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Stochastic efficiency analysis with a reliability consideration

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ABSTRACT

Stochastic Data Envelopment Analysis (DEA) models have been introduced in the literature to assess the performance of operating entities with random input and output data. A stochastic DEA model with a reliability constraint is proposed in this study that maximizes the lower bound of an entity's efficiency score with some pre-selected probability. We define the concept of stochastic efficiency and develop a solution procedure. The economic interpretations of the stochastic efficiency index are presented when the inputs and outputs of each entity follow a multivariate joint normal distribution.

1. Introduction

Data Envelopment Analysis (DEA) is a non-parametric method used to evaluate the performance of a set of operating entities called decision making units (DMUs) that consume similar inputs and create similar outputs. It has been widely applied in areas such as healthcare, agriculture and banking as well as assessing low carbon supply chains. Cooper et al. [10] provided an introduction of the various DEA models. Cook et al. [6] discussed the selection of a DEA model. The reader is referred to Cook and Seiford [5] and Liu et al. [17,18] for extensive reviews of DEA's development and applications.

Traditionally, the efficiency score of a DMU is defined as the ratio of the multiplier-weighted sum of its outputs to the multiplier-weighted sum of its inputs. The constant returns-to-scale DEA model, namely, the CCR DEA model [4], computes the efficiency index of a DMU, which is the maximum efficiency score in terms of the input and output multipliers. Any DMU with an efficiency index of one is rated as *CCR efficient* in the sense that it is not dominated by any observations or their linear combinations. The efficiency index of an inefficient DMU is less than one and reveals the proportional decrease necessary in its inputs to reach the estimated efficiency frontier, which is spanned by the efficient units.

It is widely acknowledged that variability and uncertainty are associated with the input and output data of a production process due to its inherent stochastic nature or specification errors [1]. Land et al. [14] gave convincing examples in agriculture, manufacturing, product development, education, health care and military for which it is necessary to incorporate stochastic variation of data in the concept of "efficiency". As a consequence, both multiplier and envelopment DEA models have been generalized to deal with stochastic inputs and outputs. The concepts of *dominance* and *efficiency* are extended to the stochastic domain in these models, where chance-constrained programming is applied to model the production frontier defined with stochastic inputs and outputs.

Land et al. [14] proposed a stochastic efficiency analysis formulation in envelopment form where a chance constraint is placed on every output category. In this study we focus on stochastic DEA models in multiplier form as they explicitly take into account the correlations among input and output data within every DMU, which are generally considered more important than dependencies among the observed DMUs but are ignored in envelopment models.

Cooper et al. [8,9], Huang and Li [12,13] and Li [16] developed *joint* stochastic efficiency analysis models where probabilistic efficiency dominance is established via a joint chance constraint. No computational results have been reported in the literature possibly due to the strong intractability of these models.

We next examine two multiplier form stochastic DEA models with a marginal chance constraint on every DMU. The following "satisficing" DEA model was presented in Cooper et al. [7]:

$$\pi_o^* = \max_{\mathbf{u},\mathbf{v}} P\left\{\frac{\mathbf{u}^T\tilde{\mathbf{y}}_o}{\mathbf{v}^T\tilde{\mathbf{x}}_o} \ge 1\right\}$$

s.t.

$$P\left\{\frac{\mathbf{u}^T\tilde{\mathbf{y}}_j}{\mathbf{v}^T\tilde{\mathbf{x}}_j} \le 1\right\} \ge \alpha_j, \quad j \in N,$$

$$\mathbf{u} \ge \mathbf{0},\ \mathbf{v} \ge \mathbf{0}. \tag{1}$$

In the model, it is assumed that every unit in the set of DMUs, $N = \{1, 2, \ldots, n\}$, consumes resources in m categories and creates

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products or services in s categories. P means "Probability", $\tilde{\mathbf{y}}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{sj})^T$ and $\tilde{\mathbf{x}}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{mj})^T$ represent, respectively, the vectors of stochastic output and input values of DMU $j \in N$, while $\mathbf{u} \in \mathbb{R}^s$ and $\mathbf{v} \in \mathbb{R}^m$ are non-negative virtual multipliers to be determined by solving the above model for DMU o, which is the DMU under evaluation. Throughout this paper, it is assumed that $\tilde{y}_{rj}$ and $\tilde{x}_{ij}$ are continuous random variables for any $r = 1, 2, \ldots, s$ and any $i = 1, 2, \ldots, m$. $\alpha_j \in (0, 1)$ is pre-selected and is the minimum probability required to fulfill the corresponding chance constraint.

We note that model (1) is adapted from the traditional CCR DEA model [4] and falls in the class that Charnes and Cooper [3] refer to as "P-models". As Charnes and Cooper suggested, the objective of a "P-model" can be linked to the concept of "satisficing" defined by Simon [21]. Along this perspective, the unity in the objective function of model (1) can be interpreted as an *aspiration level*, while model (1) maximizes the likelihood for the efficiency score of DMU o to achieve this aspiration level.

Assuming that the random outputs and inputs of each DMU j follow a multivariate normal distribution with a mean vector $(\overline{\mathbf{y}}_j^T, \overline{\mathbf{x}}_j^T)^T$ and a variance–covariance matrix $\mathbf{\Lambda}_j$, Olesen and Petersen [20] developed a model that optimizes the rate at which the mean input vector for the DMU under evaluation has to decrease in order to achieve efficiency. The original formulation presented by Olesen and Petersen [20] has a typo. The model after the necessary correction is presented as follows:

$$\theta_o^* = \max_{\mathbf{u},\mathbf{v}} \mathbf{u}^T \overline{\mathbf{y}}_o + \Phi^{-1}(\alpha_o) \sqrt{(\mathbf{u}^T, -\mathbf{v}^T) \mathbf{\Lambda}_o (\mathbf{u}^T, -\mathbf{v}^T)^T}$$

$$\text{s.t.}$$

$$\mathbf{v}^T \overline{\mathbf{x}}_o = 1,$$

$$\mathbf{u}^T \overline{\mathbf{y}}_j - \mathbf{v}^T \overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j) \sqrt{(\mathbf{u}^T, -\mathbf{v}^T) \mathbf{\Lambda}_j (\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0, \quad j \in N,$$

$$\mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}. \tag{2}$$

In the model, $\Phi()$ is the standard normal distribution function and $\Phi^{-1}()$ its inverse.

As will be illuminated in the next section, the stochastic efficiency index π_o^* given by model (1) is not a radial measure. In contrast, model (2) returns a radial measure θ_o^* and reduces to the CCR DEA model when there is no variability in input and output data. Consequently, (2) is a general model with CCR DEA model as a special case. However, our subsequent analysis will show that model (2) does not necessarily return a correct stochastic efficiency index. In this study, we propose a stochastic efficiency analysis model that corrects this shortcoming of model (2) using the concept of aspiration level introduced in model (1). We next analyze an example to motivate the study.

2. A motivating example

Under the assumption of joint normality model (1) can be rewritten as follows:

$$\vartheta_o^* = \max_{\mathbf{u},\mathbf{v},\vartheta} \vartheta$$

$$\text{s.t.}$$

$$\mathbf{u}^T \overline{\mathbf{y}}_o - \mathbf{v}^T \overline{\mathbf{x}}_o - \vartheta \sqrt{(\mathbf{u}^T, -\mathbf{v}^T) \mathbf{\Lambda}_o (\mathbf{u}^T, -\mathbf{v}^T)^T} \ge 0,$$

$$\mathbf{u}^T \overline{\mathbf{y}}_j - \mathbf{v}^T \overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j) \sqrt{(\mathbf{u}^T, -\mathbf{v}^T) \mathbf{\Lambda}_j (\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0, \quad j \in N,$$

$$\mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}. \tag{3}$$

where $\pi_o^* = \Phi(\vartheta_o^*)$.

Models (2) and (3) are interpreted in this section using an example of three DMUs with a single output and a single input that follow a joint normal probability distribution. As shown in Olesen and Petersen [20], each chance constraint $\mathbf{u}^T \overline{\mathbf{y}}_j - \mathbf{v}^T \overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j) \sqrt{(\mathbf{u}^T, -\mathbf{v}^T) \mathbf{\Lambda}_j (\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0$ in these two models generates a supporting hyperplane to a confidence region of DMU j at some confidence level related to the chance constraint probability level α_j. Olesen and Petersen [20] further noted that the production possibility set (PPS) is spanned by these confidence regions in the input-output space. We present the motivating example in Figs. 1 and 2 without discussing the mathematical details. The confidence region of DMU j in both figures is an ellipsoid denoted by $D_j(\alpha_j)$ $j = 1, 2, 3$, $\alpha_j > 50\%$, with the mean input and output $(\overline{x}_j, \overline{y}_j)$ of DMU j at the center, where the size of the region is derived from the probability level α_j used in the $j+1$ th chance constraint in model (3). The straight line in the two figures spanned by ellipsoid $D_1(\alpha_1)$ is the production frontier.

