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Measuring Dependence in Uncertainty Should Start in the Introduction to Probability and Statistics

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Outline

- Some previous results on measuring dependence
- General observations and rules
- Illustrations
  - Non-numeric variables
  - Numeric Variables -Dependence in Bivariate Distributions
- More Illustrations
  - Dependence in Politics
  - Local dependence in reliability systems

Introduction

- In several publications we noticed an idea how probability tools can be used to measure strength of dependence between random events
- In the present talk we propose to use it for measuring magnitude of local dependences between random variables.
- As illustration, we demonstrate how it works as a measuring tool in the complicated world of politics and in reliability models.
- Short illustration is discussed on the use of these measures in already known previously popular results for non-numeric uncertain variables.

How to INDICATE dependence?

The dependence in the world of the uncertainty is a complex concept.

- Textbooks avoid discussions in this regard.
- **In the classical approach conditional probability is used to determine if two events are dependent, or not:** *A* and *B* are independent when the probability for their joint occurrence equals to the product of the probabilities for their individual appearance, i.e. when

$$P(A \cap B) = P(A).P(B)$$

- **Otherwise, the two events are dependent.**

How to measure dependence?

- **To measure dependence between random events**

B. Dimitrov (2010, Some Obreshkov Measures of Dependence and Their Use, *Compte Rendus de l'Acad. Bulgare des Sci., v.* ***63, No.1, pp. 15-18)***

revived some measures of dependence for random events based on notion of probabilities of the events.

From that discussion and among the four proposed measures we selected the Regression coefficients as suitable measure of magnitude of dependence when the two events are dependent.

Regression Coefficients as *Measures of dependence between random events*

- **Definition 1.** ***Regression coefficient*** of the event ***A*** with respect to the event ***B*** is called the difference between the conditional probability for the event ***A*** given the event ***B***, and the conditional probability for the event *A* given the complementary event $\overline{B}$, namely

$$R_B(A) = P(A \mid B) - P(A \mid \overline{B})$$

- This ***measure of the dependence of the event A on the event B*, is *directed dependence*.**

- The regression coefficient is always defined, for any pair of events ***A*** and ***B*** (zero, sure, arbitrary).

- The regression coefficient of ***B* with respect to *A is defined symmetrically***

Properties of **Regression coefficients**

- (r1) The equality to zero $R_B(A)$ =$R_A(B)$= 0 holds only if the events ***A*** and ***B*** are independent.
- (r2) $R_A(A) = 1$; $R_{\overline{A}}(A) = -1$.
- (r3) $R_B(\sum A_j) = \sum R_B(A_j)$
- (r4) $R_S(A) = R_{\varnothing}(A) = 0$
- (r5) The regression coefficients are numbers with equal signs
- To be valid $R_B(A)$= $R_A(B)$ it is necessary and sufficient to have

$$P(A)[1 - P(A)] = P(B)[1 - P(B)]$$

Regression coefficients as measure of dependence between random events.

- **The relations**

$$R_B(A) = \frac{P(A \cap B) - P(A)P(B)}{P(B)[1 - P(B)]}$$

and

$$R_A(B) = \frac{P(A \cap B) - P(A)P(B)}{P(A)[1 - P(A)]}$$

explain when it will be $R_B(A) = R_A(B)$.

These properties, and next, may be used as exercises in the classroom.

Regression coefficients - properties

(r6) The regression coefficients and are numbers between –1 and 1, i.e. they satisfy the inequalities

$$-1 \leq R_B(A) \leq 1 \qquad -1 \leq R_A(B) \leq 1$$

(r6.1) The equality $\boldsymbol{R_B(A)}$ = 1 holds only when ***A* coincides** (is equivalent) **with the event *B***.

Then is also valid the equality $\boldsymbol{R_A(B)}$ =1;

(r6.2) The equality $\boldsymbol{R_B(A)}$ = - 1 holds only when event ***A*** **coincides** (or is equivalent) with the event $\overline{B}$ - the **complement of *B*.**

Then is also valid $\boldsymbol{R_A(B)}$ = - 1, and respectively $\overline{A} = B$.

Regression coefficients – a proposition

- **In our opinion, it is possible one event to have stronger dependence on the other than the reverse.**

- **This measure suits for measuring the magnitude of dependence between events.**

- **The distance of the regression coefficient from the zero (where the independence is) could be used to classify the strength of dependence, e,g. (taken from some textbooks)**

  - **almost independent** (when $R_A(B) < .05$) ;
  - **weakly dependent** (when $.05 < |R_A(B)| < .2$) ;
  - **moderately dependent** (when $.2 < |R_A(B)| < .45$) ;
  - **in average dependent** (when $.45 < |R_A(B)| < .8$) ;
  - **strongly dependent** (when $|R_A(B)| > .8$) ;

4. Correlation between two random events

- **Definition 3.** Correlation coefficient between two events *A* and *B* is defined by the number

$$R_{A,B} = \pm\sqrt{r_B(A)\cdot r_A(B)}$$

Its sign, plus or minus, is the sign of either of the two regression coefficients.

- An equivalent representation

$$R_{A,B} = \frac{P(A\cap B)-P(A)P(B)}{\sqrt{P(A)P(\overline{A})}\sqrt{P(B)P(\overline{B})}}$$

Correlation (properties)

- *ρ1. It is fulfilled* $R_{A,B} = 0$ *if and only if the two events A and B are independent.*
- *ρ2. It is fulfilled* $-1 \leq R_{A,B} \leq 1$.
- **ρ2.1. The equality to 1 holds if and only if the events *A* and *B* are equivalent, i.e. when A = *B*.**
- **ρ2.2. The equality** $R_{A,B} = -1$ **holds if and only if the events *A* and** $\overline{B}$ **are equivalent**

4. Correlation Properties (continued)

- **ρ3.** The correlation coefficient has the same sign as the other measures of the dependence between two random events *A* and *B* , and this is the sign of the connection.

