we pass to the limit in the weak formulation (2.2). Fix $r_0 > 0$ and observe that, since K is star-shaped with respect to 0, one has $V_{0,r_0} \subset V_{0,r}$

for all $r < r_0$. Thus,

$$\int_\Omega \nabla u_r \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in V_{0,r_0}.$$

The weak convergence of ∇u_r to ∇u_0 in $L^2(\Omega)$ allows us to pass to the limit and obtain

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in V_{0,r_0}\,,\ \text{for all } r_0 > 0. \tag{2.9}$$

From Lemma 2.3, given $v \in H^1_{\mathrm{per}}(\Omega)$ there exists a sequence of test functions $v_\varepsilon \in V_{0,\varepsilon}$ such that $v_\varepsilon \to v$ in $H^1(\Omega)$. Thus, by (2.9),

$$\int_\Omega \nabla u_0 \cdot \nabla v_\varepsilon = \int_\Omega f v_\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$, it follows that

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega f v \qquad \text{for all } v \in H^1_{\mathrm{per}}(\Omega),$$

as claimed.

Since the limiting problem has a unique solution when one imposes the zero average condition, it follows that all convergent subsequences must have the same limit. As a consequence, the original sequence converges without the need to extract a subsequence.

It remains to show that in fact $\nabla u_r \to \nabla u_0$ in $L^2(\Omega)$ as $r \to 0$. To this end we show that $\|\nabla u_r\|^2_{L^2} \to \|\nabla u_0\|^2_{L^2}$. Since $u_r - ⨍_\Omega u_r \to u_0$ in $L^2(\Omega)$,

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_r} f u_r = \int_\Omega f u_r = \int_\Omega f \left(u_r - ⨍_\Omega u_r \right) \to \int_\Omega f u_0.$$

However, from (2.3) we have

$$\int_\Omega |\nabla u_0|^2 = \int_\Omega f u_0,$$

which implies that

$$\int_\Omega |\nabla u_r|^2 \to \int_\Omega |\nabla u_0|^2.$$

Coupled with weak convergence this norm convergence implies strong convergence of ∇u_r to ∇u_0 in $L^2(\Omega)$.

b) Assume that $\int_\Omega f \neq 0$. We note here that if $\int_\Omega f \neq 0$ and one assumes a uniform bound on $\|\nabla u_r\|_{L^2}$, then one can follow the above argument (apart from obtaining the zero average condition (2.8)) to show that there is a solution of the limiting problem. But as remarked after the statement of Theorem 2.1, there can be no such solution. It follows that in this case $\|\nabla u_r\|_{L^2}$ cannot be uniformly bounded as $r \to 0$. $\square$

Remark 2.4. We note that $\|\nabla u_r\|_{L^2}$ increases as r decreases. Indeed, if $r' < r$ then $V_{0,r} \subset V_{0,r'}$. So we can take $v = u_r$ in both formulations

$$\int_{\Omega_r} \nabla u_r \cdot \nabla v = \int_{\Omega_r} fv \quad \text{and} \quad \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla v = \int_{\Omega_{r'}} fv$$

to obtain

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_r} fu_r \quad \text{and} \quad \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla u_r = \int_{\Omega_{r'}} fu_r = \int_{\Omega_r} fu_r.$$

Thus

$$\int_{\Omega_r} |\nabla u_r|^2 = \int_{\Omega_{r'}} \nabla u_{r'} \cdot \nabla u_r$$

whence

$$\|\nabla u_r\|^2_{L^2(\Omega_r)} \leq \|\nabla u_{r'}\|_{L^2(\Omega_{r'})} \|\nabla u_r\|_{L^2(\Omega_r)},$$

i.e.

$$\|\nabla u_r\|_{L^2(\Omega_r)} \leq \|\nabla u_{r'}\|_{L^2(\Omega_{r'})}.$$

2.1. Failure of 'uniform elliptic regularity'

The Poisson equation enjoys elliptic estimates on the second derivatives. Here we describe an example that shows that, for a punctured domain (with a slightly different geometry to that in (2.1)), such estimates may not be uniform with respect to the size of the hole. We consider the annulus ('punctured disc')

$$\Omega_\varepsilon = D_2 \setminus D_\varepsilon,$$

with Dirichlet conditions on the inner and outer boundary. We solve the Poisson equation in plane polar co-ordinates for radially symmetric solutions, using $'$ for $\mathrm{d}/\mathrm{d}r$:

$$\frac{1}{r}(ru')' = f(r) \qquad u(\varepsilon) = 0, \qquad u(2) = 0.$$

We take $f = 1 - (3r/4)$ so that $\int_\Omega f \,\mathrm{d}x = \int_0^{2\pi} \int_0^2 r f(r) \,\mathrm{d}r \,\mathrm{d}\theta = 0$.

