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Bioelectrical Circuits: Lecture 6

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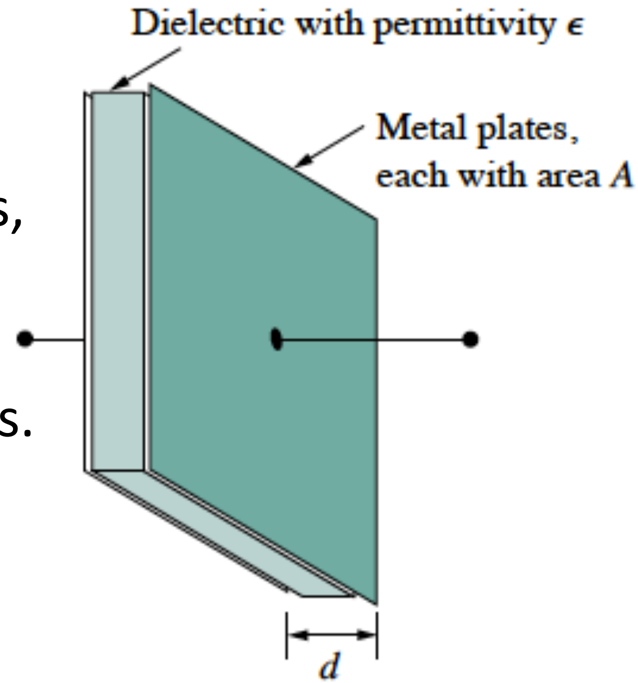
BME 205 L06

Capacitors and Inductors

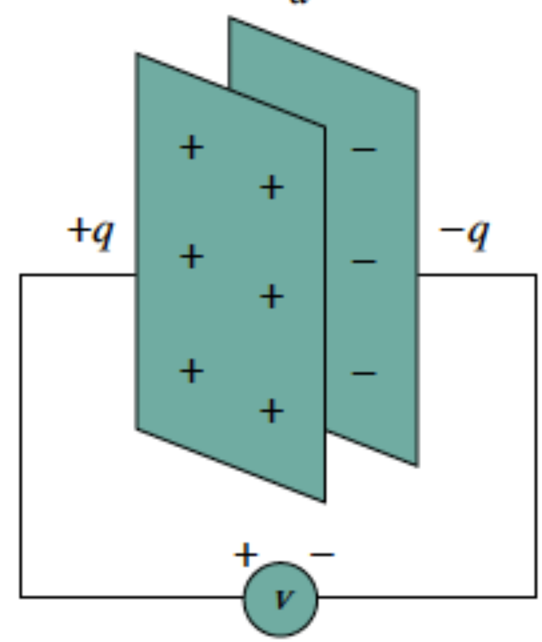
Introduction

- So far we've only looked at resistive circuits. Now we'll introduce two new and important passive linear circuit elements: the capacitor and the inductor.
- Unlike resistors, which dissipate energy, capacitors and inductors *store* energy, which can be retrieved at a later time (i.e., they have memory). For this reason, capacitors and inductors are called *storage* elements.
- With capacitors and inductors, we will be able to analyze more important and practical circuits.
- Good news: The circuit analysis techniques covered so far are equally applicable to circuits with capacitors and inductors!
- In this lecture we'll cover the basic properties of capacitors and how to combine them in series or in parallel. Then, we'll do the same for inductors. Finally, we explore how capacitors are combined with op amps to form integrators, differentiators, and analog computers.

Capacitors



- A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components, extensively used in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.
- A capacitor consists of two conducting plates (e.g. aluminum foil) separated by an insulator/dielectric (e.g. air, ceramic, paper, or mica).



- When a voltage source *v* is connected to the capacitor, the source deposits a positive charge *q* on one plate and a negative charge −*q* on the other.

- The capacitor is said to store the electric charge. The amount of charge stored, represented by *q,* is directly proportional to the applied voltage *v* so that

$$q = Cv$$

where *C,* the constant of proportionality, is known as the capacitance of the capacitor.

In words, Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates.

The unit of capacitance is the farad (F), in honor of the English physicist Michael Faraday (1791–1867).

Note that 1 farad = 1 coulomb/volt.

Capacitance depends on the physical dimensions of the capacitor. For example, for the parallel-plate capacitor, the capacitance is given by

$$C = \frac{\epsilon A}{d}$$

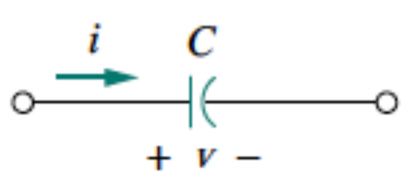
where *A* is the surface area of each plate, *d* is the distance between the plates, and is the permittivity of the dielectric material in between.

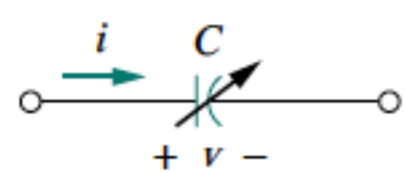
Although $C = \frac{\epsilon A}{d}$ applies to only parallel-plate capacitors, we may infer from it that, in general, three factors determine the value of the capacitance:

1. The surface area of the plates—the larger the area, the greater the capacitance.
2. The spacing between the plates—the smaller the spacing, the greater the capacitance.
3. The permittivity of the material—the higher the permittivity, the greater the capacitance.

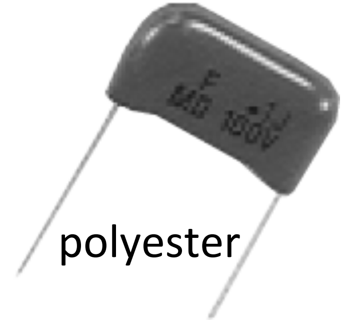
Capacitors are commercially available in different values and types.

Typically, capacitors have values in the picofarad (pF) to microfarad (μF) range. They are described by the dielectric material they are made of and by whether they are of fixed or variable type. The circuit symbols for fixed and variable capacitors are shown below. Note that according to the passive sign convention, current is considered to flow into the positive terminal of the capacitor when the capacitor is being charged, and out of the positive terminal when the capacitor is discharging.





- Common types of fixed-value capacitors are shown on the right. Polyester capacitors are light in weight, stable, and their change with temperature is predictable. Instead of polyester, other dielectric materials such as mica and polystyrene may be used. Film capacitors are rolled and housed in metal or plastic films. Electrolytic capacitors produce very high capacitance.


polyester


ceramic


electrolytic

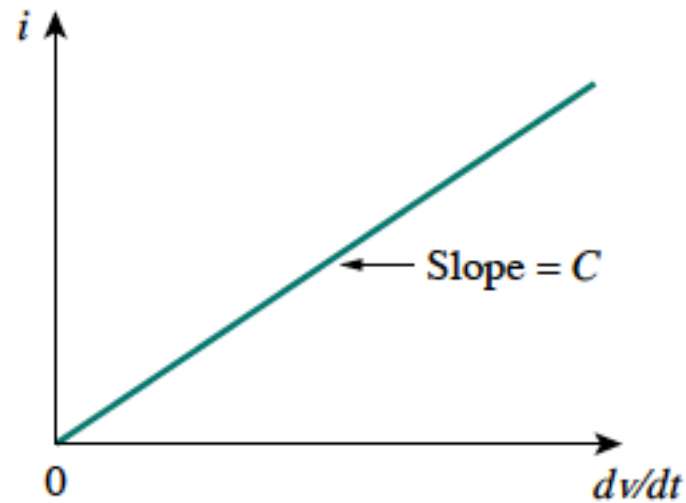
- At the bottom are the most common types of *variable* capacitors. The capacitance of a trimmer (or padder) capacitor or a glass piston capacitor is varied by turning the screw. The capacitance of the variable air capacitor (meshed plates) is varied by turning the shaft. Variable capacitors are used in radioreceivers allowing one to tune to various stations. In addition, capacitors are used to block dc, pass ac, shift phase, store energy, start motors, and suppress noise.

Current-voltage relationship

- To obtain the i-v relationship of the capacitor, we take the derivative of both sides of q = CV. Since $i = \frac{dq}{dt}$ we get

$$i = C\frac{dv}{dt}$$

This assumes the positive sign convention. The relationship is illustrated for a *linear* capacitor whose capacitance is independent of voltage. For a *nonlinear capacitor,* the i-v relationship is not a straight line. Most capacitors are linear. All of ours are linear.



To get the voltage-current relationship we integrate both sides:

$$v = \frac{1}{C}\int_{-\infty}^{t} i\, dt \quad \text{or} \quad v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$$

$$v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$$

- $v(t_0) = q(t_0)/C$ is the voltage across the capacitor at time t_0.
- This equation shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited. The instantaneous power delivered to the capacitor is

$$p = vi = Cv\frac{dv}{dt}$$

- The energy stored in the capacitor is therefore

$$w = \int_{-\infty}^{t} p\, dt = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\, dv = \frac{1}{2}Cv^2\bigg|_{t=-\infty}^{t}$$

- $v(-\infty) = 0$, because the capacitor was uncharged at $t = -\infty$. Thus,

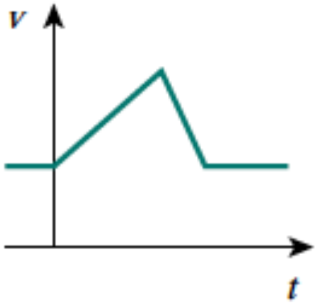
$$w = \frac{1}{2}Cv^2 \quad \text{or} \quad w = \frac{q^2}{2C} \quad \text{(using } q = vC\text{)}$$

This represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy.