- **ρ4.** The knowledge of $R_{A,B}$ allows calculating the posterior probability of one of the events under the condition that the other one is occurred. For instance, *P*(*B* | *A*) will be determined by the rule

$$P(B \mid A) = P(B) + R_{A,B}\sqrt{\frac{P(\bar{A})P(B)P(\bar{B})}{P(A)}}$$

- The net increase, or decrease in the posterior probability compare to the prior probability equals to the quantity added to P(B), and depends only on the value of the mutual correlation.

4. Correlation (continued)

- $P(B \mid \overline{A})$ = $P(B)$ - $R_{A,B}\sqrt{\dfrac{P(A)P(B)P(\overline{B})}{P(\overline{A})}}$

- **ρ5. It is fulfilled** $R_{\overline{A},B}$ = $R_{A,\overline{B}}$ = - $R_{A,B}$; $R_{\overline{A},\overline{B}}$ = $R_{A,B}$

- **ρ6.** $R_{A,A} = 1;\quad R_{A,\overline{A}} = -1;\quad R_{A,S} = R_{A,\emptyset}$ =0

- **ρ7.** ***Particular Cases*.** When $A \subset B$, then

$R_{A,B} = \sqrt{\dfrac{P(A)P(\overline{B})}{P(\overline{A})P(B)}}$; If $A \cap B = \emptyset$, then $R_{A,B} = -\sqrt{\dfrac{P(A)P(B)}{P(\overline{A})P(\overline{B})}}$

4. Correlation (continued)

- The use of the numerical values of the correlation coefficient is similar to the use of the two regression coefficients.

- As closer is $R_{A,B}$ to the zero, as "closer" are the two events *A* and *B* to the independence.

- Let us note once again that $R_{A,B}$ *= 0 if and only if the two events are independent*.

4. Correlation (continued)

- As closer is $R_{A,B}$ to 1, as “dense one within the other” are the events *A* and *B*, and when $R_{A,B}$ = 1, the two events coincide (are equivalent).

- As closer is $R_{A,B}$ to -1, as “dense one within the other” are the events *A* and $\overline{B}$, and when $R_{A,B}$ = - 1 the two events coincide (are equivalent).

- These interpretations seem convenient when conducting research and investigations associated with qualitative (non-numeric) factors and characteristics.
- Such studies are common in sociology, ecology, jurisdictions, medicine, criminology, design of experiments, and other similar areas.

4. Correlation (continued)

- Freshe-Hoefding inequalities for the Correlation Coefficient

$$\max\left\{-\sqrt{\frac{P(A)P(B)}{P(\overline{A})P(\overline{B})}},-\sqrt{\frac{P(\overline{A})P(\overline{B})}{P(A)P(B)}}\right\}\leq R(A,B)\leq \min\left\{\sqrt{\frac{P(A)P(\overline{B})}{P(\overline{A})P(B)}},\sqrt{\frac{P(\overline{A})P(B)}{P(A)P(\overline{B})}}\right\}$$

4. Correlation (continued)

- *Example 1 (continued):* We have the numerical values of the two regression coefficients and from the previous section. In this way we get

$$R_{A,B} = \sqrt{(.3368)(.2174)} = .2706.$$

- Analogously to the use of the regression coefficients, the numeric value of the correlation coefficient could be used for classifications of the degree of the mutual dependence.

- **The correlation coefficient is a number in-between the two regression coefficients. It is symmetric and absorbs the misbalance (the asymmetry) in the two regression coefficients. It is a balanced measure of dependence between the two events.**

5. Empirical estimations

- The measures of dependence between random events are made of their probabilities. It makes them very attractive and in the same time **easy for statistical estimation and practical use**.

5. Empirical Estimations (contd)

- Let in N independent experiments (observations) the random event A occurs k_A times, the random event B occurs k_B times, and the event $A \cap B$ occurs $k_{A \cap B}$ times. Then statistical estimators of our measures of dependence will be respectively:

$$\hat{\delta}(A,B) = \frac{k_{A \cap B}}{N} - \frac{k_A}{N} \cdot \frac{k_B}{N}$$

5. Empirical Estimations (contd)

- The estimators of the two regression coefficients are

$$\hat{r}_A(B) = \frac{\frac{k_{A\cap B}}{N} - \frac{k_A}{N}\cdot\frac{k_B}{N}}{\frac{k_A}{N}(1-\frac{k_A}{N})}; \qquad \hat{r}_B(A) = \frac{\frac{k_{A\cap B}}{N} - \frac{k_A}{N}\cdot\frac{k_B}{N}}{\frac{k_B}{N}(1-\frac{k_B}{N})}$$

- The correlation coefficient has estimator

$$\hat{R}(A,B) = \frac{\frac{k_{A\cap B}}{N} - \frac{k_A}{N}\cdot\frac{k_B}{N}}{\sqrt{\frac{k_A}{N}(1-\frac{k_A}{N})\frac{k_B}{N}(1-\frac{k_B}{N})}}$$

5. Empirical Estimations (contd)

- These estimators may be simplified when the numerator and denominator are multiplied by appropriate quantity. We not go into detail.
- The estimators are all consistent; the estimator of the connection δ(A,B) is also unbiased, i.e. there is no systematic error in this estimate.
- The estimators can be used in practice with reasonable interpretations and explanations

6. Some warnings and some recommendations

- **The measures of dependence between random events are not transitive.**
- It is possible *A* to be positively associated with *B,* event *B* to be positively associated with *C*, but the *A* to be negatively associated with *C*.
- Example: *A* and *B* compatible (non-empty intersection); *B* and *C* compatible, *A* and *C* - mutually exclusive, and with a negative connection.
- For non-exclusive pairs (*A, B*) and (*B, C*) every kind of dependence is possible.
- More precisions at this point deserve attention.