The other ellipsoids in the figures are adjusted confidence regions for DMU 2, the DMU under evaluation. These adjusted regions are denoted by $D_2'(q, \beta)$ with the mean output $\overline{y}_2$ and the contracted mean input $q\overline{x}_2$ from DMU 2 at the center, where $q \in (0, 1]$ is a radial contraction rate of the mean input vector $\overline{x}_2$

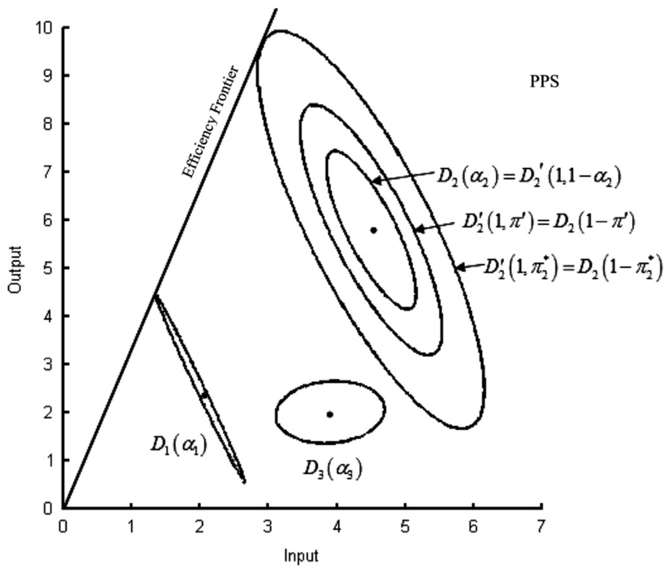

Fig. 1. Confidence regions used in models (2) and (3).

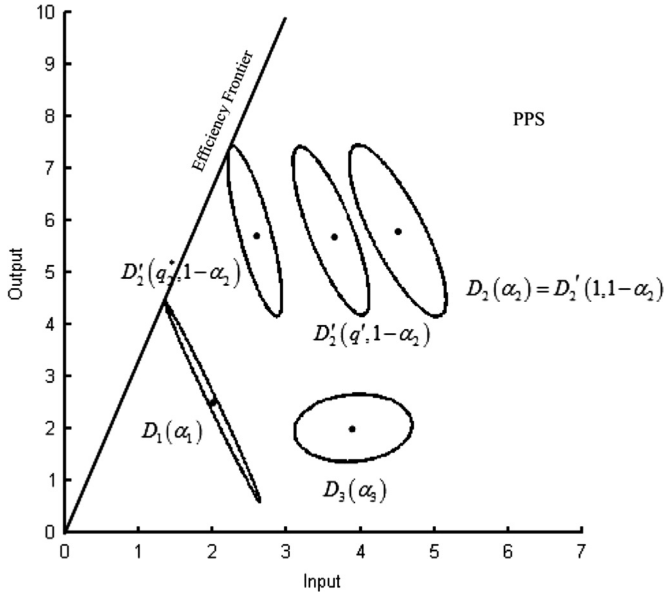

Fig. 2. Confidence regions used in the proposed model.

and β is a reliability level to be explained in detail in Section 3. As shown in that section, if $q=1$, i.e., no radial contraction is performed, then we have $D_2'(1, 1-\alpha_2) = D_2(\alpha_2)$, the confidence region of inputs and outputs derived from a chance constraint at the probability level α_2. However, if $q \in (0,1)$, then the shape of the region is changed accordingly, as illustrated in Fig. 2.

In Fig. 1 we illustrate the confidence regions used in models (2) and (3). The three concentric ellipsoids in Fig. 1 are denoted by $D_2(\alpha_2) = D_2'(1, 1-\alpha_2)$, $D_2'(1, \pi') = D_2(1-\pi')$ and $D_2'(1, \pi_2^*) = D_2(1-\pi_2^*)$, where $\pi_2^* < \pi' < 1-\alpha_2$. Note that $D_2'(1, \pi_2^*)$ is the largest adjusted confidence region obtained by decreasing the reliability level β (with $q=1$) that is still a subset of the reference technology spanned by $D_1(\alpha_1)$. This implies that π_2^* is the highest reliability level β such that the adjusted region $D_2'(1, \beta)$ overlaps with the production frontier. It is evident that the stochastic efficiency index π_o^* is not a radial measure.

On the contrary, the stochastic efficiency index θ_o^* in model (2) is a radial measure. Olesen and Petersen [20] interpreted θ_o^* as the minimum proportional decrease in the random inputs of DMU o subject to a requirement that every input output combination within the confidence region $D_o(\alpha_o)$ after the transformation stays inside the estimated PPS. By this interpretation, θ_2^* in our motivating example would be the minimum latent displacement necessary to move ellipsoid $D_2(\alpha_2)$ in Fig. 1 to overlap with the production frontier line. It is easy to infer from Fig. 1 that using model (2) and letting each confidence region $D_j(\alpha_j)$ shrink toward a point estimate will in the limit converge to the tradition CCR efficiency analysis model based on the mean values of inputs and outputs from each DMU.

However, we note that the adjusted confidence region $D_2'(q, 1-\alpha_2)$ presented in Fig. 2 is in fact contracted when the mean input contraction rate q decreases. Unfortunately, Olesen and Petersen [20] ignored the fact that a contraction of the mean input vector of an evaluated DMU will affect the shape of an adjusted confidence region. As a result, model (2) does not return the correct stochastic efficiency index for an inefficient DMU unless the inputs of that DMU are deterministic.

The model we are going to develop in this paper corrects this error by applying the concept of an aspiration level. In model (1), the aspiration level is fixed and set by the decision maker. But in our proposed model the aspiration level itself is a decision variable. In addition, a reliability chance constraint is introduced. The chance constraints and the variable aspiration level in the new model generate hyperplanes to support two types of confidence regions as shown in Fig. 2. Take $\beta = 1-\alpha_2$ as an example. The ellipsoids $D_j(\alpha_j)$ $j=1, 2, 3$ define the PPS, while decreasing the aspiration level q reduces the deviation between the production frontier and the input output combinations inside the ellipsoid $D_2'(q, 1-\alpha_2)$. q_2^* is thus the contraction rate of the mean input of DMU 2 for which the ellipsoid $D_2'(q_2^*, 1-\alpha_2)$ has one and only one intersection point with the production frontier line.

In Fig. 2 we illustrate the confidence regions used in model (4) proposed in this paper. The three ellipsoids with centers lined up at the same output level are denoted by $D_2(\alpha_2) = D_2'(1, 1-\alpha_2)$, $D_2'(q', 1-\alpha_2)$ and $D_2'(q_2^*, 1-\alpha_2)$, where $q_2^* < q_2' < 1$ and q_2^* is the smallest radial contraction rate q of the mean input from DMU 2 that keeps the confidence region $D_2'(q, 1-\alpha_2)$ as a subset of the reference technology spanned by $D_1(\alpha_1)$. Each of these adjusted confidence regions $D_2'(q, 1-\alpha_2)$ will be shown in Section 3 to be generated by a chance constraint of DMU 2 with some probability level $\beta = 1-\alpha_2$, referred to as a reliability level below.

As illustrated in Fig. 2, combining the use of the concept of an aspiration level from model (1) with the use of confidence regions in model (2) allows us to propose a model in this paper that explores two different types of confidence regions. Firstly, all DMUs contribute with the confidence regions at selected probability levels based on the non-contracted mean input output vectors. As in model (2), the PPS is the convex cone spanned by these confidence regions and enlarged by adding a certain orthant to comply with strong input and output disposability. Hence, each of these confidence regions may potentially play an active role in spanning the PPS. Secondly, we introduce a reliability confidence region, which reflects the shape and size of a confidence region for the evaluated DMU after contraction of the mean input vector with a factor q. Based on these two different sets of confidence regions we define the stochastic CCR efficiency index q_o^*, a radial measure, as the maximum contraction rate q of the mean input vector for the evaluated DMU that is necessary to move and transform the reliability confidence region $D_o'(q, \beta)$ until it either is not a proper subset of the PPS (if $\beta < 0.5$) or is entirely outside the PPS (if $\beta > 0.5$).

The contributions of our study are twofold. First, the model proposed in this paper bridges the existing models (1) and (2). Cooper et al. [7] did not interpret the stochastic efficiency index π_o^* they proposed. The motivating example in this section has illustrated that under the joint normality assumption π_o^* is the highest reliability level β necessary for an adjusted confidence region $D_o'(1, \beta)$ to overlap with the production frontier of the PPS spanned by non-adjusted confidence regions of all DMUs in consideration. Hence using the two types of confidence regions discussed above we establish a uniform framework to interpret the stochastic efficiency indices given by the three models under the multivariate joint normality assumption. Second, this proposed model complements a model in Cooper et al. [7] for characterizing behaviors of satisficing. Using a non-unity aspiration level, Cooper et al. [7] developed a variant of model (1) that can be applied to perform a trade-off analysis between optimizing and satisficing by setting the aspiration level for a stochastically inefficient entity to reach. As will be illustrated in Section 5, the model proposed in the current study can be employed to do a similar trade-off analysis by selecting the minimum probability level to achieve some aspiration level.