Then

$$(ru')' = r - \frac{3r^2}{4} \qquad \Rightarrow \qquad ru'(r) = \frac{r^2}{2} - \frac{r^3}{4} + C$$

and so

$$u'(r) = \frac{r}{2} - \frac{r^2}{4} + \frac{C}{r}.$$

Integrating again we obtain

$$u(r) = \frac{r^2}{4} - \frac{r^3}{12} - \frac{\varepsilon^2}{4} + \frac{\varepsilon^3}{12} + C\log(r/\varepsilon),$$

and the boundary condition at $r = 2$ implies that

$$C = \frac{1}{\log(2/\varepsilon)}\left[-\frac{1}{3} + \frac{\varepsilon^2}{4} - \frac{\varepsilon^3}{12}\right].$$

Rewrite the governing equation as

$$u'' + \frac{1}{r}u' = f.$$

Then $\|u''\|_{L^2}$ is bounded by $\|f\|_{L^2} + \|r^{-1}u'\|_{L^2}$. So consider

$$\frac{u'(r)}{r} = \frac{1}{2} - \frac{r}{4} - \frac{C}{r^2}.$$

As the first two terms are in L^2, we need only consider the final term. Noting that

$$\|r^{-1}u'\|_{L^2}^2 = 2\pi \int_\varepsilon^2 r(r^{-1}u')^2 \sim 2\pi C^2 \int_\varepsilon^2 \frac{1}{r^3} \sim C^2\varepsilon^{-2},$$

so $\|u\|_{\dot{H}^2} \sim \varepsilon^{-1}(-\log\varepsilon)^{-1}$ with log corrections.

One can find a similar example in the three-dimensional case, namely $f(r) = 1 - 5r^2/3$ on the spherical shell between $r = \varepsilon$ and $r = 1$.

The lack of such a bound unfortunately appears to invalidate the arguments treating a moving disc in [3] and a moving sphere in [28].

3. The Stokes equations

In this section we extend the results of the previous section to the Stokes problem

$$-\Delta \mathbf{u}_r + \nabla p_r = \mathbf{f} \text{ in } \Omega_r, \qquad \mathbf{u}_r|_{\partial K_r} = 0, \qquad \operatorname{div} \mathbf{u}_r = 0.$$

First we introduce the required spaces of vector fields. Recall that, given a space of scalar functions X, we write $\mathbb{X}$ for the two-component space $X \times X$. Define for $r \geq 0$

$$\mathbb{H}^1_{\text{per}}(\Omega_r) = \text{ the closure of } \mathbb{C}^1_{\text{per}}(\overline{\Omega}_r) \text{ in } \mathbb{H}^1(\Omega_r),$$

$$\mathbb{H}^1_{\text{per},\sigma}(\Omega_r) = \{\mathbf{v} \in \mathbb{H}^1_{\text{per}}(\Omega_r) : \ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_r\},$$

$$\mathbb{V}_{0,r} = \{\mathbf{v} \in \mathbb{H}^1_{\text{per}}(\Omega_r) : \ \mathbf{v} = 0 \text{ on } \partial K_r\},$$

and

$$\mathbb{V}_{0,r,\sigma} = \{\mathbf{v} \in \mathbb{H}^1_{\text{per},\sigma}(\Omega_r) : \ \mathbf{v} = 0 \text{ on } \partial K_r\}.$$

We observe that any function belonging to $\mathbb{V}_{0,r}$ or $\mathbb{V}_{0,r,\sigma}$ can be extended by zero inside of K_r to give a function in $\mathbb{H}^1_{\text{per}}(\Omega)$ or $\mathbb{H}^1_{\text{per},\sigma}(\Omega)$, respectively.

We will determine the asymptotic behaviour of weak solutions to the following Stokes problem when $r \to 0$:

$$-\Delta \mathbf{u}_r + \nabla p_r = \mathbf{f} \text{ in } \Omega_r, \qquad \mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}.$$

Our second convergence result is as follows. We use a colon in the left-hand side of (3.1) to denote summation in both indices,

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i,j=1}^{2} (\partial_i u_j)(\partial_i v_j).$$

Theorem 3.1. *Let $\mathbf{f} \in \mathbb{L}^2(\Omega)$. For every $r > 0$ there exists a unique solution $\mathbf{u}_r \in \mathbb{V}_{0,r,\sigma}$ of the problem*

$$\int_{\Omega_r} \nabla \mathbf{u}_r : \nabla \mathbf{v} = \int_{\Omega_r} \mathbf{f} \cdot \mathbf{v} \qquad \textit{for all } \mathbf{v} \in \mathbb{V}_{0,r,\sigma}. \tag{3.1}$$

Moreover

a) if $\int_\Omega \mathbf{f} = 0$ *then as* $r \to 0$

$$\mathbf{u}_r - \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_r \to \mathbf{u}_0 \qquad \textit{and} \qquad \nabla\mathbf{u}_r \to \nabla\mathbf{u}_0,$$

where the limits are taken in $\mathbb{L}^2(\Omega)$ *and* $\mathbf{u}_0 \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ *is the unique solution of the problem*

$$\int_\Omega \nabla\mathbf{u}_0 : \nabla\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \textit{for all } \mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \tag{3.2}$$

that satisfies $\int_\Omega \mathbf{u}_0 = 0$*;*

b) if $\int_\Omega \mathbf{f} \neq 0$ *then* $\|\nabla\mathbf{u}_r\|_{\mathbb{L}^2}$ *is unbounded as* $r \to 0$*.*

Note that if we set $\mathbf{v} = (1,0)$ and $\mathbf{v} = (0,1)$ as test functions in (3.2), then one can see immediately that for

$$\int_\Omega \mathbf{f} \neq 0$$

a solution cannot exist.

The only difference from the Poisson problem is that we now have to approximate functions in $\mathbb{H}^1_{\mathrm{per}}(\Omega)$ by functions in $\mathbb{V}_{0,r,\sigma}$, i.e. we must incorporate the divergence-free condition. If we have such approximating functions then we can use the same argument as before to show convergence of solutions to those of the limiting problem. Indeed, the Poincaré inequalities work the same way as before and if $\int_\Omega \mathbf{f} = 0$ then

$$\|\nabla\mathbf{u}_r\|_{\mathbb{L}^2} \le C\|\mathbf{f}\|_{\mathbb{L}^2},\ \forall r > 0,$$

where C is a constant independent of r.