6. Some recommendations (contd)

- One can use the measures of dependence to compare degrees of dependence.

- We recommend the use of Regression Coefficient for measuring degrees of dependence.
- For instance, let

$$| r_B(A) | \le | r_C(A) |$$

then we say that the event *A* has stronger association with *C* compare to its association with *B*.
- In this way some ranks of associations of a given event can be established for any collection of other events.

From Events to Random Variables

- The introduced measures allow **to see the interaction between** any pair of numeric **r.v.'s (X,Y) throughout the sample space**

- **Understand and use the local dependence.**

- **Let $F(x,y)=P(X \leq x, Y \leq y)$ - the joint c.d.f.**

- **Marginals $F(x) = P(X \leq x)$, $G(y)=P(Y \leq y)$.**

From Events to Random Variables

- Introduce the events
- $A=\{x \le X \le x+\Delta_1 x\}$; $B=\{y \le Y \le y+\Delta_2 y\}$,

for any $x, y \in (-\infty, \infty)$.

- **Then the measures of dependence between events *A* and *B* turn into *a measure of local dependence between the pair of r.v.'s X and Y on the rectangle***

$$D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y].$$

From Events to Random Variables

- Naturally, they can be named and calculated as follows:
- **Regression coefficient** of *X* with respect to *Y*, and of *Y* with respect to *X* on the rectangle D= [x, $x+\Delta_1 x$]×[y, $y+\Delta_2 y$]. By Definition 1 we get

$R_Y((X,Y)$ в $D)=$

$$\frac{\Delta_D F(x,y)-[F(x+\Delta_1 x)-F(x)][G(y+\Delta_2 y)-G(y)]}{[F(x+\Delta_1 x)-F(x)]\{1-[F(x+\Delta_1 x)-F(x)]\}}$$

- Here by $\boldsymbol{\Delta_D F(x,y)}$ is denoted the **two dimensional finite difference** for the function $F(x,y)$ on rectangle ***$D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$.***

From Events to Random Variables

- Namely

$$\Delta_D F(x,y) = F(x+\Delta_1 x, y+\Delta_2 y) - F(x+\Delta_1 x, y) - F(x, y+\Delta_2 y) + F(x, y).$$

- In an analogous way is defined $R_X((X,Y)$ **в** $D)$**.** Just denominator in the expression is changed respectively.

- **Correlation coefficient** $R_Y((X,Y)$ **в** $D)$ between the r.v.'s ***X*** **and *Y* on rectangle** $D=[x, x+\Delta_1 x]\times[y, y+\Delta_2 y]$ can be presented in similar way by the **use of Definition 2**. We omit detailed expressions as something obvious.

From Events to Random Variables

- The local dependence at a value (*X=i, Y=j*) **for a pair of discrete distributed r.v. (*X,Y*)**.
- **Regression coefficient** of *X* with respect to *Y*, and of *Y* with respect to *X* at a value (*X=i, Y=j*) is determined by the rule
- $R_Y(X=i,Y=j) =$

$$\frac{P(X=i,Y=j)-P(X=i)P(Y=j)}{P(X=i)[1-P(X=i)]}$$

- The local **correlation coefficient** values of the two r.v.'s

$$R_{X,Y}(X=i,Y=j) = \frac{p(i,j)-p_{i.}p_{.j}}{\sqrt{p_{i.}(1-p_{i.})}\sqrt{p_{.j}(1-p_{.j})}}$$

Categorical variables

As another illustration of the measures of dependence we analyze an example from Alan Agresti ***Categorical Data Analysis*, 2006.** The table represents data about the yearly income of people and the job satisfaction.

Job Satisfaction

Income US $$	Very Dissatisf	Little Satisfied	Moderately Satisfied	Very Satisfied	Total Marginally
< 6,000	20	24	80	82	206
6,000–15,000	22	38	104	125	289
15,000-25,000	13	28	81	113	235
> 25,000	7	18	54	92	171
Total	62	108	319	412	901

Categorical variables (continued)

Table 2: Join and marginal probability distributions (Income, Job Satisfaction) $P_{i,j}, P_{i..}, P_{.j}$

Job Satisfaction					
Income US $$	**Very Dissatisfied**	**Little Satisfied**	**Moderately Satisfied**	**Very Satisfied**	**Total (marginal) distribution**
< 6,000	.02220	.02664	.08879	.09101	.22864
6,000–15,000	.02442	.04217	.11543	.13873	.32075
15,000-25,000	.01443	.03108	.08990	.12542	.26083
> 25,000	.00776	.01998	.05993	.10211	.18978
Total (marginal) distribution	.06881	.11987	.35405	.45727	1.00000

Categorical Variables (Regr. Coeff. 1)

Tab 3: Empirical regression coefficient between particular levels of income and job satisfaction $r_{Satisfaction_j}(IncomeGroup_i)$

Income US $$	Job Satisfaction: Very Dissatisfied	Little Satisfied	Moderately Satisfied	Very Satisfied
< 6,000	0.100932704	-0.00727	0.034281	-0.05456
6,000–15,000	0.036663063	0.035276	0.00817	-0.03199
15,000-25,000	-0.05489976	-0.00176	-0.0107	0.024782
> 25,000	-0.08269601	-0.02625	-0.03175	0.061768

Categorical Variables (Regr. Coeff. 2)

Table 5: Empirical regression coefficients between particular levels of job sat income r_{Income} ***(Job Satisfaction)***

