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Bioelectrical Circuits: Lecture 7

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BME 205 L07

FIRST ORDER CIRCUITS

Introduction

At this stage we have dealt with three passive elements (resistor capacitor and inductor) individually. Now we will start considering circuits with combinations of them.

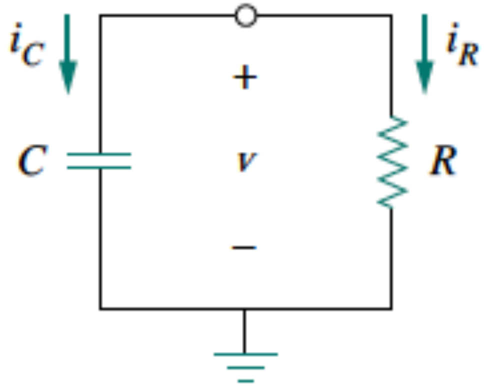
First we'll examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor. These are called *RC* and *RL* circuits, respectively. RC and RL circuits find continual applications in electronics, communications, and control systems.

- We carry out the analysis of *RC* and *RL* circuits by applying Kirchhoff's laws, as we did for resistive circuits. The only difference is that applying Kirchhoff's laws to purely resistive circuits results in algebraic equations, while applying the laws to *RC* and *RL* circuits produces differential equations.
- The differential equations resulting from analyzing *RC and RL* circuits are of the first order. Hence, the circuits are collectively known as *first-order circuits.*

Introduction

- There are two ways to excite RC and RL circuits.
- *source-free circuits:* excited by initial conditions of the storage elements in the circuits. Energy is initially stored in the capacitive or inductive element and this energy causes current to flow in the circuit and is gradually dissipated in the resistors. Although source-free circuits are by definition free of independent sources, they may have dependent sources.
- The second way of exciting first-order circuits is by independent sources.
- In this lecture, the independent sources we will consider are dc sources. (In later lectures, we shall consider sinusoidal and exponential sources.) The two types of first-order circuits and the two ways of exciting them add up to the four possible situations we will study in this lecture.

The source-free *RC* circuit



- A source-free *RC* circuit occurs when a dc source that was connected across the R and C is suddenly disconnected.
- The energy already stored in the capacitor is released to the resistor(s).
- Consider a series combination of a resistor and an initially charged capacitor, as shown. (The resistor and capacitor may be the equivalent resistance and equivalent capacitance of combinations of resistors and capacitors.)
- We want to determine the circuit response, v(t) across the capacitor. Recall v(t) this has to be continuous.
- Since the capacitor is initially charged, we can assume that at time *t* = 0, the initial voltage is $v(0) = V_0$

with the corresponding value of the energy stored as $w(0) = \frac{1}{2} C V_0^2$

Applying KCL at the top node of the circuit $i_C + i_R = 0$

By definition, $i_C = C\,dv/dt$ and $i_R = v/R$. Thus,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

or

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a *first-order differential equation*, since only the first derivative of *v* is involved. To solve it, we rearrange the terms as

$$\frac{dv}{v} = -\frac{1}{RC}\,dt$$

Integrating both sides, we get

$$\ln v = -\frac{t}{RC} + \ln A$$

where ln *A* is the integration constant. Thus, $\ln\frac{v}{A} = -\frac{t}{RC}$

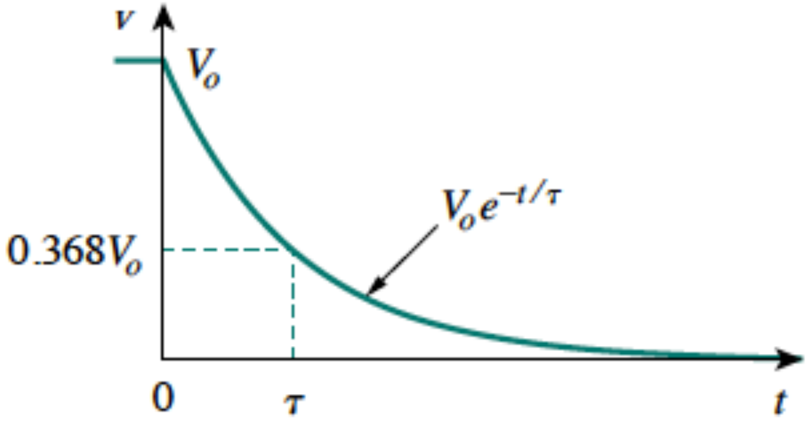
Taking powers of *e* produces $v(t) = Ae^{-t/RC}$

But from the initial conditions, $v(0) = A = V_0$. Hence,

$$v(t) = V_0e^{-t/RC}$$

- This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.
- The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- The natural response is illustrated below. Note that at $t = 0$, we have the correct initial condition V_0. As t increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by the lower case Greek letter tau, τ .

The time constant of a circuit is the time required for the response to decay by a factor of $1/\boldsymbol{e}$ or 36.8 % of its initial value.



- This implies that at $t = \tau$, $V_0 e^{-\tau/RC} = V_0 e^{-1} = 0.368 V_0$

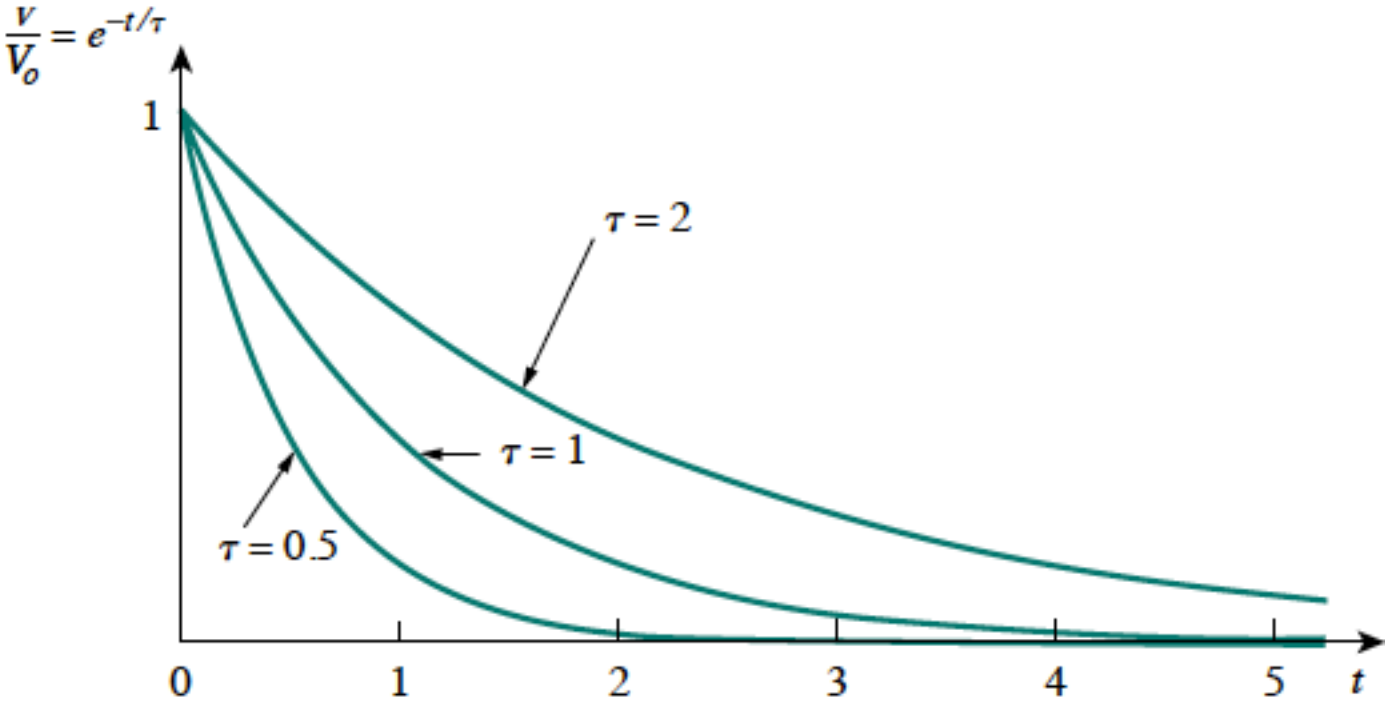
or $\tau = RC$

- In terms of the time constant, we can write the equation as

$$v(t) = V_0 e^{-t/\tau}$$

- It's easy to show with a calculator that the voltage $v(t)$ is less than 1 percent of V_0 after 5τ (five time constants). Thus, it is customary to assume that the capacitor is "fully discharged" (or charged) after 5 time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.
- Notice that for every time interval of τ , the voltage is reduced by 36.8 percent of its previous value, $v(t + \tau) = v(t)/e = 0.368v(t)$, regardless of the value of t .
- Observe that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.

- A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit is regarded to reach steady state in five time constants.



- From the equation for v(t) we can find the current $i_R(t)$,

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}$$

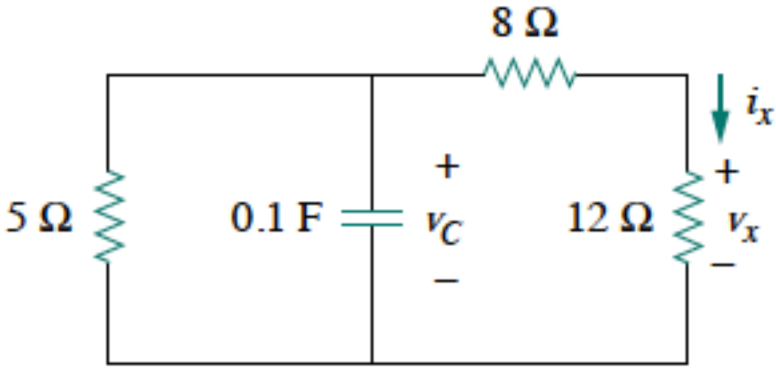
- The power dissipated in the resistor is $p(t) = v i_R = \frac{V_0^2}{R} e^{-2t/\tau}$
- The energy absorbed by the resistor up to time *t* is

$$w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R} e^{-2t/\tau} dt$$

$$= -\frac{\tau V_0^2}{2R} e^{-2t/\tau} \Big|_0^t = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad \tau = RC$$

- Notice that as $t \to \infty, w_R(\infty) \to \frac{1}{2} C V_0^2$, which is the same as $w_C(0)$, the energy initially stored in the capacitor. The energy that was initially stored in the capacitor is eventually dissipated in the resistor.
- So the key to working with a source free RC circuit is finding two things: the initial voltage V_0 across the capacitor and time constant τ.
- In finding the time constant τ = *RC*, *R* is often the Thevenin equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor *C* and find $R = R_{Th}$ at its terminals.

Example



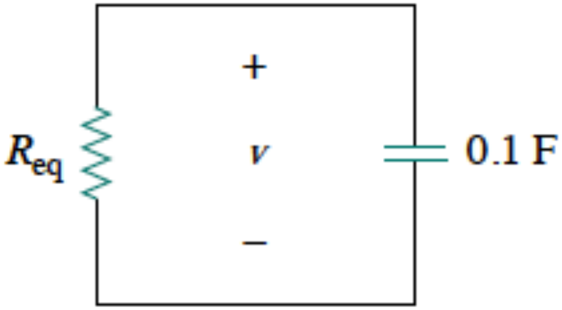
let $v_C(0) = 15$ V. Find v_C, v_x , and i_x for $t > 0$.

Solution:

We first need to make the circuit conform with the standard *RC* circuit consisting of a single R and single C. We find the equivalent or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C. From this, we can determine v_x and i_x. The 8-Ω and 12-Ω resistors in series can be combined to give a 20-Ω resistor. This 20-Ω resistor in parallel with the 5-Ω resistor can be combined so that the equivalent resistance is

$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

Hence, the equivalent circuit is as shown



- The time constant is $\tau = R_{eq}C = 4(0.1) = 0.4 \text{ s}$
- Thus,

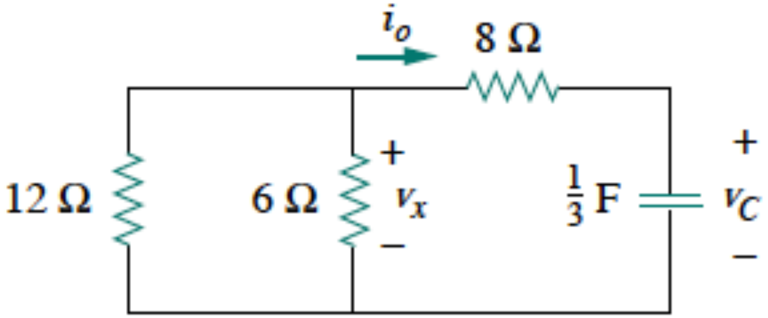
$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4} \text{ V}, \qquad v_C = v = 15e^{-2.5t} \text{ V}$$

- We can use voltage division then to get

$$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t} \text{ V}$$

- Finally, $i_x = \dfrac{v_x}{12} = 0.75e^{-2.5t} \text{A}$

Practice problem



Let $v_C(0) = 30$ V.

Determine v_C, v_x, and i_o for $t \geq 0$.

