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2D Surface Waves in Magnetohydrodynamics

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The study of nonlinear waves in water has a long history beginning with the seminal paper by Korteweg & de Vries (1895) and more recently for magnetohydrodynamics Danov & Ruderman (1983). The appearance of a Hilbert transform in the nonlinear equation for MHD distinguishes it from the water wave model description. In this paper, we are interested in examining weakly nonlinear interfacial waves in $2 + 1$ dimensions. First, we determine the wave solution in the linear case. Next, we derive the corresponding generalisation for the Kadomtsev-Petviasvilli (KP) equation with the inclusion of an equilibrium magnetic field. The derived governing equation is a generalisation of the Benjamin-Ono(BO) equation called the *Benjamin equation* first derived in Benjamin (1992) and in the higher dimensional context in Kim & Akylas (2006).

1. Introduction

The study of waves in magnetohydrodynamic approximation (MHD) applicable to highly collisional magnetised plasmas has a long history (Roberts (1985), Danov & Ruderman (1983)). There have been many interfacial flows of interest over the years in MHD which have taken up much of the literature on the subject with applications to astrophysics (Edwin & Roberts (1986)), geophysics (Barcilon & Fitzjerald (1985)) and to magnetic industrial liquids (Gerbeau *et al.* (2006)). Unlike the plasma considered in this paper, Roberts considered waves in a compressible plasma. The types of waves which we are interested in this paper are surface waves in three dimensions along a contact discontinuity. The study of surface waves in hydrodynamics began with the seminal paper by Korteweg & de Vries (1895) where they employed the approximations of long wavelength and small amplitude. Since this ground-breaking paper, a further fundamental study by Benjamin (1967) where a stratified fluid was considered leading to the celebrated Benjamin-Ono equation. The equation governing the wave propagation includes the Hilbert transform of the second derivative. Since then, there have been many instances where these two fundamentally important equations, i.e. KdV or BO, have appeared.

Benjamin further extended his analysis by expanding the dispersion equation and including weak nonlinearity, later extended for three dimensions by Kim & Akylas (2006). A new direction of development is the implementation of external fields (e.g. electric or magnetic). There has been an extensive amount of work in fluid systems with external fields (electric and magnetic) both in two and three dimensions, see e.g. Hunt (2013) for studies with electric fields, and Roberts (1985) with magnetised plasmas which deal in the case of compressible fluids.

As an introduction, let us recall briefly the paper Danov & Ruderman (1983). This work is essentially an extension of the original paper on the subject introduced by Korteweg & de Vries (1895) in fluids to include a magnetic field. A schematic view of their problem

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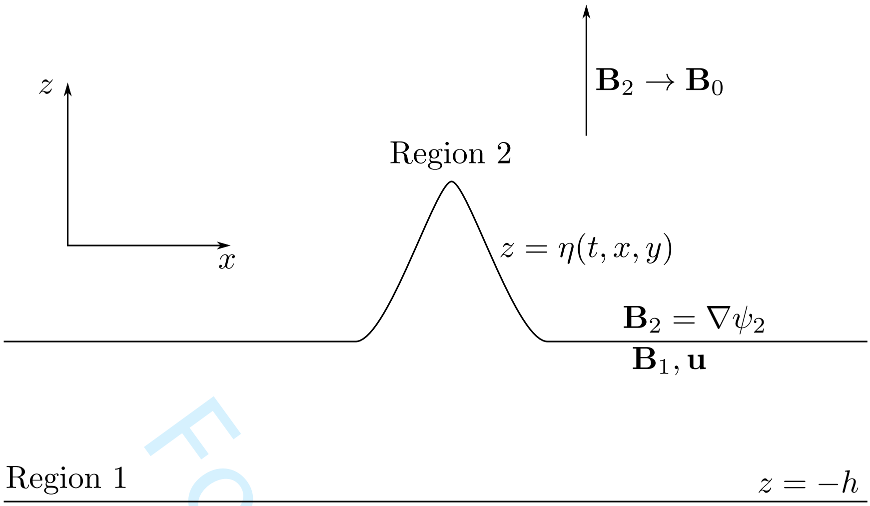

Figure 1: Set Up

set-up is given in Figure (1). The fluid (in region 1) defined by $-h \leqslant z \leqslant \eta(t,x,)$ is perfectly conducting and the region above the fluid (region 2) is perfectly insulating. We assume that fluid 2's density is small so it does not contribute to the dynamics. The magnetic field in region 2 is current free and as $z \to \infty$, the magnetic field tends to a constant. In region 2, there is no flow and the magnetic field satisfies the following conditions:

$$\nabla \cdot \mathbf{B}_2 = 0, \quad \nabla \times \mathbf{B}_2 = \mathbf{0}, \tag{1.1}$$

which means that there is a scalar potential, $\mathbf{B}_2 = \nabla\psi_2$ which satisfies the Laplace equation:

$$\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial z^2} = 0 \tag{1.2}$$

In region 1, the usual equations of MHD hold:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla\left(p + \frac{|\mathbf{B}_1|^2}{2\mu_0}\right) - g\mathbf{e}_z + \frac{1}{\rho\mu_0}(\mathbf{B}_1 \cdot \nabla)\mathbf{B}_1, \tag{1.3}$$

$$\frac{D\mathbf{B}_1}{Dt} = (\mathbf{B}_1 \cdot \nabla)\mathbf{u}. \tag{1.4}$$

In region 1, it is convenient to use stream functions from which the velocity and magnetic field can be derived:

$$\mathbf{u} = -\frac{\partial\varphi}{\partial z}\mathbf{e}_x + \frac{\partial\varphi}{\partial x}\mathbf{e}_z, \quad \mathbf{B} = -\frac{\partial\psi}{\partial z}\mathbf{e}_x + \frac{\partial\psi}{\partial x}\mathbf{e}_z. \tag{1.5}$$

Use of stream functions reduces the problems of finding the velocity and magnetic field components to just two scalar functions. The interface between the plasma and magnetic field is a tangential discontinuity. On $z = \eta(t,x)$:

$$\mathbf{B} \cdot \hat{\mathbf{n}} = 0, \quad \hat{\mathbf{n}} = \frac{-\partial_x\eta\mathbf{e}_x + \mathbf{e}_z}{\sqrt{1 + (\partial_x\eta)^2}}, \tag{1.6}$$

the continuity of total pressure across the interface:

$$\left[p + \frac{|\mathbf{B}|^2}{2\mu_0}\right]_1^2 = 0, \tag{1.7}$$

and finally the free surface equation:

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = w. \tag{1.8}$$

At the lower boundary, $z = -h$, we can state:

$$w = 0, \quad B_z = 0. \tag{1.9}$$

The weakly nonlinear analysis is carried out and the magnetic fields are perturbed from the base state, $\mathbf{B}_0$. The co-ordinates are then transformed as:

$$X = \varepsilon(x - c_0 t), \quad T = \varepsilon^2 t, \tag{1.10}$$

where c_0 is the gravity wave speed $c_0 = \sqrt{gh}$. Giving the equation for the free surface as:

$$\frac{\partial \eta_1}{\partial T} + b\eta_1 \frac{\partial \eta_1}{\partial X} + \beta \mathscr{H}\left(\frac{\partial^2 \eta_1}{\partial X^2}\right) = 0, \tag{1.11}$$

where

$$b = \frac{3gh + v_A^2 \cos^2 \alpha}{2c_0 h}, \quad \beta = \frac{v_A^2 h \cos \alpha}{2c_0}, \tag{1.12}$$

where $v_A = B_0/\sqrt{\mu_0 \rho_0}$. The operator $\mathscr{H}(\cdot)$ is known as the Hilbert transform and is defined as:

$$\mathscr{H}(f(x)) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(y)}{x - y} dy \tag{1.13}$$

Equation 1.11 derived is known as the Benjamin-Ono equation Ono (1975) and is known to have analytical solutions which decay algebraically.

2. 3-Dimensional Consideration

The problem of interest is that of an incompressible perfectly conducting fluid in a co-ordinate system (x, y, z) (see Figure 1) with z pointing upwards and with basis vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. The velocity vector in this co-ordinate system is $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$. Unlike the authors in Danov & Ruderman (1983) we do not assume that the magnetic field $\mathbf{B}$ can be expressed in terms of a stream function. The equations for incompressible MHD in the co-ordinate system are:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla\left(p + \frac{|\mathbf{B}|^2}{2\mu_0}\right) - g\mathbf{e}_z + \frac{1}{\rho_0\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} \tag{2.2}$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B}\cdot\nabla)\mathbf{u} \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

As we are going to be looking at the shallow water approximation, it is appropriate to take the density as constant. It should be noted that the current $\mathbf{J} \neq \mathbf{0}$ in the bulk of the fluid which is different to the assumptions of Melcher (1963), where he takes zero current in the bulk of the fluid.

The boundary conditions for the fluid and magnetic fields are

$$w(t,x,y,-h)=0, \tag{2.5}$$

representing impermeability at the bottom ($z=-h$). There is no magnetic field below $z=-h$ or above $z=\eta(t,x,y)$, and therefore

$$\mathbf{B}\cdot\hat{\mathbf{n}}=0 \quad \text{on} \quad z=-h \quad \text{and} \quad z=\eta(t,x,y). \tag{2.6}$$

On the free surface there is the continuity of total pressure:

$$\left[p+\frac{|\mathbf{B}|^2}{2\mu_0}\right]_2^1=0 \quad \text{on} \quad z=\eta(t,x,y). \tag{2.7}$$

The free surface equation is also used:

$$\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial y}=w \quad \text{on} \quad z=\eta(t,x,y) \tag{2.8}$$

3. Linear Theory

The linear theory is perturbed from magnetostatic equilibrium:

$$\mathbf{u}=\mathbf{0}, \quad \mathbf{B}=B_0\mathbf{e}_x, \quad \eta=0. \tag{3.1}$$

The lower boundary condition in region 1 is $\mathbf{B}\cdot\hat{\mathbf{n}}=B_z(t,x,y,-h)=0$. The pressure in the equilibrium state is given by:

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-g=0. \tag{3.2}$$

Integrating this equation, one obtains $p=-\rho_0 gz+C$. To find C, one uses the continuity of total pressure:

$$p(0)+\frac{B_0^2}{2\mu_0}=p_a \Rightarrow C=p_a-\frac{B_0^2}{2\mu_0}, \tag{3.3}$$

where p_a is atmospheric pressure. So, the equilibrium pressure is:

$$p=p_a-\frac{B_0^2}{2\mu_0}-\rho_0 gz. \tag{3.4}$$

The pressure may be written as Johnson (1997), i.e. as perturbation from the magnetostatic pressure. The non-dimensional perturbed pressure $\hat{p}$ is defined by:

$$p=p_a-\frac{B_0^2}{2\mu_0}-\rho_0 gz-\rho_0 gh\hat{p}. \tag{3.5}$$

The linear case is scaled in the following way:

$$x=h\hat{x}, \quad y=h\hat{y}, \quad z=h\hat{z}, \quad \eta=h\hat{\eta}, \quad t=\frac{h}{\sqrt{gh}}\hat{t}, \tag{3.6}$$

$$\mathbf{u}=\sqrt{gh}\hat{\mathbf{u}}, \quad \mathbf{B}=B_0\hat{\mathbf{B}}. \tag{3.7}$$