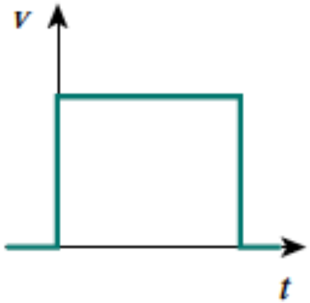
Capacitor properties

1. Note from $i = C\frac{dv}{dt}$ that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus, a capacitor is an open circuit to dc. However, if a battery (dc voltage) is connected across a capacitor, the capacitor charges.
2. The voltage on the capacitor must be continuous. i.e., The voltage on a capacitor cannot change abruptly. The capacitor resists an abrupt change in the voltage across it. According to *i = C dv/dt,* a discontinuous change in voltage requires an infinite current, which is physically impossible. For example, the voltage across a capacitor may take the form in (a), but (b) is impossible because of the abrupt change. Conversely, the current through a capacitor can change instantaneously.

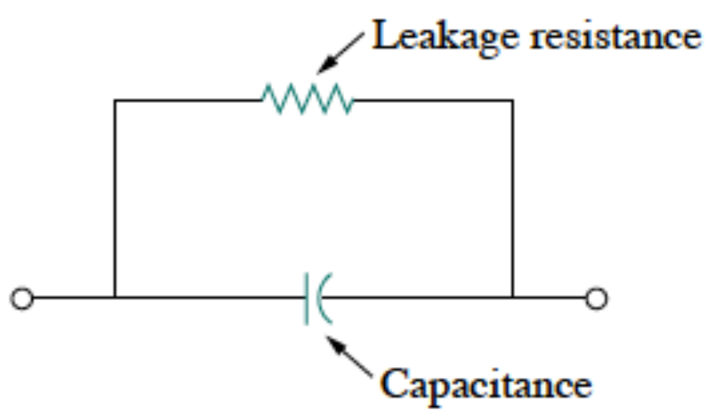
(a)



(b)



3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.
4. A real, nonideal capacitor has a parallel-model leakage resistance, as shown below. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications. For this reason, we will assume ideal capacitors in this course.



Example

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.
(b) Find the energy stored in the capacitor.

Solution:

(a) Since $q = Cv$,

$$q = 3 \times 10^{-12} \times 20 = 60 \text{ pC}$$

(b) The energy stored is

$$w = \frac{1}{2}Cv^2 = \frac{1}{2} \times 3 \times 10^{-12} \times 400 = 600 \text{ pJ}$$

Practice

What is the voltage across a 3-μF capacitor if the charge on one plate is 0.12 mC? How much energy is stored?

Example

The voltage across a 5-μF capacitor is

$$v(t) = 10 \cos 6000t \text{ V}$$

Calculate the current through it.

Solution:

By definition, the current is

$$i(t) = C\frac{dv}{dt} = 5 \times 10^{-6} \frac{d}{dt}(10 \cos 6000t)$$

$$= -5 \times 10^{-6} \times 6000 \times 10 \sin 6000t = -0.3 \sin 6000t \text{ A}$$

Practice

If a 10-μF capacitor is connected to a voltage source with

$$v(t) = 50 \sin 2000t \text{ V}$$

determine the current through the capacitor.

Example

Determine the voltage across a 2-μF capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}$$

Assume that the initial capacitor voltage is zero.

Solution:

Since $v = \frac{1}{C}\int_0^t i\,dt + v(0)$ and $v(0) = 0$,

$$v = \frac{1}{2 \times 10^{-6}} \int_0^t 6e^{-3000t}\,dt \cdot 10^{-3}$$

$$= \frac{3 \times 10^3}{-3000} e^{-3000t} \Big|_0^t = (1 - e^{-3000t}) \text{ V}$$

Practice

The current through a 100-μF capacitor is $i(t) = 50 \sin 120\pi t$ mA. Calculate the voltage across it at $t = 1$ ms and $t = 5$ ms. Take $v(0) = 0$.

Example

Determine the current through a 200-μF capacitor whose voltage is shown in (a).

Solution:

The voltage waveform can be described mathematically as

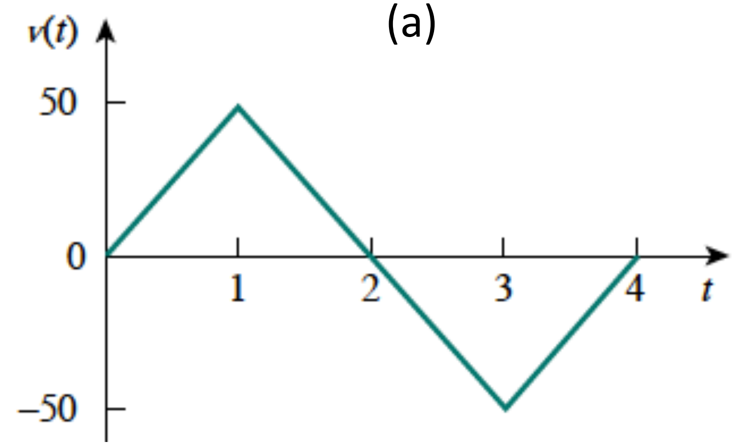
$$v(t) = \begin{cases} 50t \text{ V} & 0 < t < 1 \\ 100 - 50t \text{ V} & 1 < t < 3 \\ -200 + 50t \text{ V} & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Since $i = C\,dv/dt$ and $C = 200\ \mu$F, we take the derivative of v to obtain

$$i(t) = 200 \times 10^{-6} \times \begin{cases} 50 & 0 < t < 1 \\ -50 & 1 < t < 3 \\ 50 & 3 < t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Thus the current waveform is as shown in (b)

(a)



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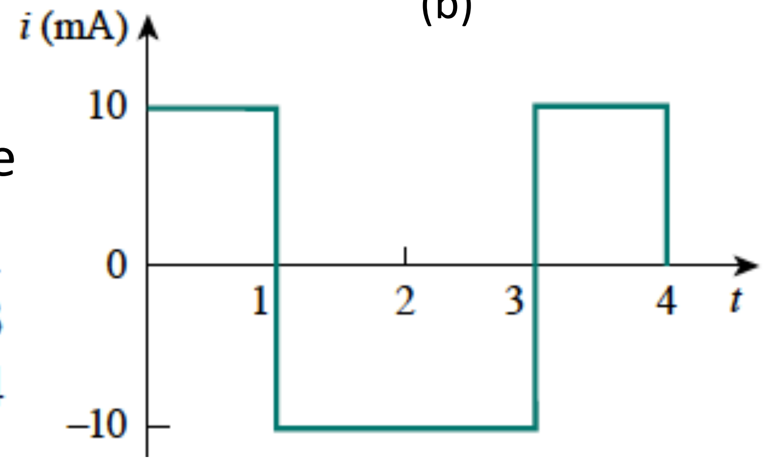
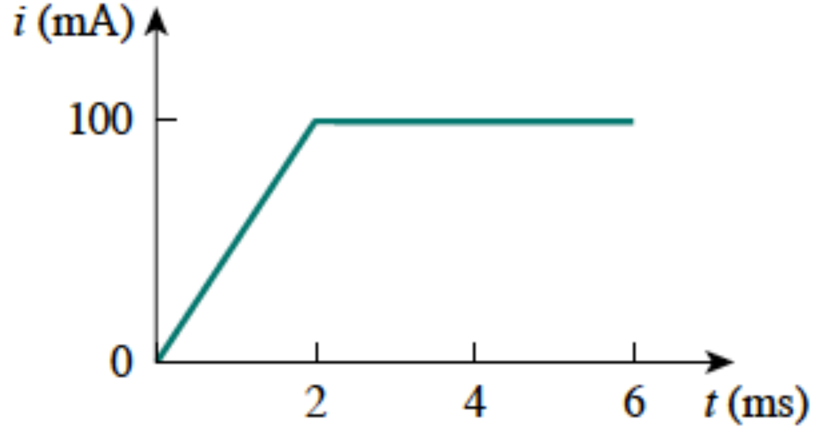


Practice problem

An initially uncharged 1-mF capacitor has the current shown in the figure. Calculate the voltage across it at t = 2 ms and t = 5 ms.



Example

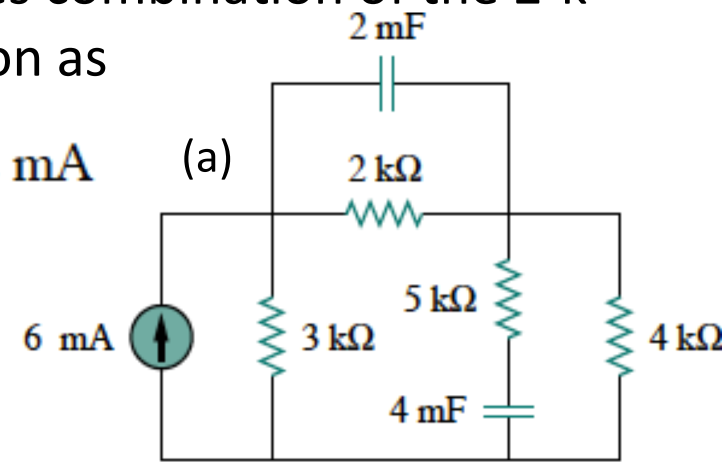
Obtain the energy stored in each capacitor in (a) under dc conditions.