Example

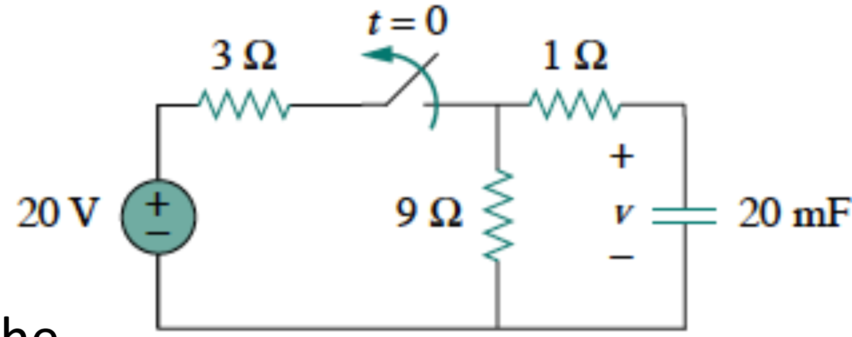
- The switch in the circuit has been closed for a long time, and it is opened at *t* = 0.
- Find *v(t)* for $t \geq 0$. Calculate the initial energy stored in the capacitor.
- Solution: For $t < 0$, the switch is closed; the capacitor is an open circuit to dc, as represented in (a).
- Using voltage division

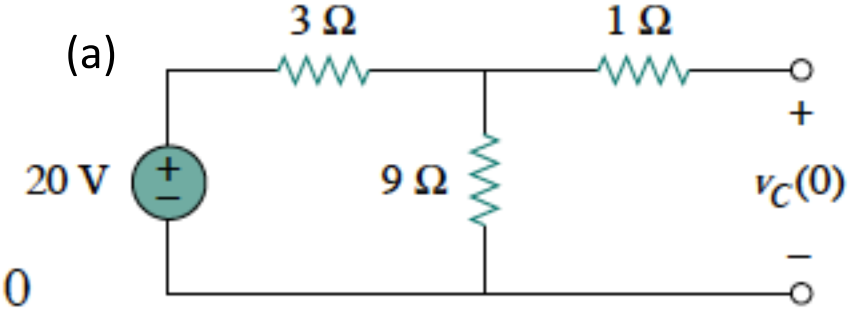
$$v_C(t) = \frac{9}{9+3}(20) = 15\text{ V}, \qquad t < 0$$

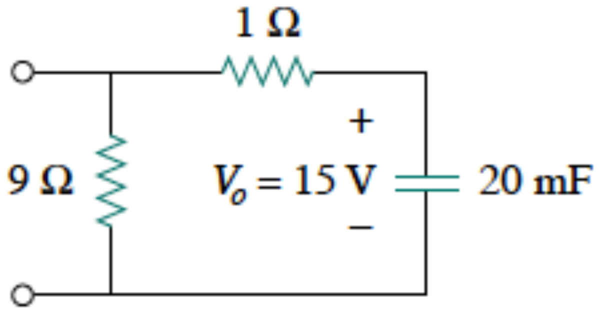
- Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$:

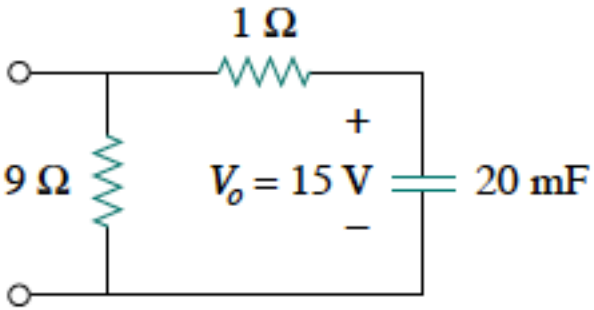
$$v_C(0) = V_0 = 15\text{ V}$$

- For $t > 0$, the switch is open, and we have the *RC* circuit shown in (b) (source free).




(a)

(b)




- The 1-Ω and 9-Ω resistors in series give

$$R_{\text{eq}} = 1 + 9 = 10\ \Omega$$

- The time constant is

$$\tau = R_{\text{eq}}C = 10 \times 20 \times 10^{-3} = 0.2\ \text{s}$$

- Thus, the voltage across the capacitor for $t \geq 0$ is

$$v(t) = v_C(0)e^{-t/\tau} = 15e^{-t/0.2}\ \text{V}$$

or

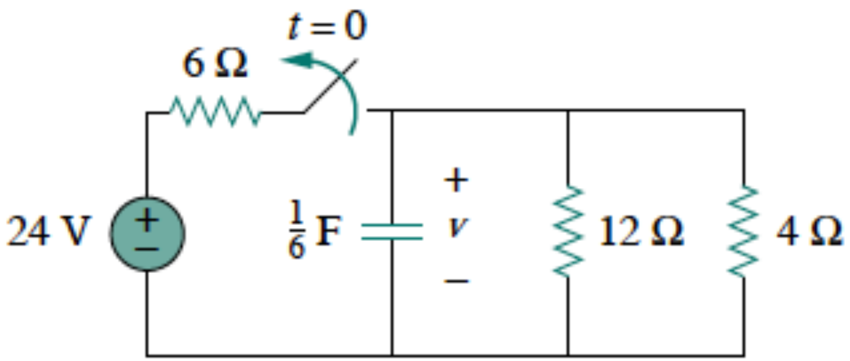
$$v(t) = 15e^{-5t}\ \text{V}$$

- The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}Cv_C^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times 15^2 = 2.25\ \text{J}$$

Practice problem

find $v(t)$ for $t \geq 0$ and $w_C(0)$.



The source-free RL circuit

- In the case of the RL circuit, shown below, we regard the current through the inductor i(t) as the circuit response. We first find i(t) because we know it has to be continuous (recall that current in an inductor can't change instantaneously). Other currents and voltages are easy to derive from i(t).
- At *t* = 0, we assume that the inductor has an initial current: $i(0) = I_0$

with the corresponding energy stored in the inductor as $w(0) = \frac{1}{2} L I_0^2$

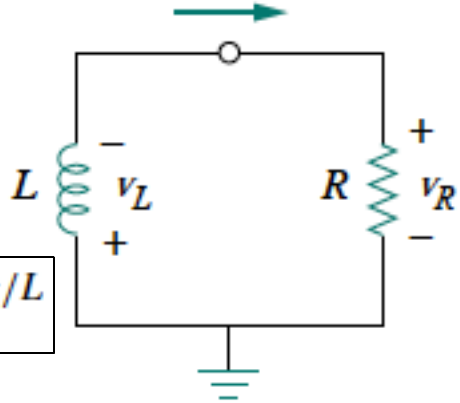
Applying KVL around the loop $v_L + v_R = 0$

But $v_L = L di/dt$ and $v_R = iR$. Thus, $L\frac{di}{dt} + Ri = 0$ or $\frac{di}{dt} + \frac{R}{L} i = 0$

Rearranging terms and integrating gives

$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt \implies \ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t \implies$$

$$\ln i(t) - \ln I_0 = -\frac{Rt}{L} + 0 \implies \ln \frac{i(t)}{I_0} = -\frac{Rt}{L} \implies \boxed{i(t) = I_0 e^{-Rt/L}}$$



- This shows that the natural response of the *RL* circuit is an exponential decay of the initial current, as shown.
- It is evident that the time constant for the *RL* circuit is

$$\boxed{\tau = \frac{L}{R}}$$

with τ again having the unit of seconds.

Thus we can write our equation as

$$\boxed{i(t) = I_0 e^{-t/\tau}}$$

We can find the voltage across the resistor from the current:

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

The power dissipated in the resistor is

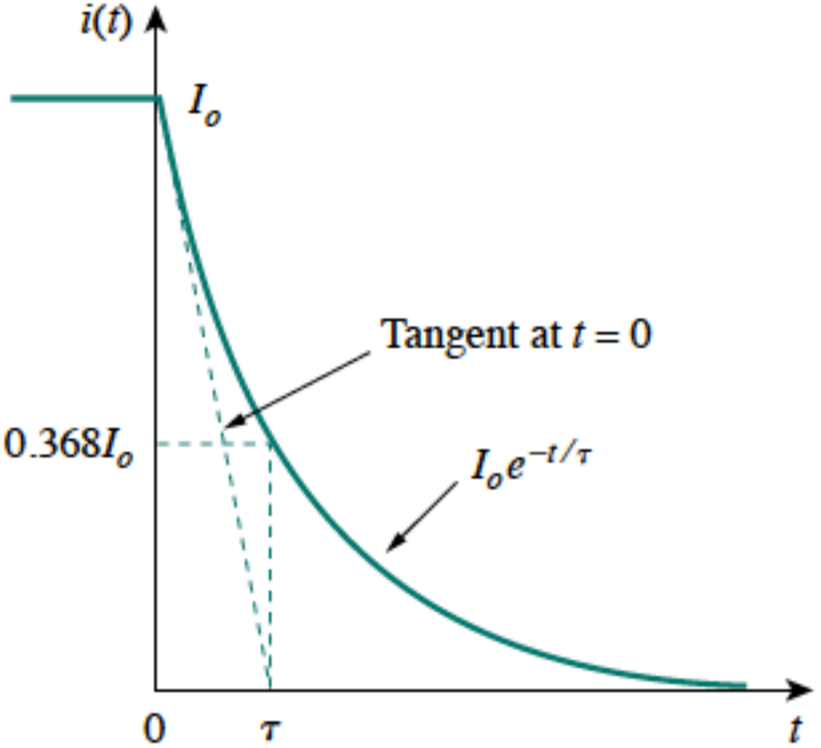
$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

so the energy absorbed by the R is

$$w_R(t) = \int_0^t p\,dt = \int_0^t I_0^2 R e^{-2t/\tau}\,dt$$

$$= -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

or $$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

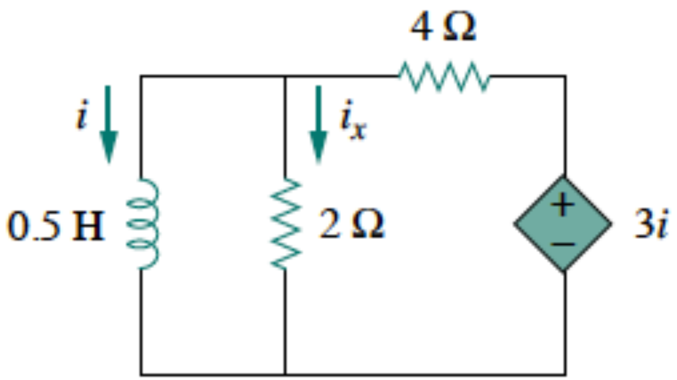


- Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2}LI_0^2$, which is the same as $w_L(0)$, the initial energy stored in the inductor. Again, the energy initially stored in the inductor is eventually dissipated in the resistor.

In summary, the key to working with a source-free RL circuit is to find:

- 1. The initial current $i(0) = I_0$ through the inductor.
- 2. The time constant τ of the circuit.
- With these two items, we obtain the response as the inductor current $i_L(t) = i(t) = i(0)e^{-t/\tau}$.
- Once we determine the inductor current i_L, other variables (inductor voltage v_L, resistor voltage v_R, and resistor current i_R) can be obtained.
- Note that in general, R in the expression for τ is the Thevenin resistance at the terminals of the inductor.

Example



Assuming that $i(0) = 10$ A, calculate

$i(t)$ and $i_x(t)$

Solution:

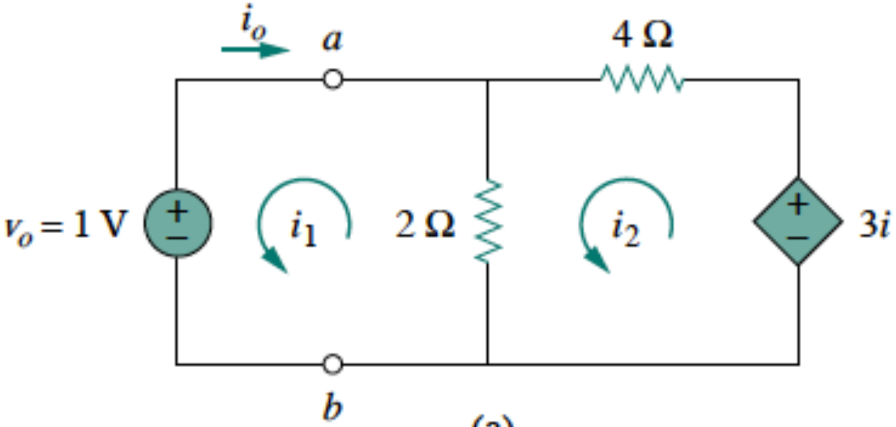
There are two ways we can solve: One way is to obtain the equivalent resistance at the inductor terminals and then use $i(t) = I_0 e^{-t/\tau}$

The other way is to start from scratch by using Kirchhoff's voltage law.

Either way, always obtain the inductor current first.

Method 1 The equivalent resistance is the same as the Thevenin resistance at the inductor terminals. Because of the dependent source, we insert a voltage source with $v_o = 1$ V at the inductor terminals *a*-*b*, as in (a).

(We could also insert a 1-A current source at the terminals.)



