2019

Bioelectrical Circuits: Lecture 4

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# BME 205 L04

CIRCUIT THEOREMS

Introduction

- We learned how to solve circuits by applying Kirchhoff's laws systematically. But it can be tedious for complex circuits.

- To handle complex circuits, theorems have been developed to simplify circuit analysis, including *Thevenin's* and *Norton's* theorems.

- These theorems are applicable to *linear circuits,* so we first discuss the concept of circuit linearity. We'll also discuss the concepts of *superposition*, *source transformation*, and *maximum power transfer*.

Linearity

- Linearity is the property of an element describing a linear relationship between cause and effect.

- Linearity combines the properties of *homogeneity* (scaling) and *additivity*.

- <u>Homogeneity:</u> if the input (also called the *excitation)* is multiplied by a constant, then the output (also called the response) is multiplied by the same constant.

- e.g. Ohm's law for a resistor: $v = iR$

- If the current is increased by a constant *k,* then the voltage increases correspondingly by *k:* $kv = kiR$

- <u>Additivity:</u> the response to a sum of inputs is the sum of the responses to each input applied separately.

- e.g. for a linear resistor, if $v_1 = i_1 R$ and $v_2 = i_2 R$ then applying $(i_1 + i_2)$ gives $v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2$

- Obviously, a resistor is a linear element.
- In general, a circuit is linear if it is both additive and homogeneous.
- We'll only deal with linear circuits, which consist of only linear elements, linear dependent sources, and independent sources.

A linear circuit is one whose output is linearly related (or directly proportional) to its input.
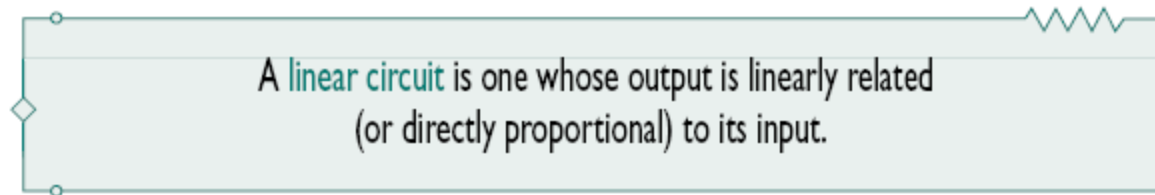
(Note that since $p = i^2R = v^2/R$ (quadratic function), the relationship between power and voltage (or current) is *nonlinear*.)

- Consider the linear circuit shown. It has no independent sources inside it. It is excited by a voltage source v_s, which serves as the input. The circuit is terminated by a load R. We may take the current i through R as the output. Suppose $v_s = 10$ V gives $i = 2$ A. According to the linearity principle, $vs = 1$ V will give $i = 0.2$ A, and $i = 1$ mA must be due to $v_s = 5$ mV.
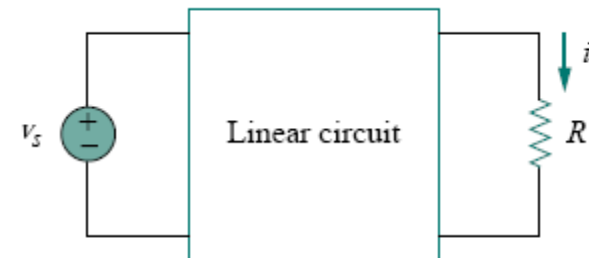
Example

- Find i_0 when v_s = 12V and v_s = 24V.
- Applying KVL to the two loops,

we obtain
$$12i_1 - 4i_2 + v_s = 0$$
$$- 4i_1 + 16i_2 - 3v_x - v_s = 0$$

But $v_x = 2i_1.$ $\implies$ $-10i_1 + 16i_2 - v_s = 0$

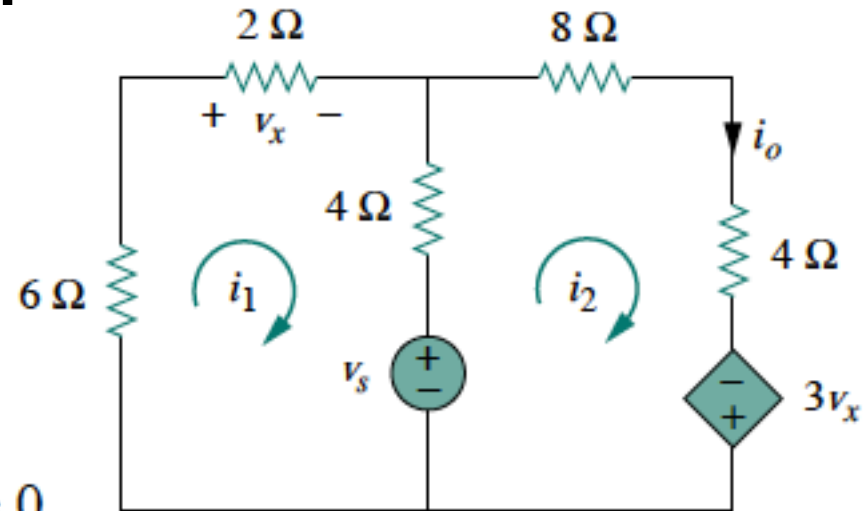add:
$$2i_1 + 12i_2 = 0 \qquad \implies \qquad i_1 = -6i_2$$

substitute:
$$-76i_2 + v_s = 0 \qquad \implies \qquad i_2 = \frac{v_s}{76}$$
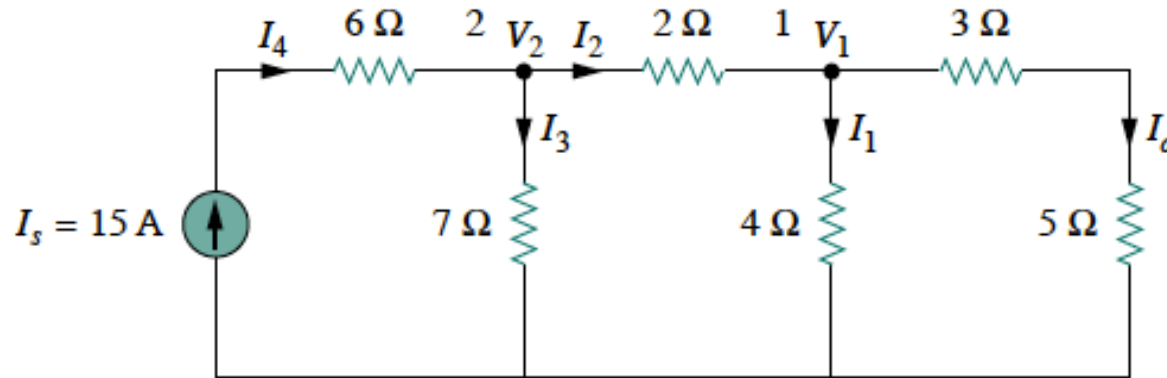
When $v_s = 12$ V,
$$i_o = i_2 = \frac{12}{76} \text{ A}$$

When $v_s = 24$ V,
$$i_o = i_2 = \frac{24}{76} \text{ A}$$

Example

Assume $I_o = 1$ A and use linearity to find the actual value of I_o

Solution:

If $I_o = 1$ A, then $V_1 = (3+5)I_o = 8$ V and $I_1 = V_1/4 = 2$ A. Applying KCL at node 1 gives

$$I_2 = I_1 + I_o = 3 \text{ A}$$

$$V_2 = V_1 + 2I_2 = 8 + 6 = 14 \text{ V}, \qquad I_3 = \frac{V_2}{7} = 2 \text{ A}$$
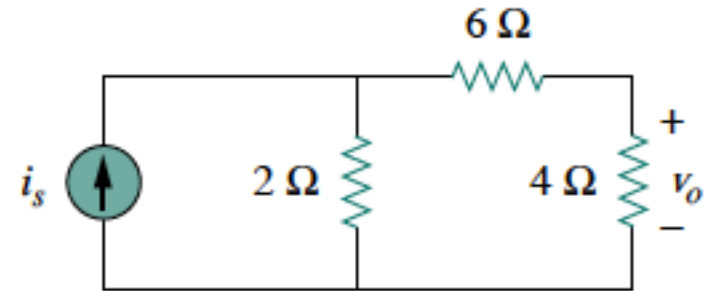
Applying KCL at node 2 gives

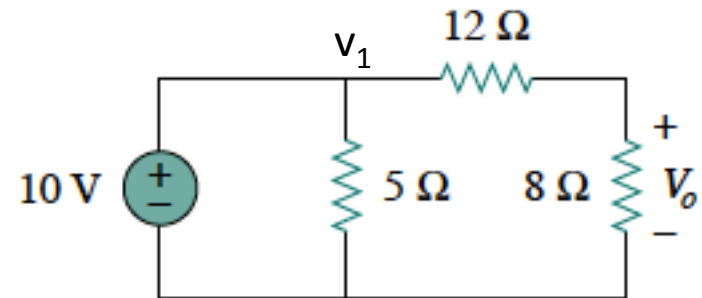$$I_4 = I_3 + I_2 = 5 \text{ A}$$

Therefore, $I_s = 5$ A. This shows that assuming $I_o = 1$ gives $I_s = 5$ A; the actual source current of 15 A will give $I_o = 3$ A as the actual value.