For the weakly nonlinear regime weak magnetic fields are required for consistency and we scale the magnetic field accordingly (see 4.4). For the linear theory however, this isn't necessary. The scaled equations are:

$$\frac{D\hat{\mathbf{u}}}{D\hat{t}} = -\hat{\nabla}\left(\hat{p} + \frac{M|\hat{\mathbf{B}}|^2}{2}\right) + M(\hat{\mathbf{B}}\cdot\hat{\nabla})\hat{\mathbf{B}} \tag{3.8}$$

$$\frac{D\hat{\mathbf{B}}}{D\hat{t}} = (\hat{\mathbf{B}}\cdot\hat{\nabla})\hat{\mathbf{u}} \tag{3.9}$$

$$\hat{\nabla}\cdot\hat{\mathbf{u}} = 0 \tag{3.10}$$

$$\hat{\nabla}\cdot\hat{\mathbf{B}} = 0 \tag{3.11}$$

$$\frac{\partial\hat{\eta}}{\partial\hat{t}} + \hat{u}\frac{\partial\hat{\eta}}{\partial\hat{x}} + v\frac{\partial\hat{\eta}}{\partial\hat{y}} = \hat{w} \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t},\hat{x},\hat{y}) \tag{3.12}$$

$$\left[\hat{p} - \hat{\eta} - \frac{M}{2} + \frac{M|\hat{\mathbf{B}}|^2}{2}\right]_1^2 = 0 \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t},\hat{x},\hat{y}) \tag{3.13}$$

$$\hat{\mathbf{B}}\cdot\hat{\mathbf{n}} = 0 \quad \text{on} \quad \hat{\eta}(\hat{t},\hat{x},\hat{y}) \tag{3.14}$$

$$\hat{B}_z = 0 \quad \text{on} \quad \hat{z} = -1 \tag{3.15}$$

$$\hat{w} = 0 \quad \text{on} \quad \hat{z} = -1. \tag{3.16}$$

Here, $M = v_A^2/gh$, $v_A = B_0/\sqrt{\mu_0\rho_0}$, $\hat{\nabla} = (\partial_{\hat{x}}, \partial_{\hat{y}}, \partial_{\hat{z}})$ and

$$\frac{D}{D\hat{t}} = \frac{\partial}{\partial\hat{t}} + \hat{\mathbf{u}}\cdot\hat{\nabla}. \tag{3.17}$$

It should be noted that the quantity M is the square of the Froude number for the Alvén velocity. From now on, hats on variable will be dropped for the sake of simplicity. Let us expand, using the following perturbation:

$$\mathbf{u} = \varepsilon\mathbf{u}_1 + o(\varepsilon), \quad \mathbf{B} = \mathbf{e}_x + \varepsilon\mathbf{B}_1 + o(\varepsilon), \quad p = \varepsilon p_1 + o(\varepsilon), \quad \eta = \varepsilon\eta_1 + o(\varepsilon).$$

The linearised equations are then:

$$\frac{\partial\mathbf{u}_1}{\partial t} = -\nabla\left(p_1 + MB_{1x}\right) + M\frac{\partial\mathbf{B}_1}{\partial x}, \tag{3.18}$$

$$\frac{\partial\mathbf{B}_1}{\partial t} = \frac{\partial\mathbf{u}_1}{\partial x}, \tag{3.19}$$

$$\nabla\cdot\mathbf{u}_1 = 0, \tag{3.20}$$

$$\nabla\cdot\mathbf{B}_1 = 0, \tag{3.21}$$

$$p_1 - \eta_1 + MB_{1x} = 0 \quad \text{on} \quad z = 0, \tag{3.22}$$

$$\frac{\partial\eta_1}{\partial t} = w_1 \quad \text{on} \quad z = 0, \tag{3.23}$$

$$B_{1z} = \frac{\partial\eta}{\partial x} \quad \text{on} \quad z = 0 \tag{3.24}$$

$$B_{1z} = 0 \quad \text{on} \quad z = -1, \tag{3.25}$$

$$w_1 = 0 \quad \text{on} \quad z = -1. \tag{3.26}$$

It is common practice to express the perturbations as Fourier transforms:

$$f(t,x,y) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(\omega,k,l)e^{i(kx+ly-\omega t)}dkdl. \tag{3.27}$$

In order to derive now the dispersion relation that will give information about the waves to be allowed to propagate. The way to obtain a linear equation for the free surface it is necessary to first derive an equation for the component B_z component of the magnetic field:

$$\frac{\partial^2 \widehat{B_z}}{\partial z^2} - |\mathbf{k}|^2 \widehat{B_z} = 0. \tag{3.28}$$

From here, using the appropriate boundary conditions, it is now simple to obtain the required dispersion relation:

$$\omega^2 = Mk^2 + |\mathbf{k}| \tanh |\mathbf{k}|, \quad \mathbf{k} = k\mathbf{e}_x + l\mathbf{e}_y. \tag{3.29}$$

The dispersion relation can be expanded for long waves $|\mathbf{k}| \ll 1$ to obtain:

$$\omega k = k^2 + \frac{M}{2}k^2 - \frac{k^4}{6} + \frac{l^2}{2}, \tag{3.30}$$

which gives rise to a linear PDE using $\omega \mapsto -i\partial_t, k \mapsto i\partial_x, l \mapsto i\partial_y$:

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \left(1 + \frac{M}{2}\right)\frac{\partial \eta}{\partial x} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3}\right] + \frac{1}{2}\frac{\partial^2 \eta}{\partial y^2} = 0. \tag{3.31}$$

This explains why the k^4 term was kept. The dispersion relation gives rise to the linear part of the final weakly nonlinear equation, had it been dropped, a mismatch of the final weakly nonlinear would have occurred. On the other hand, for short wavelength waves, the approximation $\tanh |\mathbf{k}| = 1$ is applicable, and the dispersion relation becomes:

$$\omega^2 = Mk^2 + \sqrt{k^2 + l^2}. \tag{3.32}$$

This dispersion relation gives rise to the following PDE:

$$\frac{\partial^2 \eta}{\partial t^2} = M\frac{\partial^2 \eta}{\partial x^2} - \mathscr{P}(\eta) \tag{3.33}$$

Where:

$$\hat{\mathscr{P}}(\hat{\eta}) = \sqrt{k^2 + l^2}. \tag{3.34}$$

Both the obtained governing equations (3.31 & 3.33) are the linear part of the weakly nonlinear equations to be determined in later sections. In Benjamin (1992), the weakly nonlinear equation was obtained by noting that the linear equation (Equation 3.31) is a limiting case of the weakly nonlinear version. This paper will derive the equations from a formal perspective.

It can be seen, here or in Figures 2-3 for a graphical solution, that the short waves are only weakly dispersive as $\omega \sim \sqrt{M}k + o(k)$ and the $\sqrt{k^2 + l^2}$ gives the weak dispersion whereas for the long linear MHD surface waves, there is no dispersion depending on the wave vector for a large part of the domain.