(a)

Applying KVL to the two loops:

$$2(i_1 - i_2) + 1 = 0 \quad \Longrightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \frac{5}{6}i_1$$

substituting gives

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

Hence $R_{\text{eq}} = R_{\text{Th}} = \dfrac{v_o}{i_o} = \dfrac{1}{3}\ \Omega$

The time constant is $\tau = \dfrac{L}{R_{\text{eq}}} = \dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{3}{2}$ s

Thus $i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t}$ A, $\qquad t > 0$

<u>Method 2</u> We may directly apply KVL to the circuit as in (b). For loop 1,

$$\frac{1}{2}\frac{di_1}{dt} + 2(i_1 - i_2) = 0 \quad \Longrightarrow \quad \frac{di_1}{dt} + 4i_1 - 4i_2 = 0$$

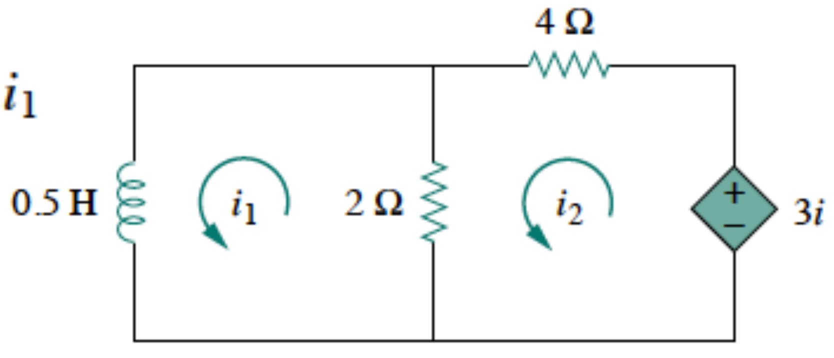
For loop 2 $\quad 6i_2 - 2i_1 - 3i_1 = 0 \quad \Longrightarrow \quad i_2 = \dfrac{5}{6}i_1$

substituting:

$$\frac{di_1}{dt} + \frac{2}{3}i_1 = 0$$

rearranging:

$$\frac{di_1}{i_1} = -\frac{2}{3}dt$$



(b)

Since $i_1 = i$, we may replace i_1 with i and integrate

$$\ln i \Big|_{i(0)}^{i(t)} = -\frac{2}{3}t \Big|_0^t \quad \text{or} \quad \ln \frac{i(t)}{i(0)} = -\frac{2}{3}t$$

Taking the powers of *e*, we finally obtain

$$i(t) = i(0)e^{-(2/3)t} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

which is the same as by Method 1. The voltage across the inductor is

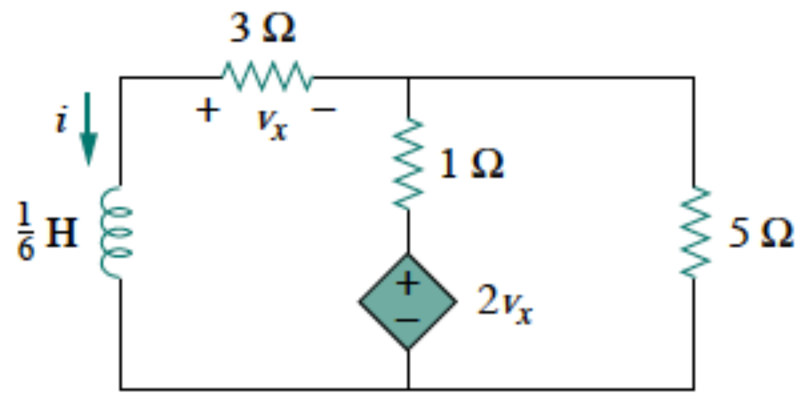
$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

Since the inductor and the 2- resistor are in parallel,

$$i_x(t) = \frac{v}{2} = -1.667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

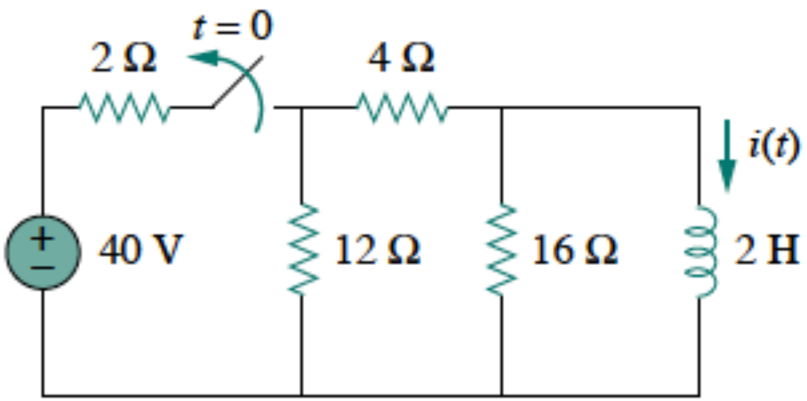
Practice problem

Find i and v_x Let $i(0) = 5$ A.



Examples

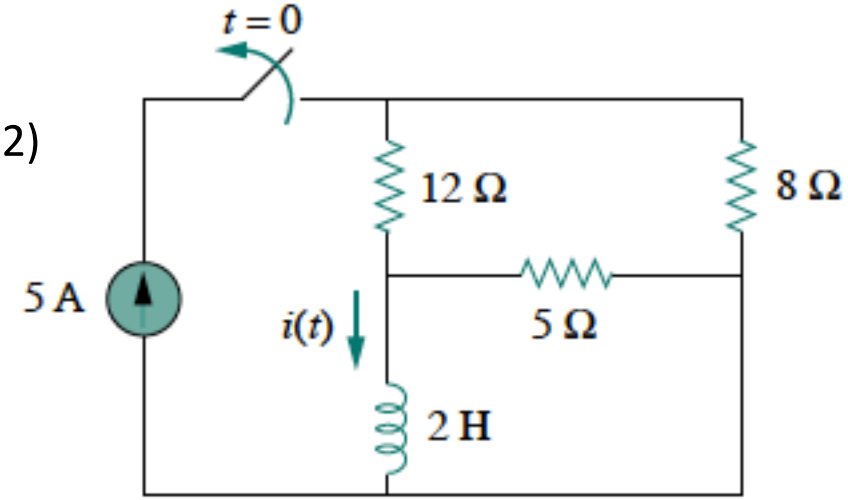
1)



Calculate $i(t)$ for $t > 0$.

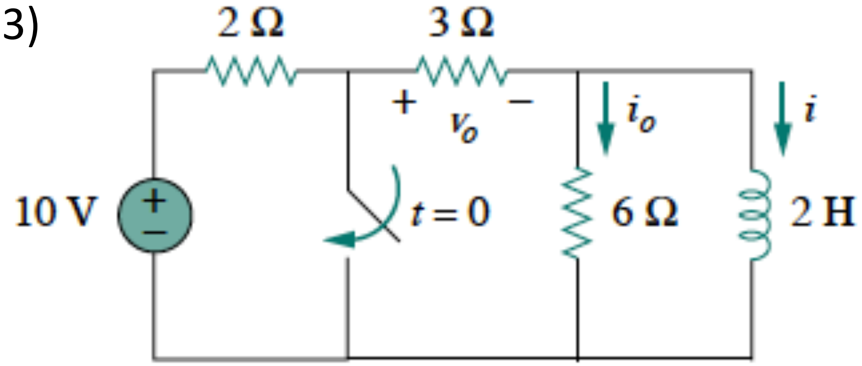
Get i(t) t<0, then close switch and get Req, hence *τ*, and finally $i(t) = i(0)e^{-t/\tau} = 6e^{-4t}$ A

2)

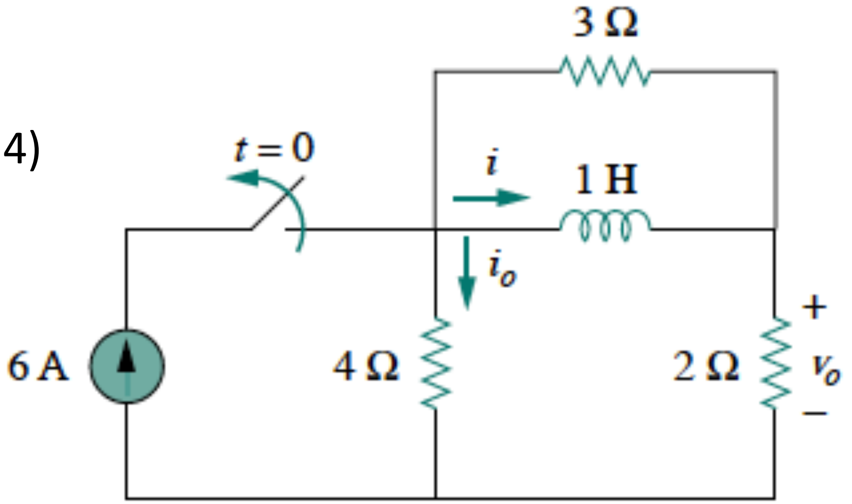


Answer: $2e^{-2t}$ A, $t > 0$.

3)



4)



$$i = \begin{cases} 4\text{ A}, & t < 0 \\ 4e^{-2t}\text{ A}, & t \geq 0 \end{cases}, \quad i_o = \begin{cases} 2\text{ A}, & t < 0 \\ -(4/3)e^{-2t}\text{ A}, & t > 0 \end{cases}$$

Solution to (3)

First find inductor current i(t) for all time, then get the other indicated quantities.
For t<0 replace inductor with a short circuit, and find i(0) = 2 A.
After t>0 the source is short-circuited, so it's a source-free circuit.
Get R_{Th}, and hence the time constant.
Then it follows that

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

$$i_o(t) = \begin{cases} 0\ \text{A}, & t < 0 \\ -\dfrac{2}{3}e^{-t}\ \text{A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\ \text{V}, & t < 0 \\ 4e^{-t}\ \text{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \text{A}, & t < 0 \\ 2e^{-t}\ \text{A}, & t \geq 0 \end{cases}$$

We notice that the inductor current is continuous at $t = 0$, while the current through the 6-Ω resistor drops from 0 to $-2/3$ at $t = 0$, and the voltage across the 3-Ω resistor drops from 6 to 4 at $t = 0$. We also notice that the time constant is the same regardless of what the output is defined to be. Figure 7.21 plots i and i_o.

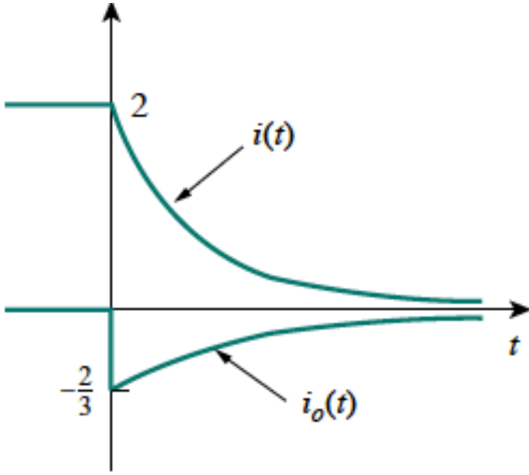
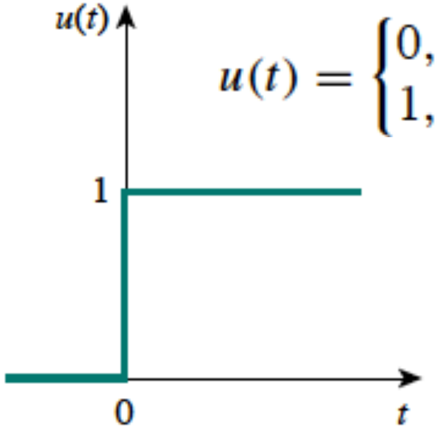


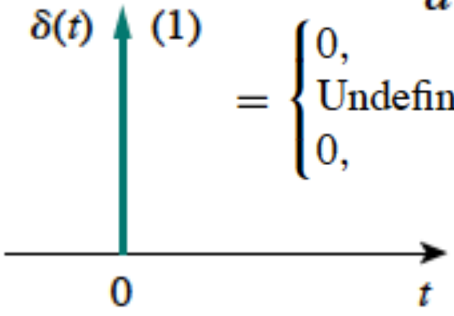
Figure 7.21 A plot of i and i_0.