Practice

Find v_0 when $i_s = 15$ A and $i_s = 30$ A.

Assume that $V_o = 1$ V and use linearity to calculate the actual value of V_o

Superposition

- If a circuit has two or more independent sources, one way to determine the value of a specific variable (voltage or current) is to use nodal or mesh analysis as in Lecture 3.

- Another way is to determine the contribution of each independent source to the variable and then add them up. The latter approach is known as *superposition* and it relies on linearity.

  The voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

- The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.

- To apply the superposition principle, we must keep two things in mind:

1. We consider one independent source at a time while all other independent sources are *turned off.* This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit). This way we obtain a simpler and more manageable circuit.

2. Dependent sources are left intact because they are controlled by circuit variables.

Superposition is applied in 3 steps:

Steps to Apply Superposition Principle:

1. Turn off all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.

2. Repeat step 1 for each of the other independent sources.

3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

NOTES

- Analyzing a circuit using superposition may involve more work - If the circuit has three independent sources, we may have to analyze three simpler circuits each providing the contribution due to the respective individual source. However, superposition does help reduce a complex circuit to simpler circuits through replacement of voltage sources by short circuits and of current sources by open circuits.

- Keep in mind that superposition is based on linearity. For this reason, it is not applicable to the effect on power due to each source, because the power absorbed by a resistor depends on the square of the voltage or current. If the power value is needed, the current through (or voltage across) the element must be calculated first using superposition.

Example

Use the superposition theorem to find v

8 Ω
6 V
4 Ω
v
3 A

Solution:
Since there are two sources let $v = v_1 + v_2$ where v_1 and v_2 are the contributions due to the 6-V voltage source and the 3-A current source, respectively.
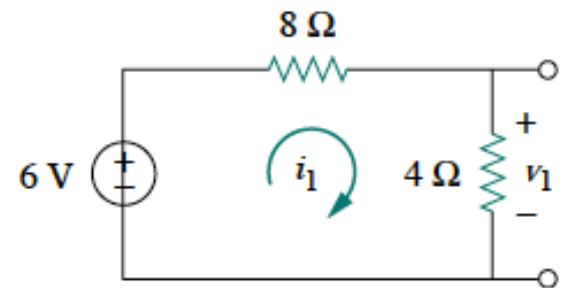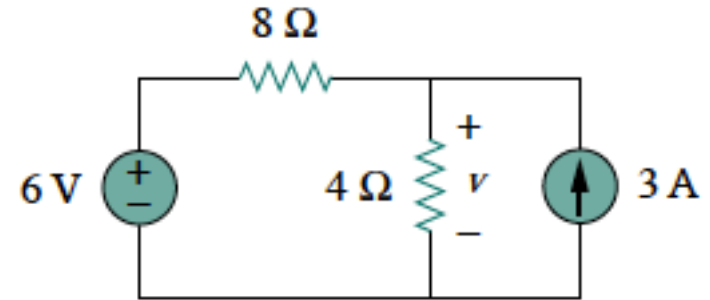
Get v_1 from (a): (voltage division)

$$v_1 = \frac{4}{4+8}(6) = 2 \text{ V}$$
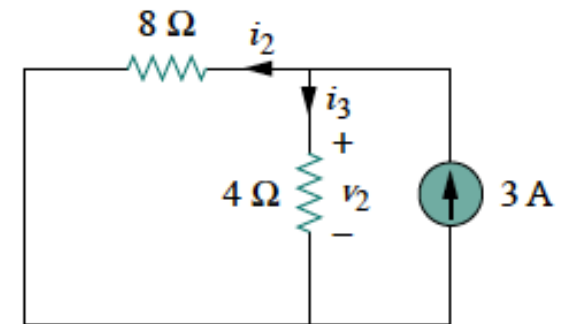
Get v_2 from (b)... (current division)

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A}$$

$$\implies \quad v_2 = 4i_3 = 8 \text{ V}$$

Then:

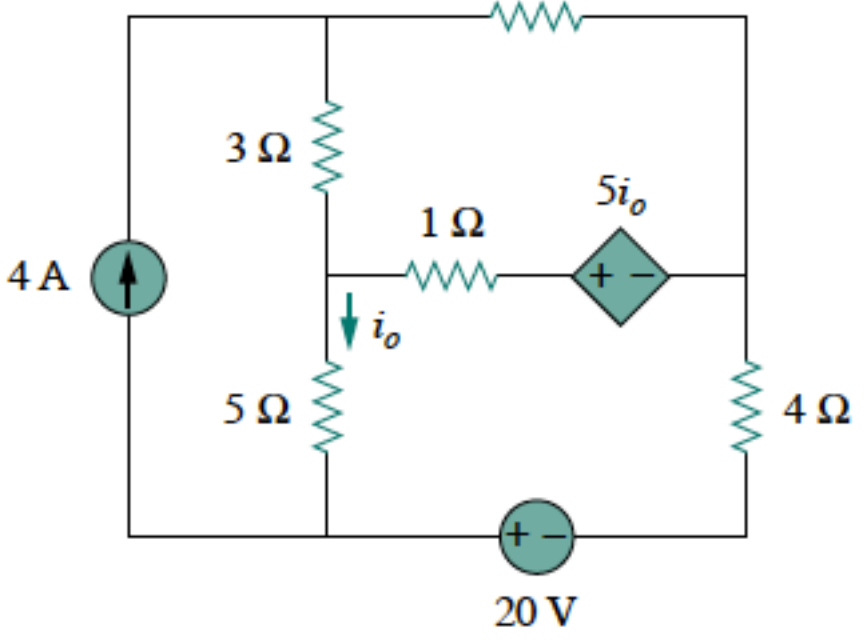$$v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$$

(a)

(b)

Examples

1) Using the superposition theorem, find v_o

2)

$2\,\Omega$

Find i_o

$3\,\Omega$

$5i_o$

$1\,\Omega$

4 A
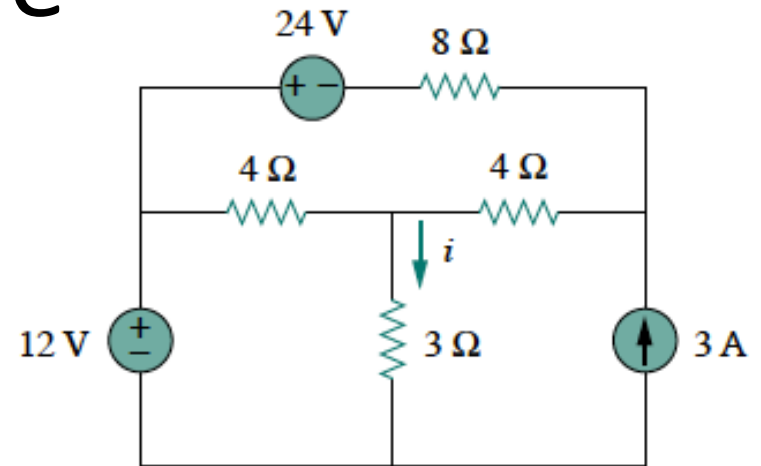
i_o

$5\,\Omega$

$4\,\Omega$

$+ -$

20 V
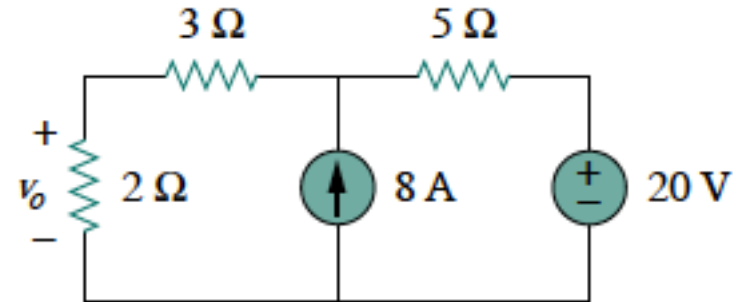
Example

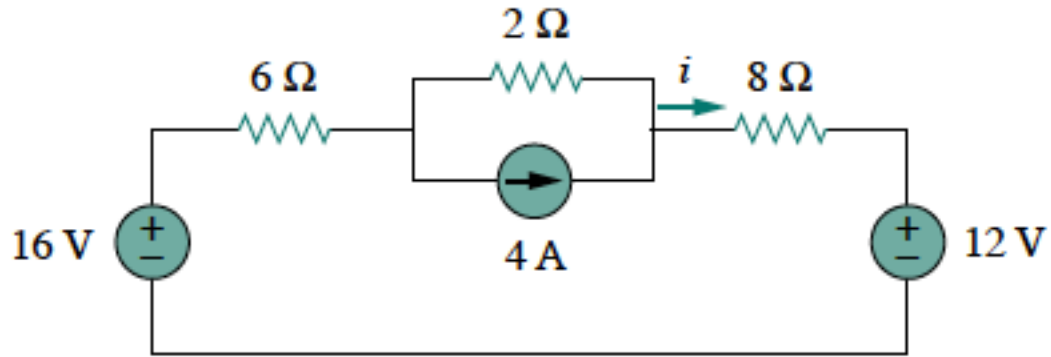- use the superposition theorem to find *i*.

