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Undergraduates’ Collective Argumentation Regarding Integration of Complex Functions Within Three Worlds of Mathematics

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UNDERGRADUATES' COLLECTIVE ARGUMENTATION REGARDING INTEGRATION OF COMPLEX FUNCTIONS WITHIN THREE WORLDS OF MATHEMATICS

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ABSTRACT

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Although undergraduate complex variables courses often do not emphasize formal proofs, many widely-used integration theorems contain nuanced hypotheses. Accordingly, students invoking such theorems must verify and attend to these hypotheses via a blend of symbolic, embodied, and formal reasoning. Using Tall's three worlds of mathematics as a theoretical lens, this research explores undergraduate student pairs' collective argumentation about integration of complex functions, with emphasis placed on students' attention to the hypotheses of integration theorems.

Data consisted of videotaped, semistructured interviews with two pairs of undergraduates, during which they collectively reasoned about thirteen integration tasks. Videotaped classroom observations were also conducted during the integration unit of the course in which these students were enrolled. Interview data were analyzed by categorizing participants' responses according to Toulmin's argumentation scheme, as well as classifying each statement as embodied, symbolic, formal, or blends of the three worlds. The student pairs' responses were further coded according to Levinson's four speaker roles in order to document how individuals contributed socially to the collective arguments, and backing statements were identified as either supporting a warrant's validity, correctness, or field.

Findings revealed that participants' nonverbal modal qualifiers and explicit challenges to each other's assertions catalyzed new arguments allowing students to reach consensus, verify conjectures, or revisit prior assertions. Hence, while existing frameworks identify two types of participation in collective argumentation, the aforementioned challenges suggest an important third type of participation. Although participants occasionally conflated certain formal hypotheses from the integration theorems, their arguments married traditional integral symbolism with dynamic gestures and clever embodied diagrams. Participants also attended to a phenomenon, referred to in the literature as *thinking real, doing complex,* in three distinct manners. First, they took care to avoid invoking attributes of real numbers that no longer apply to the complex setting. Second, they intermittently extended their real intuition to the complex setting erroneously. Third, they deliberately called upon attributes of the real numbers that were productive in describing analogous complex number operations. This three-tiered attention to the *thinking real, doing complex* phenomenon is notable because only the second type is currently documented in existing literature. Collectively, the findings suggest that instructors of complex analysis courses might wish to heavily underscore the importance of geometric interpretations of complex arithmetic early in the course and avoid utilizing acronyms that de-emphasize individual theorem hypotheses. The results also indicate that a more multimodal stance is needed when studying collective argumentation in order to capture covert aspects of students' communication.

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CHAPTER I

INTRODUCTION

Part of the inherent beauty of mathematics lies in the coherent interplay between intuitive, experientially-rooted notions, predictable symbolized manipulations, and formal axiomatic structures. Often, generalization proves to be powerful and intuitive, as one's experience with $2 + 2 = 4$ can be abstracted to tackle situations such as $2x + 2x = 4x$ and even $(2 + 2i) + (2 + 2i) = 4 + 4i$. In the world of analysis, such natural abstraction can afford students with helpful intuition during the transition from real to complex numbers. For instance, a function of one complex variable $f(z)$ is continuous if and only if its real and imaginary component functions are continuous. Additionally, familiar rules for differentiation of real-valued functions such as $\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$ generalize rather effortlessly to become analogous rules in $\mathbb{C}$ such as $\frac{d}{dz}\big(f(z) + g(z)\big) = f'(z) + g'(z)$.

However, not all mathematical concepts are as easy to generalize. As Tall (2013) discussed, "Mathematics is often considered to be a logical and coherent subject, but the successive developments in mathematical thinking may involve a particular manner of working that is supportive in one context but becomes problematic in another" (p. xv). He exemplifies this claim by illustrating how in one's everyday experiences with whole numbers, taking something away leaves him or her with less; yet subtracting a negative integer leaves one with more than he or she started with. As the mathematics education literature on the teaching and learning of complex numbers reveals, analogous difficulties are still prevalent when learning complex analysis. For instance, Danenhower (2000) identified a theme of “thinking real, doing complex” (p. 101) wherein individuals demonstrated a proclivity towards invoking attributes of real numbers that do not necessarily apply in the complex setting. For instance, one participant concluded that the function $f(z) = (2z - x)^2$ was differentiable everywhere because it was a polynomial. Additionally, Troup (2015) found further evidence of this phenomenon when undergraduates reasoned about derivatives of complex functions. For instance, participants attempted to apply the familiar conception of the derivative of a real-valued function as the slope of a tangent line to the context of the complex derivative.

It is possible, then, that undergraduates might be tempted to initially reason about integration of complex functions as area under a curve, as this is one common interpretation in the setting of certain real-valued functions. This could be especially prevalent given that even within the context of real-valued functions, the literature reveals numerous examples of students’ difficulties with integration (Grundmeier, Hansen, & Sousa, 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). However, many of these studies are now more than ten years old, and many of these studies documented the *product* of students’ deficiencies and misconceptions rather than the *process* of students’ reasoning. As such, while students might end up with faulty conclusions about integration and other subjects, their process of reasoning might actually be teeming with healthy connections to intuition or past experiences. Indeed, if nurtured properly, such connections between experientially-based intuition and formal mathematics could benefit students' reasoning in courses such as complex variables or analysis (Soto-Johnson, Hancock, & Oehrtman, 2016).

Moreover, by carefully documenting students' successful reasoning about undergraduate mathematics topics, we are able to gain insight into "what deep understanding and complex justifications are possible for students as they engage in mathematics" (Wawro, 2015, p. 355). Students' reasoning within the subject of complex variables could particularly benefit from such an investigation, as the activity within this course is often situated somewhere between formal proof and symbolic calculation. In particular, students that integrate complex functions often invoke powerful theorems, which rely on idiosyncratic hypotheses and draw on ideas from topology and real analysis. For instance, Cauchy's Integral Formula relies on the hypotheses that the function in question is analytic in a simply connected domain, and that the path used for integration is simple, closed, and positively oriented. While formal proof is typically not the focus of undergraduate courses in complex variables (Committee on the Undergraduate Program in Mathematics, 2015), application of such theorems requires that students at least recognize when these hypotheses apply. Hence it is possible that students might draw upon a combination of intuition, visualization, symbolic manipulation, and formal deduction when integrating complex functions. Accordingly, integration of complex functions serves as an appropriate topic to elicit the complex justifications that Wawro advocated for.

Integration of complex functions is also an important topic for undergraduates with respect to practical applications. For instance, it is extensively used in physics and engineering to analyze and compute flux and potential. Moreover, one can apply techniques using integration of complex functions in order to drastically simplify or enable evaluation of certain real-valued integrals. For example, one can prove $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ by reformulating the problem in terms of an integral of a complex function and applying a combination of Cauchy's Theorem and other techniques similar to those used in residue theory. Accordingly, integration of complex valued functions is a particularly useful and important branch of mathematics, and is a major focus of undergraduate courses on complex variables.

Despite the aforementioned practical and theoretical assets inherent to integration of complex functions, there exists no educational research regarding undergraduates' reasoning in this mathematical domain. This study serves to ameliorate this gap in the literature and to inform the teaching and learning of complex variables by investigating undergraduates' multifaceted argumentation about integration of complex functions. In the remainder of this chapter, I further detail the research problem, present the purpose of my study, and state my guiding research questions. I also define several important terms utilized throughout this document, and reveal the significance of my research.

Statement of the Problem

Although no educational research exists regarding students' reasoning about integration of complex functions, the literature contains several studies relevant to integration of real-valued functions. As mentioned previously, these studies primarily focus on students' various difficulties with respect to integration of real-valued functions (Grundmeier et al., 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). For instance, when asked to provide a definition of a definite integral, research participants gave examples, recited the definition of derivative, or stated some version of the Fundamental Theorem of Calculus (Grundmeier et al.; Rasslan & Tall). Grundmeier et al. and Orton also found that students struggled to connect the idea of a definite integral to a limiting process, unsure of which objects tend to zero or infinity. Participants from several studies also did not recognize when area should be counted as a negative contribution to a definite integral (Grundmeier et al.; Mahir; Rasslan & Tall). Finally, some participants attempted to translate graphs of provided functions into messy formulas and evaluate tedious antiderivatives instead of employing basic area properties from the graph (Judson & Nishimori; Mahir).

Although these studies illuminated problematic conceptions students held about real-valued integration, it is unclear to what extent such difficulties might manifest when integrating complex functions. More generally, the study of complex numbers and variables is one of the undergraduate mathematical domains that have not received much attention from mathematics education researchers. The few studies that do exist in the domain of complex variables have focused primarily on complex arithmetic and forms of a complex number (Danenhower, 2006; Karakok, Soto-Johnson, & Anderson-Dyben, 2014; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis, 2006; Soto-Johnson & Troup, 2014). Earlier research in this area by Danenhower and Panaoura et al. suggested that students struggled with when and how to use specific forms of complex numbers such as the polar form; these studies also stressed the importance of representational fluency when working with complex numbers. More recent literature in this domain has extended these findings to different populations. For instance, Karakok et al. found that a sample of in-service secondary teachers favored the

Cartesian form while working with arithmetic tasks involving complex numbers. On the other hand, undergraduates with Dynamic Geometry Environment (DGE) experience were proficient with the polar form, knew when to employ it, and could connect their algebraic and geometric reasoning (Troup, 2015).

Moving forward, a couple of recent studies have regarded more advanced topics in complex analysis such as continuity (Soto-Johnson, Hancock, & Oehrtman, 2016) and differentiation (Troup, 2015). Soto-Johnson, Hancock, and Oehrtman investigated how mathematicians reconciled formal *Conceptual Mathematics* (CM), as found in textbooks, with their own personal interpretations, or *Ideational Mathematics* (IM) (Schiralli & Sinclair, 2003), of continuity of complex functions. The authors found that mathematicians' IM incorporated domain-first reasoning that was difficult to connect rigorously to formal CM statements and definitions of continuity. This domain-first reasoning was comprised of statements articulating preservation of closeness from the domain into the codomain of a function and did not fully capture the formal definition of continuity.

In another study addressing more advanced topics in complex analysis, Troup (2015) investigated undergraduates' reasoning about the derivative of a complex function, both generally and when using the dynamic geometry software *Geometer's Sketchpad* (GSP). Troup found that through their use of GSP, participants noticed and resolved discrepancies between reasoning methods. Initially focusing on the special case of a linear function, participants discovered that linear functions always rotate and dilate circles by the same amount, regardless of location. Using GSP, they then investigated more complicated functions such as $f(z) = z^2$ and $f(z) = e^z$ to explore the rotation and dilation transformations inherent in differentiation. Studying the images of circles of varying center and radius under these two functions helped participants conclude that applying the function f to a small circle about a particular point z_0 dilated this circle by a factor of $|f'(z_0)|$ and rotated it by $Arg(f'(z_0))$. Thus participants were able to successfully develop a geometric interpretation of the derivative of a complex function using *GSP*.

As mentioned previously, my study strove to investigate students' multifaceted mathematical reasoning, particularly within the domain of integration of complex functions. This required a careful consideration about what constitutes mathematical reasoning. According to the National Council of Teachers of Mathematics (NCTM), reasoning is characterized as "the process of drawing conclusions on the basis of evidence or stated assumptions" (NCTM, 2009; p. 4). Hence, because reasoning is not directly observable as a mental process, researchers can use individuals' argumentation, including the components mentioned by the NCTM, as a window into the mind. As I detail later, such mathematical argumentation is often nuanced and can be expressed through verbal, pictorial, symbolic, and various other means (Tall, 2013).

A common model used to document individuals' argumentation was formulated by Toulmin (2003) and consists of six components: data, warrant, backing, qualifier, rebuttal, and claim. According to Toulmin, any argument is based upon the arguer attempting to convince his or her audience of some *claim* (C), or asserted conclusion. This claim is necessarily grounded in foundational evidence, or *data* (D), on which the claim is based. The arguer can then supply a *warrant* (W) justifying the link between the given data and the purported claim. A *modal qualifier* (Q) is often necessary to explicitly reference “the degree of force which our data confer on our claim in virtue of our warrant” (p. 93). Depending on the warrant provided, there might also be circumstances in which the intended claim does not hold; in this case, conditions of *rebuttal* (R) are needed to indicate when the “general authority of the warrant would have to be set aside” (p. 94).

In the mathematics education literature, participants’ mathematical argumentation has been analyzed with the aid of Toulmin’s model in several different contexts. For instance, while some researchers have chosen to analyze students’ or instructors’ mathematical arguments during an actual class session (Krummheuer, 1995, 2007; Rasmussen, Stephan, & Allen, 2004; Stephan & Rasmussen, 2002), others have used Toulmin’s framework to discuss how students examine the validity of purported written mathematical proofs (Alcock & Weber, 2005). The literature also includes studies investigating students’ argumentation in responses to written examinations (Evens & Houssart, 2004) or task-based interviews (Hollebrands, Conner, & Smith, 2010). Finally, Wawro (2015) used both in-class observations and task-based interviews to comprise a thorough case study of one student’s reasoning in linear algebra.

In the in-class setting, some researchers (Krummheuer, 1995, 2007; Rasmussen et al., 2004; Stephan & Rasmussen, 2002) felt that a reduced Toulmin model omitting the qualifier and rebuttal was appropriate, and rarely found evidence of explicit backing. Moreover, Krummheuer (2007) illuminated warrants invoked by the participants that did not even relate to the mathematical content directly, such as an appeal to the teacher’s perceived authority. However, when more formal arguments such as proofs are concerned, researchers (Alcock & Weber, 2005; Inglis, Mejia-Ramos, and Simpson, 2007; Simpson, 2015; Troudt, 2015) argued for the use of the full Toulmin model. They also mentioned that simply reading the finished product of a purported proof is inherently difficult because some of the components of the Toulmin model, such as backing and sometimes even the warrants, are implicit and cannot be elicited through real-time social discourse with the proof author. Thus it would appear that an investigation into undergraduates' nuanced argumentation about integration of complex functions should adopt the full Toulmin model and incorporate opportunities for clarification, as in an interview setting.

In order to investigate students' treatment of the idiosyncratic hypotheses from integration theorems, one needs a theoretical lens through which to rigorously study individuals' formal reasoning. However, according to the Committee on the Undergraduate Program in Mathematics (CUPM) (2015), the prerequisites for undergraduate complex variables courses "vary wildly" (p. 1) and do not necessarily include real analysis. Moreover, such courses are "typically taught without a strong emphasis on proofs" (p. 1). Thus, my lens accounted for other forms of argumentation. In particular, I adopted Tall's (2013) three worlds of mathematics as a way to theoretically orient my inquiry into undergraduates' reasoning pertaining to integration of complex functions. This perspective traces all mathematical knowledge back to three distinct but interrelated forms of thought: *conceptual embodied*, *operational symbolic,* and *axiomatic formal.*

According to Tall (2013), conceptual embodiment begins with the study of objects and their properties, progressing towards mental visualization and eventually description through increasingly subtle language. The second world of operational symbolism grows out of actions on objects and is symbolized via thinkable concepts such as number. In this world, it is possible for individuals to "conceive the symbols flexibly as operations to perform and also to be operated on through calculation and manipulation" (p. 17). This flexibility evidences what Tall describes as *proceptual thinking*, where a *procept* is a symbol operating dually as process and concept (Tall, 2008). Tall's (2013) third world is that of *axiomatic formalism*, wherein individuals build "formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (p. 17). These three worlds can also combine to form *embodied symbolic* or *symbolic formal* reasoning, as I detail in the third chapter.

As I illustrated at the beginning of this chapter, our previous experiences with mathematics can either support or create conflict with new and abstracted mathematical notions. Tall (2013) referred to the knowledge structures predicated on these prior experiences as *met-befores*. He also argued that mathematical growth can be traced back to three innate *set-befores* of recognition, repetition, and language. These set-befores foster three forms of compression: categorization, encapsulation, and definition. Through this compression, individuals build so-called *crystalline* structures, which incorporate many equivalent formulations of a mathematical object and can be unpacked in various worlds. Hence the three-worlds perspective posits that our propensity as humans for recognition, repetition, and language allows us to crystalize mathematical concepts by building upon met-befores via categorization, encapsulation, and definition. With this theoretical orientation in mind, I now explicate my study's purpose and research questions.

Despite recent research involving more advanced topics in complex analysis (Soto-Johnson et al., 2016; Troup, 2015), there remains no existing education literature regarding integration of complex functions, even though it is a central topic of any complex analysis course for undergraduates. In particular, it is unclear as of yet how undergraduate students reason algebraically, geometrically, and formally with the notion of integration of complex functions. The purpose of my qualitative research study was to explore undergraduates' multifaceted reasoning about integration of complex functions. My guiding research questions were:

Q1 How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?

Q2 How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In order to rigorously address my research questions, I enlisted the help of two pairs of undergraduate students to partake in a videotaped, semistructured (Merriam, 2009), task-based interview comprised of two 90-minute portions. To obtain a rich understanding of the context in which these participants learned about integration of complex functions, I observed and videotaped six class sessions at participants' undergraduate institution. These observations and ensuing field notes allowed me to document what mathematical content was introduced and emphasized during the integration unit in the complex variables course. They also allowed me to discern the nature of mathematical argumentation that was deemed appropriate for the complex variables course. A thorough description of my data collection and analysis procedures resides in Chapter III of this document. Next I clarify the definitions and assumptions pertinent to the formulation and investigation of my research questions.

Definitions

Notice that the purpose and research questions pertaining to this study refer to individuals' mathematical "reasoning." I also assumed that undergraduates' reasoning about integration of complex functions could be "multifaceted." In this subsection I elaborate on my chosen meanings for these and related terms within the context of this study. These meanings are either based upon constructs established by prior mathematics education researchers, or are derived from aspects of my chosen theoretical framework.

Recall from earlier in this chapter that the NCTM (2009) characterized reasoning as "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). Hence, this definition underscores the dynamic and temporal nature of reasoning as a process rather than a product. The NCTM definition also incorporates what Krummheuer (1995) named the "core" components of the Toulmin (2003) model for argumentation, namely the claim ("conclusions"), data ("stated assumptions"), and warrant ("evidence"). Given that researchers (Alcock & Weber, 2005; Inglis et al., 2007; Simpson, 2015; Troudt, 2015) have argued for the adoption of the full Toulmin model when analyzing individuals' argumentation in more advanced or formal mathematical contexts, I characterized argumentation according to all five Toulmin components. Hence, in this study, I defined *argumentation* to be the process of drawing conclusions based on data, warrants, backing, and modal qualifiers. Given that my interviews were paired, each participant's responses were heavily influenced by the other's, as well as probing from myself as the interviewer. Thus the process of argumentation during these interviews was often *collective* (Krummheuer, 1995) in the sense it emerged from the social interaction of multiple participants. Research question 1b therefore served as an inquiry into how individual students contributed socially to such collective argumentation.

As detailed in Chapter III, Tall's (2013) three-world framework posits that mathematical argumentation is supported differently within each world. For instance, in the *conceptual-embodied* world, truth is initially established in elementary geometry based on what is seen to be true by the learner visually. In contrast, within the *operational-symbolic* world, truth is established in arithmetic based on calculation. Finally, in the *axiomatic-formal* world, a statement is true either by assumption as an axiom, or because it can be proved formally from the axioms. Hence the three-world framework complements the Toulmin analysis of a mathematical argument by adding specificity with regard to the types of backing and warrants used. As such, I classified participants' Toulmin components as embodied, symbolic, formal, or various mixtures of these, as viewed through Tall's three-world lens. Therefore, in the context of this study I defined *reasoning* as mathematical argumentation within one or more of the three worlds. It is this additional world-oriented property that makes participants' reasoning "multifaceted" in the sense that I used previously.

It should also be noted that while Tall discusses many of his constructs in the context of *thinking*, my study focuses more on how individuals employ such thinking in an externally observable process of argumentation. Hence, I adopted Tall's work in the setting of reasoning as opposed to thinking. In this report, I identify participants' reasoning as *embodied*, *symbolic*, and *formal* to signify that they are operating within the conceptual-embodied, operational-symbolic, and axiomatic-formal worlds, respectively. When participants' reasoning incorporates multiple worlds, I hyphenated two or more of these labels. For instance, *embodied-symbolic* reasoning attends to aspects of both Tall's conceptual-embodied and operational-symbolic worlds, and *symbolic-formal* reasoning attends to the operational-symbolic and axiomatic-formal worlds.

At some points in this report, I also refer to several different symbolic interpretations of a complex number. For instance, a complex number can be expressed as $z = x + iy$, $z = re^{i\theta}$, in polar form as an ordered pair (r, θ), as a vector, or simply as the symbol z. In the educational literature regarding complex number arithmetic, there is not necessarily a consensus regarding what word is attached to such a symbolic characterization. For instance, Danenhower (2006) used the words "representation" and "form" interchangeably to denote each of these four ways of symbolically denoting a complex number. On the other hand, Panaoura et al. (2006) used the word "form" for these notations, and used the word "representation" as a more general classification to denote an inscription as either algebraic or geometric. For the purposes of clarity and consistency, I use *form* in this context to denote the symbolic manner in which a complex number is used, and thus I characterize $z = x + iy$ as the *Cartesian form*, $z = re^{i\theta}$ as the *exponential form,* $z = (r, \theta)$ as the *polar form*, and so on. Following Panaoura et al., I reserve the word *representation* to denote either an algebraic or geometric portrayal of a complex number.

Significance of the Research

Recently, Soto-Johnson et al. (2016) found that mathematicians drew upon a wealth of personal embodied experiences when discussing their conceptions of continuity of complex functions. Although their study pertained to the population of mathematicians, Soto-Johnson et al. hypothesized that meaningfully connecting experientially-based intuition and formal mathematics could also benefit students' reasoning in courses such as complex variables. In part, my research served to reveal how undergraduates might reconcile their met-befores with the formal idiosyncrasies present in integration theorems. One respect in which participants instantiated such reconciliations was in how they attended to the *thinking real, doing complex* phenomenon. For instance, the student pairs explicitly referenced situations in which they were purposeful about avoiding inappropriately applying attributes of real-valued functions to the structure of the complex numbers. On the other hand, they invoked productive geometric properties of vectors from multivariable calculus to enact vector addition, visualize tangent vectors, and perform other related operations in response to the tasks.

As I discuss in Chapter V, the professor of the participants' complex variables course may have contributed to the students' attention to *thinking real, doing complex* via his explicit statements referencing definitions and intuition from notions such as differentiation and integration of real functions when defining their complex analogs. However, his adoption of various acronyms in order to succinctly state multiple theorem premises might have inadvertently allowed students to not carefully attend to and separate out these individual hypotheses while evaluating integrals in practice. Hence, my inquiry into students' reasoning about integration illuminates several ways in which instructors might cultivate healthy connections between students' embodied intuition and rigorous, formal mathematics, as well as potential pitfalls to avoid in the pursuit of such endeavors.

Additionally, my study complements and extends the mathematics education literature regarding students' collective argumentation. Specifically, I illustrate proposed addendums to how collective argumentation is currently framed theoretically. One such addition is the careful consideration of students' *challenges* to both each other's and their own contributions in a collective argument. As I detail in Chapters IV and V, both types of challenges catalyzed students' corrections, modifications, or retractions of prior statements. Another feature of students' argumentation that shaped their collective reasoning process was students' *nonverbal qualifiers*, such as providing a look to either me or the other participant within a pair in order to seek validation of a particular assertion.

As such, I contend that these qualifiers, along with several other nonverbal features of communication that influenced the trajectory of the students' argumentation, suggest the need for a more multimodal framing of argumentation that transcends verbiage and inscriptions. Such attention to nonverbal aspects such as eye gaze and gesture is consistent with Nemirovsky and Ferrara's (2009) notion of a multimodal *utterance*. As I describe in Chapter V, I suggest that attending to these more covert aspects of communication could additionally shed light on K-12 students' backing, a Toulmin component that has largely been omitted from researchers' analysis involving this population of students (Krummheuer, 1995, 2007; Rasmussen, Stephan, & Allen, 2004; Stephan & Rasmussen, 2002. The importance of nonverbal and explicit verbal qualifiers in my results also corroborates previous researchers' (Alcock & Weber, 2005;

Inglis, Mejia-Ramos, & Simpson, 2007; Simpson, 2015; Troudt, 2015) contention that one should consider the full Toulmin (2003) model when analyzing undergraduate level mathematical arguments. I discuss these and other considerations in full in Chapter V.

Outline of Dissertation

In this first chapter, I motivated the rationale for and importance of my study. I also provided a brief overview of select literature that informed my research, and articulated the purpose and research questions pertinent to my work. In the next chapter, I supply a thorough review of the relevant mathematics education literature that informed my study. The third chapter begins with my researcher stance, designed to motivate my personal interest in this work and expose potential biases relevant to my experiences as a student and researcher. This third chapter also includes a detailed account of my theoretical orientation, Tall's (2013) three worlds, and its relationship to the Toulmin (2003) model of argumentation within the context of my research. A thorough description of my research methods including setting, participants, data collection and analysis procedures is also supplied in the third chapter of this document. In Chapter IV, I detail the results from the interviews conducted with both pairs of participants. Finally, in Chapter V, I situate these findings within the existing pertinent literature, proffer teaching implications and addendums for framing collective argumentation, and discuss potential avenues for future research.

CHAPTER II

LITERATURE REVIEW

Recall from the last chapter that the purpose of my research was to explore undergraduates' multifaceted reasoning about integration of complex functions. My guiding research questions were:

Q1 How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?

Q2 How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In this chapter, I synthesize the existing mathematics education literature pertinent to my research. In doing so, I discuss how this literature informed my current work, what was missing from prior research in related fields, and how my study complements and extends the existing literature. Because my study involved undergraduates' reasoning and argumentation about integration, I first review the literature involving students' reasoning about integration of real-valued functions. Next, I discuss the existing research involving the teaching and learning of complex variables and analysis, given that my work specifically focused on integration of complex functions and involved participants from a complex variables course. Finally, because I was interested in how students collectively communicate their reasoning through a mathematical argument, I discuss the literature involving Toulmin's (2003) model of argumentation, my chosen framework for data analysis.

Integration of Real-Valued Functions

While no mathematics education research exists in the domain of integration of complex functions, several researchers have investigated students' understanding of integration with respect to real-valued functions (Grundmeier, Hansen, & Sousa, 2006; Judson & Nishimori, 2005; Mahir, 2009; Orton, 1983; Palmiter, 1991; Rasslan & Tall, 2002). Note that the newest such study was published in 2009, so recent research has not considered such issues. I verified this by conducting a thorough search on online databases such as JSTOR, ERIC, Academic Search Premier, and PsycINFO. Nevertheless, many of these studies considered student participants from several different populations or groups. For instance, Judson and Nishimori (2005) compared calculus students' responses to definite integral tasks from the United States to those in Japan, and Orton (1983) solicited both high school and postsecondary participants in his study. Palmiter (1991) compared responses to definite integral tasks from calculus students who used a computer algebra system in their course to those from students in the same university who took a more traditional paper and pencil course. In this section, I review these studies and discuss how they inform my current work.

As alluded to above, nearly all of the research related to students' conceptions of integration of real-valued functions focuses on calculus students. For instance, Orton's (1983) study involved task-based clinical interviews with 110 students, aged 16-22 years from six different schools. Although the tasks represented topics from most of elementary calculus, the focus of Orton's paper was specifically the items that involved integration topics. Numerical scores, ranging from 0 to 5, were assigned to students' responses to each task in order to garner summary statistics as a representation of the 110 students' collective level of understanding. Although Orton described students' general tendencies in responding to certain tasks, he did not include any actual sample student responses.

One of Orton's (1983) primary findings was that students at both the secondary and postsecondary education levels exhibited great difficulty connecting integrals to the notion of limits. For instance, Orton scaffolded one task to help students construct a sequence of Riemann sum approximations to the area bounded by a given curve and the x-axis. A majority of students were able to recognize that their approximations were approaching a particular value yet they were unable to conclude that the limit of this sequence would yield the exact area of interest. Even though they were able to proficiently find formulas for the general term and calculate limits of explicitly provided sequences in previous tasks, students did not know to apply their knowledge about limits to the aforementioned Riemann sum task.

Interested in building upon Orton's (1983) work, Grundmeier et al. (2006) administered a written survey to 52 college students that had recently completed a calculus course covering integration theory and techniques. Grundmeier et al. investigated students' understanding of integration with respect to several criteria. For example, each student provided a formal, symbolic definition for the definite integral, as well as his or her own personal explanation of the definite integral in words. The researchers also examined students' ability to interpret and represent the graphical meaning of integration, and had students evaluate several specific definite integrals. Finally, Grundmeier et al. assessed students' ability to recognize real-world applications of integration, in the form of a true/false section, though they did not discuss any results from this portion of their survey. Ultimately, as Orton discovered, Grundmeier et al. found that students could procedurally integrate a specific function correctly, but had difficulty explaining what a definite integral is in general and how to formally define it with limits.

Unsurprisingly, the most common verbal definition of a definite integral provided by students included some mention of area under a curve, although some students confused some of the limiting aspects of this process. For instance, one sample student response was, "A definite integral is the area underneath a curve that is achieved through slicing areas and allowing $\Delta x \to \infty$" (Grundmeier et al., p.183). Five of the students made connections to antiderivatives, describing integration as reversing the process of differentiation. Others simply mentioned that a definite integral is a bounded quantity, but did not specify what the actual integral represents, and four students left the problem blank.

When asked to provide the symbolic definition of a definite integral, only one out of the 52 participants provided a complete and correct definition (Grundmeier et al., 2006). More troubling was that only 12 of the other 51 participants included some components of the correct definition, with responses such as "$\int_a^b f(x)dx = \sum f(x)\Delta x$" (p. 184). Other students merely gave an example of a definite integral of a specific function, and even more worryingly, 3 participants gave the definition of derivative instead. Finally, 9 of the 52 participants stated some interpretation of the Fundamental Theorem of Calculus, such as "$\int_a^b f(x)\, dx = F(a) - F(b)$" (p. 184).

On the computational portion of the survey, students were asked to find the definite integral of the sine function over two intervals, $[0, \pi]$ and $[0, 2\pi]$. Grundmeier et al. (2006) found that 20 of the 52 participants did not provide correct answers to either task. This was largely due to students evaluating a trigonometric function incorrectly, such as giving the wrong value for $\cos(\pi)$, or finding the wrong antiderivative. Other students broke up the second integral over $[0, 2\pi]$ into two pieces, but failed to recognize that one of these pieces contributes area negatively since the function lies below the x-axis. This trouble with identifying some area contributions in the definite integral as negative was also prevalent amongst Mahir's (2009) participants, as discussed below.

Mahir (2009) examined 62 university calculus students' procedural and conceptual knowledge related to integration of real-valued functions. According to Mahir, procedural knowledge corresponds to the use of rules, algorithms, or procedures to solve problems, while conceptual knowledge requires one to make connections between other pieces of existing knowledge and be cognizant of this connection. Students were assessed via a five-item questionnaire, which Mahir described as having two procedural questions, two questions that could be solved using either procedural or conceptual knowledge, and one purely conceptual question. In the context of this questionnaire, Mahir characterized procedural questions as ones that simply asked students to evaluate a definite integral using standard integration techniques such as trigonometric substitution. On the other hand, conceptual questions allowed students to relate the definite integral to area via an "integral-area relation" (p. 204). The purely conceptual question had students relate the graph of a derivative function $f'(x)$ to specific values of $f(x)$ using both integral-area relations and the Fundamental Theorem of Calculus.

Perhaps unsurprisingly, the participants excelled at the purely procedural questions, as 92% and 74% of the students solved these two questions correctly, respectively. However, in the two questions that could be solved using either procedural or conceptual knowledge, the students who used a procedural method tended to make computational errors, while the students who recognized a connection to area typically arrived at the correct answer and in fewer steps. Moreover, 40% of the participants did not even respond to the purely conceptual question, and the students who did respond had trouble recognizing when areas should be treated as negative contributions to the definite integral, as was the case in Grundmeier et al.'s (2006) study.

Similar to Grundmeier et al.'s (2006) results, Rasslan and Tall (2002) found that only 7 out of 41 high school participants were able to correctly state the definition of definite integral. These authors investigated English high school calculus students' *concept images* and *concept definitions* (Tall & Vinner, 1981) of the definite integral, and explored how students instantiated various concept images when evaluating specific definite integrals. As exemplified previously, students' responses to the question asking for the definition of the definite integral mostly mirrored those of Grundmeier et al.'s (2006) participants. For instance, some students substituted specific functions and evaluated their definite integrals, and others stated the Fundamental Theorem of Calculus as a procedure for calculation. Unlike Grundmeier et al.'s study, however, more than half of Rasslan and Tall's participants did not even attempt to state the definition of a definite integral. One should note that in the U.K.., the formal definition given in the participants' textbook only mentioned the definite integral as the precise area under the graph of a function, between two particular x-values. In particular, there was no mention of a limit of Riemann sums, as found in U.S. textbooks. Accordingly, no students provided an answer that alluded to this formulation of the definite integral.

Another difference between Rasslan and Tall's (2002) study and Grundmeier et al.'s (2006) was that Rasslan and Tall asked students to evaluate definite integrals with more difficult functions. For instance, one function contained the absolute value of an expression and another function was defined piece-wise. Yet still, participants made many of the same errors as Grundmeier et al.'s participants when evaluating definite integrals. Such errors included taking a derivative instead of an antiderivative, leaving the problem completely blank, or neglecting to account for the negativity of certain area contributions. As I illustrate below, such difficulties are not limited to American learners.

Judson and Nishimori (2005) administered a two-part exam, and conducted interviews with, 18 American and 26 Japanese high-school students in order to determine whether there were differences in these students' conceptual knowledge of calculus. Moreover, the authors wished to investigate any differences between the two populations' abilities to utilize algebra to solve traditional calculus problems. Judson and Nishimori deliberately picked participants in each country that represented the best high school students each country had to offer; for instance, the American students were selected from above-average high schools and from AP Calculus BC courses. AP Calculus BC is a nationally offered yearlong college-level course that provides a thorough treatment of limits, differentiation, integration, and series. In fact, passing the associated AP exam in this subject grants students college credit for a full year of calculus at most American institutions.

The authors found that American students tended to rely on calculators for computations when possible, whereas Japanese students were largely unfamiliar with using them. All students had difficulty relating definite integrals to Riemann sums, and part of the difficulty was due to students' comfort level with summation notation. This was especially the case amongst the American participants. Some of the Japanese students, similar to Mahir's (2009) participants, had difficulty finding the definite integrals of certain functions given their graph. This was primarily because they attempted to find formulas for functions depicted in the graph, rather than calculate areas of familiar shapes. Overall, Judson and Nishimori (2005) found that the American and Japanese students displayed similar levels of conceptual calculus knowledge, but the Japanese students demonstrated a stronger grasp of algebraic skills than the Americans.

In another study comparing two groups of calculus students, Palmiter (1991) found that university students that used the computer algebra system MACSYMA during their course outperformed, on a test of computational and conceptual calculus knowledge, students whose course did not use such a system. Palmiter concurred with several previous studies (Hawker, 1986; Heid, 1988; Judson, 1988) that the use of such computer algebra systems in class allows instructors to deemphasize some of the hand-calculations of limits, derivatives, and integrals, therefore freeing up class time to explore more conceptual terrain. However, while these previous studies all focused on business calculus students, Palmiter's study was concerned with students in engineering calculus.

Calculus students from Palmiter's (1991) study were randomly assigned to either a control class learning traditional paper-and-pencil calculus concepts throughout the entire 10-week quarter, or to an experimental class that used MACSYMA. Any students assigned to the experimental group, however, covered the course material in 5 weeks and were not presented the traditional techniques of integration such as integration by parts. Instead, these students had access to the MACSYMA program for homework and exams, but did not have computer access in the lecture hall during class. At the end of the 5-week period for the experimental group, and at the end of the 10-week quarter for the control group, all students took a two-part written exam testing computational calculus knowledge in one part and conceptual knowledge in the other part. Students in the experimental group were allowed to use MACSYMA during the computational portion of the exam. Following the 5-week experimental class, which used the computer algebra system, and the exam, these students learned traditional paper-and-pencil integration techniques in the final 5 weeks of the quarter. The authors believed this would ensure students would be prepared for any subsequent courses in which they did not necessarily have access to MACSYMA.

Because the analysis presented in Palmiter's (1991) article is only quantitative in nature, it is unclear whether either group of students shared the difficulty with connecting limiting procedures to the definite integral as in Orton's (1983) study. Rather, Palmiter concluded that the experimental class of students using MACSYMA performed significantly higher on both portions of the exam compared to the students from the control class. Palmiter admitted that the students in the experimental group might have outperformed the control group partly due to the fact that the experimental and control classes were taught by different instructors. Aside from having significantly higher test scores, the experimental class reported more positive affective traits than the traditional class on a post-course evaluation form given at the end of the 10-week quarter to both groups. Specifically, 85% of the students using the computer algebra system, compared with 68% of the traditional group, reported that they were confident in continuing the calculus sequence. Moreover, 95% of the students in the experimental group reported that they would sign up for another course using a computer algebra system.

Curiously, although students in the experimental course identified “concepts of calculus” as the most important idea they learned, a very close second was “techniques of integration”. Recall that the experimental group completed the techniques of integration portion of the course after the 5-week experimental portion of the course, and did not use MACSYMA during this time. This means that the students in the experimental group found the traditional computational integration techniques taught after their exam to be nearly as important as the conceptual ideas taught during the MACSYMA portion of the class. This suggests that, at least affectively, students exposed to computer algebra systems in a calculus course might still perceive traditional computations as equally important as concepts discussed in conjunction with a program such as MACSYMA.

Looking at the aforementioned studies holistically, it is evident that calculus students from many different populations struggled with the same ideas related to the definite integral. For instance, when asked to provide a definition of a definite integral, participants tended to instead give examples, recite the definition of derivative, or some version of the Fundamental Theorem of Calculus (Grundmeier et al., 2006; Rasslan & Tall, 2002). Many students also struggled to connect the idea of definite integral to a limiting process, often mixing up which objects tended to zero or infinity (Grundmeier et al., 2006; Orton, 1983). Participants from several of these studies also did not recognize when area should be counted as a negative contribution to a definite integral (Grundmeier et al., 2006; Mahir, 2009; Rasslan & Tall, 2002). Finally, when faced with a problem allowing them to either employ basic area properties from the graph of a given function or attempt to translate graphs into function formulas and evaluate tedious antiderivatives, students tended to pursue the latter, often unsuccessfully (Judson & Nishimori, 2005; Mahir, 2009).

Many of the existing studies reviewed in this section were also limited by their data collection methods. For instance, although the idea of determining students' ability to recognize applications of integration was pertinent, Grundmeier et al. (2006) only assessed this via a true/false section of a written survey, and did not even include this portion of the survey in their analysis. In fact, all of these studies make use of written surveys as the primary source of data in assessing students' knowledge. Accordingly, even though some of the studies reported results from rather large sample sizes, there are a lot of missing data. For example, as many as 40% of the participants left certain problems completely blank (Mahir, 2009). Or in the case of Rasslan and Tall's (2002) study, some students reported a correct answer to a definite integral but did not show any work. This left the authors unsure about whether the participants knew how to properly calculate definite integrals, or just used the calculator they had access to during the survey.

With the aforementioned student difficulties in mind, I was curious whether my study would illuminate similar or generalized versions of these problems with respect to integration of complex functions. For instance, if students have difficulty with limiting processes that are embedded in the definition of definite integral for a real-valued function, it is entirely plausible that they might still struggle with this idea in a more generalized and complicated setting. Moreover, when faced with the decision to either pursue a more concise and conceptual geometric solution or a tedious algebraic and procedural solution, students in a complex variables course might still be tempted to instantiate the latter. I return to these considerations in Chapter V. My research additionally had the advantage of asking participants in real time about their reasoning and having them clarify statements, so that I could ascertain their complete reasoning process. Because my study investigated participants' reasoning and argumentation with respect to integration of complex functions, I next review the mathematics education literature pertinent to the teaching and learning of complex numbers and variables.

The Teaching and Learning of Complex Numbers and Variables

Given that research in undergraduate mathematics education is a relatively young field, it is not surprising that the educational literature on the teaching and learning of complex variables and analysis is sparse. The few studies that do exist in this domain have primarily attended to complex arithmetic and forms of a complex number (Danenhower, 2006; Karakok, Soto-Johnson, & Anderson-Dyben, 2014; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis, 2006; Soto-Johnson & Troup, 2014). I begin this section with a review of these studies. Next, I discuss recent studies investigating more advanced topics such as continuity (Soto-Johnson, Hancock, & Oehrtman, in 2016) and differentiation (Troup, 2015). However, there is no existing literature regarding students' integration of complex functions, despite the fact that it is a central topic of any complex analysis course for undergraduates, and despite its applicability to flux and potential for physics majors (CUPM, 2015). In particular, it is unclear how undergraduate students reason and argue about the notion of integration of complex functions. Therefore, my current work stands to contribute to the mathematics education literature about students' reasoning in the field of complex analysis.

Complex Arithmetic and Forms of a Complex Number

Panaoura et al. (2006) investigated Greek high school students' ability to solve complex arithmetic tasks using either a primarily algebraic or primarily geometric approach. The authors indicated that complex numbers are particularly appropriate in a study about multiple representations of a mathematical concept since complex numbers inherently possess both algebraic and geometric attributes that are vital to the understanding of the subject as a whole. One of their tasks sought to determine whether students would recognize the symbolic equation $|z - 1 + i| = \sqrt{2}$ as a semicircle. Another task asked students to produce an equation that defined a particular semicircle. Panaoura et al. found that students who used a geometric approach were more successful at correctly completing tasks than those who used an algebraic approach. However, students who exhibited primarily geometric representations occasionally ran into difficulties with compartmentalization. These students struggled viewing the same complex numbers with the two different representations, yet they experienced minimal difficulty working with several different complex numbers via the same geometric representation. This indicates students' inflexibility towards using more than one type of representation in complex arithmetic tasks.

Seeking to identify potential student difficulties in a preliminary complex variables course in British Columbia, Danenhower (2006) investigated undergraduate students' willingness and ability to switch between four forms used for expressing complex numbers. Specifically, Danenhower investigated the prevalence of students' use of complex numbers in algebraic/Cartesian form $z = x + iy$, vector form $z = (x, y)$, exponential form $z = re^{i\theta}$, and symbolic form (recognizing a complex number simply by z). Note that Danenhower used the words "representation" and "form" interchangeably to denote each of these four ways of symbolically denoting a complex number, but for clarity and consistency I discuss these as *forms* in accordance with my definition from Chapter I.

Danenhower characterized students' level of understanding with respect to each form by adopting a combination of the well-established Action, Process, Object, Schema (APOS) (Breidenbach, Dubinsky, Hawks, & Nichols 1992) and reification (Sfard, 1991) frameworks. These characterizations were based, in large part, upon students' ability to: use a single form, represent an expression in different forms, translate between forms, and judge when to shift from one form to another. One pertinent result from this study was that the students held an object understanding of the algebraic and vector forms, but only a process understanding of the exponential form. In particular, Danenhower found that students tended not to employ the exponential form in multiplication, but rather persisted to use another form such as Cartesian. This is noteworthy because the exponential form lends itself naturally to a geometric interpretation of multiplication of complex numbers, as $r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$. As such, this result suggested that students had difficulty judging when to shift to the exponential form.

In another study regarding complex number arithmetic, Nemirovsky et al. (2012) investigated American preservice secondary teachers' geometric interpretations of the addition and multiplication of complex numbers. The instructor in this study provided participants with tape, string, and stick-on dots, and challenged them to use these items on a tiled floor in order to invent ways to perform complex addition and multiplication tasks. Students in the study expanded their own "realm of possibilities," (p. 291), or the collection of all possible outcomes associated with some perceptuo-motor activity (combining perception and movement), for complex number arithmetic as they utilized their environment to enact specific operations, such as multiplication by i. Accordingly, their gestures reflected an increasingly generalized conception of complex arithmetic, liberating students from a reliance on algebraic manipulation of specific examples. Moreover, these gestures allowed participants to recognize errors in their algebraic work.

Studying a different population than the aforementioned research, Karakok et al. (2014) explored three in-service high school teachers' conceptions of different forms of complex numbers as well as their ability to transition between different representations (e.g. algebraic and geometric) of these forms. These teachers, after completing three four-hour sessions of professional development on complex numbers, demonstrated an operational conception, but not a structural conception (Sfard, 1991), of the exponential form of complex numbers. According to Sfard, a conception is *operational* if it focuses on "processes, algorithms, and actions" (p.3) and *structural* if it treats a mathematical idea as an abstract object that can be manipulated in its own right. Karakok et al. also found that two of the three participants evidenced cognitive conflict when conceiving of complex numbers as vectors, particularly when treating one complex number as an operator and the other as a vector in the product of complex numbers. However, the high school teachers were more comfortable with the Cartesian form and demonstrated an ability to proficiently switch between different representations, illustrating a process/object dual conception of this form (Sfard, 1991).

In another study regarding operations on complex numbers, Soto-Johnson and Troup (2014) studied undergraduate students' diagrammatic reasoning, inscriptions, and gestures during task-based interviews involving equations with complex expressions. These students had recently completed a course in complex variables that incorporated *GeoGebra* (dynamic geometry software) labs designed to elicit a geometric understanding of arithmetic operations and conjugation of complex numbers. Findings indicated that participants tended to initially reason with algebraic inscriptions, but later proficiently switched to geometric reasoning. Moreover, these students often produced inscriptions on the board when their verbalized statements could not suffice in articulating their geometric reasoning. Finally, the nature of their gestures transformed from primarily iconic when reasoning about their geometric inscriptions, to primarily deictic when evoking previously developed reasoning. The aforementioned studies from this subsection illuminate important qualities of individuals' arithmetic and algebraic reasoning about complex numbers, but they reveal less about the axiomatic formal world (Tall, 2013). In the next subsection I discuss two studies which delve into the latter.

Continuity and Differentiation

Although much of the pertinent research in complex variables has dealt with various individuals' algebraic and geometric reasoning about the arithmetic of complex numbers, recent studies have explored more advanced topics. For instance, Soto-Johnson et al. (2016) explored the nature of, and interplay between, mathematicians' informal and formal mathematical reasoning, or *Ideational Mathematics (IM)* and *Conceptual Mathematics (CM)* (Schiralli & Sinclair, 2003), respectively, about the continuity of complex valued functions. These mathematicians evoked IM in the form of numerous metaphors capturing ideas of control and preservation of closeness, but sometimes their IM was incomplete in capturing the rigor of the precise epsilon-delta definition of continuity. In particular, these mathematicians tended to employ *domain-first* IM with respect to continuity, in the sense that their IM reasoning often began by considering objects in the domain of a function, followed by determining what happened to those objects in the codomain. This type of reasoning, such as preservation of closeness descriptions of continuity, contrasts with the formal epsilon-delta definition of continuity in the sense that the latter necessitates that one starts with an acceptable tolerance ϵ controlling closeness in the codomain before anything in the domain is considered.

In fact, when explicitly asked by the researchers, participants often did not adequately find ways to reconcile their *domain-first* IM with their formal CM statements. Hence this work suggests that while IM metaphors and descriptions can serve as helpful pedagogical tools, instructors need to be careful to be explicit about when this IM fails to fully capture the intended CM definition or concept. Although IM and CM were not the focus of my current study, my participants did exemplify IM and CM related to the continuity of complex functions, as it pertained to integration of given functions during their interviews. In doing so, these students might have alluded to some similar ideas indicating *domain-first* reasoning in the sense mentioned above, suggesting that this issue might not be unique to mathematicians like those in Soto-Johnson et al.'s (2016) study.

In another study addressing more advanced topics in complex analysis, Troup (2015) investigated the nature of undergraduates' reasoning about the derivative of a complex function. Specifically, he studied how participants expressed differentiation ideas via gesture, speech, inscriptions, and interaction with the dynamic geometry software *Geometer's Sketchpad* (*GSP*). Through their use of *GSP*, participants noticed and resolved discrepancies between reasoning methods. In particular, participants initially tended to instantiate Danenhower's (2000) "thinking real, doing complex" (p. 101) theme when reasoning about the derivative of a complex function. That is, they attempted to apply the familiar conception of derivative as the slope of a tangent line to the context of complex derivatives. When asked to describe the derivative of a complex function geometrically, participants initially tried to revert back to their understanding of the derivative of a real-valued function as slope or rate of change, but quickly found that they did not "know what slope means in complex world" (p. 178) once they started using *GSP* in their investigations.

One way in which Troup's (2015) participants were able to correctly reason about the geometric behavior of complex derivatives was to focus on the special case of a linear complex function. In doing so, participants discovered that linear functions always rotate and dilate circles by the same amount, regardless of location. Using *GSP*, they investigated more complicated functions such as $f(z) = z^2$ and $f(z) = e^z$ to explore the rotation and dilation transformations inherent in differentiation. Studying the images of circles of varying center and radius under these two functions helped participants conclude that the modulus of the derivative represents a local dilation factor. Specifically, if one considers a small circle about a particular point z_0, then applying the function f to the radius of this circle dilates this radius by a factor of $|f'(z_0)|$. Participants also eventually conceived of the argument of the derivative as a rotation angle, so that applying f to the radius of a small circle about z_0 resulted in a rotation by $Arg(f'(z_0))$. Together these properties comprise what Needham (1997) refers to as an *amplitwist*, and provide a geometric interpretation of the derivative of a complex function. Moreover, this interpretation can reveal the derivative $f'(z_0)$ as an approximation of the image $f(z_0)$.

Although mathematics education research regarding the teaching and learning of complex variables and analysis has been scarce, several aspects of the studies mentioned in this section informed my current work. Earlier research (Danenhower, 2006; Panaoura et al., 2006) suggested that students struggled with when and how to use specific forms of complex numbers, such as the polar form. These studies also stressed the importance of representational fluency with respect to working with complex numbers. More recently, in-service secondary teachers (Karakok et al., 2014) tended to favor the Cartesian form, while undergraduates with DGE experience (Troup, 2015) were more proficient with the polar form, knew when to employ it, and could connect their algebraic and geometric reasoning. Although mathematicians presumably do not struggle with transitioning between various forms and representations, Soto-Johnson, Hancock, and Oehrtman (2016) found that mathematicians' IM incorporated domain-first reasoning that was difficult to connect rigorously to formal CM statements and definitions of continuity. Though participants' choice and use of various forms of complex numbers were not central foci of my work, my study peripherally considered these aspects as part of undergraduates' argumentation about integration of complex functions. For instance, a student might choose to invoke the exponential form of a complex number when parametrizing a circular path, but might use the Cartesian form when verifying that the Cauchy-Riemann equations are satisfied for a particular function.

Another important aspect of the existing literature involves Danenhower's (2000) theme of students falling victim to "thinking real, doing complex" (p. 101). Troup (2015) also found that students succumb to this type of behavior, suggesting that it could indeed be a characteristic students instantiate when working with complex numbers and complex analysis in general. As discussed in Chapter V, my study illuminates three distinct manners in which participants attended to this phenomenon, two of which were productive to their reasoning. In order to analyze students' mathematical reasoning via an appropriate grain size, I adopted Toulmin's (2003) model of argumentation, which I discuss in the next section, along with the mathematics education literature relevant to this model.

Toulmin's Model of Argumentation

In this section, I review mathematics education literature relevant to Toulmin's (2003) model of argumentation. First, I discuss the components of the model itself, along with motivation for using such a model for analysis purposes. Next, I outline mathematics education researchers' various adaptations of this model in recent research studying mathematical argumentation at the K-12 and undergraduate levels. Finally, I review literature setting forth general model considerations pertinent to adopting Toulmin's Framework.

Recall from Chapter I that the NCTM (2009) characterized reasoning as "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). Hence, I argued that reasoning is intimately connected to the process of mathematical argumentation. One established scheme for analyzing any argument, mathematical or otherwise, was proposed by Toulmin (1958). Investigating the layout of a valid argument, according to Toulmin, requires the sophistication and attention to detail that is present in everyday legal utterances, but not necessarily the structure of formal logic. In particular, this means that mathematical arguments other than formal proof can be analyzed using Toulmin's model. Because it is tempting to associate argumentation in the field of mathematics with the notion of proof,

> The analysis of argumentation in a classroom, then, could be misleadingly understood as a treatise on proof. Therefore, one should notice that both the concept of an argument and that of argumentation need not be exclusively connected with formal logic as we know it from such proofs or as the subject matter of logic (Krummheuer, 1995, p. 235).

However, while dissecting arguments with this fine-grained lens, Toulmin stressed it is important not to lose sight of the more macro-level context in which the argument takes place. He likened an argument to a living organism with both a gross anatomical structure and a finer physiological one. As such, analyzing the finer physiological processes is most interesting and effective when this analysis is mindful of the larger organs that these finer processes take place within. Analogously, micro-arguments

> need to be looked at from time to time with one eye on the macro-arguments in which they figure; since the precise manner in which we phrase them and set them out […] may be affected by the role they have to play in the larger context (Toulmin, 2003, p. 87).

According to Toulmin (2003), any argument is based upon the arguer attempting to convince his or her audience of some *claim* (C), or asserted conclusion. This claim is necessarily grounded in foundational evidence, or *data* (D), which serves as the information on which the claim is based. However, producing these data alone often cannot convince one's audience that the conclusion holds. In this case, the arguer can supply a *warrant* (W) that justifies the link between the given data and the purported claim. While Toulmin mostly treated warrants as hypothetical, bridge-like statements, or general laws within specific disciplines, mathematics educators have broadened the scope of what can be classified as a warrant in mathematical discourse. I will discuss researchers' various modifications and adaptations of Toulmin's work in the next subsection of this literature review.

Because warrants can take on different forms and engender different levels of certainty regarding the implication of a claim based on given data, Toulmin (2003) proposed that a *modal qualifier* (Q) is often necessary to explicitly reference "the degree of force which our data confer on our claim in virtue of our warrant"(p. 93). For instance, the arguer may not be entirely confident that the claim follows necessarily from the data provided. Depending on the warrant provided, there might also be circumstances in which the intended claim does not hold; in this case, conditions of *rebuttal* (R) are needed to indicate when the "general authority of the warrant would have to be set aside" (p. 94).

According to Toulmin (2003), another potential issue surrounding an arguer's warrants is that the audience might challenge the general legitimacy of the warrant provided, or may call into question whether this warrant is actually applicable in the present context. Therefore, the warrants provided in an argument might need additional *backing* (B), or assurances confirming the warrant's authority and/or authenticity. Toulmin emphasized that the nature of the backing needed for one's warrants varies greatly depending on the field of argument, again underscoring the importance of the macro-level context in which an argument takes place. I will return to the notion of backing in the General Model Considerations subsection to highlight the various ways in which mathematics education researchers have characterized the backing of mathematical warrants. Collectively, the aforementioned components D, C, W, Q, R, and B constitute Toulmin's so-called "argument pattern," which is illustrated below in Figure 1.

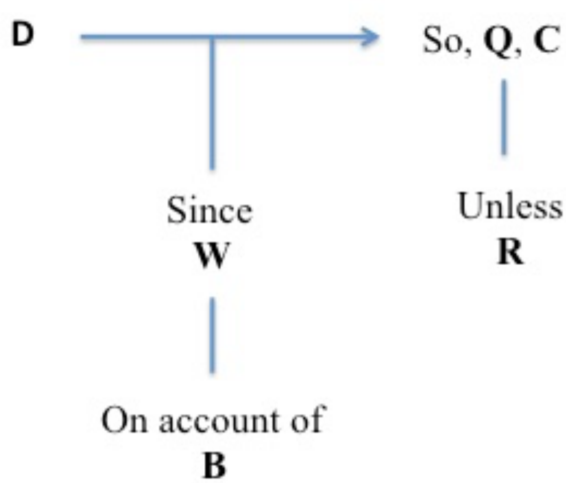


Figure 1. Toulmin's argument pattern.

With this framework in mind, I will next outline the various settings in which Toulmin's (2003) argumentation pattern has been applied to mathematics education research, and discuss how my current research fits within this context.

Toulmin's Model in Mathematics Education Research

Since the mid 1990s, Toulmin's framework has gained much popularity amongst mathematics education researchers as a way of analyzing mathematical arguments in various settings. The mathematical arguments of participants from numerous educational backgrounds have been analyzed in several different contexts. For instance, while some researchers have chosen to analyze students' or instructors' mathematical arguments during an actual class session (Krummheuer, 1995, 1997; Rasmussen, Stephan, & Allen, 2004; Stephan & Rasmussen, 2002; Wawro, 2015), others have used Toulmin's framework to discuss how students examine the validity of purported written mathematical proofs (Alcock & Weber, 2005). The literature also includes studies investigating students' argumentation in responses to written examinations (Evens & Houssart, 2004) or task-based interviews (Hollebrands, Conner, & Smith, 2010). Below I discuss the various educational settings in which Toulmin's pattern has been utilized in mathematics education research; this will allow me to illustrate how my current research complements the existing literature.

K-12 applications. Because the Toulmin model can be adapted to arguments using any level of mathematical content, researchers can examine how students argue mathematically at various points in their educational career. At the elementary school level, Krummheuer (1995) imparted Toulmin's model to investigate students' *collective argumentation*, which he characterized as "a social phenomenon when cooperating individuals tried to adjust their intentions and interpretations by verbally presenting the rationale of their actions"(p. 229), about basic arithmetic operations and properties. However, Krummheuer's study, like many others that followed, did not incorporate Toulmin's full model, as it ignores the modal qualifier (Q) and rebuttal (R) components. In fact, Krummheuer's analysis focused primarily on just the data (D), warrant (W), and conclusion (C), a subset of the model, which he referred to as the "core" of an argumentation, which serves as the "minimal form of an argumentation" (p.243). I will discuss the implications of adopting this type of a reduced model in certain settings in the General Model Considerations subsection below.

Evens and Houssart (2004) also investigated young students' argumentation without referring to qualifiers or rebuttals, citing Krummheuer's (1995) work and other similar studies as justification for not needing to adopt Toulmin's (2003) full model. Their research considered 441 11-year-olds' written responses to a single question on an assessment. This question asked students to evaluate the correctness of a hypothetical child's claim that the sequence of numbers {1, 4, 7, 10, 13, 16, …} will never contain a number that is a multiple of 3. The students were prompted to circle "yes" or "no" to indicate whether or not the hypothetical student was correct, and also provide a written explanation for their choice. Therefore, the written test question contained the data (D) as well as a potential claim (C) about the data, but the students in the study had to supply the warrants (W) and backing (B) in order to argue for or against the hypothetical child's claim.