Singularity Functions

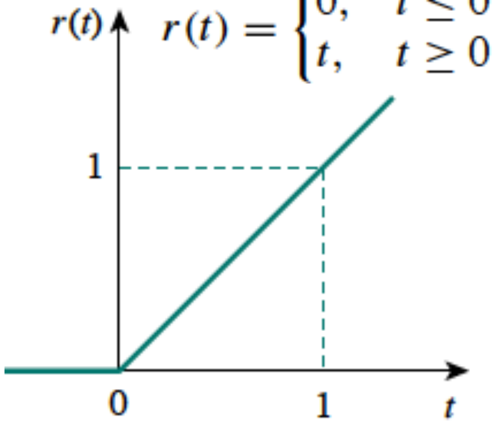
- Before getting into solving RC and RL circuits with the sudden application of independent dc sources, we need to consider the math related to the different functions the sources can take.
- Singularity functions (or "switching functions") are mathematical approximations to the switching signals that can arise in practice and are useful in describing the step response of RC and RL circuits.
- By definition, singularity functions are functions that are discontinuous or have discontinuous derivatives.
- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse,* and the *unit ramp* functions:



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

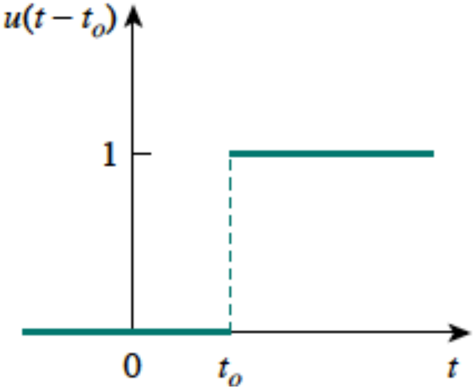


$$r(t) = \begin{cases} 0, & t \le 0 \\ t, & t \ge 0 \end{cases}$$

Unit step function

- The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1. It is dimensionless, like other mathematical functions such as sine and cosine. If the abrupt change occurs at $t = t_0$ instead of $t = 0$, the unit step function becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



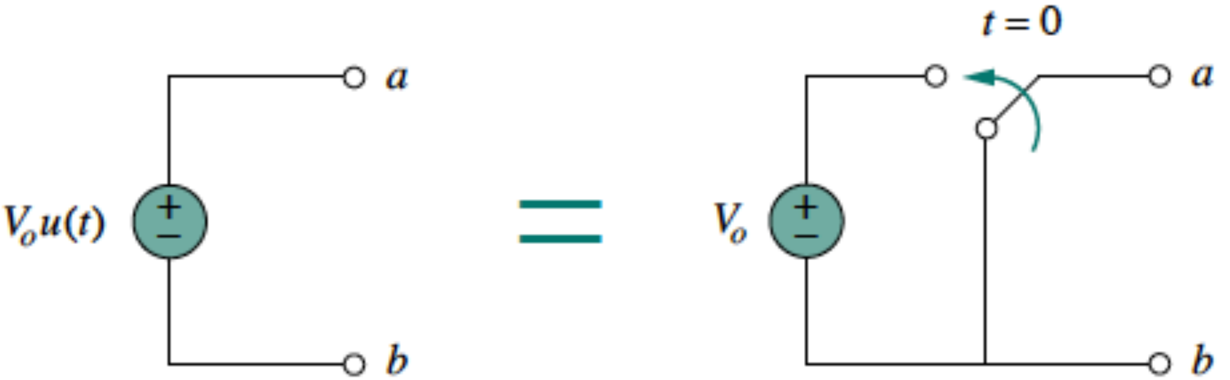
- The voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

can be expressed as

$$v(t) = V_0 u(t - t_0)$$

and if t_0= 0, realized by:

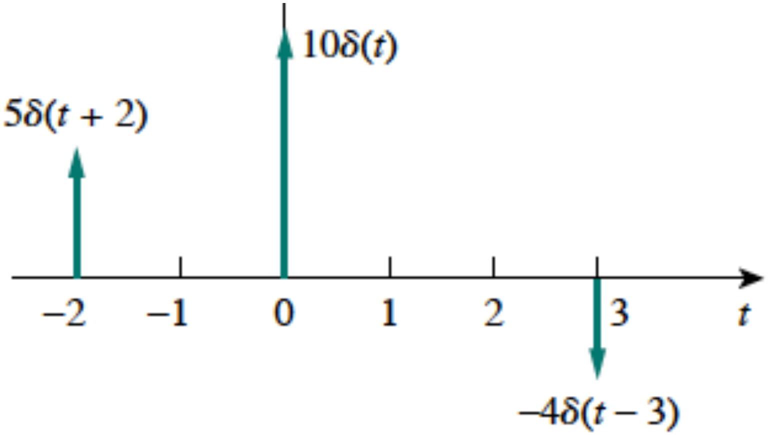


Unit impulse function

- The *unit impulse* or *delta function δ(t)* is the derivative of the unit step function. It is zero everywhere except at t = 0 where it is undefined. It may be regarded as an applied or resulting shock and visualized as a very short duration pulse of unit area:

$$\int_{0^-}^{0^+} \delta(t)\,dt = 1$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function. Likewise, the impulse functions $5\delta(t+2)$, $10\delta(t)$, and $-4\delta(t-3)$ are shown:



Note the delta function has a "sifting property:"

$$\int_a^b f(t)\delta(t-t_0)\,dt = f(t_0)$$

where $a < t_0 < b$.

Unit ramp function

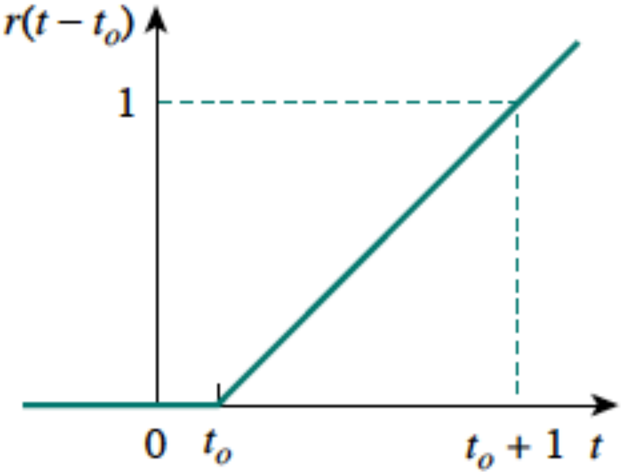
- Integrating the unit step function *u(t)* results in the *unit ramp function r(t)*:

$$r(t) = \int_{-\infty}^{t} u(t)\, dt = tu(t) \quad \text{or} \quad r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

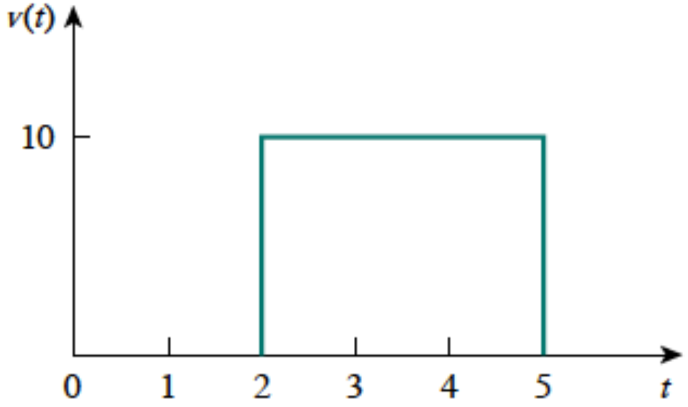
- It is zero for t<0 and has a unit slope for t>0.

- It may be delayed or advanced in the same way as u(t) or δ(t):

$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & t \geq t_0 \end{cases}$$



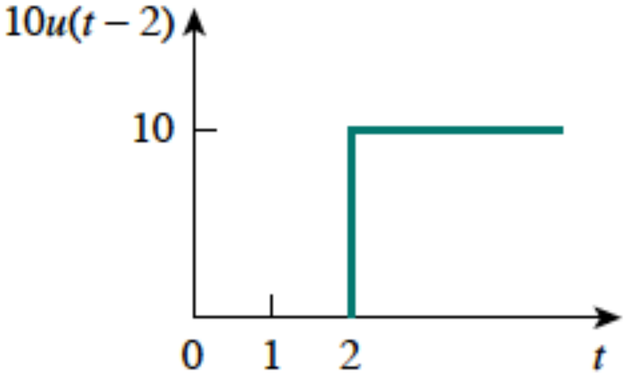
Example



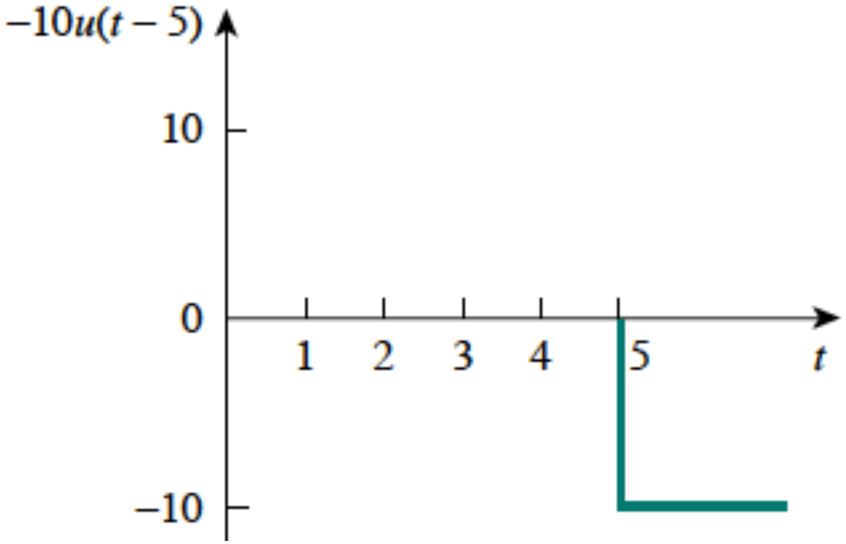
A "gate function."

can be expressed as

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

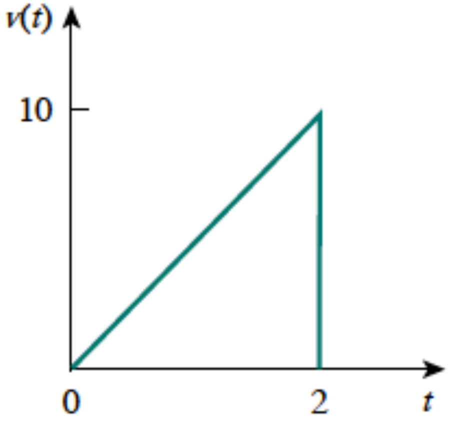


$+$



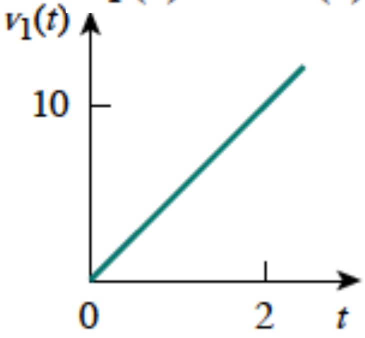
Example

Express this sawtooth in terms of singularity functions:



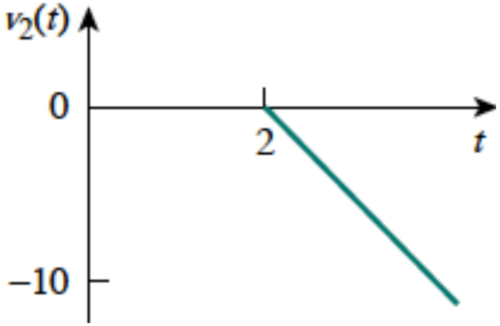
Partial decomposition:

$$v_1(t) = 5r(t)$$



+

$$v_2(t) = -5r(t-2)$$



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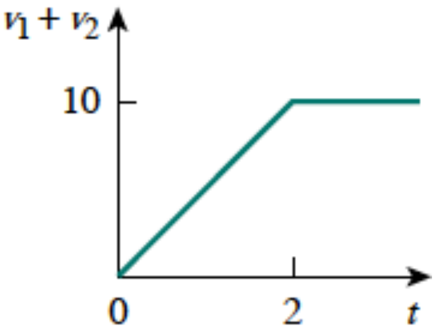
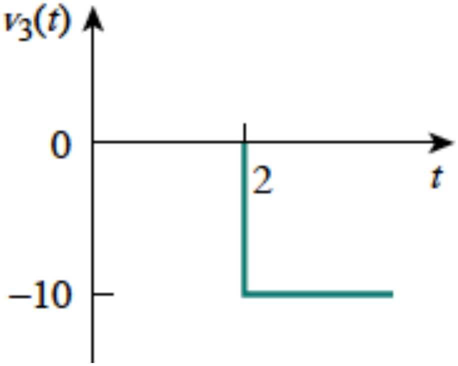


Then add third signal

$$v_3 = -10u(t-2)$$

+



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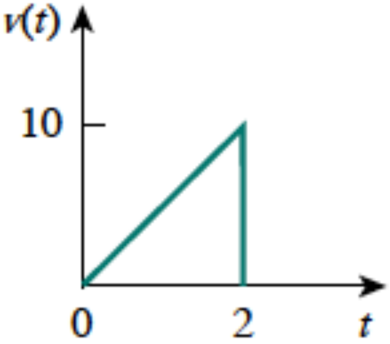
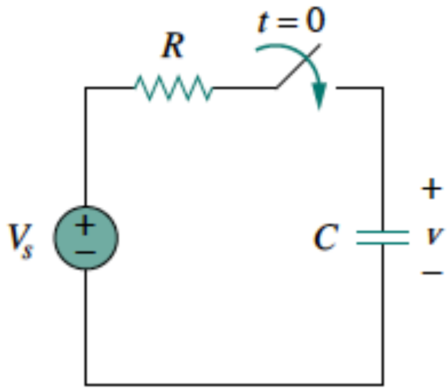


$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

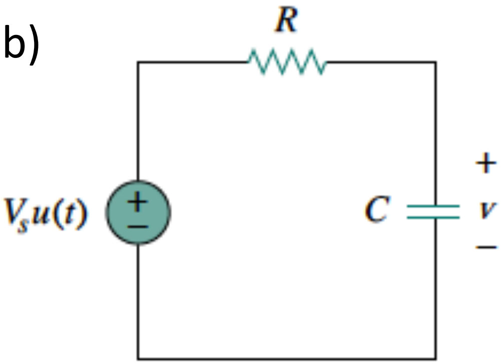
Step response of an RC circuit

- When the dc source of an *RC* circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a *step response.*
- The step response of a circuit is its behavior when the excitation is the step function (sudden application of a dc voltage or current source).
- Consider the circuits a and b, which are equivalent.
- V_s is a constant, dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined.
- We assume an initial voltage V_0 on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously, $v(0^-) = v(0^+) = V_0$ where $v(0^-)$ is the voltage across the capacitor just before switching and $v(0^+)$ is its voltage just after.