$$i = i_1 + i_2 + i_3$$

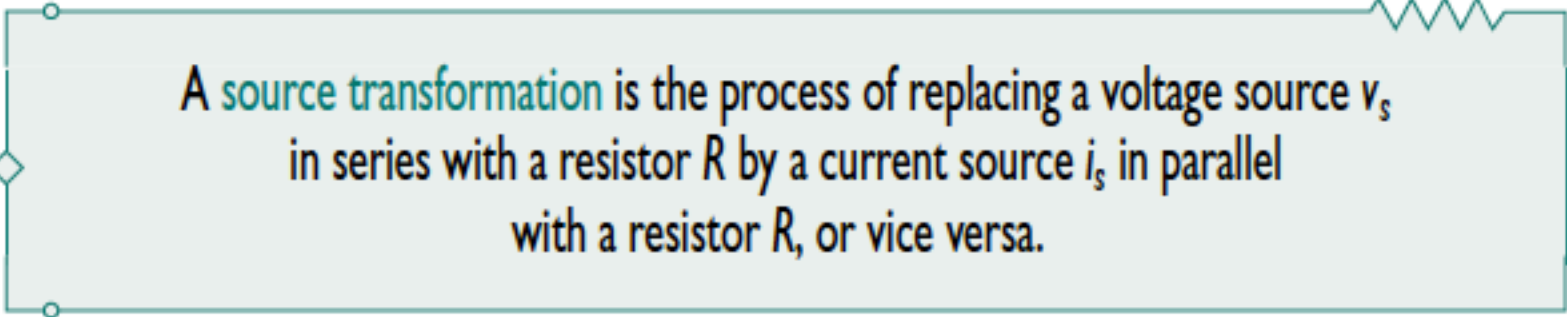where i_1, i_2, and i_3 are due to the 12-V, 24-V, and 3-A sources respectively.

Practice Problem

- Find *i* in this circuit using the superposition principle.

Source transformation

- *Source transformation* is another tool for simplifying circuits. Basic to these tools is the concept of *equivalence.*

- Recall that an equivalent circuit is one whose *v-i characteristics are* identical with the original circuit.

- In Lecture 3, we saw that node-voltage (or mesh-current) equations can be obtained by mere inspection of a circuit when the sources are all independent current (or all independent voltage) sources.

- It is therefore expedient in circuit analysis to be able to substitute a voltage source in series with a resistor for a current source in parallel with a resistor, or vice versa.

- This is known as a *source transformation.*

A source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R, or vice versa.
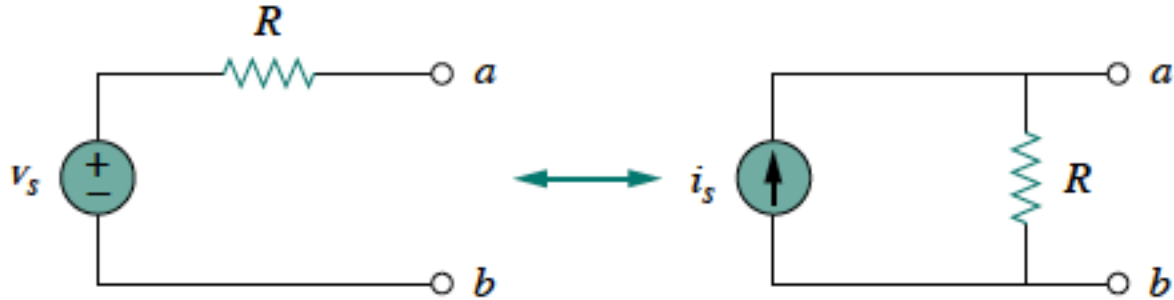
- These two circuits are equivalent – i.e. they have the same voltage-current relation at terminals *a-b*. This is easy to show:

- If the sources are turned off, the equivalent resistance at terminals *a-b* in both circuits is *R*.

- Also, when terminals *a-b* are short circuited, the short-circuit current flowing from *a* to *b* is $i_{sc} = v_s/R$ *in* the circuit on the left-hand side and $i_{sc} = i_s$ for the circuit on the right hand side.

- Thus, $v_s/R = i_s$ in order for the two circuits to be equivalent.

- Hence, source transformation requires that

$$v_S = i_S R \qquad \text{or} \qquad i_S = \frac{v_S}{R}$$

- Source transformation also applies to dependent sources, provided we carefully handle the dependent variable. A dependent voltage source in series with a resistor can be transformed to a dependent current source in parallel with the resistor or vice versa:
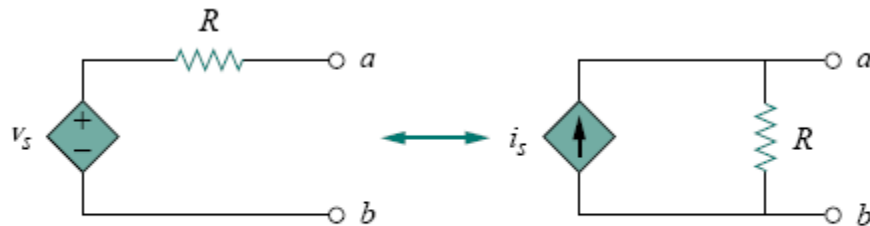

- A source transformation does not affect the remaining part of the circuit. When applicable, source transformation is a powerful tool that allows circuit manipulations to ease circuit analysis. However, we should keep the following points in mind:

1. Note that the arrow of the current source in the last two figures is directed toward the positive terminal of the voltage source.

2. Note that source transformation is not possible when $R = 0$, which is the case with an ideal voltage source. However, for a practical, nonideal voltage source, $R \neq 0$. Similarly, an ideal current source with $R = \infty$ cannot be replaced by a finite voltage source.
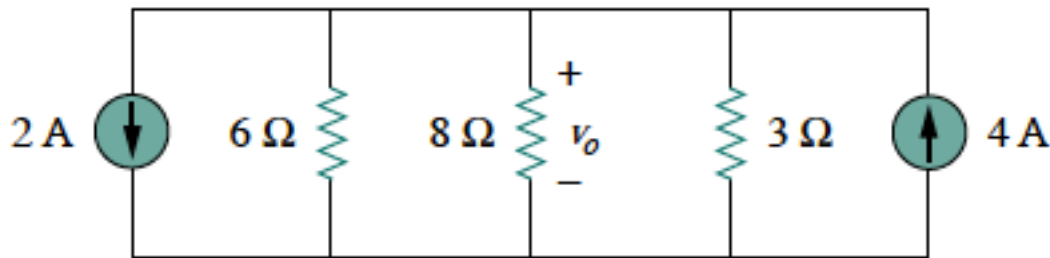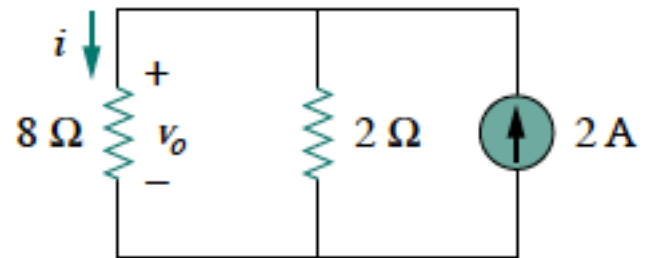
Example