Some participants' justifications in the aforementioned written task contained no warrants whatsoever, and merely restated the data as justification of the claim. Others contained explicit warrants that only constituted examples but not general statements. For instance, some responses indicated that there was a 7 in the given sequence, and provided no backing for their warrants. Still others provided explicit warrants similar to the example above, but also backed these warrants with an additional statement such as "7 is not a multiple of 3" (p. 276). There were students who provided a complete justification using two warrants, one containing reasoning about the starting point of the sequence, and another stating that each number in the sequence would be one more than some multiple of 3. However, the majority of students who provided any legitimate justification omitted one of these two warrants. Hence Evens and Houssart (2004) concluded that teachers should build on the answers that children provide, rather than modeling complete solutions to the types of problems in their study.

Krummheuer (2007) extended his prior work by elucidating specific social features inherent in the elementary mathematics classroom that thwarted the presence of mathematically based warrants and/or backing in classroom argumentation. For instance, the ways in which one teacher communicated with students as they arithmetically decomposed the number 13 resulted in a complete lack of content-related warrants on the students' part. Instead, the students' warrants supporting their claims were solely constituted by whether or not the teacher intervened after a student offered a potential solution. If the teacher did not intervene, the students used this as evidence that a student's claim followed from the data provided. In the other classroom Krummheuer analyzed, even though students were able to provide warrants to support their claims, they never explicitly backed these warrants. Thus only the aforementioned "core" version of Toulmin's model could be used to analyze students' argumentation. According to Krummheuer, this lack of backing happens often in the primary mathematics classroom. Naturally, then, the question arises as to whether the aforementioned issues involving backing and warrants are also prevalent in higher grade levels.

Fortunately, researchers have also investigated middle school and high school students' mathematical argumentation, though not necessarily during similar conditions. For instance, Weber, Maher, Powell, and Stohl Lee (2008) investigated eight middle school students' arguments about whether various hypothetical companies produced fair six-sided dice. During a summer session and as part of a larger longitudinal study, pairs of participants ran computer simulations of several companies' dice being rolled, and used the resulting output as data for their claim that either the dice were fair or unfair. The students constructed a poster with any data and written arguments they could think of in support of their claim. After a viewing period of all posters, the eight students came together to debate their conclusions. During this debate, the students initially based their conclusions on whether or not the two posters for a given company reflected the same conclusion about that company's dice, rather than the data provided by the simulation. Eventually, some students called this type of reasoning into question, and this prompted additional debate about students' warrants and backing based on ideas such as the sample size from the simulation.

Ultimately, Weber et al. (2008) hypothesized that:

> learning environments where student contributions are encouraged and not judged, sense making is encouraged and students are arbiters of what makes sense […] will invite students to attend to and challenge the arguments of others, which can make the warrants in students' discussion the objects of debate (p. 260).

Thus, the social context in which the argumentation takes place, including any classroom norms established, can influence the nature of students' justification, especially with respect to backing warrants. I return to this point in the theoretical perspective section in the next chapter.

Undergraduate applications. Given the difficulty some of the K-12 participants from the aforementioned studies displayed in constructing valid warrants and backing, perhaps a Toulmin analysis might be aided by the presence of more advanced mathematical content. At the college level, Hollebrands, Conner, and Smith (2010) explored eight students' mathematical arguments as they solved problems involving relationships about quadrilaterals in hyperbolic geometry. During each task-based interview with an individual participant, the student was given access to the dynamic geometry software *NonEuclid*. The authors found that when students provided explicit warrants, they did not use the technology provided. However, when students did not explicitly provide warrants for their claims, they did utilize the program. According to Hollebrands et al., one potential reason for this theme is that the students saw the technology as a warrant in itself.

Unlike a majority of the participants from the aforementioned K-12 studies, these geometry students additionally expressed modal qualifiers (Q) in their responses when they were uncertain about a claim. In these instances, participants tended to turn to the technology as a means of determining the correctness of a stated claim, and either accepted the claim as true or abandoned the claim, based on technological outcomes. The presentation of results in Hollebrands et al.'s (2010) study is also particularly clear, in that diagrams illustrating a participant's argument indicate when a sub-argument was prompted by the interviewer, and when a warrant was implicit as opposed to explicit.

Undertaking an emergent perspective (Cobb & Yackel, 1996) lens, Stephan and Rasmussen (2002) also studied undergraduate-level mathematical argumentation using Toulmin's (2003) model. These authors documented the emergence of several mathematical practices in a differential equations course. One key aspect of this research was that the authors argued that students' argumentation showed evidence of mathematical ideas becoming taken-as-shared. For instance, the explicit mention of certain warrants or backing in students' arguments sometimes disappeared because the underlying mathematical idea eventually stood as self-evident within the classroom culture. In other instances, components of students' argumentation shifted their role or function, yet were unchallenged by other students. For instance, a previous claim became data for a later argument during a future class session.

Building upon Stephan and Rasmussen's (2002) work, Rasmussen, Stephan, and Allen (2004) chose to re-analyze the data from the 2002 study, with a new goal of investigating how gesture and argumentation can work together to establish taken-as-shared mathematical ideas. Rasmussen et al. found that certain gesture/argument dyads not only appeared as certain mathematical practices were formed, but also reappeared at later class sessions when an older practice had to be renegotiated and different data were used in the argumentation They also found that certain dyads that began as data in one argument were used as warrants when negotiating taken-as-shared ideas such as an exact solution representing instantaneous rates of change. Given the relationship between gesture and argumentation established by Rasmussen et al., the results of my study shed light on what gesture/argumentation dyads exist in the subject area of complex analysis, as well as how they support one another. I discuss this point in detail in Chapter V.

In an in-depth case study of an undergraduate linear algebra student, Wawro (2015) applied a Toulmin analysis to investigate the ways in which this student reasoned about solutions to $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{x} = \mathbf{0}.$ Through videotaped observations during whole-class discussion and small group work, as well as individual interviews, she documented this student's mathematical argumentation regarding various equivalences in the Invertible Matrix Theorem. This consisted of *microgenetic* (Saxe, 2002) analysis of the structure of individual arguments, as well as *ontogenetic* (Saxe, 2002) analysis of a larger progression of argumentation over time. Wawro found that the student was primarily successful in his argumentation because he was "flexible in his use of symbolic representations, proficient in navigating the various interpretations of matrix equations, and explicit in referencing concept definitions within his justifications" (p. 336). Hence this study suggests an important link between representational fluency and effective mathematical argumentation. As I detail in the next chapter, this relationship between representations and argumentation is an important component of my theoretical perspective.

Research has also considered undergraduates' understanding of argumentation in formal proofs. For instance, Alcock and Weber (2005) conducted individual tasked-based interviews with thirteen undergraduate students in real analysis, and asked each student to identify a proof containing flawed argumentation as valid or invalid. The last line of this purported proof represented a true statement, but the statement did not legitimately follow from the previous lines in the proof. In other words, although the data and claim of the argument were true, the warrant provided did not connect the data to the claim in a valid manner. Specifically, the warrant implied that all increasing sequences diverge. Alcock and Weber found that only six of the thirteen participants identified the argument as invalid, and only two of these students did so based on legitimate mathematical reasoning. The authors mentioned that one potential difficulty in assessing the validity of a formal proof is that written proofs rarely explicitly state all data and warrants. Instead, the reader must often infer these details, and an unassuming reader might focus on the correct claim in the last sentence, but not notice that the implicit warrant put forth is invalid.

Originally, only six of thirteen participants rejected the proof as invalid. But when the interviewer prompted the students to reflect upon their critique and directed their attention to the last two lines of the proof, ten of the thirteen students ultimately identified the false warrant. This finding indicates that "the ability to validate proofs may be in many students' zone of proximal development and that students' abilities in this regard might improve substantially with relatively little instruction" (p. 133). Given that my study incorporated paired task-based interviews, I anticipated that one student in each group might take on the role of the "more knowledgeable other" (Vygotsky, 1978) relative to the other student. As such, the ability to validate arguments, especially assessing the validity of each other's warrants and/or appropriate backing, could potentially lie within student pairs' zone of proximal development. I briefly return to related considerations about students' zone of proximal development in Chapter V.

In the above instantiations of Toulmin's framework, it appears that the elementary nature of the mathematics content from the K-12 studies made it difficult for researchers to capture students' argumentation with the complete Toulmin model. On the other hand, when researchers studied students' conceptions regarding formal proof validity, these students had difficulty parsing the implicit warrants of proofs and therefore misclassified flawed proofs as valid (Alcock & Weber, 2005). Hence, Toulmin's model might be especially well suited for analyzing undergraduates' argumentation in courses such as complex analysis or differential equations. In such courses, students are exposed to somewhat advanced theory that they can employ as warrants and backing, but are not often required to write formal proofs. Accordingly, my work serves to complement the existing literature about undergraduates' argumentation in these types of courses that transcend elementary topics but are not proof-intensive. It is also clear from an examination of the existing literature that there is no general consensus regarding which components of Toulmin's model should be, or can be, considered in analyzing individual or collective mathematical argumentation. As such, it is worth discussing some general guidelines as to how Toulmin's model should be implemented in both the analysis and presentation of results in mathematics education studies. Finally, Wawro's (2015) study illuminates a potentially strong connection between representational fluency and effective argumentation.

General Model Considerations

As discussed at the beginning of this section, although the general structure of Toulmin's (2003) model of argumentation is applicable to a wide variety of disciplines, what constitutes appropriate justification in a given argument depends on the field and setting in which the argument is made. For instance, what suffices as a valid warrant or backing is largely field-dependent. Because of this somewhat delicate dependency, some researchers have argued that the classification of a particular statement as data, warrant, or backing within an argument is not always well defined (Simpson, 2015; Weinstein, 1990). In particular, when sub-arguments are considered together as one larger argument, a statement can take on dual meanings: the claim from one sub-argument can become the data for the next.

This field-dependency also dictates what sort of objects can be treated as warrants or backing within an argument. For instance, we have already seen an example (Krummheuer, 2007) where warrants were not content related, but rather relied upon whether or not the teacher intervened after a claim was made. Forman, Larreamendy-Joerns, Stein, and Brown (1998) contend that warrants can take the form of algorithms or formulas, such as area = length times width. Moreover, they argued that backing can take the form of convincing someone that length times width is indeed the correct algorithm for computing area of a parallelogram. However, Forman et al. accompany the plethora of other researchers (Evens & Houssart, 2004; Krummheuer, 1995, 2007; Stephen & Rasmussen, 2002; Rasmussen et al., 2004) in only adopting a partial Toulmin scheme, in the sense that they do not consider such elements as modal qualifiers (Q) and rebuttals (R).

Although Krummheuer (1995) identified the subset {D, C, W} of Toulmin's model as the core of the argument, some researchers have criticized the absence of the remaining components of the full model. According to Inglis, Mejia-Ramos, and Simpson (2007), utilizing the full Toulmin model is especially imperative when analyzing mathematical reasoning about more advanced content. They argued that modal qualifiers, in particular, play an important yet largely unrecognized role in careful mathematical argumentation, since "omitting the role of the modal qualifier in models of mathematical arguments constrains us to consider only arguments with absolute conclusions, and, consequently, to undervalue non-deductive warrants in advanced mathematics"(p. 19). Through task-based interviews with successful postgraduate mathematics students, Inglis et al. established that advanced mathematical argumentation relies heavily on these non-deductive warrants. Moreover, these warrants are themselves used to arrive at non-absolute conclusions on a regular basis. Similarly, Troudt (2015) corroborated these claims by arguing that researchers' use of the reduced model tends to "incorporate the backing into the warrant" (p. 249). In her study, Troudt additionally found that her mathematician participants' explicit modal qualifiers and backing statements provided valuable insight into the *process* by which their mathematical proofs unfolded.

Specifically, she concluded that the inclusion of qualifying and backing statements could "illuminate more patterns explaining the participants' decisions and thinking at various moments while constructing proof" (p. 251).

Aside from considering which components of the Toulmin model to include in an analysis of an argument, one must also be mindful of how such analyses are presented. For instance, when several sub-arguments are made within a larger proof, Aberdein (2005) suggested that the overall structure of the proof is more clearly elucidated when these sub-arguments are chained together in "data-conclusion pairs" (Simpson, 2015). The claim of one sub-argument in such chains becomes the data for the next. In circumstances where each sub-argument has the same modal qualifier, Aberdein suggested placing a single qualifier in the diagram rather than creating a cluttered representation with copies of the same qualifier. These examples point to a larger potential concern with adopting Toulmin's (2003) model in the analysis of mathematical argumentation. Namely, "an analysis of an argument using Toulmin's scheme does not result in a unique structure. That is, a single written proof […] might be interpreted in such a way as to produce quite different Toulmin diagrams" (Simpson, 2015, p. 7).

One way to potentially add clarity to a Toulmin analysis is to characterize specific types of warrants used in participants' argumentation. For instance, Inglis et al. (2007) classified participants' warrants under three categories: inductive, structural-intuitive, and deductive. Specifically, an inductive warrant involves evaluating one or more specific cases. Participants instantiated the structural-intuitive type of warrant via observing or experimenting with a mental structure, visual or otherwise, in the service of persuasion. Finally, a deductive warrant involved formal deductions from axioms or the use of counterexamples to argue a claim. As I discuss in the next chapter, these three types of warrants not only provide added specificity to the Toulmin analysis, but they also align with important constructs from my theoretical lens.

Mejia-Ramos (2008) also advocated for the use of the full Toulmin (2003) model when analyzing mathematical argumentation. He provided participants with conjectures including: (1) The derivative of an even function is an odd function, and (2) The product of two diagonal matrices is diagonal. Mejia-Ramos ultimately found that participants' arguments fell under three categories. The first, *inductive arguments*, included attention to special cases, much like the inductive warrants in Inglis et al.'s (2007) study. The second type of argument was *informal deductive*, and incorporated informal and sometimes pictorial justification. Finally, *formal deductive* arguments incorporated rigorous proof and are analogous to Inglis et al.'s deductive warrant classification.

Given the lack of backing discussed in many of the articles mentioned in the previous subsections, Simpson (2015) decided to more thoroughly investigate the role(s) that backing can play within the Toulmin model. By examining how earlier papers (e.g. Evens & Houssart, 2004; Inglis et al., 2007; Stephan & Rasmussen, 2002) reported the use of backing, Simpson found that there were three distinct roles for backing of warrants within an argument. Simpson denoted the first as *backing for the warrant's validity* (p. 10). This type of backing, evidenced by Evens and Houssart (2004), was invoked to explain why the warrant applies to a given argument. A second type of backing served to "highlight the logical field in which the warrants are acceptable," which Simpson characterized as *backing for the warrant's field* (p. 12). Finally, a third role of backing has been to illustrate that a given warrant is actually correct. Simpson refers to this form of backing as *backing for the warrant's correctness* (p.12).

One of the reasons why this backing characterization is important to research in the field of mathematical argumentation is that prior researchers might not have given enough attention to the *reasons* for which participants were not able to provide appropriate backing for their warrants. Rather, they have tended to simply note the absence of backing. In particular, as Simpson (2015) explains,

> A teacher may be asking a pupil to explain why their warrant applies to the situation, but the pupil may defend themselves by giving evidence that their warrant is correct. This need not mean that a student is not capable of giving an appropriate form of backing for the validity of their warrant, just that they took the enquiry to be a challenge to its correctness (p.15).

Such considerations could be especially important in real-time classroom interactions, provided that instructors are cognizant of these various types of backing.

In any case, many of the aforementioned studies using Toulmin's (2003) model of argumentation focused on in-class interactions and participation (Evens & Houssart, 2004; Krummheuer, 1995, 2007; Rasmussen et al., 2004; Stephan & Rasmussen, 2002), or assessment of purported written arguments written by someone else (Alcock & Weber, 2005; Evens & Houssart, 2004). In the in-class setting, some researchers (Krummheuer, 1995, 2007; Rasmussen et al., 2004; Stephan & Rasmussen, 2002) felt that the reduced Toulmin model was appropriate, and rarely found evidence of explicit backing, or even content-related warrants (Krummheuer, 2007). In the formal proof literature, researchers (Alcock & Weber, 2005; Inglis et al., 2007; Simpson, 2015) argued that it is important to utilize the full Toulmin model, including modal qualifiers and rebuttals. But just reading the finished product of a purported proof is inherently difficult because some of the components of the Toulmin model, such as backing and sometimes even the warrants, are implicit and cannot be elicited through real-time social discourse with the proof author.

In my research, each pair of participants had the opportunity to challenge each other's warrants, but also work together to come up with potential rebuttals. As the interviewer, I also asked probing questions with the intent of targeting specific components of the Toulmin model. For instance, I explicitly asked participants about how sure they are about a claim, hence eliciting modal qualifiers. Sometimes I also asked for clarification when an implicit warrant was called upon. Therefore, as I will discuss in the Theoretical Perspective section, the inherent social nature of my interview setting required that I adopt a theoretical lens which takes advantage of both the fact that students worked collaboratively, and the fact that I intervened, mediated, and provided scaffolding during the interview process. In the next chapter, I motivate and outline my chosen theoretical perspective as well as my methods, and discuss how the former influenced the latter.

CHAPTER III

METHODOLOGY

Recall that the purpose of this qualitative research was to explore undergraduates' multifaceted reasoning about integration of complex functions. My guiding research questions were:

Q1 How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?

Q2 How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In order to address my research questions, I observed six class sessions of an undergraduate complex variables course and conducted task-based, videotaped interviews with two pairs of students from this course. In this chapter, I first motivate my chosen theoretical orientation by providing my research stance, which conveys my personal experiences and beliefs regarding integration of complex functions, and their connection to my work. Then I detail my chosen framework of Tall's (2013) three worlds, comparing aspects to other relevant theories and frameworks. I next discuss the connection between Tall's three-world framework and Toulmin's (2003) model of argumentation, and how these molded together to comprise my theoretical lens. Subsequently, I discuss how this lens informed my coding and data analysis, and how it assisted me in answering my research questions. I then detail the methods of data collection, including a description of my participants and setting. Finally, I outline my proposed methods for data analysis, including measures taken to ensure credibility and trustworthiness of my findings.

Researcher Stance

As an undergraduate mathematics major, I took a complex analysis course not quite knowing what to expect from it. I had recently completed a course in real analysis, where we primarily focused on formal epsilon-delta proofs regarding continuity, differentiation, and Riemann integration. Accordingly, I had assumed we would approach these subjects in complex analysis with the same level of rigor and abstraction. However, to my surprise, the complex analysis course did not list real analysis as a prerequisite, and I shortly discovered that much of the class would focus more on calculation than proof. When it came time to learn integration, I found that many of the assumptions of major theorems were motivated in our textbook in a rather "hand-wavy" manner, and I did not have the intuition to visualize some concepts.

In class we sketched proofs to these theorems, and I could reproduce these basic proofs or similar ones on exams, but I rarely understood precisely *why* certain assumptions needed to be met in order for the theorem to apply. Moreover, many of the integration examples we worked through were "nice" in one way or another, so that is was not really necessary to examine all the assumptions of theorems that I applied regularly. In essence, these assumptions nearly always applied, in my experience, and thus I did not attribute much significance to them. By the end of the integration unit, I was proficient with the procedures of parametrization, partial fractions decomposition, rearranging a function to apply Cauchy's Integral Formula. But if pressed, I probably would not be able to craft a cogent argument that would justify the procedures used or theorems applied.

As a graduate student and mathematics education researcher, I have more recently had the opportunity to think more deeply about and prove the results in complex analysis that I previously took for granted. Thus, I developed an interest in how undergraduate students might argue or reason through integration problems, and what would happen if they were pressed about assumptions, warrants, and the like. Moreover, as someone who ultimately would like to teach classes such as complex variables, I had a vested interest in how instructors can help students strengthen their mathematical argumentation, even without formally proving results. In particular, I wished to discover ways that embodied, symbolic, and formal reasoning can work together to further an integration argument. Recall that embodied reasoning attends to the study of objects and their geometric properties, as well as mental visualization and description through language. On the other hand, symbolic reasoning grows out of actions on objects and is symbolized via thinkable concepts such as number. When reasoning in this world, it is possible for individuals to conceive of symbols as *procepts*, operating dually as process and concept (Tall, 2008). Formal reasoning attends to axiomatic systems articulated via set-theoretic definition, or properties that can be deduced by proof.

When I reflected on my personal experiences with evaluating integrals of complex functions, I noticed that I often drew diagrams, performed some sort of symbolic manipulations, and attended to other related theorems, all within the same problem. For instance, when faced with a particular integral, I would start by sketching a picture of the region, the contour, and any points of discontinuity for the given integrand.

I would also determine if the integrand was an analytic function, so that I could potentially apply the Cauchy-Goursat Theorem (see Appendix A). But analyticity can be determined symbolically via the Cauchy-Riemann equations, and I would compute and compare the requisite partial derivatives. I would also inspect my picture to ascertain whether other important properties held for the given contour and/or region. For instance, sometimes I needed my contour to be closed in order to apply a theorem, and I visually inspected my picture to verify that the contour started and ended at the same point. If ultimately the Cauchy-Goursat Theorem applied, I could conclude that the integral was zero.

Notice that even in this simple example, I attended to pictorial and symbolic representations, as well as the hypotheses and conclusions of major theorems. Moreover, these embodied, symbolic, and formal aspects intertwined in various ways. In order to use the Cauchy-Goursat Theorem, my contour had to be simple and closed, and this required visual inspection of my picture. Using this theorem also required that the integrand represented an analytic function, and verifying analyticity amounted to performing several symbolic manipulations.

The above example illustrates a belief I hold that even a seemingly uncomplicated integration problem can lend itself to a combination of embodied, symbolic, and formal reasoning. As a mathematics education researcher, the projects I have been involved with have all involved some level of concern for geometric, algebraic, and formal reasoning, but at times I struggled to theoretically fit each of these pieces into a cohesive whole. In the next section, I detail my choice of theoretical lens for this study, Tall's (2013) three worlds of mathematics, which I feel accomplishes such cohesion and helped me address my research questions.

Theoretical Perspective

As discussed in the introductory chapter, my study was at least partially motivated by the premise that students' prior experiences with mathematics inevitably influence how they conceive of newer and more general mathematical topics. As I alluded to previously, Tall (2013) discussed how some of these prior experiences can support students' reasoning in new situations, while others can engender cognitive dissonance. An important construct related to prior experiences is what he refers to as a *met-before*, or "a structure we have in our brains *now* as a result of experiences we have met before" (p. 23, italics in original). For instance, when we study complex numbers for the first time we are immediately introduced to a number i whose square is negative. At this point, "we experience the met-before that tells us that 'a (non-zero) square must be positive'. This 'met-before,' which is true for real numbers, forms part of our selective binding of the notion of 'number' and is usually problematic" (p. 88). Though this is merely an elementary example, similar difficulties can arise from met-befores when studying complex analysis, as with the phenomenon of *thinking real, doing complex* (Danenhower, 2000; Troup, 2015).

Such a concern for the effects of prior mathematical experiences is well documented in the mathematics education literature. For instance, a central tenet of constructivism is *equilibration* resulting from the marriage of prior knowledge and new mathematical experiences. When faced with an unfamiliar notion, one may either assimilate this idea into a more familiar category or accommodate his or her existing mental schema through a process of cognitive reorganization (Miller, 2009). Additionally, researchers have studied *transfer* of knowledge by investigating how aspects from prior experiences with mathematics carry over to new tasks or situations (Lobato, 2006; Lobato & Seibert, 2002; Wagner, 2006). But perhaps most importantly, the original formulation of met-before was influenced by the notion of *metaphor*, which several researchers have argued is central to our mathematical knowledge construction (Lakoff & Johnson, 1980; Lakoff & Nuñez, 2000; Sfard, 1994). Tall (2013) elaborated on the connection between these two ideas as follows:

> The philosophical notion of 'metaphor' and the cognitive notion of 'met-before' have much in common. Both link a new experience to an experience that is already familiar. However, the notion of 'metaphor' offers a high-level analogy to formulate a theory while the notion of 'met-before' is formulated to focus on the development of ideas *from the viewpoint of the learner* (p. 88, italics in original).

Hence Tall views metaphor as a top-down expert viewpoint of another's previous experience, whereas a met-before is a bottom-up development from the learner's perspective.

While met-befores are central to our development of mathematical knowledge, Tall (2013) also stresses the importance of three basic innate principles that guide our growth within and between three worlds of mathematical thought. I identify and describe these worlds in the next subsection, but first I detail the aforementioned innate principles. These are the *set-befores* of recognition, repetition, and language. Though animals also share the first two attributes, Tall points out that language is a uniquely human construct and is a primary means of developing formal mathematical thinking. In my study, I focused on the argumentation of multiple individuals who interact and guide each other's arguments via verbal language, written inscriptions, and gestures. As such, my study adopted a unit of analysis of student pairs.

According to Tall (2013), the aforementioned set-befores of recognition, repetition, and language enable three forms of knowledge compression: categorization, encapsulation, and definition. Through this compression, individuals build so-called *crystalline* structures, which incorporate many equivalent formulations of a mathematical object and can be unpacked in various worlds. Thus the three-worlds perspective posits that our propensity as humans for recognition, repetition, and language allows us to crystalize mathematical concepts by building upon met-befores via categorization, encapsulation, and definition. This general process of crystallization manifests itself differently within each world and between multiple worlds, as I illustrate in the next subsection. But first I outline each of the three worlds and orient them with respect to existing mathematics education frameworks.

Three Worlds of Mathematics

According to Tall (2013), by building upon our met-befores, we navigate through three distinct but interrelated worlds of mathematical thought. The first is the world of *conceptual embodiment*, which begins with the study of objects and their properties, progressing towards mental visualization and eventually description through increasingly subtle language. Because the term ‘embodiment’ can have many varied meanings (Wilson, 2002), Tall (2004a) immediately contrasted his version of embodiment against that of Lakoff and colleagues. He mentioned that Lakoff and others have argued that all mathematical knowledge is embodied, but this cannot be the case in Tall’s framework if the embodied world is but one of three distinct forms of mathematical knowledge. In particular, Lakoff (1987) distinguished between two different types of embodiment: conceptual and functional. The former involves conceiving of concepts via mental images, i.e. visuo-spatially. On the other hand, functional embodiment refers to a more automatic and possibly unconscious use of concepts requiring less effort and more closely resembling “normal functioning” (p. 13). Of these two types of embodiment, Tall (2013) chose to only consider conceptual embodiment in his three-world framework, as the name of the first world suggests. Functional embodiment, then, is reserved for the interaction between the first and second worlds, which I discuss later.

Ultimately, Tall (2013) refined Lakoff’s (1987) previous description of conceptual embodiment to refer to “the use of mental images, both static and dynamic, that arise from physical interaction with the world and become part of increasingly sophisticated human imagination” (p. 12). As such, this world includes using physical manipulatives such as base blocks, drawing geometric inscriptions that become mental pictures, and graphing functions as static images on paper. Moreover, it subsumes any dynamic visual imagery either visualized in the mind or using computer software.

This aspect involving visualization is also consistent with other researchers’ characterization of visual reasoning. For instance, Zazkis, Dubinsky, and Dautermann (1996) describe visualization as the mental construction of objects or processes associated with external objects or events. While there is no general consensus as to what exactly constitutes visualization in mathematics education research, most definitions incorporate aspects of the following definition by Presmeg (2006): “visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics”(pp.

206-207). Thus Tall's (2013) characterization of the embodied world has a visualized aspect that is well established in the literature.

Language also remains an important aspect of the embodied world, as this set-before allows for the articulation of increasingly formalized embodiments. Tall (2004b) emphasized that "A visual picture is nothing without meaning being given to what it represents. While embodiment is fundamental to human development, language is essential to give the subtle shades of meaning that arise in human thought" (p. 284). As discussed previously, language allows for definition of concepts, which is a form of compression and underpins crystallization within the embodied world.

The second world in Tall's (2013) framework is *operational symbolism*, which grows out of actions on objects and is symbolized via thinkable concepts such as number. A *thinkable* concept is attached to a specific name through the set-before of language, and over time its meaning can be refined and incorporated into knowledge structures. According to Tall, in this symbolic world, "Whereas some learners may remain at a procedural level, others may conceive the symbols flexibly as operations to perform and also to be operated on through calculation and manipulation" (pp. 16-17). When the latter is accomplished, Tall characterizes this flexibility as evidence of *proceptual thinking*, where a *procept* is a symbol operating dually as a process and as a concept (Tall, 2008). For example, consider the arithmetic expression 7 + 3. On one hand, a child might interpret this expression as instructions for a process of addition to be carried out. However, the student might instead view 7 + 3 as the number 10, the resultant concept of the sum. Over time, and using the set-befores of recognition and repetition, the child might flexibly conceive of the number 10 in many equivalent ways such as 5×2, $12 -$

2, $5 + 4 + 3 - 2$, $20 \div 2$, and even $-10i^2$. Similarly, one can conceive of an algebraic expression such as $2x + 6$ as either a procedural operation to be carried out, i.e. double the value of x and then add 6, or as a concept that can be operated on in its own right. For instance, one might multiply this concept $2x + 6$ by the aforementioned concept 10. Hence development within this world is analogous to Dubinsky's APOS framework (Breidenbach et al., 1992) and Sfard's duality principle (Sfard, 1991) in which actions are condensed into processes, which then are encapsulated into objects in their own right. In particular, Tall's crystallization within the operational symbolic world is analogous to Sfard's notion of reification.

Although the aforementioned embodied and symbolic worlds represent distinct ways of thinking, the two often interact throughout an individual's development. For instance, Tall (2013) argued that "In school mathematics, embodiment and symbolism develop in parallel, where embodied actions give rise to symbolic operations and symbolism has embodied representations" (p. 17). Tall classifies this intersection between the operational symbolic and conceptual embodied worlds as *embodied symbolic* mathematics. Subsequently, as the learner defines and deduces properties either geometrically or symbolically, he or she begins to formally think about the first two worlds. Tall argues that these intermediate territories of *embodied formal* thinking and *symbolic formal* thinking may later propel the learner into a third world of formalism. Figure 2 is a visual representation of the interactions between all three worlds.

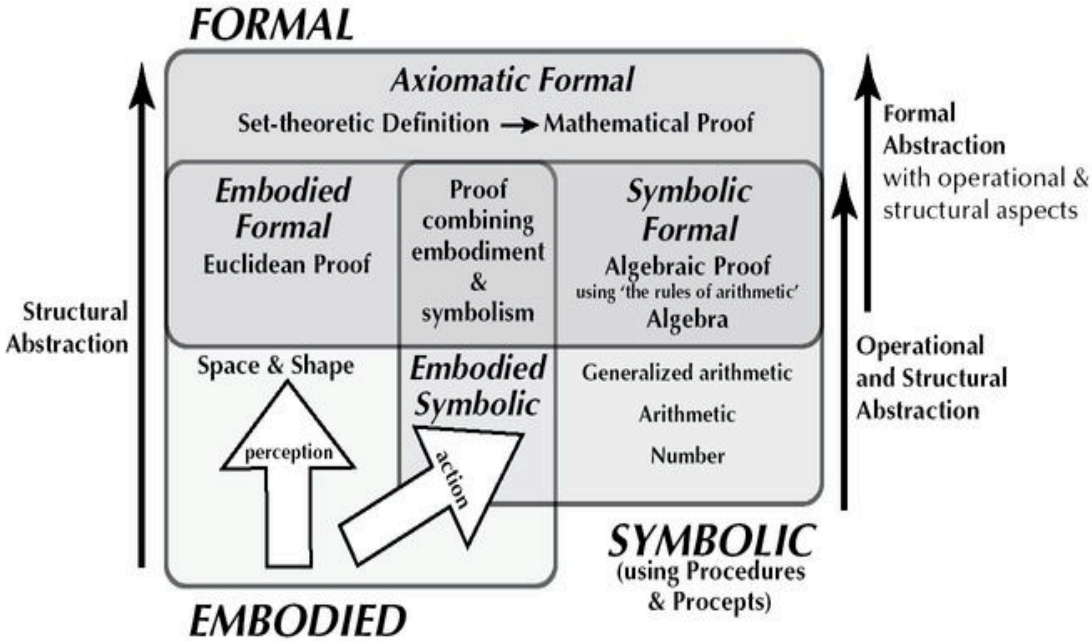


Figure 2. Tall's three worlds perspective. Taken from Tall (2013) p. 17.

A natural setting to consider the interplay between the embodied and symbolic worlds is the definite integral of a real-valued function $y = f(x)$. Following Leibniz's vision, we tend to conceive of the definite integral as the precise area under the graph of $f(x)$ from the point $x = a$ to the point $x = b$. This area is a quantity that we can see and imagine, and it can be approximated to varying degrees of accuracy by adding up the areas of rectangular strips. Leibniz eventually "envisaged the area as the sum of infinitesimally thin strips of height y and width dx and wrote the area as $\int y\, dx$ where the symbol $\int$ is an elongated S for the Latin word 'summa'" (Tall, 2009, p. 8).

Therefore, according to Tall (2009), this area is embodied as an object that can be visualized, and we can act upon this object by calculating its size using symbolism. We can blend the embodied and symbolic worlds even further by considering the area under $f(x)$ as follows. First, we can calculate the area $A(a, x)$ from some point a to a point x. In the second stage, we allow x to increase and plot the resulting area against x. If we allow the strips used to calculate $A(a, x)$ to become arbitrarily thin, we obtain a graph as in Figure 3. Figure 3 also depicts a magnification of the graph $A(a, x)$ to illustrate its local straightness. Notice that this approach ultimately involves sensing an embodied base object (the graph of $f(x)$), acting upon it (by calculating $A(a, x)$), and representing the effect of that action as another embodied object (the graph of $A(a, x)$). Thus the definite integral concept for real-valued functions lends itself to an intimate blend of embodiment and symbolism.

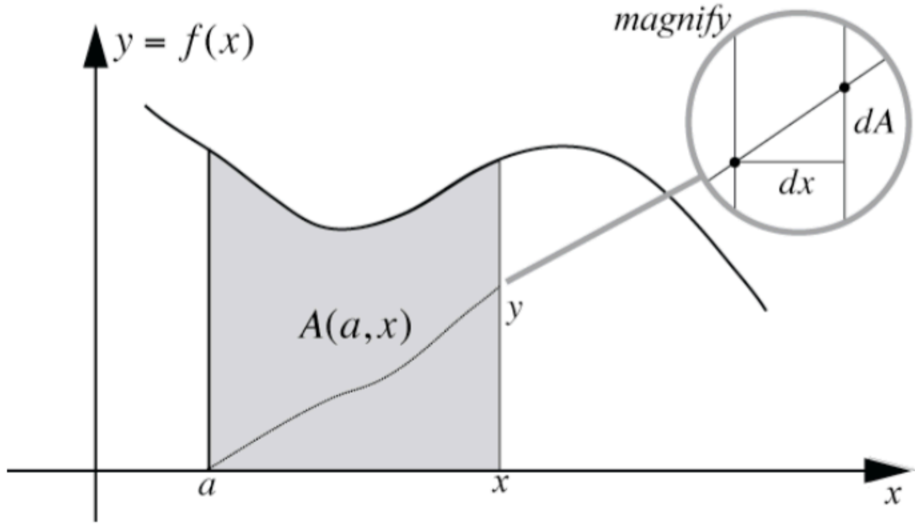


Figure 3. Graph of $A(a, x)$ and a local magnification. Taken from Tall (2009) p. 8.

Tall's (2013) third world is that of *axiomatic formalism*, wherein individuals build "formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (p. 17). In this world of thought, learners can quantify statements involving general objects and deduce further properties from a selection of axioms defining the system. Thus the individual's focus shifts from definitions based on known objects towards formal objects based on the prior definitions. For example, in symbolic formal reasoning, an individual might prove an algebraic argument or identity by simply employing rules of arithmetic. However, in the axiomatic formal world, algebraic proof would appeal to the formal group or field axioms and other results already deduced from those axioms in order to obtain the desired result.

Importantly, though it carries great utility and power, the third world is not necessarily the ultimate destination for mathematical thinking. For instance, Tall (2013) discussed how so-called *structure theorems* end up informing embodiment and symbolism in meaningful ways. For example, the structure theorem that any finite-dimensional vector space over a field F is isomorphic to F^n can be proven within the axiomatic formal world. However, as a consequence of the theorem, one can represent vectors in finite-dimensional spaces as column vectors and in turn linear maps can be expressed visually as matrices. Such matrices can then be multiplied symbolically in the usual way. Accordingly, Tall argues that such structure theorems establish single crystalline structures despite being rooted in many seemingly disparate topics. In the next subsection, I detail how Tall's three worlds framework complements Toulmin's (2003) model of argumentation, and thus how it informed my data analysis.

Connection to Toulmin's Framework

Recall from the last chapter that Toulmin's (2003) model of argumentation relies upon warrants whose role is to connect the initial data to an asserted claim. In the previous subsection, I detailed Tall's three worlds of mathematics as a theoretical framework through which mathematical development can viewed. Fortunately, these worlds can also lend additional specificity to a mathematical argument, in that "each world develops its own 'warrants for truth'" (Tall, 2004b, p. 287) in a distinctive manner. For instance, in the embodied world, truth is initially established in elementary geometry based on what is *seen* to be true by the learner visually. As the individual progresses towards more formal geometric arguments in the embodied formal world, he or she develops Euclidean proof, "which is supported by a visual instance and proved by agreed conventions, often based on the idea of 'congruent triangles'" (p.287). In contrast, within the operational symbolic world, truth is established in arithmetic based on calculation. In elementary algebra, a statement is true if one can produce the appropriate symbolic manipulations such as $(a-b)(a+b) = (a-b)a + (a-b)b = a^2 - ba + ab - b^2 = a^2 - b^2$. Finally, in the axiomatic formal world, a statement is true either by assumption as an axiom, or because it can be proved formally from the axioms.

Tall (2004b) illustrated these general classifications of truth with an example involving commutativity of vector addition:

> In the embodied world, the truth of $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ follows from the properties of a parallelogram and meaning is supported by tracing the finger along two sides to realise that the effect is the same, whichever way one goes to the opposite corner of the figure. In the symbolic world of vectors as matrices, addition is commutative because the sum of the components is commutative. At the formal level of defining a vector space, commutativity holds because it is assumed as an axiom.

This example can also be easily adapted to justify the commutativity of complex numbers in each of the three worlds, as complex numbers can be expressed in vector form and additive commutativity in $\mathbb{C}$ is one of the field axioms. Hence we have seen how each of the three worlds provides a different warrant for truth.

But more specifically, these three worlds can also correspond to particular classes of warrants mentioned in the literature review. For instance, recall from the last chapter that Inglis et al. (2007) found that participants' warrants could be classified according to three types: inductive, structural-intuitive, and deductive. The inductive warrant was comprised of evaluating specific cases, and Tall (2013) argued that such a warrant corresponds to either the embodied or symbolic worlds, depending on the nature of the case. Inglis et al.'s participants instantiated the structural-intuitive type of warrant via observing or experimenting with a mental structure, visual or otherwise, in the service of persuasion. As such, Tall associated this warrant type with the embodied world, as it "refers to thought experiments based on embodied images or calculations" (p. 343). Finally, a deductive warrant involved formal deductions from axioms or the use of counterexamples to argue a claim; hence Tall aligned such a warrant with the axiomatic formal world.

Moreover, recall from the previous chapter that Inglis et al. (2007) and Mejia-Ramos (2008) argued for the necessity of modal qualifiers in Toulmin analyses of mathematical argumentation. In particular, their studies illustrated how one's level of certainty about assertions can illuminate his or her progression towards a more formal argument. According to Tall (2013), these studies not only support the use of modal qualifiers, but they also lend credence to the three-world perspective in a Toulmin analysis. Consider the following four conjectures, the first two of which were presented to participants in Inglis et al.'s (2007) study, and the latter two of which were presented to Mejia-Ramos' participants: (A) The sum $m + n$ of two abundant numbers m, n is abundant; (B) The product mn of two abundant numbers m, n is abundant; (C) The derivative of an even function is an odd function; (D) The product of two diagonal matrices is diagonal. Although these two studies focused primarily on proof, Tall remarked that these four conjectures given to participants still lend themselves to the worlds of embodiment and symbolism:

> (A) and (B) are general properties of whole number arithmetic that benefit from theoretical symbolic arguments. (C) is a calculus problem that can be embodied as a visual picture, symbolized as a rule in calculus, or formalized in mathematical analysis. (D) is a problem in matrix algebra that is essentially symbolic but is supported by a functional embodiment to remember the formula for matrix multiplication. Each benefits from different forms of support in embodiment, symbolism, and formalism to construct a proof (p. 346).

Summarily, as I have illustrated with the aforementioned examples, Tall's three-world perspective is compatible with Toulmin's (2003) model of argumentation. Moreover, it complements the Toulmin analysis of a mathematical argument by adding specificity with regard to the types of backing and warrants used. As such, my data analysis was strengthened by classifying each Toulmin component as embodied, symbolic, formal, or various mixtures of these, as viewed through Tall's three-world lens. Given that my study considered how pairs of students reason about integration tasks, it was additionally important that I consider how each individual contributes to collective argumentation.

According to Krummheuer (1995), *collective argumentation* takes place when multiple participants construct arguments through emergent social interaction. Because of the multivoicedness of this interaction, "Disputes in parts of an argumentation might arise that could lead to corrections, modifications, retractions, and replacements. Thus, the set or sequence of statements of the finally consensual argumentation is shaped step by step by surmounting controversy" (p. 232). In my paired interviews, participants' argumentation could have additionally been influenced by my (albeit minimal) intervention. Ultimately, Krummheuer (2007) characterizes collective argumentation as a process of active participation wherein each individual participates in the production of an argument in two respects. First, he or she "produces statements that can be allocated to certain categories in the sense of Toulmin" (p. 78). In doing so, the individual participates in a second manner by invoking a particular speaking role during this interaction.

To this end, Krummheuer (2007) detailed four speaker roles, originally formulated by Levinson (1988), that he used to describe the process of collective argumentation. Levinson built on Goffman's (1981) decomposition of a speaker's utterance into two functions. The first is a function of *formulation* comprised of the syntactical form in which a statement is produced. Thus this function focuses on the specific choice of words invoked to articulate a statement. The second function regards the *content* of a contribution and is therefore semantic in nature. Krummheuer argued that a speaker need not be autonomous with respect to one or both of these functions, and thus this leads to four potential cases in this setting.

The first case coincides with the role that Levinson (1988) denoted *author*. A speaker taking on the role of an author is both syntactically and semantically responsible for his or her statement, and thus employs both the formulation and content functions. On the other hand, a speaker might claim responsibility for neither the semantic nor syntactic aspects of an utterance, in which case he or she acts as *relayer*. Alternatively, a speaker "uses the words of someone else to mean *something different* from the meaning ascribed to the utterance of the original speaker" (Krummheuer, 2007, p. 67, italics in original). In this third case, the speaker takes on the role of *ghostee*, and is autonomous with respect to the content but not the formulation of a statement. Finally, when a speaker revoices a previously mentioned idea using his or her own language, he or she is acting as *spokesman*. In this fourth case, such an individual is responsible for the syntactic, but not the semantic, aspect of an utterance. As I discuss in the next section, I adopted these four speaker roles in my study to address my second research question. In this next section, I explicate the methods pertinent to data collection and analysis for my study.

Methods

In this section, I detail the methods surrounding my study. Specifically, I thoroughly describe both the setting in which the study took place as well as the participants who consented to take part in the interviews. Next, I detail my data collection procedures, including rationale for the types of data collected. I also discuss my procedures for analyzing the data, including measures taken to ensure credible and trustworthy results. Before conducting this study, I obtained approval from the Institutional Review Board (IRB) for the methods outlined below (see Appendix B).

General Setting and Participants

The purpose of my study was to examine how undergraduates reason about integration of complex functions. As detailed in the previous section, an individual's mathematical reasoning or argumentation can be nuanced, often attending to complicated blends of embodiment, symbolism, and formalism. Accordingly, I sought to employ qualitative research methods in an effort to capture the rich intricacies of participants' mathematical argumentation. In particular, I conducted paired task-based interviews designed to elicit undergraduates' reasoning about the integration of complex functions. In order to obtain a detailed account of the manner in which my participants learned integration of complex functions, I observed their complex variables class six times during the unit on integration. I will detail both of these data sources in the next subsection, but first I describe the course setting and my participants.

Participants were selected from undergraduate students at a military academy in the United States, enrolled in the complex variables course during the spring 2015 semester. This institution, which I selected for convenience, has approximately 4000 undergraduate students, and approximately three-fourths of those students are male. In 2015 Forbes named this institution among the top ten western colleges in the nation, and in the top five public schools. Hence my participants come from an ostensibly intelligent and high-performing cross-section of undergraduate students.

The complex variables course at this institution is generally a small-enrolled course with approximately 17 students composed of primarily third and fourth year students, as was the case in the spring 2015 class. These students were primarily Caucasian, with a male to female ratio representative of the larger undergraduate population. Many of the students in this course had not taken a course in real analysis, as this was not a prerequisite for complex variables. One section of this complex variables course is offered every spring semester at the institution, and the class met on a staggered schedule alternating between two and three class sessions per week.

The instructor of this course, Dr. X., was a Visiting Scholar with expertise in complex analysis. He has published a textbook on complex analysis geared towards mathematics and engineering majors, and this book served as the official course text during the spring 2015 course. This instructor has also received several teaching awards at his home institution. Based on my classroom observations, the instructor's teaching style could best be described as lecture-based, augmented by some technology and small group work. Students sat at large tables accommodating three to four students per table.

At the end of the semester, I enlisted the participation of two pairs of students from the course to partake in semi structured, task-based interviews regarding integration of complex functions. The manner in which I selected participants is detailed in later subsections, but for now I mention general background information about these individuals. All names mentioned throughout this document are pseudonyms I have assigned to protect participants' identities, in accordance with the IRB (see Appendix B). My first pair of participants consisted of Sean and Riley, who are male and female, respectively. Sean was a fourth-year student and Riley was a second-year. The second pair of participants consisted of two males, who I refer to as Dan, a third-year student and Frank, a second-year. All four participants are Caucasian.

Data Collection Procedures

In this subsection I describe the various sources of data that I collected, as well as the purpose of these data with regard to my research questions. My study consisted of three sources of data: video-taped classroom observations, classroom observation notes, and video-taped task-based interviews. Below I detail each of these three aspects of my study. A timeline of these manners of data collection is summarized in Table 1.

Table 1

Summary Timeline

Time	Activity	Participants
March 11 – April 2	Class observations during unit on integration (6 classes)	1 researcher All students
Early May	Conduct task-based interview	1 researcher 2 pairs of consenting students

As discussed in my theoretical perspective section, the three-worlds framework predicates the growth of students' mathematical knowledge on *met-befores*, or mental structures they now have as a result of prior mathematical experiences (Tall, 2013). Accordingly, before studying participants' argumentation about integration of complex functions, I sought to first observe the context in which these students learned about such integration. Hence I sat in on the class during the integration unit of the course, which lasted six sessions during the second half of the semester. I was not an active class participant during these observations. Rather, the purpose of the classroom observations was to document what content had been presented by the instructor, and to establish a "base-line" for what students knew about integration theory before taking part in the subsequent interview. Hence these observations served to capture group characteristics and the general classroom environment, as described in the previous subsection. Summarily, these observations served to "provide some knowledge of the context or to provide specific incidents, behaviors, and so on that can be used as reference points for subsequent interviews" (Merriam, 2009, p. 119).

I had planned to personally videotape the classes I observed, but the course was video-recorded by the institution for instructional purposes. Thus the institution provided me with a copy of the recordings for these classes, in accordance with the IRB. Video-taping resulted in stronger research because it allowed me to "retain a rich record of behavior that can be reexamined again and again" (Clement, 2000, p. 577). It also allowed me to document field notes as I observed the class. These field notes, my second source of data, helped me focus on important classroom episodes from the videotaped observations in order to better summarize the classroom setting.

As such, these observation notes contain paraphrased statements and questions contributed by students and the instructor, inscriptions written by the instructor, and spontaneous connections I was able to make to prior class sessions. I also focused on how mathematical arguments were constructed during class, including the frequency and level of rigor of proofs. Although the instructor was not the focus of my research, his sequencing of events and how he taught the content likely influenced students' reasoning to some degree. For example, his linguistic formulation of certain assumptions into acronyms such as ASCODOD ("analytic in a simply-connected domain D") and SICOPOC ("simple, closed, positively oriented curve") might have potentially influenced students' attention to the hypotheses of major theorems in some way. As such, my classroom observations, videos and notes could serve as triangulation of interview findings.

The third component of the data I collected was in the form of a videotaped, task-based, semi-structured interview consisting of two 90-minute portions. According to Patton (2002),

> We interview people to find out from them those things we cannot directly observe […] We cannot observe how people have organized the world and the meanings they attach to what goes on in the world […] The purpose of interviewing, then, is to allow us to enter into the other person's perspective (pp. 340-341).

Because my research questions sought to ascertain the nature of students' mathematical reasoning, including thought processes and visualizations that are scarcely directly observable, interviewing was a crucial aspect to data collection. After the integration unit and my class observations were complete, I enlisted the help of the course instructor to select a subset of four students (two pairs) to take part in these interviews. This subset of four was purposefully sampled (Patton, 2002) because I hoped to interview students who could cogently articulate their thoughts and work well together. In order to ensure such a selection, I corresponded with the instructor of the course to get an idea about which students might reason similarly or work well together.

In particular, I directed the instructor to send me participant suggestions based on the following criteria. First, I requested that both students from each pair come from the same classroom group, so that they would be comfortable discussing complex variables content with one another aloud. Additionally, I wanted pairs of students to be relatively heterogeneous with respect to their current course grade, so that I did not interview just the top two or bottom two students in the class. As much as possible, I wanted my participants to be a representative cross-section of the larger class with respect to their mathematical argumentation and demographics. The instructor did not inform me about any particular student's course grades or perceived abilities. Rather, he merely sent me a list of names based on the above criteria that we had discussed. I then scheduled interviews with consenting participants to take place at their institution several days after their final exams. This was done to ensure that all course content had been covered, that participants would hopefully remember all pertinent integration material, and that the interview would take place in an environment familiar to the students.

During these videotaped interviews, I asked the pair of students to work together to solve a sequence of tasks related to integration of complex functions. I read tasks aloud verbally so as not to overtly suggest any particular representation or world. Each paired interview was comprised of two portions. I designed the first portion to elicit participants' foundational understandings with respect to integration of complex functions, including parametrization of paths, the Fundamental Theorem for Line Integrals, and embodied interpretations of these and related concepts. The second portion of the interview was primarily dedicated to evaluating specific integrals, some of which were intended to be familiar to the students and some unfamiliar. However, some tasks were crafted to be intentionally open-ended. The aforementioned classroom observations allowed me to discern which types of problems had been discussed in class, leading to increased credibility of my findings. Appendix C lists the tasks from the first and second portions of the interview.

Most of these tasks lent themselves to multifaceted responses with respect to Tall's (2013) three worlds framework, encouraging a mixture of embodied, symbolic, and formal argumentation. Participants were explicitly asked to communicate with one another aloud and write down their thoughts on the accompanying whiteboards. While the students worked on the tasks, I encouraged them to elaborate on their discoveries, theories, ideas, reasoning, and conjectures. Such probing allowed me to encourage the students to think aloud, to request clarification about their remarks, and to establish a rich and credible account of their argumentation.

In the next subsection, I detail the mathematical content discussed during my class observations, as well as relevant student and instructor comments made during whole-class discussions which informed the interview component of my study. This portrayal, in conjunction with the above information, engenders a *rich, thick description* (Merriam, 2009) of my setting, hence bolstering the credibility of my research. In particular, Merriam points out that such descriptions can "contextualize the study such that readers will be able to determine the extent to which their situations match the research context, and, hence, whether findings can be transferred" (p. 229).

Class Setting

A typical day in this complex variables class commenced with the instructor asking students if they had any homework questions. During this time, other students were occasionally selected to present their solutions; selection was randomly decided using a basic computer program containing all students' names. The instructor then typically lectured on new content for a while, introducing major theorems and sometimes sketching the proofs. Periodically, he directed students to practice problems in groups and then randomly selected a student or group to present a solution to the class. Below I briefly detail each of the six class sessions that I observed. Table 2 summarizes this information, displaying important concepts from each day.

On the first day of the integration unit, the instructor motivated integration of complex functions by first garnering student input about how integration behaves for real-valued functions. Students quickly brought up line integrals and parametrization, and at one point all students spoke the words "area under the curve!" in unison when asked about a geometric interpretation for real integration. The instructor then introduced complex integration by arguing that in the complex case, a path can also be divided up into pieces, and that the integral behaves like a "sum of vector multiplications." Hence, according to the instructor, this is where one can take advantage of the vector form of complex numbers, along with its multiplicative structure.

Table 2

Class Observation Summary

Class Session	Important Concepts
1. March 11	Introduction to integration; Fundamental Theorem for Line Integrals; $\int_a^b (c+id)f(t)dt = (c+id)\int_a^b f(t)dt$
2. March 16	Notation such as $C_1^+(0)$; $\int_a^b f(z)dz = \int_a^b f(z(t))z'(t)dt$; M-L Inequality; $\left\|\int_a^b f(t)dt\right\| \leq \int_a^b \|f(t)\|dt$
3. March 18	Proof sketches from session 2; Cauchy-Goursat Theorem;
4. March 20	Examples involving Cauchy-Goursat; examples involving partial fractions and decomposition of regions
5. March 31	Introduced 'simple' and 'contour' terminology; proof sketches of antiderivative theorems; ASCODOD and SICOPOC abbreviations; Cauchy's Integral Formula
6. April 2	Examples using Cauchy's Integral Formula; sketched proof of Cauchy's Integral Formula

The next portion of the lecture was dedicated to developing basic integration properties. For instance, the instructor pointed out that integrating a vector function amounted to integrating each component function; if $f(t) = u(t) + iv(t)$ then $\int_a^b f(t)dt = \int_a^b udt + i\int_a^b vdt$. He then listed two theorems from the textbook, without proof. The first was the Fundamental Theorem for Line Integrals, which he claimed "followed from the definition as in Calc 3." The second theorem stated that $\int_a^b (c +$

$id)f(t)dt = (c + id)\int_a^b f(t)dt$, and one student immediately questioned in disbelief, "There's a formal proof for that?!" Such a question seemed to indicate that at least some students relegated this theorem to solely the operational-symbolic world (Tall, 2013) and hence did not feel a formal proof was necessary.

During the final portion of the first class, the instructor had students evaluate the integral $\int_0^{\frac{\pi}{2}} e^t \cos t \, dt$ using two applications of integration by parts. He then quickly computed the integral $\int_0^{\frac{\pi}{2}} e^{t+it} dt$ on the board using the Fundamental Theorem for Line Integrals, in order to illustrate to the class that "Some things are easier in complex!"

On the second day of the integration unit, the instructor introduced the notation $C_1^+(0)$ to indicate a circle of radius 1 centered about the origin, with positive (i.e. counterclockwise) orientation. Later, he established the property that $\int_a^b f(z)dz = \int_a^b f(z(t))z'(t)dt$ by partitioning the interval $[a, b]$ into n segments and illustrating how $z(t)$ maps each of these segments. He then evaluated the integral $\oint_{C_5^+(0)} \frac{1}{z} dz$ using this result, concluding with an answer of $2\pi i$. At this point, the instructor alluded to a later connection about winding number, but did not elaborate much on this. Finally, the instructor wrote two theorems on the board, one of which was the M-L inequality (see Appendix A). The other stated that $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$. The class was informed that the latter result would be proven during the class session, and that the former would follow immediately.

As promised, the third class started with a sketch of the proof of the aforementioned inequality. Following this proof, the instructor acknowledged that "we haven't done too many proofs in this class." He then briefly mentioned that the proof of the M-L inequality follows rather easily from this result. Next the instructor stated the Cauchy-Goursat Theorem (see Appendix A) and asked students if they were familiar with the notion of a simply connected set. Very few students indicated that they were, so the instructor informed the class that this was equivalent to a region being homeomorphic to a disk. More informally, he characterized such regions as having no holes. He then sketched a proof of the theorem using Green's Theorem (see Appendix A).

During the fourth class, the instructor provided another formulation of simply connected, namely that a region is simply connected if its complement on the Riemann sphere is path connected. [Here I note that neither this characterization nor the version referring to homeomorphisms appeared to be explicitly used in subsequent class discussions, which I observed.] After reminding students of the statement of the Cauchy-Goursat Theorem, the instructor then had students evaluate the example integral $\oint_{C_1^+(0)} \frac{1}{z}\,dz$, which had already been introduced on Day 2. At this point, some students incorrectly concluded that this integral should be 0. The instructor then cautioned the students that this integral is not necessarily 0 because $C_1^+(0)$ is not simply connected in this case. Next, the instructor had students evaluate a similar integral, $\oint_{C_1^+(3)} \frac{1}{z}\,dz$. This time, the Cauchy-Goursat Theorem applied, as $C_1^+(3)$ did not include the origin.

The instructor then demonstrated how the integral $\oint_{C_3^+(1)} \frac{1}{z}\,dz$ could be evaluated, namely by rewriting $C_3^+(1) = \Gamma_1 + \Gamma_2$ where Γ_i were two semicircular paths. This allowed the original region to be broken into two simply connected regions. Afterwards, the students were informed that they would have to write the General Cauchy-Goursat Theorem (for multiply connected domains) word-for-word on the next test. Next, the instructor walked students through the evaluation of the integral $\oint_{C_{10}^{+}(0)} \frac{1}{(z-3)(z+4)} dz$, which utilized a partial fractions decomposition. Finally, Day 4 ended with the instructor stating the theorem $\int_{\Gamma} f(z)dz = -\int_{-\Gamma} f(z)dz$.

The fifth day of observations occurred after students' spring break (during which time no classes occurred). This session seemed to appeal most to the formal world of mathematics (Tall, 2013) out of the sessions I observed, as the majority of the class was dedicated to proving several theorems and corollaries. However, I note that these proofs were really just sketches, and replaced formal appeals to epsilon-delta continuity with discussions of "smallness." Before proving the theorems, which I discuss below, the instructor introduced some new terminology, such as a *simple* curve. He also defined a *contour* to be a path that is differentiable.