Solution:

Under dc conditions, we replace each capacitor with an open circuit, as shown in (b). The current through the series combination of the 2-k and 4-k resistors is obtained by current division as

$$i = \frac{3}{3+2+4}(6\text{ mA}) = 2\text{ mA}$$

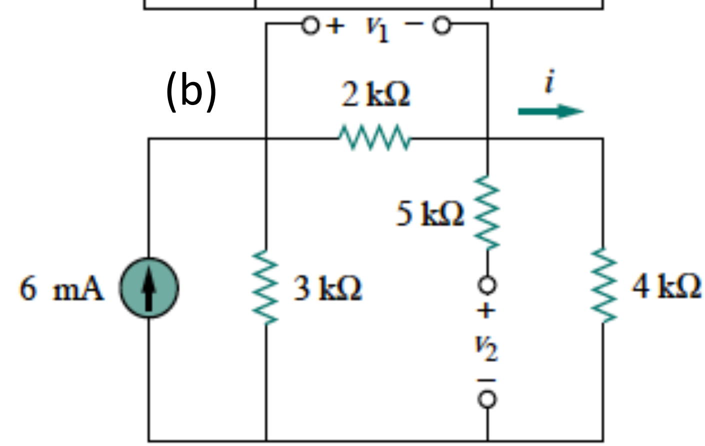
(a)



Hence, the voltages v_1 and v_2 across the capacitors are

$$v_1 = 2000i = 4\text{ V}$$
$$v_2 = 4000i = 8\text{ V}$$

(b)



and the energies stored in them are

$$w_1 = \frac{1}{2}C_1 v_1^2 = \frac{1}{2}(2\times 10^{-3})(4)^2 = 16\text{ mJ}$$
$$w_2 = \frac{1}{2}C_2 v_2^2 = \frac{1}{2}(4\times 10^{-3})(8)^2 = 128\text{ mJ}$$

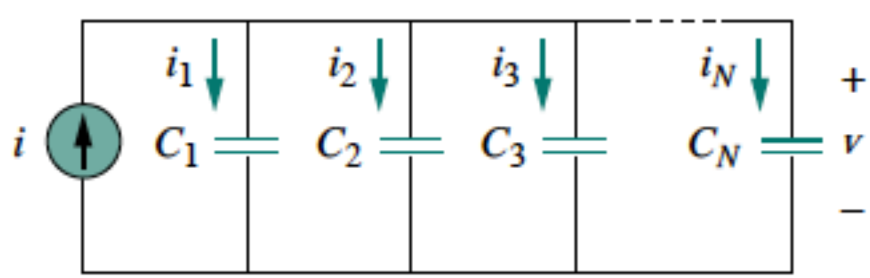
Practice problem

Under dc conditions, find the energy stored in the capacitors

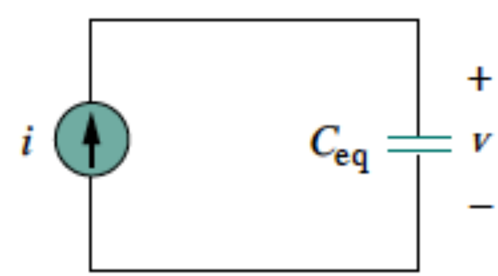
Series and Parallel Capacitors

- We know from resistive circuits that series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors. We desire to replace these capacitors by a single equivalent capacitor C_{eq}.
- In order to obtain the equivalent capacitor C_{eq} of N capacitors in parallel, consider the circuit in (a). The equivalent circuit is in (b). Note that the capacitors have the same voltage v across them. Applying KCL:

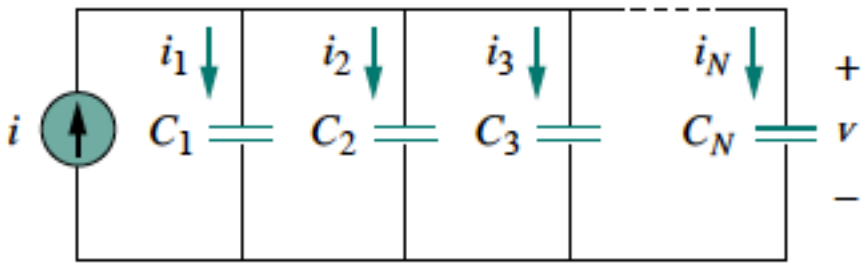
$$i = i_1 + i_2 + i_3 + \cdots + i_N$$



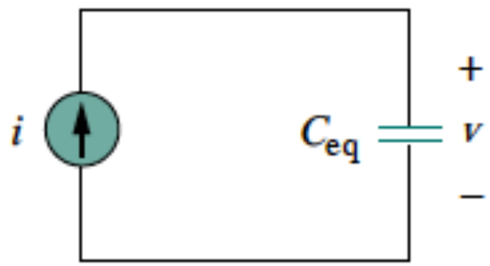
(a)



(b)



(a)



(b)

But $i_k = C_k\, dv/dt$. Hence,

$$i = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt}$$

$$= \left(\sum_{k=1}^{N} C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

where

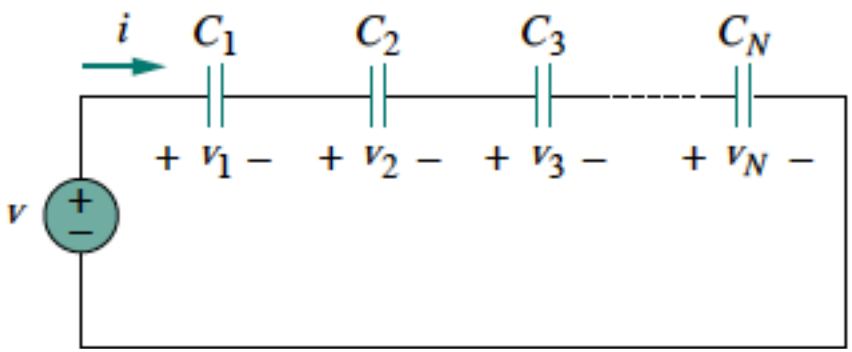
$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

The equivalent capacitance of *N* parallel-connected capacitors is the sum of the individual capacitances.

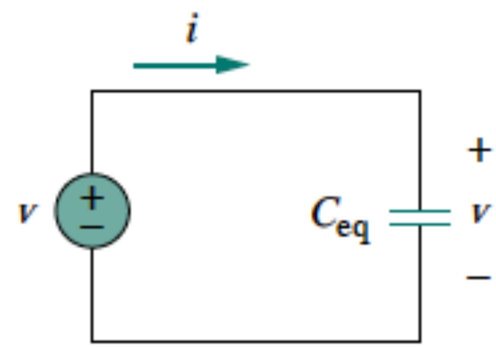
We now obtain C_{eq} of *N* capacitors connected in series by comparing the circuit in (a) with the equivalent circuit in (b). Note that the same current *i* flows (and consequently the same charge) through the capacitors. Applying KVL to the loop in (a),

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

(a)



(b)



But $v_k = \dfrac{1}{C_k}\displaystyle\int_{t_0}^{t} i(t)\,dt + v_k(t_0)$. Therefore,

$$v = \frac{1}{C_1}\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)\,dt + v_2(t_0) \;+\cdots+ \frac{1}{C_N}\int_{t_0}^{t} i(t)\,dt + v_N(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}\right)\int_{t_0}^{t} i(t)\,dt + v_1(t_0) + v_2(t_0) \;+\cdots+ v_N(t_0)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i(t)\,dt + v(t_0)$$ where

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}}$$

The initial voltage $v(t_0)$ across C_{eq} is required by KVL to be the sum of the capacitor voltages at t_0. So,

$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For N = 2 (i.e., two capacitors in series), we have

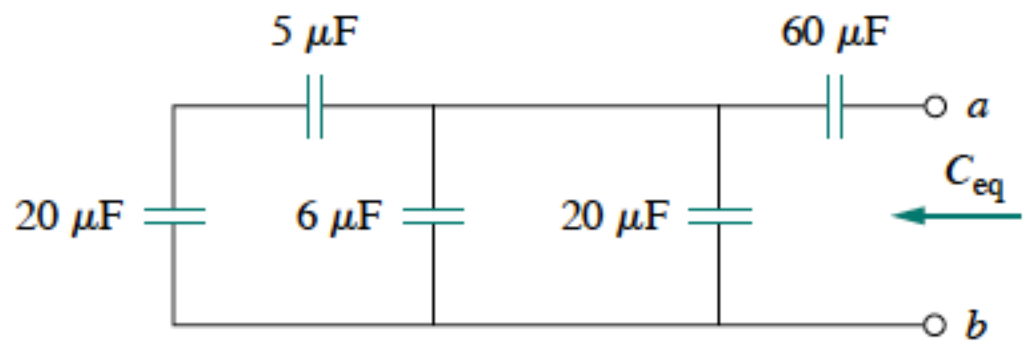
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

Example

- Find the equivalent capacitance seen between terminals *a* and *b* of the circuit