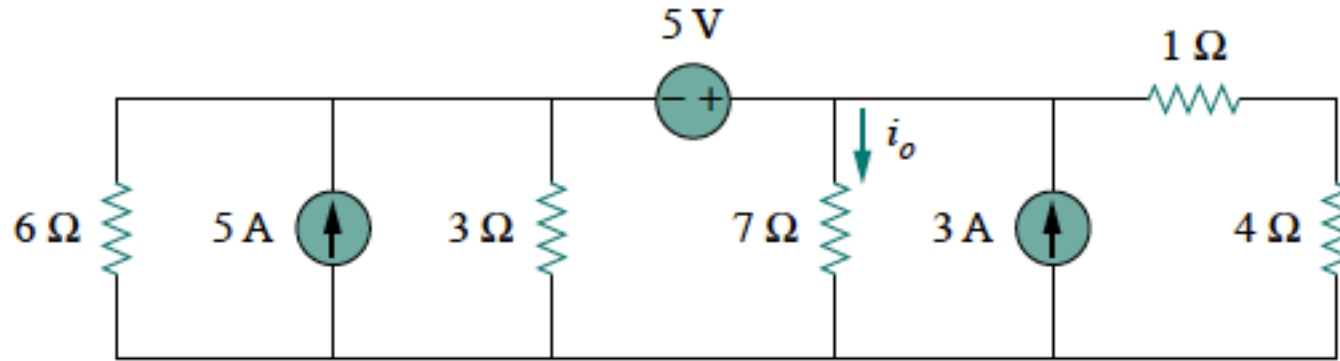Use source transformation to find v_o
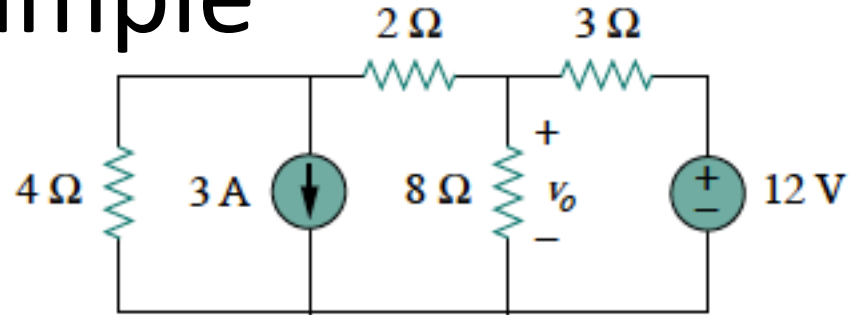
three transforming steps:
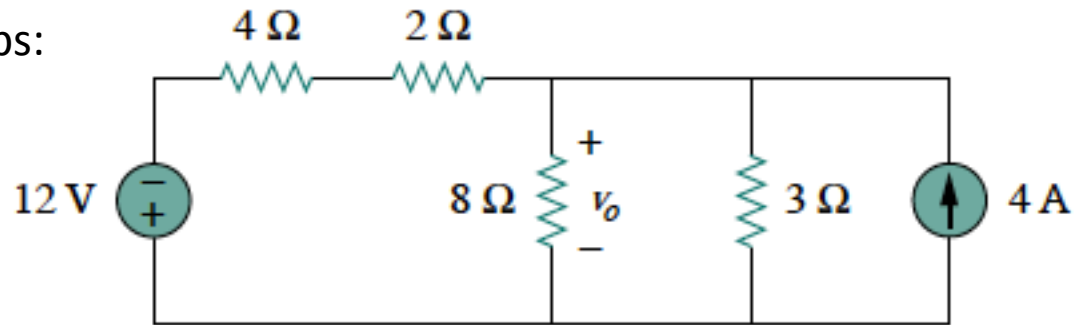
(a)

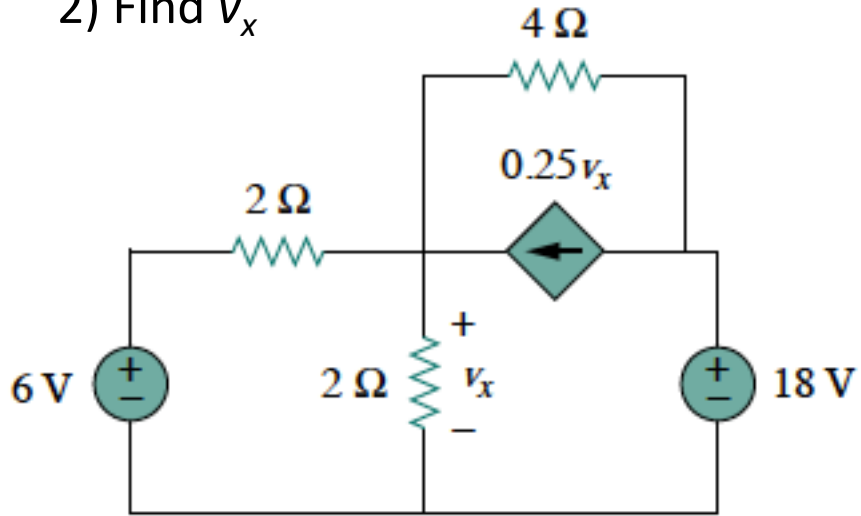(b)

(c)

$i=2(2/10)=0.4$ A $\rightarrow$ $v_o=3.2$ V

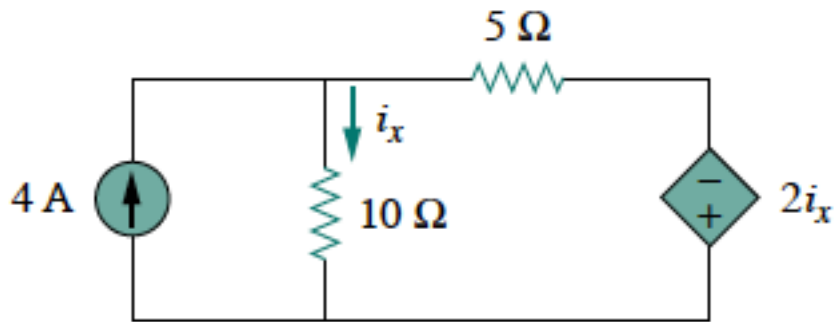# Practice examples

1) Find i_0 using source transformation.

Practice examples

2) Find v_x

Practice examples

Thevenin's Theorem

- It often occurs in practice that a particular element in a circuit is variable (usually called the *load*) while other elements are fixed.

- As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load. Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

- According to Thevenin's theorem, the linear circuit (a) can be replaced by that in (b).

(a)

Linear two-terminal circuit

I a

$+$
V Load
$-$

b

(b)

R_{Th} I a

V_{Th}

$+$
V Load
$-$

b
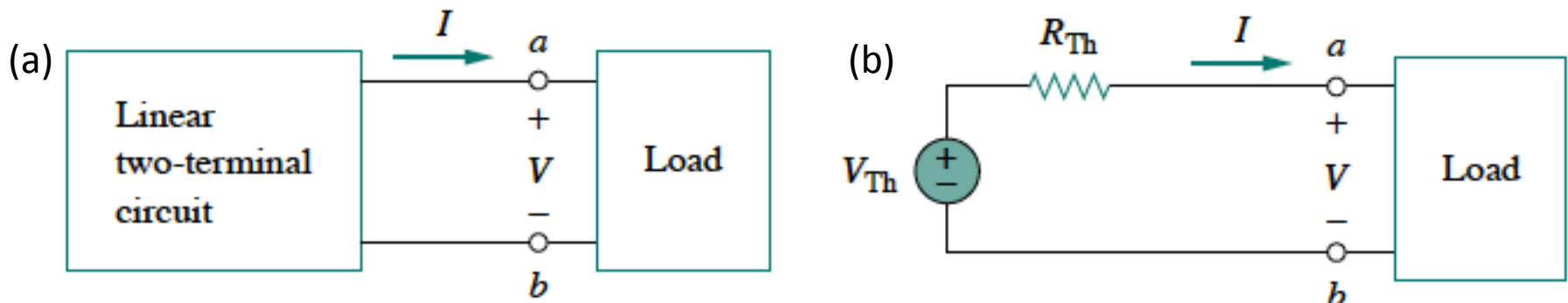
(The load may be a single resistor or another circuit.) The circuit to the left of the terminals *a-b* is known as the *Thevenin equivalent circuit.*

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.

- Two circuits are said to be *equivalent if they have the* same voltage-current relation at their terminals.

- What will make the last two circuits (a) and (b) equivalent?

- If the terminals *a-b* are made open-circuited (by removing the load), no current flows, so that the open-circuit voltage across the terminals *a-b* in (a) must be equal to the voltage source V_{Th} in (b) since the two circuits are equivalent. Thus V_{Th} is the open-circuit voltage across the terminals as shown below; that is, $V_{Th} = v_{oc}$
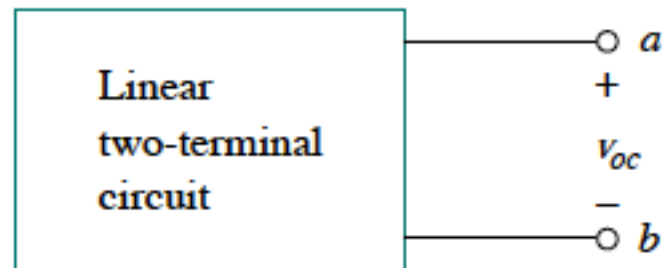
$V_{Th} = v_{oc}$

- Again, with the load disconnected and terminals *a-b* open-circuited, we turn off all independent sources. The input resistance (or equivalent resistance) of the dead circuit at the terminals *a-b* in (a) must be equal to R_{Th} in (b) because the two circuits are equivalent.