Next, a student asked about a homework question involving integration of a constant function over a closed triangular path. The instructor drew a picture of the path and briefly described how to parametrize the three sides of the triangle, and told students to finish the problem on their own at home. The instructor then introduced the following theorem about the existence of an antiderivative: *Suppose f is ASCODOD. If $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ where $z_0, z \in D$, and $\int_{z_0}^{z} f(\zeta)d\zeta$ is the integral over any path from z_0 to z and lying in D, then $F'(z) = f(z)$*. Before writing the acronym ASCODOD, the instructor paused to inform the class that this meant *analytic on a simply connected domain D.*

This was not the only time the instructor introduced an abbreviation to represent a rather complicated set of assumptions. He also did this later in this fifth class with the abbreviation SICOPOC (simple, closed, positively oriented contour). As this notation was used often in the course, I was consequently able to bring this language up during my interviews with participants. When the instructor finished writing the statement of the theorem, several students were surprised to see the lack of restrictions on the path mentioned. For instance, one student remarked, "It can really be *any* path?!" and another student asked, "It can cross over itself?" This demonstrates that at least some students in the class were attentive to the assumptions in the integration theorems thus far.

After sketching the proof of the aforementioned theorem, the instructor introduced the following corollary*: If f is ASCODOD and C is SICOPOC in D, and $z_0, z_1 \in D$, and $F'(z) = f(z)\ \forall z \in D$ then $\int_{z_0}^{z_1} f(\zeta)d\zeta = F(z_1) - F(z_0)$.* He then proved this result using the previous theorem, and closed class by stating Cauchy's Integral Formula: Suppose $f(z)$ is ASCODOD, and C is SICOPOC in D, and $z_0 \in Int(C)$. Then $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = f(z_0)$.

During the sixth and final class I observed, the instructor presented the class with the example $\int_C z^i dz$ where C represents the positively oriented semicircular arc from $z = 1$ to $z = -1$. To evaluate this integral, the instructor employed a new branch cut along the negative imaginary axis so as not to intersect the chosen path. The instructor then reminded students of the Cauchy Integral Formula introduced at the end of the previous class session. He emphasized its utility in that it can be used to evaluate integrals without parametrizing paths. He then illustrated this utility with the example $\int_{C_1^+(0)} \frac{\cos z}{z} dz$, because parametrizing this function does not result in a "friendly" expression. However, by applying the theorem, the integral is quickly seen to be $2\pi i$.

The instructor then quickly worked through several additional examples with students, including $\int_{C_1^+(0)} \frac{\sin z}{z} dz$, $\int_{C_1^+(0)} \frac{\sin z}{z-\frac{\pi}{4}} dz$, and $\int_{C_1^+(0)} \frac{\cos z}{z-6} dz$, the last of which is zero by the Cauchy-Goursat Theorem because the point $z = 6$ does not lie inside the path $C_1^+(0)$. Finally, the instructor sketched a proof of the Cauchy Integral Formula using the Extended Cauchy-Goursat Theorem, the M-L Inequality, and an informal continuity argument avoiding $\epsilon - \delta$ statments. In the next subsection, I discuss my data analysis procedures pertaining to these class observations and the other data sources.

Data Analysis Procedures

Given the interview was the primary setting where I could directly detail participants' reasoning about integration of complex functions, these interviews comprised my primary source of data analysis. The other two data sources of videotaped classroom observations and field notes served to contextualize and sometimes triangulate the interview findings, as well as to provide a rich description of the classroom setting described earlier. Hence, the following details refer primarily to my analysis of the interview data. The six steps comprising my interview data analysis for each task are summarized in Table 3 and are detailed afterwards. At the end of this subsection, I also discuss measures taken to ensure the credibility and trustworthiness of my findings.

Given that my research was qualitative in nature, I used qualitative analysis methods to analyze the data, and utilized software such as *Microsoft Excel* to organize my data. As discussed in the theoretical perspective section, Tall's (2013) three-world lens emphasizes the set-before of language as one of three basic innate principles that guide our growth within and between the three worlds. Accordingly, I began the data analysis for the student interviews by transcribing each participant's exact verbiage word-by-word in *Excel*. I also documented any written inscriptions or diagrams produced by participants during the interviews, and noted and described any important gestures made by the participants. I chose to document gestures because they can complement and corroborate students' verbal or written statements, and generally can act as a window into what students are thinking (Keene, Rasmussen, & Stephan, 2012). In the *Excel* document, responses were broken up into natural segments and time stamped for ease of location at a later time. I formed these segments primarily according to extended pauses in verbiage or when a segment lasted longer than roughly one minute. For later reference, I characterize the above portion of the analysis as stage 1.

Table 3

Interview Analysis Summary

Step	Description
1. Transcription	Document participants' exact verbiage; provide rich description of gesture and written inscriptions
2. Code Toulmin components	Classify participants' arguments for each task according to data, warrant, backing, rebuttal, qualifier, and claim as in Toulmin's (2003) model
3. Code for speaker roles	Categorize participants' speaking roles as that of author, relayer, ghostee, or spokesman (Levinson, 1988, as cited in Krummheuer, 2007)
4. Code for three worlds	Further classify participants' arguments from step 2 according to Tall's (2013) three worlds framework
5. Code for backing types	Refine coded arguments from step 3 by categorizing backing according to the types identified by Simpson (2015)
6. Thematic analysis	Reflect on the coded data from steps 1-5 to identify common themes within and across interviews

In the second stage of data analysis, the participants' responses to each of the tasks were coded according to the Toulmin (2003) model of argumentation. Specifically, I identified the data, warrant, backing, modal qualifier, rebuttal, and claim according to the definitions provided in Chapter 2. There were inevitably instances in which several sub-arguments were weaved together to represent participants' reasoning with respect to certain tasks; in these cases, I documented and coded these sub-arguments as data-conclusion pairs chained together as recommended in the literature (Aberdein, 2005; Simpson, 2015).

Stage 3 of analysis consisted of identifying participants' social roles within each collective argument, in an effort to better address my research questions. To do this, I adopted the four speaker roles, outlined previously in this chapter, that were originally formulated by Levinson (1988) and discussed later by Krummheuer (2007). Recall that Krummheuer argued that a speaker need can be autonomous with respect to one, both, or neither of two functions regarding the formulation and content of an utterance. Specifically, a speaker taking on the role of an *author* is both syntactically and semantically responsible for his or her statement, and thus employs both the formulation and content functions. I therefore coded a participant's statement as being authored by that individual if he or she was the first to mention a particular idea in such a formulation.

If a participant claimed responsibility for neither the semantic nor syntactic aspects of an utterance, I classified him or her as a *relayer* of that utterance. In particular, this occurred if one participant recycled a previous statement made by the other participant, and did not apply this statement in a new and different manner conceptually. For example, a participant might restate, in very similar wording, the other student's prior observation that the integrand of a particular function is analytic everywhere. Alternatively, if a participant used "the words of someone else to mean *something different* from the meaning ascribed to the utterance of the original speaker"

(Krummheuer, 2007, italics in original), then I coded this role as *ghostee*. Finally, when a speaker re-voiced a previously mentioned idea using his or her own language, I coded this response under the role of *spokesman*. For instance, a participant might reword a previously vague articulation of the definition of continuity of a function. Together, these four types of speaker roles helped me characterize the social nature of individual participants' contributions within collective argumentation. Accordingly, this stage of analysis assisted me in answering my research questions.

I commenced the fourth stage of analysis by classifying components of the participants' Toulmin argumentation according to Tall's (2013) three worlds framework. For instance, a theoretical warrant citing a proven theorem was characterized under the axiomatic-formal world, whereas an algebraic warrant involving the rules of arithmetic was associated with the operational-symbolic world. A warrant subsumed under the conceptual-embodied world could consist of a visual representation. Similarly, a pictorial form of backing or a gesture referential to a diagram or physical concept fell under the embodied world, and backing in the form of algebraic inscriptions served the symbolic world. Backing in the formal world sometimes consisted of convincing someone that certain hypotheses of a prominent theorem applied, or the statement of a field axiom to support an operation conducted on complex numbers.

Recall from Chapter I that in this report, I identified participants' reasoning as *embodied*, *symbolic*, and *formal* to signify that they were operating within the conceptual-embodied, operational-symbolic, and axiomatic-formal worlds, respectively. When participants' reasoning incorporated multiple worlds, I hyphenate two or more of these labels, such as *embodied-symbolic* reasoning that attends to aspects of both Tall's conceptual-embodied and operational-symbolic worlds. For instance, an appeal to the Cauchy-Riemann equations as a condition for analyticity has a formal aspect to it, but the equations might be verified for a specific function by symbolically computing partial derivatives and verifying that $u_y = v_x$. Thus the backing would ultimately be characterized as *symbolic- formal*. Alternately, an iconic gesture (McNeill, 1992) representing a rotation, produced while verbally discussing multiplication by the number i, might be classified as *embodied-symbolic.* In such a case, I treated the gesture as external evidence of a visualization originating from algebraic operations.

In the fifth stage of analysis, I further classified the coded backing components of participants' argumentation from stage 3 by using the backing categories established by Simpson (2015). Specifically, recall that Simpson delineated three forms of backing to support a warrant: *backing for the warrant's validity,* to explain why the warrant applies to a given argument; *backing for the warrant's field,* to "highlight the logical field in which the warrants are acceptable" (p. 12); and *backing for the warrant's correctness,* to illustrate that a given warrant is actually correct. For example, a participant instantiating backing for a warrant's validity could show that a given function satisfies particular conditions such as analyticity in order to apply a particular integration theorem, i.e. the warrant. Identifying the types of backing my participants used in their argumentation served to help rigorously characterize their reasoning about integration of complex functions.

Finally, through many viewings of the video data, as well as reviewing and interpreting the coded reasoning data, I conducted a thematic analysis (Creswell, 2013) to inductively determine aggregate categories that emerged within and across the two paired interviews. Such an analysis was conducted after primary coding, and was utilized to help “winnow” (p. 186) the data into more manageable chunks. After the interviews were coded and analyzed as previously described, I returned to relevant episodes of the classroom observation video data and field notes in order to either substantiate or negate certain findings from the aforementioned student interview analysis. As a hypothetical example, if a participant provided a rebuttal within his or her argument in the form of a counterexample discussed in class during my observation period, then this would strengthen my understanding of the nature of that student’s reasoning about integration.

Such triangulation amongst multiple sources of data used to confirm emerging findings served to establish credibility and trustworthiness of my study (Merriam, 2009; Patton, 2002). To establish additional trustworthiness, I also maintained a researcher’s journal (Merriam, 2009) documenting various coding decisions I made during the data analysis process. Credibility in this study is bolstered by the inclusion of my researcher’s stance earlier in this chapter as a means of elucidating the inherent reflexivity regarding my role as the primary instrument of data collection in this qualitative research (Merriam, 2009). Finally, I met with my research advisor regularly to discuss my coding for a subset of the data to ensure credible results. A sample excerpt from my codebook for Riley and Sean’s interview is provided in Appendix D. In the next chapter, I detail my results from the interview data.

CHAPTER IV

RESULTS

In this chapter, I explicate the nature of my four participants' nuanced reasoning with respect to integration of complex functions. Specifically, I detail these two pairs of students' collective argumentation as they respond to the integration tasks alluded to in Chapter III and listed in Appendix C. Accordingly, this chapter serves to address the aforementioned guiding research questions:

Q1 How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?

Q2 How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

My presentation of these pairs' reasoning is organized by task, with Dan and Frank's response to each task followed by Riley and Sean's. Because I treat reasoning in the context of this study as collective argumentation within one or more of Tall's (2013) three worlds, I format my results within each task according to argument. Included in my account of each collective argument are: pertinent excerpts of the participants' interview transcript; a Toulmin (2003) diagram summarizing the argument; and figures illustrating participants' gestures or inscriptions, often for the purpose of documenting embodied reasoning. At the end of each task's results, I provide brief summaries comparing and contrasting the two student pairs' responses.

Throughout the transcript pieces presented in this chapter, 'Int.' signals statements that I stated aloud as the interviewer, while 'D,' 'F,' 'R,' and 'S' stand for Dan, Frank, Riley, and Sean, respectively. Bracketed phrases represent non-verbal events such as gestures or written inscriptions produced by the participants. In discussing Dan and Frank's reasoning about each task, I reference line numbers from their transcript excerpts and refer to various components of the Toulmin diagrams I constructed based on my interpretation of their responses. I also convey individual participants' speaker roles germane to each Toulmin component in the collective argument. In the Toulmin diagrams, italicized statements represent participants' exact verbiage from the transcript, while non-italicized statements more succinctly summarize participants' reasoning or deduce implicit Toulmin components based on their verbiage, gestures, and inscriptions, or lack thereof. A parenthetical 'F,' 'D,' 'R,' or 'S' placed prior to the italicized verbiage indicates a statement that the respective participant individually contributed. Horizontal and vertical lines show how argumentation components are linked within a collective argument or subargument. Following the format of Wawro (2015), I represent shifts in the Toulmin categorization from one type of component to another (such as claim to data) in the figures by a diagonal line.

Part I

Recall from Chapter III that I designed the first portion of the interview to elicit participants' foundational understandings with respect to integration of complex functions, including parametrization of paths, the Fundamental Theorem for Line

Integrals, and geometric interpretations of these and related concepts. At the end of this first portion, I asked them about the integral of a specific function (see Appendix C).

Task 1 – Dan and Frank

Parametrization is a central concept in the definition and evaluation of integrals of complex functions, and typically instructors introduce integration of complex functions using related notions from multivariable calculus. Accordingly, I began part 1 of the interview by asking Dan and Frank how they generally think about parametrization (lines 1-3). Because this question was not designed to elicit an explicit argument, I focus the discussion of this brief task on only the participants' use of Tall's (2013) three worlds. Dan responded first to this question, and provided a solely embodied explanation of parametrization (lines 4-5). Specifically, his dynamic verbiage characterized parametrization as a means to describe the motion of an object over time, and his tracing gesture (see Fig. 4) complemented his verbiage by illustrating one such hypothetical path.

Figure 4. Dan's gesture tracing a hypothetical path to illustrate parametrization.

Afterwards, Frank elaborated on Dan's physical description to include the language of functions (lines 6-7). Due to the fact that he did not write any inscriptions nor gesture while speaking, it is difficult to ascertain with certainty which world or worlds

Frank was operating within at this moment. In particular, while he alluded to a function of a single variable as a representation of a path, a function can be represented in many manners (Hitt, 1998). Accordingly, while his mention of a path in two or three dimensions suggests some level of embodiment, from his statement alone, one cannot clearly identify whether or not Frank meant a symbolic representation of a function. Finally, I commented that while usually this "single variable" (line 7) tends to represents time, it need not in general (line 9).

Int: Ok so the first question is relatively general but it deals with topics in integration, at least tangentially. So what do you guys believe the word "parametrization" signifies when describing a complex path integral?
D: I'd say you're finding out some way to describe how the path, like how an object travels along a path [traces a hypothetical path with his marker in the air, while saying "travels along a path"] with respect to time, right?
F: Sure, yeah. Uh, I guess, some function that represents the curve with respect to a single variable, in three dimensions or two dimensions, whichever is relevant.
Int: Ok yeah great. And often that's time, but it certainly doesn't have to be.

Task 1 – Riley and Sean

Riley and Sean's response to Task 1 was quite similar to Dan and Frank's in many respects. Just as Dan began their response with an embodied description considering the motion of a path over time, Riley began with "I always think of parametrization in terms of […] time" (line 6). She elaborated with a more symbolic statement about a single-variable description, as opposed to one involving the two variables x and y (lines 6-7). Subsequently, I asked Sean if he wanted to add anything to Riley's response (line 8), something I did not need to do with Dan and Frank.

Int: So there's— I kind of intended to break this up into 2 sessions, just based on the type of material in there, but um, they're not necessarily of equal length, and like I said, it's possible that you will get through things a lot faster than the time allotted. I just wanted to make sure there's enough there. So just starting off with a general question, um, what do you believe the word 'parametrization' signifies when describing a complex path integral?

R: Uh, so, I always think of parametrization in terms of, uh, like time. So it allows you, basically, to describe a path in terms of just one variable instead of x and y. So yeah, ok.

Int: Ok, anything you want to add [Sean]?

S: I mean, it's like what we did in Calc 3, or Multivariable Calculus. You just set up a path through x, y, and z. It's just a path through the complex plane of x and y [traces hypothetical path through the air with his pen]. It's kind of necessary if you do complex integrals through a "two-dimensional" space.

Sean added that "it's like what we did in Calc 3" (line 9). In particular, he likened a path in the multivariable calculus setting to "a path through the complex plane of x and y" (line 10). Just as Dan did, Sean traced a hypothetical path through the air, using the tip of his whiteboard marker (see Fig. 5), depicting an embodied visualization of this path (lines 10-11). Sean closed by stating that parametrization is necessary when evaluating complex integrals "through a 'two-dimensional' space" (line 11), and he gestured "air quotes" while he said "two-dimensional." Ultimately, the primary difference between Riley and Sean's response versus Dan and Frank's was that Sean was the only person to mention a complex integral.

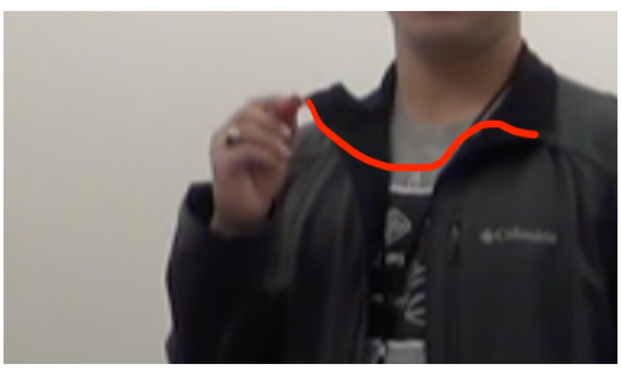


Figure 5. Sean's gesture tracing a hypothetical path to illustrate parametrization.

Task 1 Summary

During this first task, neither pair of participants incorporated an official argument in their respective responses. However, Dan and Sean produced nearly identical embodied tracing gestures depicting a hypothetical path to illustrate the notion of parametrization. Neither pair of students discussed parametrization in much detail or in much formality, but I anticipated this might be the case given the relatively informal nature of many undergraduate complex variables courses.

Task 2 – Dan and Frank

The second task also served as a warm-up to the eventual evaluation of specific integrals, and required participants to provide a short but precise argument about how to represent $z(t) = e^{it}$ as a position vector of a moving point in the complex plane (lines 1-3). I anticipated that this task would complement participants' descriptions from the first task, although this was not the direct purpose of this task. I also explicitly asked Dan and Frank to identify the two components of their vector as part of their response (line 5). Frank began the pair's argument by relaying my spoken task setup as a symbolic datum (line 4), neither modifying the syntactic nor semantic nature of the given statement.

Int: Ok great— so something a little bit more specific then, but kind of related. Explain how you would represent $z(t) = e^{it}$ as a position vector of a moving point in the complex plane, using vector component notation.
F: [Writes $z(t) = e^{it}$] Okay.
Int: And so, at some point, talk about what the two components would be.
F: Sure.
D: So that's just a circle.
F: So we can just use, um, Euler's identity for this. So we'd end up with e^{it} is $cos\ t + i\ sin\ t$ [writes $e^{it} = cos\ t + i\ sin\ t$]. So in vector form, we're going to have cosine t, sine t [writes $< \cos t, \sin t >$ on board]. The i just tells us that it's basically the y-axis [traces vertical line in the air with marker pen]. Um, so that's how I'd represent it. Do you want to—?
D: No, that's— yeah I agree.
Int: Cool.

Using this datum, Dan authored a claim that $z(t)$ represents a circular path. Accordingly, he used embodied-symbolic reasoning, in that his verbiage connected a symbolic expression to a geometric object. With his prior symbolic datum and Dan's claim in mind, Frank proceeded as author to articulate a suggestion for a formal warrant, Euler's identity (line 8). Using formal-symbolic reasoning, he elaborated the statement of the formal warrant by writing the symbolic inscriptions $e^{it} = \cos t + i\, sin\ t$ (lines 8-9).

Subsequently, Frank evidenced symbolic reasoning to author a claim that the equivalent vector form of this result has real and imaginary components $\cos t$ and $\sin t$, respectively (lines 9-10). Finally, he clarified the warrant's connection to the claim by highlighting that the imaginary component could be identified as the term containing i (line 10). In discussing the imaginary axis, which he referred to as "basically the y-axis" (line 10), Frank traced an ostensibly visualized imaginary axis in the air with the palm of his hand. This gesture (see Fig. 6) and accompanying verbiage appeared to suggest embodied-symbolic reasoning, in that Frank's gesture enacted a visualized geometric object and his verbiage connected this geometric object to his symbolic inscriptions. Dan and Frank's argument for Task 2 is summarized in Figure 7.

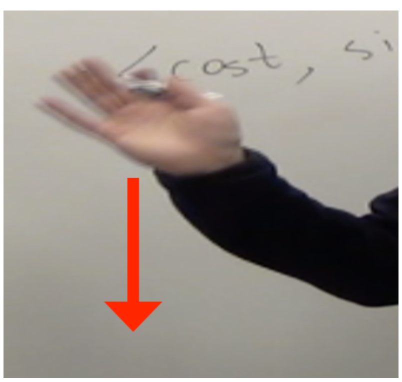


Figure 6. Frank's gesture tracing a hypothetical imaginary axis in Task 2.

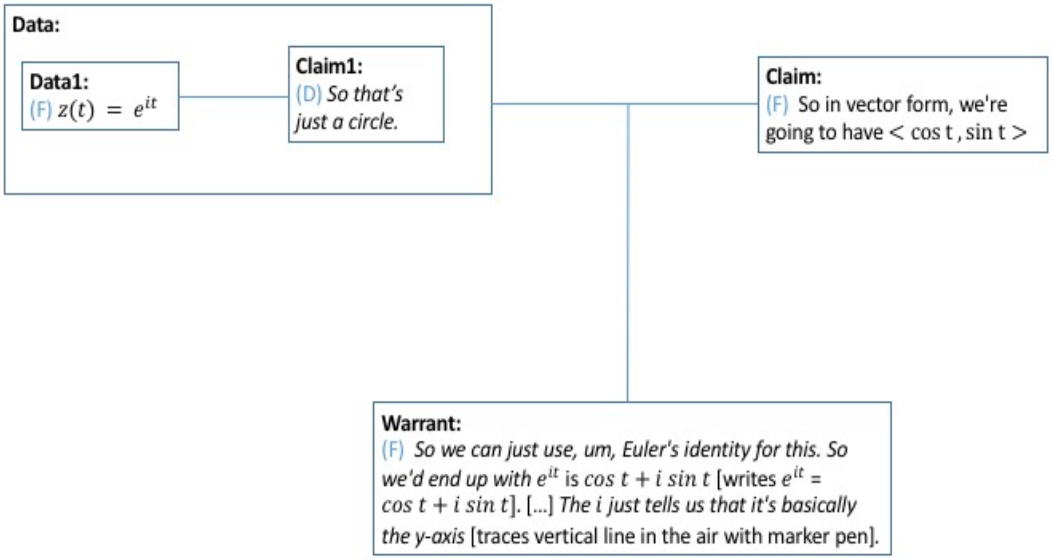


Figure 7. Dan and Frank's Toulmin diagram for Task 2.

Task 2 – Riley and Sean

Exactly as Frank began Task 2, Sean relayed the given datum that $z(t) = e^{it}$ by writing this information as a symbolic inscription on the whiteboard (line 5). Sean immediately rewrote this symbolic expression as $\cos(t) + i\, sin(t)$, implementing Euler's identity as a warrant for his subsequent claim (line 5). In particular, as spokesman, he characterized $z(t)$ as a "unit vector," and symbolically claimed $v = < x(t), y(t) > = < \cos t\,, \sin t >$ (lines 5-6). Because Sean had written several statements with only minimal accompanying verbiage, and Riley had not spoken at all, I reminded them that I wanted them to verbalize their thoughts and discuss the tasks with each other, when possible (lines 7-9). I then brought their attention back to the task at hand by reminding them that Sean had written the two vector components (line 10).

Sean continued by sketching an Argand Plane and unit circle on the whiteboard (line 11). He authored an embodied claim that the circle he just sketched is "just a unit circle, radius 1" (line 11). As spokesman, he also clarified that the two functions $\sin t$ and

$\cos t$ collectively describe an arbitrary point using the vector v (lines 11-13). Sean's clarification instantiated embodied-symbolic reasoning, in that he discussed how the symbolic component functions comprise the vector drawn on their diagram. Sean's sketch, including the vector v, is depicted in Figure 8 below.

Int: Mhm, yeah ok. Sounds good. Um, okay so something a little bit more specific. So, so why don't you explain how you would represent $z(t) = e^{it}$ as a position vector of a moving point in the complex plane using vector component notation. And in doing so, what are the two components? So if you want me to repeat anything any point— or of course feel free to write things down.

S: [Writes $z(t) = e^{it} = cos(t) + i\,sin(t)$]. So as a unit vector: [writes $\vec{v} = < x(t), y(t) > = < cost, sint >$].

Int: So just— as much as possible, while you're thinking things through and writing things down, just uh— if you have any thoughts about what you're thinking during the process, just feel free to mention those. And as much as possible, discuss things with each other.

Int: So you have your two components here.

S: [Draws Argand plane and unit circle] Yep, this is just a unit circle, radius 1. And these [points to component function inscriptions] describe a random point with this vector [draws in position vector labeled v].

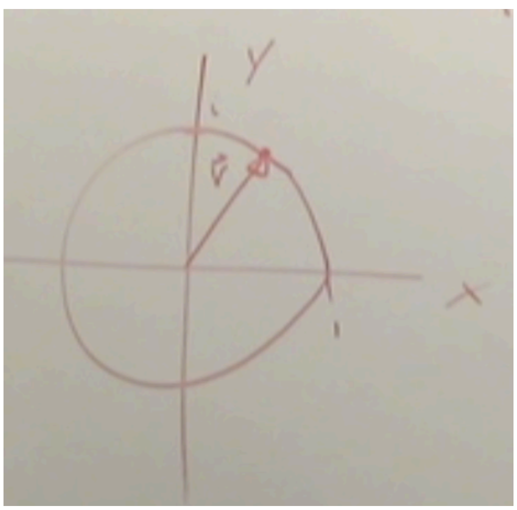


Figure 8. Sean's initial diagram of the unit circle and vector v in Task 2.

R: I mean, it corresponds to just, a normal unit circle— you know, you have $cos(a)$ is this side [traces finger along real direction] of it [the vector v] and $sin(a)$ is this side of it [traces finger along imaginary direction].

S: [Draws in an angle for the vector v and labels it "t". Writes $0 \leq t \leq 2\pi$.]

R: So it's not that much of a leap to put it in the complex plane and be like, "Hey so now instead of having just x and y in the normal unit circle, you are dealing with *x* and *y*, and *y* happens to be the imaginary component, so—

Int: Alright. Ok perfect. Yeah and you [Sean] have t going from 0 to 2π there.

S: Yeah.

Afterwards, Riley began to articulate a second warrant for Sean's claim. Using embodied reasoning, she discussed how a "normal unit circle" in $\mathbb{R}^2$ has x-component $\cos{(a)}$ and y-component $\sin{(a)}$ (lines 14-16). While identifying the two components, she traced her finger along the horizontal axis and the vertical axis in their diagram, respectively (see Fig. 9). Notice the similarity between Riley's gestures and Frank's from Figure 9. In particular, both Riley and Frank used their hand to trace along a vertical axis to illustrate the imaginary component of the vector form of $z(t)$. The primary distinction between their corresponding gestures was that Frank did not have a diagram to reference. Accordingly, Riley's gesture referenced motion along an existing diagram, while Dan's had to incorporate visualization of the complex plane as well.

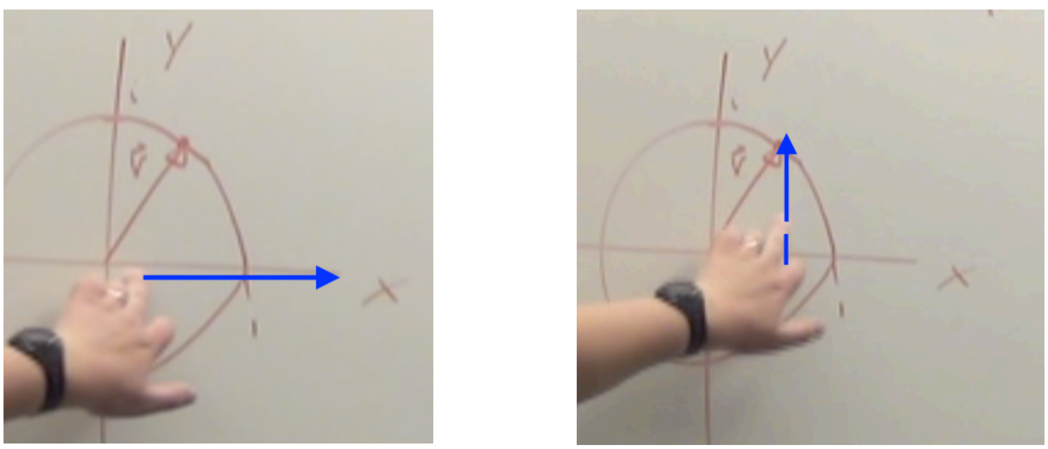


Figure 9. Riley's gestures tracing along the real axis (left) and imaginary axis (right).

Meanwhile, Sean drew additional geometric inscriptions on their diagram, namely an angle for the vector v, which he labeled t (line 17; see Fig. 10). He also wrote symbolic inscriptions clarifying that $0 \leq t \leq 2\pi$. Using formal-embodied reasoning, Riley continued to articulate her warrant, explaining that her characterization of the unit circle in $\mathbb{R}^2$ generalized naturally to the complex plane (lines 18-20). As such, her warrant represents an instantiation of Danenhower's (2000) "Thinking Real, Doing

Complex." While Danenhower's notion is usually discussed in a pejorative connotation, it should be noted that Riley's application here was actually appropriate and seemingly helpful for her in the transition from $\mathbb{R}^2$ to $\mathbb{C}$. Riley and Sean's argument for Task 2 is summarized in Figure 11.

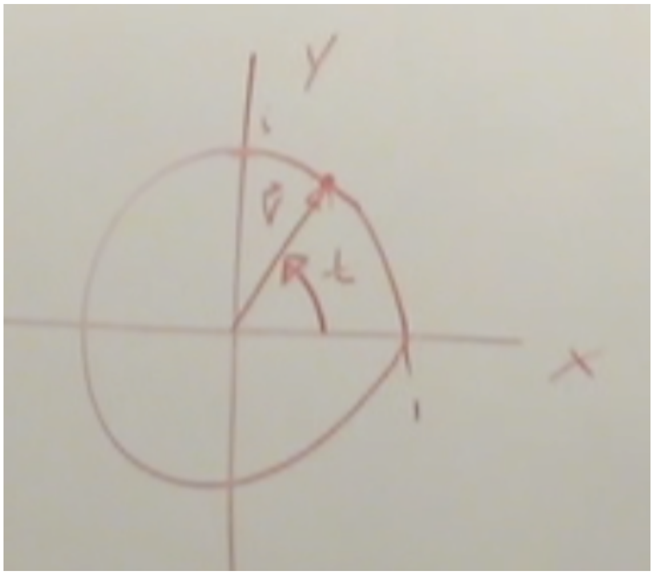


Figure 10. Sean's revised diagram with angle t in Task 2.

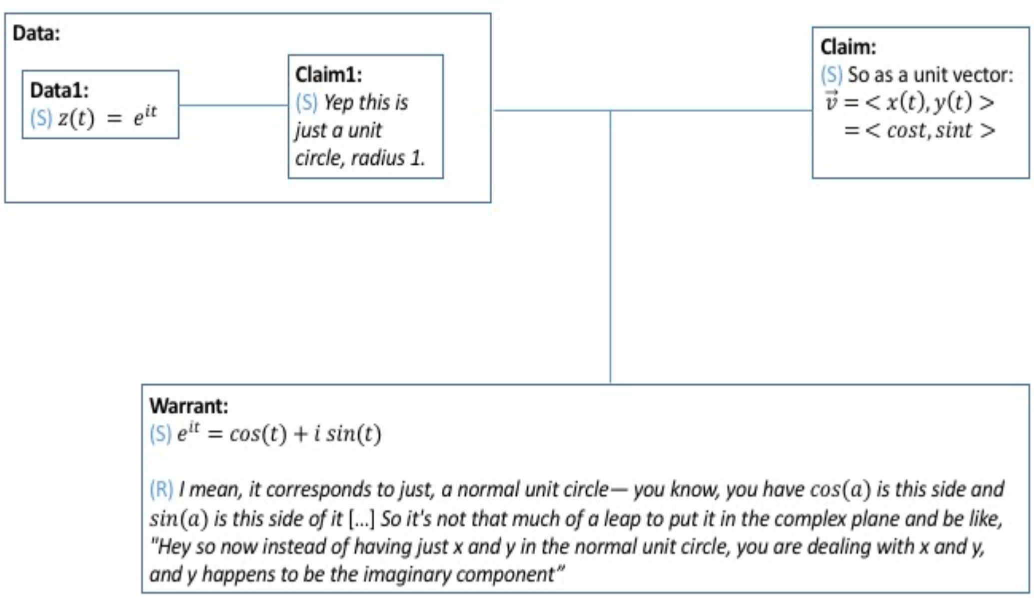


Figure 11. Riley and Sean's Toulmin diagram for Task 2.

Task 2 Summary

One notable difference between Dan and Frank's response to Task 2 and Riley and Sean's was that the latter pair incorporated more embodied reasoning, instantiated primarily in their diagram of the circular path and corresponding vector $\vec{v}$. Both Frank and Riley produced similar tracing gestures to complement their verbiage when discussing the two vector components. However, Riley gestured about both components, while Frank's gesture only alluded to the imaginary axis. Finally, although both pairs of students recognized $z(t) = e^{it}$ as a circle, only Riley and Sean explicitly identified the radius as having unit length. The two responses were rather similar otherwise.

Task 3 – Dan and Frank

Task 3 required participants to provide a physical description, along with a diagram, of the derivative $\frac{dz}{dt}$ at a point for a generic function $z = f(t)$ (lines 1-2). Accordingly, I expected their response to primarily incorporate embodied reasoning. Ultimately, this third task resulted in two arguments from Dan and Frank, the first of which is depicted in Figure 12. Dan began this first argument by authoring a claim that $\frac{dz}{dt}$ represents the "amplitwist," (lines 4-5) a notion discussed in Dan and Frank's class.

Int: Sounds good. Ok so more generally, if $z = f(t)$ is a parametrized curve, what does dz/dt represent physically at each point? And then maybe you could think about how we might draw this.
Int: So at any point, too, if you want me to repeat questions or anything, just ask.
D: So at any point, it'd represent the— I guess we called it the amplitwist in our class. So it's the change in length of the vector, and change of direction of the vector.
F: Right, that it's changing its argument and its magnitude. Because you can't really say velocity, I guess, in the context of complex numbers [looks uncertain about this statement and seems to look at me for validation], would be my understanding. But that doesn't necessarily work. Um, so I guess we should write this [turns to board to start writing $z =$].
Int: Yeah, so z equals $f(t)$.

Hence, Dan's reasoning about this claim could best be identified as embodied, in that his verbiage described changing the length and direction of a vector as geometric attributes. Acting as spokesman, Frank re-voiced Dan's claim about the amplitwist to include the words "argument" and "magnitude" (line 6). Frank continued with a warrant, "Because you can't really say velocity […] in the context of complex numbers" (lines 6-7). Due to Frank's hesitation about hastily generalizing properties of real-valued functions to complex functions, it appeared that he attempted to avoid the issue of *thinking real, doing complex* (Danenhower, 2000). After authoring this warrant, Frank looked to me for validation, and appeared uncertain about his statement. After a pause, he articulated this uncertainty with the phrase, "would be my understanding" (line 8).

Continuing to express doubt about their argument, Frank continued, "But that doesn't necessarily work. Um, so I guess we should write this" (lines 8-9). It is unclear from just this passage whether Frank meant "that" as the warrant or the claim, but because he suggested that he and Dan write down some inscriptions to assist their reasoning (lines 8-9), I did not interject. Frank's suggestion that they write down some inscriptions catalyzed the beginning of a second argument, Argument 2, which is depicted in Figure 14. Using the given datum that $z = f(t)$, Frank symbolically concluded that $\frac{dz}{dt} = f'(t)$, while Dan drew a coordinate plane with real and imaginary axes (line 11). Argument 1 is summarized in Figure 12.

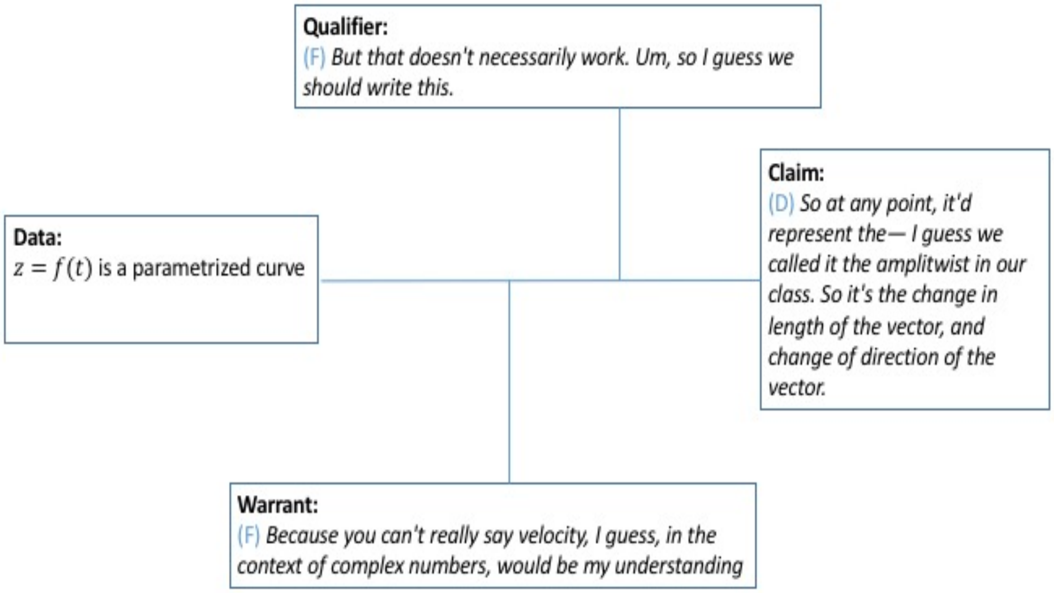


Figure 12. Toulmin diagram for Dan and Frank's Argument 1, Task 3.

F: Yeah, so [writes $z = f(t)$, as Dan draws coordinate plane] we have [writes $dz/dt = f'(t)$].
D: So if we have, like some type of path— I don't know, I'll just draw a path [draws squiggly path] and then we're trying to evaluate $f'(t)$ at a point, I guess that would just be some type of— hmm [pauses] I'm not sure how I'd describe it.
D: I think it would just be like—
F: I mean, wouldn't it still be tangential?
F: Cuz this— this [points to $f'(t)$ inscription] is going to be another vector. Right? Cuz this is just a vector [points to $f(t)$] so this is a vector [points to $f'(t)$]. If I had to guess, I would say this is tangential to the curve.
F: I'll use a different color [grabs new marker and starts drawing tangent vector to curve] So something like that. Um, but I'm honestly not sure. I think of it more in terms of an amplitwist than a [air quotes gesture with fingers] velocity vector.
D: Yeah—
F: I don't know if that's honestly correct.

Next, Dan sketched a path representing $z = f(t)$, instantiating embodied reasoning by authoring a diagrammatic datum (line 12; see Fig. 13). However, he was unable to fully articulate a claim regarding how this sketch helped depict the nature of $\frac{dz}{dt}$, as evidenced by the qualifying phrase "I'm not sure how I'd describe it" (lines 13-14). As Dan further attempted to articulate a claim (line 15), Frank interjected with a question as to whether $\frac{dz}{dt}$ represented something tangential (line 16). Frank's verbiage, especially his choice of the word "still" (line 16), suggested that he potentially invoked prior knowledge about real-valued functions to offer this conjecture. He authored an embodied-symbolic warrant to connect his prior symbolic inscriptions to his conjecture describing a physical property on the drawn diagram (line 17).

Subsequently, Frank authored backing for this warrant's correctness by elaborating that $f(t)$ is a vector, and hence $f'(t)$ is a vector (lines 17-18). Using a new colored marker, he drew in a tangent vector to Dan's curve (see Fig. 13) and reformulated his prior conjecture as a tenuous claim (lines 18-21). I say "tenuous" because Frank revealed that he was not certain of his conclusion, admitting "I'm honestly not sure" (line 21). Moreover, he mentioned that he thinks of $\frac{dz}{dt}$ more in terms of an amplitwist than as a velocity vector, and while saying the word "velocity" he gestured using "air quotes" to indicate a potentially loose interpretation of the word. Dan agreed (line 23), and Frank once again expressed uncertainty about the claim (line 24).

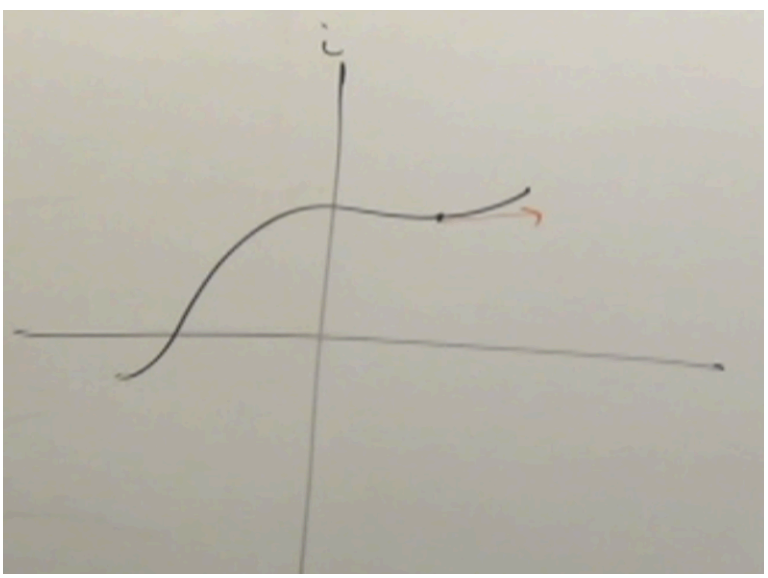


Figure 13. Sketch of $z = f(t)$ and Frank's tangent vector dz/dt in Task 3.

Perhaps comforted by the fact that Frank was also not sure how to proceed, Dan authored a second warrant: "we're taking the derivative of a function of time, not a—"

and Frank finished his sentence with "Not of a complex, yeah" (lines 25-26). This joint warrant appeared to represent symbolic reasoning, in that they used the nature of the symbolic inscription of the function to decide whether $\frac{dz}{dt}$ should be represented as an amplitwist or as a tangent vector. This realization also prompted Dan to conclude with more certainty that $\frac{dz}{dt}$ indeed represents a tangent vector, and he pointed to the recently drawn tangent vector in their diagram (line 27). Frank now agreed to this and with more certainty as well (line 28).

25 *D:* Yeah cuz we're taking the derivative of a function of time, not of a—
26 *F:* Not of a complex, yeah.
27 *D:* So I guess it *would* be just tangential, like that [points to picture on board].
28 *F:* Yeah.
29 *Int:* So what sort of object, then, is that? So you said it's tangential somehow? What sort of an object is
30 that? Are we talking about—
31 *F:* We're talking about another vector.
32 *D:* Yeah a vector.
33 *F:* Yeah, I mean sure it's a point, right? Cuz points and vectors are the same in complex numbers. But
34 yeah, I would think of it as a vector tangential to the curve at that point.
35 *Int:* Ok, so the orange thing you drew in there.
36 *F:* Yes.

Because neither Dan nor Frank had explicitly referred to a tangent vector, only of an object "tangential" to the curve, I asked a clarifying question about what type of object $\frac{dz}{dt}$ was (lines 29-30). Frank clarified that "We're talking about another vector" (line 31), and Dan quickly agreed (line 32). Finally, Frank discussed how one could also think of the object as a point, given that "points and vectors are the same in complex numbers" (line 33), but he thought of it as a tangent vector in this instance. I asked if his description corresponded to the orange vector drawn in the diagram (line 35), and Frank confirmed this (line 36). Argument 2 is summarized in Figure 14.

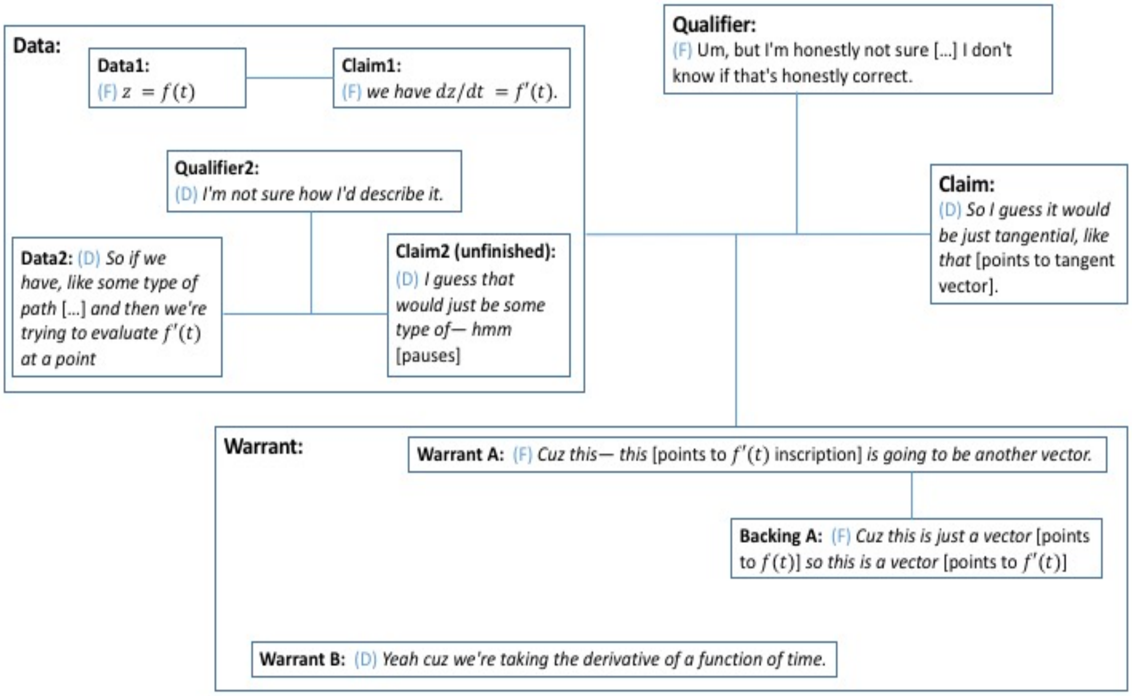


Figure 14. Toulmin diagram for Dan and Frank's Argument 2, Task 3.

Task 3 – Riley and Sean

As he did at the beginning of Task 2, Sean relayed the task information that I read aloud by writing corresponding symbolic inscriptions on the white board as data (lines 1-2). This began the first of two arguments related to this task. As spokeswoman, Riley implemented embodied-symbolic reasoning to reiterate that z is a parametrized curve and sketched such a curve on the board (lines 3-4; see Fig. 15). Using these data, she authored a claim that "dz/dt is sort of breaking it into little chunks" (line 5).

Int: Ok sounds good. Um, ok so next, if $z = f(t)$ is a parametrized curve [Sean writes $z = f(t)$], what does dz/dt represent physically [Sean writes dz/dt] at each point, and how would you draw this?
R: So let's see. So you have, like, z is a [draws curve on board]— it's like some curve, right? It's parametrized. Um so like, dz/dt is sort of breaking it into little chunks. If we're at this point [draws a point on the curve], I guess [pauses] um, it would just be a little directional kind of infinitesimal um, pointer, [draws in tangent vector at this same point] that says where we're going along this curve. So if your curve is oriented this way—
S: Tangent vector.
R: [Draws arrows on curve] then dz/dt would look like a little vector pointing off [gesture with hand of a tangent vector] to where the next, uh, z is. It's not like actually a tangible concept, because it's infinitely small, but that's how I think of it.

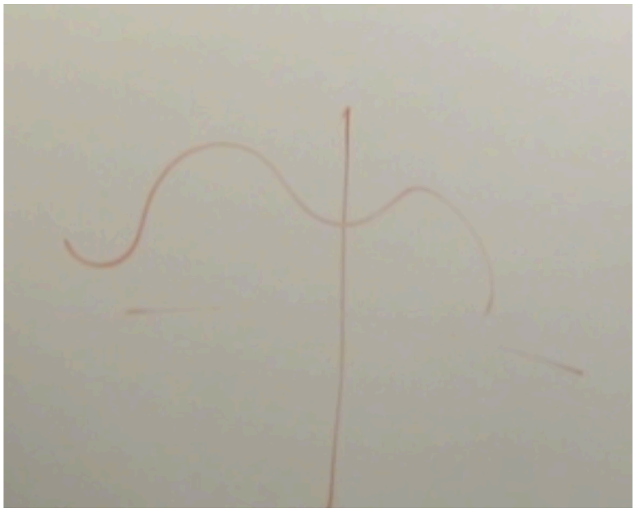


Figure 15. Riley’s initial sketch of the curve $z = f(t)$ in Argument 1, Task 3.

Next, Riley plotted a specific point on the curve as embodied datum, and concluded that dz/dt represented “a little directional kind of infinitesimal um, pointer” (lines 4-6). She drew in a small tangent vector at this same point (see Fig. 16), and provided an embodied addendum that this vector “says where we’re going along this curve” (line 6). Riley also qualified this assertion with the phrase “I guess” (line 5). Using embodied reasoning, she considered orienting the path as a datum, and drew in directional arrows on her path to indicate this orientation (lines 6-9; see Fig. 16).

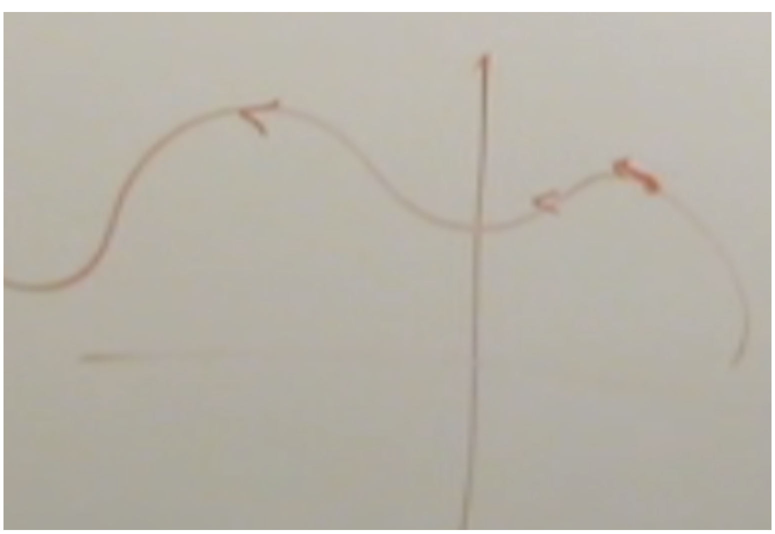


Figure 16. Riley’s revised sketch including path orientation and a tangent vector.

Meanwhile, as spokesman, Sean succinctly re-voiced Riley's description of dz/dt with the phrase "Tangent vector" (line 8). Because she was mid-sentence, Riley did not acknowledge Sean's comment, but instead articulated an embodied claim as spokesman. Specifically, she stated that "dz/dt would look like a little vector pointing off to where the next, uh, z is" (lines 9-10). She also provided an embodied gesture as she spoke the words "a little vector pointing off," using her open hand to point in a hypothetical direction based off an ostensibly visualized path (see Fig. 17). I assume she is visualizing a different path because her gestured vector points in the opposite direction of her drawn path's orientation, and she did not produce this gesture in close proximity to the actual diagrammatic inscriptions (though it is hard to tell this in Fig. 17). Riley closed Argument 1 by authoring an embodied qualifier that "It's not actually a tangible concept, because [dz/dt] is infinitely small, but that's how I think of it" (lines 10-11). Argument 1 is summarized in Fig. 18.

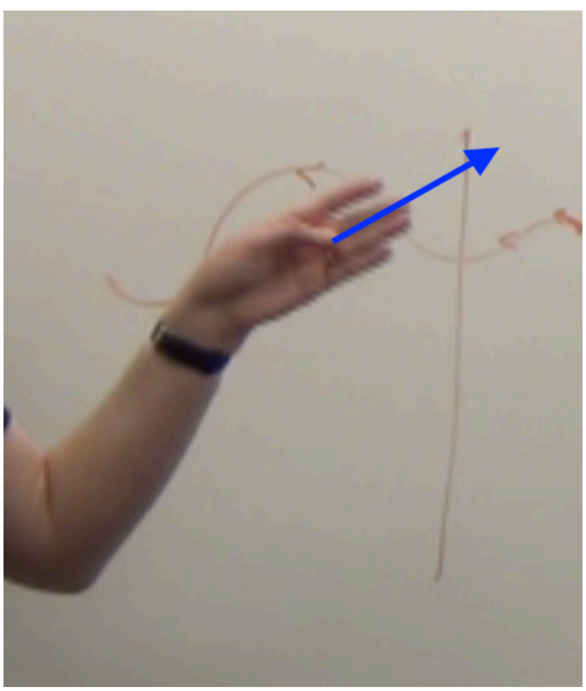


Figure 17. Riley's gesture for "a little vector pointing off" in Argument 1, Task 3.

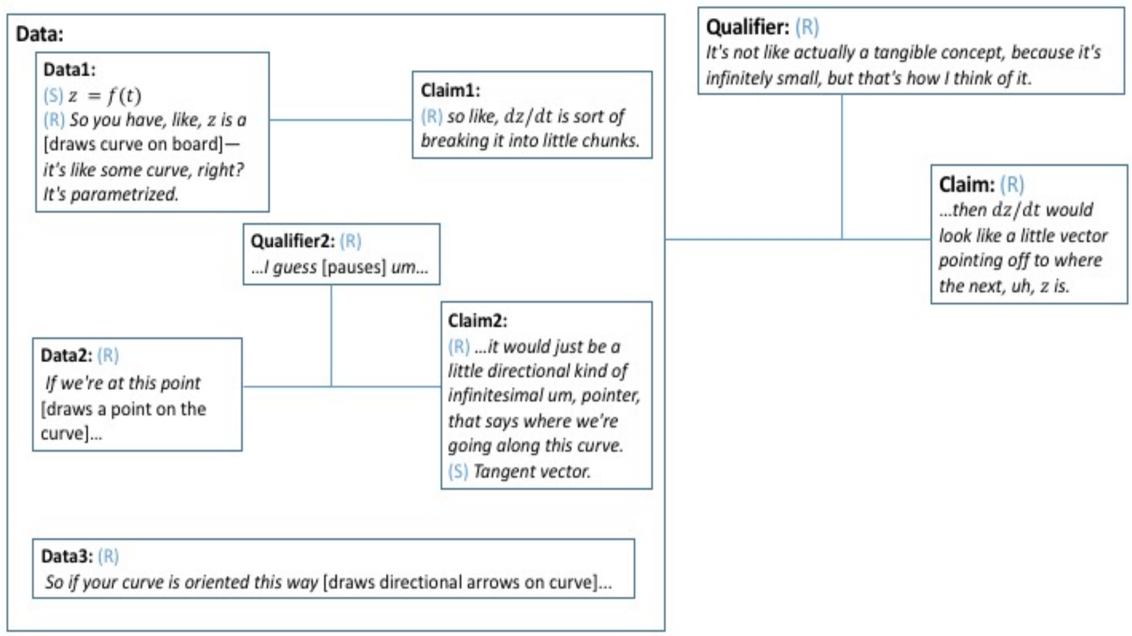


Figure 18. Toulmin diagram for Argument 1, Task 3.

S: Yeah if you think about tangent vectors— so, for this [points at diagram from Task 2] we would get, like, $z'(t) = -\sin t + i\cos t$ [writes this], and then our unit vector is, like, in this direction [draws in green unit vector on circle]. And we could call our tangent vector *T*, I guess.
R: Yeah, it's going to be, like— it should be parallel to the slope of the line at that point [traces finger along her path for this task].
Int: And so if you— if this is representing, like, the physical path of an object or something, does that tangent vector tell you anything physically about what's going on then?
S: Velocity.
Int: So it's your velocity vector?
S: Yeah.

Afterwards, Sean began a second argument by authoring an embodied datum considering what a tangent vector would look like in their diagram from Task 2, which was still on the board (line 12). Switching to embodied-symbolic reasoning, he authored a claim that $z'(t) = -\sin t + i\cos t$ (lines 12-13). Using this claim as datum, he authored a new embodied claim concerning the direction of the tangent vector (lines 13-14). He drew in a green tangent vector on their previous diagram of the circular path to illustrate this claimed direction (lines 13-14; see Fig. 19). As spokesman, he then labeled this tangent vector T (line 14).

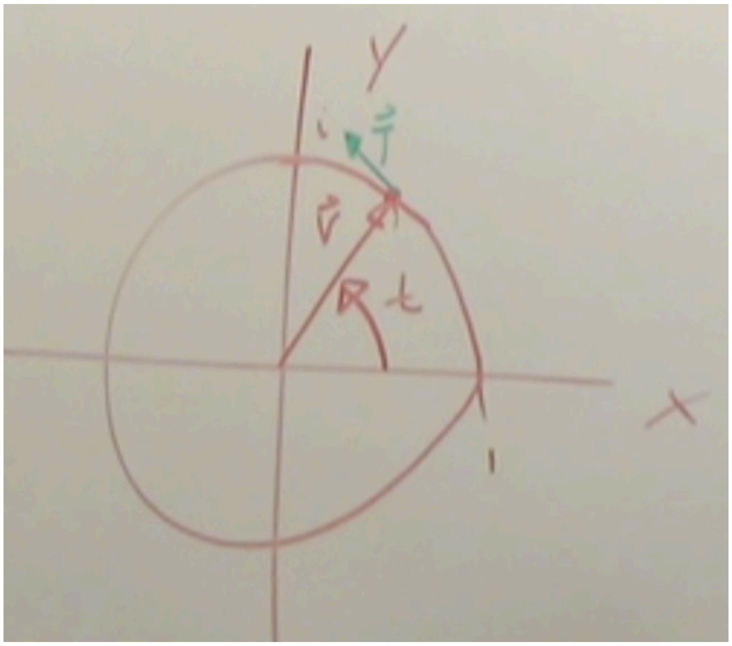


Figure 19. Sean's added green tangent vector T on a diagram from the previous task.