The remainder of the paper is organized as follows. In the next section, we introduce the model and provide an economic interpretation of the stochastic efficiency index. This development is followed by a solution procedure proposed in Section 4 for arbitrary probability distributions of input and output levels. An illustrative example is analyzed in Section 5. Concluding remarks are made in the last section.

3. Stochastic efficiency analysis

We present the stochastic DEA model in this section. The production possibility set underlying the model and the stochastic efficiency index for multivariate normally distributed inputs and outputs are also interpreted.

3.1. Model

We start with model (1). As remarked in Cooper et al. [7], the model is always feasible. Let vector $\mathbf{A} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The optimal objective function value π_o^* is the probability for the efficiency score of DMU o to exceed unity with the optimal virtual multipliers. Note $\pi_o^* \le 1-\alpha_o$. Cooper et al. [7] thus defined DMU o to be stochastically efficient (which we call *CHL efficient*below) if and only if $\pi_o^* = 1-\alpha_o$. It is easy to see that there exists at least one DMU $j \in N$ with $\pi_j^* = 1-\alpha_j$ for a given vector $\mathbf{A}$.

In model (1), unity is chosen as the aspiration level. Specifying the aspiration level as a decision variable, we develop a stochastic DEA model below:

$$q_o^* = \max_{q \in R, \mathbf{u}, \mathbf{v}} q$$
$$\text{s.t.}$$
$$P\left\{ \frac{\mathbf{u}^T \tilde{\mathbf{y}}_o}{\mathbf{v}^T \tilde{\mathbf{x}}_o} \ge q \right\} \ge \beta_o,$$

$$P\left\{\frac{\mathbf{u}^T\tilde{\mathbf{y}}_j}{\mathbf{v}^T\tilde{\mathbf{x}}_j}\le 1\right\}\ge\alpha_j,\quad j\in N,$$
$$\mathbf{u}\ge\mathbf{0},\mathbf{v}\ge\mathbf{0}. \tag{4}$$

Here β_o is given and q is a decision variable. For any set of virtual multipliers, q is the maximum value that bounds the multiplier weighted output–input ratio of DMU o from below with a probability of at least β_o. The model seeks virtual multipliers to maximize q, which is equivalent to maximizing the aspiration level to be achieved with a probability of β_o or above. We hence call the first chance constraint in the model a *reliability constraint* and β_o the *reliability level*.

Since all constraints in model (4) are satisfied by $q=0$, $\mathbf{u}=\mathbf{0}$ and $\mathbf{v}>\mathbf{0}$, the model is always feasible with $q_o^*\ge 0$. Evidently, if the input and output values are constant for any DMU j, model (4) reduces to the *deterministic* DEA model, namely the CCR DEA model. We therefore call (4) the *stochastic* CCR DEA model.

Note that the reliability constraint can be rewritten as $P\{\mathbf{v}^T\tilde{\mathbf{x}}_o/\mathbf{u}^T\tilde{\mathbf{y}}_o\le 1/q\}\ge\beta_o$ and $\mathbf{v}^T\tilde{\mathbf{x}}_o/\mathbf{u}^T\tilde{\mathbf{y}}_o$ is a loss function. We can therefore interpret $1/q$ as the Value-at-Risk (VaR) at the confidence level of β_o and thus treat model (4) as a VaR minimization problem [15].

It is evident that the stochastic efficiency index q_o^* may be sensitive to the threshold probability levels β_o and α_j. Increasing β_o or α_j tightens the corresponding chance constraint. Therefore, q_o^* is non-increasing in the parameters β_o and α_j, $\forall j$. We note that q_o^* may exceed unity if $\beta_o<1-\alpha_o$. To render the definition of the stochastic efficiency index consistent with that of its deterministic counterpart, we require $\beta_o\ge 1-\alpha_o$.

Definition 1. Given $\mathbf{A}$ and β_o, DMU o at the reliability level of β_o is (i) stochastically CCR efficient if and only if $q_o^*=1$; (ii) stochastically CCR inefficient if $q_o^*<1$; (iii) stochastically pseudo-efficient if $q_o^*=\max_{j\in N}\{q_j^*\}<1$.

By the above definition, a DMU is stochastically pseudo-efficient if it has the highest stochastic efficiency index and none of the DMUs is stochastically efficient.

Cooper et al. [7] noted that chance constrained programming makes it possible to interpret an inefficient DMU as a satisficing efficient unit with some probability of occurrence. The reader is referred to [7] for insightful discussions of "satisficing" and "inefficiency". We note that the results of model (4) could be interpreted in a similar way. For instance, suppose that DMU o is stochastically efficient, i.e., $q_o^*=1$, with a very low reliability level β_o, which implies a high risk of failing to achieve the aspiration level. A higher reliability level β_o is preferred in a less risky efficiency evaluation. But it would render $q_o^*<1$. DMU o could be deemed as satisficing efficient (acceptably inefficient) if q_o^* is not far below 1. In Section 5 an example illustrates that a trade-off analysis between optimizing (inefficiency) and satisficing can be made by changing the reliability level in model (4).

Unlike model (2), model (4) does not require any specific probability distributions. The proposition below shows that model (2) is a special case of model (4) under joint normality.

Proposition 1. *Given* $\mathbf{A}$ *and* $\beta_o=1-\alpha_o$, $q_o^*=\theta_o^*$ *is true if* $(\tilde{\mathbf{y}}_j^T,\tilde{\mathbf{x}}_j^T)^T$ *follows a multivariate normal distribution for any* $j\in N$, *and (i)* $q_o^*=1$, *or (ii)* $q_o^*<1$ *but the input vector of DMU o is deterministic.*

Proof. Under the joint normality assumption, each chance constraint on DMU j in model (2) is equivalent to $P\{\mathbf{u}^T\tilde{\mathbf{y}}_j/\mathbf{v}^T\tilde{\mathbf{x}}_j\le 1\}\ge\alpha_j$. Let q_o^*, $\mathbf{u}^*$ and $\mathbf{v}^*$ be an optimal solution to model (4). If $q_o^*=1$, then we have $(\mathbf{u}^*)^T\bar{\mathbf{y}}_o-(\mathbf{v}^*)^T\bar{\mathbf{x}}_o+\Phi^{-1}(\alpha_o)\sqrt{[(\mathbf{u}^*)^T,(-\mathbf{v}^*)^T]\mathbf{\Lambda}_j[(\mathbf{u}^*)^T,(-\mathbf{v}^*)^T]^T}=0$. It is easy to verify that $\mathbf{u}^*/[(\mathbf{v}^*)^T\bar{\mathbf{x}}_o]$ and $\mathbf{v}^*/[(\mathbf{v}^*)^T\bar{\mathbf{x}}_o]$ is feasible to model (2) with the objective function value of 1, which is the maximum value possible. $q_o^*=\theta_o^*=1$ hence follows. Now we consider the case where $\tilde{\mathbf{x}}_o=\bar{\mathbf{x}}_o$ is deterministic. Let $\mathbf{\Lambda}_o'$ be the variance–covariance matrix of the output vector $\tilde{\mathbf{y}}_o$. $(\mathbf{u}^*)^T\bar{\mathbf{y}}_o-q_o^*(\mathbf{v}^*)^T\bar{\mathbf{x}}_o+\Phi^{-1}(\alpha_o)\sqrt{(\mathbf{u}^*)^T\mathbf{\Lambda}_o'\mathbf{u}^*}=0$ implies that $\mathbf{u}^*/[(\mathbf{v}^*)^T\bar{\mathbf{x}}_o]$ and $\mathbf{v}^*/[(\mathbf{v}^*)^T\bar{\mathbf{x}}_o]$ is feasible to model (2)with the objective function value $\theta=q_o^*$. Assume that there exists a feasible solution $\mathbf{u}'$ and $\mathbf{v}'$ to model (2) with an objective function value $\theta'>\theta$. It follows that θ', $\mathbf{u}'$ and $\mathbf{v}'$ is feasible to model (4), which contradicts the knowledge that q_o^* is optimal. □

3.2. Stochastic CCR efficiency index

We now interpret the stochastic CCR efficiency index q_o^*. Similar to Olesen and Petersen [20], we assume in this sub-section that the outputs and inputs of each DMU j follow a known $s+m$ dimensional multivariate normal distribution with a mean vector $(\bar{\mathbf{y}}_j^T,\bar{\mathbf{x}}_j^T)^T$ and a variance–covariance matrix $\mathbf{\Lambda}_j$ of full rank. This assumption is adopted because close form expressions of the chance constraints in model (4) may not exist or may make it hard to interpret the stochastic efficiency index if inputs and outputs follow probability distributions other than normality. As noted by Cooper et al. [7], the selection of the normal distribution is not so restrictive since normal approximation is readily acceptable in many situations. We further require $\alpha_j\ge 50\%$ for DMU j.

3.2.1. Production possibility set

Note that model (4) and model (2) have identical chance constraints on DMUs under the joint normality assumption. The study in Olesen and Petersen [20] suggested that the production possibility set (PPS) defined by these chance constraints is spanned by the confidence regions for the DMUs in consideration. We next briefly summarize the relevant results. The reader is advised to consult Olesen and Petersen [19,20] for details.