To deal with the divergence-free issue, we consider the following divergence problem for $g \in L^2(\Omega)$, and $\int_\Omega g = 0$:

$$\begin{cases} \operatorname{div}\mathbf{h} = g & \text{in } \Omega, \\ \mathbf{h} \in \mathbb{H}^1_0(\Omega). \end{cases} \tag{3.3}$$

When Ω is star-shaped with respect to every point of $D_R(x_0) := x_0 + D_R$ with $\overline{D}_R(x_0) \subset \Omega$, the existence of a solution $\mathbf{h}$ of this problem is proved in [6, Lem. III.3.1] together with the inequality

$$\|\mathbf{h}\|_{\mathbb{H}_0^1(\Omega)} \leq C\|g\|_{L^2(\Omega)},$$

where the constant C depends on R and the diameter of Ω. Note that the divergence problem does not have a unique solution, since by adding any divergence-free function that vanishes on the boundary to the function $\mathbf{h}$ one would get another solution. Nevertheless, for more general bounded domains, for instance, those satisfying the cone condition, the following result is true (cf. [6, Thm III.3.1, Rmk. III.3.1]).

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\Omega = \cup_{j=1}^n U_j$, where each U_j is star-shaped with respect to some open disc D_j with $\overline{D_j} \subset U_j$. Then, given $g \in L^2(\Omega)$ with $\int_\Omega g = 0$, there exists at least one solution* $\mathbf{h}$ *to* (3.3) *satisfying*

$$\|\mathbf{h}\|_{\mathbb{H}_0^1(\Omega)} \leq C^* C\|g\|_{L^2(\Omega)},$$

where C depends on n, the diameter of Ω and the smallest radius of the discs D_j. The constant C^* is the maximum of

$$C_1 = 1 + \left(\frac{|U_1|}{|F_1|}\right)^{1/2}$$

and

$$C_k = \left(1 + \left(\frac{|U_k|}{|F_k|}\right)^{1/2}\right) \prod_{i=1}^{k-1} \left(1 + \left(\frac{|\mathcal{U}_i \setminus U_i|}{|F_i|}\right)^{1/2}\right), \quad k \geq 2,$$

where $\mathcal{U}_i = \cup_{s=i+1}^n U_s$ and $F_i = U_i \cap \mathcal{U}_i$.

We are going to apply this theorem to the domain Ω_ε. Strictly, Theorem 3.2 applies to Ω_ε with Dirichlet boundary conditions on the lateral boundaries, but since the resulting function h belongs to $H_0^1(\Omega_\epsilon)$, it can trivially be extended periodically to produce a function in $H^1_{\text{per}}(\Omega_\epsilon)$.

Remark 3.3. It is not difficult to see that the constant in the inequalities can be bounded independently of ε, as follows. Since ∂K is C^1, we can find a covering $(U_0, \ldots, U_N)$ of Ω by open sets, constructed by joining points on $\partial\Omega$

to 0, such that each $U_i \backslash K$ is star-shaped with respect to all points in a disc D_i (see Figure 3), and $U_i \cap (\cup_{j=i+1}^{N} U_j)$ has a non-zero area. For $\varepsilon \in (0,1)$ set $U_i^\varepsilon = U_i \backslash K_\varepsilon$. Since K is star-shaped with respect to 0, the open sets $(U_i^\varepsilon)_{i=0,\dots,N}$ cover Ω_ε, and each U_i^ε is star-shaped with respect to D_i (not depending on ε). Moreover, the areas $|U_i^\varepsilon \cap (\cup_{j=i+1}^{N} U_j^\varepsilon)|$ are bounded above by the areas corresponding to $\varepsilon = 1$. Therefore, we see that the constants in Theorem 3.2 can be bounded independently of ε, as claimed.

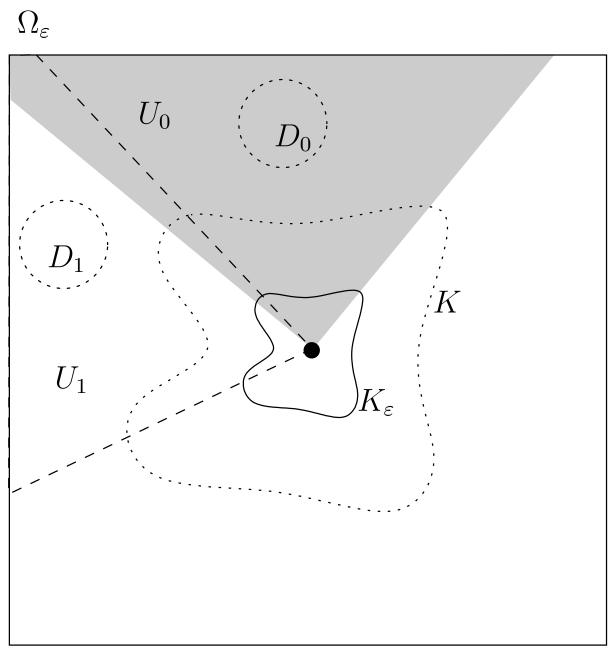

Figure 3: Two sets U_0 (greyed) and U_1 (dashed boundary) to illustrate that the constant in Theorem 3.2 can be taken to be bounded independently of ε.

We now prove the required lemma on the approximation of functions in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ by functions in $\mathbb{V}_{0,\varepsilon,\sigma}$.