(a)



(b)



Applying KCL, $C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$ or $\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$

(or KVL,
gives the same!)

where *v* is the voltage across the capacitor. For *t* > 0, $\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$

Rearranging: $\frac{dv}{dt} = -\frac{v - V_s}{RC}$ or $\frac{dv}{v - V_s} = -\frac{dt}{RC}$

Integrating:

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

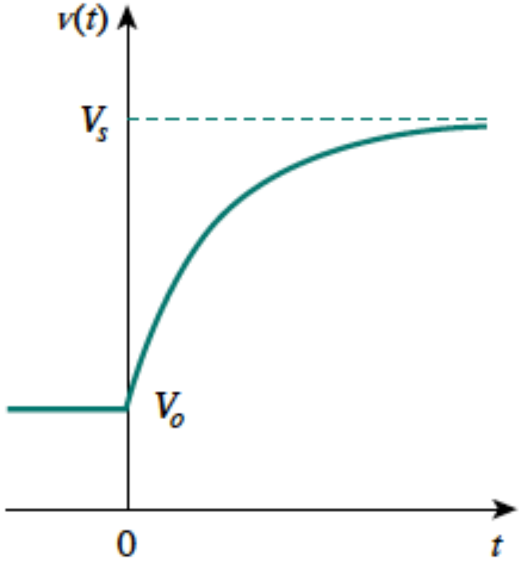
or $$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \qquad \tau = RC$$

$v - V_s = (V_0 - V_s)e^{-t/\tau}$ or $v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \qquad t > 0$

Thus, $$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

- This is known as the *complete response* of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged.
- The plot to the right is a case where $V_s > V_0$.



- If the capacitor is initially uncharged ($V_0=0$):

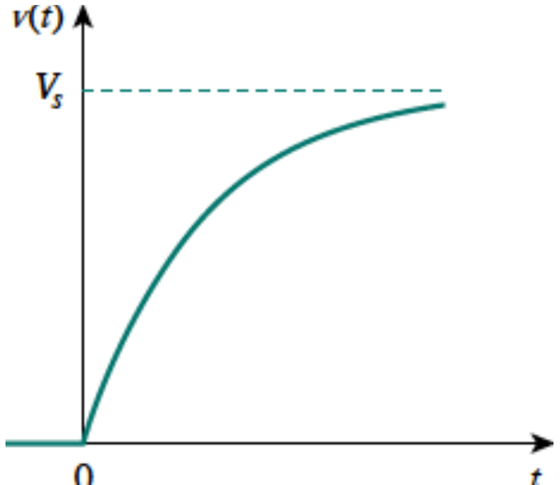
$$v(t) = V_s(1 - e^{-t/\tau})u(t)$$

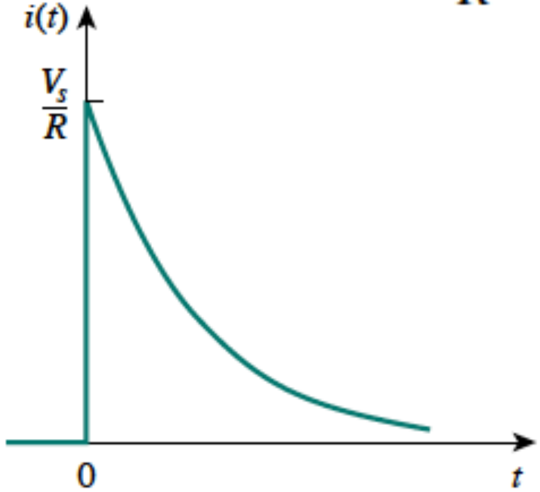
or

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s(1 - e^{-t/\tau}), & t > 0 \end{cases}$$

The current through the capacitor is obtained by using $i(t) = C\,dv/dt.$:

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \qquad \tau = RC, \quad t > 0 \quad \text{or} \quad i(t) = \frac{V_s}{R}e^{-t/\tau}u(t)$$





Short-cut to finding RC/RL step response

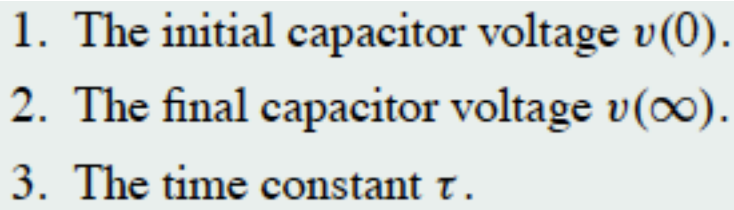
- It is evident that $V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0$ has two components:

$$v = v_f + v_n$$ where $$v_f = V_s$$

and $$v_n = (V_0 - V_s)e^{-t/\tau}$$

- We already came across v_n, the natural response of the circuit. Since this part of the response will decay to almost zero after five time constants, it is also called the *transient* response because it is a temporary response that will die out with time.
- Now, v_f is known as the *forced* response because it is produced by the circuit when an external "force" is applied (a voltage source in this case). It represents what the circuit is forced to do by the input excitation. It is also known as the *steady-state response*, because it remains a long time after the circuit is excited.
- The complete response is a sum of the forced and natural response.
- So: $$v(t) = \underset{\text{steady-state}}{v(\infty)} + [\underset{\text{initial}}{v(0)} - v(\infty)]e^{-t/\tau}$$

- Thus to find the step response of an RC circuit requires 3 things:



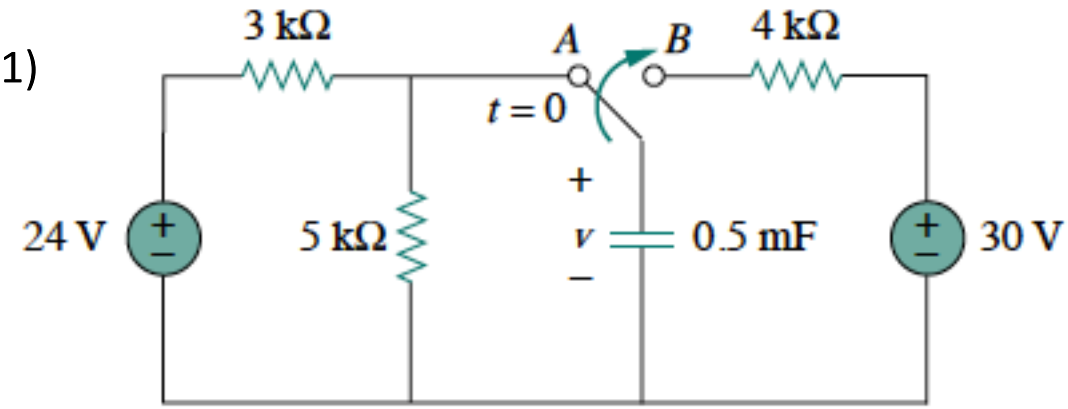
1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

- We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$. Once these are determined, we obtain the response using $v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$
- Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that the above equation becomes $v(t) = v(\infty) + [v(t_0) - v(\infty)]e^{-(t-t_0)/\tau}$

where $v(t_0)$ is the initial value at $t = t_0^+$

Examples

1)



Determine *v(t)* for *t* > 0 and calculate its value at *t* = 1 s and 4 s.

By voltage division $v(0) = v(0^-) = v(0^+) = 15 \text{ V}$

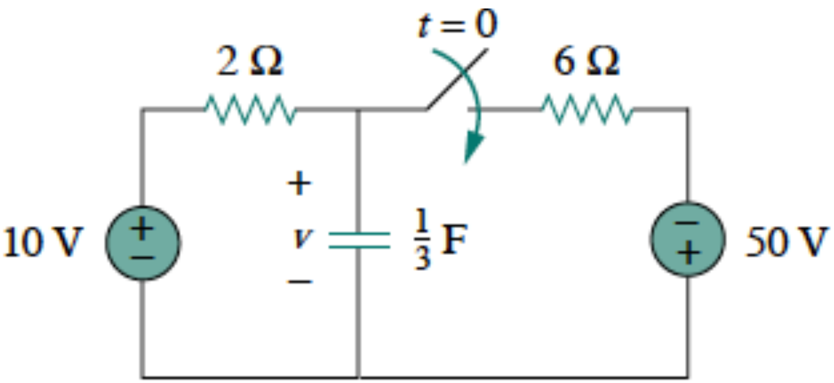
For t>0, capacitor acts like open circuit (dc) $v(\infty) = 30 \text{ V}.$ $\tau = R_{\text{Th}}C = 2 \text{ s}$

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = (30 - 15e^{-0.5t}) \text{ V}$$

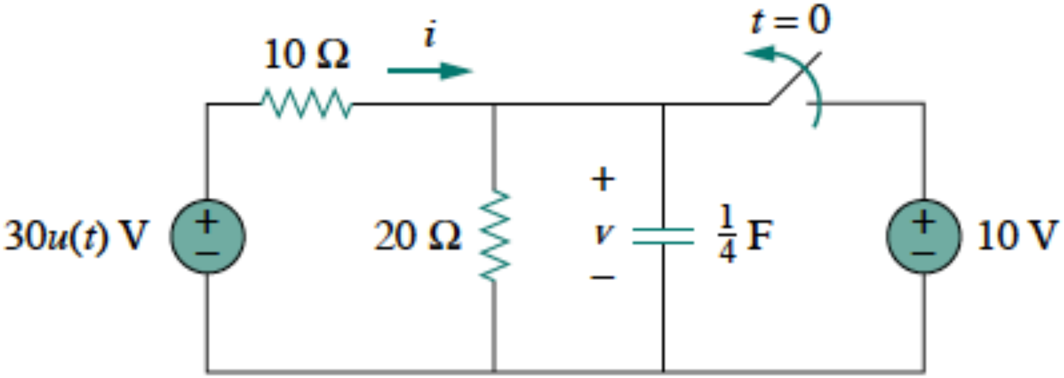
2) Practice:

Find $v(t)$ for $t > 0$

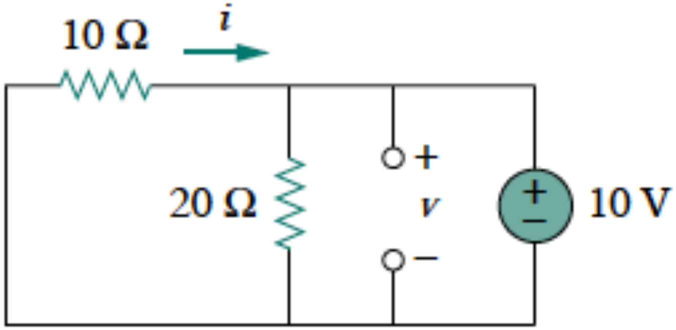


Example

Find i and v

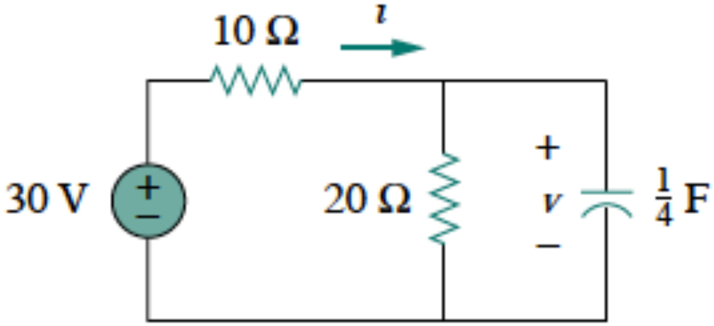


For t < 0 circuit behaves like:



v(t)=10V
i(t)=-1A
t<0

For t > 0:



$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$= 20 + (10 - 20)e^{-(3/5)t} = (20 - 10e^{-0.6t})\text{ V} \quad t \geq 0$$

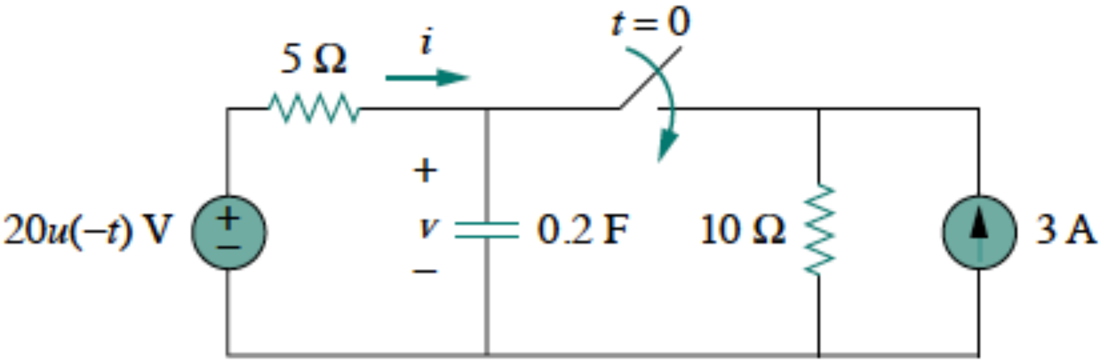
$$i = \frac{v}{20} + C\frac{dv}{dt} \quad = 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\text{ A}$$

Notice that v is continuous but i is not.