4. Magnetic KP Equation

The linear free surface profile, $\eta_1(x, y)$ arising from a moving pressure distribution moving with speed U say can be computed using the techniques of section 3. One can plot $\eta_1(x, 0)$ against U/U_min and see a singularity appear at $U/U_min = 1$, see (Hunt (2013), section 2.3.2). Therefore, it is necessary to move to weakly nonlinear theory. The derivation will follow the method laid out by Johnson (1997). Starting with the equations (2.1)-(2.8), introduce the following scaling

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = h\hat{z}, \quad t = \frac{L}{\sqrt{gh}}\hat{t}, \quad u = \sqrt{gh}\hat{u}, \tag{4.1}$$

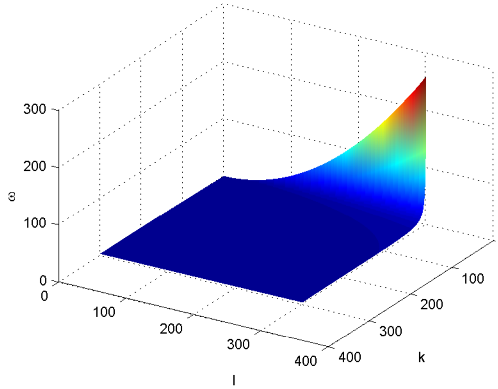

Figure 2: Long Waves

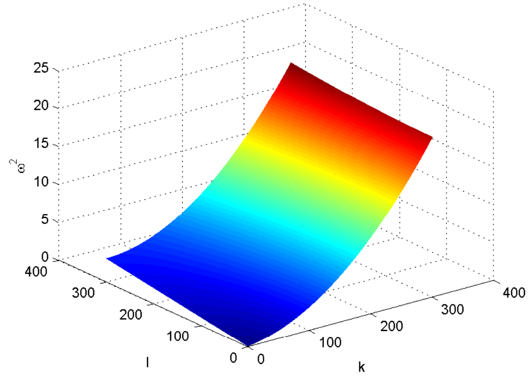

Figure 3: Short Waves

$$v = \sqrt{gh}\hat{v}, \quad w = \frac{h\sqrt{gh}}{L}\hat{w}, \quad \mathbf{B} = B_0\hat{\mathbf{B}}, \quad \eta = a\hat{\eta}. \tag{4.2}$$

As the perturbations in the free surface are going to be small it follows that there can really be no large quantities. To this end the pressure is scaled $p \mapsto \varepsilon p$ in order to obtain a consistent system of equations. The process of choosing the scaling which yields the most terms is often referred to as the *principle of least degeneration*. In the process of non-dimensionalsation, there are two parameters which appear:

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{L^2}. \tag{4.3}$$

In order to balance the nonlinearity and the dispersion take $\alpha = \beta = \varepsilon$ and make the following change of variables, as suggested by Johnson (1997):

$$X = \hat{x} - \hat{t}, \quad T = \varepsilon\hat{t}, \quad y = \sqrt{\varepsilon}Y, \quad v = \sqrt{\varepsilon}V, \quad M = \varepsilon\bar{M}. \tag{4.4}$$

Implicitly, $M \ll 1$, which implies that $B_0^2/\mu\rho \ll gh$ which means that the magnetic field must be small. The equations (2.1)-(2.8) now become:

$$\frac{\partial u}{\partial X} + \varepsilon\frac{\partial V}{\partial Y} + \frac{\partial w}{\partial z} = 0, \tag{4.5}$$

$$\varepsilon\frac{\partial u}{\partial T} - \varepsilon\frac{\partial u}{\partial X} + \varepsilon^2\left(u\frac{\partial u}{\partial X} + \varepsilon V\frac{\partial u}{\partial Y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial}{\partial X}\left(\varepsilon p + \frac{\varepsilon\bar{M}|\mathbf{B}|^2}{2}\right) + \\ + \varepsilon\bar{M}\left(B_x\frac{\partial B_x}{\partial X} + \sqrt{\varepsilon}B_y\frac{\partial B_x}{\partial Y} + B_z\frac{\partial B_x}{\partial z}\right), \tag{4.6}$$

$$\varepsilon^{\frac{5}{2}}\frac{\partial v}{\partial T} - \varepsilon^{\frac{3}{2}}\frac{\partial V}{\partial X} + \varepsilon^{\frac{5}{2}}\left(u\frac{\partial V}{\partial X} + \varepsilon V\frac{\partial V}{\partial Y} + w\frac{\partial V}{\partial z}\right) = -\sqrt{\varepsilon}\frac{\partial}{\partial Y}\left(\varepsilon p + \frac{\varepsilon\bar{M}|\mathbf{B}|^2}{2}\right) + \\ + \varepsilon\bar{M}\left(B_x\frac{\partial B_y}{\partial X} + \sqrt{\varepsilon}B_y\frac{\partial B_y}{\partial Y} + B_z\frac{\partial B_y}{\partial z}\right), \tag{4.7}$$

$$\varepsilon^3\frac{\partial w}{\partial T}-\varepsilon^2\frac{\partial w}{\partial X}\left(u\frac{\partial w}{\partial X}+\sqrt{\varepsilon}V\frac{\partial w}{\partial Y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial}{\partial z}\left(\varepsilon p+\frac{\varepsilon\bar{M}|\mathbf{B}|^2}{2}\right)+$$
$$+\varepsilon\bar{M}\left(B_x\frac{\partial B_z}{\partial X}+\varepsilon B_z\frac{\partial B_y}{\partial Y}+B_z\frac{\partial B_z}{\partial z}\right), \quad (4.8)$$

$$\varepsilon\frac{\partial B_x}{\partial T}-\frac{\partial B_x}{\partial X}=\varepsilon^{\frac{3}{2}}\frac{\partial}{\partial Y}(uB_y-\sqrt{\varepsilon}VB_x)-\sqrt{\varepsilon}\frac{\partial}{\partial z}(\sqrt{\varepsilon}wB_x-uB_z), \quad (4.9)$$

$$\varepsilon\frac{\partial B_y}{\partial T}-\frac{\partial B_y}{\partial X}=\varepsilon\frac{\partial}{\partial z}(VB_z-wB_y)-\varepsilon\frac{\partial}{\partial X}(uB_y-\sqrt{\varepsilon}VB_x), \quad (4.10)$$