Lemma 3.4. *If* $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ *then there exists a sequence* $\mathbf{v}_\varepsilon \in \mathbb{V}_{0,\varepsilon,\sigma}$ *such that*

$$\mathbf{v}_\varepsilon \to \mathbf{v} \quad \textit{in} \quad \mathbb{H}^1(\Omega) \qquad \textit{as} \qquad \varepsilon \to 0.$$

Proof. Let ϕ_ε be the function introduced in the proof of Lemma 2.3. We first assume that $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \cap \mathbb{L}^\infty(\Omega)$. Then for ε small $\phi_\varepsilon \mathbf{v} \in \mathbb{V}_{0,\varepsilon}$. Since

$\mathrm{div}(\mathbf{v}) = 0$ it follows

$$\mathrm{div}(\phi_\varepsilon \mathbf{v}) = \nabla\phi_\varepsilon \cdot \mathbf{v}.$$

Moreover,

$$\int_{\Omega_\varepsilon} \nabla\phi_\varepsilon \cdot \mathbf{v} = \int_{\Omega_\varepsilon} \mathrm{div}(\mathbf{v}\phi_\varepsilon) = 0.$$

Noting that also that $\nabla\phi_\varepsilon \cdot \mathbf{v}$ belongs to $L^2(\Omega)$, it follows that it satisfies the conditions required by Theorem 3.2, and so the divergence problem

$$\begin{cases} \mathrm{div}\,\mathbf{h}_\varepsilon = -\nabla\phi_\varepsilon \cdot \mathbf{v} & \text{in } \Omega_\varepsilon, \\ \mathbf{h}_\varepsilon \in \mathbb{H}_0^1(\Omega_\varepsilon), \end{cases}$$

has a solution $\mathbf{h}_\varepsilon$ satisfying

$$\|\mathbf{h}_\varepsilon\|_{\mathbb{H}_0^1(\Omega_\varepsilon)} \leq C\|\nabla\phi_\varepsilon \cdot \mathbf{v}\|_{L^2(\Omega_\varepsilon)},$$

where C depends only on p and Ω. Setting $\mathbf{v}_\varepsilon = \mathbf{h}_\varepsilon + \phi_\varepsilon \mathbf{v}$ it is clear that $\mathbf{v}_\varepsilon \in \mathbb{V}_{0,\varepsilon,\sigma}$ and, by Lemma 2.3, recalling that $\nabla\phi_\varepsilon \to 0$ in $L^2(\Omega)^2$ we obtain

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ in } \mathbb{H}^1(\Omega) \text{ as } \varepsilon \to 0.$$

It remains only to prove that a function in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ can be approximated by functions in $\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega) \cap \mathbb{L}^\infty(\Omega)$ which will allow us to conclude via a diagonal argument.

Let $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ supposed to be extended by periodicity to $\mathbb{R}^2$. Let ϱ_n be a standard mollifier, i.e. $\varrho_n(x) = n^2\varrho(nx)$ where ϱ is a C^∞ function with support in the unit disc and such that

$$\varrho \geq 0\,, \quad \int_{\mathbb{R}^2} \varrho = 1.$$

Then set

$$\mathbf{v}_n(x) = \varrho_n * \mathbf{v}(x) = \int_{\mathbb{R}^2} \varrho_n(y)\mathbf{v}(x-y)\mathrm{d}y.$$

It is clear that $\mathbf{v}_n$ is periodic in x – with the same period as $\mathbf{v}$ – divergence free, smooth (and thus in $\mathbb{L}^\infty(\Omega)$) and, as $n \to \infty$,

$$\mathbf{v}_n,\ \nabla\mathbf{v}_n \to \mathbf{v},\ \nabla\mathbf{v} \text{ in } \mathbb{L}^2(\Omega) \text{ and } \mathbb{L}^2(\Omega)^2, \text{ respectively.}$$

This completes the proof. $\square$

To prove Theorem 3.1 we essentially recapitulate the proof of Theorem 2.1 in this new setting.

Proof (Theorem 3.1). Define

$$\tilde{\mathbf{u}}_r = \mathbf{u}_r - \fint_\Omega \mathbf{u}_r.$$

Then from the Poincaré–Wirtinger inequality, $\|\tilde{\mathbf{u}}_r\|_{\mathbb{H}^1(\Omega_r)}$ is uniformly bounded. Therefore for a subsequence $\nabla \mathbf{u}_r = \nabla \tilde{\mathbf{u}}_r \rightharpoonup \nabla \mathbf{u}_0$ in $\mathbb{H}^1(\Omega)$ and $\tilde{\mathbf{u}}_r \to \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$, where $\mathbf{u}_0$ satisfies $\int_\Omega \mathbf{u}_0 = 0$.

For a fixed r_0, $\forall r < r_0$ one has $\mathbb{V}_{0,\sigma,r_0} \subset \mathbb{V}_{0,\sigma,r}$. Thus

$$\int_\Omega \nabla \mathbf{u}_r : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,\sigma,r_0}.$$

Passing to the limit in r we obtain

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{V}_{0,\sigma,r_0}. \tag{3.4}$$

Let $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ and let $\mathbf{v}_\varepsilon$ be the approximating sequence from Lemma 3.4. Then for $\varepsilon \leq r_0$ we have

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v}_\varepsilon = \int_\Omega \mathbf{f} \cdot \mathbf{v}_\varepsilon$$

and passing to the limit in ε we obtain

$$\int_\Omega \nabla \mathbf{u}_0 : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$$

as required. (This is (3.2).)