To corroborate Sean's claim, Riley authored an embodied warrant that the tangent vector "should be parallel to the slope of the line at that point" (line 15). While she spoke these words, she also produced a tracing gesture along her drawn path (see Fig. 20). Although this gesture did not refer to Sean's vector T drawn on the Task 2 diagram, it appeared to embody a universal quantifier, signifying the slope of the (tangent) line at every point along her oriented path. Because Riley and Sean had not explicitly provided a physical interpretation of dz/dt, and because of Riley's qualifying statement from Argument 1 that dz/dt "is not like actually a tangible concept" (line 10), I asked them about the physical meaning (lines 17-18). Sean quickly replied with an embodied claim, "Velocity" (line 19). To make sure I correctly understood him, I re-voiced his response with the question, “So it’s your velocity vector?” (line 20), and he confirmed this (line 21). Argument 2 is summarized in Figure 21.

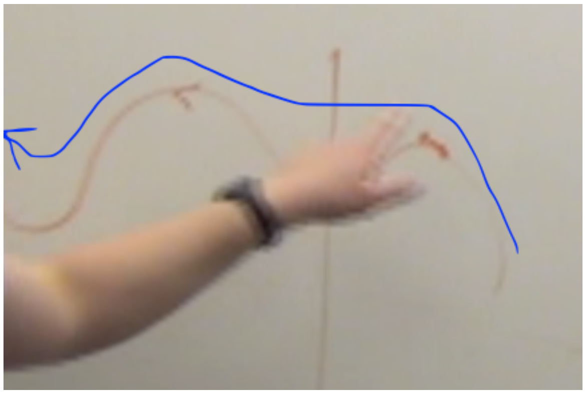


Figure 20. Riley’s tracing gesture along her diagram in Argument 2 of Task 3.

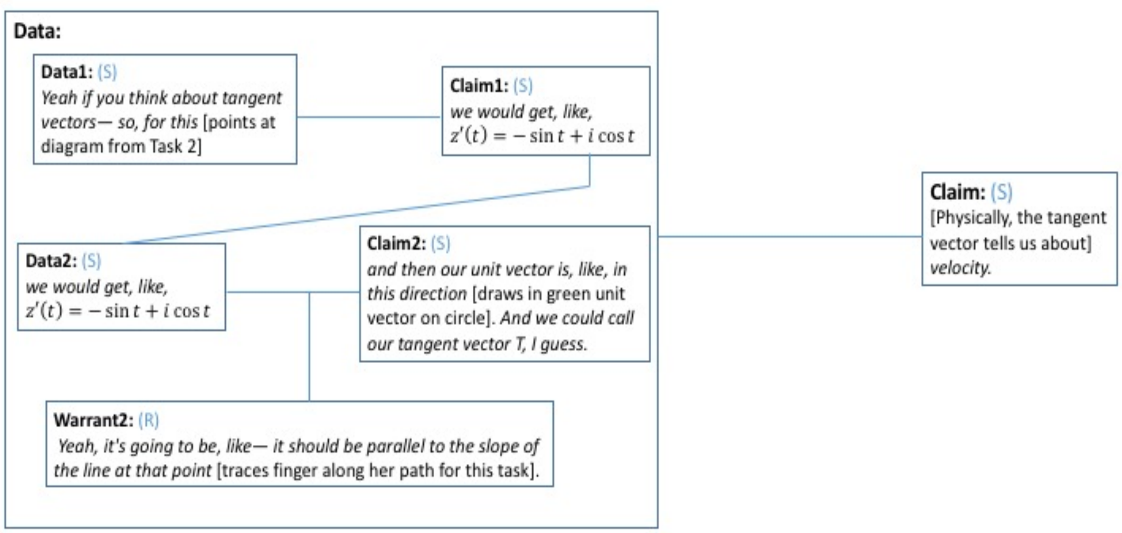


Figure 21. Toulmin diagram for Riley and Sean’s Argument 2, Task 3.

Task 3 Summary

Note that there are several key differences between Dan and Frank’s response to Task 3 versus Riley and Sean’s. One primary distinction is that Dan and Frank spoke about dz/dt as an amplitwist, which was an instantiation of “Thinking Real, Doing Complex” (Danenhower, 2000), while Riley and Sean did not. Another difference between the participants' responses was that Sean chose to draw a tangent vector on the Task 2 diagram. Finally, Riley provided several embodied gestures to accompany her and Sean's verbiage and diagrams, whereas Dan and Frank did not.

Task 4 – Dan and Frank

The fourth task (see Appendix C) required participants to supply a geometric interpretation of the identity $\int_a^b \frac{dz}{dt} dt = f(b) - f(a)$, where $z = f(t)$ is a parametrized curve described as a complex function of t (lines 1-4). Because this task explicitly asked for a "geometric interpretation," I anticipated that this task would elicit primarily embodied and embodied-symbolic reasoning. However, Dan and Frank's first argument consisted nearly entirely of symbolic reasoning. A Toulmin diagram for Argument 1 is depicted in Figure 22.

Frank began their response by writing a symbolic inscription corresponding to the identity that I read aloud to them, however he initially denoted the function using a capital letter F. Shortly after, he changed his mind and rewrote the statement using a lower-case f, explaining that "we don't need an antiderivative" (line 6). When I asked him why he originally thought about an antiderivative (line 7), Frank clarified that he initially interpreted my verbiage as an integral of the function $z = f(t)$, as opposed to dz/dt. Treating the integrand as $f(t)$, he presumed that my statement "$f(b) - f(a)$" used an antiderivative F of the function f, and instead wrote "$F(b) - F(a)$" (lines 8-9). But after realizing this discrepancy between my intended symbolism and his initial interpretation of the task, he concluded that "we obviously don't need an antiderivative" (line 10). Frank authored a warrant for this claim as well, explaining "because we're integrating dz/dt with respect to time" (lines 9-10).

Int: Ok cool. Sounds good. Um, ok so um, keeping with this same general sort of setup, um, we're considering $z = f(t)$ is a parametrized curve described as a complex valued function of t. How would you provide a geometric interpretation of the identity— and the identity is— the integral from a to b of $dz/dt\ dt = f(b) - f(a)$?
F: [Writes the identity on the board, and uses $F(t)$ instinctively].
F: Not a capital F, cuz we don't need an antiderivative. [Erases $F(b) - F(a)$ and rewrites with f]
I: Well so what made you think of antiderivative off the bat?
F: Well so at first I made the faulty assumption that we're actually integrating the function z, not dz, in which case we'd need some kind of antiderivative of the function f. But because we're integrating dz/dt with respect to time, we obviously don't need an antiderivative.
F: Uh, ok. So—
D: Shouldn't it just be the same as integrating dz?
F: Yeah so it is effectively equivalent to doing this [writes $\int_a^b dz = z\,\big|_a^b = f(b) - f(a)$]. But in terms of a geometric interpretation of that?

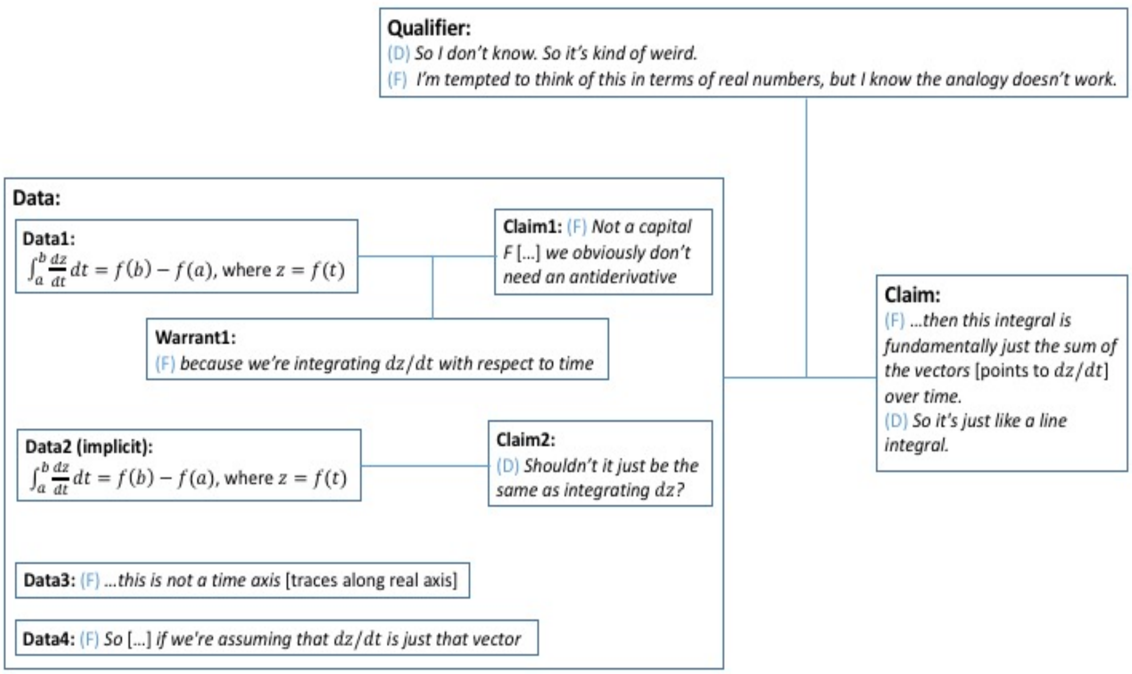


Figure 22. Toulmin diagram for Dan and Frank's Argument 1, Task 4.

At this time, Dan used symbolic reasoning to author a claim that evaluating the integral of dz/dt with respect to time is equivalent to integrating dz (line 12). Frank agreed with Dan and symbolically argued as spokesman that $\int_a^b dz = z\,\big|_a^b = f(b) - f(a)$, but expressed difficulty in providing a geometric interpretation of this (lines 13-14). Because Dan and Frank did not provide a geometric interpretation of the identity and

Frank expressed some confusion about doing so, I directed the pair's attention to their last diagram (lines 15-16).

Int: Yeah. So you have your picture, for instance, over there. Um is there some way you could releate this—
F: Ok, I guess the complexity is this is not a time axis [tracing gesture along the real axis], um— [pauses]
D: Hmm, yeah that's interesting. Cuz I mean, it's— it's like the same. You're integrating— yeah, like with respect to time, you're integrating your function. But then you're also doing just an integral of z as it goes from a to b. So I don't know, so it's kind of weird. Um, hm—
F: I guess— so basically we're adding up all the derivatives over some interval in time. Um, I'm tempted to think of this in terms of real numbers, but I know the analogy doesn't work.
F: So if we're— if we're assuming that dz/dt is just that vector [points to orange tangent vector from diagram in Task 3], then this integral is fundamentally just the sum of the vectors [points to dz/dt] over time.
D: So it's just like a line integral.
F: Yeah so it's basically just a line integral. Um, that'd be my interpretation of it.

In response, Frank discussed the difficulty with using their same picture from Task 3, namely that "the complexity is [that] this is not a time axis" (line 17). As he said these words, he traced along the real axis in the diagram from Task 3 using his hand, with the tip of the whiteboard marker taking on a referential role (see Fig. 23). This gesture and corresponding verbiage referential to the geometric diagram comprised the first instance of embodied reasoning during Task 4. After a pause of several seconds, Dan articulated an oddity about their prior claim that $\int_a^b \frac{dz}{dt} dt = \int_a^b dz$. In particular, he recognized that they integrated with respect to time on one hand, but also used the same bounds of a and b to integrate with respect to $z = f(t)$ (lines 19-21). As a result, Dan qualified his uncertainty with the statement, "So I don't know, so it's kind of weird" (line 21).

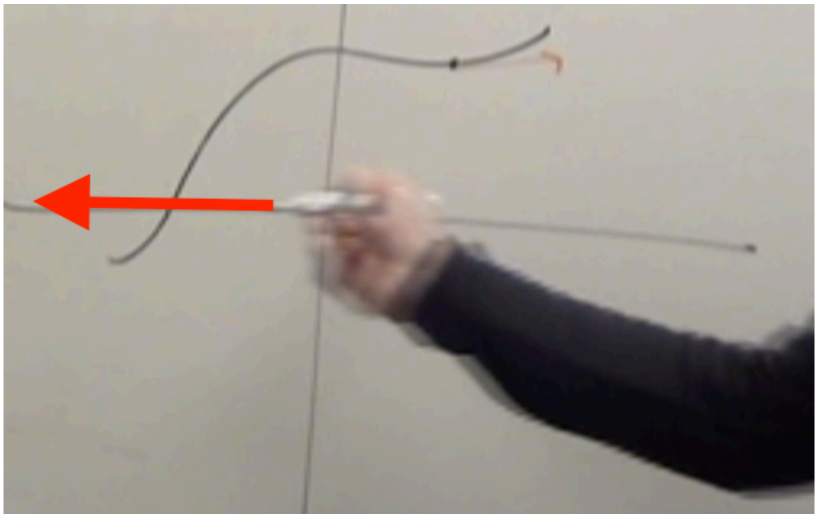


Figure 23. Frank's tracing gesture along the real axis during Argument 1 in Task 4.

Following another pause lasting several seconds, Frank authored a new claim, "so basically we're adding up all the derivatives over some interval in time" (line 22). However, he qualified this assertion with the statement, "I'm tempted to think of this in terms of real numbers, but I know the analogy doesn't work." Accordingly, Frank's qualifier represents a deliberate attempt to avoid erroneously applying properties of real numbers to the complex world, i.e. *thinking real, doing complex* (Danenhower, 2000). Proceeding as spokesman, Frank re-voiced his aforementioned claim using a new embodied datum that the orange tangent vector from their previous diagram from Task 3 represents a generic $\frac{dz}{dt}$ vector (lines 24-26). Subsequently, Dan authored a follow-up claim characterizing Frank's description as "just like a line integral" (line 27), and Frank agreed with this alternate characterization (line 28).

Though Dan and Frank discussed their geometric interpretation of the integral portion of the identity, they had not provided such an interpretation about the quantity $f(b) - f(a)$ that this integral equaled. As such, I asked them to consider this other portion of the identity geometrically (lines 29-31). This question prompted a second argument, Argument 2, which Frank began by labeling values for the endpoints a and b in the diagram (line 32; see Fig. 24), despite his previous recognition that the horizontal axis in that diagram was "not a time axis" (line 17).

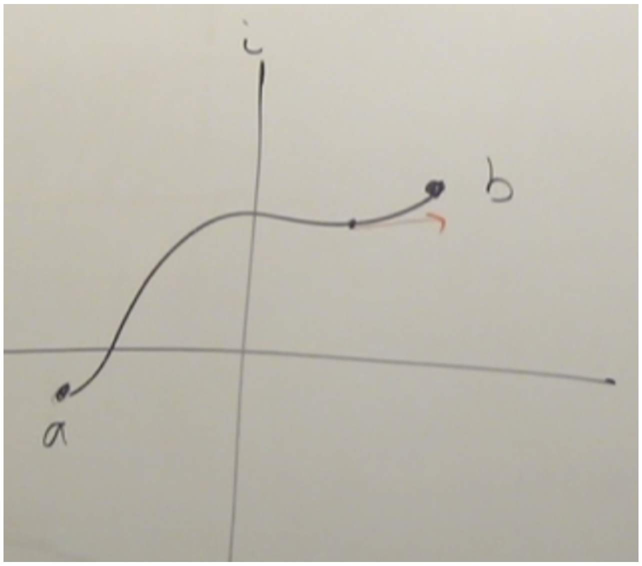


Figure 24. Frank's labels for points a and b during Argument 2 in Task 4.

Int: So, but you have the identity— so you're kind of thinking— it seems like you're mainly focusing on the integral part of this. But notice— so you have that integral equals $f(b) - f(a)$. What does $f(b) - f(a)$ look like in your picture?

F: So that'd be— just assuming that the endpoints represent a and b [labels the endpoints] then this integral is just the same as the function evaluated at b minus the function evaluated at a.

D: So you have to look at it in like some other plane. [Draws a second Argand plane] So like you can call it the $u - v$ plane. Then $f(b)$ could be— I don't know, it depends on your function. $f(b)$ could be that [draws a point in 4th quadrant] and $f(a)$ could be something here [draws a point in 2nd quadrant]. Um and then you'd just be taking the difference between those two points.

F: Right

D: Like, you'd have to look at a completely different graph.

F: …the complexity being that like, adding up the vectors is effectively the same as adding up a bunch of different complex numbers. So yeah, it's the same; it's just evaluating those two points [Dan labels the two points as $f(a)$ and $f(b)$], and subtract— um finding the difference between the two.

Perhaps realizing the inherent contradiction in Frank's datum, Dan authored a claim that they would need a second, "u-v plane" (lines 34-35; see Fig. 26) to depict $f(b) - f(a)$ geometrically. Using embodied reasoning to supply another datum, Dan drew such a plane and plotted hypothetical points representing $f(b)$ and $f(a)$ (lines 35-

37). Using this datum, Dan re-voiced Frank's prior claim that the value of the integral is $f(b) - f(a)$, arguing as spokesman that "you'd just be taking the difference between those two points" (line 37). He then reiterated that "you'd have to look at a completely different graph" (line 39). Afterwards, Frank authored a warrant articulating the equivalence of vector addition and the addition of complex numbers (lines 40-41). He also provided backing for this warrant's validity by affirming that the warrant applies to this task, stating "So yeah, it's the same; it's just evaluating those two points and […] finding the difference between the two" (lines 41-42). Frank articulated this backing in the role of spokesman because the latter portion reiterated the semantic content from claims 1 and 3 using slightly different wording. Argument 2 is summarized in Figure 25.

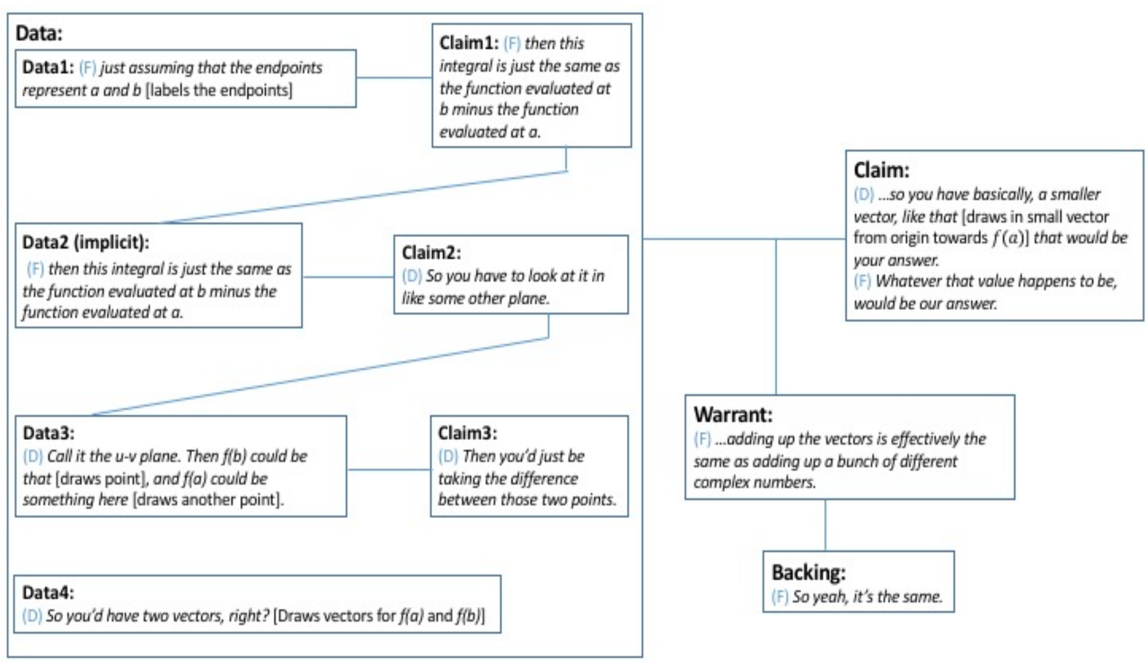


Figure 25. Toulmin diagram for Dan and Frank's Argument 2, Task 4.

Although Dan and Frank both described the difference $f(b) - f(a)$ as vector "addition" and plotted the points $f(b)$ and $f(a)$ on a new *u-v* plane, they had not provided a geometric depiction of the result of this difference on their diagram.

Accordingly, I asked them to explicitly draw this portion of the result using their diagram (line 43). In response, Dan provided an embodied datum by drawing in position vectors corresponding to the points $f(b)$ and $f(a)$ (line 45). As spokesman, he once again reiterated that the result is the difference of $f(b)$ and $f(a)$, however these objects now explicitly represented vectors (line 46). Using embodied reasoning, Dan described the resultant vector and drew in what he thought to be its location on the *u*-*v* plane (lines 46-48; see Fig. 26). Frank agreed with this claim, and added that they could not provide the exact coordinates of this resultant vector without knowing the coordinates of $f(a)$ and $f(b)$ (line 49). Note from Figure 26 that Dan's resultant vector is incorrect, both in terms of magnitude and direction. The correct result should have considerably longer length and lie in the second quadrant. However, I did not make this known during the interview, as it was not my goal to ensure that participants arrived at a correct answer.

Int: So geometrically, can you think of a way to draw what that difference would be?
F: Sure
D: Yeah. So you'd have two vectors, right? [draws vector from origin to $f(a)$ and similar for $f(b)$]. And then— so then all you have to do is just difference between those two, so you have basically, a smaller vector, like that [draws in small vector from origin towards $f(a)$ but shorter] that would be your answer.
F: Whatever *that* value happens to be, would be our answer.
Int: Ok, cool. Um let's see. So do you recognize that identity that has a familiar name?
F: The Fundamental—
D: I mean there's the theorem that says like, you have—
F: I mean, this is one of the Fundamental Theorems of Calculus, so— one of our fundamental integral theorems.

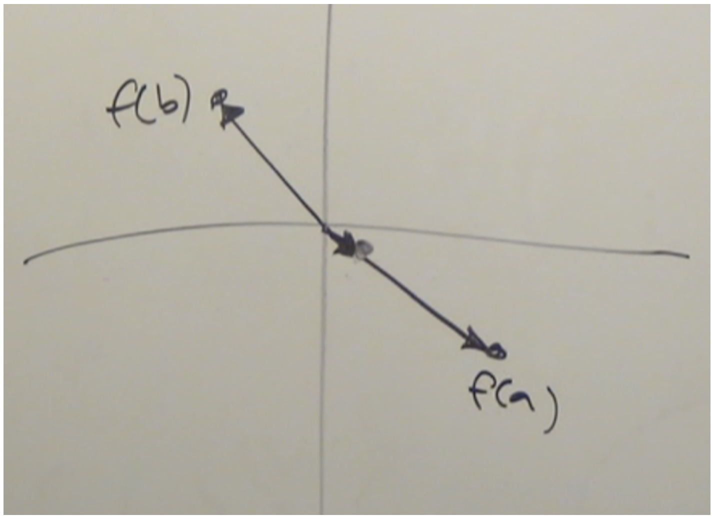


Figure 26. Dan's depiction of the vector difference $f(b) - f(a)$ in Argument 2, Task 4.

Sensing that the pair had concluded their argument, I asked them if they recognized the identity by a familiar name (line 50). Both Dan and Frank appeared to recognize the result, and Frank identified it as the Fundamental Theorem (lines 51-54). Dan also claimed that this result was equivalent to "a thing we talked about earlier" (line 55). This assertion catalyzed a third argument related to this task, which I refer to as Argument 3 and depict in Figure 27. Dan continued by authoring a symbolic datum considering the definition of a contour integral in the special case where $f\big(z(t)\big) = z(t)$ (lines 55-57). Unsure of this statement, he also provided a qualifier, asking Frank, "Is that right?" (line 57).

D: Yeah. And that's also a thing we talked about earlier [writes $\int z\, dz$]. Like if you have some type of complex function and you just parametrize it, and then um put in the parametrization, then you'd have $z(t)\; dz/dt$ right? [Writes $z(t)\; dz/dt$ below his last inscription] Is that right?
F: Yeah. And then you integrate that with respect to time [writes in dt at the end of Dan's integral].
D: I should say $dz(t)/dt$ [changes z to $z(t)$ in previous inscription, and writes in an integral sign]
F: Yeah.
D: So that's pretty much what you did.
F: So if this function is just 1, [writes in a 1 in the integrand of the original inscription $\int_a^b \frac{dz}{dt}\, dt$] then basically we have that [points back to Dan's latest inscriptions].

Frank affirmed Dan's datum and authored a symbolic claim that they needed to integrate Dan's inscription with respect to time (line 58). This prompted Dan to revise his previous inscription by replacing z with $z(t)$ (line 59). In doing so, he acted as spokesman because he altered the syntactic structure of his prior statement while keeping the semantic nature intact. Identifying the resulting inscription $\int z(t)\frac{dz}{dt}\,dt$ as equivalent to the integral from the statement of the task, Dan claimed "So that's pretty much what you did" (line 61). Frank clarified this assertion by authoring a symbolic warrant identifying the task integral as the special case where $z(t) = 1$ (lines 62-63). Summarily, this third argument served to conclude that the identity from Task 4 could be thought of as a special case of the definition of the contour integral applied to the special case $f(z(t)) = 1$.

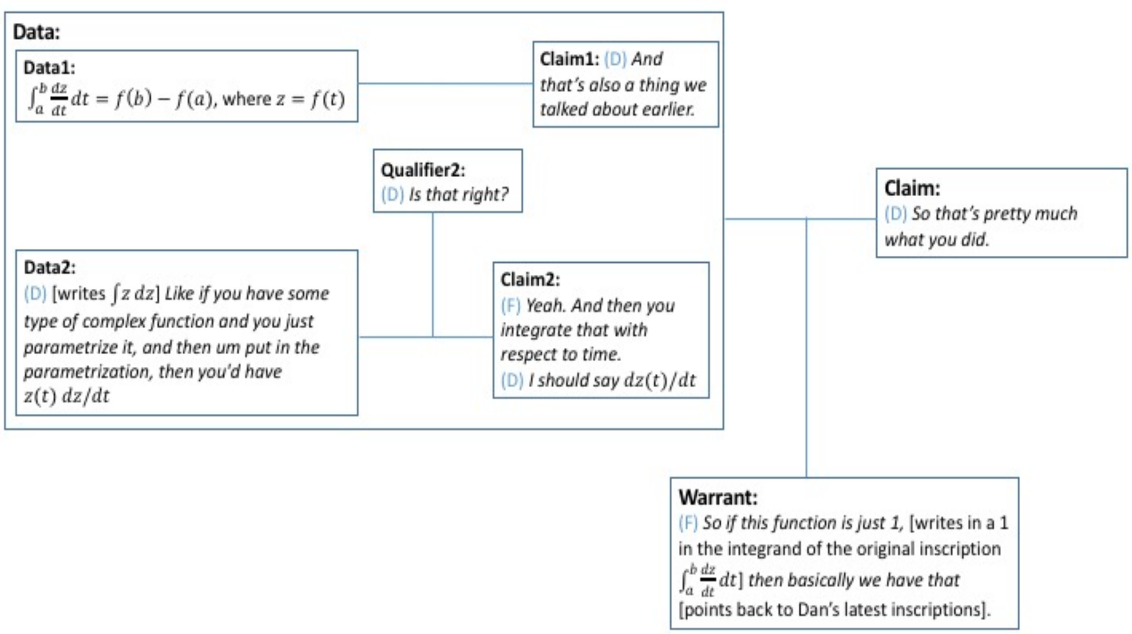


Figure 27. Toulmin diagram for Dan and Frank's Argument 3, Task 4.

Task 4 – Riley and Sean

Task 4 began as I read the task aloud while Sean, as spokesman, wrote symbolic inscriptions corresponding to the provided assumptions (lines 6-9). Exactly as Frank did initially, Sean instinctively wrote the function f as F after I read the task identity (line 9). I finished reading the task by directing Riley and Sean to identify the result by name if they recognized it (lines 10-11). Sean quickly wrote "F.T.C." under the identity, and as spokesman, claimed that this is the Fundamental Theorem of Calculus (line 12).

Int: Cool. Um, ok great. So keeping this sort of thing in mind, we have this $z = f(t)$ thing, again, a parametrized curve. And I guess we can keep the pictures up— maybe, well why don't you decide whether you want to pick stuff. Um, so it's still going to be somewhat related here, but— so maybe we could erase the $z = e^{it}$ example, for instance.
S: [Erases most of writing on board.]
Int: Ok, so again, we have $z = f(t)$ is a parametrized curve described as a complex valued function of t [Sean writes $z = f(t) = x(t) + i\,y(t)$]. How would you provide a geometric interpretation of the identity— and I'll give you this identity here— so we have $\int_a^b \frac{dz}{dt}\,dt = f(b) - f(a)$.
S: [Writes $\int_a^b \frac{dz}{dt}\,dt = F(b) - F(a)$ on the board]
Int: And if you recognize what the identity is, maybe by name, then maybe speak a little bit about what that is.
S: [Writes F.T.C. under the identity] It's the Fundamental Theorem of Calculus, which just says— So for like the— I guess the Calc 1 version is just [writes "Calc I"], like, [the integral of] $f(x)\,dx$ from point a to point b, and you know that the antiderivative of f, capital F— or the antiderivative of little f is capital F—
R: Then the derivative of capital F is little f. [Sean writes $F'(x) = f(x)$]
S: [Writes $\int_a^b f(x)dx = F(b) - F(a)$] Then you can use this identity. So this [the task version] is the exact same thing. If um, this [underlines F] is the antiderivative, that would mean big $F'(t) = f(t) = z(t)$ [writes this statement on board], then you can use this [points to the task identity]. You can say, "What's my path? What's my ending point and starting point?" so we can find an antiderivative with respect to t, then just plug in the points and subtract [points to $f(b) - f(a)$ from the task identity] .

As a symbolic warrant, Sean began to author the "Calc I version" of the theorem (lines 12-14). In stating this version of the theorem, he clarified that "the antiderivative of little f is capital F" (lines 14-15). As spokesman, Riley re-voiced this clarification as an equivalent statement, "the derivative of capital F is little f," which Sean wrote symbolic inscriptions for (line 16). Sean finished writing the "Calc I" version of the Fundamental Theorem, and stated that "this [the task identity] is the exact same thing" (lines 17-18).

As such, he authored backing for their warrant's validity by describing why the Fundamental Theorem of Calculus is essentially the same as the task identity. In particular, Sean wrote symbolic inscriptions for what it means to be an antiderivative in the context of Task 4 (lines 18-19). Employing embodied-symbolic reasoning, he described how the identification of one's path and endpoints allow the evaluation of the antiderivative at those endpoints (lines 20-21). Argument 1 is summarized in Figure 28.

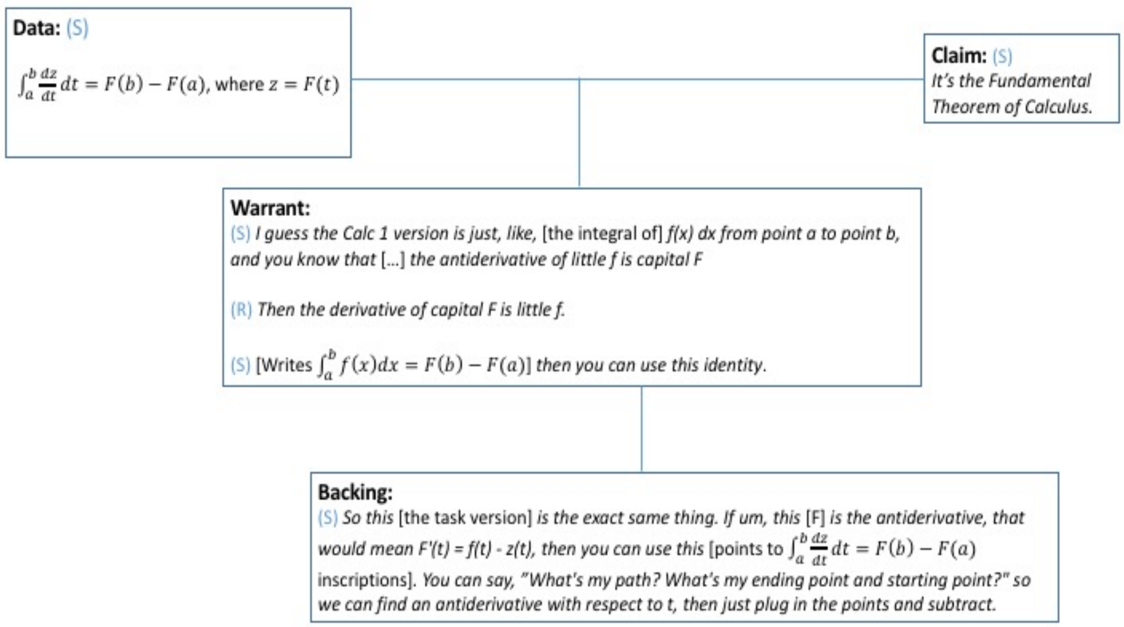


Figure 28. Toulmin diagram for Riley and Sean, Argument 1, Task 4.

Int: Ok perfect, so yeah I saw you had the capital F notation before, so I was glad that you explained what that is and everything. So ok, we'll come back a little later, um, to talk a little more about antiderivatives and things of that nature. But, ok cool, so anything you wanted to add to that, Riley?
R: Um, let's see— so it's going to be— I think this has to be true [points to the task identity] for any path between these two points?
S: Mhm.
R: It has to always be true.
Int: Yeah so we're really talking about any path from a to b there.
R: Yeah so it's nice because it makes it more flexible, since like if we're working here [in the Calc 1 case; points to circled Calc 1 version on board] if you're only in one dimension, there's only one way to get between, between the two points.
S: Yeah.
R: But in 2 dimensions, you can take any path you'd like [gestures complicated path in the air with finger], cuz it works for any path.
S: And there are like, fairly technical things. Like you have to assume this $[f]$ is continuous on the interval here [points to a and b] and you have to assume that the path here is piecewise smooth, or something like that. So there's no special— the antiderivative is defined so there's no like, breaks, or anything. So you have to make sure it's a "well-behaved" [air-quotes gesture] path.
R: But generally those are the ones we're working with, so—
S: Yeah, then of course, you have to distinguish between this thing [points to task identity] like, the integral of a complex function [writes $\int f(t)\, dt$], versus the integral of, not a real variable [points to $\int f(t)\, dt$ inscription he just wrote] but actually a complex variable [writes $\int f(z)\, dz$]. And that's when it gets a little more involved, but I guess that's later [in the interview].

Because Riley did not provide much input in Argument 1, I asked her directly if she had any additional comments about this argument (lines 22-24). I also alluded to the nature of Task 13 by telling Riley and Sean that we would return to the notion of an antiderivative of a complex function later in the interview. Riley responded to my follow-up question with a qualifier expressing some degree of uncertainty about whether the task identity is true for any path between points *a* and *b* (lines 25-26). Sean assured her that it is true for any path (line 27), and as spokesman, Riley re-voiced this statement as a claim (line 28). This segment incorporated embodied, symbolic, and formal reasoning, as it entailed a universal statement about the relationship between the symbolic identity and the embodied path. I additionally clarified that the initial statement of the Task did not specify any particular path (line 29).

In response, Riley claimed that this generality with respect to path choice "makes it more flexible," and began to author an embodied warrant to support her assertion.

Specifically, she discussed how in the real-valued setting, one can only approach a given point from the left or the right (lines 30-32). She then compared this setting to its two-dimensional analog, wherein “you can take any path you’d like” (lines 34-35) as she used her finger to incorporate an embodied tracing gesture illustrating a hypothetical path through the air (see Fig. 29).

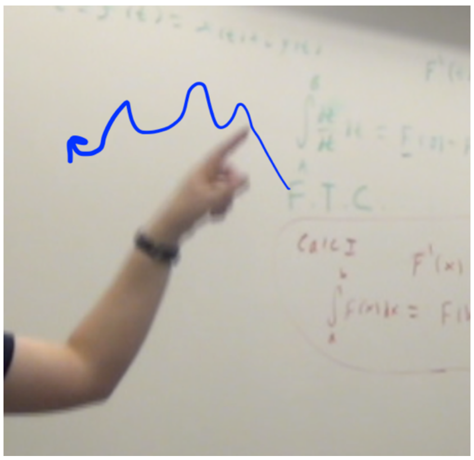


Figure 29. Riley’s tracing gesture for a hypothetical path in two dimensions.

Subsequently, Sean authored formal-embodied backing for their warrant’s correctness by discussing various assumptions needed to apply the Fundamental Theorem of Calculus (lines 36-39). He characterized these assumptions as “fairly technical” (line 36), and argued that they collectively ensure that the path is “well-behaved” (line 39). Riley provided an embodied addendum backing the warrant’s field, as she appealed to the fact that “generally those are the [paths] we’re working with” (line 40). Sean closed Argument 2 with formal-symbolic backing for their warrant’s validity. He described the importance of distinguishing between the integral of a real-valued function and that of a complex function $f(z)$, thereby identifying conditions under which the warrant applies or does not (lines 41-44). Argument 2 is summarized in Figure 30.

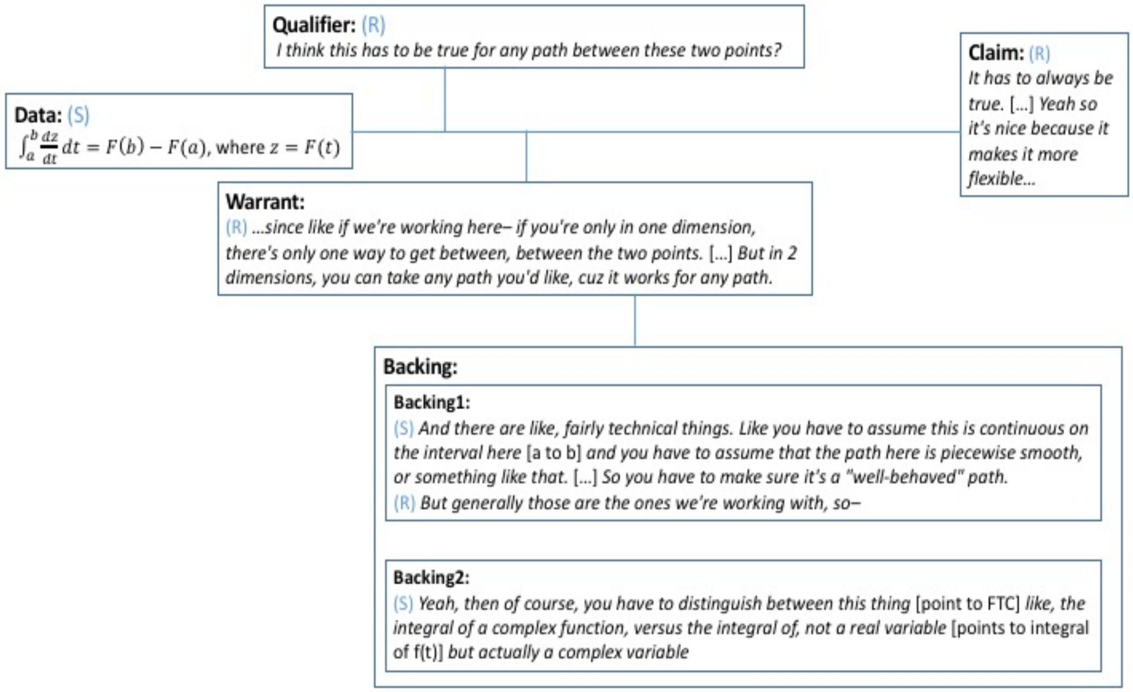


Figure 30. Toulmin diagram for Riley and Sean, Argument 2, Task 4.

Int: Yeah. Well so, um, going back to the picture that you kind of drew there, in the orange, before then [in Task 3]. Um, does this identity have any sort of meaning there in that context with the physical interpretation?

R: Uh, it does but I don't remember what it is. I mean, so for Calc 1, it would be the area below the curve. But that's not the case for um, like with complex variables. Um, but I don't know, do you [Sean] remember what the— what the physical interpretation is [traces finger along the orange path from Task 3] for the integral?

S: Um, so this is like, your position vector [draws new vector from origin, labels it r] and this is like your velocity vector [points to tangent vector] so like r is just— so v is just dr/dt [writes $v = dr/dt$]. If you're going to integrate from, say time 1 to time 2, like, $v(t)\,dt$ [writes $\int_{t_1}^{t_2} v(t)dt$] then you're going to get your change in position, so it would be [writes $= \Delta r = r(t_2) - r(t_1)$]

R: So it's like length of the curve?

S: Um yeah. It's like— well no, um, like length of the curve would be like if you integrate absolute value of this [points to $v(t)$ writing on board] like the magnitude of that. Then you get your arclength. But if you're integrating this, then you just do change in position [points to $r(t_2) - r(t_1)$]. So like, say we have, like, r_2 and r_1 [draws in position vectors for r_2, r_1], you know the velocity curve, and you say "ok, integrate $v(t)$ from t_2 to t_1" [points to tip of r_2 vector, then r_1] and all that is just the change from here to here [points again to $r(t_2) - r(t_1)$]

R: So it's displacement versus distance, or whatever?

S: Yeah. Mhm. [Writes 'displacement' above $\int_{t_1}^{t_2} v(t)dt = \Delta r = r(t_2) - r(t_1)$ inscriptions]

Although Riley and Sean discussed important and interesting aspects of the task identity in Arguments 1 and 2, they had not provided a physical interpretation of this identity. As such, I asked a follow-up question in an effort to elicit such an interpretation, using their diagram from Task 3 (lines 45-47). Riley qualified Argument 3 by sharing that she did not remember a physical interpretation of this identity, though she acknowledged its existence (line 48). Applying embodied reasoning, she alluded to the common “Calc I” characterization of integration as “area below the curve,” but claimed, “that’s not the case for […] complex variables” (lines 48-49). Hence, Riley exemplified an explicit attempt to avoid an inappropriate application of *thinking real, doing complex* (Danenhower, 2000). As spokeswoman, she re-directed my question to Sean, and produced an embodied tracing gesture along the opposing direction of their original orange path from Task 3 (lines 49-51; see Fig. 31).

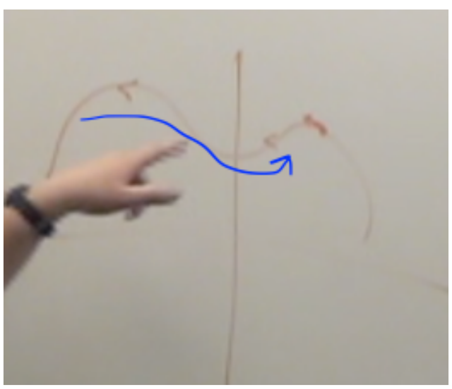


Figure 31. Riley’s tracing gesture as she said “physical interpretation” in Argument 3.

Sean responded by authoring an embodied datum. He drew a position vector $\vec{r}$ corresponding to the point on their orange curve where they previously drew a representative tangent vector, and labeled this vector $\vec{v}$ (lines 52-53; see Fig. 32). With this tangent vector in mind, he authored a symbolic claim that $v = dr/dt$ (line 53). Continuing with symbolic reasoning, Sean authored a datum considering the integral $\int_{t_1}^{t_2} v(t)dt$ (lines 53-54). Employing embodied reasoning, he supplied a warrant that this integral yields a “change in position.” This warrant supported the symbolic claim that $\Delta r = r(t_2) - r(t_1)$, which Sean provided as spokesman (lines 54-55).

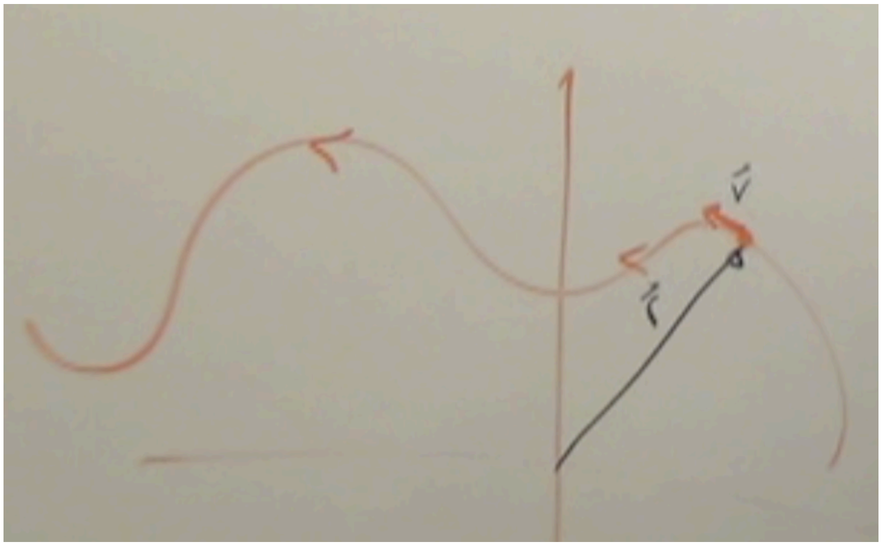


Figure 32. Sean’s revised diagram including position vectors r and v, Argument 3.

Riley asked Sean if an equivalent interpretation of this integral would be “length of the curve” (line 56). Sean initially agreed with this interpretation (line 57), but quickly changed his mind, and challenged Riley’s assertion. He authored an embodied-symbolic claim that arc length is instead obtained by integrating the “absolute value” of $v(t)$ (lines 57-58). Subsequently, Sean provided embodied-symbolic backing for his previous warrant’s correctness. He began this backing by relaying his stance that integrating $v(t)$ alone results in a change of position, and pointed to his previous symbolic inscriptions. Next, Sean drew in a second position vector $\vec{r}_2$ and relabeled his original vector $\vec{r}$ to be $\vec{r}_1$ (lines 59-60; see Fig. 33).

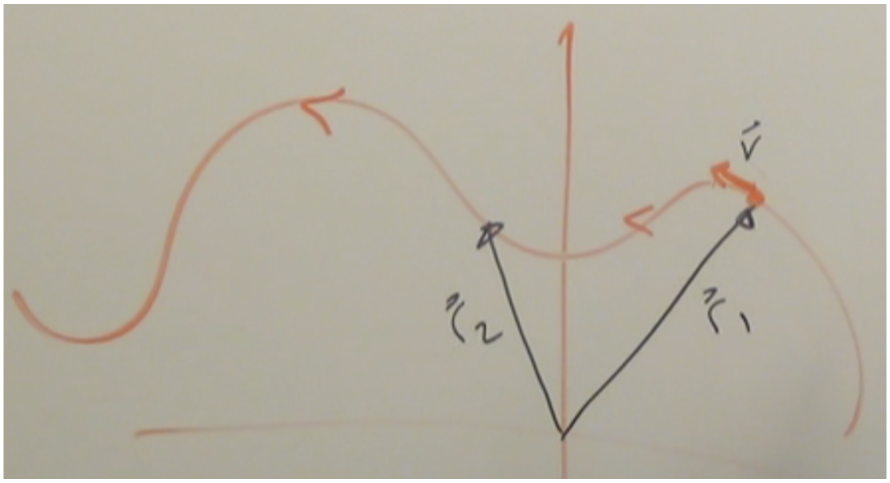


Figure 33. Sean's position vectors r_1 and r_2, Argument 3, Task 4.

Sean once again concluded that integrating from time t_2 to t_1 yields a change in position, and he pointed to the tips of $\vec{r}_2$ and $\vec{r}_1$ as he specified these two respective times (lines 61-62). Note Sean's apparently accidental transposition of these two times, as the times should actually range from t_1 to t_2. As spokesman, Riley succinctly re-voiced Sean's backing with the embodied statement, "So it's displacement versus distance, or whatever?" (line 63). Sean affirmed her summary and labeled his recent symbolic transcriptions with the word "displacement" (line 64). A summary of Argument 3 is depicted in Figure 34.

S: Then you could do the magnitude of the speed, and you do [writes $\int_{t_1}^{t_2} |v(t)|\, dt$] which of course is going to be, like, the really—
R: So this would be like the length of the curve? [Draws arrow from $\int_{t_1}^{t_2} |v(t)|\, dt$ inscription to their diagram, then traces along the path with dotted line]
S: Yeah, actual arclength.

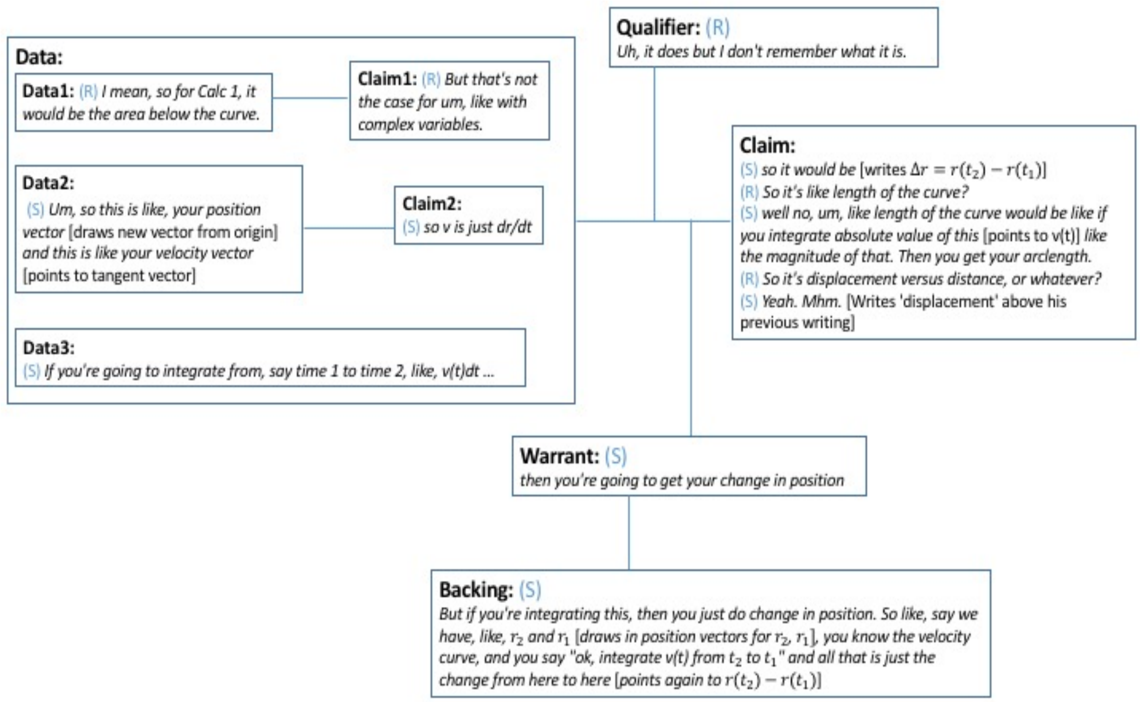


Figure 34. Toulmin diagram for Riley and Sean, Argument 3, Task 4.

The previous distinction between displacement and arc length in Argument 3 catalyzed a short follow-up argument, Argument 4, as follows. Implementing embodied-symbolic reasoning, Sean began to re-voice his previous assertion that integrating $|v(t)|$ yields the length of the curve (lines 65-66). Before finishing his thought, Sean qualified this claim with the phrase, “Which of course, is going to be” (lines 65-66). As spokesman, Riley finished Sean’s claim, but phrased it as a question (line 67). She very explicitly linked the symbolic and embodied worlds by drawing an arrow from Sean’s symbolic inscriptions to their path diagram, and traced along the path using “dotted” line segments as she said “length of the curve” (lines 67-68; see Fig. 35). As spokesman, Sean affirmed her claim, calling the result “actual arc length” (line 69). Argument 4 is summarized in Figure 36.

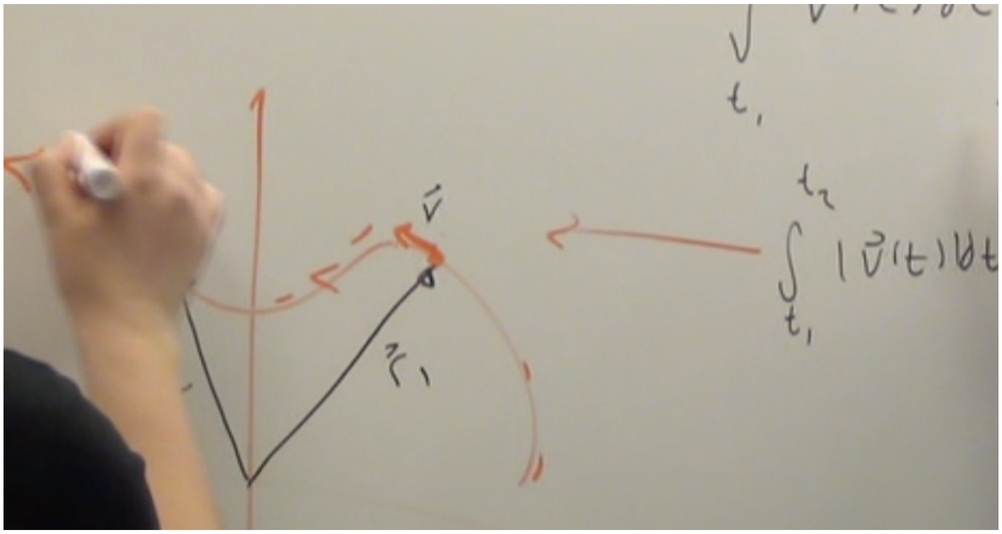


Figure 35. Riley's connection between symbolism and geometry for arc length.

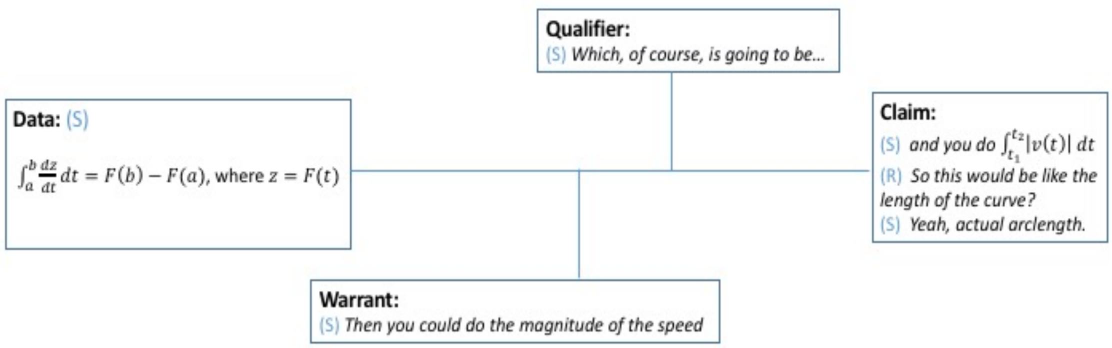


Figure 36. Toulmin diagram for Riley and Sean, Argument 4, Task 4.

Following her and Sean's brief discussion about arc length, Riley redirected their attention back to the original task (line 70). As spokesman, she sought to clarify that the symbolic inscriptions $\int_{t_1}^{t_2} v(t)dt = \Delta r = r(t_2) - r(t_1)$ represented a distance (lines 70-71). She re-drew their previous orange path, labeled the distance between starting and ending points, and drew an arrow from the symbolic inscriptions to this new diagram (see Fig. 37). Accordingly, she once again elucidated the connection between her and Sean's symbolic and embodied representations in a very explicit manner. As before, she did so in the form of an embodied-symbolic claim.

R: And this is like— this one's [the task question] like this distance here? [Draws arrow from $\int_{t_1}^{t_2} v(t)dt = \Delta r = r(t_2) - r(t_1)$ inscription to a redrawn version of their diagram, with the distance from start to end point labeled]
S: Mhm [writes $= \int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + \dot{y}^2}\, dt$, then labels the two aforementioned scenarios as arc length and displacement, resp.]
S: Of course x dot is just dx/dt. Physics notation. Yeah something like that. Yeah so you get displacement here [points to $\int_{t_1}^{t_2} v(t)dt = \Delta r = r(t_2) - r(t_1)$] , and you get the speed there [points to $\int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + \dot{y}^2}\, dt$].
Int: So you have your r_1 and r_2 vectors drawn in there, so what does the $r(t_2) - r(t_1)$ look like in that picture?
S: Just put a line from here to there [draws in displacement vector]
Int: Yeah, cool. Anything else you wanted to add about that?
R: Nope.

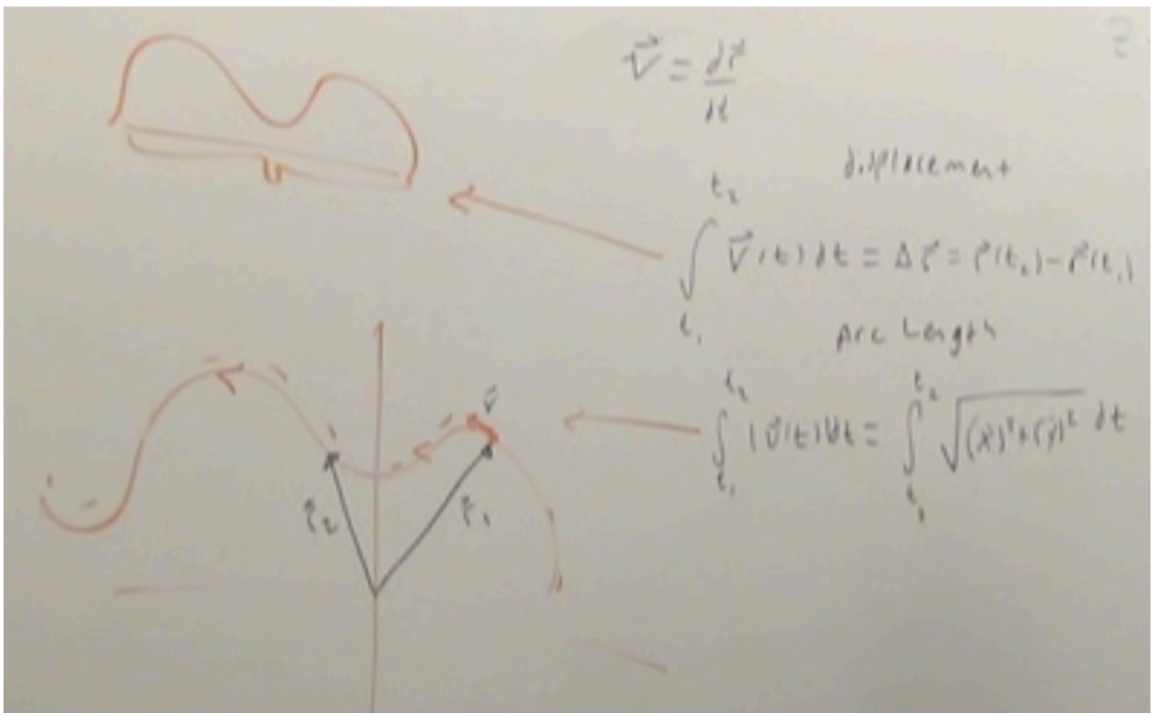


Figure 37. Riley's connection between symbolism and geometry for displacement.