$$\varepsilon\frac{\partial B_z}{\partial T}-\frac{\partial B_z}{\partial X}=\varepsilon\frac{\partial}{\partial z}(\sqrt{\varepsilon}wB_x-uB_z)-\varepsilon^2\frac{\partial}{\partial Y}(VB_z-wB_y), \quad (4.11)$$

$$\frac{\partial B_x}{\partial X}+\sqrt{\varepsilon}\frac{\partial B_y}{\partial Y}+\frac{1}{\sqrt{\varepsilon}}\frac{\partial B_z}{\partial z}=0, \quad (4.12)$$

$$\varepsilon p-\varepsilon\eta-\frac{\varepsilon\bar{M}}{2}+\frac{\varepsilon\bar{M}|\mathbf{B}|^2}{2}=0, \quad z=\varepsilon\eta(T,X,Y), \quad (4.13)$$

and

$$\varepsilon\frac{\partial\eta}{\partial T}-\frac{\partial\eta}{\partial X}+\varepsilon u\frac{\partial\eta}{\partial X}+\varepsilon^2\frac{\partial\eta}{\partial Y}=w, \quad z=\varepsilon\eta(T,X,Y). \quad (4.14)$$

The nonlinear governing equations are perturbed as:

$$p,u,V,w,\eta=p_0,u_0,V_0,w_0,\eta_0+\varepsilon p_1,\varepsilon u_1,\varepsilon V_1,\varepsilon w_1,\varepsilon\eta_1+o(\varepsilon), \quad (4.15)$$
$$B_x=1+\varepsilon B_{1x}+\varepsilon^2B_{2x}+o(\varepsilon^2), \quad (4.16)$$
$$B_y,B_z=\varepsilon^{\frac{3}{2}}B_{1y},\varepsilon^{\frac{3}{2}}B_{1z}+o(\varepsilon^{\frac{3}{2}}). \quad (4.17)$$

Considering the lowest order $O(\varepsilon)$ of equations (2.1)-(2.8) equations, we obtain:

$$\frac{\partial u_0}{\partial X}+\frac{\partial w_0}{\partial Z}=0, \quad \frac{\partial u_0}{\partial X}=\frac{\partial p_0}{\partial X}, \quad \frac{\partial p_0}{\partial z}=0, \quad \frac{\partial B_{1x}}{\partial X}=\frac{\partial w_0}{\partial z}, \quad (4.18)$$

$$\frac{\partial V_0}{\partial X}=\frac{\partial p_0}{\partial Y}, \quad \frac{\partial B_{1y}}{\partial X}=\frac{\partial V_0}{\partial X}, \quad \frac{\partial B_{1z}}{\partial X}=\frac{\partial w_0}{\partial X}. \quad (4.19)$$

The boundary conditions read as:

$$p_0=\eta_0, \quad w_0=-\frac{\partial\eta_0}{\partial X}, \quad \text{on} \quad z=0. \quad (4.20)$$

The solutions of these equations in dimensionless form are:

$$p_0=u_0=-B_{1x}=\eta_0, \quad w_0=-(1+z)\frac{\partial\eta_0}{\partial X}, \quad \frac{\partial V_0}{\partial X}=\frac{\partial\eta_0}{\partial Y}. \quad (4.21)$$

The next order equations are straightforward:

$$\frac{\partial u_1}{\partial X}+\frac{\partial V_0}{\partial Y}+\frac{\partial w_1}{\partial z}=0, \quad (4.22)$$
$$\frac{\partial\eta_0}{\partial T}-\frac{\partial u_1}{\partial X}+\eta_0\frac{\partial\eta_0}{\partial X}=-\frac{\partial p_1}{\partial X}, \quad (4.23)$$
$$-(z+1)\frac{\partial^2\eta_0}{\partial X^2}=\frac{\partial p_1}{\partial z}. \quad (4.24)$$

Along with the boundary conditions:

$$p_1 + \bar{M}B_{1x} = \eta_1, \quad \frac{\partial \eta_0}{\partial T} - \frac{\partial \eta_1}{\partial X} + 2\eta_0 \frac{\partial \eta_0}{\partial X} = w_1. \tag{4.25}$$

Integrating equation (4.24) and using the first equation of (4.25) shows that:

$$p_1 = \frac{1}{2}\left[1 - (1+z)^2\right]\frac{\partial^2 \eta_0}{\partial X^2} + \eta_1 + \bar{M}\eta_0. \tag{4.26}$$

To find w_1, insert (4.26) into (4.23), substitute $\partial_X u_1$ into (4.22), and, integrate it w.r.t. z to have:

$$w_1 = -\frac{1}{3}\frac{\partial^3 \eta_0}{\partial X^3} - \frac{\partial \eta_1}{\partial X} + \eta_0 \frac{\partial \eta_0}{\partial X} - \bar{M}\frac{\partial \eta_0}{\partial X} - \frac{\partial \eta_0}{\partial T} - \frac{\partial V_0}{\partial Y}. \tag{4.27}$$

The appearance of the $\partial_X^3 \eta_0$ term shows why the k^4 term must be kept in the dispersion relation. Inserting (4.27) into the second equation of (4.25) and differentiating w.r.t. X gives:

$$\frac{\partial}{\partial X}\left[\frac{\partial \eta_0}{\partial T} + \frac{3}{2}\eta_0\frac{\partial \eta_0}{\partial X} + \frac{\bar{M}}{2}\frac{\partial \eta_0}{\partial X} + \frac{1}{6}\frac{\partial^3 \eta_0}{\partial X^3}\right] + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial Y^2} = 0. \tag{4.28}$$

It may be noted that a co-ordinate transformation may remove the term $\bar{M}\partial_X \eta_0/2$. This term may be regarded as setting the correct wave speed. After applying the co-ordinate transformation (4.28) reduces to the well-known Kadomtsev-Petviashvilli(KP) equation. This is done by moving back into the scaled co-ordinates defined by:

$$X = \hat{x} - \hat{t}, \quad T = \varepsilon \hat{t}, \quad Y = \frac{\hat{y}}{\sqrt{\varepsilon}} \tag{4.29}$$

KP is known to be integrable and has analytical solutions (Johnson (1997)). From these solutions of the KP equation one can now obtain solutions to (4.28) via co-ordinate transformation.