Practice problem

Find $i(t)$ and $v(t)$ for all time.

Note that $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Also, $u(-t) = 1 - u(t)$.



Step response of an RL circuit

- We will use our short-cut to find the inductor current *i* in the RL circuit with step input, shown in its two equivalent forms below.
- We break i into natural and forced components:

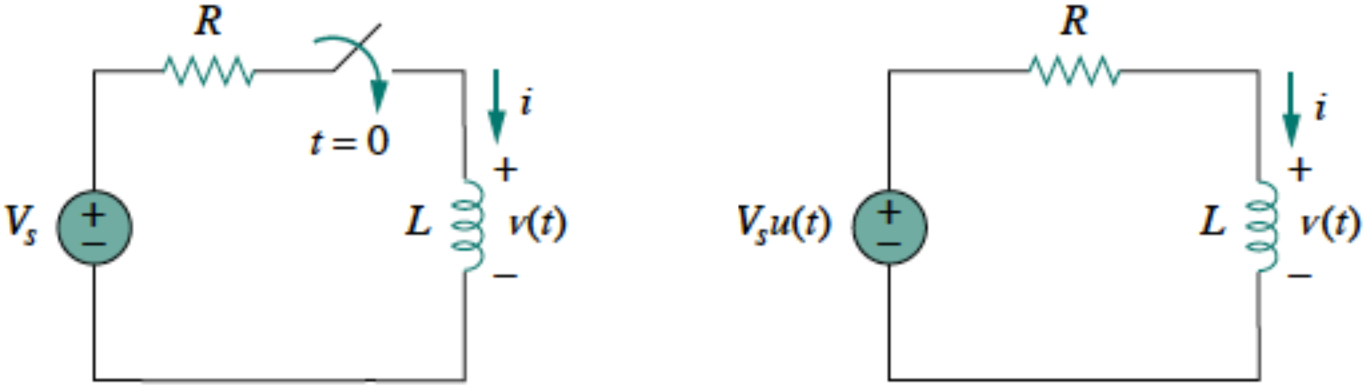
$$i = i_n + i_f$$

- We know the natural response is always a decaying exponential:

$$i_n = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R}$$

(A = some constant)

- The forced response is the current after a long time, when the natural response has died away, and the L is a short circuit: $i_f = \frac{V_s}{R}$



- So the complete response is $i = Ae^{-t/\tau} + \frac{V_s}{R}$
- We now determine the constant *A* from the initial value of *i*. Let I_0 be the initial current through the inductor, which may come from a source other than V_s . Since the current through the inductor cannot change instantaneously, $i(0^+) = i(0^-) = I_0$
- Thus at t=0,

$$I_0 = A + \frac{V_s}{R} \quad \Longrightarrow \quad A = I_0 - \frac{V_s}{R}$$

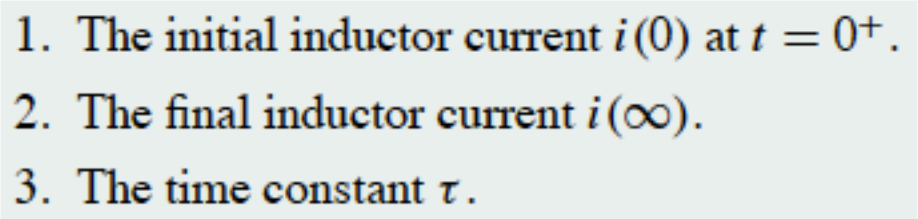
- substituting:

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-t/\tau}$$

- which may be written

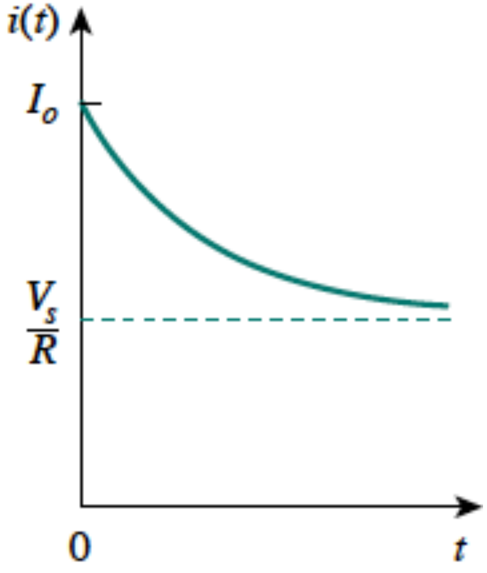
$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

- So to find RL step response we just need:



1. The initial inductor current $i(0)$ at $t = 0^+$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

(Remember: only for step responses)



- Again, if the switching takes place at time $t = t_0$ instead of $t = 0$,

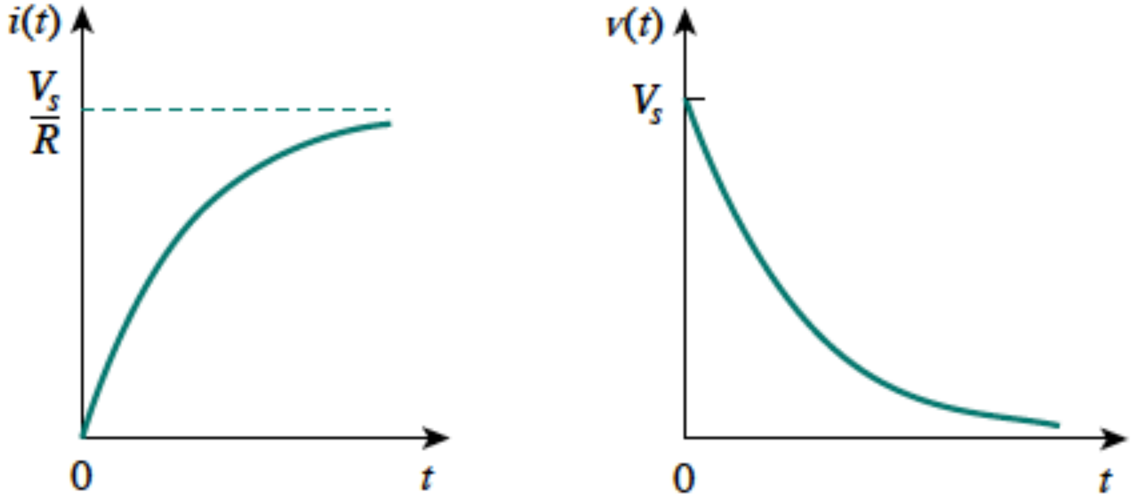
$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

- If $I_0 = 0,$

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \quad \text{or} \quad i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

- This is the step response of the *RL* circuit. The voltage across the inductor is obtained using $v = L di/dt$:

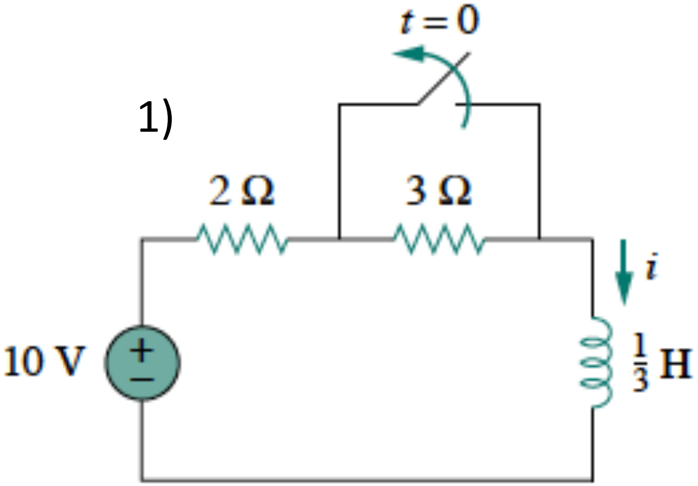
$$v(t) = L\frac{di}{dt} = V_s\frac{L}{\tau R}e^{-t/\tau}, \qquad \tau = \frac{L}{R}, \quad t > 0 \qquad \text{or} \qquad v(t) = V_s e^{-t/\tau}u(t)$$



Examples

Find i(t) in all circuits

1)



$$i(0^-) = \frac{10}{2} = 5\text{ A}$$

By continuity of i:

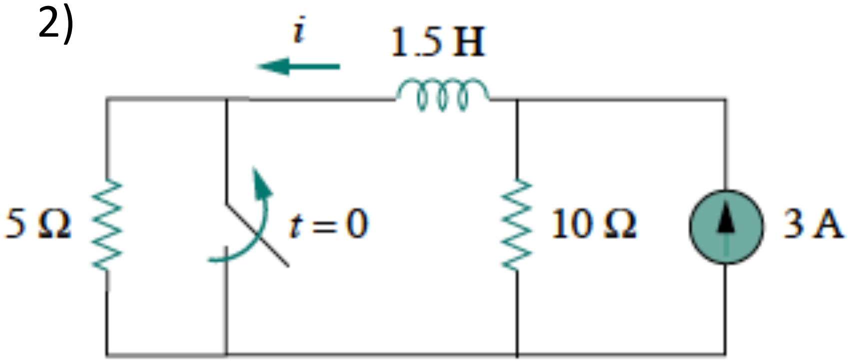
$$i(0) = i(0^+) = i(0^-) = 5\text{ A}$$

$$i(\infty) = \frac{10}{2+3} = 2\text{ A} \qquad R_{\text{Th}} = 2 + 3 = 5\ \Omega$$

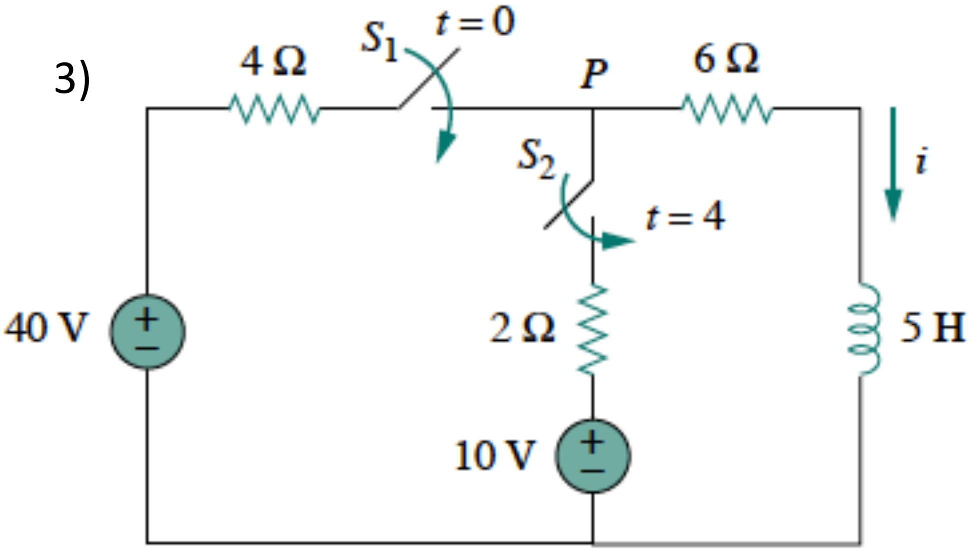
$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}\text{ s}$$

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 2 + (5-2)e^{-15t} = 2 + 3e^{-15t}\text{ A}, \qquad t > 0 \end{aligned}$$

2)



3)



Solution for (3)

We need to consider the three time intervals $t \leq 0$, $0 \leq t \leq 4$, and $t \geq 4$ separately. For $t < 0$, switches S_1 and S_2 are open so that $i = 0$. Since the inductor current cannot change instantly,

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \leq t \leq 4$, S_1 is closed so that the 4-Ω and 6-Ω resistors are in series. Hence, assuming for now that S_1 is closed forever,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ A}, \qquad R_{\text{Th}} = 4 + 6 = 10\ \Omega$$

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{5}{10} = \frac{1}{2} \text{ s}$$

Thus,

$$\begin{aligned} i(t) &= i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \\ &= 4 + (0-4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \qquad 0 \leq t \leq 4 \end{aligned}$$

