Finite Element Operators in the Vertical

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25th ALADIN Workshop & HIRLAM All Staff Meeting
Helsingør, Denmark
16 April 2015

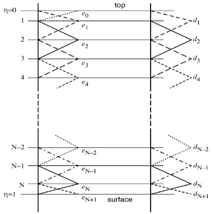
- The purpose of the present work is to provide a vertical finite element technique making use of **analytical properties of B-splines**

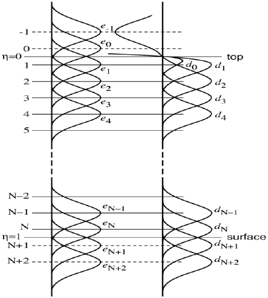
- This technique can be a solution to **solve constraints**
  - invertibility between integral and derivative: d and w
  - C1 constraint

- Cooperation in VFE with
  - Mariano Hortal, Juan Simarro (AEMET)
  - Petra Smolíková (CHMI)
  - Jozef Vivoda (SHMI)

VFE in hydrostatic model

Splines have been implemented successfully on IFS **hydrostatic model** by A. Untch and M. Hortal with linear and cubic B-splines using **Galerkin method**. All variables are kept at **full levels**, no staggering of variables is used





In **non-hydrostatic model** there is a constraint between vertical operators (**C1**) which is very desirable to satisfy in order to reduce the Helmholtz equation to a single variable $\hat{d}$

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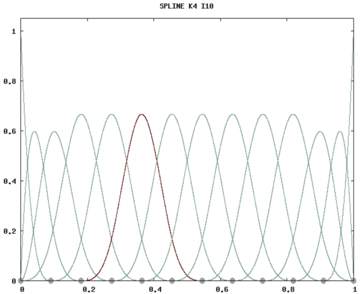
P. Smolíková and J. Vivoda have developed a FE discretization with **B-splines** (computed with the **de Boor** algorithm) and Galerkin method. The C1-constraint is relaxed by an **iterative method**

$$\boxed{N_{ik} = (t - t_i)\frac{N_{i,k-1}}{\Delta_{i,k-1}} + (t_{i+k} - t)\frac{N_{i+1,k-1}}{\Delta_{i+1,k-1}}}$$

where $\Delta_{ik} := t_{i+k} - t_i$



0th-order B-splines are

$$N_{i1}(t) = \begin{cases} 1 & t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

knots t_i are a non-decreasing sequence of points "related" to levels

Analytical VFE

VFE operators based on **analytical properties of B-splines** instead of Galerkin method. B-splines are a **partition of unity**, constants can be written as a linear combination of basis functions

$$\sum_i N_{ik}(t) = 1$$

B-splines are well suited also in view of their properties under **derivation** and **integration**

$$\frac{\partial}{\partial t} N_{ik} = (k-1)\left[\frac{N_{i,k-1}}{\Delta_{i,k-1}} - \frac{N_{i+1,k-1}}{\Delta_{i+1,k-1}}\right]$$

$$\int_0^t N_{ik} = \frac{\Delta_{ik}}{k} \sum_{i \le s} N_{s,k+1}$$

VFE integral and derivative

as a consequence we have the **commutative diagram** where $\sim$ identify functions that differ by a constant. S_k is the space spanned by B-splines

$$
\begin{tikzcd}
S_k \arrow[r, shift left, "\int_0 , \int_1"] & S_{k+1}/\sim \arrow[l, shift left, "\partial"]
\end{tikzcd}
$$

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\begin{tikzcd}[column sep=huge, row sep=huge]
S_k \arrow[r, shift left, "\int_0 {,} \int_1"] \arrow[d, shift right, "\mathcal{P}_k"'] & S_{k+1}/\sim \arrow[l, shift left, "\partial"] \arrow[d, shift right, "\mathcal{P}_{k+1}"'] \\
\mathcal{F}_{(1:L)} \arrow[u, shift right, "\mathcal{Q}_k"'] \arrow[r, dotted, shift left, "\widehat{\int_0} {,} \widehat{\int_1}"] & \mathcal{F}_{(1:L+1)}/\sim \arrow[u, shift right, "\mathcal{Q}_{k+1}"'] \arrow[l, dotted, shift left, "\widehat{\partial}"]
\end{tikzcd}
$$

it ensures **invertibility** between integral and derivative operators in grid-point space $\widehat{\partial}, \widehat{\int_0}, \widehat{\int_1}$, what can be seen as a constraint