Since the limiting problem has a unique solution when one imposes the zero average condition, it follows that all convergent subsequences must have the same limit. As a consequence, the whole original sequence converges toward $\mathbf{u}_0$.

To see that $\nabla \mathbf{u}_r \to \nabla \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$ we show that $\|\nabla \mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega)} \to \|\nabla \mathbf{u}_0\|^2_{\mathbb{L}^2(\Omega)}$. Since $\mathbf{u}_r - \fint_\Omega \mathbf{u}_r \to \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$,

$$\int_{\Omega_r} |\nabla \mathbf{u}_r|^2 = \int_{\Omega_r} \mathbf{f} \cdot \mathbf{u}_r = \int_\Omega \mathbf{f} \cdot \mathbf{u}_r = \int_\Omega \mathbf{f} \cdot \left(\mathbf{u}_r - \fint_\Omega \mathbf{u}_r \right) \to \int_\Omega \mathbf{f} \cdot \mathbf{u}_0.$$

But from (3.2) we have

$$\int_\Omega |\nabla \mathbf{u}_0|^2 = \int_\Omega \mathbf{f} \cdot \mathbf{u}_0,$$

which implies that

$$\int_\Omega |\nabla \mathbf{u}_r|^2 \to \int_\Omega |\nabla \mathbf{u}_0|^2.$$

Coupled with weak convergence this implies strong convergence of $\nabla \mathbf{u}_r$ to $\nabla \mathbf{u}_0$ in $\mathbb{L}^2(\Omega)$. □

4. The time-dependent Navier–Stokes equations

In this section we tackle the vanishing obstacle problem for the Navier–Stokes equations. The corresponding problem in a two-dimensional exterior domain (i.e. $\mathbb{R}^2 \setminus K_r$) was analysed in [14] with the initial condition for the velocity corresponding to a fixed initial vorticity (independent of r). Here, by considering a periodic domain and suitable initial data we provide a less technical proof by using arguments along the lines of the previous sections. Let us observe that the setting here is simpler due to the fact that the velocity is bounded in L^2.

We consider weak solutions to the following Navier–Stokes problem

$$\begin{cases} \partial_t \mathbf{u}_r - \Delta \mathbf{u}_r + (\mathbf{u}_r \cdot \nabla)\mathbf{u}_r + \nabla p_r = \mathbf{f} \ \text{ in } \Omega_r \times (0,\infty), \\ \operatorname{div} \mathbf{u}_r = 0 \text{ in } \Omega_r \times (0,\infty), \\ \mathbf{u}_r = 0 \text{ in } \partial K_r \times (0,\infty), \\ \text{periodic}, \\ \mathbf{u}_r(0) = \mathbf{u}_r^0 \text{ in } \Omega_r, \end{cases} \tag{4.1}$$

and show that they converge to periodic solutions of the equations on Ω. Note that in this section we do not require that $\int_\Omega \mathbf{f} = 0$.

We introduce the spaces

$$\mathbb{H}_{r,\sigma} = \text{ the closure of } \{\mathbf{v} \in \mathbb{C}^1_{\text{per}}(\overline{\Omega}_r) : \ \mathbf{v} = 0 \text{ on } \partial K_r, \ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_r\} \\ \text{in } \mathbb{L}^2(\Omega_r)$$

and

$$\mathbb{H}_\sigma = \mathbb{H}_{0,\sigma} = \{\mathbf{v} \in \mathbb{L}^2_{\text{per}}(\Omega) : \ \operatorname{div} \mathbf{v} = 0\}.$$

We can now prove our convergence result for time-dependent Navier–Stokes solutions.

Theorem 4.1. *Let $T > 0$, $\mathbf{u}_r^0 \in \mathbb{H}_{r,\sigma}$ and $f \in \mathbb{L}^2((0,T) \times \Omega)$. For every $r > 0$ there exists a unique weak solution $\mathbf{u}_r$ of problem (4.1), i.e. a unique $\mathbf{u}_r \in L^2(0,T;\mathbb{V}_{0,r,\sigma}) \cap L^\infty(0,T;\mathbb{H}_{r,\sigma})$ with $\partial_t \mathbf{u}_r \in L^2(0,T;\mathbb{V}'_{0,r,\sigma})$, such that*

$$\langle \partial_t \mathbf{u}_r, \mathbf{v}\rangle + \int_{\Omega_r} \nabla \mathbf{u}_r : \nabla \mathbf{v} + \int_{\Omega_r} [(\mathbf{u}_r \cdot \nabla)\mathbf{u}_r] \cdot \mathbf{v} = \int_{\Omega_r} \mathbf{f} \cdot \mathbf{v} \\ \textit{for all } \mathbf{v} \in \mathbb{V}_{0,r,\sigma}, \tag{4.2}$$

$$\mathbf{u}_r(0) = \mathbf{u}_r^0. \tag{4.3}$$

In addition, $\mathbf{u}_r$ satisfies the energy inequality

$$\|\mathbf{u}_r(t)\|^2_{\mathbb{L}^2(\Omega_r)} + \int_0^t \|\nabla \mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega_r)} \leq C(T) \left(\|\mathbf{u}_r^0\|^2_{\mathbb{L}^2(\Omega_r)} + \int_0^t \|\mathbf{f}\|^2_{\mathbb{L}^2(\Omega_r)} \right). \tag{4.4}$$