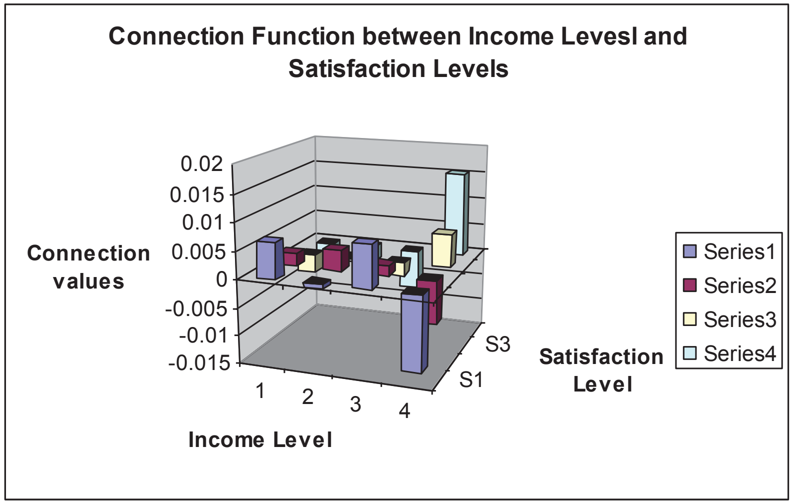
	Job Satisfaction			
Income US $$	**Very Dissatisfied**	**Little Satisfied**	**Moderately Satisfied**	**Very Satisfied**
< 6,000	0.03667013	-0.00435	0.044454	-0.07677
6,000–15,000	0.01078257	0.017082	0.008576	-0.03644
15,000-25,000	-0.01824561	-0.00096	-0.01269	0.0319
> 25,000	-0.03446045	-0.01801	-0.04723	0.099694

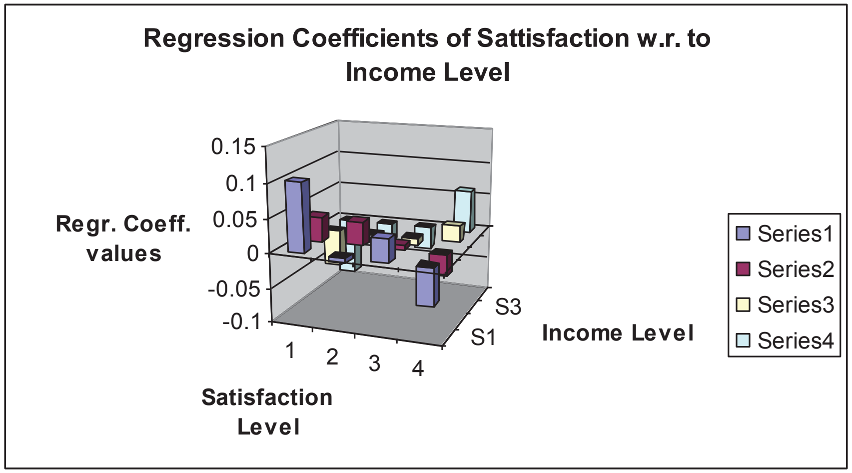
Categorical Variables (Correl. Coeff.)

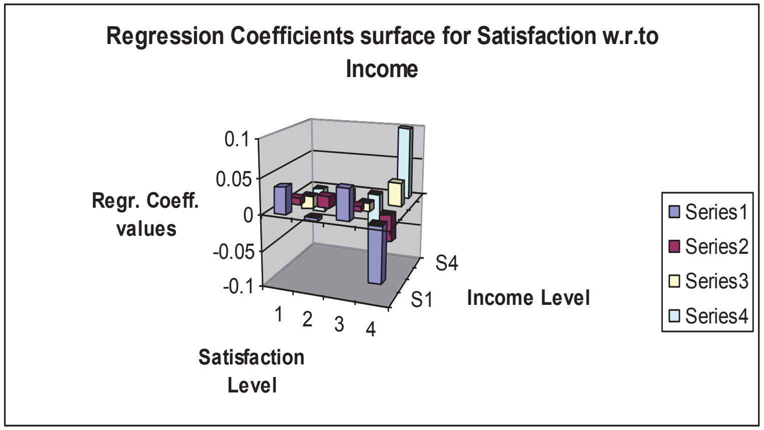
Tab 6: Empirical correlation coefficient between particular income and job satisfaction levels *$R(Income_i, Satisfaction_j)$*

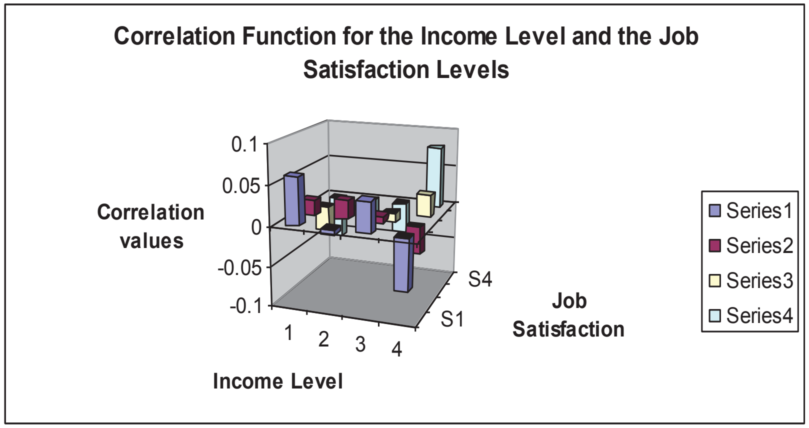
Job Satisfaction				
Income US $$	**Very Dissatisfied**	**Little Satisfied**	**Moderately Satisfied**	**Very Satisfied**
< 6,000	0.060838	- 0.005623	0.039037	- 0.064721
6,000–15,000	0.019883	0.024548	0.008371	- 0.034144
15,000-25,000	- 0.031649	- 0.001302	- 0.011653	0.028117
> 25,000	- 0.053383	- 0.02174	- 0.038723	0.078472