VFE integral and derivative: invertibility between gw and d

We can make use of $\widehat{\partial}, \widehat{\int_1}$ operators to have an invertible-full-level representation of vertical divergence and vertical velocity

$$\mathtt{d} = -\frac{p}{mR_dT}\,\partial_\eta\,\mathtt{gw} \qquad\qquad \mathtt{gw} = \vec{v}_s\vec{\nabla}\phi_s - \int_1^\eta \frac{mR_dT}{p}\mathtt{d}\,d\eta'$$

Spectral integral and derivative

- -NELEV
- 0
- NFLEVG+1

we make a **periodic extension** at the upper levels of the atmosphere and apply **fourier analysis** using the basis functions

$$\boxed{\begin{aligned} &1, x \\ &\lambda_n := e^{\frac{2\pi i n x}{L}} \end{aligned}}$$

$$\begin{aligned} \partial_x x &= 1 \\ \partial_x \lambda_n &= \frac{2\pi i n}{L} \lambda_n \end{aligned} \qquad \begin{aligned} \int_*^x 1 &= x \\ \int_*^x \lambda_n &= \frac{L}{2\pi i n} \lambda_n \end{aligned}$$

the **vertical operators** that appear in semi-implicit nh-model are

derivative:

$$\begin{aligned} \partial^* &:= \pi^* \frac{\partial}{\partial \pi^*} \\ \mathcal{L}^* &:= \partial^*(\partial^* + 1) \end{aligned}$$

integration:

$$\begin{aligned} \mathcal{G}^* f &:= \int_{\pi^*}^{\pi_s^*} f \frac{d\pi^*}{\pi^*} \\ \mathcal{S}^* f &:= \frac{1}{\pi^*} \int_0^{\pi^*} f d\pi^* \\ \mathcal{N}^* f &:= \frac{1}{\pi_s^*} \int_0^{\pi_s^*} f d\pi^* \end{aligned}$$

constraints:

C1	C2
$\mathcal{A}_1^* \equiv 0$	$\mathcal{T}^* \equiv 1$

$$\begin{aligned} \mathcal{A}_1^* &:= \mathcal{G}^*\mathcal{S}^* - \mathcal{G}^* - \mathcal{S}^* + \mathcal{N}^* \\ \mathcal{T}^* &:= \frac{g^2}{c^2 N^2} \mathcal{L}^* \left[\mathcal{S}^*\mathcal{G}^* - \frac{c_{pd}}{c_{vd}} [\mathcal{G}^* + \mathcal{S}^*] \right] \end{aligned}$$

C1 constraint: factorization

Factorization of C1-constraint $\boxed{(\mathcal{G}^* - 1)(\mathcal{S}^* - 1) = (1 - \mathcal{N}^*)}$ allows us to make the chain of operators in functional space (where C1 is always guaranteed)

$$\begin{tikzcd}
H_k \arrow[rr, "1-\mathcal{N}^*"] \arrow[dr, "\mathcal{S}^*-1"'] & & H_k \\
& K_k \arrow[ur, "\mathcal{G}^*-1"'] &
\end{tikzcd}$$

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& K_k \arrow[ur, "\mathcal{G}^*-1"'] \arrow[dd, shift right, "\mathcal{P}_{Kk}"'] & \\
\mathcal{F}_{(0:L)} \arrow[uu, "\mathcal{Q}_{Hk}"] \arrow[rr, dotted, "\widehat{1-\mathcal{N}^*}"] \arrow[dr, dotted, "\widehat{\mathcal{S}^*-1}"'] & & \mathcal{F}_{(0:L)} \\
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\end{tikzcd}
$$

and in **grid-point space** $\widehat{\mathcal{X}^*} := 1 + \widehat{\mathcal{X}^* - 1}$

C1 constraint: factorization

the basis functions are not exactly B-splines. Coordinate is $t := \frac{\pi^*}{\pi^*_s}$

$$\begin{array}{lr} H_k : & (\partial^* + 1)\, N_{ik} \\ K_k : & -\,\partial^* N_{ik} \end{array}$$

to compute vertical laplacian $\widehat{\mathcal{L}^*}$ we simply use the spaces S_k and $\mathcal{L}^* S_k$, it's only a first attempt because is not obvious how to satisfy C2-contraint simultaneoulsy with C1-constraint

C1 constraint: coding

nh-vfe operators are defined in setup inside SUVERT and **encapsulated at low level**

invertibility		C1-constraint			
VERINT	$\widehat{\int_0}, \widehat{\int_1}$	SIGAM	$\widehat{\mathcal{G}*}$	SISEVE	$\widehat{\mathcal{L}*}$
VERDER	$\widehat{\partial}$	SITNU	$\widehat{\mathcal{S}*}, \widehat{\mathcal{N}*}$		

we have found that C1 constraint is guaranteed up to $\max|\widehat{\mathcal{A}_1^*}_{ij}| \simeq 10^{-13}$