Furthermore, if $\mathbf{u}_r^0 \rightharpoonup \mathbf{u}^0$ in $\mathbb{L}^2(\Omega)$ as $r \to 0$, then

$$\mathbf{u}_r \to \mathbf{u} \textit{ strongly in } \mathbb{L}^2(0,T;\mathbb{H}_\sigma) \textit{ and weakly in } L^2(0,T;\mathbb{H}^1_{\text{per},\sigma}(\Omega)),$$

where $\mathbf{u}$ is the unique weak solution of the Navier–Stokes problem

$$\langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega [(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \textit{ for all } \mathbf{v} \in \mathbb{H}^1_{\text{per},\sigma}(\Omega), \\ \mathbf{u}(0) = \mathbf{u}^0.$$

Remark 4.2 (Dimension 3). This theorem also holds in dimension 3, modulo the following modifications: $\mathbf{u}_r$ and $\mathbf{u}$ are no longer ensured to be unique, and the convergence $\mathbf{u}_r \to \mathbf{u}$ only holds up to a subsequence as $r \to 0$.

Proof. The proof of existence of weak solutions follows by using the Galerkin method and, since we are in dimension two, the uniqueness is also standard. The energy inequality, which follows formally from the differential inequality

$$\partial_t \|\mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega_r)} + 2\|\nabla \mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega_r)} \le \|\mathbf{f}\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}_r\|^2_{\mathbb{L}^2(\Omega_r)}$$

using the Gronwall lemma, follows rigorously from the same limiting Galerkin procedure, with an energy inequality obtained for each approximation. (See Constantin & Foias [2], Galdi [5], or Robinson [26], for example.)

We split the proof of convergence into three steps. Briefly, we will obtain estimates for the solution $\mathbf{u}_r$ independent of r, show that $\mathbf{u}_r$ converges to a limit in various senses, and show this is sufficient to pass to the limit in the weak formulation of the problem.

Step 1: Estimates. From the energy inequality (4.4) we already know that

$$\mathbf{u}_r \text{ is bounded in } L^\infty(0,T;\mathbb{H}_\sigma) \cap L^2(0,T;\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)) \tag{4.5}$$

uniformly for $r > 0$. Recall that $\mathbf{u}_r$ has been extended by zero inside K_r.

We need some strong convergence in order to pass to the limit in the nonlinear term. To this end, we first estimate the time derivative of $\mathbf{u}_r$ from (4.2). Observe that

$$\int_{\Omega_r} [(\mathbf{u}_r \cdot \nabla)\mathbf{u}_r] \cdot \mathbf{v} = -\int_{\Omega_r} [(\mathbf{u}_r \cdot \nabla)\mathbf{v}] \cdot \mathbf{u}_r, \quad \text{for all } \mathbf{v} \in \mathbb{V}_{0,r,\sigma}.$$

Thus, for any $\mathbf{v} \in \mathbb{V}_{0,r,\sigma}$

$$\begin{aligned}
|\langle \partial_t \mathbf{u}_r, \mathbf{v}\rangle| &= \Big| -\int_{\Omega_r} \nabla \mathbf{u}_r : \nabla \mathbf{v} + \int_{\Omega_r} [(\mathbf{u}_r \cdot \nabla)\mathbf{v}] \cdot \mathbf{u}_r + \int_{\Omega_r} \mathbf{f} \cdot \mathbf{v} \Big| \\
&\le C(\|\nabla \mathbf{u}_r\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}_r\|_{\mathbb{L}^2(\Omega)} \|\mathbf{u}_r\|_{\mathbb{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbb{L}^2(\Omega)}) \|\mathbf{v}\|_{\mathbb{H}^1(\Omega)} \\
&\le C(\|\mathbf{u}_r\|_{\mathbb{H}^1(\Omega)} + \|\mathbf{f}\|_{\mathbb{L}^2(\Omega)}) \|\mathbf{v}\|_{\mathbb{H}^1(\Omega)}, \ \text{a.e. } t,
\end{aligned} \tag{4.6}$$

where we have used the interpolation inequality

$$\|\mathbf{u}\|_{\mathbb{L}^4(\Omega)} \le C \|\mathbf{u}\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)} \|\mathbf{u}\|^{\frac{1}{2}}_{\mathbb{H}^1(\Omega)}$$

and that $\mathbf{u}_r$ is uniformly bounded in $L^\infty(0,T;\mathbb{H}_\sigma)$.

Next, we claim that

$$\|\mathbf{u}_r(\cdot+h)-\mathbf{u}_r(\cdot)\|^2_{L^2(0,T-h;\mathbb{L}^2(\Omega))}\leq Ch.$$

Indeed,

$$\begin{aligned}
&\|\mathbf{u}_r(\cdot+h)-\mathbf{u}_r(\cdot)\|^2_{L^2(0,T-h;\mathbb{L}^2(\Omega))}\\
&\qquad=\int_0^{T-h}\langle\mathbf{u}_r(t+h)-\mathbf{u}_r(t),\mathbf{u}_r(t+h)-\mathbf{u}_r(t)\rangle dt\\
&\qquad=\int_0^{T-h}\langle\int_t^{t+h}\partial_t\mathbf{u}_r(s)ds,\mathbf{u}_r(t+h)-\mathbf{u}_r(t)\rangle dt\\
&\qquad=\int_0^{T-h}\int_t^{t+h}\langle\partial_t\mathbf{u}_r(s),\mathbf{u}_r(t+h)-\mathbf{u}_r(t)\rangle dsdt.
\end{aligned}$$