- **Solution:**
- The 20-μF and 5-μF

capacitors are in series; their equivalent capacitance is

$$\frac{20 \times 5}{20 + 5} = 4\ \mu\text{F}$$

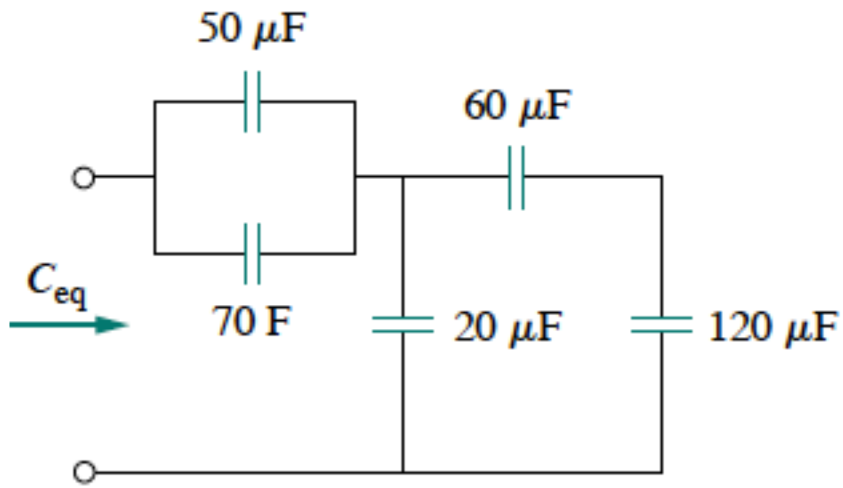
- This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is $4 + 6 + 20 = 30\ \mu\text{F}$

- This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

$$C_{\text{eq}} = \frac{30 \times 60}{30 + 60} = 20\ \mu\text{F}$$

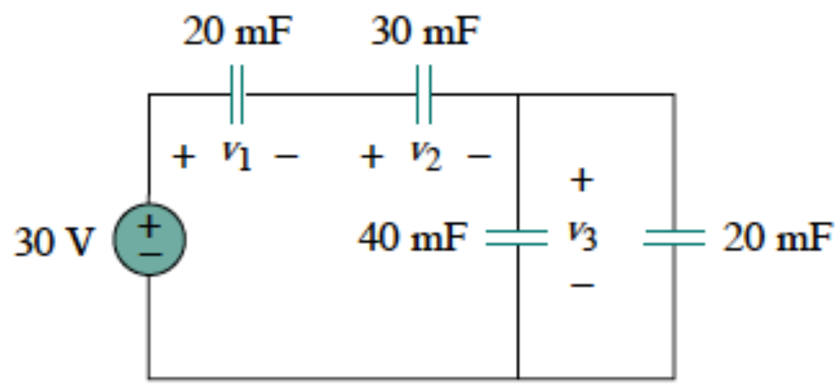
Practice problem

Find the equivalent capacitance seen at the terminals of the circuit



Example

- For this circuit, find the voltage across each capacitor.
- **Solution:**
- We first find the equivalent capacitance *C*eq. The two parallel capacitors can be combined to get 40+20 = 60 mF.



- This 60-mF capacitor is in series with the 20-mF and 30-mF capacitors. Thus,

$$C_{eq} = \frac{1}{\frac{1}{60} + \frac{1}{30} + \frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

- The total charge is $q = C_{eq} v = 10 \times 10^{-3} \times 30 = 0.3 \text{ C}$
- This is the charge on the 20-mF and 30-mF capacitors, because they are in series with the 30-V source. (charge acts like current, since *i* = *dq/dt* .) Therefore,

$$v_1 = \frac{q}{C_1} = \frac{0.3}{20 \times 10^{-3}} = 15 \text{ V} \qquad v_2 = \frac{q}{C_2} = \frac{0.3}{30 \times 10^{-3}} = 10 \text{ V}$$

- Having determined $v1$ and $v2$, we now use KVL to determine $v3$ by

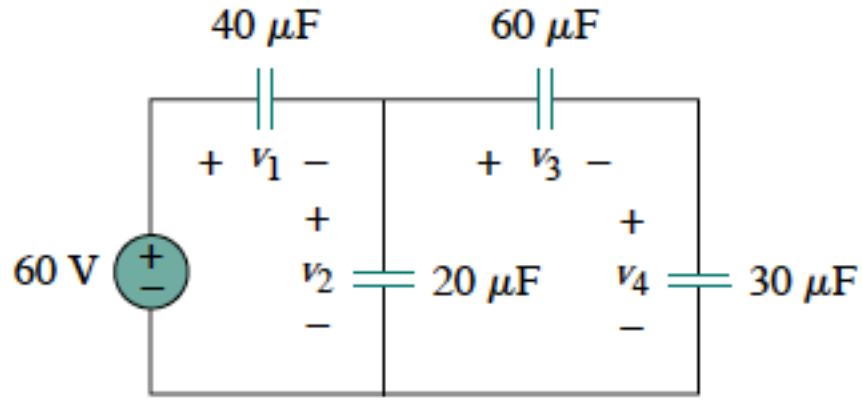
$$v_3 = 30 - v_1 - v_2 = 5 \text{ V}$$

- Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage $v3$ and their combined capacitance is 40 + 20 = 60 mF. This combined capacitance is in series with the 20-mF and 30-mF capacitors and consequently has the same charge on it. Hence,

$$v_3 = \frac{q}{60 \text{ mF}} = \frac{0.3}{60 \times 10^{-3}} = 5 \text{ V}$$

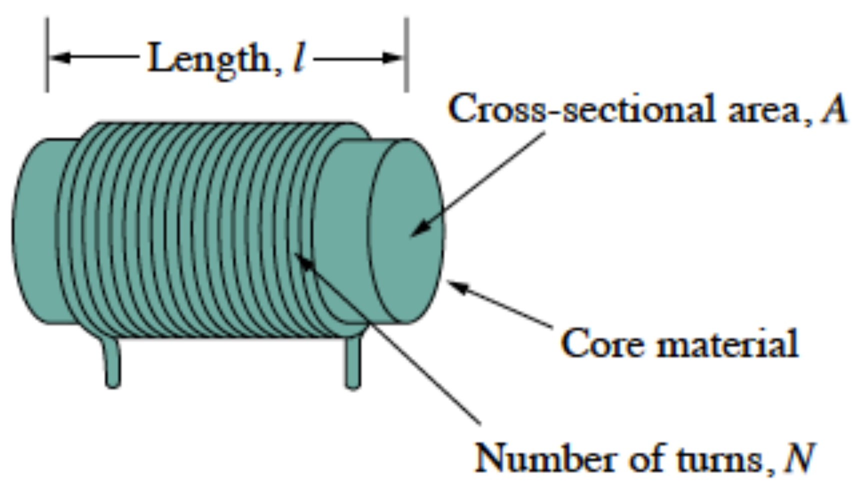
Practice problem

Find the voltage across each of the capacitors



Inductors

- An inductor is a passive element designed to store energy in its magnetic field. Inductors find numerous applications in electronic and power systems. They are used in power supplies, transformers, radios, TVs, radars, and electric motors.
- Any conductor of electric current has inductive properties and may be regarded as an inductor. But in order to enhance the inductive effect, a practical inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown below.



- If current is allowed to pass through an inductor, it is found that the voltage across the inductor is directly proportional to the time rate of change of the current. Using the passive sign convention,

$$v = L\frac{di}{dt}$$

where *L* is the constant of proportionality called the inductance of the inductor. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797–1878).

- Clearly 1 henry = 1 volt-second per ampere.
- Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it.
- The inductance of an inductor depends on its physical dimension and construction. Formulas for calculating the inductance of inductors of different shapes are derived from electromagnetic theory. For example, for the solenoid inductor

$$L = \frac{N^2 \mu A}{\ell}$$

where *N* is the number of turns, *l* is the length, *A* is the cross-sectional area, and *μ* is the permeability of the core.

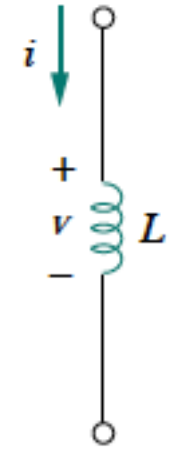
- So inductance can be increased by increasing the number of turns of coil, using material with higher permeability as the core, increasing the cross-sectional area, or reducing the length of the coil.
- Like capacitors, commercially available inductors come in different values and types. Typical practical inductors have inductance values ranging from a few microhenrys (μH), as in communication systems, to tens of henrys (H) as in power systems. Inductors may be fixed or variable. The core may be made of iron, steel, plastic, or air. The terms *coil* and *choke* are also used for inductors. Common inductors are shown below.

solenoidal wound inductor

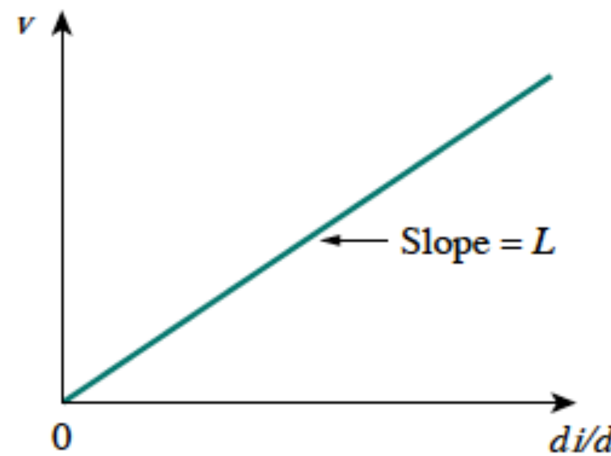
toroidal inductor

chip inductor

- The circuit symbol for inductors are shown on the right, following the passive sign convention.



The voltage-current relationship for an inductor is shown graphically for an inductor whose inductance is independent of current. Such an inductor is known as a *linear inductor*. For a *nonlinear inductor*, inductance varies with current. We will assume linear inductors.



- The current-voltage relationship is obtained as

$$di = \frac{1}{L} v \, dt$$

Integrating gives $i = \frac{1}{L} \int_{-\infty}^{t} v(t) \, dt$ or $i = \frac{1}{L} \int_{t_0}^{t} v(t) \, dt + i(t_0)$

where $i(t_0)$ is the total current for $-\infty < t < t_0$ and $i(-\infty) = 0$.