- Thus, R_{Th} is the input resistance at the terminals when the independent sources are turned off, as shown below; that is, $R_{\text{Th}} = R_{\text{in}}$
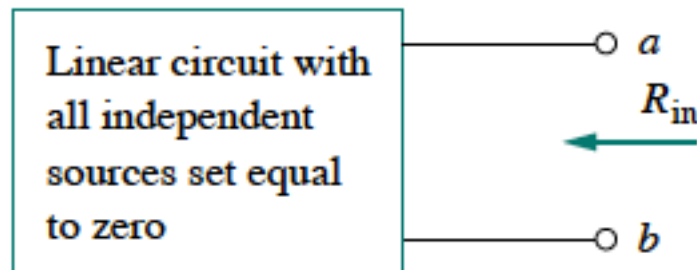
$$R_{\text{Th}} = R_{\text{in}}$$

To apply this idea in finding R_{Th}, we need to consider two cases:

CASE 1 If the network **has no dependent sources**, we turn off all independent sources. R_{Th} is the input resistance of the network looking between terminals a and b, as shown above.

CASE 2 If the network **has dependent sources**, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by circuit variables.

Case 2 cont'd: We apply a voltage source v_0 at terminals a and b and determine the resulting current i_0. Then $R_{Th} = v_0 / i_0$, as shown below
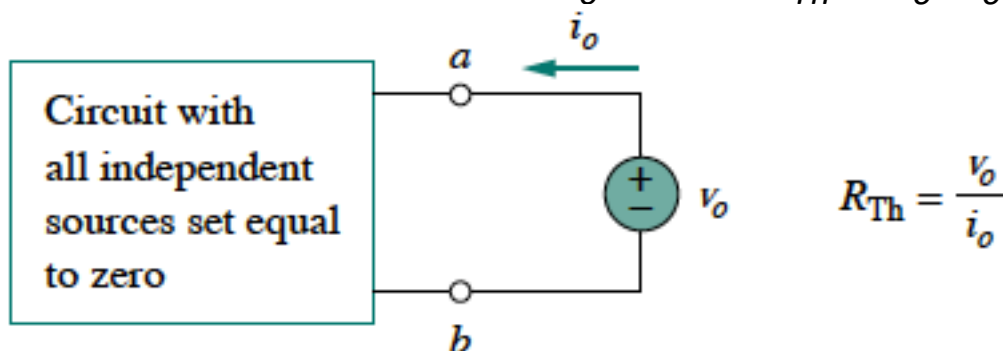

Alternatively, we may insert a current source i_0 at terminals a-b as shown below and find the terminal voltage v_0 .
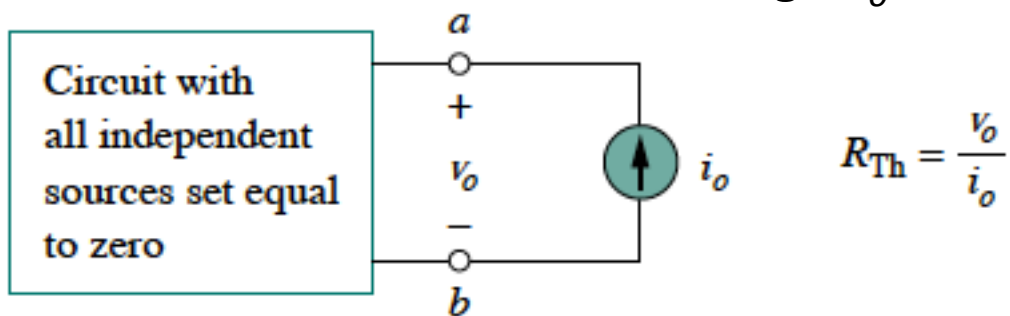
Again $R_{Th} = v_0 / i_0$. Either of the two approaches will give the same result. We may assume any value of v_0 and i_0 . For example, we may use $v_0 = 1$ V or $i_0 = 1$ A, or even use unspecified values of v_0 or i_0 .

- Thevenin's theorem is very important in circuit analysis. A large circuit may be replaced by a single independent voltage source and a single resistor. This is a powerful tool in circuit design.
- As mentioned earlier, a linear circuit with a variable load can be replaced by the Thevenin equivalent, exclusive of the load. The equivalent network behaves the same way externally as the original circuit. Consider a linear circuit terminated by a load R_L :
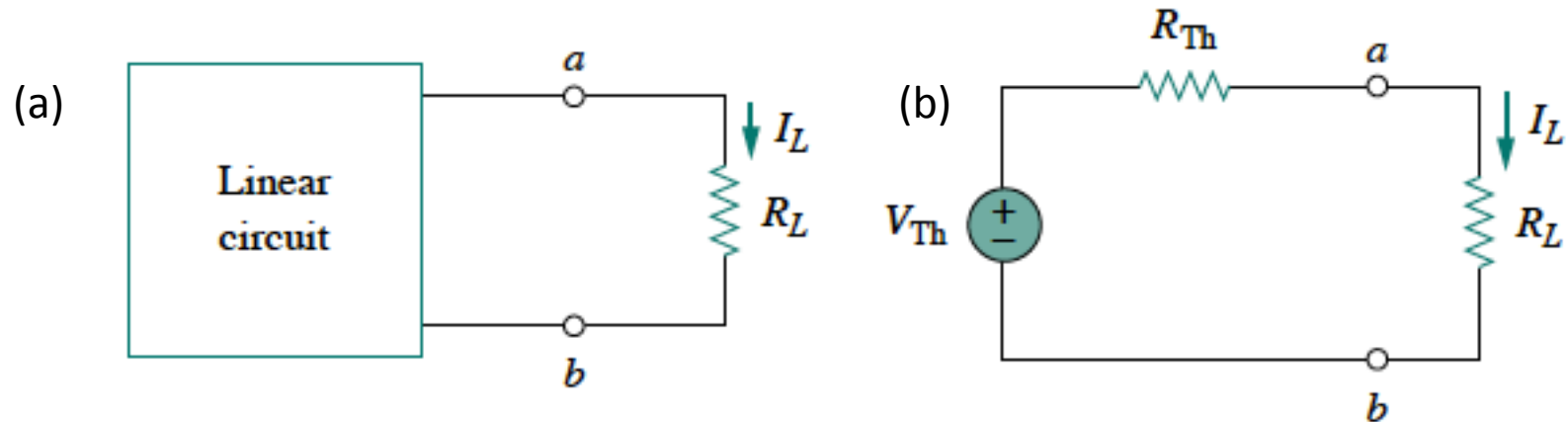

- The current I_L through the load and the voltage V_L across the load are easily determined once the Thevenin equivalent of the circuit at the load's terminals is obtained, as shown in (b). We obtain

$$I_L = \frac{V_{Th}}{R_{Th} + R_L}$$

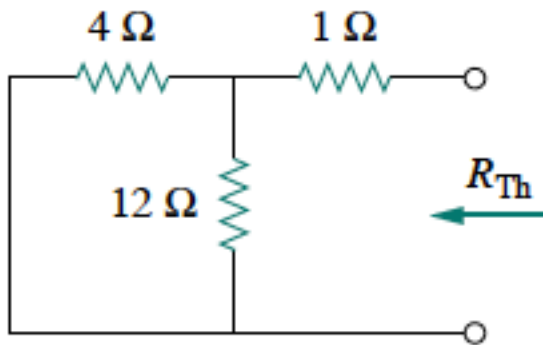$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

Example

- Find the Thevenin equivalent circuit of the circuit shown, to the left of the terminals *a-b*. Then find the current through *RL = 6, 16,* and 36 ohms.
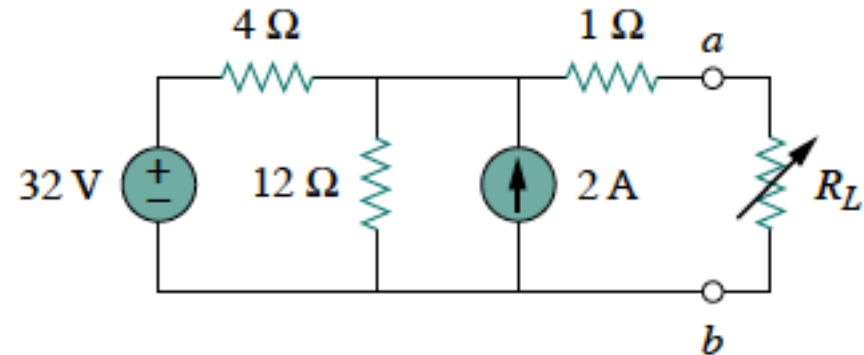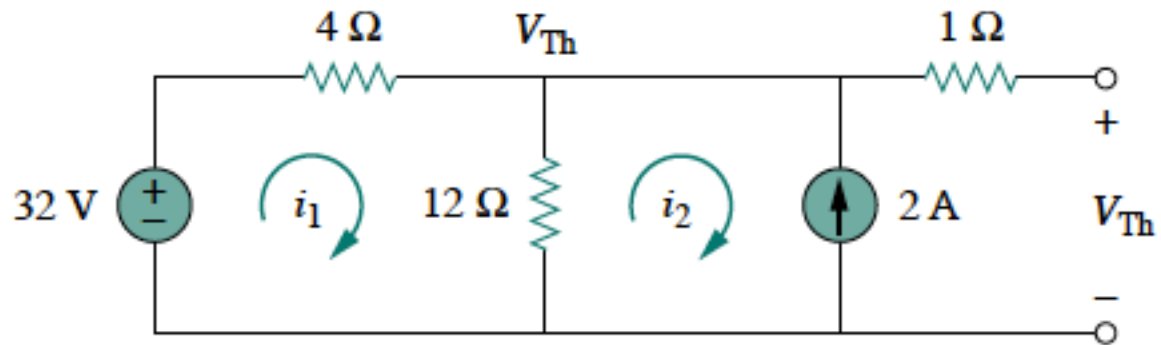