Sean agreed, and continued to clarify the distinction between the integrals of $v(t)$ and $|v(t)|$ (line 73). Specifically, he rewrote $\int_{t_1}^{t_2} |v(t)|\, dt$ as $\int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + \dot{y}^2}\, dt$, and labeled these inscriptions with the words "arc length" (lines 73-74). As spokesman, Sean clarified that the symbolic "dot" notation represents a derivative with respect to time, as used in physics contexts (line 75). Implementing embodied-symbolic reasoning, he relayed the prior claim that $\int_{t_1}^{t_2} v(t)dt = \Delta\, r = r(t_2) - r(t_1)$ represents displacement and $\int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + \dot{y}^2}\, dt$ represents arc length, though he accidentally said "speed" in the latter case (lines 75-77). Sean's earlier label of "arc length" written above these symbolic inscriptions, as well as the content of Argument 4 allow me to confidently conclude that he indeed misspoke when saying "speed" here.

As a quick follow-up question, I asked Riley and Sean how to connect their symbolic inscriptions for the original task identity to their recently drawn diagram with position vectors $\vec{r}_2$ and $\vec{r}_1$ (lines 78-79). Although they previously provided an embodied interpretation of the task identity as displacement, they had not drawn a geometric interpretation for the $f(b) - f(a)$ portion of the identity. In response to my question, Sean authored an embodied-symbolic claim that $r(t_2) - r(t_1)$ could be represented geometrically as a displacement vector between the two corresponding points along the path. He drew this displacement vector on their diagram, as depicted in Figure 38. Argument 5 is summarized in Figure 39 afterwards.

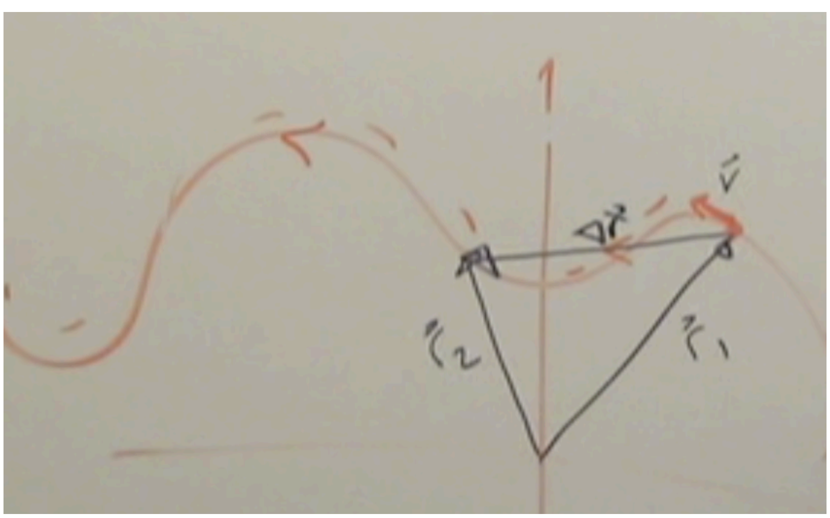


Figure 38. Sean's geometric inscriptions for displacement vector Δr, Argument 5.

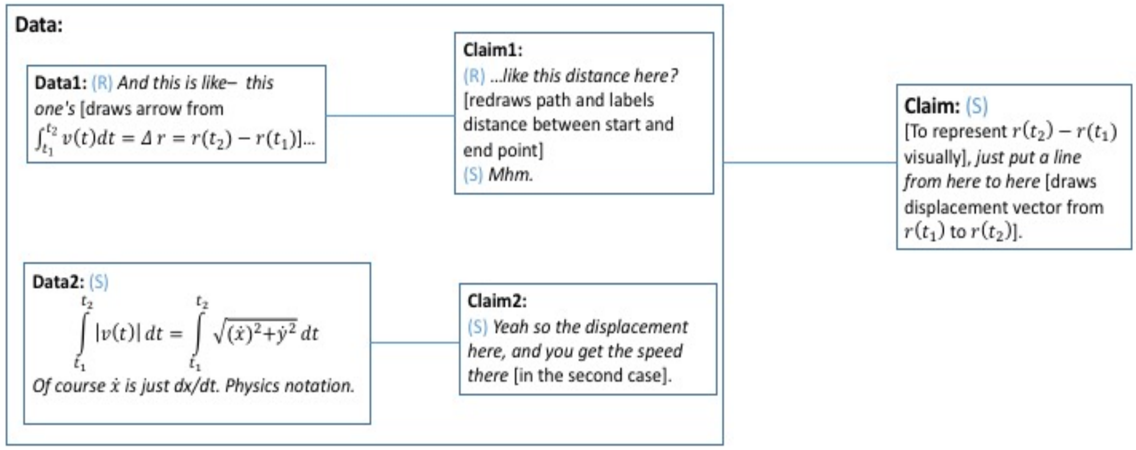


Figure 39. Toulmin diagram for Riley and Sean, Argument 5, Task 4.

After Riley and Sean's brief Argument 5, I asked one additional follow-up question to make sure neither of them had anything else to add about this task. Riley did not wish to add anything else to her response, but Sean discussed a short hypothetical scenario that comprised Argument 6. He began with a warrant that this scenario represented an analogous physics situation (line 83). Specifically, he authored an embodied datum considering a scenario in which the function $f(t)$ represented velocity rather than position, in which case dz/dt would represent acceleration (lines 83-84). Employing embodied-symbolic reasoning, he authored a claim that in this case, the task identity would represent "change in velocity" (line 84). This brief Argument 6 is summarized below in Figure 40.

S: Nothing specific. Same thing but with more physics stuff, if this were acceleration and that were speed, or velocity then you could find change in velocity as well. Just some more physical applications of it.
Int: Ok cool. Um, so— yeah and you mentioned that was the Fundamental Theorem. Ok yeah, perfect. Ok so why don't we erase all this stuff here?

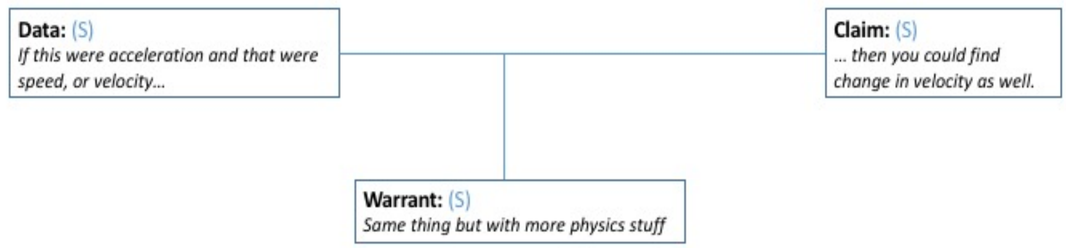


Figure 40. Toulmin diagram for Riley and Sean, Argument 6, Task 4.

Task 4 Summary

Overall, Riley and Sean appeared to exhibit more embodied reasoning than Dan and Frank during Task 4, evidenced in part by Riley's tracing gestures in Figures 29 and 31. Another distinctive aspect of Riley and Sean's response was Riley's explicit connections between the embodied and symbolic worlds, wherein she drew arrows illustrating the relationship between her and Sean's symbolic inscriptions, and the embodied diagrams they drew. A symbolic difference between the pairs' responses existed in Sean's Newtonian "dot" notation for time derivatives, which Dan and Frank did not incorporate. One noteworthy similarity between both pairs was that they each explicitly articulated a desire to avoid inappropriate applications of *thinking real, doing complex* (Danenhower, 2000) during this task. However, both pairs also provided backing for a warrant's validity that likened the Task 4 identity to the Fundamental Theorem in Calculus I. Accordingly, they also instantiated *thinking real, doing complex* in a manner that they felt suitably extended results from $\mathbb{R}^n$ to $\mathbb{C}$.

Task 5a – Dan and Frank

Task 5 (see Appendix C) required participants to consider the integral of a specific function for the first time in the interview. In part a, I asked Dan and Frank about the analyticity of this function, $f(z) = \overline{z}$. After writing the formula for this function on the board as a symbolic datum (line 4), Frank immediately authored an initial claim that $f(z)$ is not analytic everywhere (line 7). Dan agreed and added that this function is only differentiable on the [real and imaginary] axes, though the upward inflection in his voice suggested some uncertainty about this (line 8). Next, Frank relayed Dan's claim and refined his own previous claim from line 7 by conjecturing that the function is analytic nowhere (line 9). Like Dan, however, Frank posed this claim more as a question, and subsequently looked over to me as if seeking validation of their claim. With no response from me, Frank then qualified the remainder of the argument with the statement, "I'd need to confirm that" (lines 9-10).

Int: Great. Alright, so let's consider a specific function now. And we'll talk about some different things. [D and F erase board.] Yeah so you can kind of erase that. Alright so let's consider the function $f(z) = \overline{z}$.
F: [Writes $f(z) = \overline{z}$]
Int: And first of all, I want you to discuss whether this function is analytic or not. If so, where it's analytic, and if not, why it wouldn't be analytic.
F: It's not analytic everywhere; I can tell you that.
D: Yeah it's only differentiable on the axes?
F: Yeah it's differentiable on the axes and maybe analytic nowhere? [Looks to me for validation]. I'd need to confirm that.

After this dialogue, Dan authored a suggestion about using the limit definition of derivative in order to support their prior claim about differentiability (lines 11-14). By expressing $f'(z)$ as $\lim\limits_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}$, Dan provided a second datum for their argument and invoked formal-symbolic reasoning because he invoked the formal limit definition of derivative in the service of symbolic manipulations. However, it should be noted that this limit represents $f'(z_0)$, not $f'(z)$. I did not mention this error to Frank and Dan, so as not to interrupt their reasoning process. Proceeding as spokesman, Dan used symbolic reasoning to rewrite $\overline{z}$ as $x - iy$ (line 15). Frank then authored a symbolic portion of their warrant about substituting, presumably for $x - iy$, during their limit calculations (line 17). Dan then suggested that they approach some general point z_0 along two paths, a horizontal line and a vertical line (lines18-20), as he elaborated on their warrant pertaining to their limit characterization of $f'(z)$. He used the palm of his hand to gesture what the two paths of approach would look like, illustrating embodied reasoning (see Figure 41).

Figure 41. Dan's gestures representing a horizontal linear path of approach (left) and a vertical linear path (right) during Argument 1 for Task 5a.

D: So you could just take the limit of the um—
F: Right.
D: So $f'(z)$ equals, um, $f(z_0) - f(z)$ over $z_0 - z$ [writes this difference quotient, then writes in $\lim_{z \to z_0}$ as if he originally forgot it].
D: So if you're to look at this function-- so this is the same as $x - iy$ [writes it next to $\overline{z}$]. Um we could—
F: We could just substitute. So if you want to show that there's issues at zero, for example, or yeah—
D: We should approach it, like, from some random horizontal line [moves palm horizontally from left to right along the board] and some vertical line [moves palm vertically from up to down along the board] —

Frank proceeded as ghostee by rephrasing Dan's suggestion in terms of approaching along the real axis (line 21). Though his subsequent symbolic inscriptions (lines 21-28) supported his eventual intended claim (to follow), the semantic meaning of those inscriptions would not correspond to his verbiage about the real axis unless $z_0 = 0$. Rather, the inscriptions are consistent with Dan's formulation, wherein we approach z_0 along the horizontal line $y = y_0$ and then along the vertical line $x = x_0$. In any case, Frank instantiated symbolic reasoning comprised of function notation, as well as embodied reasoning comprised of language about paths of approach, to reach a claim that $\lim\limits_{x+iy_0 \to x_0+iy_0} \frac{(x_0-iy_0)-(x-iy_0)}{(x_0+iy_0)-(x+iy_0)} = 1$ (lines 21-28). In particular, because Dan and Frank chose to approach the point z_0 along the horizontal line $y = y_0$, which Frank mistakenly referred to as the real axis, Frank substituted $x + iy_0$ for z, $x_0 + iy_0$ for z_0, $x_0 - iy_0$ for $f(z_0)$, and $x - iy_0$ for $f(z)$ in the original difference quotient. As Frank algebraically simplified his new expression, Dan silently authored symbolic inscriptions to set up their second limit, approaching z_0 along the vertical line $x = x_0$ (line 25).

F: [Talking at same time] Yeah if you approach any point from the real, we can say the limit as $x + iy_0$ approaches $x_0 + iy_0$, so we're approaching along the real axis, of – let's see, $f(z_0)$ is $x_0 - iy_0$ [writes $x_0 - iy_0$]. And we subtract x um, hang on, what do I do? $f(z) - f(z_0)$ which is– [points to z-bar and mumbles as writes $f(z_0) - f(z)$] Oh hang on. I did something silly. [Erases his limit.]

D: [Dan starts writing $\lim\limits_{x+iy \to x_0+iy_0} \frac{x_0-iy_0-x_0+iy}{x_0-iy_0-x_0+iy_0}$ shortly after Frank starts line 21].

F: Oh right, these we're approaching the real axis. So it's just $x_0 - iy_0$ minus, uh, $x - iy_0$, all over, z_0 is just $x_0 + iy_0$, minus z is $x + iy_0$ [writes $\frac{(x_0-iy_0)-(x-iy_0)}{(x_0-iy_0)-(x+iy_0)}$].

F: Which simplifies to – $iy_0 + i\,y_0$, so we end up with $\frac{x_0-x}{x_0-x} = 1$ [writes this]. And if we approach that same point from the imaginary axis, I think we'll end up with minus i. [Writes $\lim\limits_{x_0+iy \to x_0+iy_0}$] Dan am I doing this right?

After Frank reached his claim that $\lim\limits_{x+iy_0 \to x_0+iy_0} \frac{(x_0-iy_0)-(x-iy_0)}{(x_0+iy_0)-(x+iy_0)} = 1$ (line 28), he conjectured that the other limit, approaching z_0 along the "imaginary axis," should be – i (lines 28-29). Note again that this limit should approach along the vertical line $y = y_0$, and their inscriptions support this latter path. At this point, Frank asked Dan if he was "doing this right" (lines 29-30). But Dan had already independently simplified his limit expression to $\frac{i(y-y_0)}{i(y_0-y)}$, which he concluded yielded a limit of -1 (line 31). When Dan communicated this result to Frank (line 33), Frank maintained that the limit should be $-i$. Dan pointed to his inscriptions on the board and argued that "the i's cancel" (line 35), but again Frank questioned the result, and wanted to "double check that" (line 36). Thus, Frank proceeded to run through a nearly identical calculation (lines 36-42) as Dan's, and concluded that $\lim\limits_{x_0+iy \to x_0+iy_0} \frac{(x_0-iy_0)-(x_0-iy)}{(x_0+iy_0)-(x_0+iy)} = -1$ (line 42) using the aforementioned embodied and symbolic reasoning from his other limit calculation. Specifically, his embodied reasoning consisted of language describing geometric paths of approach pertaining to limits, and his symbolic reasoning consisted of the associated symbolic manipulations that followed from the choice of path. Summarily, these two limit calculations yielded two different limits as Frank and Dan approached $z_0 = x_0 + iy_0$ along a horizontal line and a vertical line.

D: [Has been writing $= \frac{i(y-y_0)}{i(y_0-y)} = -1$.] I think so. What did you end up getting for that?
F: One.
D: I think I ended up with -1 for mine. I just did a – so you approached from some type of—
F: Oh you approached from— yeah it should be minus i, I think?
D: The i's cancel and just [gestures a flip flop with index and middle finger]
F: Do they? Ok let me double check that. You end up with, uh, same thing. $f(z_0)$: $x_0 - iy_0$ [writes this] minus $f(z)$ [writes $x_0 - iy$], all over—the naught is $(x_0 + iy_0) - (x_0 + iy)$ [writes this], which simplifies down to— [points at the x_0's] the x_0's cancel.
F: So we have minus i times $(y - y_0)$. Do we? Hang on— [points at the various terms] No we don't. We have minus $i(y_0 + iy)$ [writes it] all over— on the bottom we have $i(y_0 - y)$ [writes these]. Oh yeah, I think you're right. And this is just $-i(y_0 - y)$over $i(y_0 - y)$ [writes this and cancels the $y_0 - y$] so minus one.

Using formal reasoning, Dan subsequently noticed that they could have used the Cauchy-Riemann equations (line 43) to investigate analyticity. This observation catalyzed a follow-up argument that I describe below, but first, both Dan and Frank identified their previous limit argument as more formal (lines 45-46). Frank then acted as spokesman by succinctly recapitulating their limit argument (lines 46-48). He explained that they approached the point z_0 from two different paths and obtained two different limits, and used this summary as a datum to claim that $f(z)$ is differentiable nowhere. Frank's summary contained embodied-symbolic reasoning, in that the phrase "approach any point from two different directions" described physical motion towards an object using visualized processes, while the limit answers represented the product of a symbolic manipulation. Again, I take this process to be visualized because Dan and Frank did not draw a diagram depicting these paths, and Dan's gestures from Figure 41 indicate an external window into such a visualization of these linear paths.

D: Yeah and you could also just use the Cauchy-Riemann equations.
F: We could've. Yeah we were silly.
D: I guess this [limits way] is more formal.
F: Yeah, more formal proof I guess, but-- Yeah so I guess if you approach any point from two different directions, um, you end up with different values for the derivative. So I guess it's differentiable nowhere. Um, and therefore it's not analytic.
D: Because we chose an arbitrary x_0 and y_0.
F: Yeah.

Finally, Frank stated the pair's overall claim that $f(z)$ is analytic nowhere (line 48), using the relationship between differentiability and analyticity as a warrant to support this assertion using formal reasoning. Invoking formal-symbolic reasoning, Dan also clarified that this warrant supported their conclusion because they utilized an arbitrary point z_0 in their argument (line 49). Thus, this statement served as backing for the warrant's field, in that it underscored their limit argument's generality as appropriate for the mathematical setting. Argument 1 is summarized in Figure 42.

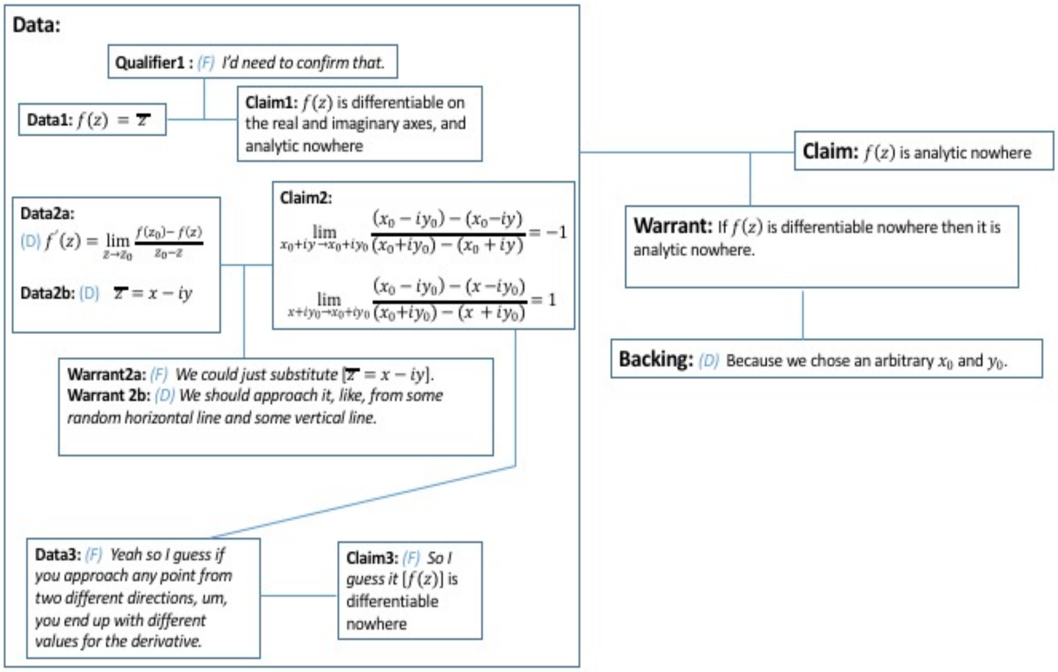


Figure 42. Toulmin diagram for Dan and Frank's Argument 1, Task 5a.

D: Yeah and you could also just use the Cauchy-Riemann equations.
F: We could've. Yeah we were silly.
D: I guess this [limits way] is more formal.
F: Yeah, more formal proof I guess, but-- Yeah so I guess if you approach any point from two different directions, um, you end up with different values for the derivative. So I guess it's differentiable nowhere. Um, and therefore it's not analytic.
D: Because we chose an arbitrary x_0 and y_0.
F: Yeah.
Int: So Dan, you said something about using the Cauchy-Riemann equations also. You mind maybe talking about how you'd do that?
D: Yeah so-- just erase this [erases board]
D: Ok so that's [writes $U_x = V_y$ and $U_y = -V_x$]. So um, like [writes $f(z) = \bar{z} = x - iy$]. So we have um, U-- the partial of x, is 1. And V sub y is -1 [writes $U_x = 1, V_y = -1$]. So that doesn't hold [draws a big X next to the two differing results]. And the bottom one-- we have zero equals zero, so that does hold. But because the first Cauchy-Riemann doesn't hold, you can just say it's not analytic anywhere.
F: Yeah that's probably an easier way of looking at it than what we did.
D: Yeah.
Int: Yeah that stuff comes in handy a lot [laughs]. But yeah it's good to think of it in both ways, though.

As mentioned previously, Dan's comment in line 43 catalyzed a second argument wherein I asked the pair to think about this task using the Cauchy-Riemann equations (lines 51-52). I analyzed what followed as a separate argument, Argument 2 (see Fig. 43).

Because my question was primarily directed at Dan, he acted as author for the duration of this brief argument. Dan began by writing the Cauchy-Riemann equations in their general form (line 54), using formal-symbolic reasoning. He then relayed their previous symbolic data from Argument 1 that $f(z) = \bar{z} = x - iy$ (line 54).

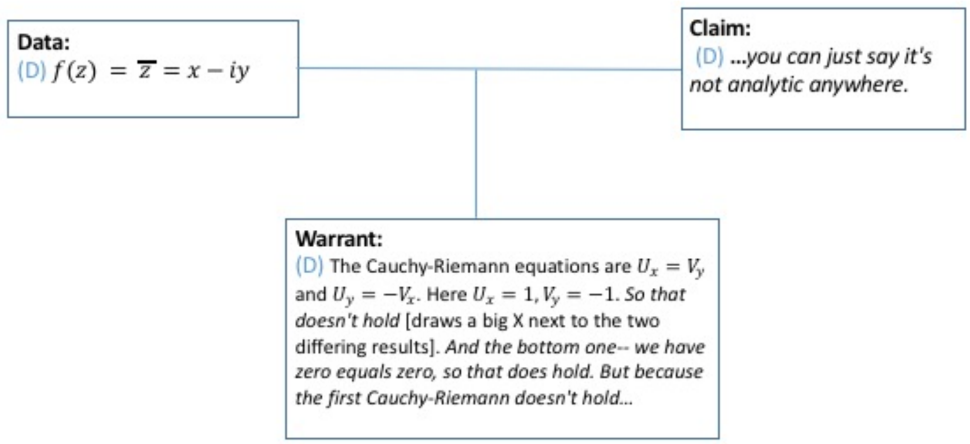


Figure 43. Toulmin diagram for Dan and Frank's Argument 2, Task 5a.

Next, Dan identified the real component function to be $U(x, y) = x$ and used symbolic reasoning to calculate $U_x = 1$. Similarly, he determined V_y to be -1, and concluded that the Cauchy-Riemann equations do not hold for this function (lines 55-56). This required formal-symbolic reasoning, in that he used the symbolic fact that U_x and V_y did not agree to relate back to the formal nature of the Cauchy-Riemann equations as necessary and sufficient conditions for differentiability. Dan determined that while $U_y = 0 = V_x$ and thus the second Cauchy-Riemann equation is satisfied (lines 56-57), this is not enough to make the function analytic anywhere (lines 57-58). Lines 57-58 explicitly indicate Dan's use of the Cauchy-Riemann equations as his formal-symbolic warrant for the claim that f is not analytic anywhere. Finally, both Dan and Frank agreed that this second argument represented an easier way to determine that the function was not analytic (lines 59-60).

Task 5b – Dan and Frank

After Dan and Frank concluded that the function $f(z) = \bar{z}$ is not analytic anywhere, I asked them if it was possible to integrate this function over the path L, a circle of radius R traversed counterclockwise (lines 1-2). The first argument for this task began with Frank proceeding as spokesman, writing my verbal description of the path using the symbolism C_R^- (line 3). Note that the path should be positively oriented, so Frank's path inscription should have read C_R^+; Dan and Frank recognized this error at a later point. Using the function formula and path description as data, Frank claimed that it would not be permissible to use "the Fundamental Integration Theorem" (lines 3-4). From the context, it appears that he meant either the Cauchy-Goursat Theorem or Cauchy's Integral Formula. In particular, he explained that this theorem required the function to be analytic in some simply-connected domain (line 4). This requirement served as a formal warrant because Frank provided formal conditions, which prevent the theorem from holding based on the given data.

Int: Ok so keeping the same function in mind. So we decided it's not analytic anywhere. Is it possible to find the integral of this function over L, where L is a circle of radius R traversed counterclockwise?
F: [Writes $f(z) = \bar{z}, L = C_R^-$] I mean, we certainly couldn't do it with the Fundamental Integration Theorem, because we require that the function is analytic in some simply connected domain. [Traces a circle clockwise with index finger in the air]. Um, but if we parametrized the curve [said with upward inflection as if unsure]. Can we do that?
D: I don't remember.
F: We could give it a shot. So it's a circle of radius R.
D: I mean, you'll get an answer but I don't know if it's--
F: I don't know if it's going to be valid or not [looks over at Int.].

Without the ability to invoke a powerful theorem directly, Frank hesitantly authored a claim that the pair parametrize the path instead (lines 5-6). Immediately afterwards, Frank questioned this claim, asking Dan "Can we do that?" (line 6). The pair continued to express uncertainty about this approach (lines 7-10), arguing that although they might obtain an answer, they might be unsure of its validity. Nevertheless, Frank suggested that they persist with his plan, and relayed a portion of their previous datum (line 8). As in Task 5a, Frank looked at me for validation after the pair expressed the aforementioned uncertainty in the form of an extended qualifier (line 10).

At this point, I redirected the conversation back to Dan and Frank by asking if they required any special properties about the function or domain when they used parametrization in the past (lines 11-12). Essentially, this probing question served to elicit their met-befores (Tall, 2013) related to parametrization in the hopes that doing so would drive their argument forward. In response, Dan authored a warrant for Frank's claim that they could parametrize *L*, arguing that they "just did it" in the past (line 13). Frank elaborated that the only times they could not freely parametrize were when the function had discontinuities (lines 14-16). Frank supported this rebuttal with an embodied example (lines 14-16), as his verbiage "pass through the negative real axis" described motion through a geometric location on an ostensibly visualized diagram. At this point, Dan and Frank had not drawn any such diagram on the board.

Int: Well so, you've parametrized functions in the past to find an integral. Were you using any nice properties about the function or the domain? Do you remember if there's anything that you needed?
D: I don't think so. I think we just did it, as far as I can remember.
F: Yeah um, I think the only issues that came up were with if there were discontinuities. Like if you tried to use the logarithm function and you tried to pass through the negative real axis, that would be a problem.
D: But it's not discontinuous.
F: Right. It's continuous everywhere, it's just not differentiable anywhere. So this should be doable. Um, let's give it a shot.
F: So it's oriented negatively, and radius R. So it would be--
D: $Re^{-i\theta}$
F: [Writes $Re^{-i\theta}$] Yeah. Which is just R— Cosine of negative theta is cosine of theta, right?
D: Would we even need to break it up like that? Can't you just plug it in?
F: Yeah we could do it like that.

With this rebuttal in mind, Dan observed that the function from this task is not discontinuous, and Frank added as spokesman that the function is continuous everywhere (lines 17-18). Hence, this statement about continuity served as backing for Dan's warrant's validity, in that the continuity of f prevented any issues brought up in Frank's rebuttal, thus supporting the applicability of the original warrant. Having convinced themselves that the path could be parametrized, Dan and Frank proceeded to write L as $Re^{-i\theta}$ (lines 21-24). However, because of the importance placed on continuity in Argument 1, I asked them to provide additional support for their assertion that $f(z)$ is continuous (lines 25-26). This began a new argument, which I refer to as Argument 2. Argument 1 is summarized in Figure 44.

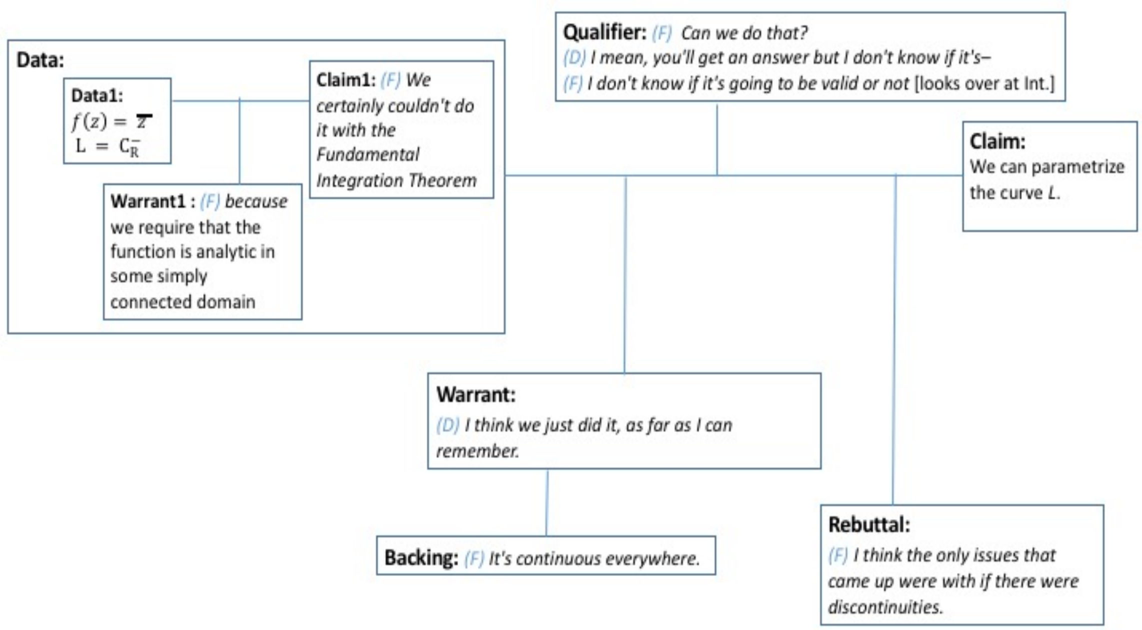


Figure 44. Toulmin diagram for Dan and Frank's Argument 1, Task 5b.

Dan began this continuity argument by relaying a symbolic datum from Task 5a, writing the inscription $x - iy$ (line 27). After briefly looking at his symbolic inscription, he mentioned that "you would never be dividing by zero, so I mean you can plug in any x

value and any *y* value" (lines 27-28). Although Frank agreed (line 29) with this warrant, Dan also added that "there's a formal way you could prove it" (line 30). In response, Frank authored the definition of continuity for real-valued functions (lines 31-33). I characterized this reasoning as formal-symbolic because Frank wrote symbolic inscriptions that corresponded with a formal definition of continuity. Dan then provided backing for this warrant's validity with the statement, "It seems like it's pretty clear that that would happen for this function" (line 34), which served to underscore the warrant's applicability to the situation at hand.

However, Frank was unsure that this characterization of continuity transferred to complex functions (lines 35-36). This consideration provided qualification for this sub-argument, in that Frank expressed uncertainty about the backing for the warrant's validity. Moreover, this statement seemed to represent symbolic-formal reasoning, as it considered the generalization of a symbolic definition of continuity to a different formal context. Following this qualifier, Frank mentioned that they had not discussed continuity at length in their complex variables course, but focused more on differentiability (line 37). Dan elaborated, "we just kind of looked at something and said, 'Look it's clearly continuous' or 'It's discontinuous at this point'" (lines 38-39).

Int: Well, so real quickly, going back. Before you pursue that. You said something about you were pretty sure that the function's continuous at least? Why do you think it's continuous?
D: Um, so if you do, I don't know [writes $x - iy$] I mean there's no— you would never be dividing by zero, so I mean you can plug in any x value and any y value and it's--
F: Right.
D: I mean, there's a formal way you could prove it.
F: I mean, I know that in real numbers, one of the proofs for continuity is if the limit as um, x approaches c minus [writes $lim_{x \to c^-} f(x)$] of f(x) equals the limit as x approaches c plus of $f(x)$ [writes $lim_{x \to C^+} f(x)$] equals $f(c)$ [writes $= f(c)$, then we know it's continuous.
D: It seems like it's pretty clear that that would happen for this function.
F: Right. I just don't know what the analog is necessarily in terms of transferring that to complex numbers, or if it's really different.
F: We didn't really talk about continuity much. We talked more about differentiability.
D: I think we just kind of looked at something and said look it's clearly continuous, or it's discontinuous at this point.
F: Right. I mean, there's no issue with like zero here, because, so what. It's not like we're dividing by it, so I can't see any immediate discontinuities glaring out at me. [Erases continuity inscriptions]

Acting as spokesman, Frank used Dan's clarification as an opportunity to re-voice Dan's previous warrant about avoiding division by zero (lines 40-41). Accordingly, Dan's elaboration in lines 38-39 served as backing for this warrant's field. If, in their complex variables course, it was sufficient to simply look at a function's formula and draw conclusions about continuity, then the absence of any division by zero or similar symbolic issues was enough to conclude that $f(z) = \bar{z}$ is continuous. Hence Dan and Frank concluded Argument 2 with the claim that no obvious discontinuities exist. Curious if the pair had considered using the component functions u and v, I directed them (lines 42-43) to provide an alternate argument, which I refer to as Argument 3. Argument 2 is summarized in Figure 45.

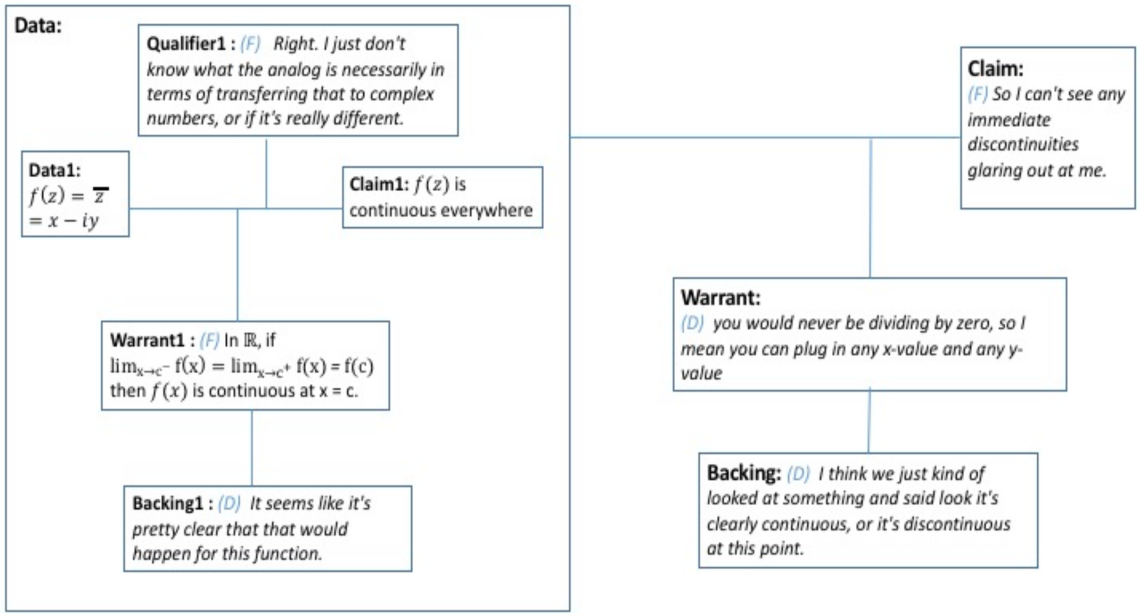


Figure 45. Toulmin diagram for Dan and Frank's Argument 2, Task 5b.

Int: Yeah so it certainly feels continuous. Um maybe if you— does it help to consider the component functions u and v at all?
D: Sure. So yeah, so if you look at x. So x is clearly continuous and y is clearly continuous as a single variable. So the--
F: Their sum has to be continuous, since the sum of two continuous functions is continuous.
Int: Yeah, so I don't know if you guys talked about this in class but there's a pretty well-known result that if the component functions u and v are continuous, then—
F: Then the sum is continuous.
Int: [...] Yeah then the actual function is continuous.
F: Yeah I think we talked about that. Yeah we did talk about that. Yeah I mean we've seen that in other contexts, too. If you have two complex functions themselves that are continuous, then their sum is certainly continuous. So yeah it would hold for the components as well.

Rather immediately and as author, Dan used symbolic reasoning to identify the real component function as x, and claimed that x and y are "clearly continuous" (lines 44-46). As Dan began to use these data to formulate another statement, Frank interrupted and claimed, "Their sum has to be continuous" (line 46). I then began to remind them of the result that if a complex function's component functions are continuous, then the function itself is continuous (lines 47-50). However, Frank interrupted my conclusion as well and reiterated, "Then the sum is continuous" (line 49). Thus, it appeared that he was quite certain that this was an acceptable warrant for their claim that the function is continuous. Indeed, using formal-symbolic reasoning, he followed this warrant with backing for its validity (line 53) by arguing that this more general property certainly applied to these particular component functions. Argument 3 is summarized in Figure 46 below.

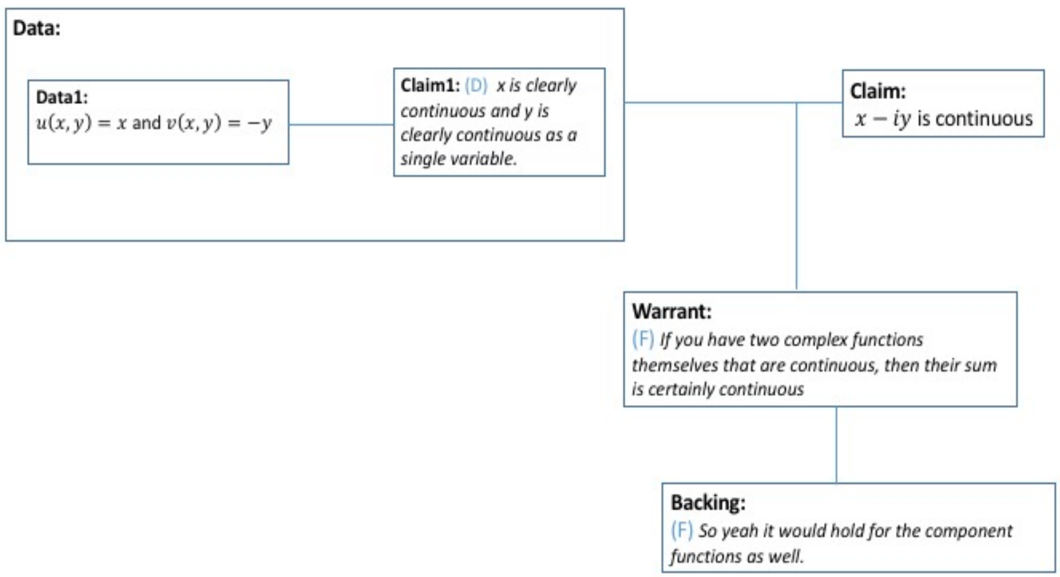


Figure 46. Toulmin diagram for Dan and Frank’s Argument 3, Task 5b.

Int: Yeah so anyway, continue with—
F: So we're trying to integrate this thing [writes $\int \bar{z}\, dz$] from— are we just doing any two points a to b? Is that what—
Int: So it's over this circle of radius R.
F: So from 0 to 2pi, is how we'll define it [writes 0, 2π off to the side].
Int: And I guess I didn't specify a center of the circle. [D writes in C_R^- under the integral symbol] You can pick whatever center you want. And it's oriented counterclockwise. I believe that's what you called a positive orientation.
F: Oh it's oriented counterclockwise! Ok
D: Ohhh! Oops. [Erases ‘-‘ and puts in ‘+’]
F: Then it's just $Re^{i\theta}$, which is even easier [erases previous parametrization].

Not wanting to interrupt the natural flow of their original argument too extensively, I asked them to proceed in their evaluation of the integral of this function (line 54). This signaled the beginning of Argument 4. Frank proceeded to relay the integral he and Dan were evaluating (lines 55-58). As I pointed out that they could pick a center for the circular path, Dan relayed their prior symbolic representation for the path, C_R^- (line 59). At this point, I chose to remind them (lines 60-61) that the path was oriented counterclockwise because Dan had repeated their previous error of denoting negative orientation. Dan and Frank were both surprised to hear this (lines 62-63), perhaps because they misinterpreted my original prompt. Dan altered his symbolic inscription to reflect this change (line 63). Using this revised inscription as a datum, Frank authored a claim that L can be parametrized as $Re^{i\theta}$, and qualified their revised task as “even easier” than previously anticipated (line 64).

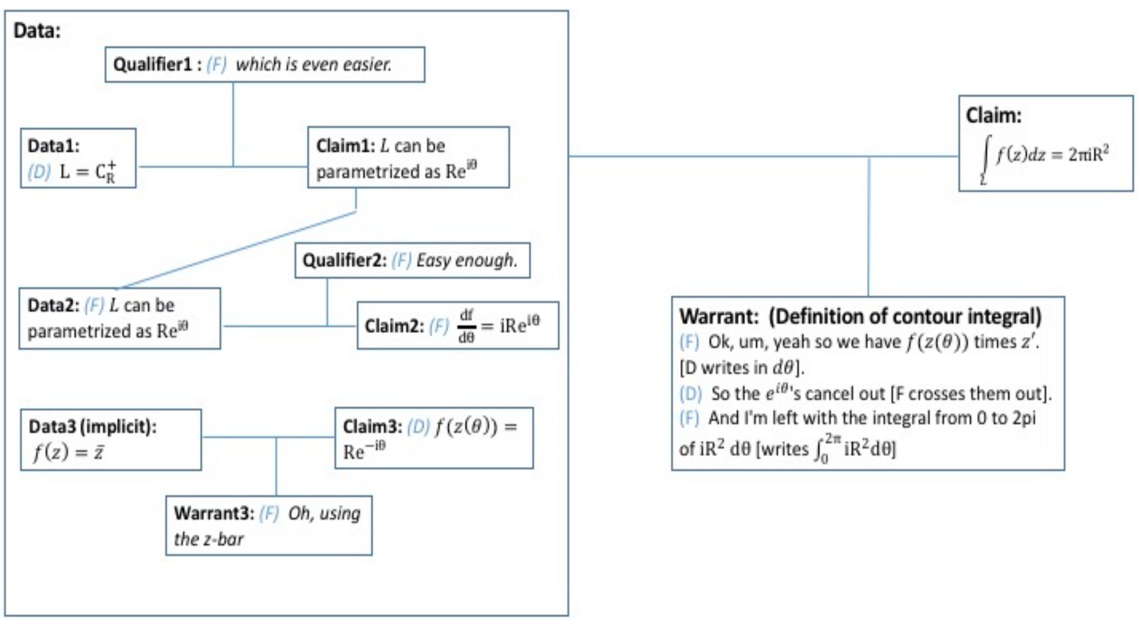


Figure 47. Toulmin diagram for Dan and Frank’s Argument 4, Task 5b.

Next, Frank applied symbolic reasoning to the previous claim, used now as a datum, to author a new claim that $\frac{df}{d\theta} = iRe^{i\theta}$ (lines 65-66), and qualified this as “easy enough.” Dan followed this with another symbolic claim that “z prime will just be $Re^{-i\theta}$” (line 67). However, because of what he said directly afterwards in line 69, I interpreted this claim to be that $f\big(z(\theta)\big) = Re^{-i\theta}$. Moreover, Frank’s clarification about the warrant, "Oh, using the z-bar," further suggests that Dan meant $f\big(z(\theta)\big)$ as opposed to z'. Frank additionally relayed Dan's claim in line 70.

F: So that's just $Re^{i\theta}$ [writes this below L =]. Uh if we take df/dt, because we're going to need that— [writes df/dt] that's just— oh sorry, $df/d\theta$, that's just $iRe^{i\theta}$ [writes this]. Easy enough.
D: And then then z prime will just be $Re^{-i\theta}$. Because, so we would need to plug in—
F: Oh, using the z bar.
D: I mean cuz you have the [writes $f(z(t))$] that's going to be [writes $Re^{-i\theta}$].
F: $iRe^{-i\theta}$. Or I'm sorry, $Re^{-i\theta}$. I agree with you there.
D: Ok so our integral is going to be from 0 to 2pi [writes $\int_0^{2\pi} Re^{i\theta}\, iRe^{-i\theta}$]
F: So are we defining our discontinuity to occur on the positive real axis?
D: Just the normal argument.
F: In which case, wouldn't we have to go from -pi to pi?
F: Oh yeah this works this works. I'm being silly. Ok, um, yeah so we have $f(z(\theta))$ times z'. [D writes in $d\theta$] I agree with that so far.
D: So the $e^{i\theta}$'s cancel out [F crosses them out]
F: And I'm left with the integral from 0 to 2pi of $iR^2\, d\theta$ [writes $\int_0^{2\pi} iR^2 d\theta$], which is just—
D: $2\pi iR^2$.
F: [Writes $i2\pi R^2$ and boxes it]. $i2\pi R^2$.

Frank clarified their choice of branch cut (line 72), using embodied reasoning as supported by the fact that he referred to a geometric location on a visualized Argand Plane. I say "visualized" here because Dan and Frank never drew a geometric diagram during this argument. Afterwards, Frank expressed concern about potentially having to alter their parametrization to make θ range from $-\pi$ to π, but quickly dismissed this concern (lines 74-75). Frank and Dan then continued to apply the definition of a contour integral as a warrant (lines 75-78), which I considered formal-symbolic reasoning because it relates the specific symbolic nature of the given function and parametrization to a formal definition. After algebraically simplifying their setup, Dan and Frank obtained an answer of $i2\pi R^2$, establishing their final claim (lines 79-80). Argument 4 is summarized in Figure 47.

Task 5c – Dan and Frank

Task 5c required participants to explicitly comment on whether $\int_L \bar{z}\, dz$ depends on the radius of the circular path. Dan initially answered "I don't think so" but clarified that "you just plug in your radius for R" (line 3), indicating symbolic reasoning related to their aforementioned result $i2\pi R^2$. Because Dan's two statements seemed to contradict one another, I echoed what I interpreted to be Dan's intended meaning (line 4), and Dan affirmed my statement (line 5). Proceeding as spokesman, Frank agreed that the integral depends on the radius (line 6), and his corresponding pointing gesture towards their prior inscription $i2\pi R^2$ suggested symbolic reasoning. In an effort to explain why he and Dan attained a particular symbolic answer, Frank additionally authored a formal-symbolic warrant for this assertion (lines 7-8), which attended to the analyticity of the function.

Int: Sounds good. And so does that depend on— so if I change the radius of that circle, does that change the answer of the integral then?
D: Um no, I don't think so. I mean, you just plug in your radius for R.
Int: Right. So your answer depends on R. So in that sense.
D: Right
F: Yeah I mean it's certainly going to depend on the radius [points to the inscription of the boxed answer $i2\pi R^2$]. And the reason why it's not— the reason why it is dependent on the radius is because it's a non-analytic function.
F: Were it analytic, then we wouldn't have to worry about it.
D: Ohhhh.
F: Cuz then the, uh, Cauchy-Goursat theorem would apply, right?
D: Yeah so it'd be zero or it'd be--
F: $2\pi i$. Yeah, depending on the number of discontinuities. But yeah in this case, a function of R [points to R in the inscription with their answer from Task 5b]. Yeah that makes sense.

Frank then authored a rebuttal articulating how the argument would change if $f(z)$ was analytic (lines 9-13). This rebuttal consisted of the hypothetical datum that $f(z)$ was analytic, which Frank used to claim that "we wouldn't have to worry about it" (line 9), likely meaning that the integral in question did not depend on the radius of the circle. At this point, Dan appeared to realize what Frank had in mind, as he exclaimed "Ohhhh" (line 10). Frank proceeded with a formal warrant for this claim, arguing that the Cauchy-

Goursat Theorem applied in this case (line 11), but qualified this assertion with the word "right?" (line 11).

Dan continued with the warrant by discussing the possible resulting symbolic values of the integral (line 12), but Frank interrupted and concluded that the answer depended on the number of discontinuities (line 13). Because Dan and Frank discussed the symbolic possibilities for an integral as dictated by a formal theorem, I characterized this reasoning as formal-symbolic. As spokesman, Frank re-voiced the pair's claim that the integral of the provided function $f(z)$ depended on the radius R, concluding that it was a "function of R" (lines 13-14). Argument 1 is summarized in Figure 48 below.

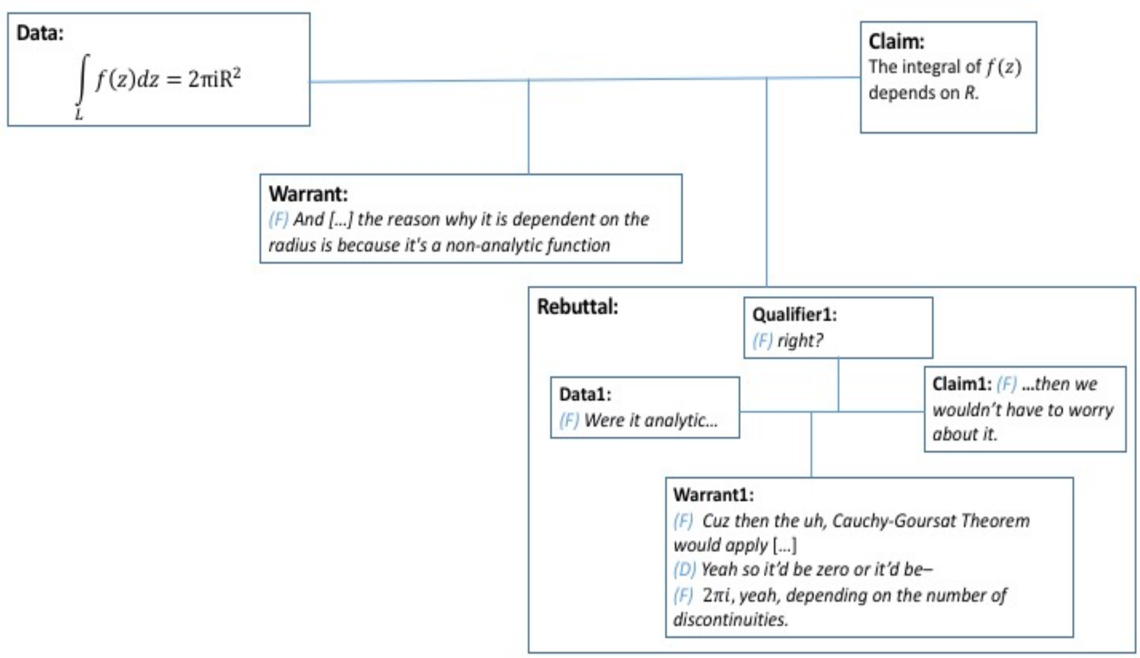


Figure 48. Toulmin diagram for Dan and Frank's Argument 1, Task 5c.

Probing further, I asked Dan and Frank where they chose to center the circle in this task (line 15). This question prompted a second argument, Argument 2, about Task 5c. Frank responded with embodied-symbolic reasoning, relating the geometric location of the center of the circular path to a symbolic inscription describing the path as $L =$

$C_R^+(0)$ (line 16). Because Frank's revised inscription clarified the semantic information used previously, but using new syntax, it appeared he acted as spokesman in this dialogue. Next, I asked if changing the center of the circle would affect the value of the integral (lines 17-18). Frank claimed that it would not, and that the integral "should simplify down to the same result" (line 19), but qualified this assertion with the phrase "um, I mean, I would imagine" (line 19).

Int: And so where did you choose to have the circle centered here [in Task 5b]?
F: We centered at zero [writes in (0) so that $L = C_R^+(0)$ now]
Int: So if we changed the center to be at maybe a different location, would that make this that different overall?
F: No it should simplify down to the same result, um, I mean, I would imagine.
D: Yeah I would agree.
Int: Well so if we did choose the center to be something different, not that we have to do the whole thing out, but how would the parametrization change? So what would L look like?
F: So we'd just have to shift it, right?
D: Yeah. So you'd say, like *a*— so say your point is centered at *a* right? It [points to $Re^{i\theta}$ inscription] would just be $a + Re^{i\theta}$.
F: Or if it was some complex number we would write it as $(a + bi) + Re^{i\theta}$ [writes this inscription]. Which we could write this out [points to $Re^{i\theta}$ inscription] using, uh, Euler's identity to break up the real and imaginary parts. Basically it's just some point [points to $a + bi$ inscription] plus the circle [traces a circular path in the air with his marker].

Dan agreed with this claim (line 20), but neither participant proceeded to elaborate on their assertion, so I asked them to at least consider how the parametrization for L would change (lines 21-22). In response, Frank authored the tentative suggestion that "we'd just have to shift it, right?" (line 23). Dan agreed and, as spokesman, provided a new symbolic parametrization $a + Re^{i\theta}$ (lines 24-25) using the datum that the circular path is centered at some point a.

I assumed that this point a was a complex number, but Frank responded with a rebuttal that considered an alternate case of the circle centered at some $a + bi$, and adjusted the symbolic inscriptions for the parametrization accordingly (line 26).

Using this other parametrization as a datum, Frank pointed to the $e^{i\theta}$ portion and claimed that they could algebraically expand this expression, invoking Euler's Identity as a warrant (lines 27-28). Finally, Frank articulated their revised parametrization as spokesman, using the phrase "some point plus the circle" (lines 28-29). While saying "some point," he pointed at the symbolic inscription $a + bi$, and while saying "plus the circle" he traced a circular path in the air with his marker pen (see Fig. 49). Accordingly, Frank's summary remark seems to indicate embodied-symbolic reasoning, in that he related the symbolic inscriptions $(a + bi)$ and $e^{i\theta}$ to a point in the Argand plane and a dynamic enactment of a circular parametrized path, respectively. Argument 2 is depicted in Figure 50.

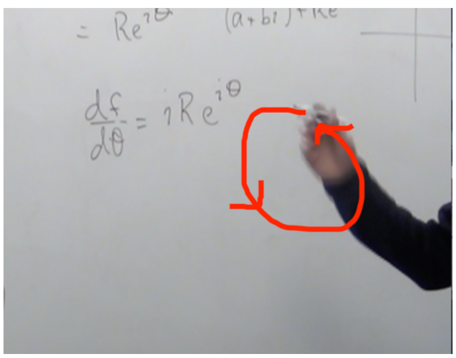


Figure 49. Frank's circular path gesture during Argument 2 for Task 5c.

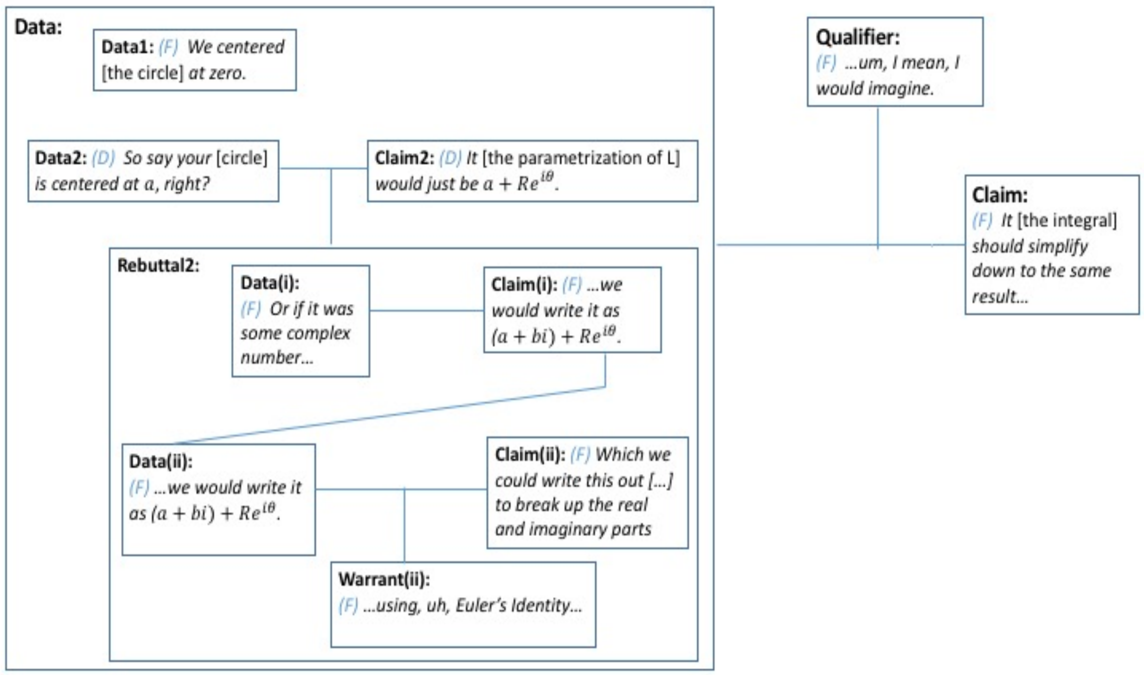


Figure 50. Toulmin diagram for Dan and Frank's Argument 2, Task 5c.