Power & Energy in the Inductor

- The inductor is designed to store energy in its magnetic field. The power delivered to the inductor is

$$p = vi = \left(L\frac{di}{dt}\right)i$$

The energy stored is

$$w = \int_{-\infty}^{t} p\,dt = \int_{-\infty}^{t} \left(L\frac{di}{dt}\right) i\,dt$$

$$= L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty)$$

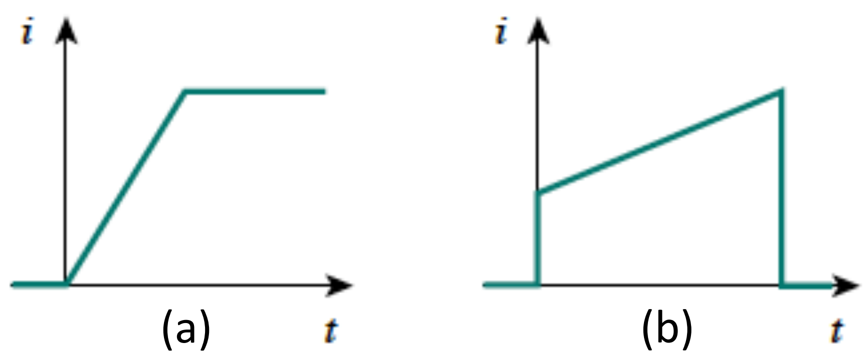
Since $i(-\infty) = 0$,

$$\boxed{w = \frac{1}{2}Li^2}$$

We should note the following important properties of an inductor.

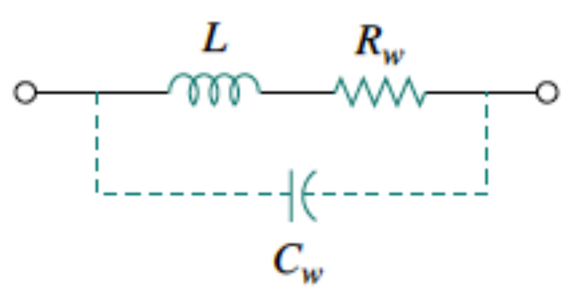
1. Note from $v = L\, di/dt$ that the voltage across an inductor is zero when the current is constant. Thus, An inductor acts like a short circuit to dc.

2. An important property of the inductor is its opposition to the change in current flowing through it. The current through an inductor cannot change instantaneously. A discontinuous change in the current through an inductor requires an infinite voltage, which is not physically possible. Thus, an inductor opposes an abrupt change in the current through it. For example, the current through an inductor may take the form shown in (a), but not that in (b) in real-life situations due to the discontinuities. However, the voltage across an inductor can change abruptly.



(a) (b)

3. Like the ideal capacitor, the ideal inductor does not dissipate energy. The energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.
4. A practical, nonideal inductor has a significant resistive component, as shown below. This is due to the fact that the inductor is made of a conducting material such as copper, which has some resistance. This resistance is called the *winding resistance Rw,* and it appears in series with the inductance of the inductor. The presence of *Rw* makes it both an energy storage device and an energy dissipation device. Since Rw is usually very small, it is ignored in most cases. The nonideal inductor also has a winding capacitance Cw due to the capacitive coupling between the conducting coils. Cw is very small and can be ignored in most cases, except at high frequencies. We will assume ideal inductors in this course.



Example

The current through a 0.1-H inductor is $i(t) = 10te^{-5t}$ A. Find the voltage across the inductor and the energy stored in it.

Solution:

Since $v = L\,di/dt$ and $L = 0.1$ H,

$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1 - 5t) \text{ V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t} \text{ J}$$

Practice

If the current through a 1-mH inductor is $i(t) = 20 \cos 100t$ mA, find the terminal voltage and the energy stored.

Example

Find the current through a 5-H inductor if the voltage across it is

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Also find the energy stored within $0 < t < 5$ s.

Solution:

Since $i = \frac{1}{L}\int_{t_0}^{t} v(t)\, dt + i(t_0)$ and $L = 5$ H,

$$i = \frac{1}{5}\int_0^t 30t^2\, dt + 0 = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\, dt = \int_0^5 60t^5\, dt = 60\frac{t^6}{6}\bigg|_0^5 = 156.25 \text{ kJ}$$

Alternatively, we can obtain the energy stored using Eq. (6.13), by writing

$$w\big|_0^5 = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0) = \frac{1}{2}(5)(2 \times 5^3)^2 - 0 = 156.25 \text{ kJ}$$

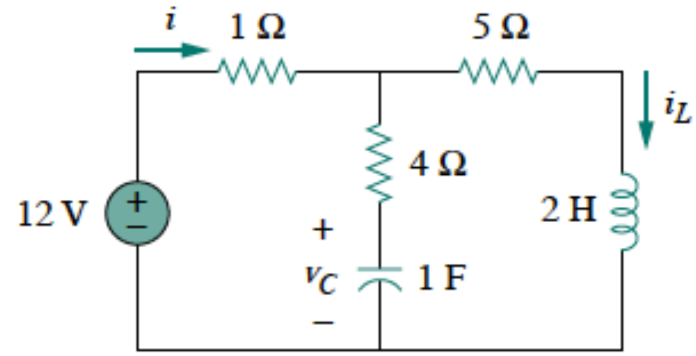
as obtained before.

Practice

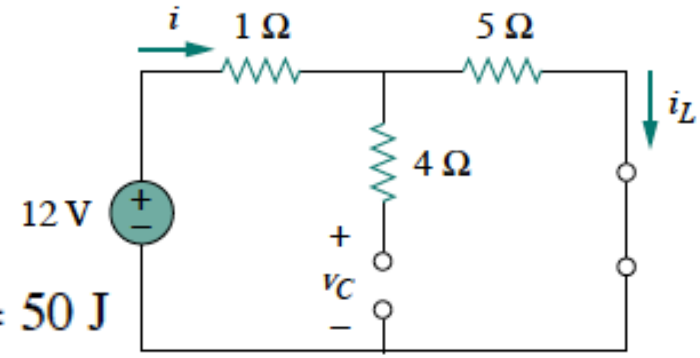
The terminal voltage of a 2-H inductor is $v = 10(1-t)$ V. Find the current flowing through it at $t = 4$ s and the energy stored in it within $0 < t < 4$ s. Assume $i(0) = 2$ A.

Example

In this circuit, under dc conditions, find: (a) *i*, *vC*, and *iL*, (b) the energy stored in the capacitor and inductor.



Solution: (a) Under dc conditions, we replace the capacitor with an open circuit and the inductor with a short circuit.



$$i = i_L = \frac{12}{1+5} = 2\text{ A}$$

The voltage *vC* is the same as the voltage across the 5- resistor. Hence, $v_C = 5i = 10\text{ V}$

(b) energy in capacitor $w_C = \frac{1}{2}Cv_C^2 = \frac{1}{2}(1)(10^2) = 50\text{ J}$

energy in inductor $w_L = \frac{1}{2}Li_L^2 = \frac{1}{2}(2)(2^2) = 4\text{ J}$

Series and Parallel Inductors

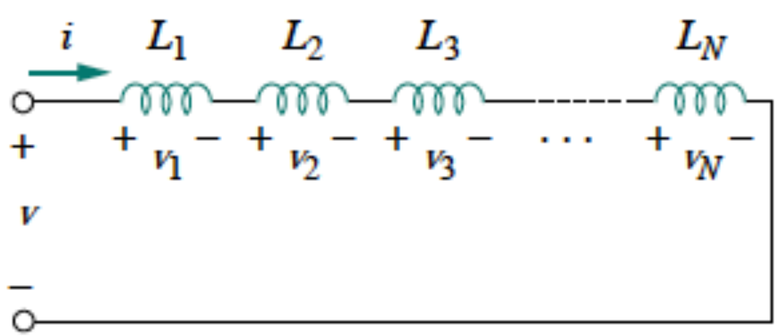
- How do we find the equivalent inductance of a series-connected or parallel-connected set of inductors?
- Consider a series connection of *N* inductors shown in (a), with the equivalent circuit in (b). The inductors have the same current through them. Applying KVL to the loop,

$$v = v_1 + v_2 + v_3 + \cdots + v_N$$

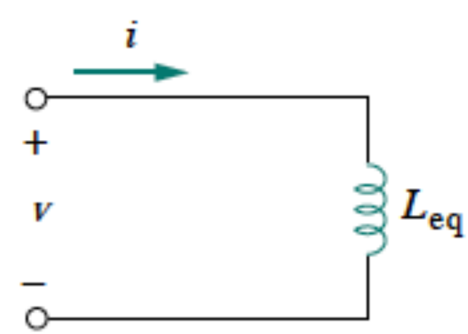
Substituting $v_k = L_k\, di/dt$ results in

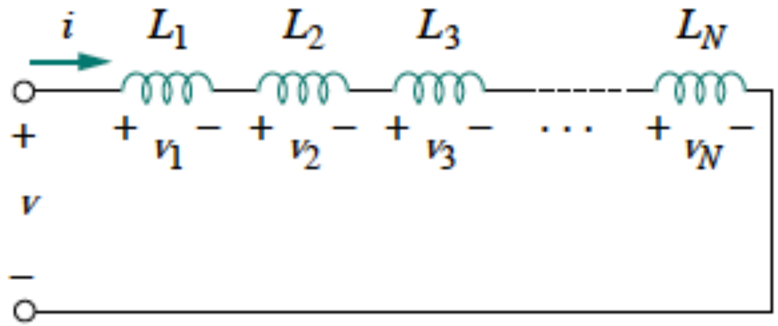
$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$$