Note that we have used that $\int_{\Omega_r}\mathbf{w}\cdot\mathbf{v}=\langle\mathbf{w},\mathbf{v}\rangle$ for $\mathbf{w}\in\mathbb{H}_{r,\sigma}$ and $\mathbf{v}\in\mathbb{V}_{0,r,\sigma}$. As $\mathbf{u}_r(t+h)-\mathbf{u}_r(t)\in\mathbb{V}_{0,r,\sigma}$ a.e. t, we can use estimate (4.6). Thus, by applying Young inequality and Fubini Theorem, we arrive at

$$\begin{aligned}
&\|\mathbf{u}_r(\cdot+h)-\mathbf{u}_r(\cdot)\|^2_{L^2(0,T-h;\mathbb{L}^2(\Omega))}\\
&\leq\int_0^{T-h}\int_t^{t+h}(\|\mathbf{u}_r(s)\|_{\mathbb{H}^1(\Omega)}+\|\mathbf{f}(s)\|_{\mathbb{L}^2(\Omega)})\|\mathbf{u}_r(t+h)-\mathbf{u}_r(t)\|_{\mathbb{H}^1(\Omega)}dsdt\\
&\leq\int_0^{T-h}\int_t^{t+h}(\|\mathbf{u}_r(s)\|^2_{\mathbb{H}^1(\Omega)}+\|\mathbf{f}(s)\|^2_{\mathbb{L}^2(\Omega)}\\
&\qquad\qquad+\|\mathbf{u}_r(t+h)\|^2_{\mathbb{H}^1(\Omega)}+\|\mathbf{u}_r(t)\|^2_{\mathbb{H}^1(\Omega)})dsdt\\
&\leq(\|\mathbf{f}\|^2_{L^2(0,T;\mathbb{L}^2(\Omega))}+3\|\mathbf{u}_r\|^2_{L^2(0,T;\mathbb{H}^1(\Omega))})h\\
&\leq Ch
\end{aligned}$$

where C is independent of r. The claim is proved.

Step 2: Convergence of $\mathbf{u}_r$. Since $\mathbf{u}_r$ is bounded in $L^\infty(0,T;\mathbb{L}^2(\Omega))$,

$$\|\mathbf{u}_r(\cdot+h)-\mathbf{u}_r(\cdot)\|_{L^2(0,T-h;\mathbb{L}^2(\Omega))}\to 0 \text{ as } h\to 0 \text{ uniformly in } r,$$

and $\mathbb{H}^1_{\text{per},\sigma}(\Omega)\subset\subset\mathbb{H}_\sigma$, we can apply Theorem 3 from [30, p. 80] and conclude that

$$\mathbf{u}_r \text{ is relatively compact in } L^2(0,T;\mathbb{H}_\sigma).$$

Hence, up to a subsequence, it holds

$$\begin{aligned}&\mathbf{u}_r \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)) \text{ and}\\&\mathbf{u}_r \to \mathbf{u} \text{ in } L^2(0,T;\mathbb{H}_\sigma)\end{aligned}$$

By interpolation and the Hölder inequality,

$$\begin{aligned}\int_0^T \|\mathbf{u}_r - \mathbf{u}\|^2_{\mathbb{L}^4(\Omega)} &\le C\int_0^T \|\mathbf{u}_r - \mathbf{u}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}_r - \mathbf{u}\|_{\mathbb{H}^1(\Omega)}\\&\le C\Big(\int_0^T \|\mathbf{u}_r - \mathbf{u}\|^2_{\mathbb{L}^2(\Omega)}\Big)^{\frac{1}{2}}.\end{aligned}$$

Thus, we infer in addition that

$$\mathbf{u}_r \to \mathbf{u} \text{ in } L^2(0,T;\mathbb{L}^4(\Omega)). \tag{4.7}$$

Step 3: Passage to the limit in the weak formulation. By using that, for a fixed r_0, $\forall r < r_0$ one has $\mathbb{V}_{0,r_0,\sigma} \subset \mathbb{V}_{0,r,\sigma}$, multiplying (4.2) by $\xi \in C_0^\infty[0,T)$ and integrating in time, we have

$$\begin{aligned}-\int_0^T\int_\Omega \mathbf{u}_r\cdot\mathbf{v}\xi' + \int_0^T\int_\Omega \nabla\mathbf{u}_r : \nabla\mathbf{v}\xi - \int_0^T\int_\Omega [(\mathbf{u}_r\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_r\xi&\\= \int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{v}\xi + \int_\Omega \mathbf{u}_r^0\cdot\mathbf{v}\xi(0)&\end{aligned}$$

for all $\mathbf{v} \in \mathbb{V}_{0,r_0,\sigma}$ and $\xi \in C_0^\infty[0,T)$.

The weak convergences are sufficient to pass the limit in the linear terms. To show the convergence of the nonlinear term, we re-write

$$\begin{aligned}\int_0^T\int_\Omega &[(\mathbf{u}_r\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_r\xi - [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\xi\\&= \int_0^T\int_\Omega [((\mathbf{u}_r-\mathbf{u})\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_r\xi + [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot(\mathbf{u}_r-\mathbf{u})\xi.\end{aligned}$$

We prove that the first term on the right-hand side goes to zero; the convergence of the second term is proved similarly. By using the Hölder inequality in space and then in time, we have

$$\Big|\int_0^T\int_\Omega[((\mathbf{u}_r-\mathbf{u})\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}_r\xi\Big|$$
$$\le\int_0^T\|\mathbf{u}_r-\mathbf{u}\|_{\mathbb{L}^4(\Omega)}\|\nabla\mathbf{v}\|_{\mathbb{L}^2(\Omega)}\|\mathbf{u}_r\|_{\mathbb{L}^4(\Omega)}\|\xi\|_{L^\infty(0,T)}$$
$$\le C\Big(\int_0^T\|\mathbf{u}_r-\mathbf{u}\|^2_{\mathbb{L}^4(\Omega)}\Big)^{\frac12}\Big(\int_0^T\|\mathbf{u}_r\|^2_{\mathbb{H}^1(\Omega)}\Big)^{\frac12},$$

where we have used the embedding $\mathbb{H}^1(\Omega)\subset\mathbb{L}^4(\Omega)$. The convergence follows from convergence (4.7) and estimate (4.5).