Task 5a – Riley and Sean

Sean began the response to Task 5a as spokesman as he wrote symbolic inscriptions characterizing $f(z)$ as $\bar{z} = x - iy$ (line 2). Riley authored a formal claim that "to be analytic it has to be differentiable everywhere," and Sean agreed (lines 4-5). She qualified their argument by stating that she recalled this function as not analytic but could not remember why (lines 8-9). Sean assisted by authoring a formal-symbolic warrant appealing to the Cauchy-Riemann equations (line 10). He elaborated this warrant by symbolically identifying $u(x, y) = x$ and $v(x, y) = iy$, both of which he classified as continuous (lines 12-13). Setting $u_x = v_y$ and $u_y = -v_x$, he concluded the function is not differentiable and thus "not analytic anywhere," citing that $u_x = 1$ and $v_y = -1$ (lines 13-15). Argument 1 is summarized in Figure 51.

Int: Ok so we're going to consider for this task the function $f(z) = z$ conjugate.
S: [Writes $f(z) = \bar{z} = x - iy$]
Int: So first of all, is this function analytic? If so, where is it analytic? Or if not, how do you know?
R: Ok so to be analytic it has to be differentiable everywhere.
S: Mhm.
R: Um—
S: So you can use—
R: I mean, I remember that it's not, but I don't remember why. So let's see if we can figure this out. [Draws an Argand Plane]
S: Well there's a test. We can use the Cauchy-Riemann equations if we wanted to.
R: Oh that's true.
S: Yeah, so we can say that's u [labels the x term with u], that's v [labels the iy term with v]. So we can say, first of all the partials are always continuous, cuz these partials— so that's good. Then you set $u_x = v_y$, and there's $u_y = v_x$. u_x is 1, v_y is negative 1. Nope! Ok it's not differentiable, and therefore cannot be analytic anywhere. Which is a very quick test.

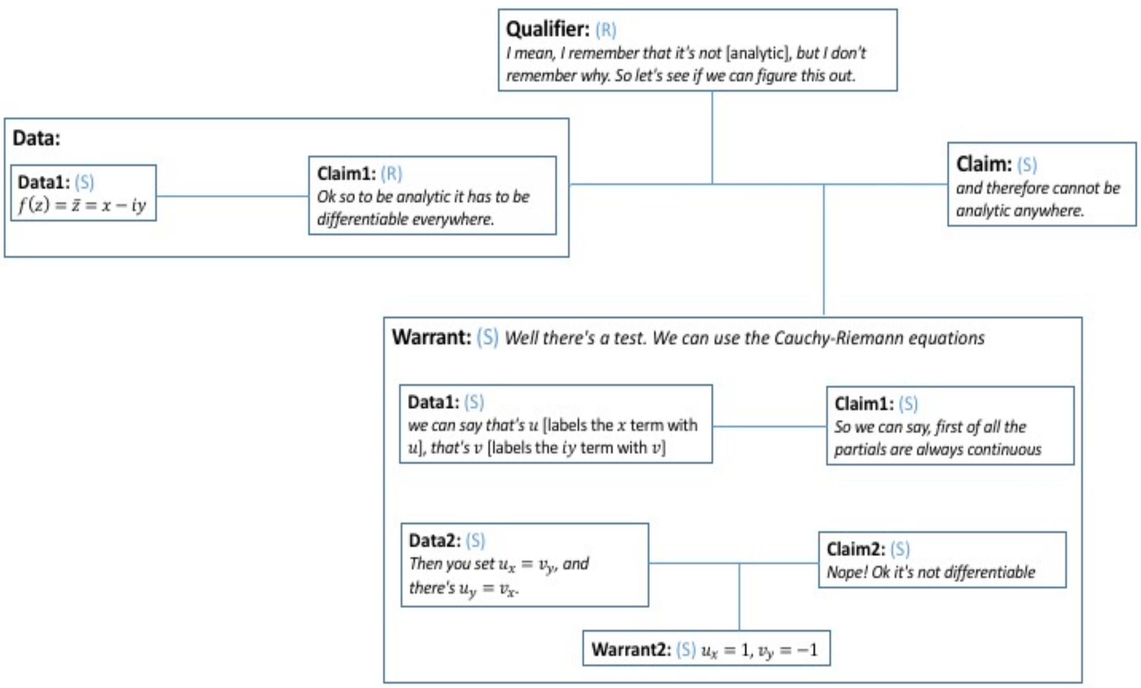


Figure 51. Toulmin diagram for Riley and Sean's Argument 1, Task 5a.

Following Argument 1, I asked Riley and Sean a follow-up question about why they immediately wrote the function as $x - iy$ (lines 16-17). Riley authored a formal-symbolic datum that they were using the Cauchy-Riemann equations to test the function's differentiability, and claimed that they needed f to be expressed in terms of its component functions $u(x, y)$ and $v(x, y)$ (lines 18-21). As spokeswoman, Riley curiously referred to this symbolic form as “vector notation” (line 21), and claimed that $u(x, y) = x$ and $v(x, y) = y$. Note that in both Arguments 1 and 2, Sean and Riley respectively identified $v(x, y)$ incorrectly, as Sean claimed $v(x, y) = -iy$ and Riley claimed $v(x, y) = y$, when in fact it should be $v(x, y) = -y$. Riley also referred to $x - iy$ as z, but revised her symbolism to “$f(z)$ […] Or like w or something” (line 24). As a symbolic warrant for her choices of u and v, Riley clarified that “conjugate z is just, um, the negative of the […] imaginary component, for whatever z was” (lines 22-23). She closed Argument 2 with a formal-symbolic claim relaying the ease of invoking the Cartesian form when evaluating the Cauchy-Riemann equations to test for differentiability (lines 25-28). Argument 2 is depicted in Figure 52.

Int: Ok yeah, great. Um, so I noticed you wrote the function immediately too as $x - iy$. Is there any particular reason why you wanted to work with that representation?
R: Yeah so if you want to use like, the Cauchy— uh, is it the Cauchy-Riemann?— differentiability um—
S: "Tests."
R: conditions, yeah, then we're going to need z to be represented as uh, like a function, well, of $u(x)$ and v— uh, $u(x, y)$ and $v(x, y)$. Um and so we can put that in vector notation or whatever, and so it turns out— so, conjugate z is just, um, the negative of the y component or the— the imaginary component, for whatever z was. And so if we're just writing conjugate of z, then it's $z = x - iy$, where $x = u(x, y)$ and I guess this— this should probably be $f(z)$ right? Or like w or something. Um, and y is going to be $v(x, y)$ so if we write it that way it's easier for us to use those conditions for differentiability.
Int: Ok sounds good.
R: So it's an easier way to represent it.

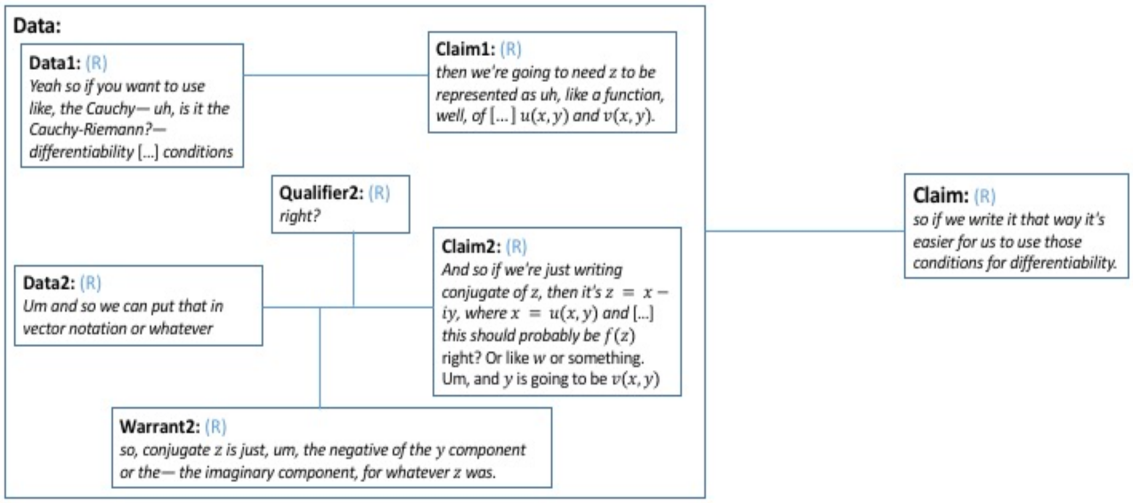


Figure 52. Toulmin diagram for Riley and Sean's Argument 2, Task 5a.

Task 5b – Riley and Sean

Riley began the pair's response to the second portion of Task 5 as spokeswoman, producing an embodied diagram of the circular path L (line 3; see Fig. 53). She then quickly authored a claim that it is not possible to integrate $f(z) = \bar{z}$ along the path L, and cited the formal-symbolic warrant that this function is not analytic (line 4). As spokesman, Sean wrote a symbolic inscription echoing the integral in question (line 5), which prompted me to clarify that I had not specified a center of the circle, but that he and Riley could center it at zero (lines 6-7). At this time, Riley repeated her claim and warrant (line 9), and Sean qualified this claim-warrant pair by remarking, "That's what I think. I just want to make sure" (line 10).

Int: Yeah, just curious. Ok so, keeping this example in mind, is it possible to find the integral of this function around the path L where L is a circle of radius r traversed counterclockwise?
R: [Draws circle and labels orientation and radius] Counterclockwise, radius R.
R: It's not an analytic function, so— it shouldn't be, should it?
S: [writes $\int_{C_R^+(0)} \bar{z}\, dz$] Uhh—
Int: And I didn't specify necessarily that it would be centered at 0, so you're free to choose— if you'd like it to be centered at 0, that's totally fine with me.
S: Hmm, that's a good question.
R: I don't think you can do it, because it's not analytic.
S: That's what I think. I just want to make sure.
Int: Well so why do you say that? What is— what does it being analytic—
R: Actually, well, could we parametrize it? Cuz that doesn't have to be analytic, it just has to be continuous.
S: We *could*. I remember it not being continuous anywhere, from re-reading it. Like on the test, it's not— like you end up with—
R: Is it uh [pauses].

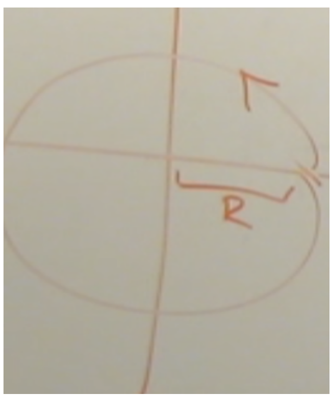


Figure 53. Riley's sketch of circular path, Argument 1, Task 5b.

Because neither Riley nor Sean had provided any further explanation about why the absence of analyticity prevented them from integrating this function, I asked them to elaborate on this connection (line 11). This prompted Riley to challenge her previous claim by authoring a formal-symbolic rebuttal suggesting parametrization (lines 12-13). However, Sean claimed that he remembered this function as discontinuous when it appeared on a previous test in the course (lines 14-15). This first argument is summarized below in Figure 54.

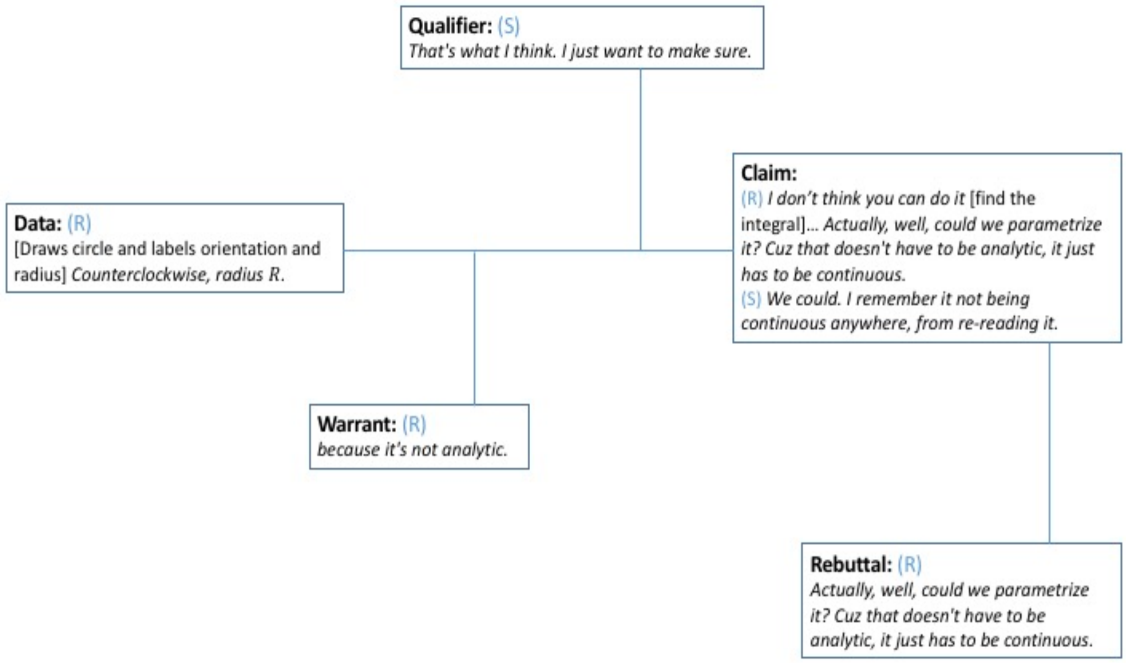


Figure 54. Toulmin diagram for Riley and Sean's Argument 1, Task 5b.

Given that Sean and Riley attempted to simply recall from a previous test the continuity of this function, I directed them to revisit its continuity together (line 17). This began Argument 2, in which Sean authored an embodied-symbolic datum considering a limit approaching the origin along the real and imaginary axes (lines 18-19). While articulating the "two different paths" (line 18), Sean produced a pair of embodied gestures illustrating these two manners of approaching the origin (see Fig. 55). Riley then challenged some of Sean's symbolism in his limit inscriptions, and encouraged him as spokeswoman to rewrite (x, y) as $(x, 0)$ given that $y = 0$ along the real axis (lines 20-23). Sean authored a formal claim that the function is not continuous anywhere, and cited a formal-symbolic warrant that the two aforementioned limits yield "different values" (line 24). Sean then further revised his symbolism as spokesman to account for approaching a general point (x_0, y_0) rather than $(x_0, 0)$ (lines 24-26).

Int: Well so how could you determine if that function was continuous or not?
S: Let me do— what, I take a limit as it approaches, say, 0 from like two different paths. We could say like, limit as (x, y) approaches (0,0) [writes this limit] for— for like the *x*-axis. Then do it again for—
R: Wait you're probably going to want to— so if you're approaching it from the *x*-axis then *y* is always going to be 0, so do like, limit as $(x, 0)$ approaches (0,0).
S: Yeah.
R: Just for the notation, yeah.
S: So then I could do like, show different values and be like "ok it can't be continuous anywhere." Actually no, it would be any point in general. That's just 0. So you would say limit as (x, y) goes to (x_0, y_0) [revises previous limit inscription] of—
R: Of *z* conjugate, right?
R: So like— cuz you end up breaking this into two ones right? Like you approach from two different paths. So this is the general limit. But then you say limit as like (x_0, y) approaches (x_0, y_0) [writes this limit off to the side], um sorry my handwriting is too large but—
S: I think we're going differentiability right now. Because we're going to go—
S: Um, yeah. So for continuity, I know one's an epsilon-delta proof, which I don't want to do. So I know it was like, limit at a point exists, the function at a point exists, the two are equal. So we'd have to show the limit as you approach two different paths is not the same as the limit value at a point.
R: I have not done very much with limits since like Calc 1. Even then, we sort of brushed through it, so—

Figure 55. Sean's gestures for approaching along two paths, Argument 2, Task 5b.

Next, Riley suggested writing $\bar{z}$ in their inscription in order to clarify what they were taking the limit of, and qualified this addendum with "right?" (line 27). As spokeswoman, she again changed the symbolism corresponding to their limits, claiming that (x_0, y) should approach (x_0, y_0) in accordance with their embodied-symbolic warrant that "you approach from two different paths" (lines 28-30). At this time, Sean further illustrated their confusion by authoring a symbolic rebuttal that their work corresponded to differentiability and not continuity (line 31). He acknowledged that one could demonstrate continuity via a formal epsilon-delta proof, but he instead invoked a formal-symbolic warrant that in order for $f(z)$ to be continuous at z_0, $\lim_{z \to z_0} f(z)$ must equal $f(z_0)$: “limit at a point exists, the function at a point exists, the two are equal” (lines 32-33).

Note that there are multiple ways for this continuity equality to be violated, such as the limit not existing or the limit not equaling $f(z_0)$. Despite their prior symbolic attempts at the former, Sean chose to discuss the latter, and claimed that “we'd have to show the limit as you approach two different paths is not the same as the limit value at a point” (lines 33-34). Riley closed Argument 2 by authoring a qualifier expressing uncertainty with their statements about continuity due to her inexperience with limits (lines 35-36). Argument 2 is summarized in Figure 56 below.

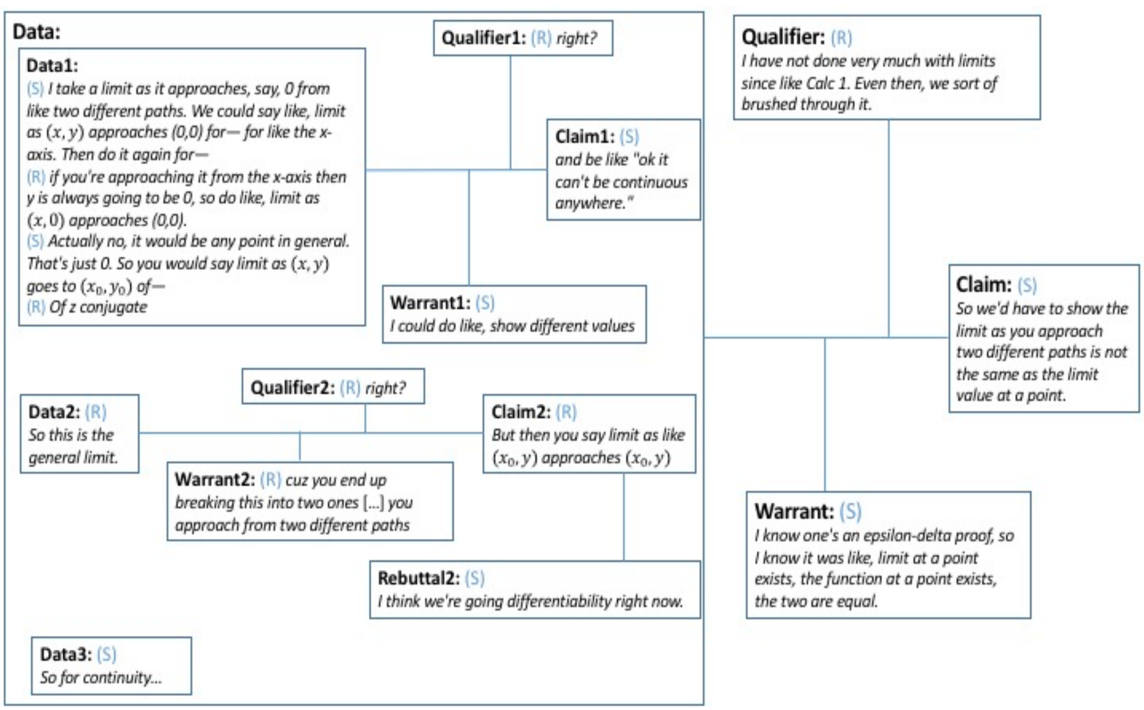


Figure 56. Toulmin diagram for Riley and Sean’s Argument 2, Task 5b.

Given Sean's apparent conflation with differentiability conditions, as well as his and Riley's difficulty with the symbolism in their limit statements, I asked Sean to clarify whether he was showing the function was continuous or discontinuous (lines 37-39). Riley clarified that they were attempting to show the function is discontinuous, and authored a brief Argument 3 in support. Specifically, as spokeswoman, she re-voiced the previous requirement for continuity that the limit as one approaches z_0 along any path must exist (lines 40-41). She further claimed that the formal Cauchy-Riemann equations hold due to this same type of limit property, though her articulation of this connection was fairly nebulous (lines 41-42). Finally, she authored a formal-embodied claim that a discontinuous function has the property that "there will always be at least two paths that converge to different limits" (lines 43-45). Note that this once again attends to the existence of the limit rather than whether the limit equals $f(z_0)$. Argument 3 is depicted in Figure 57.

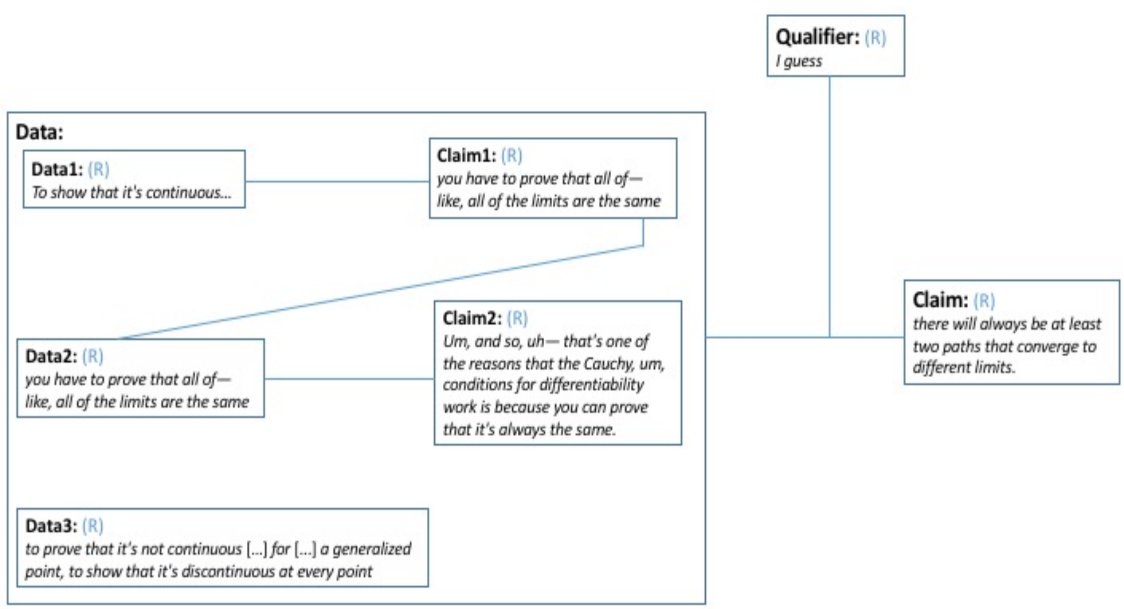


Figure 57. Toulmin diagram for Riley and Sean's Argument 3, Task 5b.

Int: Well so you were saying, Sean, that if you look at two paths and the limits don't agree, then— were you trying to say that you need to check two paths for showing that it's continuous or showing that it's discontinuous?
R: To show that it's discontinuous. To show that it's continuous, you have to prove that all of— like, all of the limits are the same. Um, and so, uh— that's one of the reasons that the Cauchy, um, conditions for differentiability work is because you can prove that it's always the same.
R: Um but yeah, to prove that it's not continuous, all you have to do is show that for some like— for I guess a generalized point, to show that it's discontinuous at every point, there will always be at least two paths that converge to different limits.

With this general approach in mind, I redirected Riley and Sean's attention to the specific function at hand and asked them if they thought this function was continuous or not (lines 46-47). This catalyzed Argument 4, which Riley began with an uncertain "hmm" (line 48). I reminded them that they already determined f was not analytic (line 49). Sean relayed his recollection of the function not being differentiable (line 51), but Riley authored a formal claim cautioning that a lack of differentiability does not imply discontinuity (line 52). This prompted Riley to consider a symbolic datum of parametrizing the circular path as $z = re^{i\theta}$, in which case she symbolically claimed $\bar{z} = re^{-i\theta}$ (lines 54-55).

Int: Ok yeah. Well so do you have hunch as to whether this function here is going to be continuous or discontinuous?
R: Hmm.
Int: So you determined it wasn't analytic right?
R: Yeah, um [pauses]
S: I definitely remember that it's definitely not [pauses] differentiable.
R: Well it's definitely not differentiable, but that doesn't mean it's not continuous.
S: Yeah. Um—
R: Um, can we [pauses] So if we look at this one more— z equals, let's see, $re^{i\theta}$ [writes this], right? Then wouldn't z conjugate just be $re^{-i\theta}$ [writes this]?
S: Mhm.
R: Um, that seems like it ought to be continuous, because [pauses]
Int: Why does it seem like that would be continuous?
R: Um [pauses]
S: Because polar notation is just— you're just going on the unit circle, or a circle with radius r, but now you're clockwise instead of counterclockwise, as you go from $\theta = 0$ to 2π. So it does seem like it should be continuous.
Int: So your hunch is that it's continuous.
R: Yes.

With this parametrization in mind, Riley symbolically claimed that it "seems like it ought to be continuous" (line 57). She began to explain why, but paused long enough for Sean to step in and author his own embodied warrant that reversing the orientation of a circle should not affect its continuity (lines 60-62). Riley agreed with this conclusion (line 64), closing Argument 4, which is summarized in Figure 58 below.

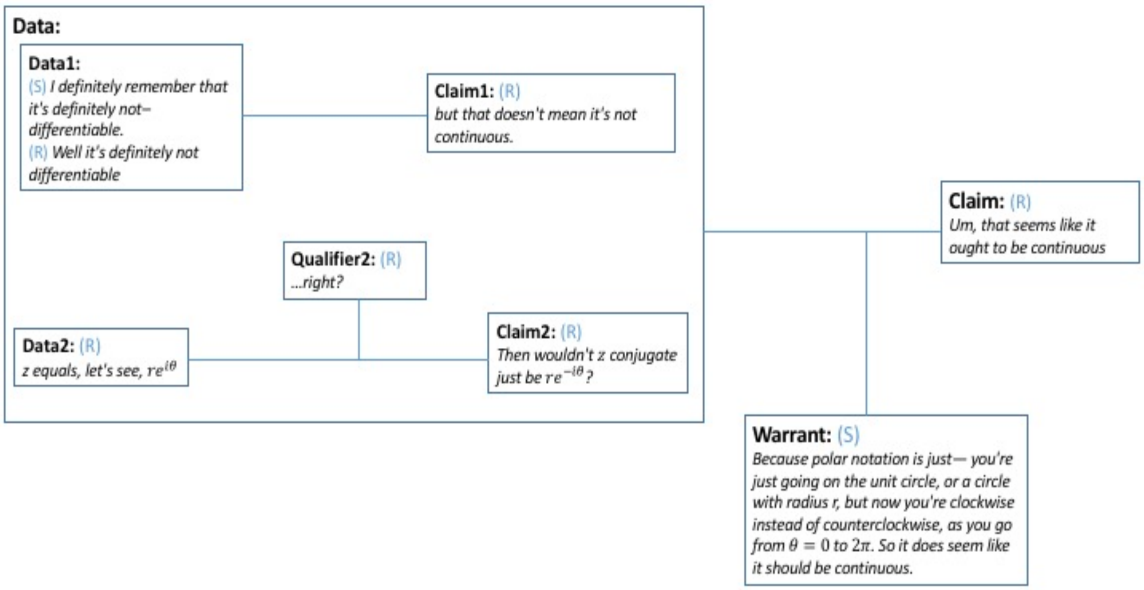


Figure 58. Toulmin diagram for Riley and Sean's Argument 4, Task 5b.

Because Riley and Sean abandoned their previous limit inscriptions, I asked them if they wished to revisit this prior reasoning (lines 65-66), and this resulted in a long pause from both participants (line 67). I took this to mean Riley and Sean did not wish to pursue their limit inscriptions. Due to the amount of time already spent on determining the function's continuity, I provided a rather large hint about considering the component functions u and v (lines 68-69). Even so, Riley only hesitantly claimed that the continuity of u and v should determine the continuity of f, as indicated by her qualifier "I mean, I guess [...] right?" (lines 70-71). She also claimed that this implication meant that $f(z) = \bar{z}$ is continuous (lines 71-74), and authored a symbolic warrant comparing u and v for the functions z and $\bar{z}$ (lines 76-77). This argument implicitly rested on the continuity of $g(z) = z$, so I asked Riley and Sean explicitly if they believed that this identity function is continuous, and they confirmed that they did (lines 78-79). Argument 5 is summarized in Figure 59.

Int: You mentioned some things about limits. Do you want to pursue some of those limit things, or is there maybe something else that you can consider? It's up to you guys.
R: Um [~30 second pause]
Int: So you also have your component functions u and v, right? Do those tell you anything about whether the whole function will be continuous or not?
R: I mean I guess if both u and v are continuous then it would make sense that the function would be continuous, right? So, um [pauses] so yeah I think it's continuous. So that would make this continuous. Um—
Int: So you think u and v are continuous functions?
R: Uh, yeah.
Int: Ok.
R: Just cuz, like, u is— u is the same for both z and z conjugate. And v is just negative, um— so just negating should not make it not continuous.
Int: And you're quite certain that $f(z) = z$ [not the same f as the task function] is continuous?
R: Yes. [Sean also nods his head in agreement].

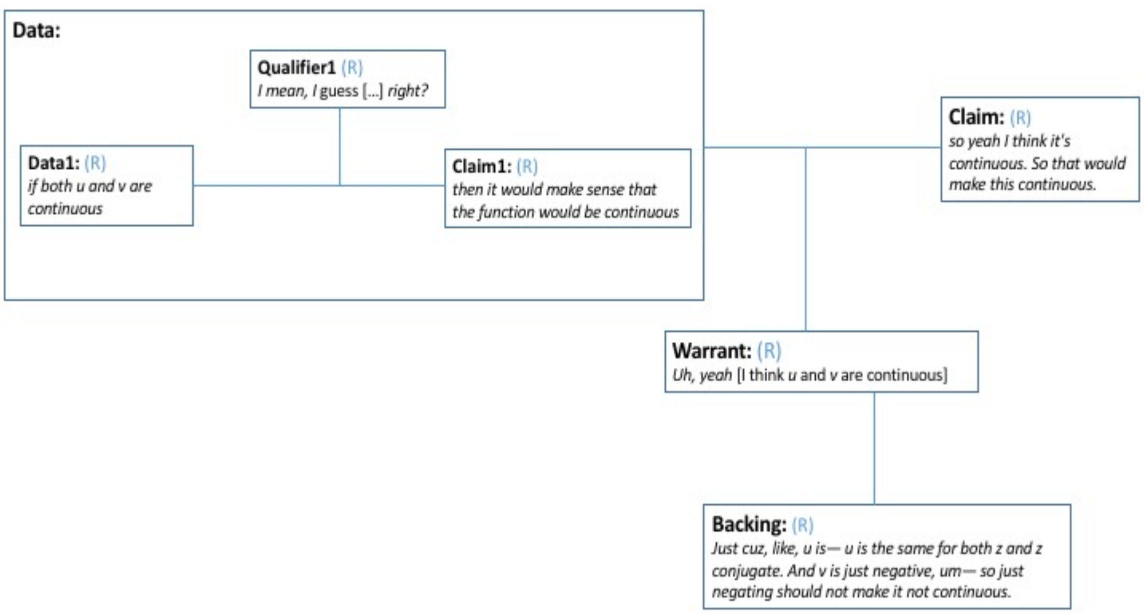


Figure 59. Toulmin diagram for Riley and Sean's Argument 5, Task 5b.

Int: Did you guys, in the course, show things about continuity very much? In terms of specific functions or—?
S: No.
R: No, not really.
S: We did a lot of problems on differentiability, like we tried to do there [points to board work with C-R inscriptions] and analyticity, but never— not continuity.
Int: Ok.
S: I think he said it's more for like, a "complex analysis" like, kind of a real analysis-based course. Like those kind of topics that we would just do that for a one-semester course.
Int: Yeah, that makes sense. Well yeah, so, as you said though, Riley, if you have— if the component functions u and v are continuous, then in fact that's a condition for the whole function being continuous. It's an if and only if thing, even. So if the function is continuous, then u and v are going to be continuous, vice versa too. So that's a nice way to go about that here.

Due to Riley and Sean's difficulty with determining continuity in this task, I asked them if they had carefully shown whether particular functions were continuous or not during their course (lines 80-81). This question was not meant to induce another argument, but rather to put their struggle in context; as such, this portion of the interview did not constitute an argument. Rather, Riley and Sean both denied discussing continuity of specific functions in their course, and Sean clarified that they instead focused more on differentiability and analyticity (lines 84-85). Sean explained that Professor X justified this choice of omission based on the fact that such material is often covered as part of a real-analysis-based "complex analysis" course rather than just complex variables (lines 87-88). To conclude the discussion on continuity, I informed Riley and Sean that indeed the continuity of u and v implies the continuity of f (lines 89-92).

Int: But you also could— you know, I won't have you go through all that, but you could sort of pursue this by looking at some limit ideas too. But it's kind of more Calc 3 than what we're interested in here. But anyway, so you determined the function is continuous. But um, remember the original question was to see if we could find this integral here of z conjugate. So we decided the function is continuous, so now how would we pursue maybe going about finding the integral?

R: Could we use, um, parametrizing it and say, um— cuz I don't remember what the exact, uh, conditions are, but basically say like, [integral of] $f(z(t))$ times $z'(t)\ dt$ [writes $\int_a^b f(z(t))z'(t)dt$]?

S: Mhm.

R: Because we already um— so we have it parametrized, right [points to $re^{i\theta}$ inscription]? Like assuming we're on a circle centered at the origin [points to diagram of circular path], then— I mean if we're not then it's not that difficult, but— Um, because then [pauses] Well let's see, um—

S: Well the differentiability constraint could come in.

R: I don't think it does. I think it just has to be continuous. But like I said, I don't remember the exact restrictions— but I don't think it had to be analytic.

S: Ok [long pause]. So we should be fine.

With the continuity of f resolved, I redirected Riley and Sean's attention to the integral of this function (lines 93-97). Riley began the ensuing Argument 6 by articulating a symbolic warrant appealing to parametrization (line 98). She qualified this suggestion by admitting, "I don't remember what the exact, uh, conditions are" (lines 98-99), but claimed they could evaluate the symbolic integral $\int_a^b f(z(t))z'(t)dt$ (line 99). Sean agreed (line 100), and Riley relayed their previous parametrization $z = re^{i\theta}$ as a symbolic datum (line 101). She also relayed the embodied datum of their circular path, and authored an embodied-symbolic claim that if the circle were centered at a location other than the origin, "it's not that difficult" to adjust their inscriptions accordingly (lines 102-103).

In response, Sean challenged her assertion in the form of a formal rebuttal in which he cautioned that the function's (lack of) differentiability might preclude them from pursuing this method (line 104). However, Riley maintained that she did not believe this differentiability was germane, but qualified this response by relaying her previous acknowledgement of not knowing the necessary conditions for parametrization (lines

105-106). Sean conceded, and claimed, “we should be fine” (line 107), thereby concluding Argument 6, which is depicted in Figure 60.

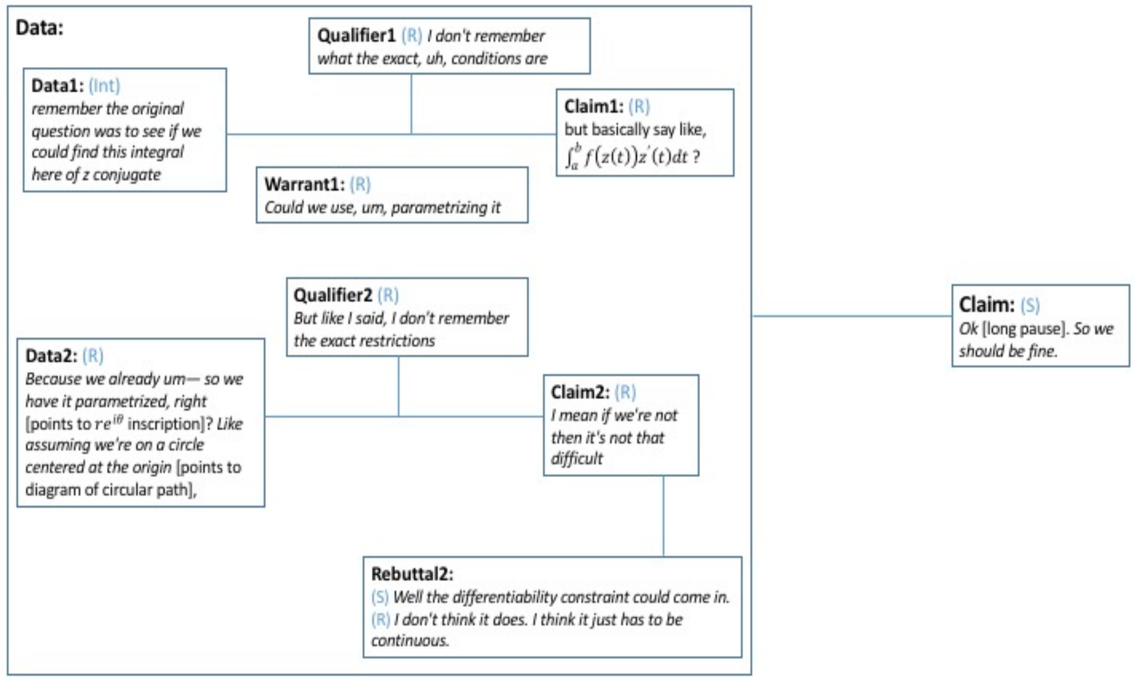


Figure 60. Toulmin diagram for Riley and Sean’s Argument 6, Task 5b.

Afterwards, Sean symbolically set up their integral as $\int_0^{2\pi} Re^{-i\theta}(iR)e^{-i\theta}d\theta$ and simplified this to become $-iR^2\int_0^{2\pi} e^{-2i\theta}d\theta$ (lines 108-109). Note that Sean incorrectly computed $z'(\theta) = iRe^{-i\theta}$ instead of $iRe^{i\theta}$. He then evaluated this integral by symbolically taking an antiderivative of the integrand and employing the Fundamental Theorem of Calculus to obtain $\frac{-iR^2}{-2i}e^{-2i\theta}\Big|_0^{2\pi} = \frac{R^2}{2}[e^{-4\pi i} - 1]$ (line 109). Next, Sean authored a symbolic warrant that $e^{-4\pi i} = \cos(-4\pi) - i\sin(4\pi)$ (lines 109-110), though Riley apparently did not realize that Sean had implicitly applied the identity $\sin(-\theta) = -\sin(\theta)$ in his inscription (line 111). Nonetheless, Sean and Riley jointly concluded that the integral vanishes “just like the last way was” (line 111-113). However, Riley qualified their conclusion by questioning their correctness (line 114). Because Sean did not respond to Riley's question, and because be previously set up the integral incorrectly due to misidentifying $z'(\theta)$, I tried to draw their attention to this setup by asking a follow-up question about using the Fundamental Theorem (line 115). This led to a new argument, as detailed below; Argument 7 is summarized in Figure 61.

108 *S:* Let's just set it up [writes $\int_0^{2\pi}$]. So $Re^{-i\theta}$, times [writes integrand of $Re^{-i\theta}(iR)e^{-i\theta}d\theta$, then writes =
109 $-iR^2\int_0^{2\pi}e^{-2i\theta}d\theta = \frac{-iR^2}{-2i}e^{-2i\theta}\Big|_0^{2\pi} = \frac{R^2}{2}[e^{-4\pi i}-1]$]. Then that's going to be [points to $e^{-4\pi i}$ then
110 draws arrow from this term and rewrites as $\cos(-4\pi) - i\sin(4\pi)$].
111 *R:* Well it's going to be— it's negative 4 [likely referring to $i\sin(4\pi)$]. It's going to be the same as zero
112 just like the last way was, right?
113 *S:* [Agrees and writes "= 0"].
114 *R:* The question is, is that correct?
115 *Int:* Well, so it looked like you used, uh, the Fundamental Theorem somewhere? Or an antiderivative?

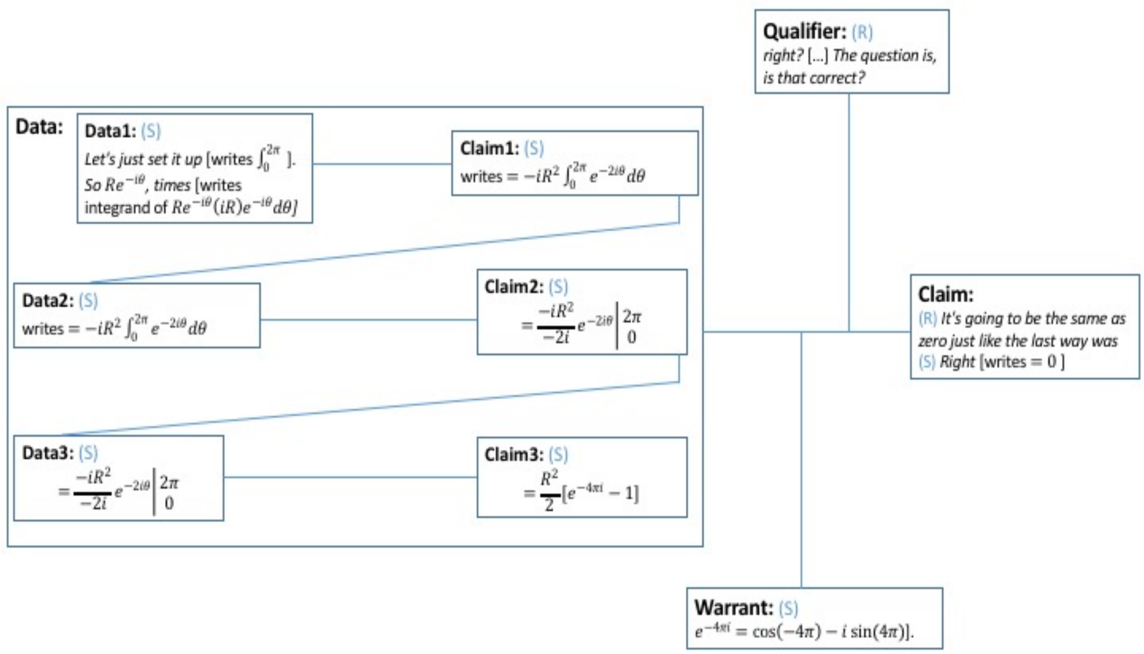


Figure 61. Toulmin diagram for Riley and Sean's Argument 7, Task 5b.

In response, Riley asked me to illuminate what step in their calculation I was referring to, and then redirected my question to Sean (lines 117-121). Employing symbolic reasoning, Sean clarified that the integrand was a function of θ (line 122), and Riley acted as spokeswoman to add that "these are real variables, just going from 0 to

2π" (line 124). Riley additionally authored a formal-symbolic warrant that Sean was previously using the "Calc 1 version" and that this technique is "not specific to complex" (lines 123-124). She drew an arrow between Sean's previous symbolic inscriptions to indicate where he had implicitly utilized the theorem (line 127), and Sean provided backing for their warrant's validity by explaining "$e^{-2i\theta}$ is well-defined and definitely differentiable" (line 128). Accordingly, Sean claimed as spokesman that they could take an antiderivative (lines 128-129). Argument 8 is summarized in Figure 62.

R: Like uh, where do you mean?
Int: Like in this third step there.
R: Here? [points to $\frac{-iR^2}{-2i}e^{-2i\theta}\Big|_0^{2\pi}$]
Int: Yeah.
R: So here, he used— I guess you used the Fundamental Theorem of Calculus?
S: But now this is just a function of theta [points to $-iR^2\int_0^{2\pi}e^{-2i\theta}d\theta$]
R: Yeah so in this case, it's not— like it's not specific to complex. It's just, um, like he was saying with the Calc 1 version, because we're uh— like these are real variables, just going from 0 to 2π. Um, but yeah as far as that integration, we have the Fundamental Theorem of Calculus.
Int: So that's what you're using there? In that third step there?
R: Yeah, here? Between here and here? [Draws arrow between $-iR^2\int_0^{2\pi}e^{-2i\theta}d\theta$ and $\frac{-iR^2}{-2i}e^{-2i\theta}\Big|_0^{2\pi}$]
S: Yeah. Cuz $e^{-2i\theta}$ is well-defined, and definitely differentiable, so you just use the antiderivative of that.

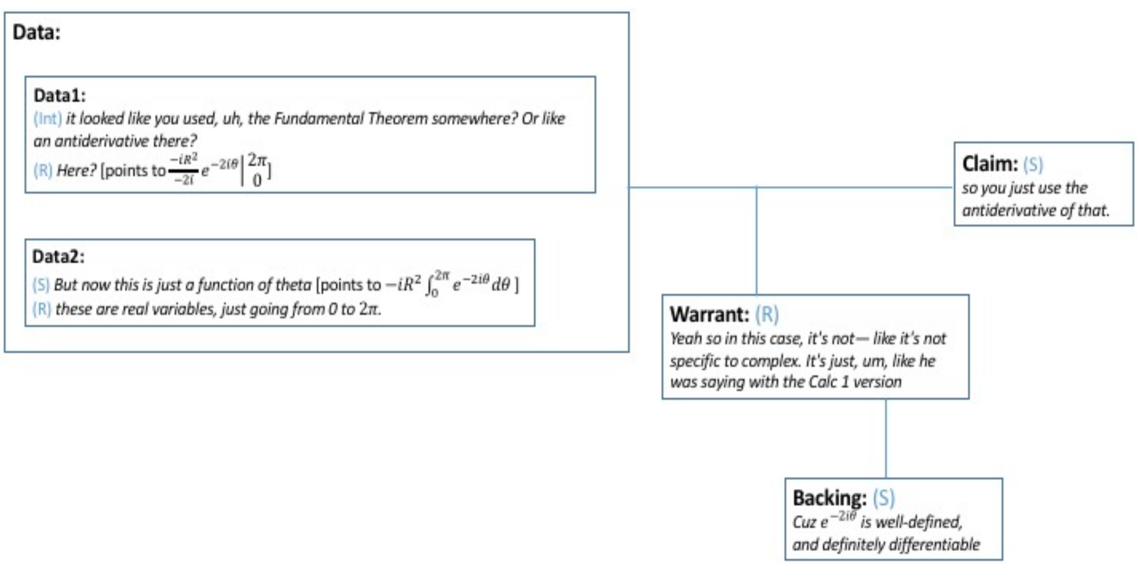


Figure 62. Toulmin diagram for Riley and Sean's Argument 8, Task 5b.

Because neither Sean nor Riley discovered Sean's aforementioned differentiation error, I pressed Sean on his backing by reminding them that they decided $f(z)$ was not differentiable anywhere in Task 5a (lines 130-134). Riley seemed to recognize a potential problem with this (line 135) and Sean claimed "that's where the disconnect comes" (line 136). Riley also added that this disconnect made her doubt whether they could use parametrization to evaluate the integral because "it probably has to do with […] those endpoints," perhaps alluding to the connection between path-independence and analyticity (lines 137-138). Argument 9 is summarized in Figure 63 below.

Int: So $e^{-2i\theta}$, you were saying, is differentiable?
S: Mhm.
R: Yeah, and you can, you know, integrate it. Um—
Int: Well so, before, you were saying that it didn't satisfy the Cauchy-Riemann equations, but now you're saying it is differentiable?
R: Hmm, yeah that's true. Mmmm
S: That's where the disconnect comes.
R: Yeah that's why I'm not sure whether we can use this or not [points $\bar{z} = re^{-i\theta}$] because it probably has to do with like [pauses] those endpoints. Um—

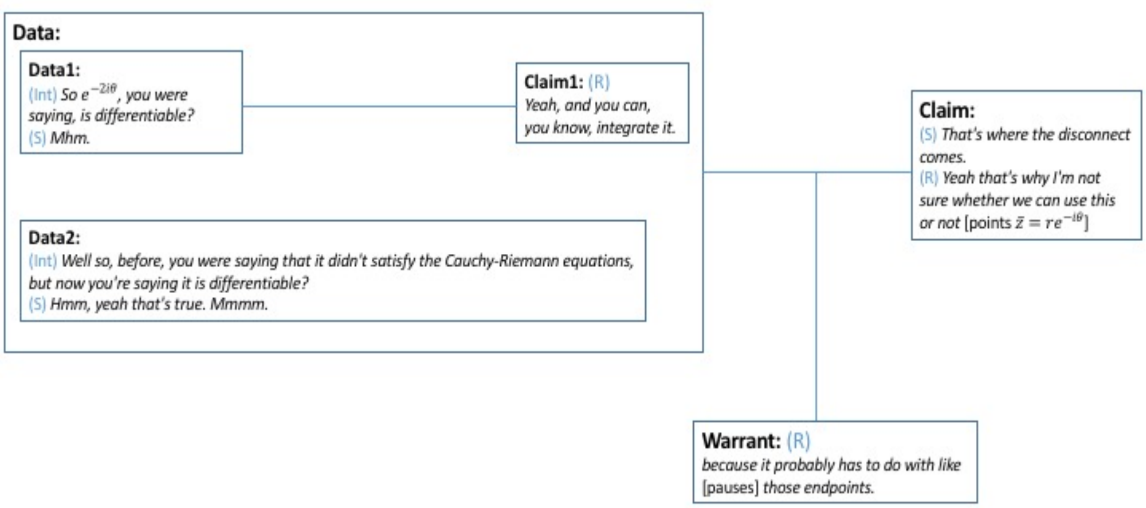


Figure 63. Toulmin diagram for Riley and Sean's Argument 9, Task 5b.

Likely due to their doubt surrounding whether a function's differentiability impacts one's ability to parametrize and use the Fundamental Theorem, Riley pursued a purely embodied approach to evaluate the integral. In particular, she began Argument 10 by recalling from their work in Task 4 that “the integral […] was a, like in the physical manifestation, it was basically displacement between a and b,” but qualified this with the words “I mean, I guess […] right?” (lines 139-140). She reproduced a diagram similar to the one drawn in their response to Task 4, as an embodied datum (line 142; see Fig. 64). Riley then qualified her datum by questioning the labeling of her endpoints a and b (line 142), and pondered how to adjust her diagram to account for the fact that their path in this task is a circle meeting “at the same point” (lines 142-144).

R: Uh, yeah so [pauses] I mean, I guess we said that the integral, right, um, was a, like in the physical manifestation, it was basically displacement between *a* and *b*, right?
S: Mhm.
R: So, I don't know. [Draws path from *a* to *b*] It probably matters here which one's *a* and *b* right? So we said it was just this. So if we do that just with *z* conjugate, our *a* and *b*, um, around a curve at the same point, right? Around a circle? So it would be— so it would be 0 still. It just means you started and ended at the same point. So you don't go anywhere. So if the integral is just like a physical— is just a way of mathematically representing displacement, then it should still be 0.

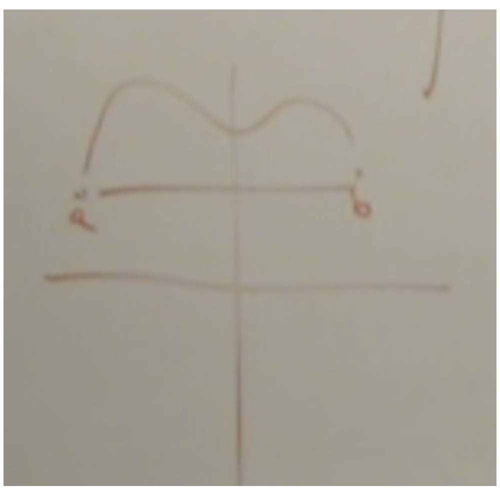


Figure 64. Riley’s diagram for displacement in Argument 10, Task 5b.

With this embodied characterization of integration as displacement in mind, Riley authored a claim that the integral “would be 0 still” (line 144), and cited an embodied warrant that “you don’t go anywhere” because the circle starts and ends at the same point (lines 144-146). Recall that in Task 4, Riley and Sean integrated the derivative of a parametrized path to obtain displacement, whereas in Task 5b the integrand is not the derivative of the circular path, let alone the circular path itself. Hence, Riley appeared to conflate certain embodied aspects of these two tasks during Argument 10. Rather than addressing this issue directly, Sean chose to describe an alternate embodied interpretation of the integral in Task 5b, which manifested as Arguments 11-13. Argument 10 is summarized in Figure 65.

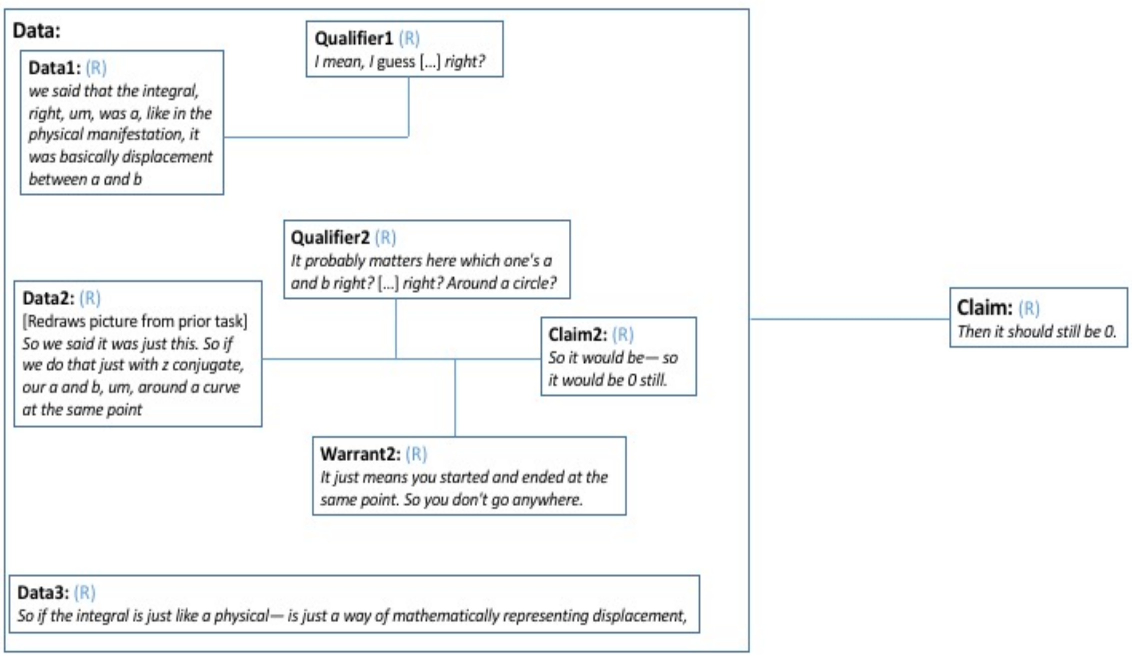


Figure 65. Toulmin diagram for Riley and Sean's Argument 10, Task 5b.

Sean began his embodied Argument 11 by relaying the circular path and authoring an embodied-symbolic datum considering $\Delta r = \int v(t)dt$, where $v(t)$ represents velocity (lines 147-149). Next, he plotted a point z_1 on the circular path in the first quadrant along with its conjugate and corresponding tangent vector (lines 149-152; see Fig. 66). Sean labeled points z_2 and $\overline{z_2}$ at the tips of the tangent vectors corresponding to $\overline{z_1}$ and z_1, respectively (lines 152-153; see Fig. 67). He articulated an embodied warrant regarding the orientation of the tangent vectors to author an embodied claim that these two vectors sum to "a little vertical vector," which he sketched off to the right of his diagram (lines 153-154; see Fig. 67).

S: The way I'm thinking about it is— if we assume that, again, the integral— like we have our circle there [draws circular path on a new Argand plane] and we interpreted, mapping, we did like [writes $\Delta r = \int v(t)dt$] like the velocity, so we go like maybe— so here's a point here [plots point on circle in first quadrant], so this is *z-bar* [plots the conjugate of the first point], so there's its little, like tangential vector [sketches small tangent vector at the conjugate point]. Then you go down here, a point on this map so this is like z_1 and $\bar{z}_1$, and that's really z_2 and $\bar{z}_2$ [plots these points at the tip of the tangent vectors for $\bar{z}_1$ and z_1, resp.], so tangent vector's there [draws it], so those two add up to, like a little vertical vector [draws upward-facing vector to right of diagram].

S: And if you go here, so like z_3 [plots point in second quadrant], this would be $\bar{z}_3$ [plots this conjugate], and that's a point z_4 [draws tangent vector for $\bar{z}_3$] , so this is also $\bar{z}_4$ [draws tangent vector for z_3] , and if you have a point there, a point there, the integral is just adding them all vectorially pretty much, so these we have a little negative vector there [draws downward facing vector at left of diagram]. So just from symmetry, like, you add them all. On this side [points to right half of figure], it's all little upward arrows, and this side's all downward arrows [points to left half of figure], and they all cancel out pretty much to get zero. That's what I'm thinking of.

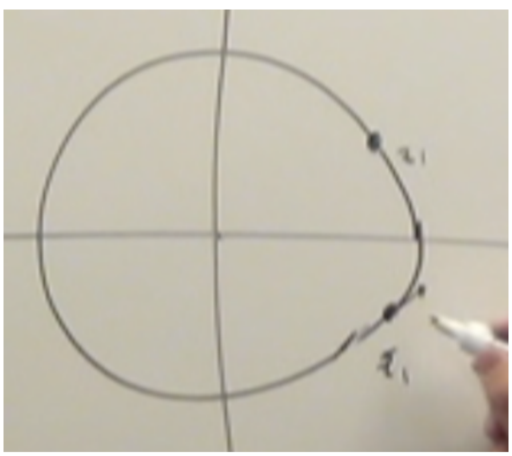


Figure 66. Sean's sketch including z_1 and its conjugate, Argument 11, Task 5b.

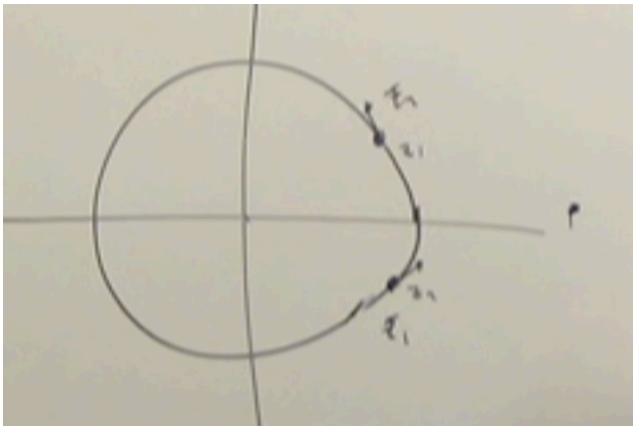


Figure 67. Sean's z_2, its conjugate, and "little vertical vector," Argument 11, Task 5b.