Categorical variables - Graphic comparison









Local dependence structure: The simplest Bivariate Poisson distribution with dependent components

- The bivariate discrete distribution presented by the three positive parameters (*λ, μ, ν*) family

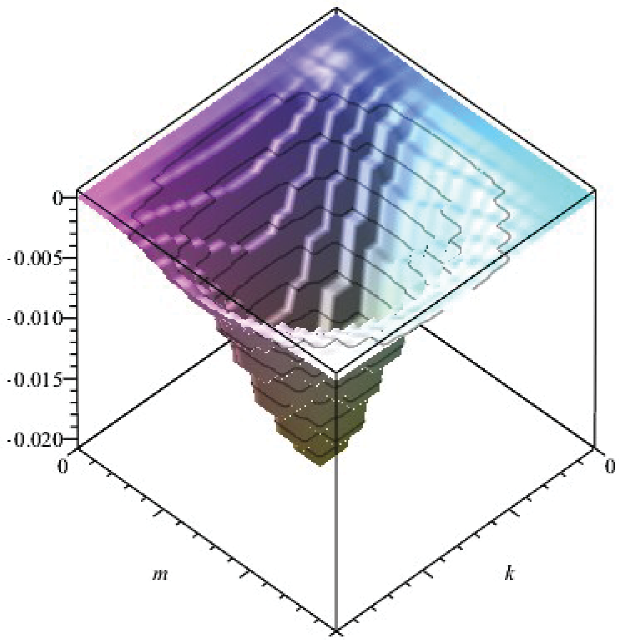
$$P(X = x, Y = y) = e^{-\lambda-\mu-\nu}\frac{\lambda^x}{x!}\cdot\frac{\mu^y}{y!}\sum_{k=0}^{\min(x,y)}\binom{x}{k}\binom{y}{k}k!\left(\frac{\nu}{\lambda\mu}\right)^k$$

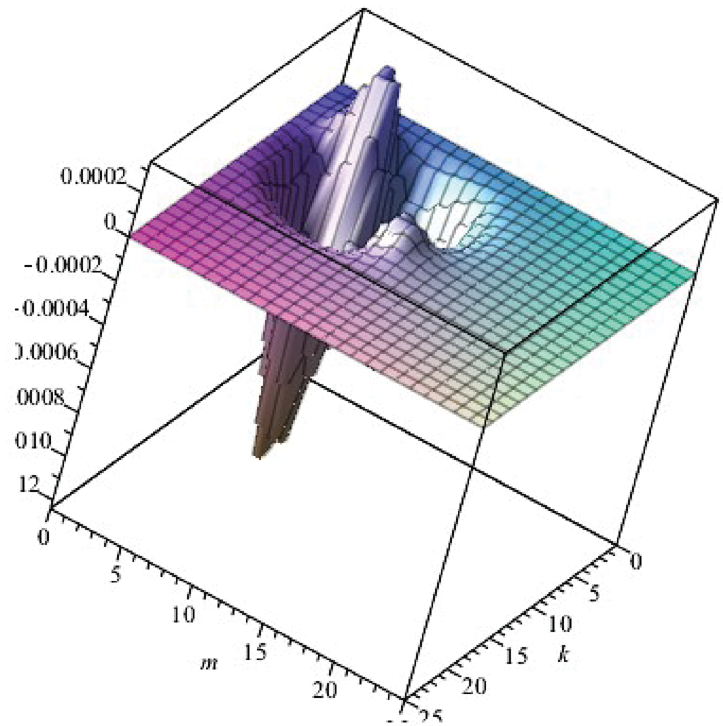
- **Here *x, y* = 0, 1, 2, … are the possible values of the variables *X* and *Y*. If M_1, M_2, and M_3 are three independent Poisson distributed r.v.'s with parameters *λ, μ*, and ν correspondingly, then dependence between X and Y comes from the fact that X is distributed as the sum $X = M_1 + M_3$, and $Y = M_2 + M_3$. Inclusion of M_3 in both sums makes them dependent.**
- The marginal distributions of *X* and *Y* are Poisson with parameters *λ+ μ,* and *λ+ ν* respectively

The Bivariate Poisson distribution with dependent components

- We avoid explicit cumbersome expressions for $\delta_{X,Y}(x,y)$, the two regression coefficient functions $R_X(Y; x,y)$ and $R_X(Y; x,y)$, and the correlation function $\rho_{X.Y}(x,y)$
- Local Dependence at each point (x,y) with integer coordinates is programmed for the values ***λ*=3, *μ*=2 and *ν*=5**
- The graphs of these functions are shown on next graphs.
- As the ancient Greek geometers use to say, **just watch and conclude at what point what kind of dependence works, and what is its strength.**

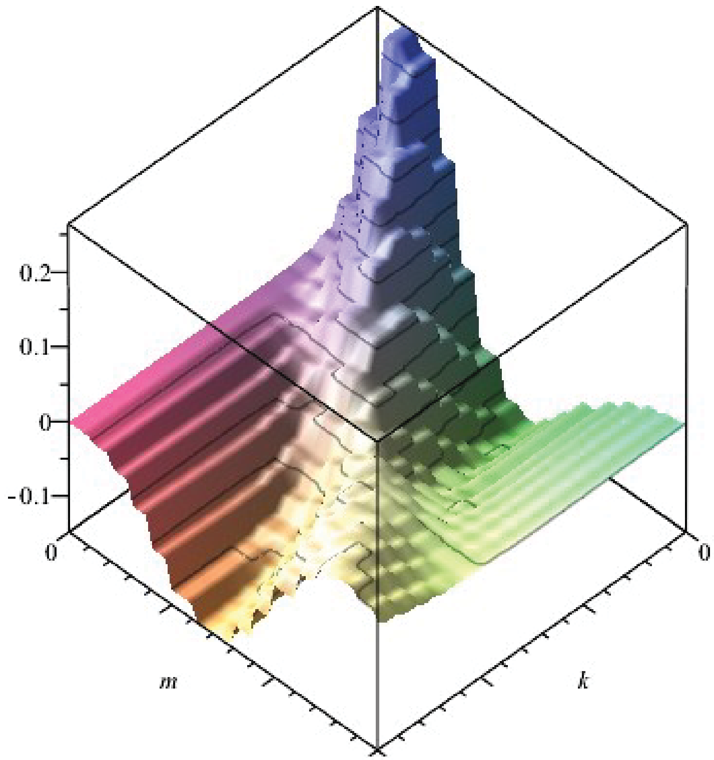
The Bivariate Poisson distribution with dependent components