For $t \geq 4$, S_2 is closed; the 10-V voltage source is connected, and the circuit changes. This sudden change does not affect the inductor current because the current cannot change abruptly. Thus, the initial current is

$$i(4) = i(4^-) = 4(1 - e^{-8}) \simeq 4 \text{ A}$$

To find $i(\infty)$, let v be the voltage at node P

Using KCL, $$\frac{40-v}{4}+\frac{10-v}{2}=\frac{v}{6} \quad \Longrightarrow \quad v=\frac{180}{11}\text{ V}$$

$$i(\infty)=\frac{v}{6}=\frac{30}{11}=2.727\text{ A}$$

The Thevenin resistance at the inductor terminals is

$$R_{\text{Th}}=4 \parallel 2+6=\frac{4\times 2}{6}+6=\frac{22}{3}\ \Omega$$

and $$\tau=\frac{L}{R_{\text{Th}}}=\frac{5}{\frac{22}{3}}=\frac{15}{22}\text{ s}$$

Hence $$i(t)=i(\infty)+[i(4)-i(\infty)]e^{-(t-4)/\tau}, \qquad t\geq 4$$

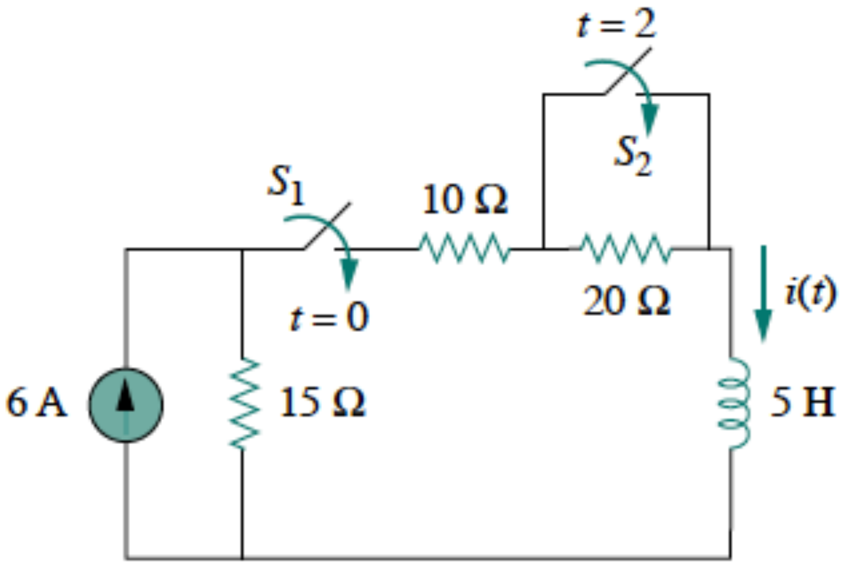
We need $(t-4)$ in the exponential because of the time delay. Thus,

$$i(t)=2.727+(4-2.727)e^{-(t-4)/\tau}, \qquad \tau=\frac{15}{22}$$

Putting all this together,

$$i(t)=\begin{cases}0, & t\leq 0\\ 4(1-e^{-2t}), & 0\leq t\leq 4\\ 2.727+1.273e^{-1.4667(t-4)}, & t\geq 4\end{cases}$$

Practice problem

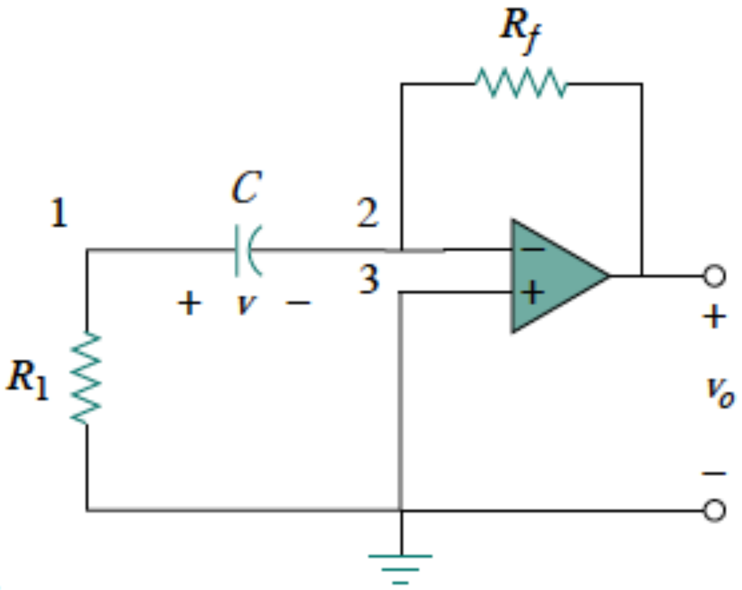


First order Op Amp circuits

- An op amp circuit containing a storage element will exhibit first-order behavior. Differentiators and integrators are examples of first-order op amp circuits. Again, for practical reasons, inductors are hardly ever used in op amp circuits; therefore, the op amp circuits we consider here are of the *RC* type.
- As usual, we analyze op amp circuits using nodal analysis. Sometimes, the Thevenin equivalent circuit is used to reduce the op amp circuit to one that we can easily handle. The following three examples illustrate the concepts. The first one deals with a source-free op amp circuit, while the other two involve step responses. The three examples have been carefully selected to cover all possible *RC* types of op amp circuits, depending on the location of the capacitor with respect to the op amp; that is, the capacitor can be located in the input, the output, or the feedback loop.

Example

- find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ kΩ, $R_1 = 20$ kΩ, and $C = 5\ \mu$F.



one way to solve is by nodal analysis:

$$\frac{0 - v_1}{R_1} = C\frac{dv}{dt}$$

v_2 is zero (ideal op amp), so $v_1 = v$

and $$\frac{dv}{dt} + \frac{v}{CR_1} = 0$$

So $$v(t) = V_0 e^{-t/\tau}, \qquad \tau = R_1 C$$

$v(0) = 3 = V_0$ and $\tau = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1.$ so $v(t) = 3e^{-10t}$

KCL at node 2 $C\frac{dv}{dt} = \frac{0 - v_o}{R_f}$ or $v_o = -R_f C\frac{dv}{dt}$

Hence $v_o = -80 \times 10^3 \times 5 \times 10^{-6}(-30e^{-10t}) = 12e^{-10t}$ V, $\quad t > 0$

Short-cut method

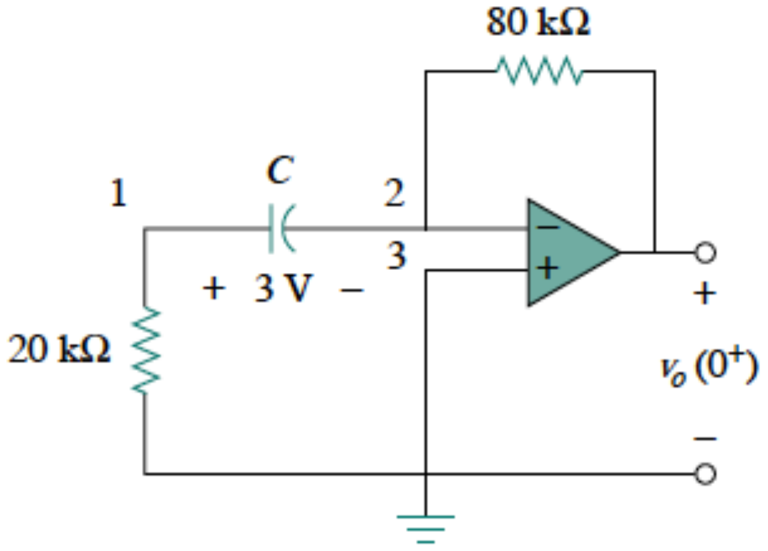
- We need to find $v_o(0+)$, $v_o(\infty)$, and τ . Since $v(0+) = v(0-) = 3$ V, we apply KCL at node 2 in circuit (a):

$$\frac{3}{20{,}000} + \frac{0 - v_o(0^+)}{80{,}000} = 0 \quad \rightarrow \quad v_o(0^+) = 12\ \text{V}.$$

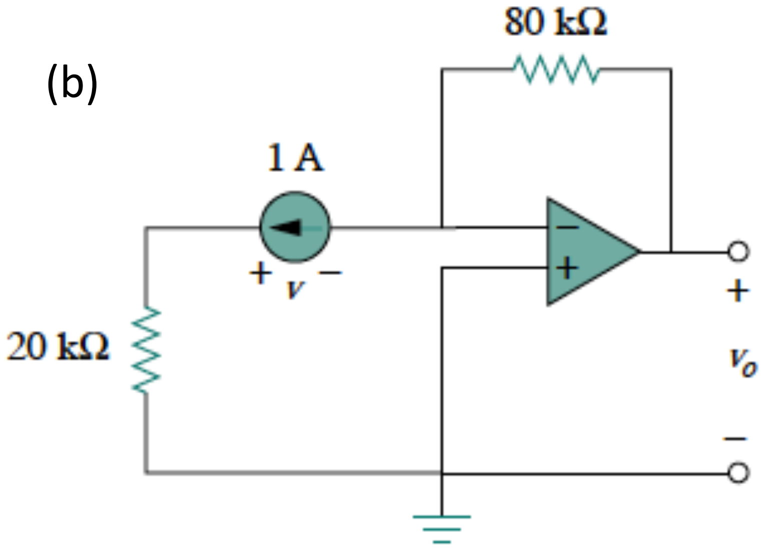
- Since the circuit is source free, $v(\infty) = 0$V. To find τ , we need the equivalent resistance *R*eq across the capacitor terminals. If we remove the capacitor and replace it by a 1-A current source (cct b), we can applying KVL to the input loop:

$$20{,}000(1) - v = 0 \qquad \Longrightarrow \qquad v = 20\ \text{kV}$$

(a)



(b)



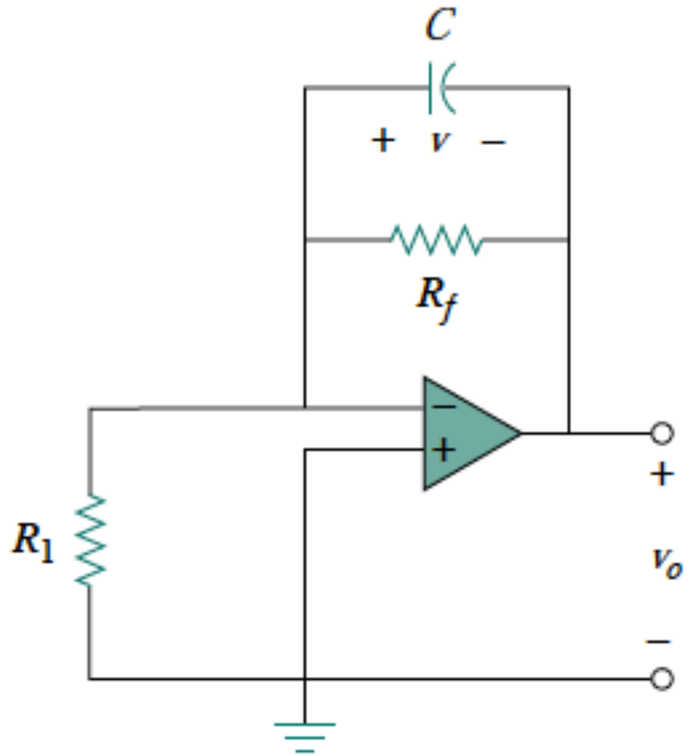
Then $R_{eq} = \frac{v}{1} = 20\ \text{k}\Omega$

and $\tau = R_{eq}C = 0.1$. Thus,

$$v_o(t) = v_o(\infty) + [v_o(0) - v_o(\infty)]e^{-t/\tau}$$
$$= 0 + (12 - 0)e^{-10t} = 12e^{-10t}\ \text{V}, \qquad t > 0 \qquad \text{as before.}$$

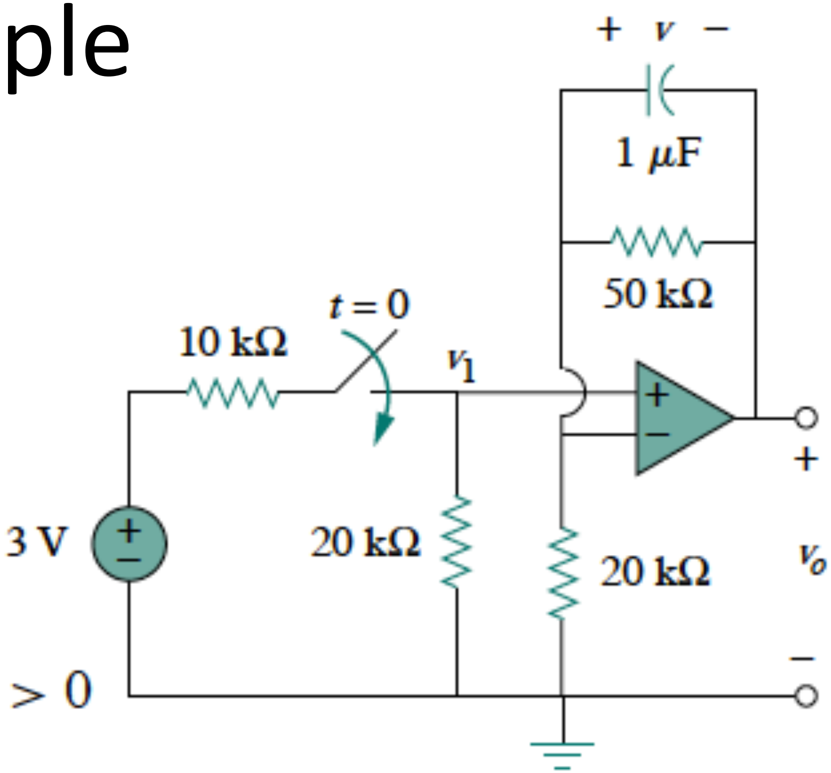
Practice problem

For this op amp circuit, find v_o for $t > 0$ if $v(0) = 4\text{V}$. Assume that $R_f = 50\ \text{k}\Omega$, $R_1 = 10\ \text{k}\Omega$, and $C = 10\ \mu\text{F}$.