We find R_{Th} by turning off the 32-V voltage source and the 2-A current source

$$R_{Th} = 4 \| 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\,\Omega$$

To find V_{Th}, consider

(a)

(b) $V_{Th} = 30$ V

The Thevenin equivalent circuit is shown in Fig. 4.29. The current through R_L is

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$,

$$I_L = \frac{30}{10} = 3 \text{ A}$$

When $R_L = 16$,

$$I_L = \frac{30}{20} = 1.5 \text{ A}$$

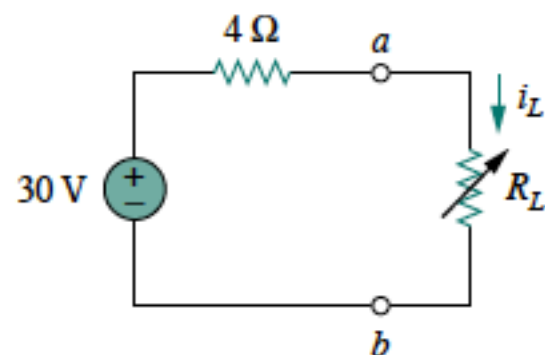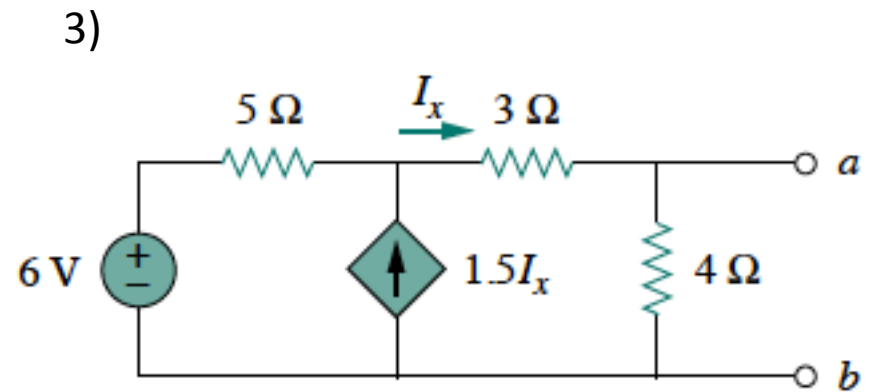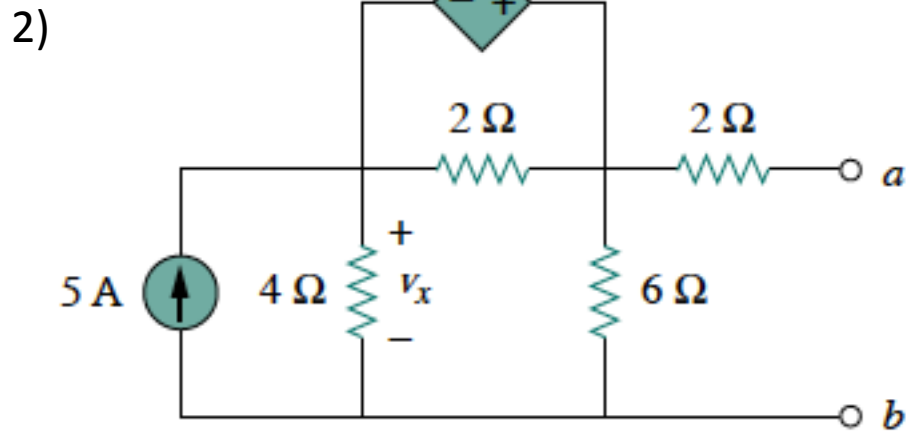When $R_L = 36$,
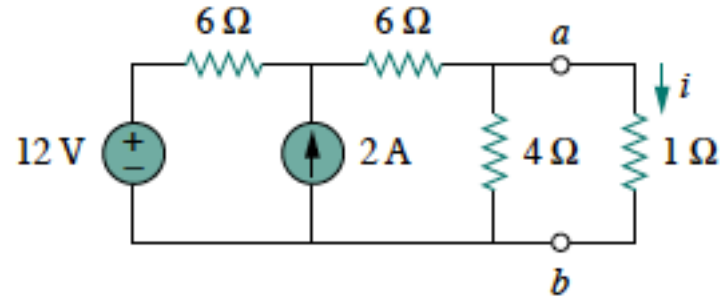
$$I_L = \frac{30}{40} = 0.75 \text{ A}$$

Figure 4.29

Examples/Practice

1) Find Equivalent circuit to the left of terminals a-b

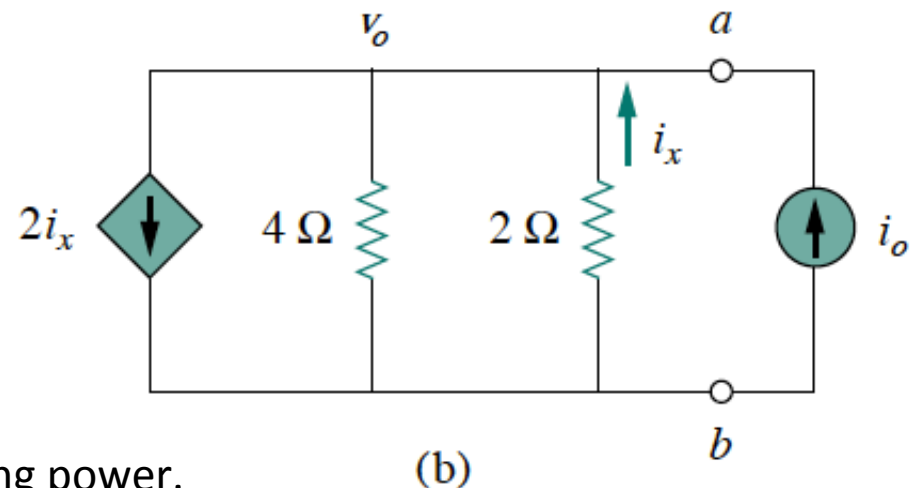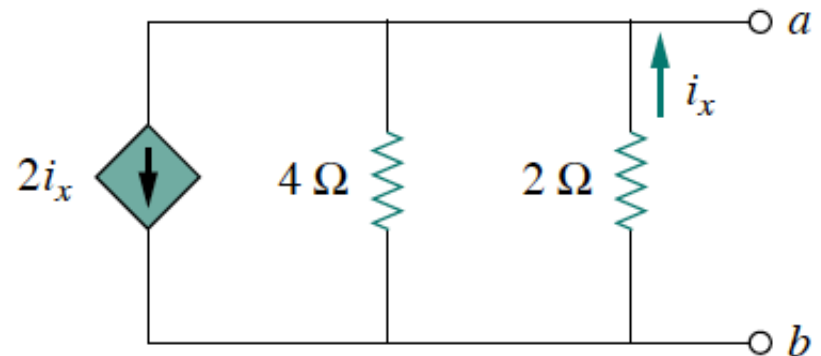2)

3)

Example

Find the Thevenin equivalent.

Since it has no independent sources, $V_{Th}=0$. To find R_{Th} apply a current source at the terminals as in (b). Nodal analysis gives:

$$i_o + i_x = 2i_x + \frac{v_o}{4}$$

but

$$i_x = \frac{0 - v_o}{2} = -\frac{v_o}{2}$$

so

$$v_o = -4i_o$$

Thus

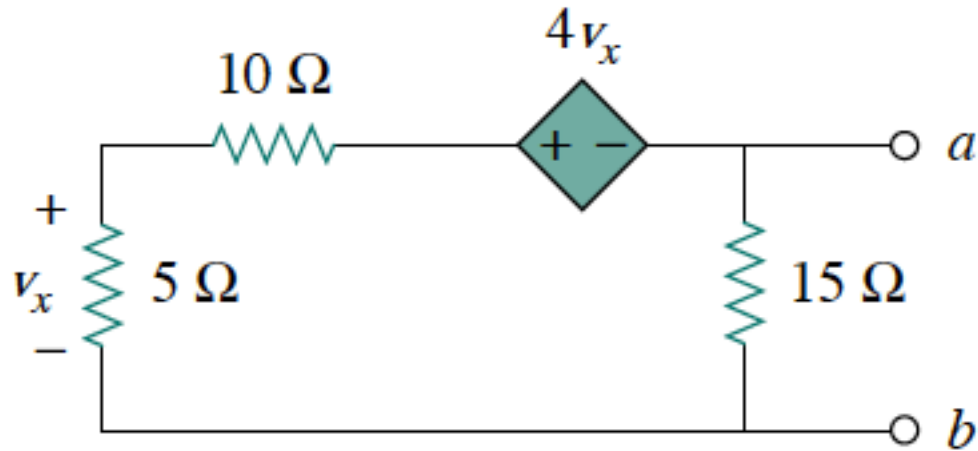$$R_{Th} = \frac{v_o}{i_o} = -4 \ \Omega$$

(b)

Negative resistance tells us the circuit is supplying power.