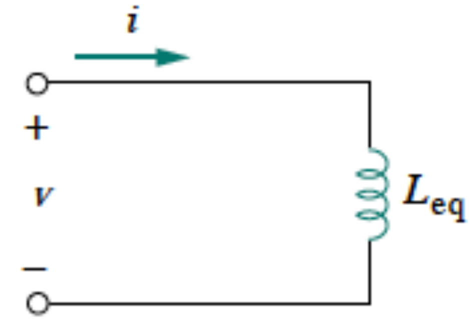
(a)



(b)



(a) 

(b) 

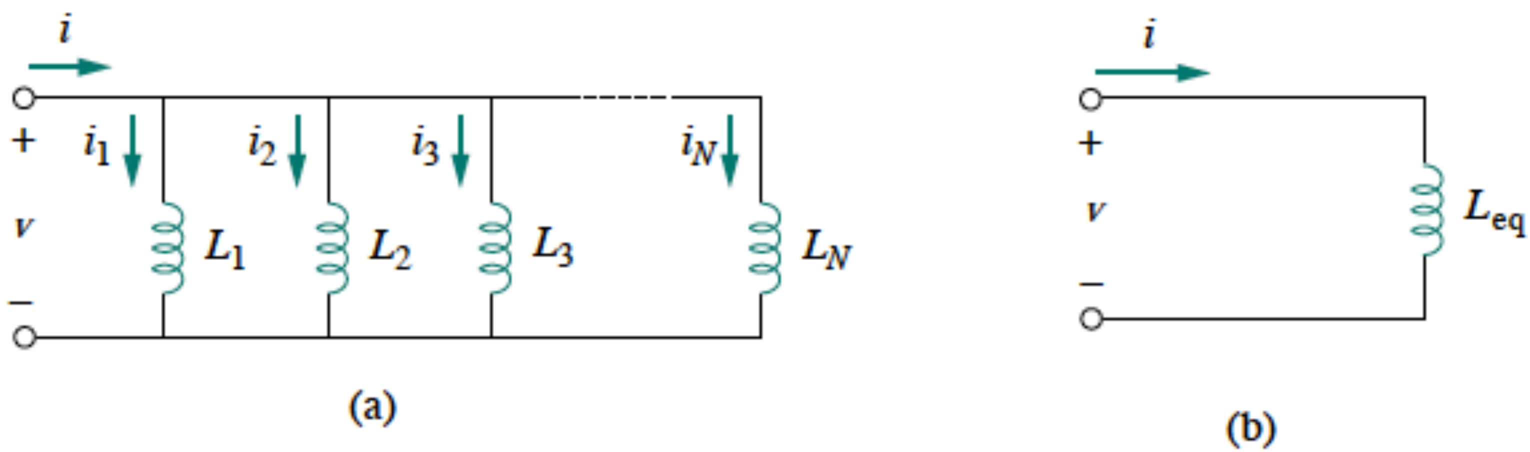
$$v = (L_1 + L_2 + L_3 + \cdots + L_N)\frac{di}{dt}$$

$$= \left(\sum_{k=1}^{N} L_k\right)\frac{di}{dt} = L_{eq}\frac{di}{dt}$$

where $$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N$$

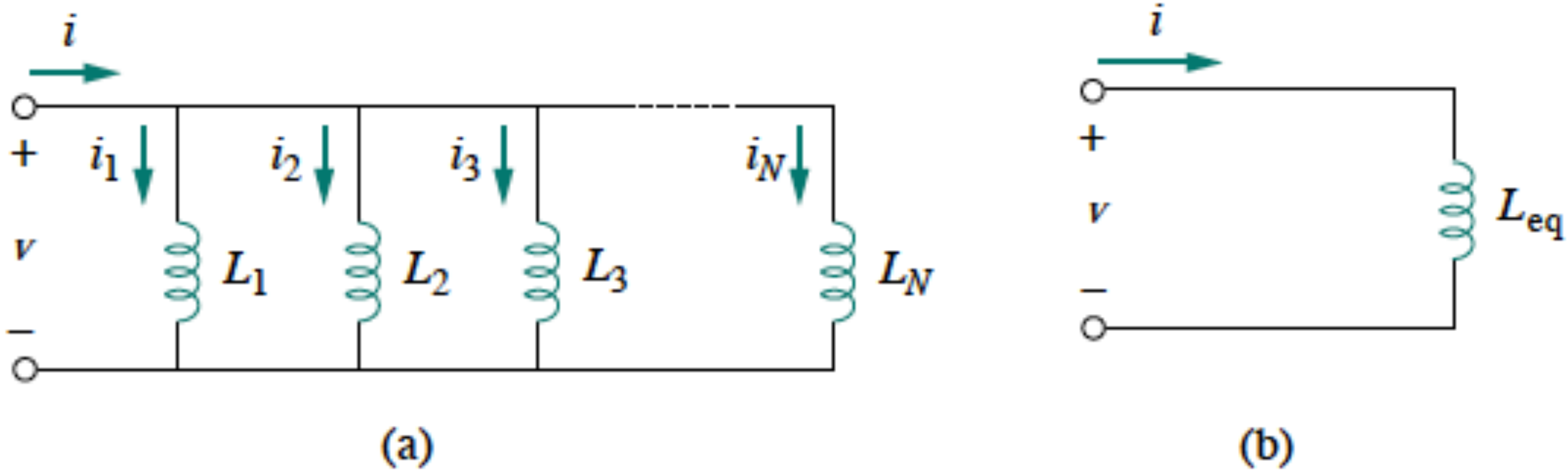
Thus, the equivalent inductance of series-connected inductors is the sum of the individual inductances.

Now consider a parallel connection of *N* inductors (a), with the equivalent circuit in (b). The inductors have the same voltage across them. Using KCL,



$$i = i_1 + i_2 + i_3 + \cdots + i_N$$

$$\text{But } i_k = \frac{1}{L_k}\int_{t_0}^{t} v\, dt + i_k(t_0);$$



$$i = \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + i_2(t_0)$$

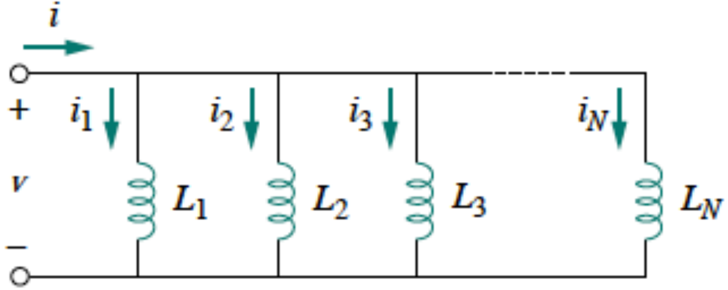
$$+ \cdots + \frac{1}{L_N}\int_{t_0}^{t} v\,dt + i_N(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right)\int_{t_0}^{t} v\,dt + i_1(t_0) + i_2(t_0)$$

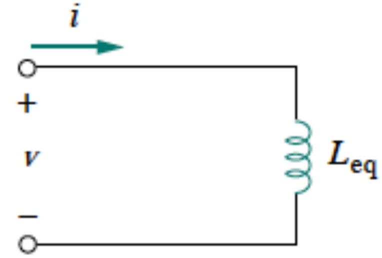
$$+ \cdots + i_N(t_0)$$

$$= \left(\sum_{k=1}^{N}\frac{1}{L_k}\right)\int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0) = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,dt + i(t_0)$$

$$\text{where } \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$$



(a)



(b)

$$i = \frac{1}{L_{eq}} \int_{t_o}^{t} v\,dt + i(t_o)$$

The initial current $i(t_0)$ through L_{eq} at $t = t_0$ is expected by KCL to be the sum of the inductor currents at t_0. Thus,

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

- So, the equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances.
- Note that the inductors in parallel are combined in the same way as resistors in parallel. For two inductors in parallel *(N = 2)*, our equation becomes

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \qquad \text{or} \qquad L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$$

To summarize...