Subsequently, Sean continued plotting similar vectors z_3 and z_4, their conjugates, and corresponding tangent vectors (lines 155-157; see Fig. 68). Authoring an embodied-symbolic warrant that "the integral is just adding them all vectorially pretty much," Sean concluded that the vector sum of these second two tangent vectors produces "a little negative vector" and drew this resultant vector at the left of the diagram (lines 157-159; see Fig. 69). He used these two example resultant vectors to author a general warrant that, continuing in this manner, all pairwise vector sums on the right half of the diagram would result in an upward-facing vector and those on the left half would result in a downward-facing one (lines 159-161). Accordingly, "just from symmetry," Sean authored a claim that "they all cancel out […] to get zero" (line 161). Argument 11 is summarized in Figure 70.

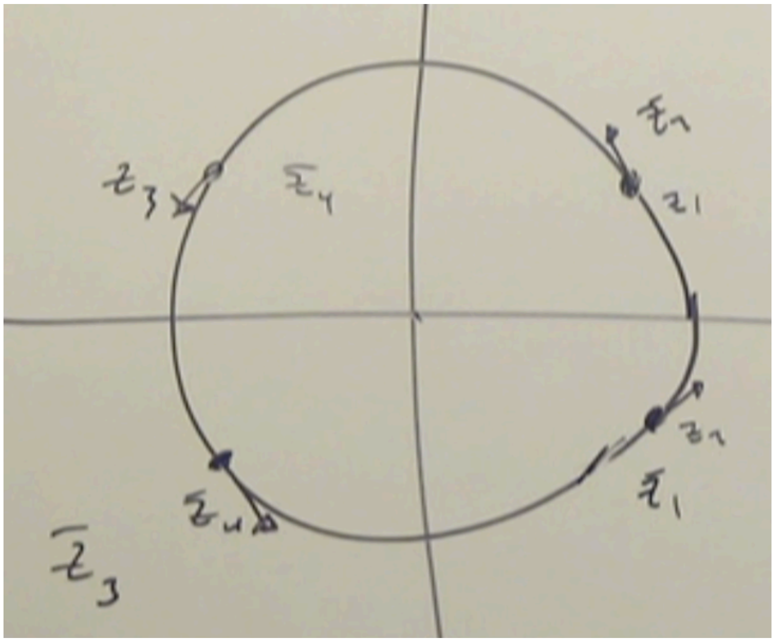


Figure 68. Sean's z_3, z_4 and their conjugates, Argument 11, Task 5b.

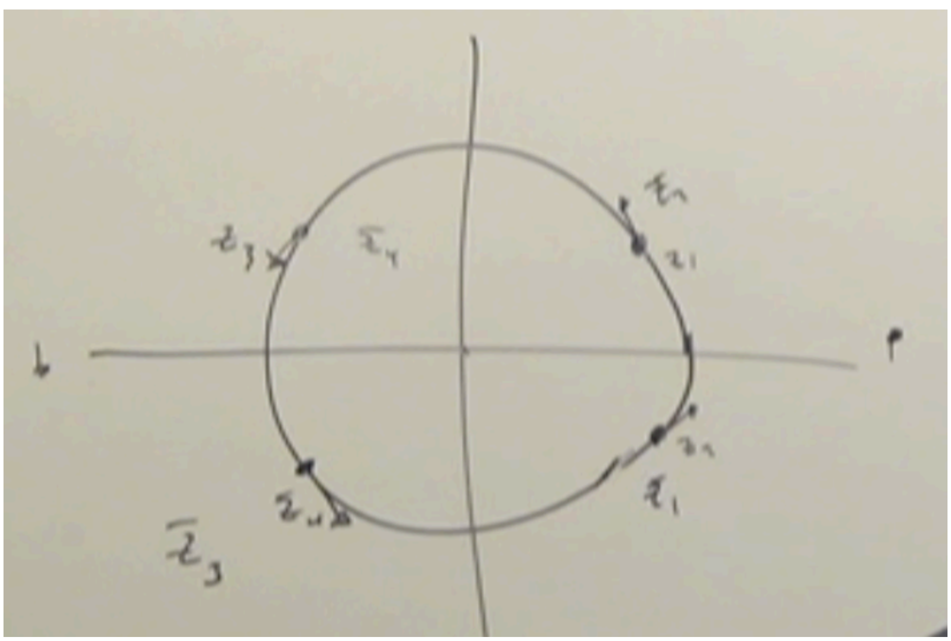


Figure 69. Sean's downward resultant vector at left, Argument 11, Task 5b.

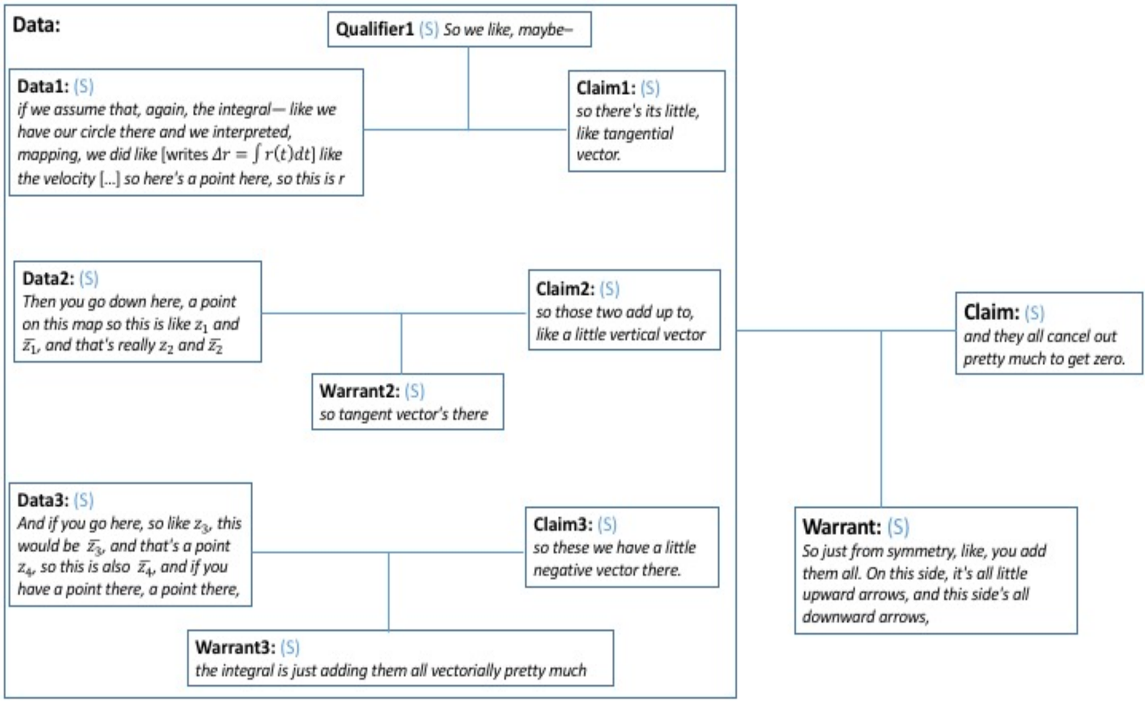


Figure 70. Toulmin diagram for Riley and Sean's Argument 11, Task 5b.

Because Sean did not properly plot the conjugate of z_2, I asked a follow-up question to elicit more detail from Sean about how the conjugates factored in (lines 163-164). He reiterated that he was considering the point z_1 and "a little point" just past it, then mapping them "down" via the conjugation function (lines 165-167). Sean's reiteration as spokesman caused him to realize that his arrows were actually reversed (line 167), and thus he drew vertical dotted lines from z_1 and z_2 to indicate where their respective conjugates should be (lines 168-169; see Fig. 71). Sean used the revised locations of these conjugates as an embodied warrant for a resulting embodied claim in which "our little vector" should point in the opposite direction as the one he drew previously in the fourth quadrant (lines 169-170; see Fig. 71). He qualified this assertion with the words "I guess," and concluded that the resultant vectors from summing the pairs of vectors on the left and right halves of the diagram should "just flip" directions "to get mostly the same result" (lines 170-172).

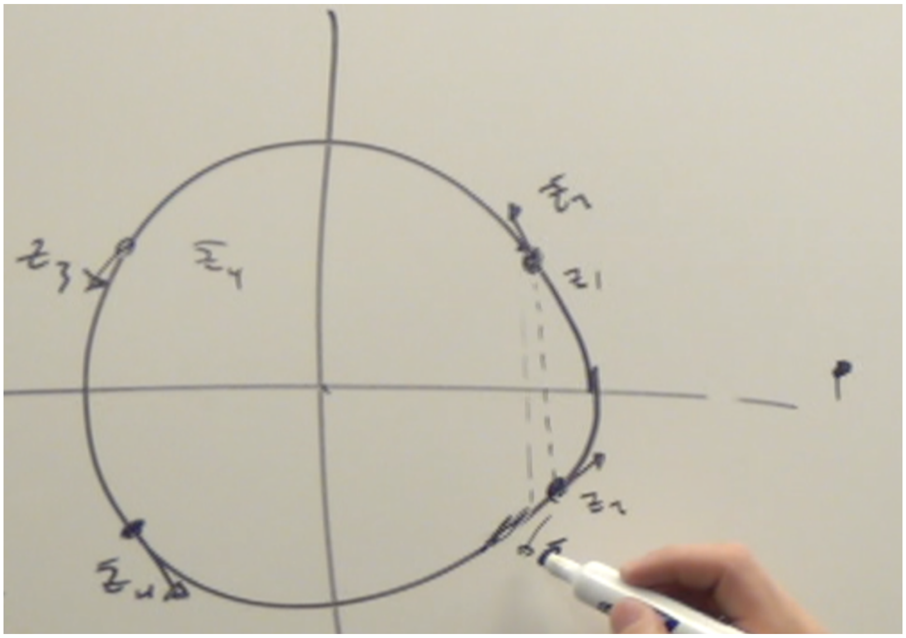


Figure 71. Sean's revised diagram for z_1, z_2 and conjugates, Argument 12, Task 5b.

Int: So does that sort of approach depend on the function that we're integrating? Right, so here, how did you use *z* conjugate in your explanation? How is that going to factor in?

S: Well, so here, so you're adding up between this point and this point [points to z_1 and its tangent vector] , a little— a little point here. So like you'd map the points down to like, there and there [points to $\bar{z}_1$ and tangent vector] [pauses] oh so you kind of have to reverse arrows. So say we're going from like, z_1 to z_2. So this maps to there, and there [draws vertical dotted lines from z_1 and z_2 to indicate where their resp. conjugates should be], so our little vector would go— I guess it would go that way [draws tangent vector in opposite direction from the original in fourth quadrant]. So you just have to flip the directions of the vectors to get mostly the same result. So they still cancel out in the end. You see what I'm saying? [talking to Riley]

Int: Do you recognize what he's doing?

R: I mean, so this is— is this specific to *z* conjugate though?

S: Yeah.

R: Um—

While articulating how the direction of these resultant should flip, Sean produced corresponding directional gestures to illustrate how the vector on the right would change from pointing up to pointing down, and similarly the vector at left should point upwards (see Fig. 72). However, note that the resultant vector from summing Sean's tangent vectors from the first and fourth quadrants should point left, not down. Applying similar reasoning to revised vectors on the left half of the diagram would also yield a resultant vector pointing left, and thus Sean's conclusion that "they still cancel out in the end" is inaccurate. I did not notify Sean of this or other related errors during this portion of the interview, though I will discuss the implications of such geometric difficulties in Chapter V. In any case, Riley expressed doubt via the qualifier, "is this specific to *z* conjugate though?" (line 174) but Sean maintained that it is (line 175). Argument 12 is summarized in Figure 73.

Figure 72. Sean's gestures for "flip the directions of the vectors," Argument 12, Task 5b.

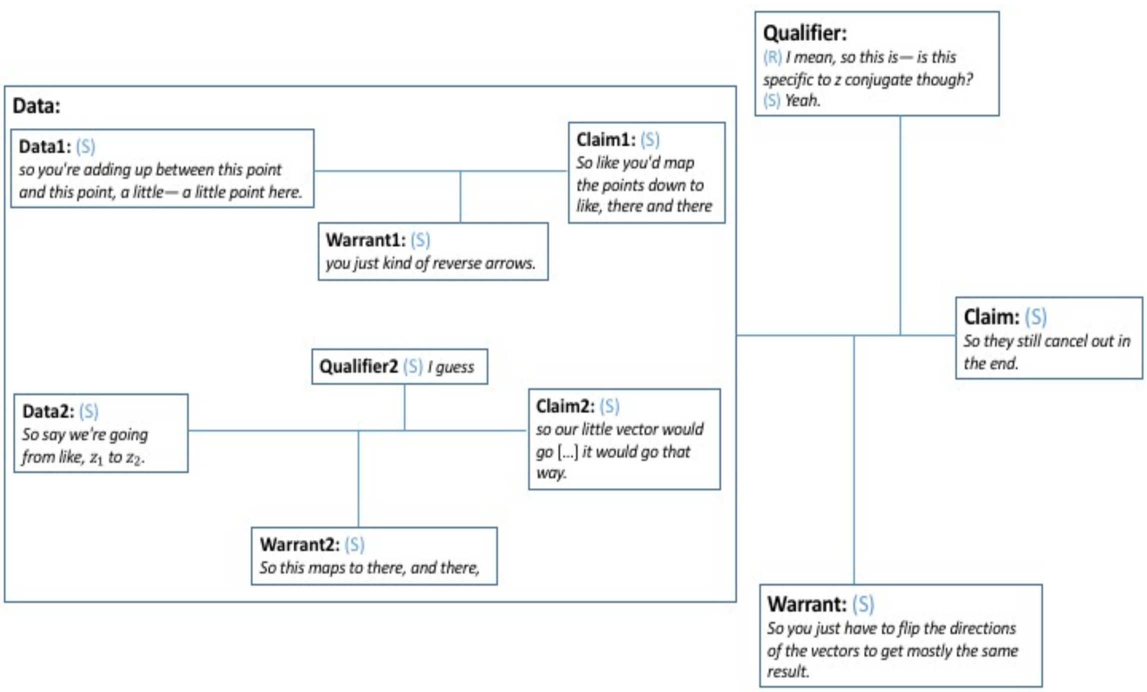


Figure 73. Toulmin diagram for Riley and Sean's Argument 12, Task 5b.

Subsequently, Sean began Argument 13 by expressing doubt about whether his embodied approach is valid. In particular, he relayed the previous formal datum that f is not differentiable, authored the symbolic inscription for a difference quotient $\frac{f(z_2)-f(z_1)}{z_2-z_1}$, and claimed that “the method kind of fails” (lines 177-179). He also qualified this conclusion with the phrase “I guess” (line 178). In response, I pushed him and Riley to explain why they believed that non-differentiability would make Sean’s method fail (line 180). As author, Riley stepped in and articulated a formal-embodied warrant that “it’s because of those infinitesimal vectors,” though she qualified this justification with her usual “right?” (line 182).

S: But the notion of— I guess that all rests on the notion of— what I'm doing is this, which we kind of ruled out [writes difference quotient $\frac{f(z_2)-f(z_1)}{z_2-z_1}$], because we said it was not differentiable. I guess the method kind of fails, then, if we want to do that.
Int: So if it's not differentiable, then what about your method doesn't work here? You said that will fail.
S: It should fail. Because it would be—
R: It's because of those infinitesimal vectors, right?
S: Yeah, it depends on what path you're approaching.
Int: So Riley, you're saying something about the infinitesimal vectors?
R: Yeah, like he's drawing like there's just one [tangent vector], or whatever, but if it's not differentiable, then we know that we can choose a couple of different paths so that they'll be different. And so they won't necessarily cancel. Like it would be path dependent.
Int: You'll get different vectors there?
R: Right. Yeah and so that would make the method fail.

Sean agreed with Riley’s warrant, and authored an embodied addendum that “it depends on what path you’re approaching” (line 183). Because I felt the link between these two statements needed more clarification, I asked Riley to elaborate on her warrant regarding the infinitesimal vectors (line 184). As spokeswoman, she provided backing for their warrant’s correctness by re-voicing Sean’s previous statement in terms of path dependence, arguing that these infinitesimal vectors might depend on the choice of path and thus yield resultant vectors that “won’t necessarily cancel” (lines 185-189). Argument 13 is summarized in Figure 74.

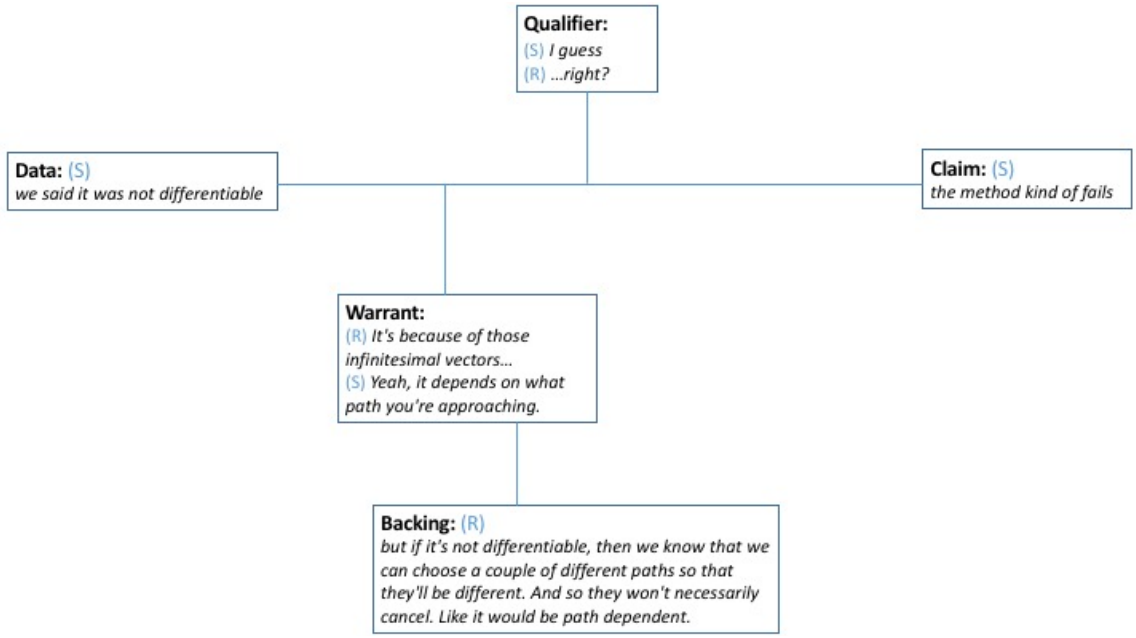


Figure 74. Toulmin diagram for Riley and Sean's Argument 13, Task 5b.

Because Riley and Sean were convinced that they could no longer pursue their embodied approach, and because they had not discovered their aforementioned error in their parametrization setup, I asked them about this previous setup (lines 190-195). Riley and Sean relayed the symbolic data of their parametrized path $z(t)$ and function $f(z)$ (lines 196-199), though Riley expressed uncertainty about their previous inscriptions, asking, "Did we do this right?" (lines 197-199). Sean still seemed relatively certain about their prior symbolism, and claimed "this is a path we can put into the function" (line 200). However, when Riley relayed the parametrized path $z = Re^{i\theta}$ as a symbolic datum, Sean realized their error and claimed, "we did this wrong" (line 202).

As spokesman, Sean rewrote their integral as $\int_{C_R^+(0)} \bar{z}\, dz$ and once again wrote the parametrized path, this time as $z(t) = Re^{i\theta}$ (lines 202-203). Recognizing his inconsistent use of the variables t and θ, he authored a symbolic warrant that "t is theta" and that $Re^{i\theta}$ is the expression he needed to input into $f(z) = \bar{z}$ (lines 203-204). Sean used this warrant to fix their previous symbolic inscriptions for $z'(\theta)$, claiming that "$Re^{-i\theta}\ dz$ turns into […] $iRe^{i\theta}$" (lines 204-206). He additionally qualified this claim by confidently asserting, "Yeah we made a stupid mistake [previously]" (line 206). Sean used their corrected symbolism to simplify the integral to $\int_0^{2\pi} 1\ d\theta$, and implemented the Fundamental Theorem as a formal-symbolic warrant to obtain an answer of $2\pi R^2 i$ (lines 206-207). He qualified both of these steps with the phrase "of course," indicating a high level of certainty. Argument 14 is summarized in Figure 75.

Int: Well so, if that's not really panning out, maybe go back to your original— the way you wrote the integral there, so you have in that first line, that $Re^{-i\theta}$ times that other stuff there. Um, and then you rewrote stuff a little bit. So you have $-iR^2$ times the integral from 0 to 2π of $e^{-2i\theta}$. So—

R: How can we do, that, um [pauses]

Int: So why don't you walk me through how you set up that first line, actually. So where do the pieces come from?

S: So first we pick our path of [pauses then points to $f(z(t))$, then $z'(t)$]

R: Wait—

S: I think— I think, do you see something wrong?

R: Did we do this right? Because— so $f(z) = z$ conjugate right?

S: Yes. I think— yeah okay, we have this is a path we can put into the function.

R: Ok, so the path is actually going to be—

S: e to the— $Re^{i\theta}$. So really— we did this wrong. So, your integral— let's focus on our curve [writes $\int_{C_R^+(0)} \bar{z}\ dz$] I'm going to say our path is [writes $z(t) = Re^{i\theta}$] $Re^{i\theta}$ from 0 to 2π. So using this, so t is theta, so the input $z(t)$ into this, so it'd be z-bar, so $Re^{-i\theta}$, dz turns into $z'(\theta)d\theta$.

R: So we're going to get Ri—

S: So $iRe^{i\theta}$. Yeah we made a stupid mistake. This is, of course, i times the integral from 0 to 2π of 1, $d(\theta)$ [writes $\int_0^{2\pi} 1\ d\theta$]. Which of course, using FTC, would be $2\pi R^2 i$. But [pauses] So we can put it there again. That's what we had before.

R: Right, um—

Int: Sorry I can't quite see what you have written there, Riley—

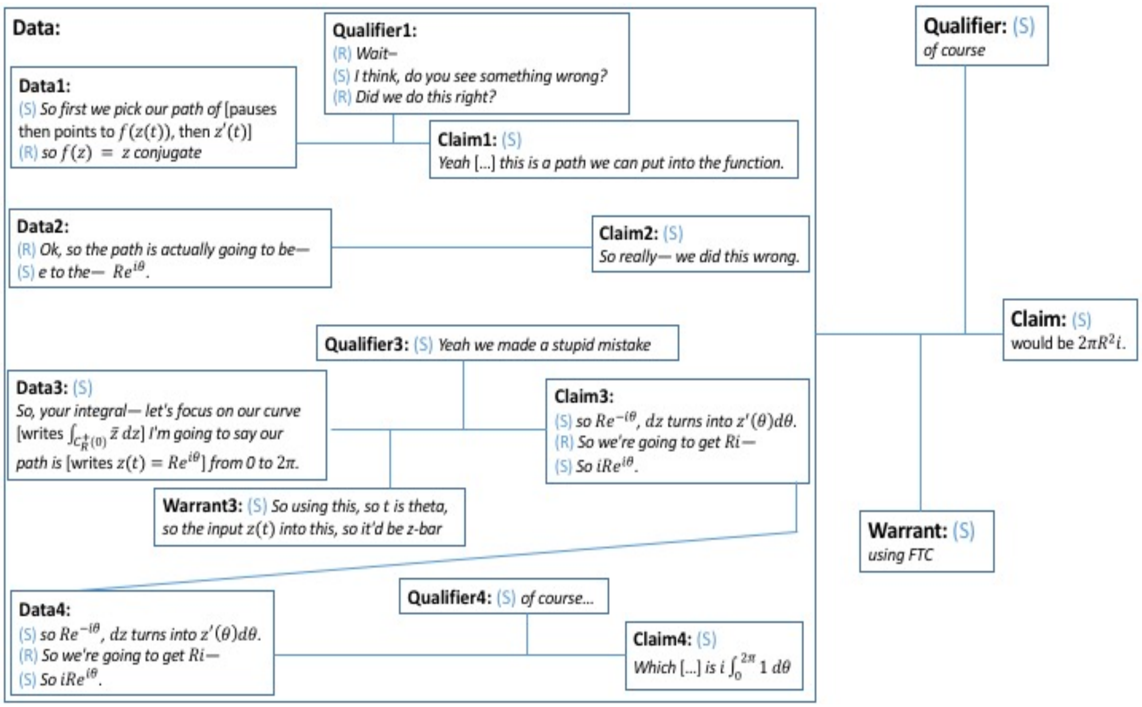


Figure 75. Toulmin diagram for Riley and Sean's Argument 14, Task 5b.

R: [Moves to left] that's ok. Um, so let's see. I mean, how sure are we that we can even use this? Because you're always— I mean, you're always going to have to do an integration— use the Fundamental Theorem of Calculus.
S: Mhm.
R: And like I said, I'm [pauses] I think that the only condition for this is continuity, but I don't actually remember. Um, so [pauses] that's fun. Is there any way we could figure that out?

Following Argument 14, Riley expressed some uncertainty about whether or not their approach was valid (lines 211-216), likely due to my previous questioning in Argument 8 about using the Fundamental Theorem. She recounted her hypothesis that "the only condition for [parametrizing] this is continuity," but once again emphasized that she "[doesn't] actually remember" (lines 215-216). Nonetheless, she used this qualification as an opportunity to move forward, asking Sean, "Is there any way we could figure this out?" (line 216). Sean looked puzzled about what Riley was referring to specifically, so I re-voiced her concern and asked both participants what conditions are required to parametrize a function (lines 217-220). Hence, Riley's qualifying question catalyzed Argument 15, as follows.

Int: Well so maybe, kind of returning real briefly to the first question I was asking. When you think of parametrization, that's kind of basically what you're using there, in that orange expression there. So what conditions does your function have to have in order to parametrize it? What do you end up with— uh, what do you need to do that? Is there anything nice about your function?

R: It's gotta be continuous. Um— I guess you could, I mean— yeah I guess you could [pauses] sort of parametrize a function that, like you'd piecewise parametrize a function that wasn't [continuous]. But it would be not very fun to work with. Um, so yeah it's gotta be continuous. It doesn't really have to be much of anything else, does it?

R: It's nice if it doesn't have sharp edges and— is a nice curve. Um, because otherwise you're going to end up probably with a bunch of, like— oh well, you know, like piecewise defined functions are like $f(z)$ equals, you know, some function of t for $t < 0$ and some other function of t for t is, uh, less than one and greater than 0, and like some other function for $t > 1$, right? So that's parametrizing that's not fun to work with, but I mean, you can still use it on that. Um, is there— I mean, are there any other conditions that you can think of?

S: Um [pauses] now that I think of it, if it's piecewise continuous you should be fine. Um—

Riley relayed their previous assertion that the function in question has "got to be continuous," but revised this claim via an embodied-symbolic rebuttal considering piecewise-continuous functions that she conceded "would be not very fun to work with" (lines 221-223). She also qualified her claim by questioning whether any other conditions applied (lines 223-224). Riley additionally authored an embodied-symbolic warrant clarifying why she intuitively felt the function should be continuous, or at least piecewise-continuous (lines 225-229). She described a function as "nice if it doesn't have sharp edges," though such a function could technically be continuous and just not differentiable. As spokeswoman, Riley repeated her thought that piecewise functions "are not fun to work with" but still can be parametrized (lines 228-229), and Sean agreed (line 231). A summary of Argument 15 is depicted in Figure 76.

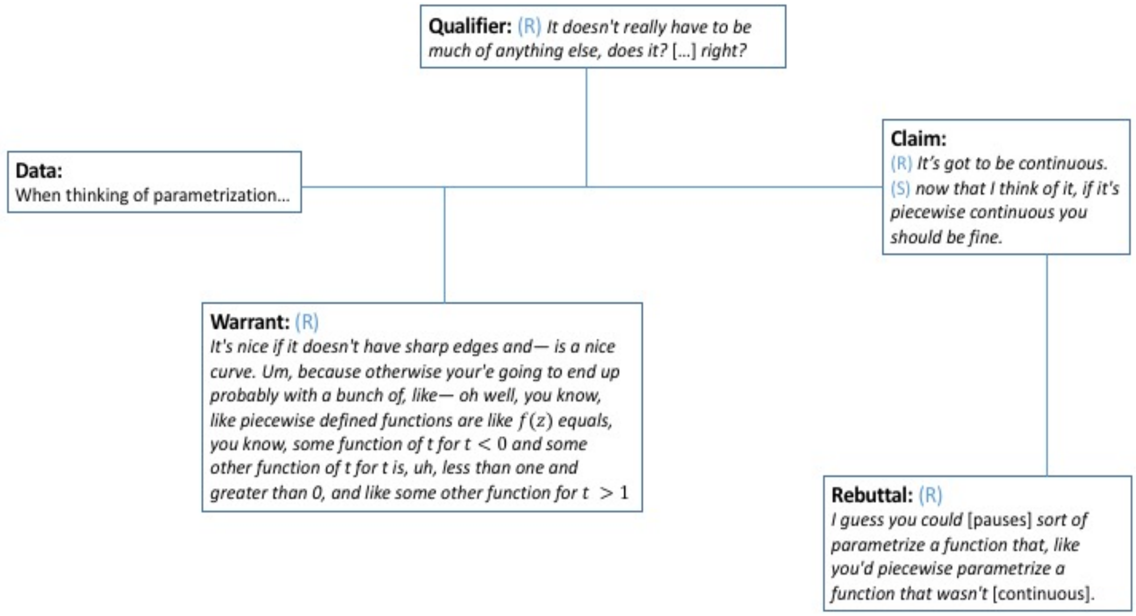


Figure 76. Toulmin diagram for Riley and Sean's Argument 15, Task 5b.

Int: Well, so you feel pretty confident with your answer then?
R: Umm, [laughs] I mean I guess the question for whether we can use this boils down to: does this $f(z)$ originally have to be differentiable? Um, and [shrugging gesture] I don't obviously remember the condition for this. I don't think it had to be, um—
Int: Well, so assuming— assuming that you don't need differentiability for like, the original statement you had in orange there, where you're integrating $f(z(t))$ times $z'(t)\ dt$, so assuming you don't need differentiability for that—
R: Assuming that, then yeah I would say so. [Laughs] "Am I confident?" "Yeah I think so!"
Int: So do you think you used differentiability anywhere else in—?
R: I mean, we used the differentiability of ultimately, of 1.
S: [Points to their final inscriptions in black with the implicit identity function integrand]
Int: Yeah.
R: Uh, I don't think we used it anywhere else, did we?
S: Mm mm [shakes head no]. No, this one's just—
R: And we don't rely on— I guess in saying that 1 is differentiable, um, we don't rely on *z* conjugate being differentiable necessarily, so—
Int: Right that's kind of a separate issue.

Because Riley seemed to waver a bit regarding her continuity conclusion and repeatedly questioned whether other conditions were required, I asked her and Sean how confident they were about their claim (line 232). Riley responded by repeating their prior concern regarding the potential need for differentiability (lines 233-235). I asked her if, aside from that issue, they felt comfortable with their conclusion, given that their symbolic inscription $f(z(t))\ z'(t)\ dt$ did not include any derivatives (lines 236-238).

After Riley confirmed she felt rather sure otherwise, I asked if there was another place in their argument where she and Sean felt differentiability was needed (lines 239-240). Riley answered that they had implicitly assumed the differentiability of the identity function 1, and acknowledged that this was separate from the differentiability of $f(z)$ (lines 241-248).

S: And I know it definitely comes into play if you want to use the Fundamental Theorem of Calculus version for the "complex function" [air quotes] where it's like, the antiderivative of this [points to integral of $\bar{z}$ in black] evaluating at a point b and a point a— like not the one we did earlier, but one that's like a function of z, and you find the "antiderivative"—
R: That, I mean that you have to have an analytic function for—
S: Yeah, I think with that one, you do.
Int: Do you have a feeling for why— I mean, without formally proving anything, do you have a feeling for why you might need the function to be analytic in order to use that sort of technique?
R: Um, so I mean, that makes it independent of path. And for this one, like $z(t)$ is a specific path.
S: Yeah, um, something to do with the Cauchy-Goursat Theorem, I think. Like in the proof, like the diagram, it doesn't matter what path it is— I forget the proof, but [it] has something to do with that.
Int: Well that's fine, we don't need to go through the formal proof but—

Though this reflection on their prior argument(s) was not an argument itself, it did set the stage for Argument 16, in which Sean authored a datum considering the role of differentiability when applying the Fundamental Theorem of Calculus and finding a complex antiderivative (lines 249-252). He qualified this statement with, “I know it definitely comes into play” (line 249), and as spokeswoman, Riley clarified that he was referring to analyticity (lines 253-254). Because Task 13 served as a venue to discuss this connection further, I did not probe much further with their claim, but I asked them to briefly explain why they thought this, given that they brought up the connection organically (lines 255-256). In response, Riley and Sean co-authored a formal-embodied warrant that analyticity allowed for path independence, and Sean briefly alluded to its applicability in the formal proof of the Cauchy-Goursat Theorem (lines 257-259). Argument 16 is summarized in Figure 77.

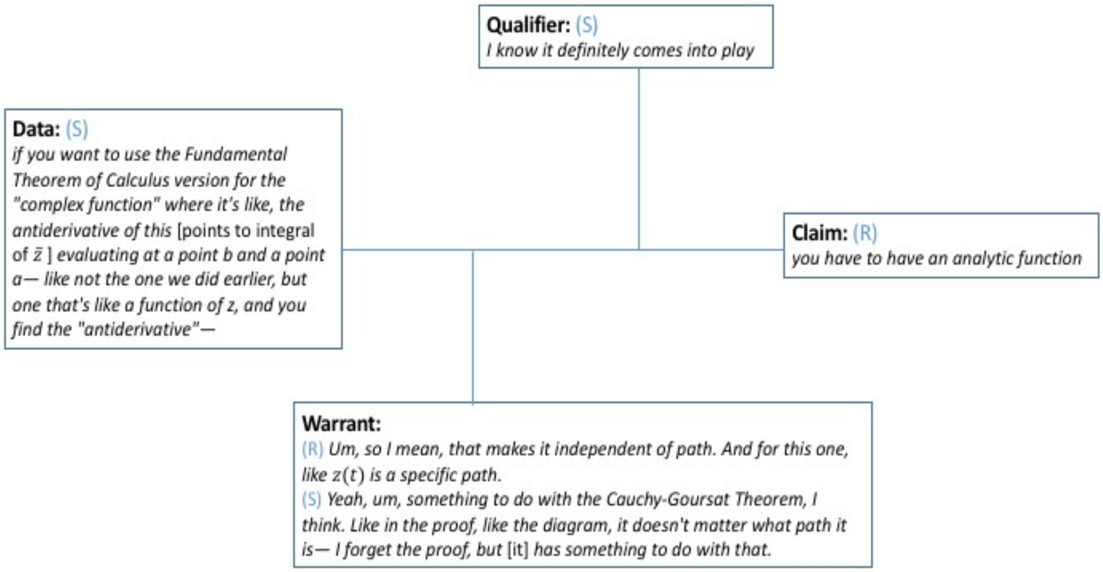


Figure 77. Toulmin diagram for Riley and Sean's Argument 16, Task 5b.

Subsequently, Sean brought up a problem on their recent final exam that dealt with integrating $1/z$ on a semicircular path from $z = -i$ to $z = i$ (lines 261-264). He drew this path, denoted gamma, as an embodied datum; this prompted Riley to remember having to "choose a different branch" (line 265). Sean continued to discuss the setup of the final exam problem. He authored a second datum that recalled the problem's two parts, one prompting them to incorporate parametrization, and the other directing them to "antiderive" using a logarithm (lines 266-268). Again, Riley stepped in and recalled that the latter approach "didn't work" (line 269), and mentioned the formal-embodied datum that $1/z$ is not analytic along the traditional branch cut on the negative real axis (lines 271-272).

Sean qualified their current argument, Argument 17, by explaining that this exam problem "makes me more confident in our answer." In particular, he authored a formal-symbolic claim that the integral in Task 5b will analogously not allow them to simply take an antiderivative, but that they can still parametrize (lines 275-276). Riley challenged this assertion, in the sense that he did not explain *why* the former method fails but the latter method "works" (lines 277-278). However, she conceded, in the form of a rebuttal, that there is no antiderivative for the given function (lines 281-285). Argument 17 is summarized in Figure 78.

S: So remember the final exam? [unintelligible] So remember he had like the little, it was integrating $1/z$? It was integrating $1/z$ from like— So on the final, we had a problem where it was like here's this [draws Argand plane] we have a point i, a point $-i$, integrate here [draws semicircle from $-i$ to i], here's our path gamma. And—

R: We had to choose a different branch, right?

S: And here was— oh, we're just going to use this function [$1/z$] and we're going to be integrating this function from there to there. And there were like two responses— one was, one— parametrize it, and get the right answer. The other one was, oh, I'll just antiderive and get the proper Log of z—

R: Oh and that didn't work because—

S: And that did not work, but then we parametrized and plugged into the integral, like—

R: Well yeah, and like $1/z$ isn't an analytic function over that specific— at least the primary branch, because it isn't analytic along the negative real axis.

S: So that kind of makes me more confident in our answer.

R: Ok, yeah.

S: So when we try to do it directly [points to $\int \bar{z}\, dz$], um, it will not work. But we actually parametrize it and put it in this—

R: Ok but that doesn't really explain the question of why does it work when you parametrize but it doesn't work when you try to use the Fundamental Theorem of Calculus, right?

S: Yeah.

Int: Why do you say that, Riley?

R: Well, I mean— So, like he said, on the final we had an example of one that worked, uh— ok, it's not an analytic function over that domain, but it did work if you parametrize it, and it doesn't work if you don't parametrize it. Um, so that gives us confidence that we can actually solve it like we did. But it doesn't really answer the question of why we can't use the Fundamental Theorem of Calculus to solve this, besides the fact that we don't have an antiderivative for it. Um—

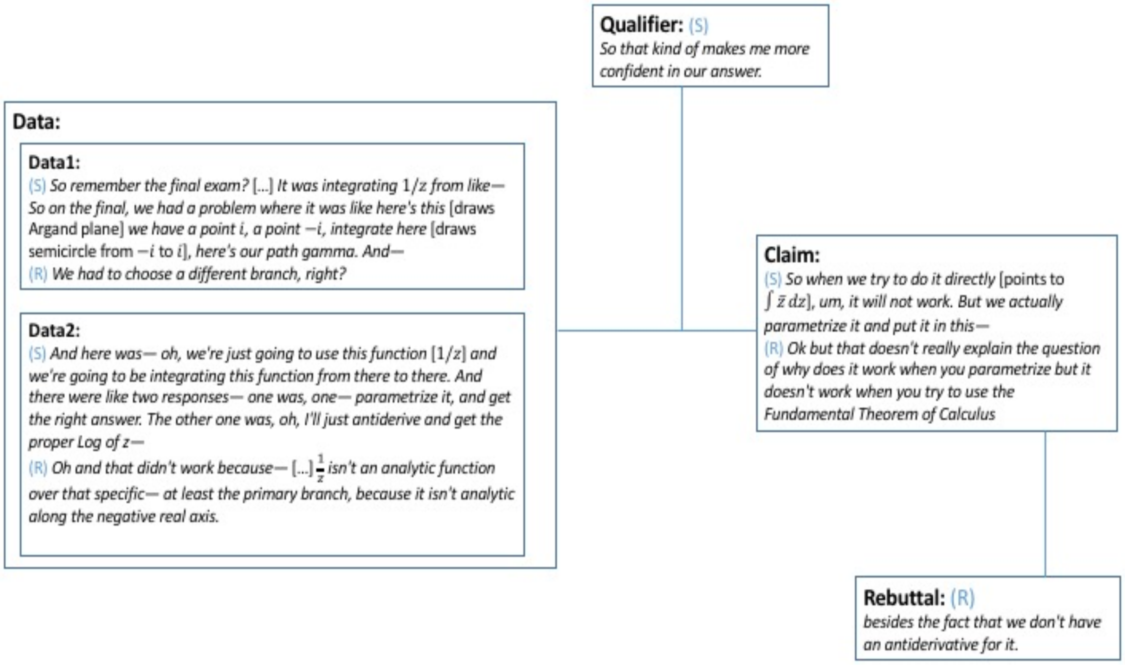


Figure 78. Toulmin diagram for Riley and Sean's Argument 17, Task 5b.

Because Riley and Sean had mentioned previously that they were unsure that their parametrization method was valid (due to their concern for applying the Fundamental Theorem to their function of θ), I directed them to revisit this avenue with their new consensus from Argument 17 in mind (lines 286-288). Accordingly, Riley and Sean began Argument 18 by clarifying that they were not trying to utilize the Fundamental Theorem of Calculus, but were rather "just parametrizing it" (lines 289-290). To set this apart from their second part of the final exam question, I re-voiced their response as "finding […] automatically an antiderivative […] and then evaluating at the endpoints]" and Riley agreed with this distinction (lines 293-295).

Int: Well, so if you did try to use the Fundamental Theorem, what sort of result would you get? Because it looked like you sort of maybe tried doing that before, but then again you kind of had the first step written incorrectly, right?
R: Yeah, so uh, I don't think we were trying to use the Fundamental Theorem of Calculus so much as—
S: Just parametrizing it.
R: We had, uh, instead of— let's see. Instead of taking $f(z(t))$ we had taken like, f of— I don't know, we had taken, uh— it was like $f'(z(t))$ or something.
Int: Oh so you're just saying if you had tried to directly go from, like, you know, finding just automatically an antiderivative for *z* conjugate, or something?
R: Right.
Int: And then evaluating at the endpoints?
S: Remember, Riley, from Calc 3, multivariable calculus—
R: Maybe [laughs].
S: So remember when we did line integrals, we did almost the same thing, like we had [writes $\int F \cdot dr$], and you would say ok, we use the path $r(t)$ and we'd say, well that's [writes $\int_{t_1}^{t_2} F(r(t)) \cdot r'(t)dt$], give us an answer?
R: Yeah.
S: But if *F* was a special conservative function, equal to the gradient of some potential function phi, then we could say— this is my point *A* and this is my point *B* [writes $\int_A^B F$] then it's just $\Phi(B) - \Phi(A)$.
R: And Φ was like an antiderivative function?
S: Exactly, kind of. It was like anti-gradient. Yeah, and so it kind of reminds me of that, since in this case *F* was something special, like it was conservative, or I guess in this case, satisfies the Cauchy-Riemann equations. Whereas no matter what, this would be a fail-proof method, like we did over there [points to other side of the board]— but when you try to use shortcut methods—
R: But *why*, is the question? Like why does one work and not the other?
S: As for why? I just wanted to give an example.

This discussion prompted Sean to recall a similar technique from multivariable calculus in which "we did almost the same thing" (line 299). In particular, Sean authored a symbolic datum considering $\int F \cdot dr$ where $r(t)$ is a path and F is considered as a function of $r(t)$ and thus the integral can be expressed as $\int_{t_1}^{t_2} F(r(t)) \cdot r'(t)dt$ (lines 299-301). Sean proferred an embodied-symbolic datum describing the specific case in which F was a "special conservative function, equal to the gradient of some potential function phi" (line 303). Under these circumstances, Sean symbolically claimed that $\int_A^B F = \Phi(B) - \Phi(A)$, where A and B are generic starting and endpoints, respectively (line 304). Riley verified with Sean his implicit symbolic warrant that Φ is antiderivative, and Sean additionally referred to this function as an "anti-gradient" as spokesman (line 306).

Sean concluded that Task 5b reminded him of this situation, and cited an embodied-symbolic warrant that in both cases F was "something special," either conservative or satisfying the Cauchy-Riemann equations, and this allowed one to take an antiderivative directly (lines 306-308). On the other hand, he referred to parametrization as the "fail-proof method" (lines 308-309). However, once again, Riley pressed Sean about *why* one method works and the other does not. Sean replied by authoring backing for his warrant's validity; he merely wanted to give an illustrative example (lines 310-311). This eighteenth and final argument from Task 5b is depicted in Figure 79.

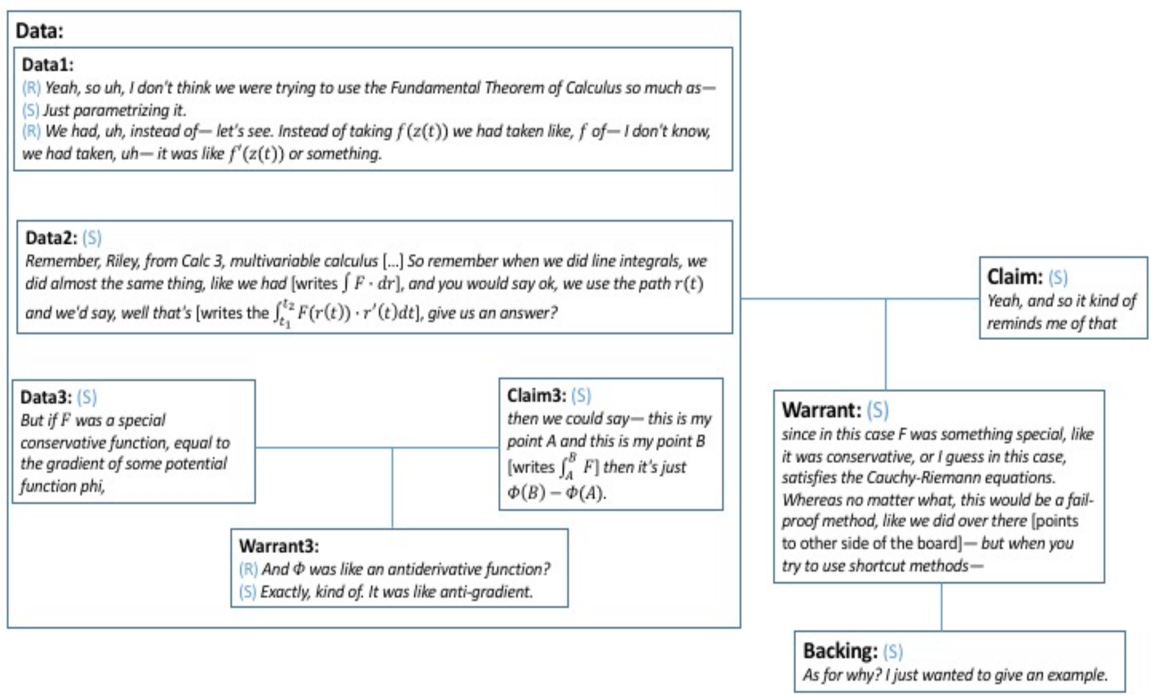


Figure 79. Toulmin diagram for Riley and Sean's Argument 18, Task 5b.

Task 5c – Riley and Sean

Because Riley and Sean had extensively discussed Task 5b, their response to Task 5c was comparatively brief. Riley claimed that changing the radius of the circular path is "not that big of a deal," and proffered a symbolic warrant that their inscriptions already were in terms of R (line 2). As spokesman, Sean clarified that "R is a variable," thus closing their response to Task 5 and the first portion of the interview. Their sole argument for Task 5c is depicted in Figure 80.

Int: Ok. And then what about changing the radius of the circle? Does that really mess things up much?
R: Not really that big of a deal because R, it's already there in—
S: R is a variable.
Int: Yeah so you already got an answer in terms of R. So not a big deal. Ok great. That's looking good. Ok, um, so that's actually the end of the first set of questions that I had. So how are we doing on time here? So if you'd like maybe we can take a second to have some water, or we could just plow ahead.
R: Sure! Can I erase the board?
Int: Yeah that sounds good.

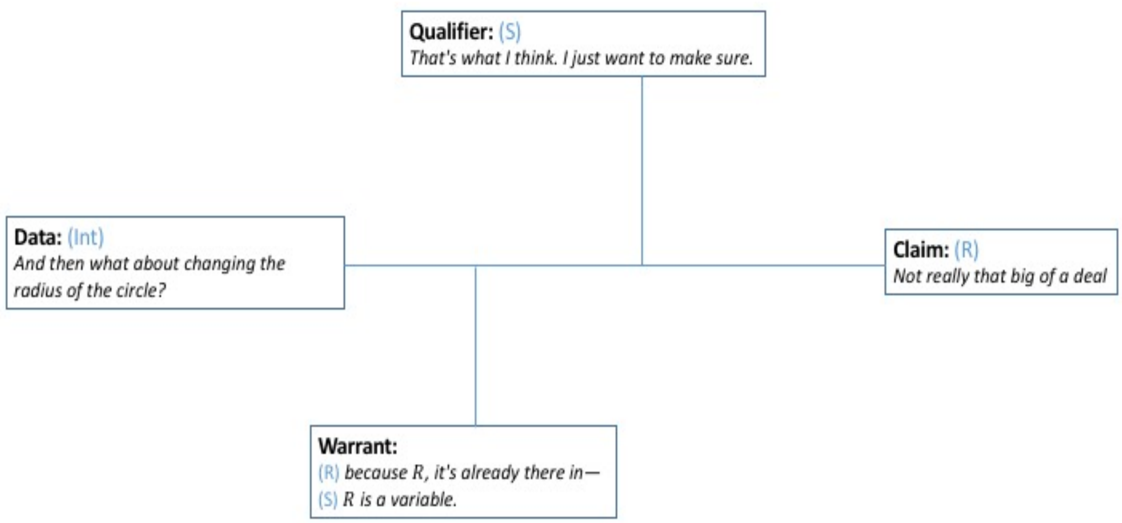


Figure 80. Toulmin diagram for Riley and Sean's Argument 1, Task 5c.

Task 5 Summary

One notable difference between Dan and Frank's response versus Riley and Sean's was that the former pair appeared to be more comfortable with limit symbolism, so much so that they chose to test the differentiability of f using the limit definition of the derivative rather than the Cauchy-Riemann equations. This made Dan and Frank's response to Task 5a longer than Riley and Sean's, but their response for Task 5b was considerably shorter than Riley and Sean's. While Dan and Frank immediately recognized that they could not use an antiderivative, quickly determined that the function in question was continuous, and correctly parametrized, Riley and Sean spent a long time struggling with the limit symbolism regarding continuity. After deeming the function continuous, Riley and Sean made a symbolic error when differentiating their parametrized $z(\theta)$ function, which resulted in several attempts wherein they claimed the integral should vanish. Throughout, they were uncertain whether their various approaches were valid, due to their lack of confidence about, and inability to recall, various assumptions for the tools they invoked.

Another reason why Riley and Sean spent more time on Task 5b was that they conflated the setting in Task 4 with that of Task 5b when trying to provide an embodied interpretation for Task 5b. In particular, they appeared to treat either the circular path $z(\theta)$ (Riley, Argument 10) or the $\bar{z}$ function (Sean, Arguments 11-13) as velocity, i.e. the *derivative* of a parametrized path, and thus their vector addition yielded a sum of zero. As in previous arguments, Riley and Sean invoked more embodied reasoning than Dan and Frank, including the aforementioned arguments attempting a purely embodied approach to integration.

Part II

Recall that the second portion of the interview was primarily dedicated to evaluating specific integrals, some of which were intended to be familiar to the students and some unfamiliar. As in the first portion of the interview, I asked follow-up questions to elicit more detail about certain components of participants' arguments. At the end of the interview, I also asked two general questions about integration that were not tied to a particular function (see Appendix C). Though the content of these last two tasks was implicitly addressed in previous tasks, these concluding questions served to corroborate and recapitulate participants' earlier statements.

Task 6 – Dan and Frank

Task 6 (see Appendix C) required participants to evaluate the integral $\int_L \frac{1}{z}\,dz$, where L denotes the unit circle $|z| = 1$ traversed counterclockwise. Dan began the pair's response by writing down the path of integration as $\mathrm{L} = C_1^+(0)$ (lines 1-3), which served as a datum. He thus acted as spokesman because he framed my spoken task using different notation. I note here that the pair refer to this path later in shorthand notation as 'C' rather than 'L.' His inscription $C_1^+(0)$ was adopted from the class's notation for a positively oriented circle of radius 1, centered about the origin, and illustrates operational-symbolic reasoning because he identified the path with a purely symbolic inscription.

Int: Ok so how would you find the integral of 1/z dz? [D writes $\int \frac{1}{z}dz$ on board] And this is over the path L [D writes in 'L' beneath the integral symbol] where L represents the unit circle |z|=1, traversed counterclockwise. So I believe— [D erases 'L' and writes $C_1^+(0)$] yeah you have your own notation for that.
D: The circle—
F: It's just the unit circle oriented counterclockwise.
Int: Yeah positive orientation, centered at 0.
F: Oh. That's easy enough. Ok so, um—
D: This is discontinuous-- there's a discontinuity inside the circle.
F: Right, so—
D: If we just write this as—
F: I'll draw it [draws the circular path]. So this is our path, and the discontinuity is going to occur at zero.

Next, Frank qualified their argument with the statement "That's easy enough," (line 8) expressing a high degree of confidence about completing the task. After staring at the symbolic inscription $\frac{1}{z}$, Dan authored a second datum that there exists a discontinuity inside the circular path (line 9). This datum appeared to be embodied-symbolic, as staring at the symbolic inscription led him to verbally relate the inscription to an imagined physical location relative to the path L. I say "imagined" here because neither participant had drawn a corresponding diagram. Frank then drew a diagram of the circular path (see Fig. 81), and indicated the discontinuity by drawing a dot at the origin, illustrating embodied reasoning (line 12).

Figure 81. Frank's diagram of the circular path during Argument 1 for Task 6.

Next, using symbolic reasoning, Dan rewrote the integral (line 13) in a form that more closely resembles the statement of Cauchy's Integral Formula. I identify Dan's role as spokesman because he modified the formulation of the original integrand while keeping the same conceptual meaning behind the inscription. At this point, Frank asked if Dan was using Cauchy's Integral Formula (line 14), clarifying the warrant for their argument. Using formal reasoning, Dan affirmed that Cauchy's Integral Formula could be invoked for this situation (line 15). Thus, Frank chose to elaborate the remainder of this warrant (lines 16-17), as he relayed Data2 and Data3 and to their eventual claim that the result is $2\pi i$ (line 17).

D: And this is just like $\frac{1}{z-z_0}dz$, where $z_0 = 0$ [writes $= \int_C \frac{1}{z-z_0}dz$, $z_0 = 0$].
F: Are you using Cauchy's Integral Formula?
D: Yeah you can just use Cauchy's Integral Formula.
F: So it's just going to be $2\pi i$ times this function [points at $\frac{1}{z-z_0}$ integrand] evaluated at the discontinuity, which is going to be just 1 [writes $2\pi i(1)$]. So it's just $2\pi i$.
D: And the reason why we used that is because z_0[points to $z_0 = 0$ inscription] was in the interior of the circle [points to the circular path in diagram].
F: Yeah.
D: And it's a discontinuity.

Frank's elaboration used formal-symbolic reasoning because he related the statement of the theorem to Dan's prior symbolic manipulation of writing $1/z$ as $1/(z - z_0)$ where $z_0 = 0$. Dan then used the phrase "And the reason why we used that is…" (line 18) to instantiate backing for their warrant's validity. This backing is detailed in lines 18-21 and represents embodied-symbolic reasoning because Dan discussed the physical location of the point z_0 relative to their drawn circular path, and pointed to two symbolic inscriptions corroborating his verbiage. Figure 82 depicts Argument 1.

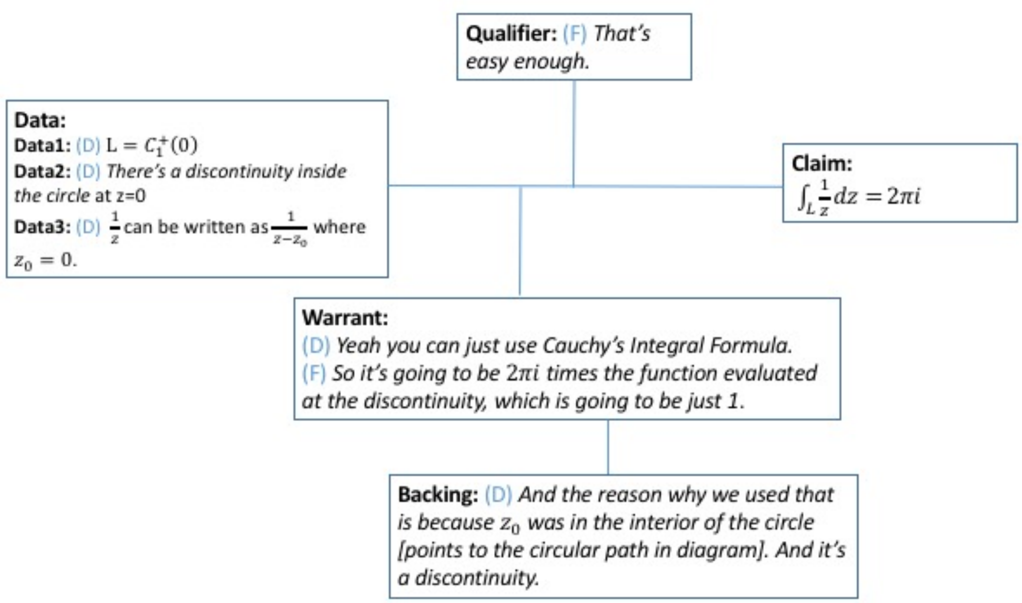


Figure 82. Toulmin diagram for Dan and Frank's Argument 1, Task 6.

22 *Int:* Ok cool. So what sort of assumptions were you using there? I believe you mentioned—
23 *D:* So—[waves hand towards picture]
24 *F*: So the function is analytic everywhere except for the point 0 [points to the diagram].
25 *D*: [Starts writing *D is simply connected* and *C is simple, closed* as soon as F starts talking]

To probe for additional clarity in Dan and Frank's argument regarding their use of Cauchy's Integral Formula, I asked them to elaborate (line 22) on the assumptions that they used in reaching their prior claim. Because Dan and Frank ended up providing another complete argument for their previous claim, I analyzed what followed as a separate argument, Argument 2. Dan began to speak (line 23) but Frank interrupted as he authored a discussion about analyticity of the integrand. He referred to this integrand as "the function" (line 24). I characterized this datum as formal embodied reasoning because Frank referred to an abstract notion of analyticity and referenced a location on the drawn diagram via his pointing gesture (see Fig. 83).

Figure 83. Frank's pointing gesture referencing the origin during Argument 2 for Task 6.