Let $c_j=\Phi^{-1}(\alpha_j)$ for any $j\in N$. Denote by $\chi^2_{(s+m)}$ a random variable following the chi-square distribution with $s+m$ degrees of freedom. The confidence region at the confidence level $\varphi_j=P(\chi^2_{(s+m)}\le c_j^2)$ is supported by the chance constraint on DMU $j\in N$:

$$D_j(\alpha_j)=\{(\mathbf{y}^T,\mathbf{x}^T)^T\in\mathbb{R}_+^{s+m}|[(\mathbf{y}-\bar{\mathbf{y}}_j)^T,(\mathbf{x}-\bar{\mathbf{x}}_j)^T]\mathbf{\Lambda}_j^{-1}[(\mathbf{y}-\bar{\mathbf{y}}_j)^T,$$
$$(\mathbf{x}-\bar{\mathbf{x}}_j)^T]^T\le c_j^2\}, \tag{5}$$

where $\mathbf{\Lambda}_j^{-1}$ is the inverse of the variance–covariance matrix $\mathbf{\Lambda}_j$. Note that a random realization of $(\tilde{\mathbf{y}}_j^T,\tilde{\mathbf{x}}_j^T)^T$ is located inside the region $D_j(\alpha_j)$ with a probability of φ_j.

As Olesen and Petersen [19] demonstrated, random realizations of DMU j that fall within the confidence region $D_j(\alpha_j)$ are positioned inside the PPS if $c_j\ge 0$, or equivalently, $\alpha_j\ge 0.5$. Therefore, the PPS for model (4), denoted by $\mathbb{Q}(\mathbf{A})$, is the envelopment of n confidence regions $D_j(\alpha_j)$, $\forall j\in N$. It follows that $\mathbb{Q}(\mathbf{A})=\{(\mathbf{y}^T,\mathbf{x}^T)^T\in\mathbb{R}_+^{s+m}|\exists(\widehat{\mathbf{y}}_j^T,\widehat{\mathbf{x}}_j^T)^T\in D_j(\alpha_j)$ and $\lambda_j\ge 0$, $j\in N$ such that $\sum_{j\in N}\lambda_j\widehat{\mathbf{x}}_j\le\mathbf{x}$ and $\sum_{j\in N}\lambda_j\widehat{\mathbf{y}}_j\ge\mathbf{y}\}$.

Denote by $\mathbb{P}(\mathbf{A})$ the set of feasible virtual multipliers under vector $\mathbf{A}$. It can be formulated as

$$\mathbb{P}(\mathbf{A})=\{(\mathbf{u}^T,-\mathbf{v}^T)^T\in\mathbb{R}^{s+m}|\mathbf{u}\ge\mathbf{0},\mathbf{v}\ge\mathbf{0}\quad\text{and}$$
$$P(\mathbf{u}^T\tilde{\mathbf{y}}_j-\mathbf{v}^T\tilde{\mathbf{x}}_j\le 0)\ge\alpha_j\quad\forall j\in N\}. \tag{6}$$

Transforming the chance constraints in Eq. (6) into deterministic equivalent constraints, we have

$$\mathbb{P}(\mathbf{A})=\{(\mathbf{u}^T,-\mathbf{v}^T)^T\in\mathbb{R}^{s+m}|\mathbf{u}\ge\mathbf{0},\mathbf{v}\ge\mathbf{0}$$

and

$$\mathbf{u}^T\bar{\mathbf{y}}_j - \mathbf{v}^T\bar{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\boldsymbol{\Lambda}_j(\mathbf{u}^T, -\mathbf{v}^T)^T} \le 0, \forall j \in N\}.$$

Since $\alpha_j \ge 50\%$, $\mathbf{u}^T\bar{\mathbf{y}}_j - \mathbf{v}^T\bar{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\sqrt{(\mathbf{u}^T, -\mathbf{v}^T)\boldsymbol{\Lambda}_j(\mathbf{u}^T, -\mathbf{v}^T)^T}$ is a convex function. It follows that $\mathbb{P}(\mathbf{A})$ is in general a convex cone.

Theorem 5 in Olesen and Petersen [19] suggests that $\mathbb{Q}(\mathbf{A})$ presented in terms of $\mathbb{P}(\mathbf{A})$ is given by

$$\mathbb{Q}(\mathbf{A}) = \{(\mathbf{y}^T, \mathbf{x}^T)^T \in \mathbb{R}_+^{s+m} | \forall (\mathbf{u}^T, -\mathbf{v}^T)^T \in \mathbb{P}(\mathbf{A}) : \mathbf{u}^T\mathbf{y} - \mathbf{v}^T\mathbf{x} \le 0\}. \quad (7)$$

As a consequence, the production frontier for $\mathbb{Q}(\mathbf{A})$ can be presented as

$$Eff\,\mathbb{Q}(\mathbf{A}) = \{(\mathbf{y}^T, \mathbf{x}^T)^T \in \mathbb{R}_+^{s+m} | \exists (\mathbf{u}^T, -\mathbf{v}^T)^T \in \mathbb{P}(\mathbf{A}) : \mathbf{u}^T\mathbf{y} - \mathbf{v}^T\mathbf{x} = 0\}. \quad (8)$$

3.2.2. Economic interpretation

The stochastic CCR efficiency index given by model (4) is next interpreted. In a way similar to derive Eq. (5), we claim that the reliability constraint defines a supporting hyperplane to the following confidence region, which is called a reliability confidence region, at the confidence level $P[\chi^2_{(s+m)} \le (\Phi^{-1}(\beta_o))^2]$ for DMU o:

$$D'_o(q, \beta_o) = \{(\mathbf{y}^T, \mathbf{x}^T)^T \in \mathbb{R}_+^{s+m} | [(\mathbf{y} - \bar{\mathbf{y}}_o)^T, (\mathbf{x} - q\bar{\mathbf{x}}_o)^T] \underline{\boldsymbol{\Lambda}}_o^{-1} [(\mathbf{y} - \bar{\mathbf{y}}_o)^T, (\mathbf{x} - q\bar{\mathbf{x}}_o)^T]^T \le (\Phi^{-1}(\beta_o))^2\}, \quad (9)$$

where $\underline{\boldsymbol{\Lambda}}_o = \mathbf{B}\boldsymbol{\Lambda}_o\mathbf{B}$ and $\mathbf{B} = [b_{gh}]$ is a $(s+m) \times (s+m)$ matrix with $b_{gh} = 0$ if $g \ne h$, $b_{gg} = 1$ if $g \le s$ and $b_{gg} = q$ otherwise. Note that $D'_o(q, \beta_o)$ is a confidence region rendered after the mean input vector of DMU o changes proportionally by q.

Given q, let $V(q, \beta_o) = \{(\mathbf{u}^T, -\mathbf{v}^T)^T \in \mathbb{R}^{s+m} | \mathbf{u} \ge \mathbf{0}, \ \mathbf{v} \ge \mathbf{0}$ and $P(\mathbf{u}^T\tilde{\mathbf{y}}_o - q\mathbf{v}^T\tilde{\mathbf{x}}_o \ge 0) = \beta_o\}$, i.e., $\mathbf{u}^T\bar{\mathbf{y}}_o - q\mathbf{v}^T\bar{\mathbf{x}}_o - \Phi^{-1}(\beta_o)\sqrt{(\mathbf{u}^T, -q\mathbf{v}^T)\boldsymbol{\Lambda}_o(\mathbf{u}^T, -q\mathbf{v}^T)^T} = 0$ holds for every vector $(\mathbf{u}^T, -\mathbf{v}^T)^T$ in $V(q, \beta_o)$. We consider $1 - \alpha_o \le \beta_o < 50\%$ and $\alpha_o, \beta_o \ge 50\%$ separately.

Suppose $1 - \alpha_o \le \beta_o < 50\%$. Given a vector $(\mathbf{u}^T, -\mathbf{v}^T)^T \in V(q, \beta_o)$, we have $\mathbf{u}^T\bar{\mathbf{y}}_o - q\mathbf{v}^T\bar{\mathbf{x}}_o \le 0$ as $\Phi^{-1}(\beta_o) < 0$. Applying Corollary 1 in Olesen and Petersen [19], $\mathbf{u}^T\mathbf{y} - \mathbf{v}^T\mathbf{x} \le 0$ follows at any realization $(\mathbf{y}^T, \mathbf{x}^T)^T \in D'_o(q, \beta_o)$. As will be shown later in Corollary 1, the reliability constraint is binding at optimality. It implies that a vector $(\mathbf{u}^T, -\mathbf{v}^T)^T$ in $V(q_o^*, \beta_o) \cap \mathbb{P}(\mathbf{A})$ (the intersection of the two sets is not empty as the model is always feasible) must be optimal. By Eqs. (7) and (8), we realize that the reliability confidence region $D'_o(q_o^*, \beta_o)$ is positioned inside the PPS $\mathbb{Q}(\mathbf{A})$. Since q_o^* is the maximum objective function value, $D'_o(q_o^*, \beta_o)$ shall overlap with the production frontier $Eff\,\mathbb{Q}(\mathbf{A})$, that is, there exist a realization $(\mathbf{y}^T, \mathbf{x}^T)^T \in D'_o(q_o^*, \beta_o)$ and $(\mathbf{u}^T, -\mathbf{v}^T)^T \in \mathbb{P}(\mathbf{A})$ such that $\mathbf{u}^T\mathbf{y} - \mathbf{v}^T\mathbf{x} = 0$.