5. Magnetic Fields in Both Regions

Consider now the problem described in Sec. 1 with a magnetic field in both regions. Region 1 is defined by $\{(x,y,z)|z > \eta(t,x,y,z)\}$, and, region 2 is defined by $\{(x,y,z)| - h \leqslant z \leqslant \eta(t,x,y,z)\}$. There is no current in region 2, and so $\nabla \times \mathbf{B}_2 = \mathbf{0}$, i.e. it can be stated that $\mathbf{B}_2 = \nabla \Phi$. The two boundary conditions which must be obeyed are:

$$\mathbf{B}_2 \to \mathbf{B}_{2,0} \quad \text{as} \quad z \to \infty \tag{5.1}$$

$$\mathbf{B_2} \cdot \hat{\mathbf{n}} = 0 \quad z = \eta(t,x,y) \tag{5.2}$$

The magnetic field in region 2 can be written in the following manner:

$$\mathbf{B}_2 = \mathbf{B}_{2,0} + \nabla \Phi \tag{5.3}$$

to yield the equation:

$$\nabla^2 \Phi = 0 \tag{5.4}$$

One of the boundary conditions which links the magnetic field in region 1 to the magnetic field of region 2 is:

$$\left[p + \frac{|\mathbf{B}|^2}{2\mu}\right]_1^2 = 0. \tag{5.5}$$

The boundary condition $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ reduces to the following:

$$\mathbf{B}_{1,0} \cdot \mathbf{e}_x \partial_x \eta = \mathbf{B}_{1,0} \cdot \mathbf{e}_z, \tag{5.6}$$

yielding a lower boundary condition.

5.1. *Linear Theory*

5.1.1. *Derivation of Dispersion Relation*

A dispersion relation is sought to gain insight into how waves will propagate in a magnetised fluid system. The derivation is very similar to the previous case with the magnetic field confined to the fluid. The equilibrium condition for the magnetic field in region 1 is given by $\mathbf{B} = B_1\mathbf{e}_x$. The scaling for the linear case is the same as in the previous case in section 3. The magnetic field in region 1 is scaled as $\mathbf{B} = B_1\hat{\mathbf{B}}$. The equilibrium pressure, in this case, is determined as before. The general kinetic pressure is $p = -\rho_0 g z + C$. The continuity of total pressure now reads:

$$p(0) + \frac{B_{1,0}^2}{2\mu_1} = p_a + \frac{B_{2,0}^2}{2\mu_2}. \tag{5.7}$$

The scaled equations are then:

$$\begin{aligned}
\frac{D\hat{\mathbf{u}}}{D\hat{t}} &= -\hat{\nabla}\left(\hat{p} + \frac{M_2|\hat{\mathbf{B}}_2|^2}{2}\right) + M_2(\hat{\mathbf{B}}_2 \cdot \hat{\nabla})\hat{\mathbf{B}}_2 \quad -1 \leqslant \hat{z} \leqslant \hat{\eta}(\hat{t}, \hat{x}, \hat{y}),\\
\frac{D\hat{\mathbf{B}}_2}{D\hat{t}} &= (\hat{\mathbf{B}}_2 \cdot \hat{\nabla})\hat{\mathbf{u}} \quad -1 \leqslant \hat{z} \leqslant \hat{\eta}(\hat{t}, \hat{x}, \hat{y}),\\
\hat{\nabla} \cdot \hat{\mathbf{u}} &= 0 \quad -1 \leqslant \hat{z} \leqslant \hat{\eta}(\hat{t}, \hat{x}, \hat{y}),\\
\hat{\nabla} \cdot \hat{\mathbf{B}}_2 &= 0 \quad -1 \leqslant \hat{z} \leqslant \hat{\eta}(\hat{t}, \hat{x}, \hat{y}),\\
\nabla^2\hat{\Phi} &= 0 \quad \hat{\eta}(\hat{t}, \hat{x}, \hat{y}) < \hat{z} < \infty,\\
\frac{\partial\hat{\eta}}{\partial\hat{t}} + \hat{u}\frac{\partial\hat{\eta}}{\partial\hat{x}} + v\frac{\partial\hat{\eta}}{\partial\hat{y}} &= \hat{w} \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t}, \hat{x}, \hat{y}),\\
\left[\hat{p} - \hat{\eta} + \frac{M|\hat{\mathbf{B}}|^2}{2}\right]_1^2 &= 0 \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t}, \hat{x}, \hat{y}).\\
\hat{B}_z &= 0 \quad \text{on} \quad \hat{z} = -1,\\
\hat{w} &= 0 \quad \text{on} \quad \hat{z} = -1,\\
\frac{\partial_{\hat{x}}\eta}{\sqrt{1 + (\partial_{\hat{x}}\hat{\eta})^2}} &= \hat{B}_{1z} \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t}, \hat{x}, \hat{y}).
\end{aligned}$$

The notation used here is:

$$M_1 = \frac{B_1^2}{gh\mu_1\rho_2}, \quad M_2 = \frac{B_2^2}{gh\mu_2\rho_2} \tag{5.8}$$

Note that in region 2 the magnetic field scales as the following:

$$\hat{\mathbf{B}}_2 = 1 + \varepsilon\nabla\hat{\Phi}, \tag{5.9}$$

where the variation from the constant field should not be large. The linearisation is now analogous to the previous case in Section 3. The no flux condition reduces to:

$$\hat{\mathbf{B}}_{1z} = \partial_x\eta = \hat{\mathbf{B}}_{2z} \tag{5.10}$$

which is equivalent to:

$$B_{1z} = B_{2z} \quad \text{on} \quad \hat{z} = \hat{\eta}(\hat{t}, \hat{x}, \hat{y}). \tag{5.11}$$

Again, it is assumed the perturbations can be expanded as Fourier transforms and an