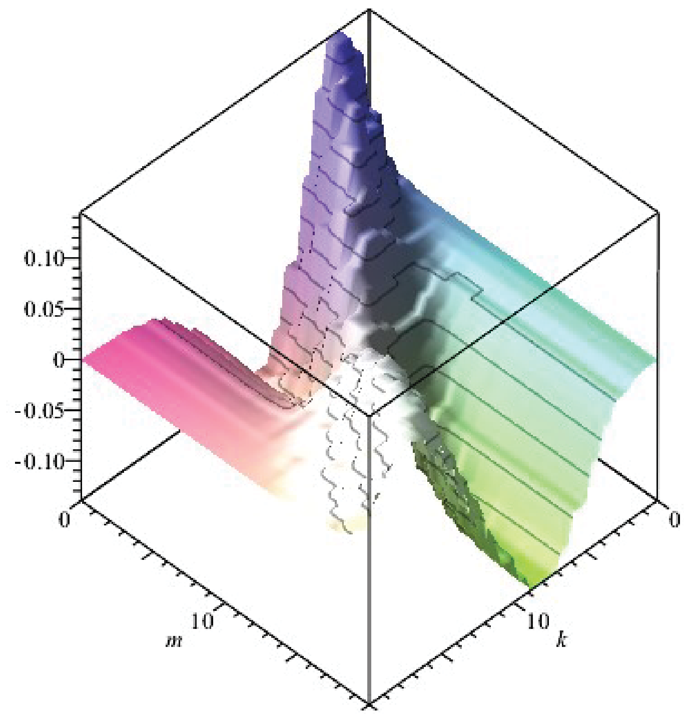


Connection function $\delta_{X,Y}(x,y)$ The correlation function $\rho_{X.Y}(x,y)$

The Bivariate Poisson distribution with dependent components



The regression coeff. F-n $R_Y(X; x,y)$



The regression coeff. F-n $R_X(Y;x,y)$

Local dependence structure in the political charts

- Esa and Dimitrov (2013a) have shown that a **multinomial model** describes the spectrum of the party's life in a country. **With *N* independent active *free* *i*ndividuals**, the coordinates of **the random vector (*X_0*, *X_1*, …, *X_r*)** represent the number of individuals members of each party. They are distributed by the multinomial law

- $P(X_0=k_0, X_1=k_1, \ldots, X_r=k_r) =$

$$\frac{N!}{k_0!k_1!\ldots k_r!}P_0^{k_0}P_1^{k_1}\ldots P_r^{k_r}$$

$$k_0+k_1+\ldots+k_r=N$$

Local dependence structure in the political charts

- The regression coefficients and correlation coefficients of the local dependence between any two components of the political life in the country are obtained from

$$P(X_i=n, X_j=m) = \frac{N!}{n!m!(N-n-m)!} P_i^n P_j^m (1-P_i-P_j)^{N-n-m}$$

$$P(X_i=n) = \frac{N!}{n!(N-n)!} P_i^{N-n} (1-P_i)^{N-n}$$

and

$$P(X_j=m) = \frac{N!}{m!(N-m)!} P_j^{N-m} (1-P_j)^{N-m}$$

Local dependence structure in the political charts –An Example

- **Let us assume that in the main model we have r=4; $\lambda_1=\lambda_2=\lambda_2=\lambda_4=1$; $\mu_1=2$, $\mu_2=3$, $\mu_3=4$, and $\mu_4=5$. Then**

$$P_0 = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)^{-1} = \frac{60}{137}$$

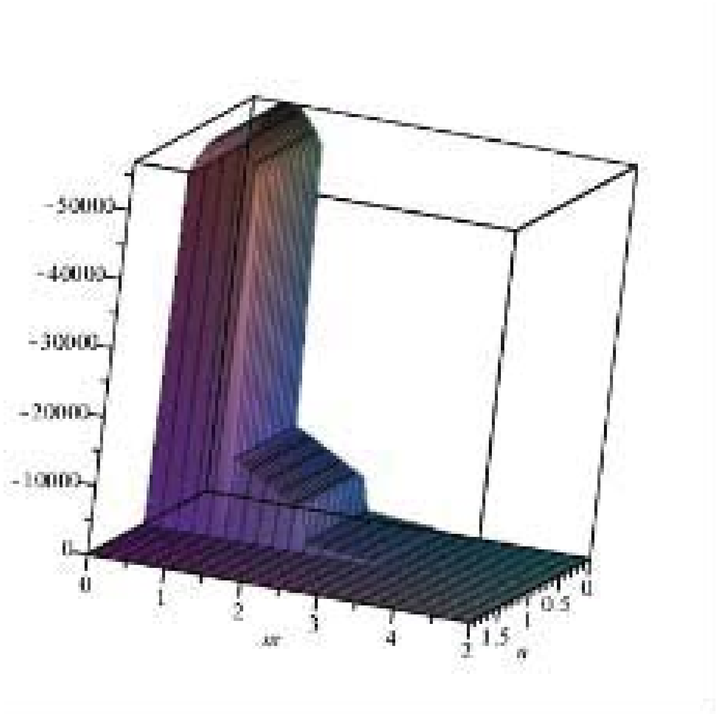
- Respectively $P_1=30/137$; $P_2=20/137$; $P_3=15/137$

  and $P_4=12/137$.