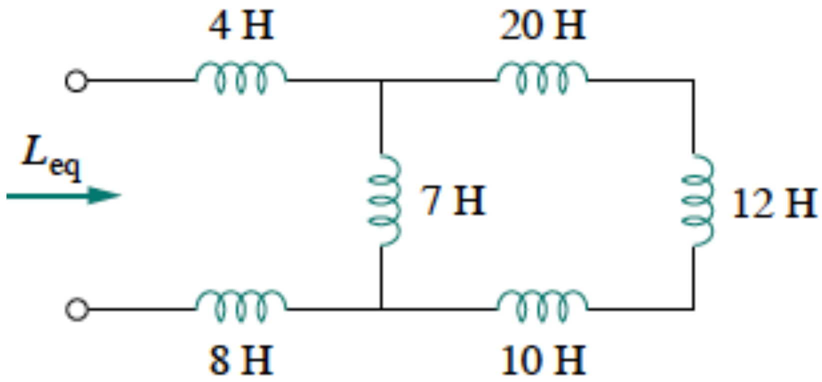
Important characteristics of the basic elements.[†]

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\, dt + i(t_0)$
p or w:	$p = i^2 R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

[†] *Passive sign convention is assumed.*

Example

Find the equivalent inductance of the circuit shown



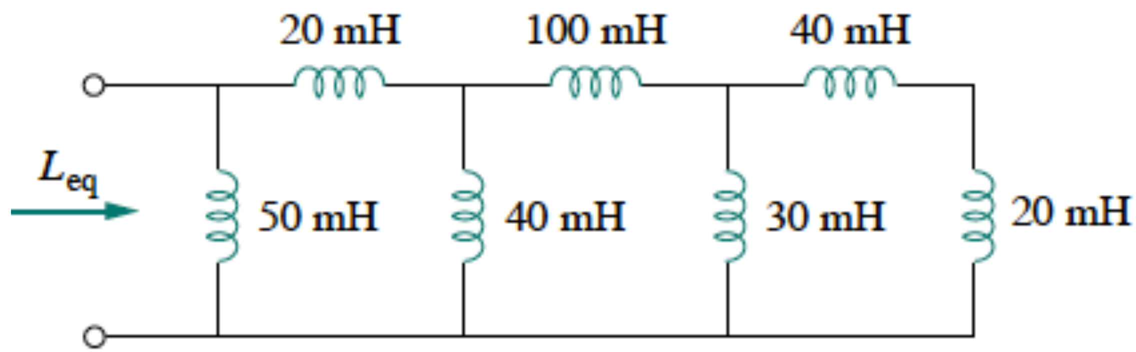
Solution:
The 10-H, 12-H, and 20-H inductors are in series; thus, combining them gives a 42-H inductance. This 42-H inductor is in parallel with the 7-H inductor so that they are combined, to give

$$\frac{7 \times 42}{7 + 42} = 6\text{ H}$$

This 6-H inductor is in series with the 4-H and 8-H inductors. Hence, $L_{eq} = 4 + 6 + 8 = 18$ H

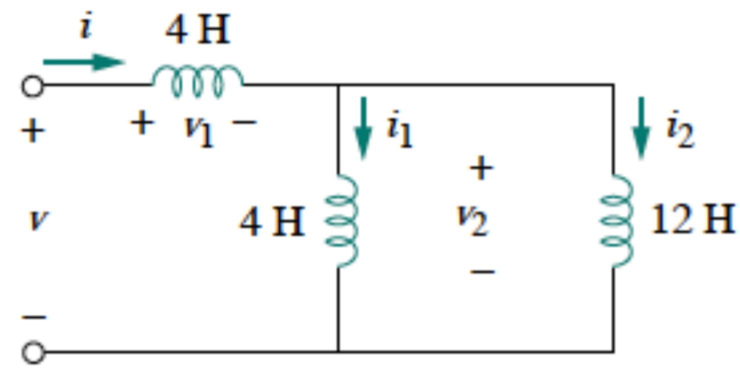
Practice problem

Calculate the equivalent inductance for the inductive ladder network



Example

For this circuit, $i(t) = 4(2 - e^{-10t})$ mA.
If $i_2(0) = -1$ mA, find: (a) $i_1(0)$; (b) $v(t)$, $v_1(t)$, and $v_2(t)$; (c) $i_1(t)$ and $i_2(t)$.



(a) From $i(t) = 4(2 - e^{-10t})$ mA, $i(0) = 4(2 - 1) = 4$ mA. Since $i = i_1 + i_2$,

$$i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5 \text{ mA}$$

(b) The equivalent inductance is

$$L_{eq} = 4 + 4 \parallel 12 = 4 + 3 = 7 \text{ H}$$

Thus, $v = L_{eq}\dfrac{di}{dt} = (7)\dfrac{d}{dt}\left(8 - 4e^{-10t}\right)$ $\quad v = (7)(-4)(-10)e^{-10t} = 280e^{-10t}$

and $v_1 = 4\dfrac{di}{dt} = (4)\dfrac{d}{dt}\left(8 - 4e^{-10t}\right) = (4)(-4)(-10)e^{-10t} = 160e^{-10t}$

Since $v = v_1 + v_2$, $\quad v_2 = v - v_1 = 280e^{-10t} - 160e^{-10t} = 120e^{-10t}$

(c) The current i_1 is obtained as

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5\text{ mA}$$
$$= -3e^{-10t}\,\Big|_0^t + 5\text{ mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\text{ mA}$$

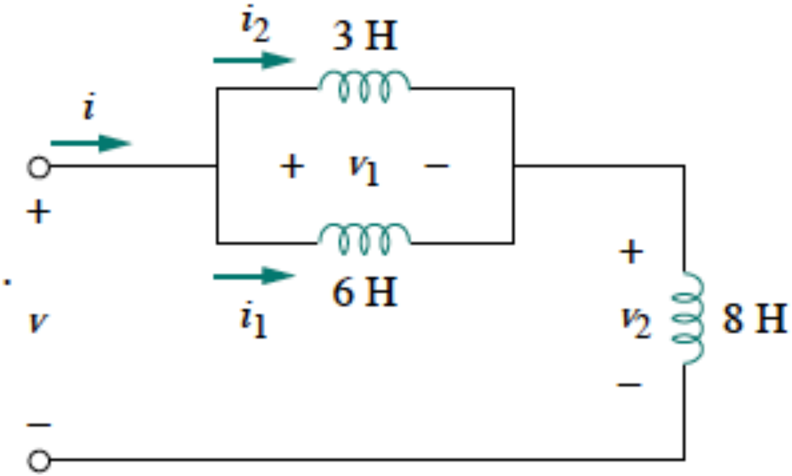
Similarly,

$$i_2(t) = \frac{1}{12}\int_0^t v_2\,dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t}\,dt - 1\text{ mA}$$
$$= -e^{-10t}\,\Big|_0^t - 1\text{ mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\text{ mA}$$

Practice

$i_1(t) = 0.6e^{-2t}$ A. If $i(0) = 1.4$ A, find:

(a) $i_2(0)$; (b) $i_2(t)$ and $i(t)$; (c) $v(t)$, $v_1(t)$, and $v_2(t)$.



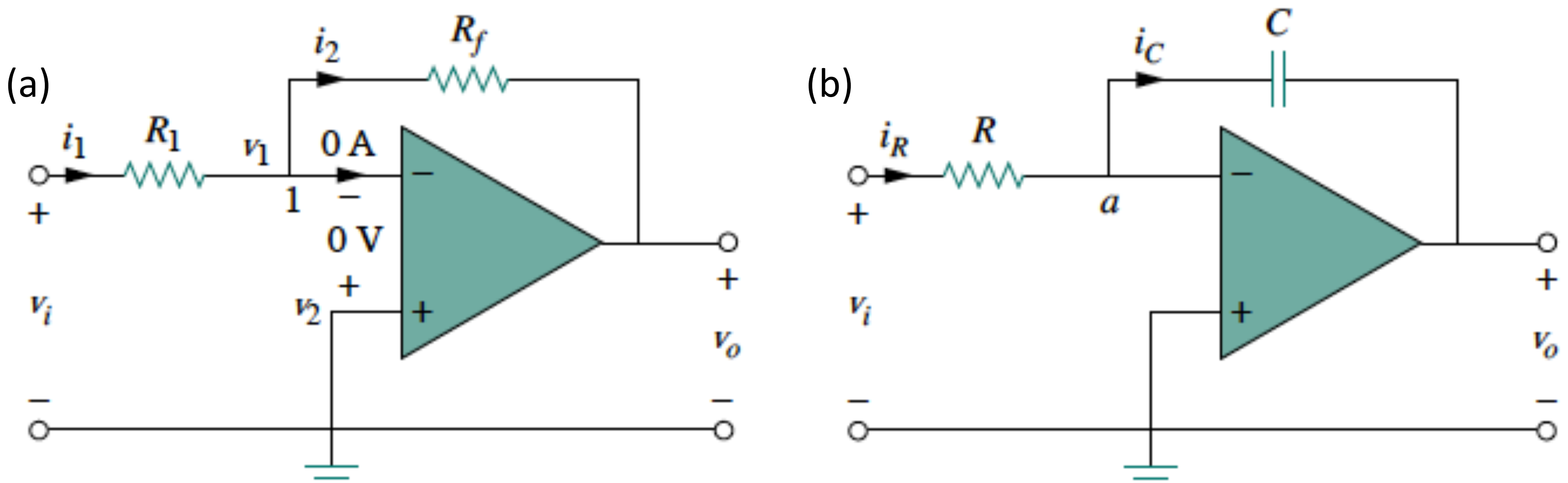
Applications

- Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit (IC) form. Unlike capacitors and resistors, inductors with appreciable inductance are difficult to produce on IC substrates.
- Therefore, inductors (coils) usually come in discrete form and tend to be more bulky and expensive. For this reason, inductors are not as versatile as capacitors and resistors, and they are more limited in applications.
- However, there are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, pick-up heads, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

- Capacitors and inductors possess the following three special properties that make them very useful in electric circuits:
- 1. The capacity to store energy makes them useful as temporary voltage or current sources. Thus, they can be used for generating a large amount of current or voltage for a short period of time.
- 2. Capacitors oppose any abrupt change in voltage, while inductors oppose any abrupt change in current. This property makes inductors useful for spark or arc suppression and for converting pulsating dc voltage into relatively smooth dc voltage.
- 3. Capacitors and inductors are frequency sensitive. This property makes them useful for frequency discrimination.
- The first two properties are put to use in dc circuits, while the third one is taken advantage of in ac circuits. We will see how useful these properties are later. For now, consider two applications involving capacitors and op amps: integrator and differentiator.