Note that $D'_o(1, \beta_o)$ is the original reliability confidence region and $D'_o(q_o^*, \beta_o)$ the one after every input of DMU o contracts by the rate q_o^*. It now becomes clear that the stochastic CCR efficiency index q_o^* at the reliability level of β_o measures the deviation between the production frontier and the reliability confidence region $D'_o(1, \beta_o)$. We can interpret q_o^* as follows.

- DMU o is stochastically CCR efficient if $q_o^* = 1$, which implies that the reliability confidence region $D'_o(1, \beta_o)$ overlaps with the production frontier $Eff\,\mathbb{Q}(\mathbf{A})$. In other words, there exists some random realization $(\mathbf{y}^T, \mathbf{x}^T)^T \in D'_o(1, \beta_o)$ that lies on the production frontier. As $P[\chi^2_{(s+m)} \le (\Phi^{-1}(\beta_o))^2] \le \varphi_o$, we know that $D'_o(1, \beta_o)$ is a subset of $D_o(\alpha_o)$. Note that the PPS $\mathbb{Q}(\mathbf{A})$ spans confidence regions $D_j(\alpha_j) \forall j$. It follows that a necessary condition for $q_o^* = 1$ is that confidence regions $D'_o(1, \beta_o)$ and $D_o(\alpha_o)$ coincide, i.e., $\beta_o + \alpha_o = 1$.
- $q_o^* < 1$ means no random realization in the reliability confidence region $D'_o(1, \beta_o)$ is efficient. q_o^* is the maximum rate q of proportional decrease in the inputs of DMU o before some input–output combination within $D'_o(q, \beta_o)$ becomes efficient. The index introduces a target reliability confidence region, $D'_o(q_o^*, \beta_o)$, which overlaps with the production frontier.

Now we consider $\alpha_o \ge 50\%$ and $\beta_o \ge 50\%$. In a way similar to the analysis for the above case, we conclude that $\mathbf{u}^T\mathbf{y} - \mathbf{v}^T\mathbf{x} \ge 0$ holds at any realization $(\mathbf{y}^T, \mathbf{x}^T)^T \in D'_o(q_o^*, \beta_o)$ for a multiplier vector $(\mathbf{u}^T, -\mathbf{v}^T)^T \in V(q_o^*, \ \beta_o) \cap \mathbb{P}(\mathbf{A})$. As a result, the target reliability confidence region $D'_o(q_o^*, \beta_o)$ is positioned outside the PPS $\mathbb{Q}(\mathbf{A})$ with some realization on the production frontier. q_o^* is always less than unity and can be regarded as the maximum rate q to decrease the mean input vector of DMU o until no input–output combination within $D'_o(q, \beta_o)$ is inefficient.

In summary, the stochastic CCR efficiency index q_o^* is the maximum contraction rate q of the input vector for DMU o that is necessary to move the reliability confidence region $D'_o(q, \beta_o)$ until (i) it is not a proper subset of the PPS if $1 - \alpha_o \le \beta_o < 50\%$ or (ii) it is entirely outside the PPS if $\beta_o \ge 50\%$.

Since $\Phi^{-1}(50\%) = 0$, the confidence region $D_j(50\%)$ of DMU j reduces to a single point $(\bar{\mathbf{y}}_j^T, \bar{\mathbf{x}}_j^T)^T$. Note that $(\bar{\mathbf{y}}_o^T, \bar{\mathbf{x}}_o^T)^T$ is also the reliability confidence region $D'_o(1, 50\%)$. It is easy to see that q_o^* coincides with the DMU's deterministic CCR efficiency index (under the assumption of no variability) when $\alpha_j = 50\%$ for any $j \in N$ and $\beta_o = 50\%$. Hence the deterministic and stochastic CCR efficiency indices can be interpreted similarly. But the latter is concerned with a set of input–output combinations of the DMU under evaluation instead of a single observation. Because q_o^* is non-increasing as β_o or α_j increases, the deterministic CCR efficiency index of a DMU o is no less than its stochastic counterpart q_o^* when $\alpha_j > 50\% \forall j \in N$ and $\beta_o > 50\%$.

Olesen and Petersen [20] interpreted θ_o^* as the maximum reduction rate in the mean inputs necessary for the confidence region $D_o(\alpha_o)$ to overlap with the estimated production frontier. However, we note that the variance–covariance matrix for the inputs and outputs of DMU o shall change accordingly as the mean inputs are displaced. Because the authors ignore this change in modeling, their analysis would not identify the true target reliability confidence region on the production frontier for an inefficient DMU and a correct stochastic efficiency index is returned by model (2) only when the inefficient DMU's input vector is deterministic.

An example of three DMUs adapted from Olesen and Petersen [20] is used to illustrate the exposition in this sub-section. Each DMU produces a single output from a single input. The input and output combination of a DMU is assumed to follow a two-dimensional normal distribution. The parameters of the distributions are given in Table 1. Let $\alpha_j = 85\%, j = 1, 2, 3$. Given j, we have $c_j = \Phi^{-1}(85\%) \approx 1.036$ and $\varphi_j = P(\chi^2_{(2)} \le 1.036^2) \approx 0.5847$. As an illustration, DMU 1's confidence region at the confidence level of

Table 1
Summary measures of the inputs and outputs.

DMU j	Mean $(\bar{y}_j, \bar{x}_j)$	Variance–covariance matrix Λ_j	
1	(5, 2)	3.604	−1.2
		−1.2	0.404
2	(5.8, 4.5)	0.148	0.264
		0.264	0.488
3	(3.3, 3.9)	0.04	0.048
		0.048	0.068

0.5847 is formulated as

$$D_1(85\%) = \left\{ (y,x) \in \mathbb{R}_+^2 \,|\, \begin{bmatrix} y-5 \\ x-2 \end{bmatrix}^T \begin{bmatrix} 3.604 & -1.2 \\ -1.2 & 0.404 \end{bmatrix}^{-1} \begin{bmatrix} y-5 \\ x-2 \end{bmatrix} \le 1.036^2 \right\}.$$

Since $m = s = 1$, each confidence region is an ellipsoid for which the center and the axes are defined by the mean vector and the variance–covariance matrix. The confidence regions $D_1(85\%)$, $D_2(85\%)$ and $D_3(85\%)$ are presented in Fig. 3.

As shown in the figure, the production frontier of the PPS spans the confidence region $D_1(85\%)$. By Eqs. (5) and (9), $D_1'(1, 15\%)$ is identical to $D_1(85\%)$ as $[\Phi^{-1}(85\%)]^2 = [\Phi^{-1}(15\%)]^2$. It follows that DMU 1 is stochastically CCR efficient at the reliability level of 15%.

Suppose now $\beta_1 = 30\%$. As presented in Fig. 4, the reliability confidence region $D_1'(1, 30\%)$ is a subset of $D_1(85\%)$. Hence, DMU 1 is stochastically inefficient at the reliability level of 30%. The production frontier is a tangent line to the target reliability confidence region $D_1'(0.6928, 30\%)$. It follows that $q_1^* = 0.6928$ when $\beta_1 = 30\%$.

Let $\beta_1 = 70\%$. Note $[\Phi^{-1}(\rho)]^2 = [\Phi^{-1}(1-\rho)]^2$ for any $\rho \in (0,1)$. By Eqs. (5) and (9), we realize $D_1'(q,\rho) = D_1(\rho) = D_1'(q, 1-\rho)$ when $q=1$ and $\rho \ge 50\%$. Therefore, the reliability confidence region $D_1'(1, 70\%)$ coincides with the region $D_1'(1, 30\%)$, which was presented in Fig. 4. As shown in Fig. 5, $D_1'(0.3308, 70\%)$ is the target reliability confidence region for DMU 1. That is, $q_1^* = 0.3308$ at the reliability level of 70%. Comparing Figs. 4 and 5 we note that (i) choosing $\alpha_j \ge 50\%$ $j \in N$, $\beta_1 < 50\%$ and solving Eq. (4) for $o=1$ is equivalent to a search for the maximum value of the radial contraction rate q of the mean input $\bar{x}_1$ such that $D_1'(q, \beta_1)$ is not a proper subset of the interior of the PPS partly spanned by the confidence regions $D_j(\alpha_j)$ for DMU $j = 1,2,3$; (ii) choosing $\alpha_j \ge 50\%$ $j \in N$, $\beta_1 > 50\%$ and solving Eq. (4) for $o=1$ is equivalent to a search for the maximum value of the radial contraction rate q of the mean input $\bar{x}_1$ such that the intersection of $D_1'(q, \beta_1)$ and the interior of the PPS partly spanned by the confidence regions $D_j(\alpha_j)$ for DMU $j = 1,2,3$ is empty.