Assuming for simplicity N=1000, we draw a **surface of the local Regression coefficient of the dominating party 1 with respect to the next leading party 2 (Fig. 1.).** It is shown on the next figure.

Next to it is the regression coefficient measure of dependence of the weakest party 4 on the strongest party 1 (Fig. 2.).

Local dependence structure in the political charts –An Example



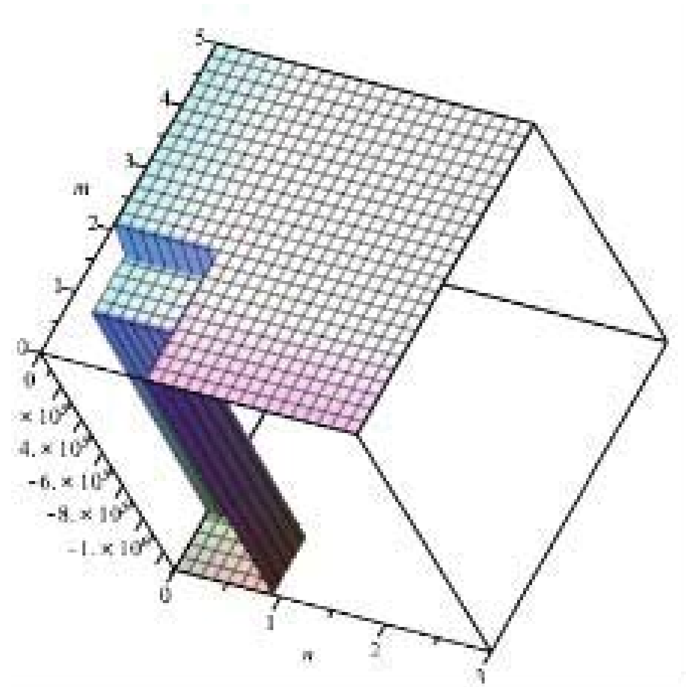


Fig. 1. Local Regression coefficient surface **dominating party 1 with respect to next leading party 2.**

Fig. 2. Regression coefficient surface - **- the weakest party 4 with respect to strongest party 1**

Local dependence structure in the political charts –An Example - discussion

- We see that **the local dependences between** parties (the strongest to next leading) **are negative when both parties have low results in votes**. **Dependence is negligible when votes are higher**.

- **Dependence of the weakest party vs. strongest one goes low flat when it gets low number of votes.**

Local dependence in reliability systems

- We focus on **two traditional systems** of independent components,

  System in series and

  System in parallel.
- **1.** We study the **regression coefficients of a component with respect to the system**, and
- **2. Regression coefficient of the system with respect to a component**

  How these measures of dependence change in time during the work of the system.
- **For simplicity consider system of just two components.**
- **REASON:** considering one component, everything else can be aggregated as a second component.

System in series.

Assume, components have **exponentially distributed live times** with parameters λ_1 and λ_2 .

Then the **reliability function** at time *t* (**event *B***) equals

r(*t*)= $e^{-(\lambda_1+\lambda_2)t}$ The probability **that component 1 functions** (**event *A***) is $e^{-\lambda_1 t}$.

The **regression coefficient of the system with respect to component 1** is

$$R_1(S) = \frac{r(t) - r(t)e^{-\lambda_1 t}}{e^{-\lambda_1 t}(1 - e^{-\lambda_1 t})} = e^{-\lambda_2 t}$$

System in series

- The **regression coefficient of the component 1 with respect to the system at time t** is given by

$$R_S(1) = \frac{r(t) - r(t)e^{-\lambda_1 t}}{r(t)[1 - r(t)]} = \frac{1 - e^{-\lambda_1 t}}{1 - e^{-(\lambda_1 + \lambda_2)t}}$$

- The **correlation coefficient between system** reliability **and the component** reliability are changing during the time according to

$$\rho_{s,1}(t) = \sqrt{\frac{e^{-\lambda_2 t}(1 - e^{-\lambda_1 t})}{1 - e^{-(\lambda_1 + \lambda_2)t}}} \quad \rho_{s,2}(t) = \sqrt{\frac{e^{-\lambda_1 t}(1 - e^{-\lambda_2 t})}{1 - e^{-(\lambda_1 + \lambda_2)t}}}$$

System in series

Notice that all dependences are positive. Graphs of these functions of local dependence in time for $\lambda_1=1$ and $\lambda_2=2$ are shown on next figures (Fig. 3 and Fig.4)

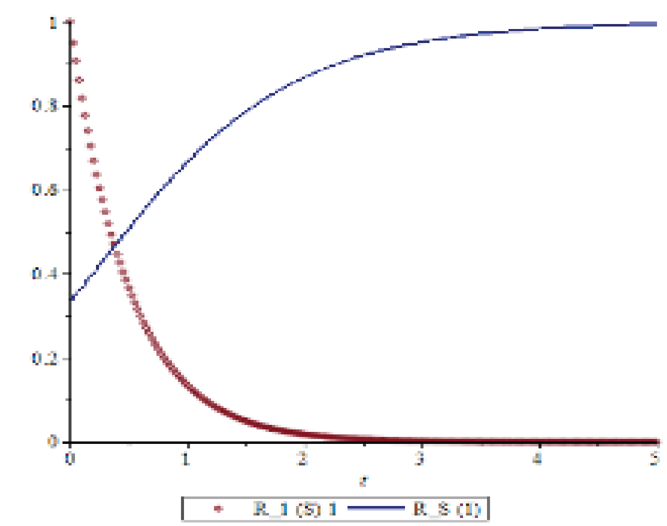


Fig. 3. Regression coeff. between between system reliability and the strongest component (λ=1) **(time dependence)**