This must come from the dependent source – we've used it to simulate negative resistance.

Practice Problem

- Obtain the Thevenin equivalent of this circuit:
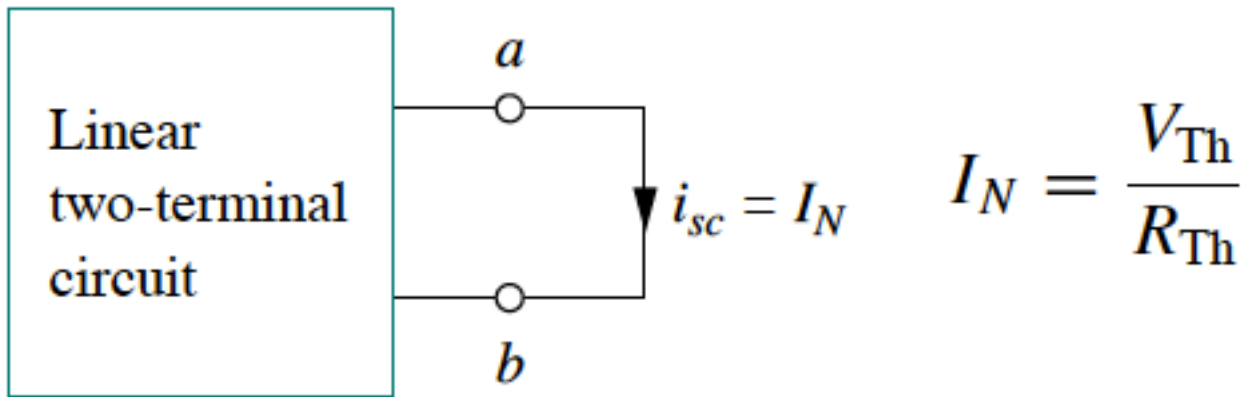
Norton's theorem

- Very similar to Thevenin:

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

Linear two-terminal circuit

$\circ$ a

$\circ$ b

can be replaced by:

I_N

R_N

$\circ$ a

$\circ$ b
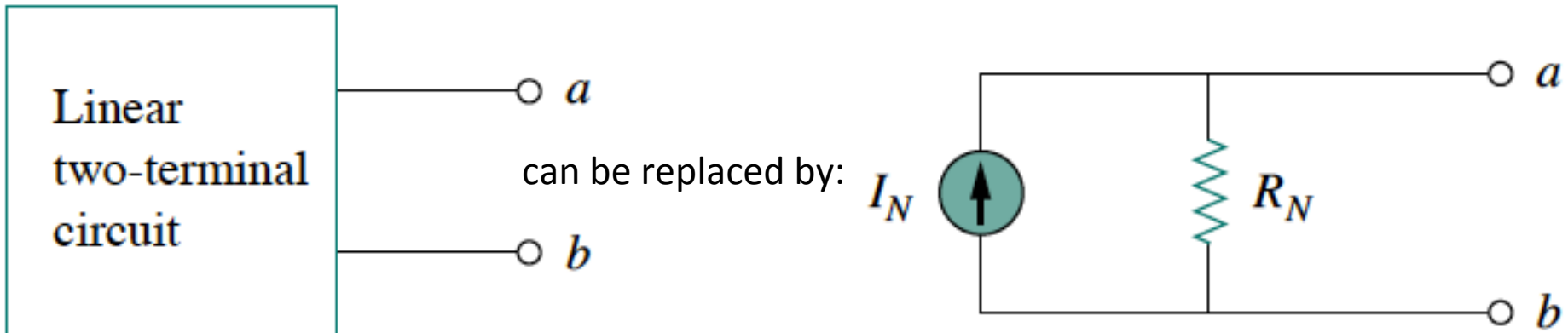
- We find R_N in the same way we find R_{Th}.
- In fact, from what we know about source transformation, the Thevenin and Norton resistances are equal; that is

$$R_N = R_{Th}$$

- To find the Norton current I_N, we determine the short-circuit current flowing from terminal a to b. The short-circuit current in the Norton equivalent is I_N, which must be the same as in the original circuit since the two circuits are equivalent. Thus:

$$i_{sc} = I_N \qquad I_N = \frac{V_{Th}}{R_{Th}}$$

- This is essentially source transformation. For this reason, source transformation is often called Thevenin-Norton transformation.
- Dependent and independent sources are treated the same way as in Thevenin's theorem.

So to determine the Thevenin or Norton equivalent circuit requires that we find:

• The open-circuit voltage v_{oc} across terminals a and b.

• The short-circuit current i_{sc} at terminals a and b.

• The equivalent or input resistance R_{in} at terminals a and b when all independent sources are turned off.

We can calculate any two of the three using the method that takes the least effort and use them to get the third using Ohm's law. Also since

$$V_{\text{Th}} = v_{oc}$$

$$I_N = i_{sc}$$

$$R_{\text{Th}} = \frac{v_{oc}}{i_{sc}} = R_N$$

the open-circuit and short-circuit tests are sufficient to find any Thevenin or Norton equivalent.

Example

8 Ω

4 Ω

2 A

12 V

5 Ω

8 Ω

Figure 4.39   For Example 4.11.

8 Ω

4 Ω

5 Ω

R_N

8 Ω

a

b

(a)

Figure 4.40

Find the Norton equivalent circuit of the circuit in Fig. 4.39.

Solution:

We find R_N in the same way we find R_{Th} in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 4.40(a), from which we find R_N. Thus,

$$R_N = 5 \parallel (8 + 4 + 8) = 5 \parallel 20 = \frac{20 \times 5}{25} = 4\,\Omega$$

To find I_N, we short-circuit terminals a and b, as shown in Fig. 4.40(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2\,\text{A}, \qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1\,\text{A} = i_{sc} = I_N$$

Alternatively, we may determine I_N from $V_{\text{Th}}/R_{\text{Th}}$. We obtain V_{Th} as the open-circuit voltage across terminals a and b in Fig. 4.40(c). Using mesh analysis, we obtain