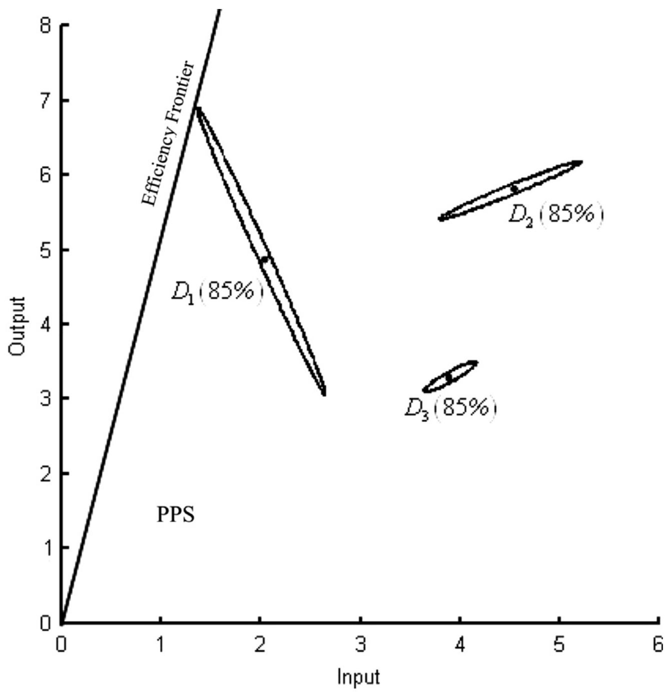

Fig. 3. Production possibility set and production frontier.

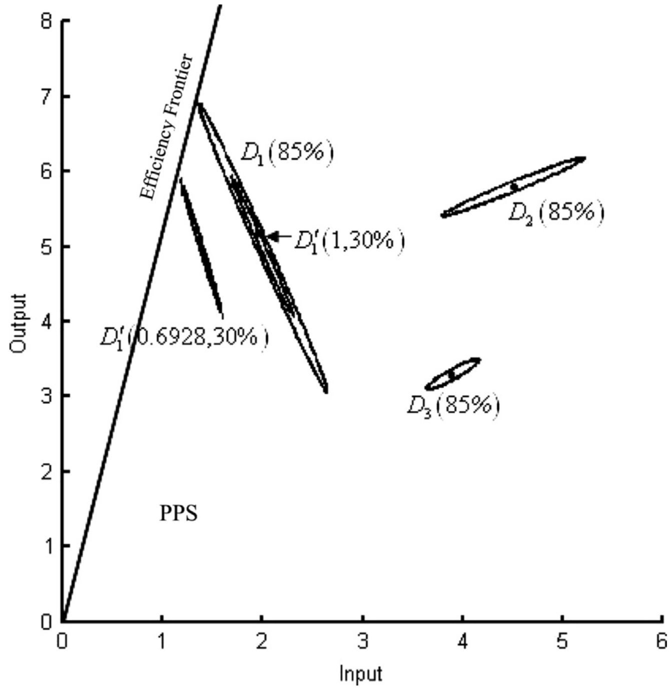

Fig. 4. Efficiency analysis of DMU 1 at $\beta_1 = 30\%$.

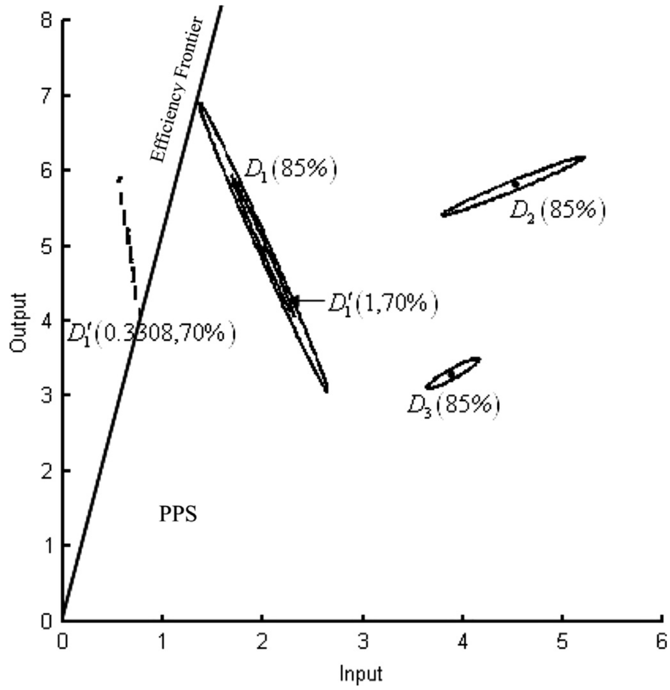

Fig. 5. Efficiency analysis of DMU 1 at $\beta_1 = 70\%$.

4. Solution approach

We note that model (4) is difficult to solve directly due to its strong nonlinearity caused by the decision variable q appearing in the reliability constraint as well as non-convexity. In this section, we develop a solution procedure for model (4) by solving a series of models in a form similar to (1), for which the deterministic equivalent formulation is relatively easier to solve at least approximately.

The next result is useful to our ensuing exposition.

Proposition 2. *Suppose that* $\mathbf{u}'$ *and* $\mathbf{v}'$ *are feasible multiplier vectors to the constraints of model (4) other than the reliability constraint.*

Let $q' = \max q$ subject to the reliability constraint with $\mathbf{u} = \mathbf{u}'$ and $\mathbf{v} = \mathbf{v}'$. It follows that $P\{(\mathbf{u}')^T \tilde{\mathbf{y}}_o / (\mathbf{v}')^T \tilde{\mathbf{x}}_o \geq q'\} = \beta_o$ always holds.

Proof. Note that there exists at least one element in $\mathbf{v}'$ that is not zero. Since $\tilde{y}_{ro}$ and $\tilde{x}_{io}$ are continuous random variables, $P\{(\mathbf{u}')^T \tilde{\mathbf{y}}_o / (\mathbf{v}')^T \tilde{\mathbf{x}}_o \geq q\}$ in terms of q can be interpreted as a continuous mapping into (0, 1). By the Intermediate Value Theorem [11], there exists some value q' such that $P\{(\mathbf{u}')^T \tilde{\mathbf{y}}_o / (\mathbf{v}')^T \tilde{\mathbf{x}}_o \geq q'\} = \beta_o$. It is easy to verify that $q' = \max q$ subject to the reliability constraint. □

The corollary below is trivial to prove.

Corollary 1. *Let q_o^*, $\mathbf{u}^*$ and $\mathbf{v}^*$ be optimal to model (4). It follows that $P\{(\mathbf{u}^*)^T \tilde{\mathbf{y}}_o / (\mathbf{v}^*)^T \tilde{\mathbf{x}}_o \geq q_o^*\} = \beta_o$ holds.*

We now introduce the following programming problem with a given parameter η:

$$\gamma_o^*(\eta) = \max_{\mathbf{u},\mathbf{v}} P\left\{\frac{\mathbf{u}^T \tilde{\mathbf{y}}_o}{\mathbf{v}^T \tilde{\mathbf{x}}_o} \geq \eta\right\}$$
$$\text{s.t.}$$
$$P\left\{\frac{\mathbf{u}^T \tilde{\mathbf{y}}_j}{\mathbf{v}^T \tilde{\mathbf{x}}_j} \leq 1\right\} \geq \alpha_j, \quad j \in N,$$
$$\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}. \tag{10}$$

The optimality condition of model (4) is presented below.

Theorem 1. *$\gamma_o^*(q_o^*) = \beta_o$ is the sufficient and necessary optimality condition for model (4).*

Proof. We first show that the optimality condition is necessary. Let $\mathbf{u}^*$, $\mathbf{v}^*$ and q_o^* be optimal to model (4). It follows that $\mathbf{u}^*$ and $\mathbf{v}^*$ are feasible to model (10) for any η and $\gamma_o^*(q_o^*) \geq \beta_o$. Suppose that for model (10) the optimal multipliers $\mathbf{u}'$ and $\mathbf{v}'$ are such that $\gamma_o^*(q_o^*) = P\{(\mathbf{u}')^T \tilde{\mathbf{y}}_o / (\mathbf{v}')^T \tilde{\mathbf{x}}_o \geq q_o^*\} > \beta_o$. It is easy to see that $q = q_o^*$, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{v} = \mathbf{v}'$ would be feasible to model (4), which contradicts Proposition 2.

Next we show that the optimality condition is sufficient. Assume that $\gamma_o^*(q) = \beta_o$ holds. Suppose that $\mathbf{u}^*$, $\mathbf{v}^*$ and q_o^* is the optimal solution to model (4) with $q_o^* > q$. By Corollary 1, $P\{(\mathbf{u}^*)^T \tilde{\mathbf{y}}_o / (\mathbf{v}^*)^T \tilde{\mathbf{x}}_o \geq q\} > \beta_o$ and hence $\gamma_o^*(q) > \beta_o$, which contradicts our assumption. □

A solution procedure is developed to solve the optimality condition $\gamma_o^*(q_o^*) = \beta_o$.