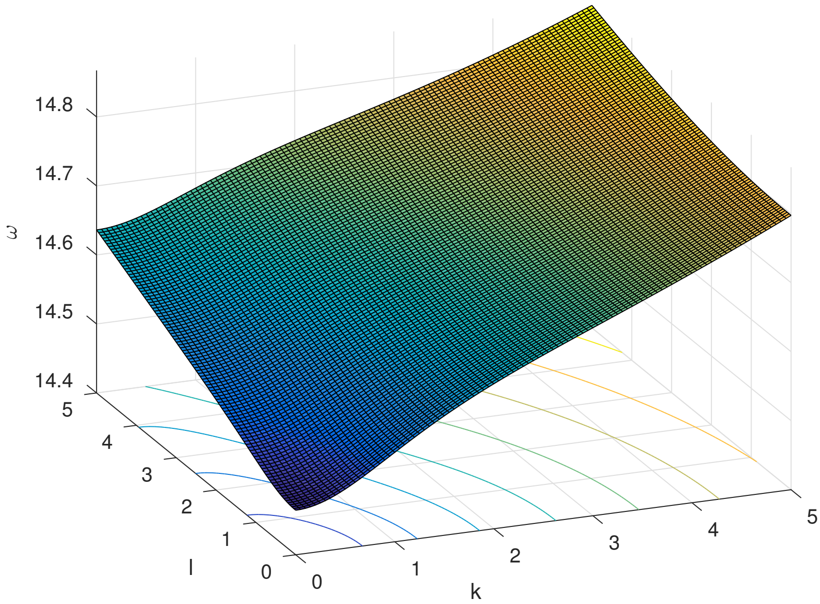

Figure 4: Dispersion Relation for Infinite β

equation for B_{1z} is found which is mathematically similar to that found previously (equation 3.28). Finally, the corresponding dispersion relation is then:

$$\omega^2 = M_2 k^2 + |\mathbf{k}| \tanh |\mathbf{k}| + M_1 k^2 \tanh |\mathbf{k}|. \tag{5.12}$$

The long wave approximation of the dispersion relation is:

$$\omega k = \left(1 + \frac{M_2}{2}\right) k^2 + \frac{l^2}{2} - \frac{k^4}{6} + M_1 k^2 |k|. \tag{5.13}$$

Note that this dispersion relation is only valid when $M_1 \ll 1$, as otherwise it can be ignored. The requirement that $M_2 \ll 1$ was required for mathematical consistency. To obtain a result where $M_1, M_2 \not\ll 1$ would require different techniques employed in this paper. The approach taken here for the small magnetic field is analagous is the assumption of small surface tension in Benjamin (1992). This dispersion relation has appeared several times in the literature before. The first was in Kim & Akylas (2006) and the second was in Hunt (2013). The form of the equations is exactly the same but the coefficients are different reflecting the application. Upon using the substitutions $\omega \mapsto -i\partial_t, k \mapsto i\partial_x, l \mapsto i\partial_y$ as before, the following linear PDE is obtained governing long-wavelength *linear* MHD waves propagating along a contact discontinuity in magnetised plasma:

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + \left(1 + \frac{M_2}{2}\right)\frac{\partial \eta}{\partial x} + \frac{1}{6}\frac{\partial^3 \eta}{\partial x^3} + M_1 \mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right)\right] + \frac{1}{2}\frac{\partial^2 \eta}{\partial y^2} = 0. \tag{5.14}$$

This equation represents the linear part to the weakly nonlinear equation in the small amplitude and long wavelength. The nonlinear part of the equation is derived in section 4.

5.1.2. *Wave Profiles from a Moving Pressure Distribution*

This section demonstrates a linear 2D wave with a moving pressure distribution to give an idea of some of the linear wave profiles. It is possible to obtain wave profiles for steady linear waves by the use of a moving pressure distribution, $P(x, y)$ on the surface of the plasma. In order to do this move to a frame where the plasma is moving uniformly with a velocity $\mathbf{u} = U\mathbf{e}_x$. The scaling for the linear theory is just the same as in Section 3 given by (3.6-3.7) and as a result the scaled velocity is perturbed from $F = U/\sqrt{gh}$. The linearised governing equations become:

$$
\begin{aligned}
F\frac{\partial \mathbf{u}_1}{\partial x} &= -\nabla(p_1 + M_2 B_{2x}) + M_2\frac{\partial \mathbf{B}_2}{\partial x},\\
F\frac{\partial \mathbf{B}_2}{\partial x} &= \frac{\partial \mathbf{u}_2}{\partial x},\\
\nabla\cdot\mathbf{B}_2 &= 0,\\
\nabla\cdot\mathbf{u}_2 &= 0,\\
F\frac{\partial \eta_1}{\partial x} &= u_{1z} \quad \text{on} \quad z = 0,\\
p_1 - \eta_1 + M_2 B_{2x} &= \hat{P} + M_1 B_{1x} \quad \text{on} \quad z = 0,\\
B_{2z} &= 0 \quad \text{on} \quad z = -1,\\
w_1 &= 0 \quad \text{on} \quad z = -1,\\
B_{1z} &= B_{2z} \quad \text{on} \quad z = 0.
\end{aligned}
$$

Following the previous linear calculations given in section 3, an equation is now derived for the z-component of the magnetic field equation (3.28). From the solution of equation (3.28) all other variables can be easily obtained. Finally, in terms of the moving pressure on the surface, the Fourier transform for the free surface is:

$$
\hat{\eta} = \frac{|\mathbf{k}|\hat{P}}{(F^2k^2 - M_2k^2)\coth(|\mathbf{k}|) + |\mathbf{k}|(M_1k^2 - 1))} \tag{5.15}
$$

There is an interesting feature of figure 5 in the number of ripples in the wave, which is suggestive the the decay is algebraic. It can be shown that increasing the Froude number to the minimum of the dispersion relation, the amplitude becomes unbounded which is why the need for weakly nonlinear theory.