$$i_3 = 2\,\text{A}$$

$$25i_4 - 4i_3 - 12 = 0 \qquad \Longrightarrow \qquad i_4 = 0.8\,\text{A}$$

and

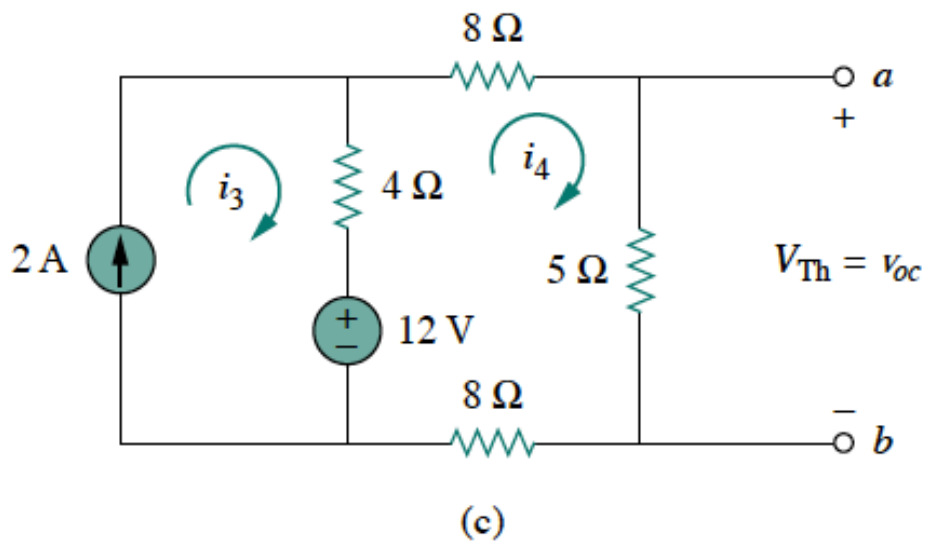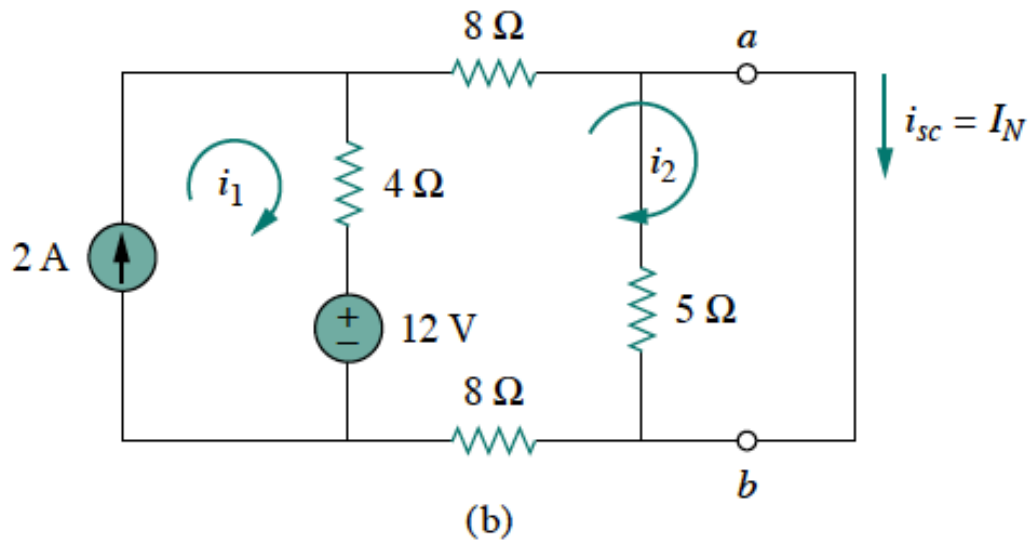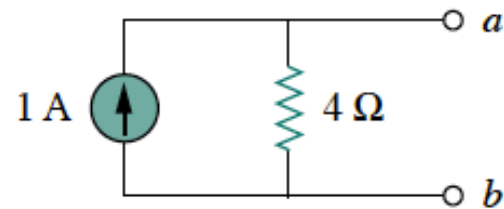$$v_{oc} = V_{\text{Th}} = 5i_4 = 4\,\text{V}$$
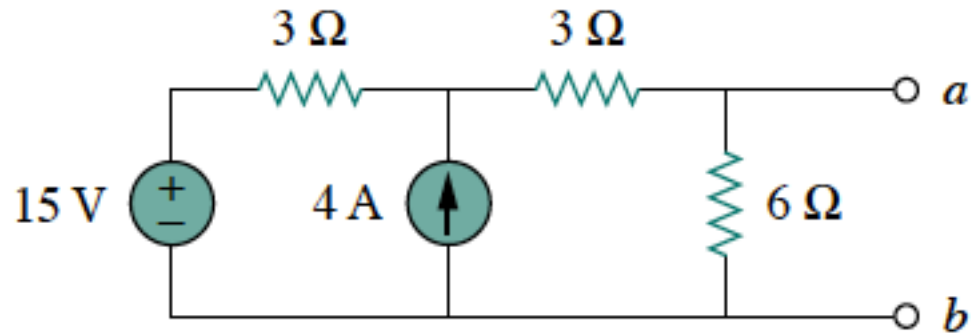
(b)

(c)

Figure 4.40

Thus,

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1 \text{ A}$$
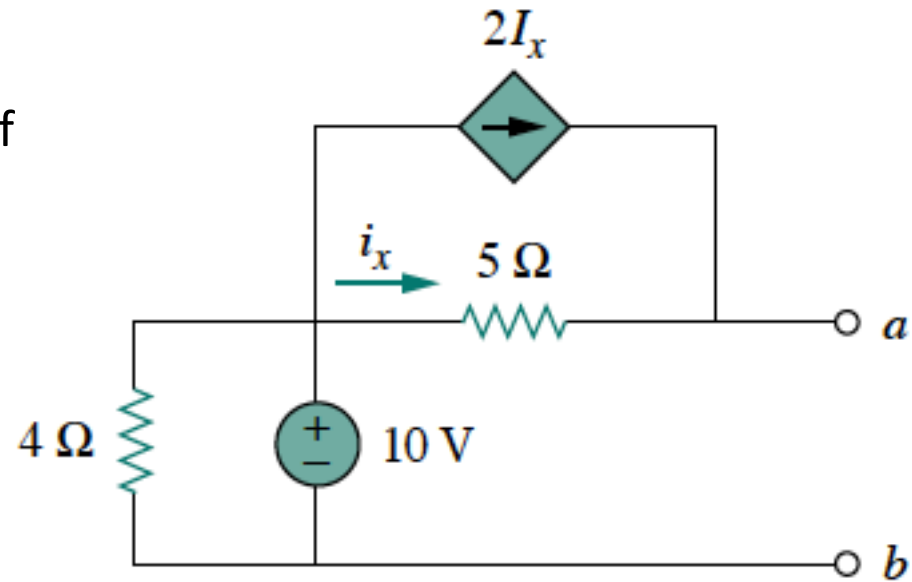
and the Norton equivalent cct is:

Practice Problem
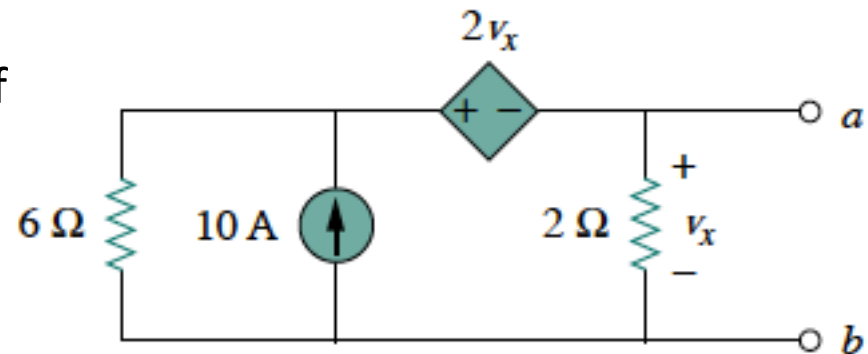
- Find the Norton Equivalent circuit

Examples

1) Using Norton's theorem, find R_N and I_N of this circuit between terminals a & b

2) Using Norton's theorem, find R_N and I_N of this circuit between terminals a & b

Maximum Power Transfer

- In many practical situations, a circuit is designed to provide power to a load. While for electric utilities, minimizing power losses in the process of transmission and distribution is critical for efficiency and economic reasons, there are other applications in areas such as communications where it is desirable to maximize the power delivered to a load.

- How do we deliver the maximum power to a load when given a system with known internal losses?

- The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance R_L.

- If the entire circuit is replaced by its Thevenin equivalent except for the load as shown, the power delivered to the load is

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$
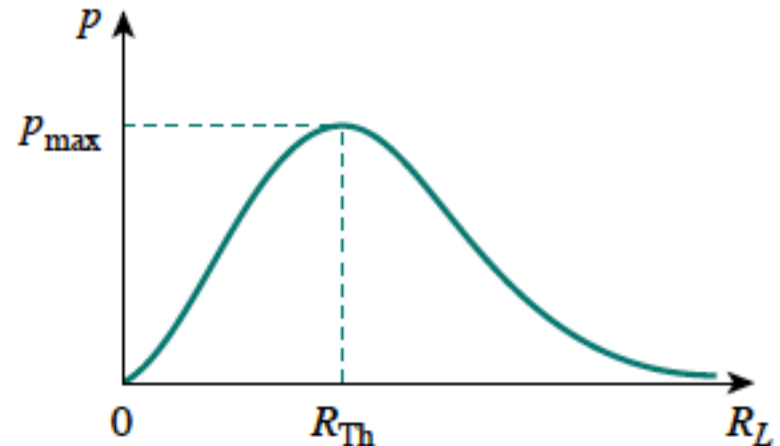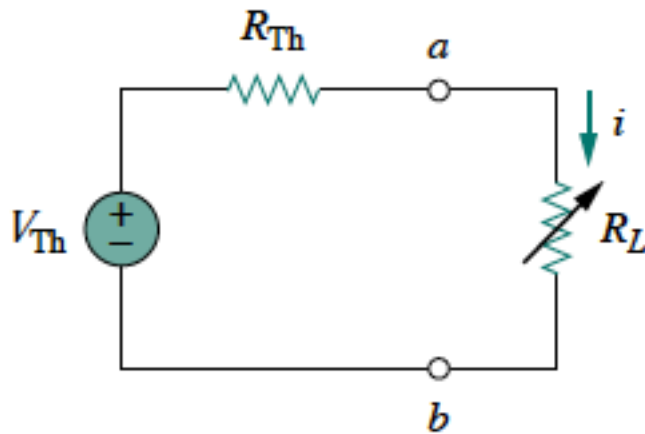
- For a given circuit, V_{Th} and R_{Th} are fixed. By varying the load resistance R_L, the power delivered to the load varies as sketched above. The power is small for small or large values of R_L but maximum for some value of R_L between 0 and ∞.

- It's easy to show that this maximum power occurs when $R_L = R_{Th}$. This is known as the *maximum power theorem.*