Algorithm: Solving model (4)

Step 1: Let $k=1$ and $\eta^{(1)}$ be sufficiently small.
Step 2: Solve model (10) with $\eta = \eta^{(k)}$. Denote by $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$ the vectors of optimal multipliers. We require that not all elements in $\mathbf{u}^{(k)}$ are zero when $k=1$.
Step 3: If $\gamma_o^*(\eta^{(k)}) - \beta_o < \delta$ (a pre-selected tolerance), then stop and return $\eta^{(k)}$ as q_o^*. Otherwise, let $\eta^{(k+1)}$ be the value such that $P\{(\mathbf{u}^{(k)})^T \tilde{\mathbf{y}}_o / (\mathbf{v}^{(k)})^T \tilde{\mathbf{x}}_o \geq \eta^{(k+1)}\} = \beta_o$, increase k to $k+1$ and go to Step 2.

The initial value of η should be chosen such that $\gamma_o^*(\eta^{(1)})$ is 1 or close to 1. We can set $\eta^{(1)} = 0$ if inputs and outputs are all positive random variables. At the kth iteration of the algorithm, the sub-problem (10) is solved for $\gamma_o^*(\eta^{(k)})$. The optimal multipliers obtained are then used to generate $\eta^{(k+1)}$. This process repeats until $\gamma_o^*(\eta^{(k)})$ and β_o become sufficiently close.

The algorithm yields two sequences of numbers $\{\eta^{(k)}\}$ and $\{\gamma_o^*(\eta^{(k)})\}$. The next proposition characterizes the sequence $\{\eta^{(k)}\}$.

Proposition 3. *Iteration sequence $\{\eta^{(k)}\}$ is monotone increasing in k.*

Proof. Note that $\gamma_o^*(\eta^{(1)})$ is sufficiently close to 1. Since $P\{(\mathbf{u}^{(1)})^T \tilde{\mathbf{y}}_o / (\mathbf{v}^{(1)})^T \tilde{\mathbf{x}}_o \geq \eta^{(2)}\} = \beta_o < 1$, we have $\eta^{(1)} < \eta^{(2)}$. Suppose that the algorithm does not terminate at the kth iteration. Since $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$ are feasible to model (10), we have $\gamma_o^*(\eta^{(k)}) > \beta_o + \delta$, $P\{(\mathbf{u}^{(k)})^T \tilde{\mathbf{y}}_o / (\mathbf{v}^{(k)})^T \tilde{\mathbf{x}}_o \geq \eta^{(k+1)}\} = \beta_o \leq \gamma_o^*(\eta^{(k+1)})$. Hence, $\eta^{(k)} < \eta^{(k+1)}$ unless $\gamma_o^*(\eta^{(k)}) - \beta_o < \delta$ and therefore the sequence stops. □

The next corollary is natural.

Corollary 2. *Iteration sequence $\{\gamma_o^*(\eta^{(k)})\}$ is monotone decreasing in k.*

According to the Monotone Convergence Principle [11], iteration sequence $\{\gamma_o^*(\eta^{(k)})\}$ converges to β_o.

Now we consider a special case where the inputs and outputs of some DMUs are all deterministic. Let J be the set of all such DMUs. The chance constraint in model (10) on any DMU $j \in J$ changes to $\mathbf{u}^T \mathbf{y}_j / \mathbf{v}^T \mathbf{x}_j \leq 1$ or $\mathbf{u}^T \mathbf{y}_j - \mathbf{v}^T \mathbf{x}_j \leq 0$, where again $\mathbf{y}_j$ and $\mathbf{x}_j$ denote deterministic output and input vectors. If $o \notin J$, then the algorithm presented above is still applicable. Otherwise, the following single problem is solved for the efficiency index q_o^*:

$$q_o^* = \max_{\mathbf{u},\mathbf{v}} \frac{\mathbf{u}^T \mathbf{y}_o}{\mathbf{v}^T \mathbf{x}_o}$$
$$\text{s.t.}$$
$$P\left\{\frac{\mathbf{u}^T \tilde{\mathbf{y}}_j}{\mathbf{v}^T \tilde{\mathbf{x}}_j} \leq 1\right\} \geq \alpha_j, \quad j \notin J,$$
$$\frac{\mathbf{u}^T \mathbf{y}_j}{\mathbf{v}^T \mathbf{x}_j} \leq 1, \quad j \in J,$$
$$\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}.$$

By the Charnes–Cooper transformation of linear fractional programming problems [2], the above model can be rewritten as

$$q_o^* = \max_{\mathbf{u},\mathbf{v}} \mathbf{u}^T \mathbf{y}_o$$
$$\text{s.t.}$$
$$\mathbf{v}^T \mathbf{x}_o = 1,$$
$$P\left\{\frac{\mathbf{u}^T \tilde{\mathbf{y}}_j}{\mathbf{v}^T \tilde{\mathbf{x}}_j} \leq 1\right\} \geq \alpha_j, \quad j \notin J,$$
$$\mathbf{u}^T \mathbf{y}_j - \mathbf{v}^T \mathbf{x}_j \leq 0, \quad j \in J,$$
$$\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}. \tag{11}$$

The iterative algorithm presented earlier in this section, which we call Algorithm 1 can be applied here by replacing model (10) with model (11).

We note that the algorithms developed in this section are applicable to general probability distributions. The sub-problem (10) can be solved in a way similar to model (1). Cooper et al. [7] derived deterministic quadratic programs equivalent to model (1), respectively, under two assumptions:

- Stochastic outputs and inputs are related only through a single normally distributed factor.
- Input and output values are random variables following a multivariate normal distribution.

In the next section, we will illustrate how to derive a deterministic equivalent problem for model (10). We recommend that the reader refer to Cooper, Huang, and Li [7] for details.

5. An illustrative example

We now evaluate the performance of a subset of the selected gas stations studied by Suyoshi [22] to illustrate model (4) and the algorithms developed in the previous section. In the computational studies, the linear problems are solved in Lindo What'sBest! 10.0, an Excel spreadsheet add-in for mathematical programming, while Algorithms 1 is coded using Microsoft VBA (Visual Basic for Applications).

Sueyoshi [22] used a data set generated in summer 1998 to predict future operational performance of sixty selected gas stations in Tokyo, Japan. The three inputs in the data set are the number of employees; the space size of a gas station; and the monthly operational cost. The input values were observed in summer 1998 and assumed to be deterministic. The two outputs chosen are the sales of gasoline and petrol to be realized in winter 1998. The output levels were unknown at the time and a manager in a Japanese petroleum firm was asked to provide the most likely estimate, the optimistic estimate and the pessimistic estimate for either output of each gas station. Under the assumption that a random output level is independent and follows a particular beta distribution used in PERT/CPM Sueyoshi [22] applied these estimates to approximate the means and variances of the outputs. (We note that Eq. (18) in [22] has typos. It should read $b_{rj}^2 = [(OP_{rj} - PE_{rj})/6]^2$.) Furthermore, the author adopted the *single factor symmetric disturbance* assumption, i.e., the component of any output determined solely by a single underlying random factor ξ is formulated as

$$\tilde{y}_{rj} = \overline{y}_{rj} + b_{rj}\xi,$$

for $j = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, s$, where ξ follows the standard normal distribution. Note that $\overline{y}_{rj}$ is the expected value of $\tilde{y}_{rj}$, while b_{rj} is the standard deviation. We would like to point out that the assumptions of an independent PERT-beta distribution and a single factor symmetric distribution are inconsistent, while the author did not motivate or justify these assumptions. Despite these problematic assumptions we choose the data set presented in [22] as an illustrative example because there are very few applications of stochastic DEA models available in the literature.

In the computational study, we run models (1) and (4) on this data set. Only the twenty gas stations classified as "large" (1st to 20th DMUs in Tables 1 and 2 of [22]) are assessed.

Let $\overline{\mathbf{y}}_j = (\overline{y}_{1j}, \overline{y}_{2j}, \ldots, \overline{y}_{sj})^T$ and $\mathbf{b}_j = (b_{1j}, b_{2j}, \ldots, b_{sj})^T$. $\overline{\mathbf{x}}_j = (\overline{x}_{1j}, \overline{x}_{2j}, \ldots, \overline{x}_{mj})$ denotes the vector of deterministic input values for DMU j.