C1 constraint: B matrix

there's still the problem of to obtain real and positive **eigenvalues of B matrix** that appears in the inversion of Helmholtz equation [KY]

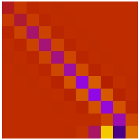
SUNHSI
SUNHBMAT

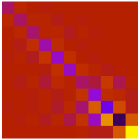
$$\mathtt{B} = C^2 \left[1 - \beta^2 \Delta t^2 \frac{C^2}{H^2} \frac{T^*}{T_a^*} \widehat{\mathcal{L}^*} \right]^{-1} \left[1 + \beta^2 \Delta t^2 N^2 \frac{T^*}{T_a^*} \widehat{\mathcal{T}^*} \right]$$

NPDVAR=2
NVDVAR=3, 4

a symmetric matrix has non-negative real eigenvalues, so if `B` were one of that class their eigenvalues would be as required. A good trial can be symmetrize $\widehat{\mathcal{L}^*}$ as much as possible due to the dependency of $\mathcal{T}^*$ on $\mathcal{L}^*$

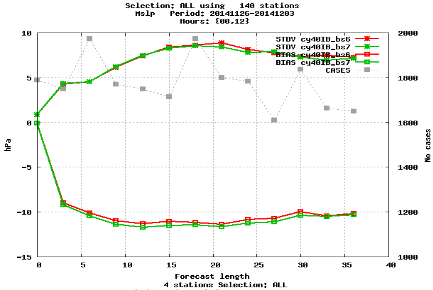
the subroutine **`SISEVE`** acting on the identity matrix gives a matrix representation $\widehat{\mathcal{L}^*}$. In a FE construction is good to symmetrize it at the lowest full-level where it shows his higher asymmetry

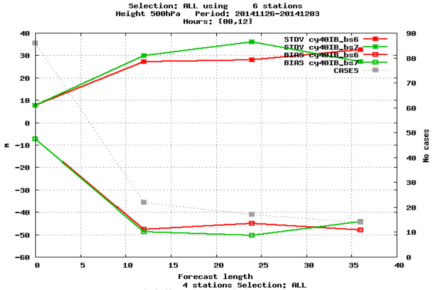


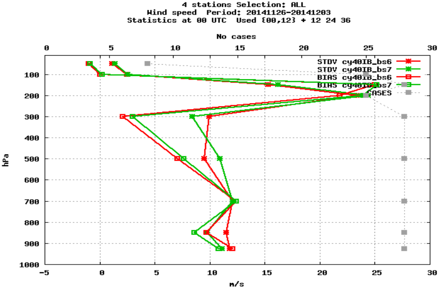


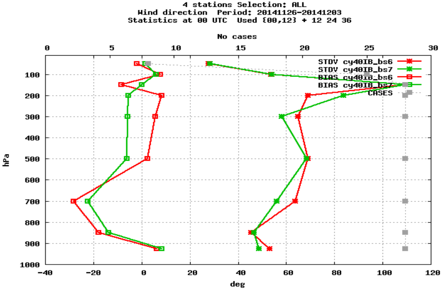
C1 constraint: first test

Iberian peninsula 26nov14 to 3dec14 LL36 LPC_FULL.T. **FD vs FD + FE GSNL**









Future Work

- Adapt non-linear model to be consistent with $\mathcal{G}^*, \mathcal{S}^*, \mathcal{N}^*$ operators. This can be done through compatible integral operators

$$
\begin{aligned}
\sum_n \widehat{\mathcal{G}^*}_{ln} f_n &= \sum_n \mathtt{RINTEG}_{ln} \left(\frac{m^*}{\pi^*}\right)_n f_n \\
\sum_n \widehat{\mathcal{S}^*}_{ln} f_n &= \frac{1}{\pi^*_l} \sum_n \mathtt{RINTES}_{ln}\, m^*_n f_n \\
\sum_n \widehat{\mathcal{N}^*}_n f_n &= \frac{1}{\pi^*_s} \sum_n \mathtt{RINTEN}_n\, m^*_n f_n
\end{aligned}
$$

- Test $\mathcal{G}^*, \mathcal{S}^*, \mathcal{N}^*$ operators in full-nhvfe scheme

- Test invertible integral and derivative operators in current code. Surface data is needed for invertible derivation!

$$
\begin{aligned}
&\mathtt{RINTE} \quad \text{is } (L+1) \times L \\
&\mathtt{RDERI} \quad \text{is } L \times (L+1)
\end{aligned}
$$

- Study the impact of the choice of knots in the quality of solutions. They should be at the maxima of basis functions (nice talks with Jozef) but not very close to boundaries in order to avoid high Δ_{ik}

Thank you for your attention!