Passing to the limit in r we obtain

$$-\int_0^T\int_\Omega\mathbf{u}\cdot\mathbf{v}\xi'+\int_0^T\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}\xi-\int_0^T\int_\Omega[(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\xi$$
$$=\int_0^T\int_\Omega\mathbf{f}\cdot\mathbf{v}\xi+\int_\Omega\mathbf{u}^0\cdot\mathbf{v}\xi(0)$$

for all $\mathbf{v}\in\mathbb{V}_{0,r_0,\sigma}$ and $\xi\in C_0^\infty[0,T)$.

Next, we argue as in the Stokes problem by using the approximation from Lemma 3.4. Given $\mathbf{v}\in\mathbb{H}^1_{\text{per},\sigma}(\Omega)$ there exist $\mathbf{v}^\varepsilon\in\mathbb{V}_{0,\varepsilon,\sigma}$ such that $\mathbf{v}^\varepsilon\rightharpoonup\mathbf{v}$ in $\mathbb{H}^1_{\text{per},\sigma}(\Omega)$. Thus, for $\varepsilon\le r_0$ one has

$$-\int_0^T\int_\Omega\mathbf{u}\cdot\mathbf{v}^\varepsilon\xi'+\int_0^T\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}^\varepsilon\xi-\int_0^T\int_\Omega[(\mathbf{u}\cdot\nabla)\mathbf{v}^\varepsilon]\cdot\mathbf{u}\xi$$
$$=\int_0^T\int_\Omega\mathbf{f}\cdot\mathbf{v}^\varepsilon\xi+\int_\Omega\mathbf{u}^0\cdot\mathbf{v}^\varepsilon\xi(0).$$

Passing to the limit in ε we get

$$-\int_0^T\int_\Omega\mathbf{u}\cdot\mathbf{v}\xi'+\int_0^T\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}\xi-\int_0^T\int_\Omega[(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\xi$$
$$=\int_0^T\int_\Omega\mathbf{f}\cdot\mathbf{v}\xi+\int_\Omega\mathbf{u}^0\cdot\mathbf{v}\xi(0)\qquad(4.8)$$

for all $\mathbf{v}\in\mathbb{H}^1_{\text{per},\sigma}(\Omega)$ and $\xi\in C_0^\infty[0,T)$.

In particular, since $\mathbf{u} \in L^2(0,T;\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)) \cap L^\infty(0,T;\mathbb{H}_\sigma)$ we can take $\xi \in C_0^\infty(0,T)$ in (4.8) and deduce that $\partial_t \mathbf{u} \in L^2(0,T;(\mathbb{H}^1_{\mathrm{per},\sigma}(\Omega))')$, whence $\mathbf{u}$ satisfies

$$\langle \partial_t \mathbf{u}, \mathbf{v}\rangle + \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$.

It remains only to prove that $\mathbf{u}(0) = \mathbf{u}^0$. To see this, multiply the previous equality by $\xi \in C_0^\infty[0,T)$ and integrate in time, to obtain

$$-\int_0^T \int_\Omega \mathbf{u}\cdot\mathbf{v}\xi' + \int_0^T \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\xi - \int_0^T \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}]\cdot\mathbf{u}\xi \\ = \int_0^T \int_\Omega \mathbf{f}\cdot\mathbf{v}\xi + \int_\Omega \mathbf{u}(0)\cdot\mathbf{v}\xi(0)$$

for all $\mathbf{v} \in \mathbb{H}^1_{\mathrm{per},\sigma}(\Omega)$ and $\xi \in C_0^\infty[0,T)$. Comparing with (4.8) we conclude that $\mathbf{u}(0) = \mathbf{u}^0$. Notice also that $\mathbf{u} \in C([0,T];\mathbb{H}_\sigma)$.

Since the limiting problem has a unique solution, it follows that all convergent subsequences must have the same limit. As a consequence, the whole original sequence converges toward $\mathbf{u}$. □

5. Conclusions

We have analysed three models in a simple but unusual geometry, the 'punctured periodic domain', showing that the influence of the obstacle K_r evaporates in the limit as $r \to 0$.

Some interesting open problems remain. While the lack of a bound on the average of the solution u_r over Ω (in both the Poisson and Stokes problems) that is uniform in r appears initially to be only a mathematical curiosity, such a bound is central to tackling the stationary Navier–Stokes problem in this geometry.

The fact that there is no 'uniform elliptic regularity' for the Laplacian or Stokes operator in this geometry means that the important 'vanishing tracer' problem (cf. [3, 28]) also remains open. Recently, Lacave & Takahasi

[19] obtained a partial result in the two-dimensional case assuming that the density of the solid is independent of r. They employed some optimal L^p–L^q decay estimates of the semigroup associated to the fluid-rigid body system. Another recent result in the two-dimensional and three-dimensional cases was given by He & Iftimie [7, 8] where was considered the diameter of the rigid body going to zero, that the initial velocity has bounded energy, and that the density of the rigid body goes to infinity.

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