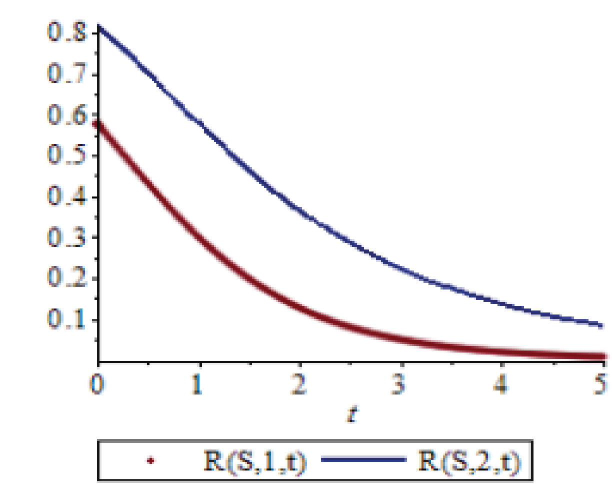


Fig. 4. Regression coefficients between system reliability and weakest component (λ=2) **(time dependence)**

System in series (discussion)

The Regr. coeff. for the weakest component 2 w.r. to system and system w.r. to component , decrease when the time increases, and behave similarly;

The regression coefficients between the system and the strongest component behave different:
Local dependence **$R_1(S)$ approaches 0** with the time (**system becomes independent on component 1 with the time increase**);

The local dependence $R_S(1)$ of strongest component 1 on the system reliability approaches 1 with the time increase (Fig.3).

System in parallel

Assume again **both components lives exponential** with parameters λ_1 and λ_2.

The **reliability function** at time t (event B) equals

$r(t) = 1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})$,

- The probability that component 1 functions (event A)

  is $e^{-\lambda_1 t}$

- The **regression coefficient** of the **system with respect to component 1** is

$$R_1(S) = \frac{1-r(t)}{1-e^{-\lambda_1 t}} = 1-e^{-\lambda_2 t}$$

System in parallel

- The regression coefficient of the **component 1 with respect to the system at time t is given by**

$$R_S(1) = \frac{e^{-\lambda_1 t} - r(t)e^{-\lambda_1 t}}{r(t)[1-r(t)]} = \frac{e^{-\lambda_1 t}}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}$$

- The correlation coefficient between system reliability and the component reliability in time is

$$\rho_{S,1}(t) = \sqrt{\frac{e^{-\lambda_1 t}(1-e^{-\lambda_2 t})}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}} \quad ; \rho_{S,2}(t) = \sqrt{\frac{e^{-\lambda_2 t}(1-e^{-\lambda_1 t})}{1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}}$$

System in parallel – an example

- **All dependences are positive.** Graphs of these functions of local dependence in time for **λ_1=1 and λ_2=2** are shown on Fig. 5 and Fig .6.

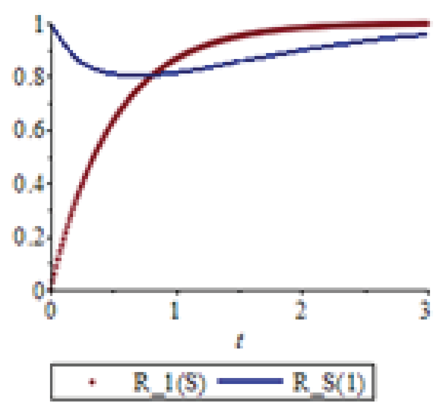


Fig. 5. Regression coefficients $\boldsymbol{R_1(S)}$, the thicker line, $\boldsymbol{R_S(1)}$ - the thinner curve

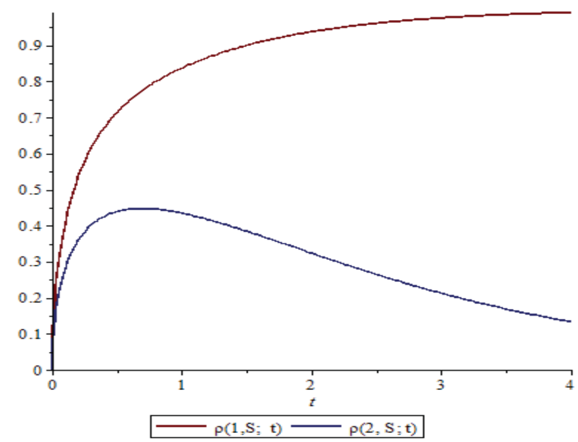


Fig. 5. Correlation coefficients $\boldsymbol{\rho_1(S)}$**,** the thicker line, and $\boldsymbol{\rho_S(1)}$ (the thinner)

CONCLUSIONS

- We discussed measures of dependence between two random events.
- These measures are equivalent, and exhibit natural properties.
- Their numerical values may serve as indication for the magnitude of dependence between random events.
- These measures provide simple ways to detect independence, coincidence, degree of dependence.
- If either measure of dependence is known, it allows better prediction of the chance for occurrence of one event, given that the other one occurs.

CONCLUSIONS

- We extend the use of these measures from events to local dependence between random variables
- Our study of the local dependence is on rectangles where interval values of the random variables meet. It exhibits different behavior than the global dependence.
- The local dependences are universally valid and can be continued for higher dimensions.
- Numerical illustrations (for politics and reliability systems, and non-numeric social study) confirm our expectations.
- We show that **local dependence can be essentially different on different areas in the field.**

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