5.2. *Weakly Nonlinear Theory*

Consider weakly nonlinear theory of MHD surface wave propagation in the equilibrium defined by (figure 1) Scale the magnetic fields in the following way, following Johnson (1997): $\mathbf{B}_1 = B_{10}\hat{\mathbf{B}}_1$ and $\mathbf{B}_2 = B_{20}\hat{\mathbf{B}}_2$. Expand the pressure in region 1 as

$$
p = p_a - \rho_0 g z - \frac{B_{10}^2}{2\mu_1} + \frac{B_{20}^2}{2\mu_2} + \rho_0 g h \hat{p}. \tag{5.16}
$$

The total pressure becomes:

$$
p + C - \varepsilon\hat{\eta} + \frac{M_1|\hat{\mathbf{B}}_1|^2}{2} = \frac{M_2|\hat{\mathbf{B}}_2|^2}{2}, \tag{5.17}
$$

where $M_{1,2} = v_{A1,2}^2/gh$. Here $v_{A1,2} = B_{1,20}^2/2\rho\mu_{1,2}$. The transformation $p \mapsto \varepsilon p$ is used. For brevity, drop again the hats and set $M_{1,2} = \varepsilon\bar{M}_{1,2}$ resulting in

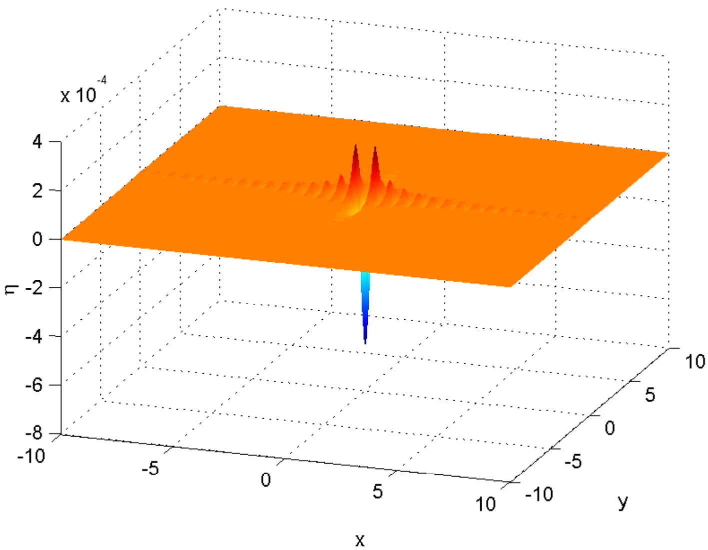

Figure 5: Travelling Linear Wave, $F = 1, M_2 = 1, M_1 = 2$

$$\varepsilon p - \varepsilon\eta + \frac{\varepsilon \bar{M}_1 |\mathbf{B}_1|^2}{2} = \frac{\varepsilon \bar{M}_2 |\mathbf{B}_2|^2}{2}. \tag{5.18}$$

The governing equation for the magnetic field in region 1 becomes:

$$\frac{\partial^2 \Phi}{\partial X^2} + \varepsilon \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{5.19}$$

This reduces the equation into a 2D Laplace equation, differentiating w.r.t. z yields a Laplace equation for B_{1z}, with lower boundary condition $B_{1z}(z = 0) = \partial_x \eta$. The solution for this problem is:

$$B_{1z} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{z \partial_X \eta}{(X - X')^2 + z^2} dX' \tag{5.20}$$

To find B_{1X}, one integrates w.r.t. z and then differentiates w.r.t. X and set $z = 0$ to obtain:

$$B_{1X}(z = 0) = -\mathscr{H}(\partial_X \eta) \tag{5.21}$$

Differentiating the above w.r.t. X and inserting it in eq. (4.19), followed by setting $z = 0$ shows that:

$$\frac{\partial B_{1X}}{\partial X} = \mathscr{H}\left(\frac{\partial^2 \eta_0}{\partial X^2}\right). \tag{5.22}$$

Just as in Section 4, perturbation of the constant magnetic field in the $\mathbf{e}_x$-direction are introduced. Expand as

$$p = p_0 + \varepsilon p_1 + o(\varepsilon), \tag{5.23}$$
$$\eta = \eta_0 + \varepsilon \eta_1 + o(\varepsilon), \tag{5.24}$$
$$B_{1x,2x} = 1 + \varepsilon B_{1x1,2x1} + o(\varepsilon). \tag{5.25}$$

Gathering the expanded terms results in

$$\varepsilon(p_0 + \varepsilon p_1) - \varepsilon(\eta_0 + \varepsilon\eta_1) + \frac{\varepsilon \bar{M}_1}{2}(1 + 2\varepsilon B_{1x1}) = \frac{\varepsilon \bar{M}_2}{2}(1 + 2\varepsilon B_{2x1}). \tag{5.26}$$

The part of this equation with $O(\varepsilon)$ is:

$$p_0 - \eta_0 = 0. \tag{5.27}$$

The next order, $O(\varepsilon^2)$, is:

$$p_1 - \eta_1 + \bar{M}_1 B_{1x1} = \bar{M}_{2x1}. \tag{5.28}$$

From this point onwards a mathematically analogous analysis can be carried out as in the one-field case (see section 4). The final governing equation is:

$$\frac{\partial}{\partial X}\left[\frac{\partial\eta_0}{\partial T} + \frac{\bar{M}_1}{2}\frac{\partial\eta_0}{\partial X} + \frac{3}{2}\eta_0\frac{\partial\eta_0}{\partial X} + \frac{1}{2}\frac{\partial^3\eta_0}{\partial X^3} + \frac{\bar{M}_2}{2}\mathscr{H}\left(\frac{\partial^2\eta_0}{\partial X^2}\right)\right] + \frac{1}{2}\frac{\partial^2\eta_0}{\partial Y^2} = 0. \tag{5.29}$$

The one-dimensional version of this equation, known as the Benjamin equation, was derived in Benjamin (1992) using a different method than the one used here. The 2D Benjamin equation has appeared in several instances, in interfacial flows examined in Kim & Akylas (2006) and in electrohydrodynamics, Hunt (2017).

6. Discussions and Conclusions

The new result presented in this paper is the derivation of an equation that describes the derivation of an equation that describes the weakly nonlinear wave propagation in 2+1 dimensions. The equation has been investigated elsewhere in different circumstances. Nevertheless, some important results have been found by other authors. In Zaiter (2009), the author proves existence of solutions of (5.29) as well as the algebraic decay of solutions. In (Kim & Akylas (2006)), they study bifurcation of steady lump solutions and also prove instability results concerning transverse stability. As of writing, there are no known analytical solutions to the 1D Benjamin equation.

A numerical solution of the Benjamin equation would provide valuable insight but that is not within the scope of the present paper.

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