Example

Determine $v(t)$ and $v_o(t)$.



Solution:

This problem can be solved in two ways, just like the previous example. However, we will apply only the second method. Since what we are looking for is the step response, we can write

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}, \qquad t > 0$$

where we need only find the time constant τ , the initial value $v(0)$, and the final value $v(\infty)$. Notice that this applies strictly to the capacitor voltage due a step input. Since no current enters the input terminals of the op amp, the elements on the feedback loop of the op amp constitute an *RC* circuit, with

$$\tau = RC = 50 \times 10^3 \times 10^{-6} = 0.05$$

For $t < 0$, the switch is open and there is no voltage across the capacitor. Hence, $v(0) = 0$. For $t > 0$, we obtain the voltage at node 1 by voltage division as

$$v_1 = \frac{20}{20 + 10} 3 = 2 \text{ V}$$

Since there is no storage element in the input loop, v_1 remains constant for all t. At steady state, the capacitor acts like an open circuit so that the op amp circuit is a noninverting amplifier. Thus,

$$v_o(\infty) = \left(1 + \frac{50}{20}\right) v_1 = 3.5 \times 2 = 7 \text{ V}$$

But $v_1 - v_o = v$ so that $v(\infty) = 2 - 7 = -5 \text{ V}$

Plugging these values into our general equation

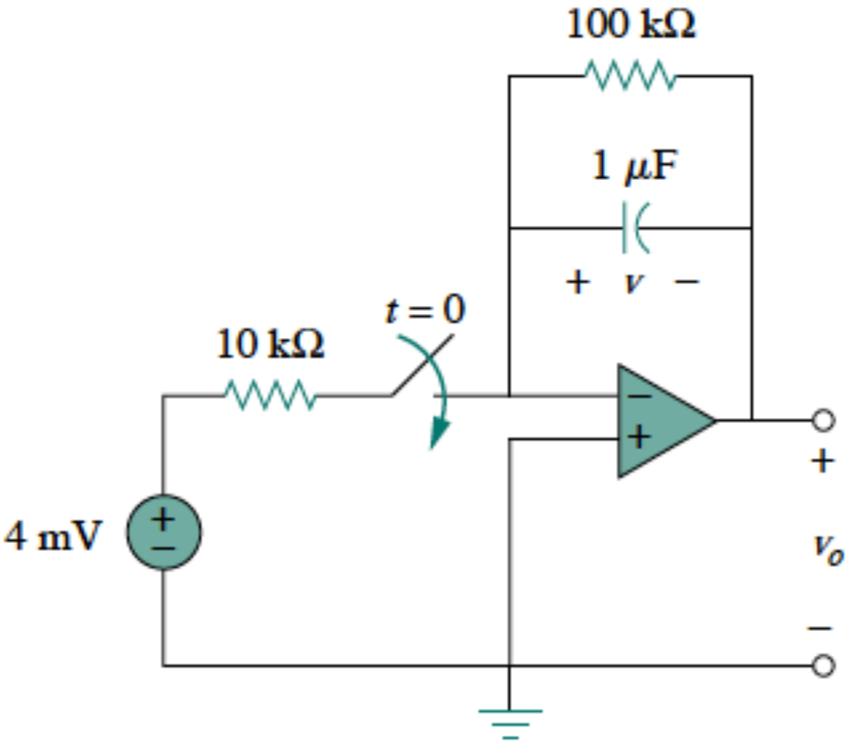
$$v(t) = -5 + [0 - (-5)]e^{-20t} = 5(e^{-20t} - 1) \text{ V}, \qquad t > 0$$

from which we easily get v_o:

$$v_o(t) = v_1(t) - v(t) = 7 - 5e^{-20t} \text{ V}. \qquad t > 0$$

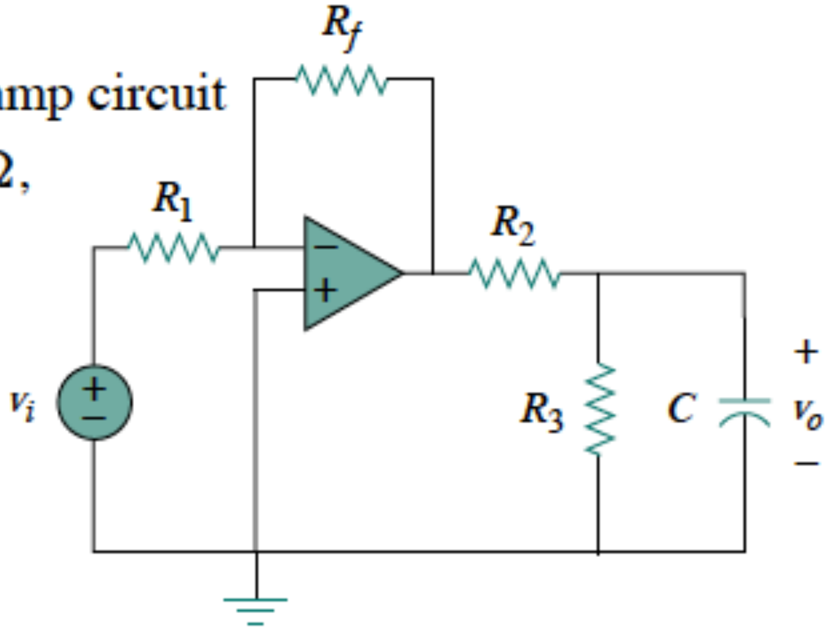
PRACTICE

Find $v(t)$ and $v_o(t)$ in the op amp circuit



Example

Find the step response $v_o(t)$ for $t > 0$ in the op amp circuit Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 50$ kΩ, $R_2 = R_3 = 10$ kΩ, $C = 2\ \mu$F.



Solution:

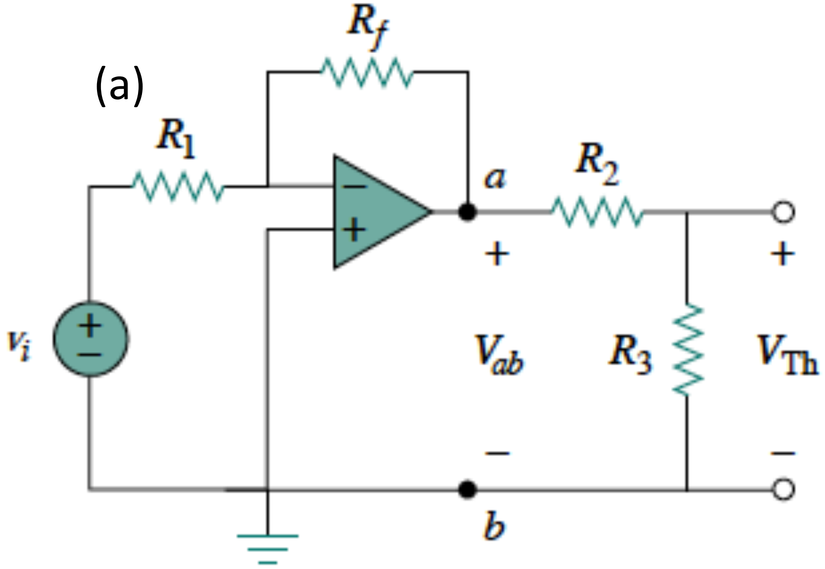
Notice that the capacitor is located in the output of the op amp. Again, we can solve this problem directly using nodal analysis. However, using the Thevenin equivalent circuit may simplify the problem.

We temporarily remove the capacitor and find the Thevenin equivalent at its terminals. To obtain V_{Th}, consider the circuit in (a).

Since the circuit is an inverting amplifier,

$$V_{ab} = -\frac{R_f}{R_1} v_i$$

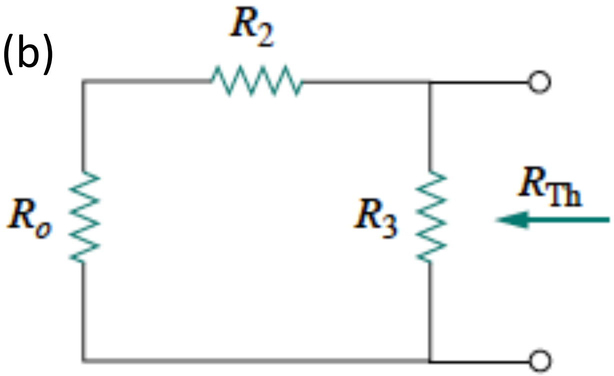
(a)



By voltage division, $V_{\mathrm{Th}} = \frac{R_3}{R_2 + R_3} V_{ab} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i$

To obtain R_{Th}, consider the circuit in (b), where *Ro* is the output resistance of the op amp. Since we are assuming an ideal op amp, $R_o = 0$, and

(b)



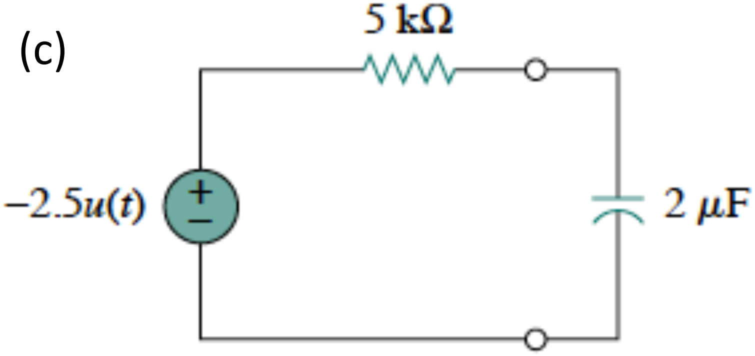
$$R_{\mathrm{Th}} = R_2 \parallel R_3 = \frac{R_2 R_3}{R_2 + R_3}$$

Substituting the given numerical values,

$$V_{\mathrm{Th}} = -\frac{R_3}{R_2 + R_3} \frac{R_f}{R_1} v_i = -\frac{10}{20} \frac{50}{20} 2u(t) = -2.5u(t)$$

$$R_{\mathrm{Th}} = \frac{R_2 R_3}{R_2 + R_3} = 5\ \mathrm{k}\Omega$$

(c)



The Thevenin equivalent circuit is shown in (c), which is the one we know. Hence, the solution is

$$v_o(t) = -2.5(1 - e^{-t/\tau})\, u(t)$$

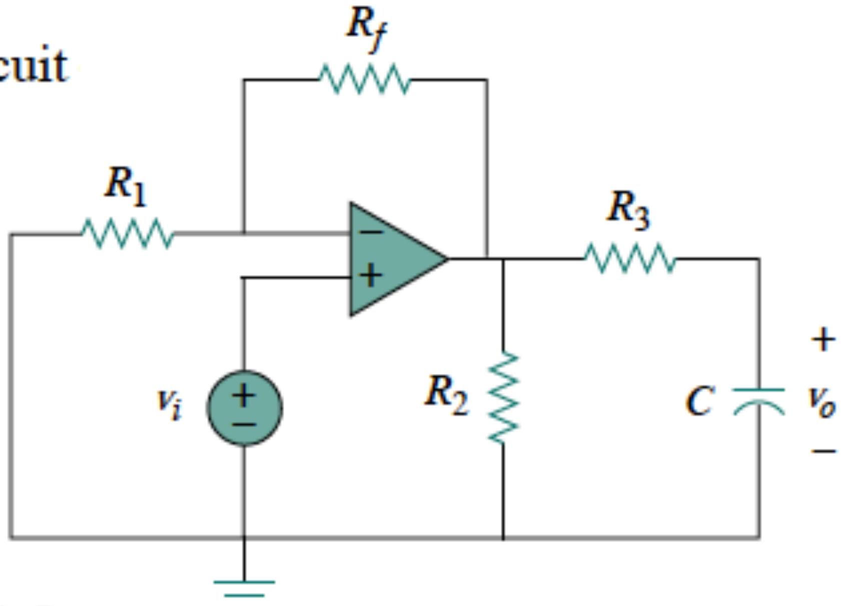
where $\tau = R_{\mathrm{Th}} C = 5 \times 10^3 \times 2 \times 10^{-6} = 0.01.$

Thus the step response is

$$v_o(t) = 2.5(e^{-100t} - 1)\, u(t)\ \mathrm{V}$$

Practice problem

Obtain the step response $v_o(t)$ for the circuit



Let $v_i = 2u(t)$ V, $R_1 = 20$ kΩ, $R_f = 40$ kΩ,
$R_2 = R_3 = 10$ kΩ, $C = 2$ μF.