Proceeding in a way analogous to Cooper, Huang and Li ([7]), we can obtain a linear programming model equivalent to model (1) under the single factor symmetric disturbance assumption:

$$\begin{aligned}
\kappa_o^* = \max_{\mathbf{u},\mathbf{v}}\ & \mathbf{u}^T\overline{\mathbf{y}}_o - \mathbf{v}^T\overline{\mathbf{x}}_o \\
\text{s.t.}\ & \\
& \mathbf{u}^T\mathbf{b}_o = 1, \\
& \mathbf{u}^T\overline{\mathbf{y}}_j - \mathbf{v}^T\overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\mathbf{u}^T\mathbf{b}_j \le 0 \quad j \in N, \\
& \mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}.
\end{aligned}$$

Similarly, a linear programming model equivalent to model (10) can be derived:

$$\begin{aligned}
\omega_o^*(\eta) = \max_{\mathbf{u},\mathbf{v}}\ & \mathbf{u}^T\overline{\mathbf{y}}_o - \eta\mathbf{v}^T\overline{\mathbf{x}}_o \\
\text{s.t.}\ & \\
& \mathbf{u}^T\mathbf{b}_o = 1, \\
& \mathbf{u}^T\overline{\mathbf{y}}_j - \mathbf{v}^T\overline{\mathbf{x}}_j + \Phi^{-1}(\alpha_j)\mathbf{u}^T\mathbf{b}_j \le 0 \quad j \in N, \\
& \mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}.
\end{aligned} \tag{12}$$

Remark 1. Note $\pi_o^* = \Phi(\kappa_o^*)$ and $\gamma_o^*(\eta) = \Phi(\omega_o^*(\eta))$. When Algorithm 1 is applied, model (12) is solved at the kth iteration with $\eta = \eta^{(k)}$, while the optimal virtual multiplier vectors, denoted by $\mathbf{u}^{(k)}$ and $\mathbf{v}^{(k)}$ are used to compute $\eta^{(k+1)}$ as

$$\eta^{(k+1)} = \frac{\Phi^{-1}(\beta) + (\mathbf{u}^{(k)})^T\overline{\mathbf{y}}_o}{(\mathbf{v}^{(k)})^T\overline{\mathbf{x}}_o}.$$

Remark 2. Since model (1) was solved as a linear program and Algorithm 1 iteratively solved a series of linear programs, the true values of π_o^* and q_o^* were returned in this computational study for every DMU o.

The computational results of model (1) are presented in Table 2 with $\alpha = \alpha_j = 95\%$, 90%, 80%, $\forall j \in N$. It is easy to see that for any of these values of α stations 6, 15, 17, and 20 are deemed CHL efficient.

Applying Algorithm 1, we assess the efficiency of each gas station by solving a series of linear programming problems. Table 3 gives q_o^* of the twenty "large" gas stations with combinations between $\alpha = \alpha_j = 95\%$, 90%, 80%, $\forall j \in N$ and $\beta = \beta_o = 1 - \alpha$, 50%, and α. It is obvious that q_o^* decreases with α and β. We note that all CHL efficient units are stochastically CCR efficient when $\beta = 1 - \alpha$. If $\beta = \alpha$ or $\beta = 50\%$, none of the units is stochastically CCR

Table 2
Stochastic outputs related through a single normally distributed factor: π_o^* values (%).

α	80%	90%	95%
DMU 1	2.139	0.589	0.172
DMU 2	0.404	0.101	0.028
DMU 3	0.000	0.000	0.000
DMU 4	2.351	0.418	0.075
DMU 5	0.257	0.055	0.013
DMU 6	20.000	10.000	5.000
DMU 7	7.479	3.004	1.245
DMU 8	0.388	0.081	0.019
DMU 9	0.000	0.000	0.000
DMU 10	0.001	0.000	0.000
DMU 11	17.485	8.332	4.000
DMU 12	0.000	0.000	0.000
DMU 13	0.000	0.000	0.000
DMU 14	1.078	0.314	0.100
DMU 15	20.000	10.000	5.000
DMU 16	0.000	0.000	0.000
DMU 17	20.000	10.000	5.000
DMU 18	0.265	0.074	0.023
DMU 19	0.112	0.009	0.001
DMU 20	20.000	10.000	5.000

Table 3
Stochastic outputs related through a single normally distributed factor: q_o^* values.

α	80%			90%			95%		
β	20%	50%	80%	10%	50%	90%	5%	50%	95%
DMU 1	0.957	0.927	0.896	0.956	0.911	0.865	0.955	0.898	0.840
DMU 2	0.926	0.891	0.857	0.927	0.876	0.824	0.928	0.863	0.798
DMU 3	0.866	0.834	0.803	0.867	0.820	0.773	0.869	0.809	0.749
DMU 4	0.970	0.948	0.926	0.965	0.932	0.900	0.961	0.920	0.878
DMU 5	0.928	0.897	0.867	0.928	0.882	0.836	0.929	0.870	0.811
DMU 6	1.000	0.966	0.931	1.000	0.949	0.897	1.000	0.935	0.870
DMU 7	0.979	0.949	0.920	0.979	0.935	0.891	0.980	0.923	0.867
DMU 8	0.943	0.917	0.890	0.942	0.902	0.863	0.941	0.891	0.840
DMU 9	0.847	0.822	0.796	0.845	0.808	0.770	0.844	0.797	0.749
DMU 10	0.868	0.836	0.804	0.870	0.821	0.773	0.871	0.809	0.748
DMU 11	0.997	0.969	0.943	0.997	0.955	0.916	0.997	0.944	0.894
DMU 12	0.887	0.866	0.844	0.883	0.851	0.820	0.880	0.840	0.799
DMU 13	0.888	0.862	0.835	0.886	0.847	0.808	0.885	0.836	0.786
DMU 14	0.942	0.909	0.876	0.943	0.894	0.844	0.944	0.881	0.818
DMU 15	1.000	0.967	0.934	1.000	0.951	0.901	1.000	0.937	0.875
DMU 16	0.834	0.816	0.798	0.827	0.800	0.773	0.821	0.787	0.754
DMU 17	1.000	0.970	0.940	1.000	0.955	0.910	1.000	0.943	0.886
DMU 18	0.919	0.884	0.849	0.923	0.870	0.818	0.925	0.859	0.793
DMU 19	0.942	0.920	0.898	0.937	0.904	0.871	0.933	0.891	0.850
DMU 20	1.000	0.975	0.950	1.000	0.962	0.924	1.000	0.952	0.904

efficient, but station 20 appears to be stochastically pseudo-efficient.

Next the same data set is used to demonstrate the application of model (4) to perform a trade-off analysis between optimizing and satisficing. Applying the model, the manager of each gas station o seeks the highest aspiration level to be achieved with a chosen probability β_o. In light of the economic interpretation of the stochastic CCR efficiency index, this is equivalent to finding the maximum radial input contraction rate necessary for DMU o to become stochastically efficient with a reliability level β_o.

Take station 15 as an instance. Our computational results presented in Table 3 suggest $q_{15}^* = 1.0$ when $\beta_{15} = 10\%$ and $\alpha = 90\%$. That is, station 15 is stochastically CCR efficient at the reliability level of 10%. Following our analysis in Section 3, we infer that the reliability confidence region $D'_{15}(1.0, 10\%)$ shall be large so that some realizations are on the production frontier, which implies that the risk of failing to achieve the aspiration level of 1.0 is high. The manager of the gas station may prefer a higher reliability level β_{15} and therefore a smaller reliability confidence region in order to perform a less risky efficiency evaluation. Increasing β_{15} renders $q_{15}^* < 1$ and requires that the inputs of the gas station be cut to $q_{15}^* \times 100\%$ of the original levels so as to remain efficient and thus stay in business. Recall that given α_j $\forall j$, q_{15}^* is non-increasing in β_{15}. As β_{15} increases, the performance of the gas station should improve, if feasible, in order to generate the desired outputs using less inputs.

Given $\alpha = \alpha_j = 90\%$ $\forall j$, we obtain $q_{15}^* = 0.9732$ at $\beta_{15} = 30\%$ and $q_{15}^* = 0.8871$ at $\beta_{15} = 95\%$. An input (cost) reduction rate of 0.8871 with a reliability level of 95% seems preferred. However, our analysis in Section 3 indicates that the target reliability confidence region $D'_{15}(0.8871, 95\%)$ is outside the production possibility set. The manager of station 15 as a satisficer may thus argue that it is too costly or technically challenging to make changes to the process necessary to be efficient while cutting the inputs to 88.71% of the current levels. Instead, the manager may be satisfied with reducing the inputs to 97.32% of the current levels with a reliability of 30% if the necessary changes to the process are easy to implement.

6. Concluding remarks

It is critical to consider data uncertainty and variability when assessing the performance of DMUs. A chance-constrained efficiency analysis model with a reliability constraint has been proposed in this paper. This new model links the formulations developed by Olesen and Petersen [20] and Cooper et al. [7], and can be applied to perform a trade-off analysis between optimizing and satisficing.

For multivariate joint normal inputs and outputs the stochastic efficiency index introduced in this study is shown to be a radial measure that can be interpreted in a way similar to the deterministic CCR efficiency index. The chance constraints in the proposed model support two types of confidence regions in the input–output space. Every DMU contributes a confidence region with its non-contracted mean input and output vectors at the center. These confidence regions span the production possibility set. The reliability constraint generates a hyperplane to support a reliability confidence region of the DMU under evaluation based on the mean output vector and contracted mean input vector as well as the reliability level chosen. The stochastic efficiency index is the maximum contraction rate such that the reliability region is either not a proper subset of the production possibility set (if the reliability level is less than 0.5) or completely out of the production possibility set (if the reliability level is greater than 0.5).

In this study, we have suggested a solution method that determines the stochastic CCR efficiency index for a DMU by generating and solving sub-problems iteratively. We realize that this method cannot guarantee a global optimum in instances where the sub-problems are not convex programs. This snag is common for stochastic DEA models [23]. The task of developing a more effective algorithm is left for future research.

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