Integrator

- Integrator and differentiator op amp circuits often involve resistors and capacitors; inductors (coils) tend to be more bulky and expensive.
- The op amp integrator is used in numerous applications, especially in analog computers.
- An integrator is an op amp circuit whose output is proportional to the integral of the input signal.
- If the feedback resistor R_f in the familiar inverting amplifier of (a) is replaced by a capacitor, we obtain an ideal integrator, as shown in (b).



- At node *a* in fig (b), $i_R = i_C$
- But

$$i_R = \frac{v_i}{R}, \qquad i_C = -C\frac{dv_o}{dt}$$

- Substituting these:

$$\frac{v_i}{R} = -C\frac{dv_o}{dt} \quad \text{or} \quad dv_o = -\frac{1}{RC}v_i\,dt$$

- Integrating both sides gives

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(t)\,dt$$

- To ensure that $v_o(0) = 0$, it is always necessary to discharge the integrator's capacitor prior to the application of a signal. Assuming $v_o(0) = 0$,

$$v_o = -\frac{1}{RC}\int_0^t v_i(t)\,dt$$

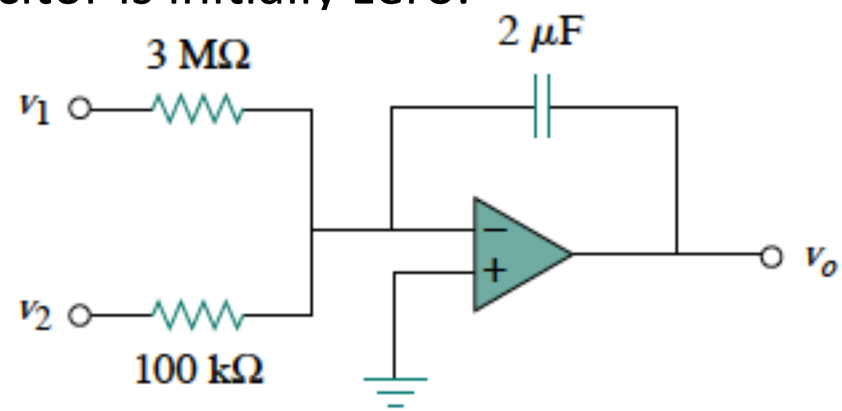
which shows that the circuit in Fig. (b) provides an output voltage proportional to the integral of the input.

In practice, the op amp integrator requires a feedback resistor to reduce dc gain and prevent saturation. Care must be taken that the op amp operates within the linear range so that it does not saturate.

Example

If $v1$ = 10 cos 2t mV and $v2$ = 0.5t mV, find vo in the op amp circuit. Assume that the voltage across the capacitor is initially zero.

Solution: This is a summing integrator



$$v_o = -\frac{1}{R_1 C}\int v_1\,dt - \frac{1}{R_2 C}\int v_2\,dt$$

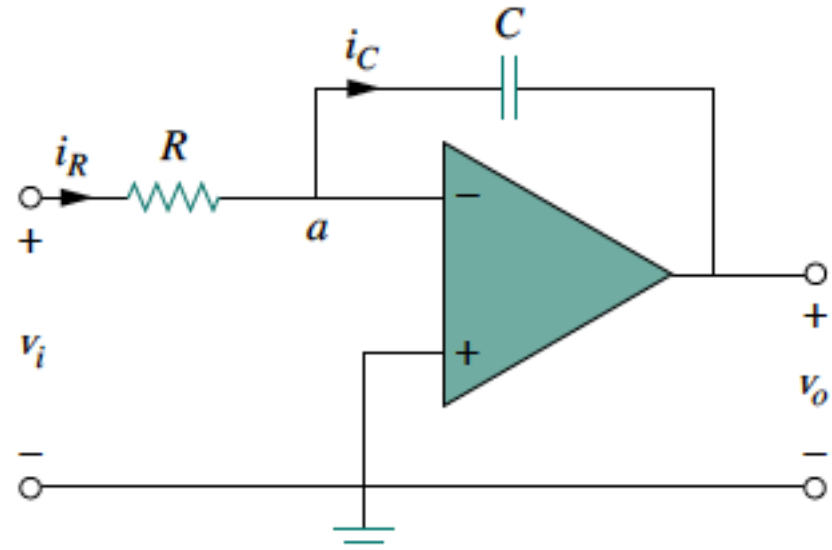
$$= -\frac{1}{3\times 10^6 \times 2\times 10^{-6}}\int_0^t 10\cos 2t\,dt$$

$$-\frac{1}{100\times 10^3 \times 2\times 10^{-6}}\int_0^t 0.5t\,dt$$

$$= -\frac{1}{6}\frac{10}{2}\sin 2t - \frac{1}{0.2}\frac{0.5t^2}{2} = -0.833\sin 2t - 1.25t^2 \text{ mV}$$

Practice

The integrator in Fig. (b) of two slides ago has R = 25 k, C = 10 μF. Determine the output voltage when a dc voltage of 10 mV is applied at t = 0. Assume that the op amp is initially nulled.



Differentiator

- A differentiator is an op amp circuit whose output is proportional to the rate of change of the input signal.
- If the input resistor in our original inverting amp circuit is replaced by a capacitor, the resulting circuit is a differentiator. Applying KCL at node *a*,

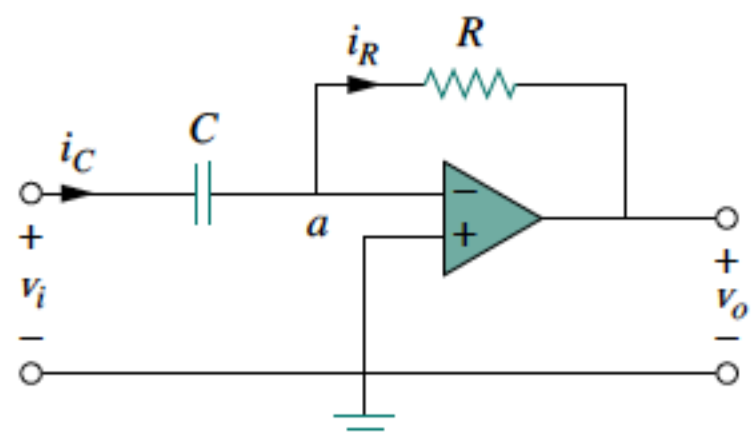
$$i_R = i_C$$

- But $i_R = -\frac{v_o}{R}, \qquad i_C = C\frac{dv_i}{dt}$
- Substituting:

$$v_o = -RC\frac{dv_i}{dt}$$

showing the output is the derivative of the input.

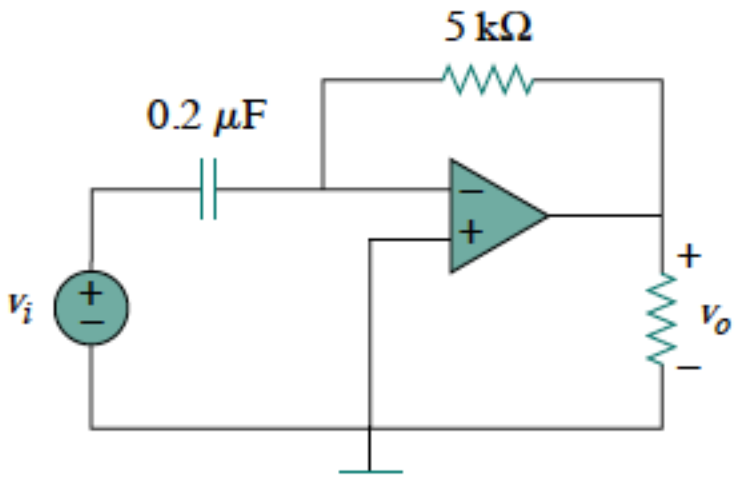
Note: seldom used in practice due to noise.



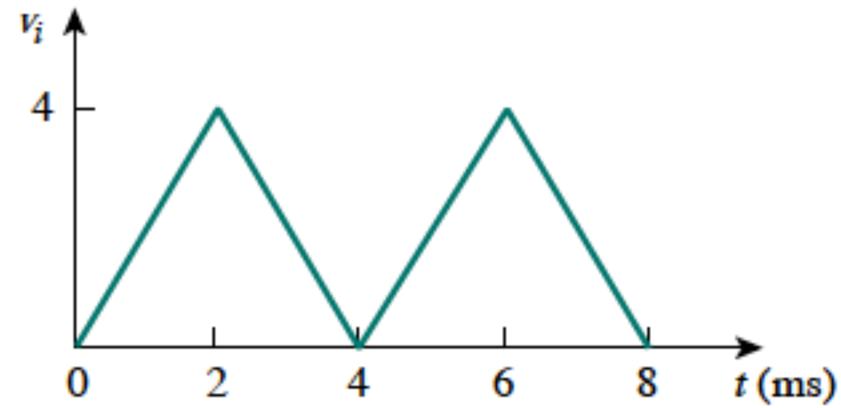
Example

- Sketch the output voltage for the circuit in (a), given the input voltage in (b). Take *vo* = 0 at *t* = 0.

(a)



(b)



- This is a differentiator with $RC = 5 \times 10^3 \times 0.2 \times 10^{-6} = 10^{-3}$ s

For $0 < t < 4$ ms

$$v_i = \begin{cases} 2t & 0 < t < 2 \text{ ms} \\ 8 - 2t & 2 < t < 4 \text{ ms} \end{cases}$$

So the output is:

$$v_o = -RC\frac{dv_i}{dt} = \begin{cases} -2 \text{ mV} & 0 < t < 2 \text{ ms} \\ 2 \text{ mV} & 2 < t < 4 \text{ ms} \end{cases}$$