Shortly after Frank started verbalizing this datum, Dan began writing *D is simply-connected* and *C is simple, closed* (line 25), which represents formal reasoning due to its attention to abstract assumptions related to Cauchy's Integral Formula. Because Dan, as author, did not verbalize what information he used to make this assertion, it appeared that he implicitly reasoned about Data1 from Argument 1 and the integrand $\frac{1}{z}$. A version of

Dan’s written statement ultimately ended up serving as data for Argument 2, but first it was challenged as described below.

Frank continued to speak while he watched Dan finish writing the inscriptions from line 25, and appeared to be ready to use Dan’s inscriptions as data for a claim, as signified by the words “and because of those properties, we can—” (line 27). However, Frank’s line of reasoning was interrupted when he expressed uncertainty about Dan’s assertion regarding the existence of a simply-connected domain (lines 27-28). At this point, Dan authored a formal warrant for his previous assertion by explaining that we can just assume a simply connected domain exists (line 29). Using embodied reasoning, Frank then drew a domain (see Fig. 84) within the interior of the circular path on the previous diagram (line 30).

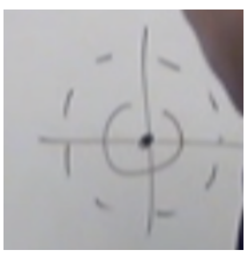


Figure 84. Frank’s proposed domain during Argument 2 for Task 6.

26 *F*: And so when we evaluate it along the path [circular tracing gesture around the drawn path], because--
27 because of those properties [facing and reading what D is writing], we can—Hang on, is there a
28 simply connected domain?
29 *D*: I mean, we can just assume that there is one, yeah.
30 *F*: [Starts to draw in a domain inside the circle C]
31 *D*: If it's greater than the circle [points to the picture as F is drawing a domain inside the circle].

Noticing that Frank drew a domain that was not simply connected (as the path *C* enclosed both points of the domain and points in the domain’s complement), Dan added, “If it’s greater than the circle” (line 31) and pointed at Frank’s proposed domain. This addition instantiated embodied reasoning because Dan’s verbiage imposed a constraint on the existence of a hypothetical domain, concerning its position relative to the drawn diagram. His pointing gesture further suggested that this constraint was necessary in order to avoid coming up with a problematic domain like the one Frank drew.

Realizing his previous error, Frank agreed with Dan's addendum to the warrant in line 31, and hence to his written assertion in line 25. Taking on the role of spokesman, he re-voiced Dan's written inscription from line 25, with added detail. Specifically, he surmised in lines 32-34 that the curve *L*, which they denoted *C*, is a simple closed curve and there exists a simply connected domain (previously denoted as *D* by Dan) that contains the curve. Frank used this finalized data as the basis for their warrant, Cauchy's Integral Formula (lines 35-39).

F: Oh ok [looks down as if looking for an eraser] yeah that's true. Yeah so C is a simple closed, positively oriented curve. There exists some simply connected domain [erases the picture with wrong domain] that connects the c-- no, not that connects the curve, but that contains the curve. Um, and the only discontinuity occurs at the point 0, which Dan chose to represent as z_0. Um, because of that, I mean, it relates to the Cauchy-Goursat Theorem and specifically, we can use Cauchy's Integral Formula to give us the result: um, $2\pi i$ times whatever function is on the numerator [points in circular motion to the integrand] in this form, evaluated at the discontinuity, in which case it's just one because of the function [points at integrand]. So $2\pi i$ times one is just $2\pi i$, our result. If there were multiple discontinuities, then it would just be, $2\pi i$ plus $2\pi i$ plus $2\pi i$, over and over for the number of discontinuities. But in this case, we just have the one.

This warrant is comprised of three different types of reasoning with respect to Tall's (2013) three worlds. In particular, lines 35-37 instantiate formal reasoning as an appeal to a major theorem. Next, "$2\pi i$ times whatever […] the discontinuity" (lines 37-38) represents symbolic formal reasoning as it relates the statement of the theorem to the symbolic nature of the specific integrand given. Finally, "in which case it's just one because of the function [points at integrand]" (lines 38-39) exemplifies symbolic reasoning due to a symbolic evaluation of a particular function $f(z) = 1$ within the integrand.

Frank also provided backing for this warrant's validity in lines 34-35 by reiterating the datum regarding the discontinuity at $z_0 = 0$. Ultimately, Dan and Frank reached their concluding claim that the integral results in $2\pi i$ (line 39). Frank concluded with a rebuttal that considered a situation in which their claim would not hold, namely if there were "multiple discontinuities" (lines 39-41). This rebuttal represents formal symbolic reasoning because it appeals to a variation of the theorem that allows for multiply-connected domains, and relates this to a hypothetical symbolic answer. Argument 2 is summarized in Figure 85.

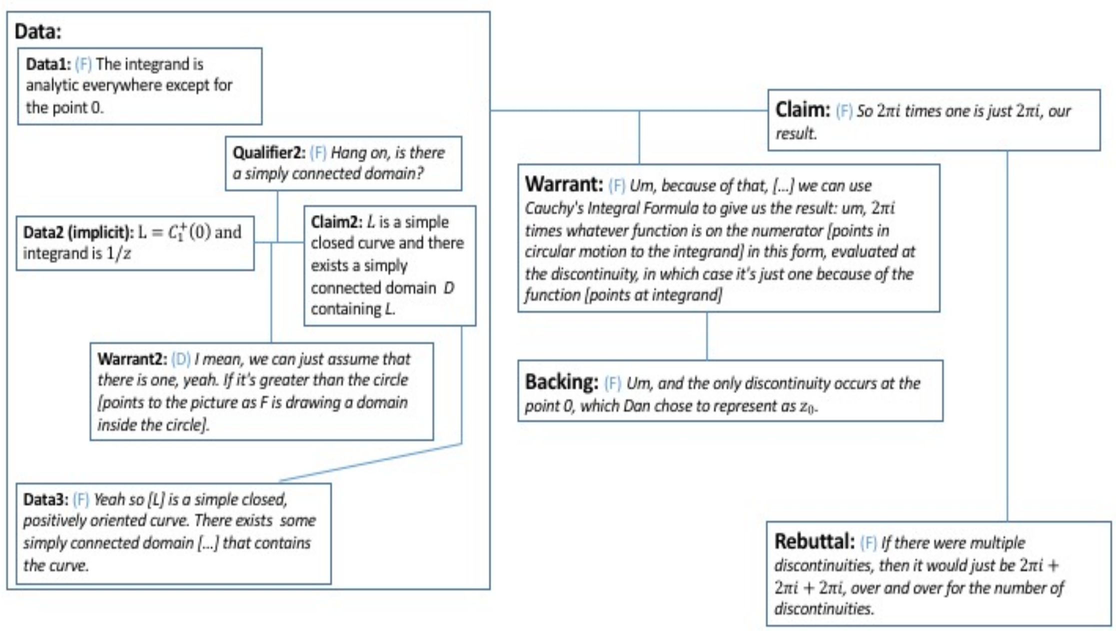


Figure 85. Toulmin diagram for Dan and Frank's Argument 2, Task 6.

As a follow up to this task, I asked Dan and Frank about evaluating the same integral by parametrizing the path instead of invoking a major theorem (lines 42-47), thereby prompting a new argument, Argument 3. Both Dan and Frank seemed quite confident that they would obtain the same answer as before (lines 48-49), but Frank decided to work out the details (line 50). In beginning to parametrize the circle as $Re^{i\theta}$, Frank stopped after writing R and observed that here $R = 1$ so he did not need to include an R in his symbolic expression (lines 50-51). Thus, he seemed to implicitly use the datum that $L = C_1^+(0)$.

At this point, Dan questioned whether $\frac{1}{z}$ and $\bar{z}$ are "the same thing" (line 52), authoring a potential connection to the function from Task 5. Frank quickly responded, as spokesman, to instead represent $\frac{1}{z}$ as z^{-1} using symbolic reasoning (line 53). However, Dan was committed to pursuing his aforementioned connection to $\bar{z}$, and wrote some supporting algebraic inscriptions as Frank watched (line 54). Frank then changed his mind and relayed Dan's conjecture (lines 55-56), using the symbolic warrant that $z^{-1} = cos\theta - isin\theta$ to claim, incorrectly, that indeed $\frac{1}{z} = \bar{z}$. Perhaps noticing the implications of what they just concluded, Frank re-voiced Claim 2 to include mention of a discontinuity (line 57). At this point, Dan and Frank quickly erased their inscriptions and appeared hesitant to elaborate on this claim any further (line 58).

Int: Ok. Given that we just talked about a somewhat similar idea, but using the parametrization, would it be possible to do this same sort of problem in that method too?
F: Sure.
Int: And you'd get the same answer right?
F: Yeah. Do you want us to do that or?
Int: Well so it's up to you. It sounds like you'd be pretty confident that you'd get the same answer.
D: Yeah I know it would be the same thing.
F: Yeah it would be the same thing.
F: But I mean like, here— [writes R] The circle is just— actually, you don't even need an R cuz it's a radius of one.
D: Wait can't you just, um— wasn't 1/z the same thing as the z bar?
F: It's z to the negative one [writes z^{-1}].
D: Because, wait— [writes 1/z] 1 over z [multiplies top and bottom by $\bar{z}$ in writing].
F: Yeah that's just z bar [pauses] Yeah, it is z bar because that's just cosine of theta minus i sin of theta [writes $cos\theta - isin\theta$ next to z^{-1}], which is z bar [writes $= \bar{z}$].
F: So basically it's like z bar with a discontinuity at zero.
D: I don't know [they both erase their inscriptions quickly].
F: Anyway, we got a result from the last integral was $2\pi i R^2$ [writes $2\pi i R^2$] for the integral of z-bar along that circle [writes bounds of 0 to 2π, then erases $\bar{z}$.] Let me write this properly. [Writes in $f(\bar{z})$] times some path. You know, I'm not writing this very nicely but [erases the integral] the last integral, when we evaluated the function, uh, z bar, the conjugate of z, along the same circle, this was our result. [points to the $2\pi i R^2$ inscription].
F: And fundamentally, this is the same thing [points to the 1/z] as z-bar, just with a discontinuity at zero. Um, so, if we give a radius of 1 [writes 1 under the R^2 and draws an arrow pointing to R^2] , we'll end up with the same result of $2\pi i$ [looks at me for approval]. So I think that kind of shows that you can do it two ways [starts to erase board].

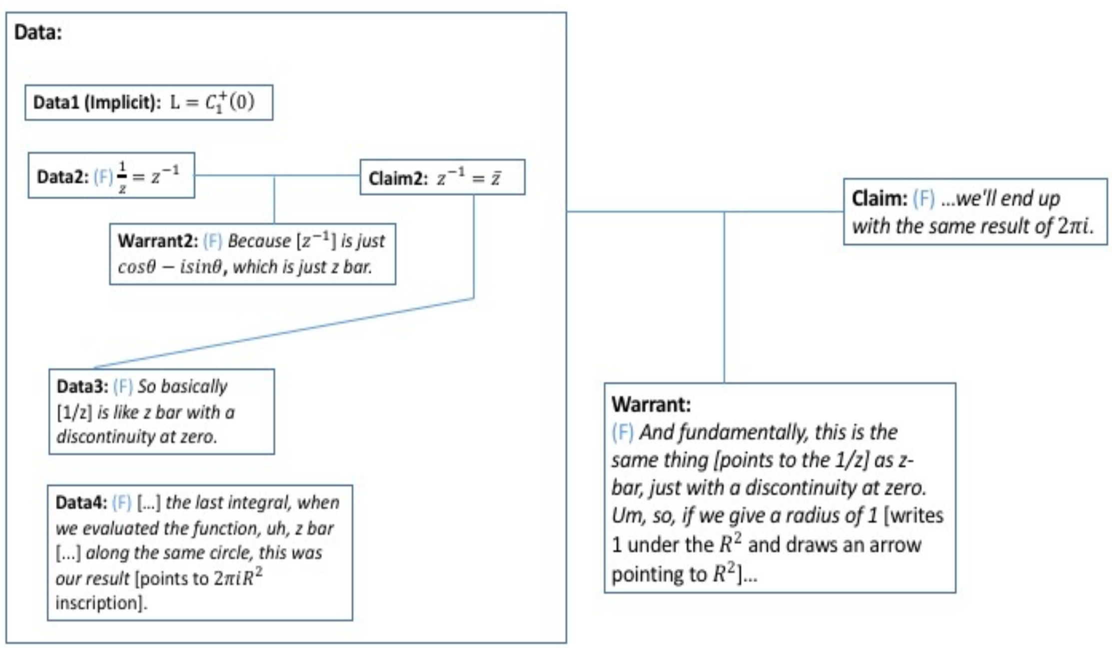


Figure 86. Toulmin diagram for Dan and Frank's Argument 3, Task 6.

Seemingly convinced that $\frac{1}{z} = \bar{z}$, but uninterested in providing further justification for this claim, Frank alluded to previous work from Task 5 (lines 59-63). In particular, he relayed their previous answer of $2\pi i R^2$ for the integral of $\bar{z}$ using the same circular path. He then used the recently established datum that $\frac{1}{z}$ is essentially $\bar{z}$ with a discontinuity as a warrant for their claim that ultimately they will get the same answer of $2\pi i$ as they did using Cauchy's Integral Formula (lines 64-66). As part of the elaboration for this warrant, Frank symbolically reasoned that using a radius of 1 for R in their previous answer $2\pi i R^2$ yields an answer of $2\pi i$. Argument 3 is summarized in Figure 86.

During Argument 3, it was unclear how the pair distinguished between $\frac{1}{z}$ and $\bar{z}$, especially when z does not lie on the unit circle. As such, I asked Dan and Frank to elaborate on their assertion that $\frac{1}{z} = \bar{z}$ (lines 68-70). I refer to their response to this follow-up question as Argument 4. This time, Dan re-voiced Frank's previous symbolic warrant (Warrant2 in Argument 3), acting as spokesman and using the extra datum that $z = e^{i\theta}$ (lines 71-73). However, Dan additionally provided symbolic backing for this warrant's correctness, clarifying that the claim holds because in this case the radius has unit length (line 74). He also qualified this backing with the word "right?" (line 74), expressing potential uncertainty or seeking affirmation from Frank. Frank did agree with this backing (line 76), so Dan continued with a symbolic rebuttal considering a hypothetical case wherein $R \neq 1$ (lines 77-78). Argument 4 is depicted in Figure 87.

Int: Well so you said that 1 over z and z-bar are fundamentally the same. Could you speak a little bit more about how they're related? Like, so z-bar certainly looks different than 1/z. What exactly is the difference between those two?

D: I think we'd have to assume that you have— So represent z as e to the i theta [writes $z = e^{i\theta}$]. Right, and then you have $1/z = e^{-i\theta}$ [writes this] which is cosine theta - i sin (theta) , which is z bar [writes these equalities].

D: Um, but that's assuming that you have a radius equal to one, right? [writes in a small 1 in front of the $e^{i\theta}$].

F: Yeah.

D: Because otherwise you'd have one over R, one over R [writes in 1/R in front of $cos(\theta)$ and $isin\theta$ terms]. So if our radius is one— [F starts talking]

F: Yeah if we're only on the unit circle, then they're fundamentally the same. The only issue, I guess, would be at the negative real axis? With the argument function. But because that's always an issue, we, you know— because we choose to usually define $-\pi < \theta \leq \pi$ [writes this] then that shouldn't be a problem.

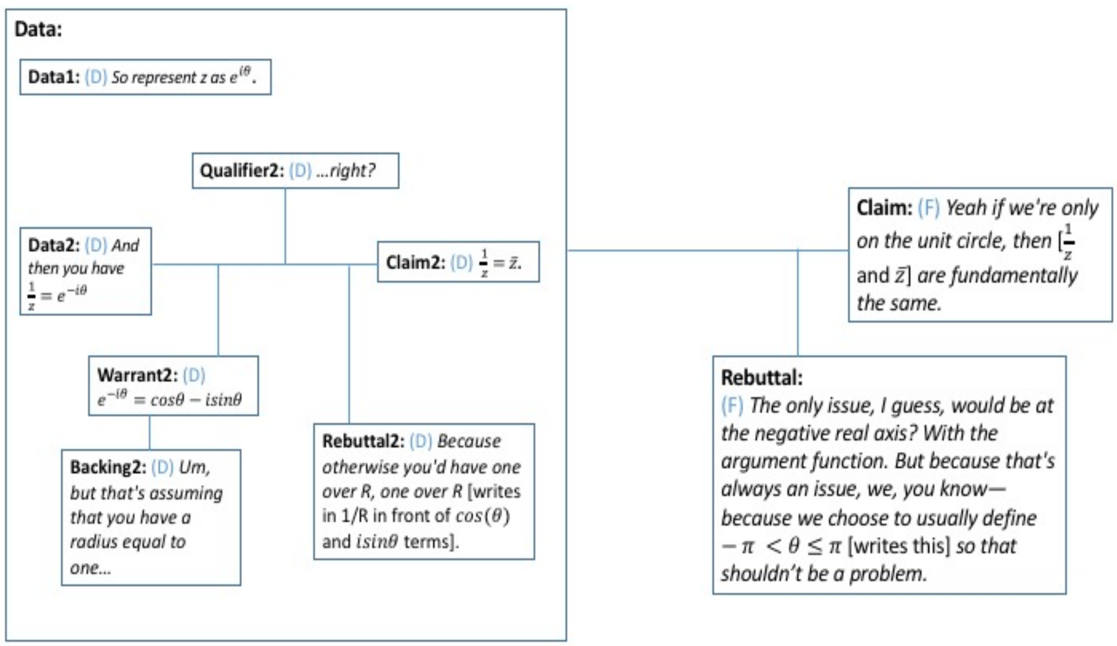


Figure 87. Toulmin diagram for Dan and Frank's Argument 4, Task 6.

Next, Frank provided their claim as spokesman, incorporating the aforementioned backing with the phrase "if we're only on the unit circle" (line 79). This claim represents embodied-symbolic reasoning, in that Frank used the geometric location of *z* on the unit circle to conclude the equality of two symbolic representations. Finally, Frank closed out Argument 4 by authoring a rebuttal considering a potential issue with the standard choice of branch cut for the argument function along the negative real axis (lines 79-82).

However, Frank ultimately decided that their choice of branch cut would not be problematic after all, and recalled a conversation Dan and Frank had with their professor about their class project as a way of exemplifying the rebuttal (lines 83-87). As part of this elaboration, Frank employed embodied reasoning by referencing the geometric location of the negative real axis (line, and using his hand to trace along a hypothetical negative real axis of a presumably visualized complex plane (see Fig. 88).

F: Because remember when we gave our thing on the Gauss-Lucas Theorem? We were talking about how 1 over arg of some complex number omega is the same as negative arg of omega [writes $\frac{1}{\arg(\omega)} = -\arg(\omega)$]. And Dr. X pointed out that that's only true assuming we're not on the negative real axis. We have kind of the same thing going on here [points back at the task inscriptions]. Um, I think. Because we're inverting it and basically saying it's equivalent to the negative.

F: Um, so I think, like, as long as we, um, don't have to deal with that, then it's not a problem. But given that we usually treat the negative real axis as our discontinuous region [traces a hypothetical negative real axis on board with four fingers and palm facing down], um, it shouldn't be a problem.

Int: Well so in the last example, it sounded like you guys were integrating from 0 to 2π for your values of θ, then?

F: Yeah.

Int: Is that going to present any sort of trouble, given where your branch cut is there? [Pause] You're saying you want to avoid that negative real axis right?

F: Sure. I mean, I guess we could just as easily integrate from negative pi to pi.

Figure 88. Frank’s gesture tracing along the negative real axis during Argument 4.

In response, I reminded Frank and Dan that their values of θ in their last parametrization ranged from 0 to 2π, and asked them if this would be problematic given Frank’s concern for their choice of branch cut (lines 91-95). Frank quickly decided that any potential issue could be avoided by simply integrating from $-\pi$ to π instead (line 96). I then asked if making this change would affect their previous answer, and Frank indicated that he did not believe it would (lines 97-98). Nevertheless, he asked Dan if the pair should verify this (lines 98-99). Following a clarifying question from Dan, Frank proceeded to parametrize the circular path using these adjusted values for θ, and thus began a fifth argument which is depicted in Figure 89.

Int: And would that change anything?

F: It shouldn't, no. Not as far as I can tell. Do you want to try parametrizing it?

D: This one?

F: Sure. I mean the circle is just e to the i times theta [writes $e^{i\theta}$]. And if we choose theta to be greater than pi, or sorry negative pi, and less than or equal to pi [writes $-\pi < \theta \leq \pi$] um, well then we know $dz/d\theta$ is just $ie^{i\theta}$ [writes $dz/d\theta = ie^{i\theta}$]. So the function we end up with, the integral from—if we're choosing to go from minus pi to pi, one over z is the same as $e^{-i\theta}$. And we have, times $ie^{-i\theta}\, d\theta$ [writes out this integral as talking].

F: And so these—sorry this one's positive [erases the negative sign from the second $e^{i\theta}$]. Those cancel each other and we're left with the integral from -pi to pi of $i\, d\theta$ [writes it], which is just i times theta, evaluated from minus pi to pi, which is i times $(\pi - (-\pi))$, which is 2 pi i [writes these symbolically]. So yeah, even with the different parametrization, or the different endpoints and starting points, it still works the same way.

Int: Ok great. Yeah so it's probably good that you got the same thing for both methods, right? [All laugh]. Ok cool. [F erases board]. Alright.

Argument 5 began as Frank expressed z as $e^{i\theta}$, a symbolic datum (line 100). Frank acted as spokesman in articulating this datum, in that he modified the syntactic structure of Dan's previous statement (Data1, Argument 4) but retained the same semantic meaning. He continued with another symbolic datum articulating the revised range of values for θ (lines 100-101), and used these two data to conclude that $dz/d\theta = ie^{i\theta}$. Frank also relayed Dan's datum from a previous argument (Data2, Argument 4) that $\frac{1}{z} = e^{-i\theta}$ (line 103).

Next, Frank used these data to instantiate the definition of contour integral $\int_L f(z)dz = \int_a^b f(z(t))\, z'(t)dt$ as a warrant, so that $\int_{C_1^+(0)} \frac{1}{z} dz = \int_{-\pi}^{\pi} e^{-i\theta} ie^{i\theta}\, d\theta$ (lines 102-104). Frank continued to simplify this symbolic expression and use the Fundamental Theorem for Line Integrals to conclude that the integral still yielded a result of $2\pi i$ (lines 105-108). Thus, Frank reached the claim that "even with a diferent parametrization, [...] it still works the same way" (lines 108-109).

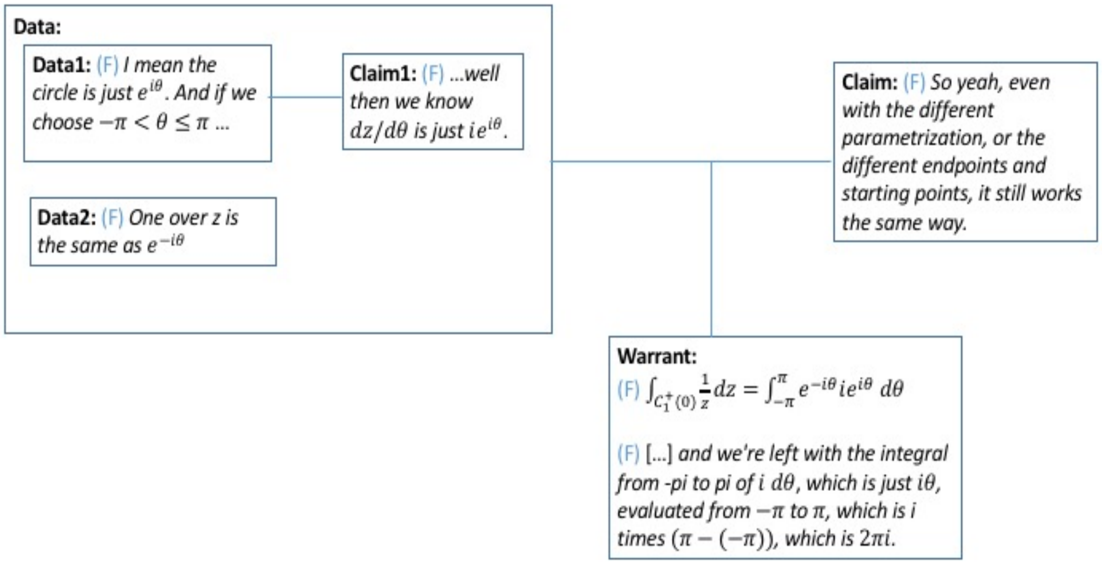


Figure 89. Toulmin diagram for Dan and Frank's Argument 5, Task 6.

Task 6 – Riley and Sean

Riley and Sean pursued a markedly different approach to Task 6 than Dan and Frank, resulting in only one argument that did not call upon Cauchy's Integral Formula. As I read the task aloud, Sean symbolically relayed the data comprised of the integral $\int_L \frac{1}{z} dz$ and the path |z|= 1 (line 4). He also authored an embodied datum by drawing the circular path on an Argand plane (see Fig. 90). As spokesman, Sean then symbolically rewrote L as $C_1^+(0)$, and I acknowledged this alternate symbolism from their class (lines 4-5). Riley agreed, but Sean made sure to document that this was Dr. X's notation, as if indicating that he did not hold any agency when using it (lines 6-8).

Int: Ok so more integration stuff. And I think given what we just discussed, I think this one shouldn't be too bad for you. So how would you find the integral of $\frac{1}{z}\,dz$, and this is over the path L, if L represents the unit circle $|z| = 1$, traversed counter-clockwise?
S: [Writes $\int_L \frac{1}{z} dz$, |z|= 1, and draws the path on an Argand plane. Changes notation of L to $C_1^+(0)$]
Int: Yeah, and I realize you guys have your own notation for that too, so—
R: Yeah.
S: Dr. X's notation. I'll document that [laughs].
Int: Right [laughs].
S: Ok so kind of like we did last time, I think I'm pretty sure that— I normally call it using the antiderivative, but—
R: [Draws in orientation on path] There's no branch we can choose, right, so that it's going to be analytic over the entire path?
S: Yeah, so I'd just say, $z = e^{i\theta}$ [writes this]. Therefore $z'(\theta) = ie^{i\theta}$ [writes this]. So we say the integral from 0 to 2π, because theta is of course from these values [writes $0 \leq \theta \leq 2\pi$], 1 over $e^{i\theta}$, times $ie^{i\theta}\,d\theta$ [writes $\int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta$]. Which then you're going to get $i \int_0^{2\pi} d\theta$ [writes this], $2\pi i$ [writes $= 2\pi i$], which gives the well-known result of integral of 1 over z minus a pole [points to $\int_L \frac{1}{z} dz$ inscription], a circle centered around it [points to diagram of path], gives you $2\pi i$ [points to $2\pi i$ inscription].

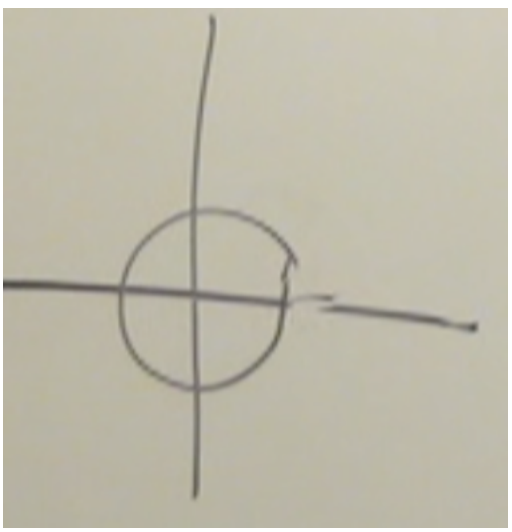


Figure 90. Sean's initial diagram for the path L in Task 6.

Sean proceeded as spokesman, indicating that they could apply an antiderivative, as in the last task (lines 9-10). He also qualified this suggestion with the phrase, "I think I'm pretty sure that…" (line 9). However, Riley challenged Sean as she authored a warrant: "There's no branch we can choose […] so that [the integrand] is going to be analytic over the entire path" (lines 11-12). Invoking embodied reasoning, Riley also revised Sean's initial diagram of the circular path to include a positive orientation (see Fig. 91).

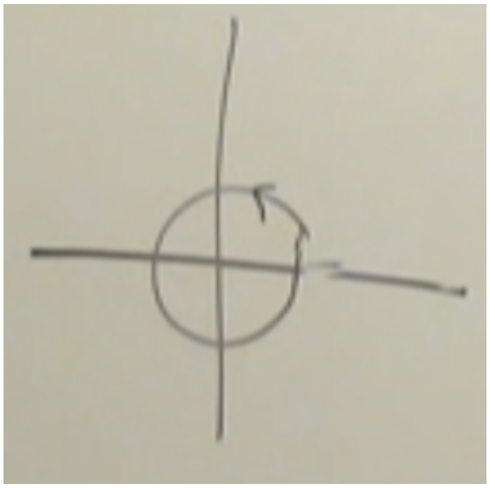


Figure 91. Riley's counterclockwise orientation to the path L in Task 6.

Sean conceded, and used their warrant to author an alternate approach implementing parametrization. Specifically, he first used embodied-symbolic reasoning to conclude that $z = e^{i\theta}$ as a parametrization of their path (line 13). Using this now as a datum, he further concluded that $z'(\theta) = ie^{i\theta}$, using symbolic reasoning (line 13). As spokesman, Sean implemented embodied-symbolic reasoning to re-write the original integral, incorporating this new parametrization. The embodied aspect of this rewriting came from the decision to allow theta to vary from 0 to 2π, a decision qualified by the phrase, "theta is of course from these values" (lines 13-15). Sean symbolically simplified this integral to obtain $i\int_0^{2\pi} d\theta$, and claimed that they obtained the "well-known result" of $2\pi i$ (lines 16-18). This sole argument for Task 6 is summarized in Figure 92.

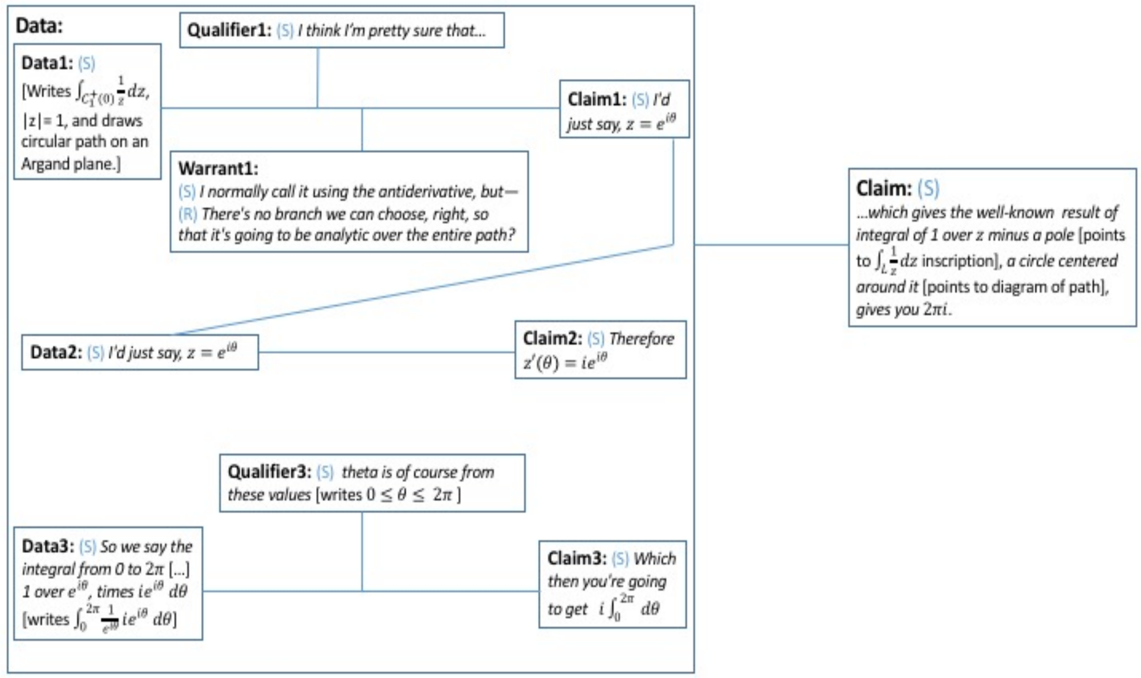


Figure 92. Toulmin diagram for Riley and Sean, Task 6.

Task 6 Summary

As stated previously, Riley and Sean chose not to invoke Cauchy's Integral Formula to evaluate the task integral, unlike Dan and Frank. Accordingly, Dan and Frank partook in several follow-up arguments concerning the idiosyncratic hypotheses of the theorem, whereas Riley and Sean provided a more succinct response comprised of just one argument. This difference in approach led Dan and Frank to also supply more embodied reasoning in the form of an extra diagram illustrating Frank's proposed simply-connected domain. Other distinct embodied reasoning included Dan's tracing gesture along the real axis, which neither Riley nor Sean incorporated into their response. Another consequence of Dan and Frank pursuing a more formal response was that their arguments including several backing statements, whereas Riley and Sean's response contained no backing.

Task 7 – Dan and Frank

Task 7 (see Appendix C) required participants to consider how the value of the integral of the same function $f(z) = \frac{1}{z}$ from Task 6 would change, if at all, by altering the radius of the circular path L to be 2 instead of 1 (lines 1-2). Dan and Frank's response to this task occurred over two arguments, which I present in Figures 93-94. At the commencement of Argument 1, both Dan and Frank immediately claimed that their result would be the same as in Task 6 (lines 3-4). Dan authored a warrant for this claim, suggesting that they employ Cauchy's Integral Formula, but forgot the name of the theorem (lines 5-7).

I was initially surprised that Dan forgot the name of this theorem, given that he and Frank just invoked this result in Task 6. However, once Frank started to elaborate the statement of the result (lines 8-9), it became clear that they were implementing a more general version of this result, namely the Cauchy Integral Formula for Derivatives. While Frank used formal-symbolic reasoning to write symbolic inscriptions corresponding to this formal theorem, Dan claimed, "So it doesn't matter how big your circle is. It just matters how many discontinuities are inside the circle" (lines 7-8). As such, Dan acted as spokesman by re-voicing the aforementioned claim that their answer would be the same as in the last task, with added detail that underscored the use of their warrant. He qualified this assertion with the word "Right?" (line 8), suggesting potential uncertainty about the claim.

Int: Let's see. So what if, now this circle L now had radius 2 centered around the origin, and traversed counterclockwise still? How would that affect your answer, if at all?
D: It wouldn't affect it; it'd be the same.
F: It wouldn't.
D: You'd just use the, um— what is the name of the formula?
F: Cauchy's Integral Formula?
D: Yeah. Cauchy's Integral Formula. So it doesn't matter how big your circle is. It just matters how many discontinuities are inside the circle. [F is writing $\frac{n!}{2\pi i}$ on board as D talking] Right?
F: Which tells us [writes $\frac{n!}{2\pi i}$ at this point]. Yeah I think it's this. Times the integral over circle of radius R [writes integral symbol and the path as C_R^+]. And it's what, f—
D: $f(z)$ over $(z - z_0)$, the quantity to the $n + 1$, and that equals $f(z_0)$ [Frank writes $f^n(z_0)$] or yeah, $f^n(z_0)$.
F: Right, so the radius itself [points to the R in C_R^+ inscription] is irrelevant, um, when we use Cauchy's Integral Formula. And even if we parametrized it, it'd end up cancelling, um, in the parametrization [looks at me for reassurance]. So no, it wouldn't affect anything.

Continuing to employ formal-symbolic reasoning, Dan and Frank worked together to articulate the rest of the statement of Cauchy's Integral Formula (lines 9-12). Note that Dan and Frank intended for the symbolic inscription $f^n(z_0)$ to represent the n^{th} derivative of the function f, evaluated at the point z_0, in accordance with the theorem, and not the expression $[f(z_0)]^n$. While I did not call attention to this notational ambiguity during the interview, Dan and Frank explicitly referenced this symbolism in Argument 2 in discussions about derivatives. It is also worth noting that throughout Dan and Frank's inscriptions and verbiage related to this task, they never incorporated the particular radius of 2 specified in the task. Rather, Dan and Frank referenced a general radius R, arguing that "the radius itself is irrelevant" (line 13). Frank's claim about the irrelevancy of the radius suggests a speaker role of spokesman, in that he essentially rephrased Dan's assertion from line 7.

Perhaps because I previously asked them to use parametrization to verify their answer in Task 6, Frank authored a second warrant involving parametrization to support their claim about the radius (lines 14-15). As he had done several times previously, Frank looked over at me after articulating this second warrant, as if seeking validation or reassurance. Still curious about why Dan and Frank chose to invoke the generalized Cauchy Integral Formula for Derivatives when the task did not mention any such derivatives, I asked them about the meaning of n in their inscriptions (line 16). This follow-up question prompted a second argument as detailed below.

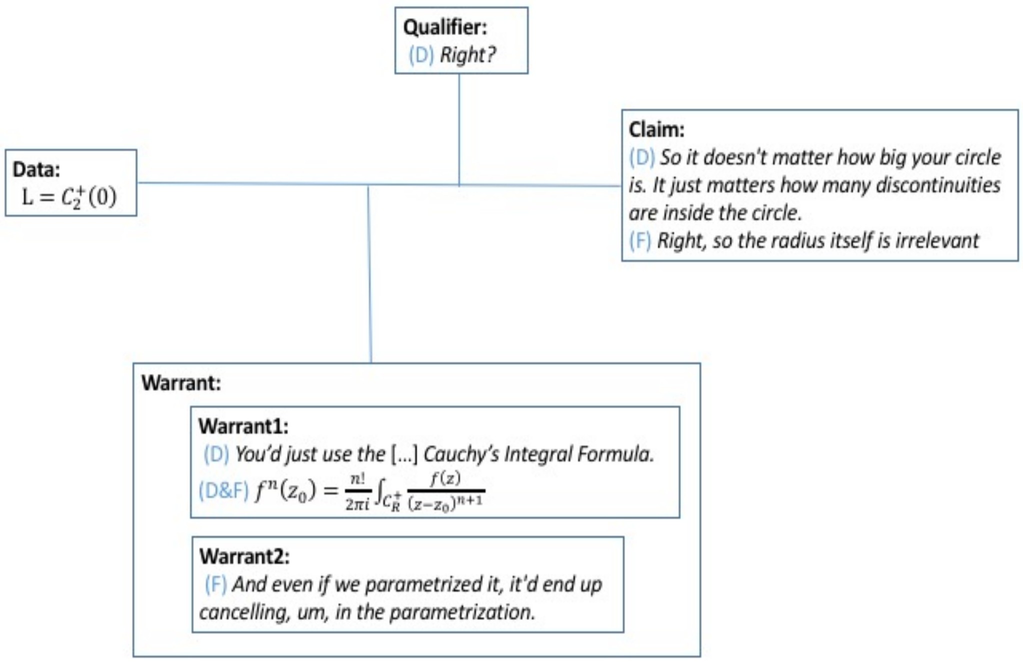


Figure 93. Toulmin diagram for Dan and Frank's Argument 1, Task 7.

Frank seemed to interpret this question as an inquiry about their chosen value for n in this task, claiming that $n = 0$, and qualified this claim with the phrase "in this instance" (line 17). Using symbolic reasoning, Dan provided a partially articulated warrant for this claim (line 18), and Frank elaborated that "we're evaluating the function, not like the *derivative* of the function at any point" (line 19). Frank continued to use symbolic reasoning to discuss the algebraic implications of taking on this value of n, adopting Claim1 as a datum for a second claim that the exponent $n + 1$ in the denominator $(z - z_0)^{n+1}$ is simply 1 (lines 19-21).

Using the aforementioned conclusions as data, Frank authored the remaining details of their warrant. In particular, he discussed how the symbolic expressions in the theorem simplify when $n = 0$, the integrand is $1/z$, and hence the function f in the statement of the theorem is $f(z) = 1$ (lines 21-25). Interspersed into Frank's articulation of this warrant was backing for the warrant's validity. Specifically, Frank mentioned that "$f(z)$ is not discontinuous at z_0" (lines 22-23), thus verifying a condition for applicability of the Cauchy Integral Formula as a warrant.

Int: So what's the n here that you're using?

F: The n [points to the n in the $\frac{n!}{2\pi i}$ inscription]— so in this case, our n is just 0, because um—

D: $z - z_0$.

F: Right, we're evaluating the function, not like the *derivative* of the function at any point. And if n is zero then the exponent on the entire denominator is one [points to the $n + 1$ in the inscription of $\int_{C_R^+} \frac{f(z)}{(z-z_0)^{n+1}}$] which corresponds to our situation. So yeah, basically what that's telling us is, in our case, the integral on some circle of radius R [writes integral over C_R^+] of some function $f(z)$, noting that $f(z)$ is not discontinuous at z_0, over $z - z_0$, equals 2pi i times $f(z_0)$ [writes this on board]. So regardless of our R value, the discontinuity always occurs at zero. This [points to $f(z)$] is just going to be one, because that's our function, 1 over z. So it's always just going to be $2\pi i$ times 1, which is $2\pi i$.

Int: Cool, alright great. [F and D erase board]

I note here that Frank likely meant "discontinuous" when he said "not discontinuous" (line 23). I base this assumption on the fact that Frank later said that "regardless of the R value, the discontinuity always occurs at zero" (line 24), and in this case $z_0 = 0$. In any case, Frank used the aforementioned warrant to author a claim that the integral of $1/z$, regardless of the choice of R in the path $C_R^+(0)$, is "always just going to be [...] $2\pi i$" (lines 25-26). Argument 2 is summarized in Figure 94.

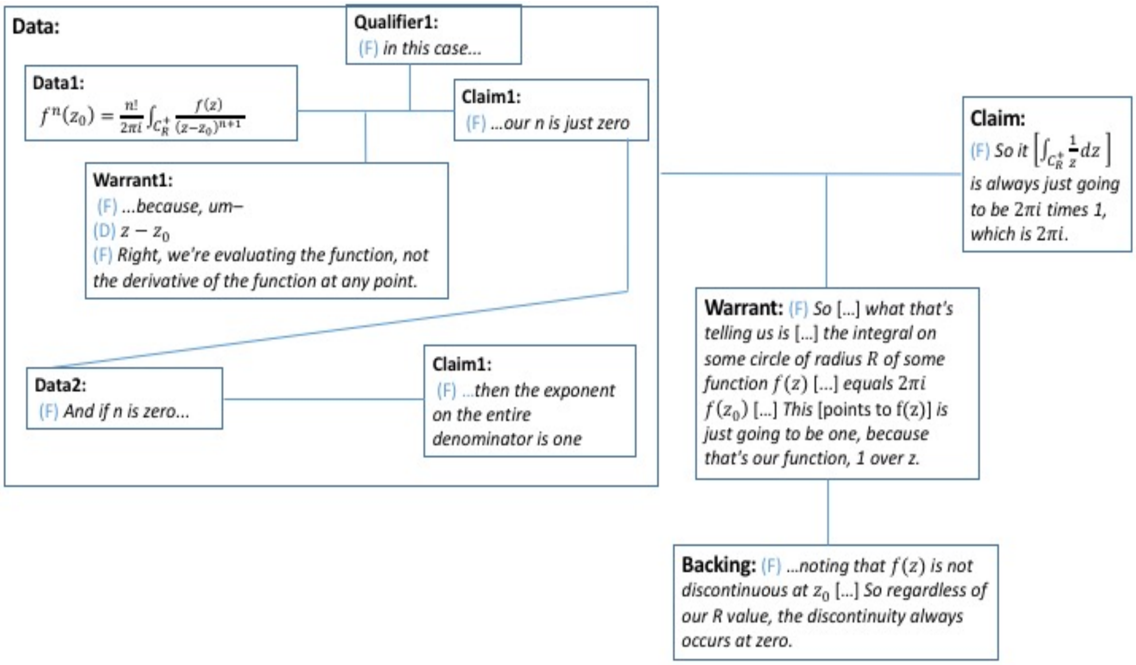


Figure 94. Toulmin diagram for Dan and Frank's Argument 2, Task 7.

Task 7 – Riley and Sean

As with Task 6, Riley and Sean chose not to pursue a formal approach to Task 7. In particular, they did not invoke Cauchy's Integral Formula like Dan and Frank. Rather, Sean revised their existing inscriptions from Task 6 to account for a circle with radius 2 rather than 1 (lines 3-5). This revision incorporated purely symbolic reasoning, as he did not alter the diagram depicting the path L.

Int: Cool, sounds good. And you seem pretty confident about that. Ok cool. So what if we now change L to be a circle of radius 2?
S: Ok, so then we say that's 2 [changes label for L], that's 2 [changes $|z| = 1$ to $|z| = 2$], that's 2 [places 2 in his prior symbolic parametrization], 2 there, 2 there [places 2 in the integral setup and evaluation], and we get the same.
Int: So we still get $2\pi i$.
R: Right and you can generalize that. For any circle of radius R, it's always going to cancel.
Int: Right. You read my mind! [Sean writes in R everywhere in place of 2].
R: [Laughs].

Using these symbolic revisions as a warrant, Sean authored the claim that “we get the same [answer of $2\pi i$]” (lines 5-6). Riley followed this claim with formal-symbolic backing for the warrant’s correctness, providing a general justification that they would obtain the same answer using a circle of any radius R. More specifically, she argued that the function and the dz portions of the integral are “always going to cancel,” leaving the integral of a constant (line 7). Meanwhile, Sean once again replaced the value of the radius in all symbolic inscriptions, this time with R (line 8). This concluded their sole argument regarding Task 7, which is depicted below in Figure 95.

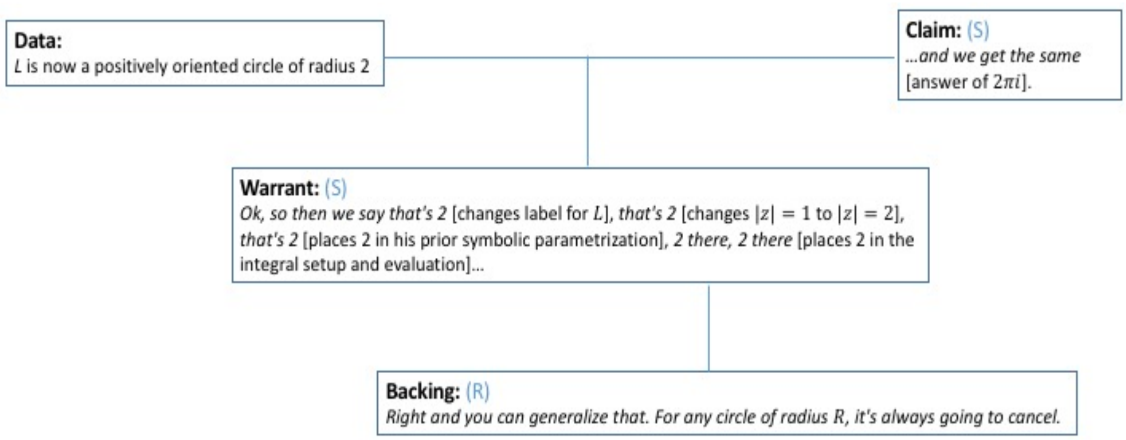


Figure 95. Toulmin diagram for Riley and Sean’s argument in Task 7.

Task 7 Summary

Once again, the prominent distinguishing factor between Riley and Sean’s versus Dan and Frank’s response to Task 7 was the absence of Cauchy’s Integral Formula. This indicated a general lack of formal reasoning in Riley and Sean’s response compared to Dan and Frank’s, aside from Riley’s general backing statement. Curiously, Dan and Frank chose to essentially rework the task from scratch, rather than alter their previous inscriptions as Riley and Sean chose to do. One quality apparent in both Dan and Frank and Riley and Sean’s responses was that neither pair invoked any embodied reasoning during Task 7. Because both pairs rather immediately identified that the value of the integral would not change by modifying the circle's radius, it is relatively unsurprising that no one felt the need to revise any embodied diagrams when altering various symbolic inscriptions.

Task 8 – Riley and Sean

Unfortunately, after asking Dan and Frank my follow-up question to Task 7, I forgot to ask them about Task 8 (see Appendix C), which required participants to discuss how reversing the orientation of a path affects the value of the integral. Accordingly, I only discuss the results of this task for Riley and Sean, and acknowledge the omission here as a potential limitation of my study. After I introduced the task (line 1), Sean relayed the last portion of my question about the clockwise orientation (line 2), and I confirmed (line 3). He then authored an embodied-symbolic claim that linked the counterclockwise datum to a new symbolic manifestation of z: $z = -Re^{i\theta}$ (line 4).

Int: And then, so what if we still keep that same path, but now we're traversing it clockwise?
S: Clockwise?
Int: Yeah.
S: We'll say z equals, now– [writes in a negative sign in the $z = Re^{i\theta}$ inscription]
R: So we're going 2π to 0, right? Which would just make it negative.
S: We can do that. Yeah, [erases the negative sign he just wrote] because then say theta begins at 2π and goes to 0, so we just say, 2π, 0 [writes in new limits of integration in each integral inscription from the last task]— which of course just gives us the negative of our value [writes a negative sign in front of the previous answer of $2\pi i$].
Int: Ok. I believe you guys formally proved the result in class, but is there kind of a quick way you can think about why it ends up— why it ends up having the negative there? I mean you can say stuff that—

At this time, Riley stepped in and authored an embodied-symbolic datum that the clockwise orientation corresponds to a reversal of the limits of integration from Task 7 (line 5). She used this datum to author a symbolic claim that this reversal of limits "would just make it negative" (line 5). Sean agreed, and erased his negative sign from the

$z = -Re^{i\theta}$ inscription he just wrote (line 6). As spokesman, he proceeded to re-voice what Riley proposed regarding the values of theta, both verbally and by adjusting the symbolic Task 7 inscriptions on the whiteboard (lines 6-8). Using formal-symbolic reasoning, he concluded that reversing the limits of integration yields "the negative of our [previous] value," and drew a negative sign in front of their $2\pi i$ answer from Task 7 (lines 8-9). Sean also qualified this assertion with the phrase, "of course," expressing a high degree of certainty (line 8). This first argument pertinent to Task 8 is depicted below in Figure 96.

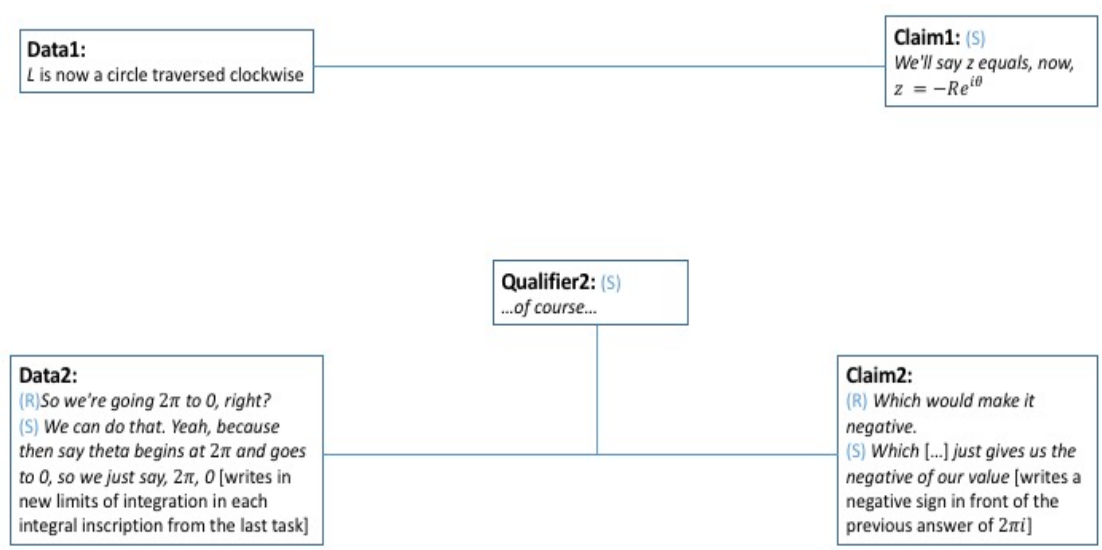


Figure 96. Toulmin diagram for Riley and Sean, Argument 1, Task 8.

Because neither Riley nor Sean discussed why swapping the limits of integration yielded a negative value in this context, I asked them to clarify this point (lines 10-12). Although I did not ask them for a formal proof, Riley acknowledged that they had proved the result in class, but she did not "remember the formal proof at all" (lines 13-14). However, Sean proceeded to explain his conception of the result. Employing embodied-symbolic reasoning, he described breaking up the original path into paths C_1 and C_2 such that $C_2 = -C_1$. As spokesman, he clarified that C_1 could be expressed as the set $\{x(t), y(t) | a \leq t \leq b\}$ (lines 16-17). Similarly, Sean authored a symbolic claim that C_2 "would be the same path but you just switch your limits," and wrote the corresponding inscription C_2: $\{x(t), y(t) | b \leq t \leq a\}$ (lines 17-18).

R: Yeah I don't remember what the— we did formally prove it in class. I don't remember the formal proof at all.

S: Yeah so like, there's a path C_1 and a path C_2. And C_2 is just a negative C_1 [writes $-C1 = C2$]. And then like, basically, it's like— so like C_1 is your set of points that make a path [writes C_1: $\{x(t), y(t) | a \leq t \leq b\}$], and so like C_2 would just be the same path but you just switch your limits [writes C_2: $\{x(t), y(t) | b \leq t \leq a\}$]. You go like— you start at $t = b$ and then go to $t = a$.

S: It's like we're going from $t = a$ to $t = b$ [tracing gesture left to right], and when you go backwards it's $t = b$ to $t = a$ [tracing gesture right to left]. Same curve but just running in reverse. Cuz you usually think of a particle, saying $z(t)$ is like from $t = a$ to $t = b$ is like a particle moves this way [traces path on board with finger]. Well if you reverse the path and run t backwards you just go here, going backwards.

Int: But as far as the integral there, though. Yeah I guess it really boils down to, you had it in your bounds, you changed somehow from 2π to 0. So how does that integral from 2π to 0 relate to the integral from 0 to 2π?.

Subsequently, Sean re-voiced the last portion of his claim as an embodied warrant. Specifically, he contrasted "going from $t = a$ to $t = b$" against "going from $t = b$ to $t = a$," and produced respective tracing gestures from left to right and right to left while verbalizing these scenarios (lines 19-20; see Fig. 97). As spokesman, he summarized this embodied warrant as, "Same curve but just running in reverse" (line 20). Sean further supported his assertion by authoring embodied backing for his warrant's field. He discussed how "you usually think of a particle" and its motion along a path, which he traced in the air with his finger (lines 20-22; see Fig. 98). On the other hand, he explained that "run[ning] t backwards" reverses the motion of this particle, and produced another tracing gesture to illustrate such a process (lines 22-23; see Fig. 98).

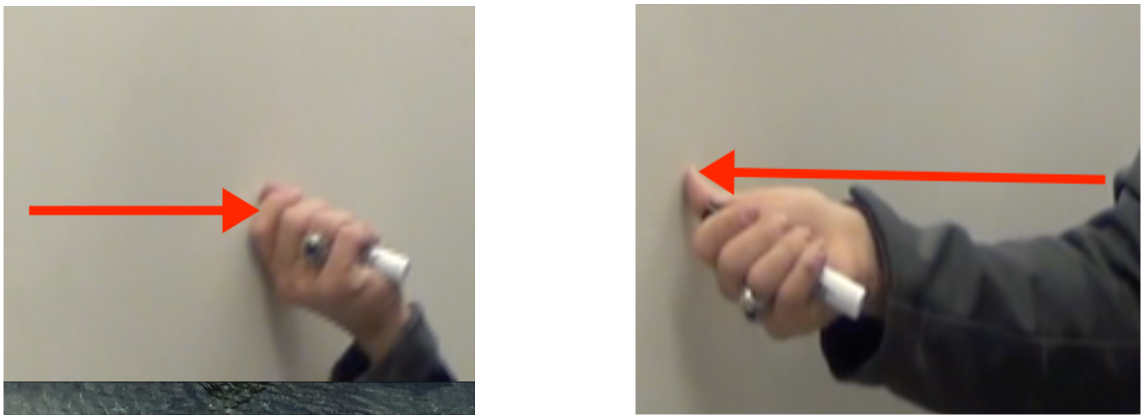


Figure 97. Sean's gestures for $t = a$ to $t = b$ (at left) and $t = b$ to $t = a$ (at right).

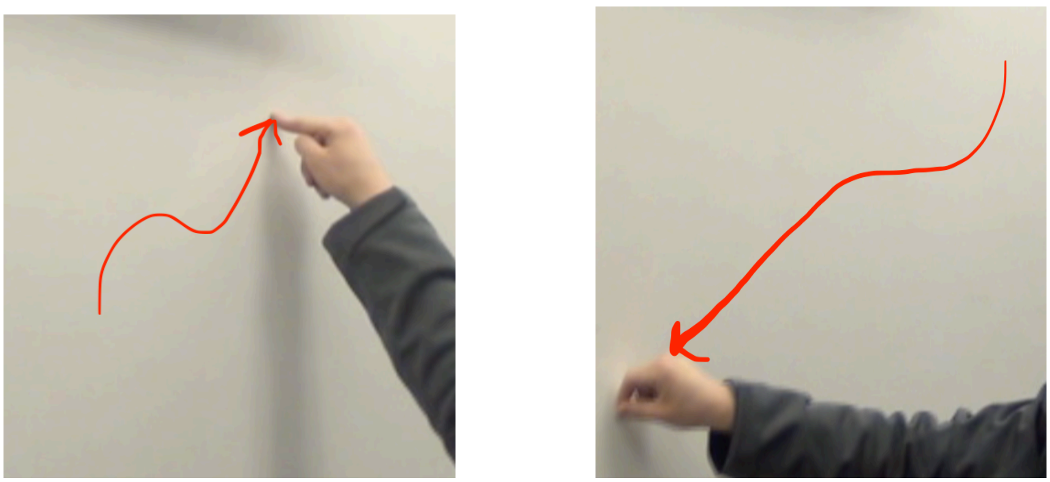


Figure 98. Sean's gestures a particle's motion (at left) and reversing this path (at right).

Although Sean provided an extensive embodied account of reversing a path's orientation, I pressed him and Riley to more explicitly connect this embodiment back to their symbolic inscriptions (lines 24-26). This follow-up question catalyzed a third argument, as detailed below. Argument 2 is summarized in Figure 99.

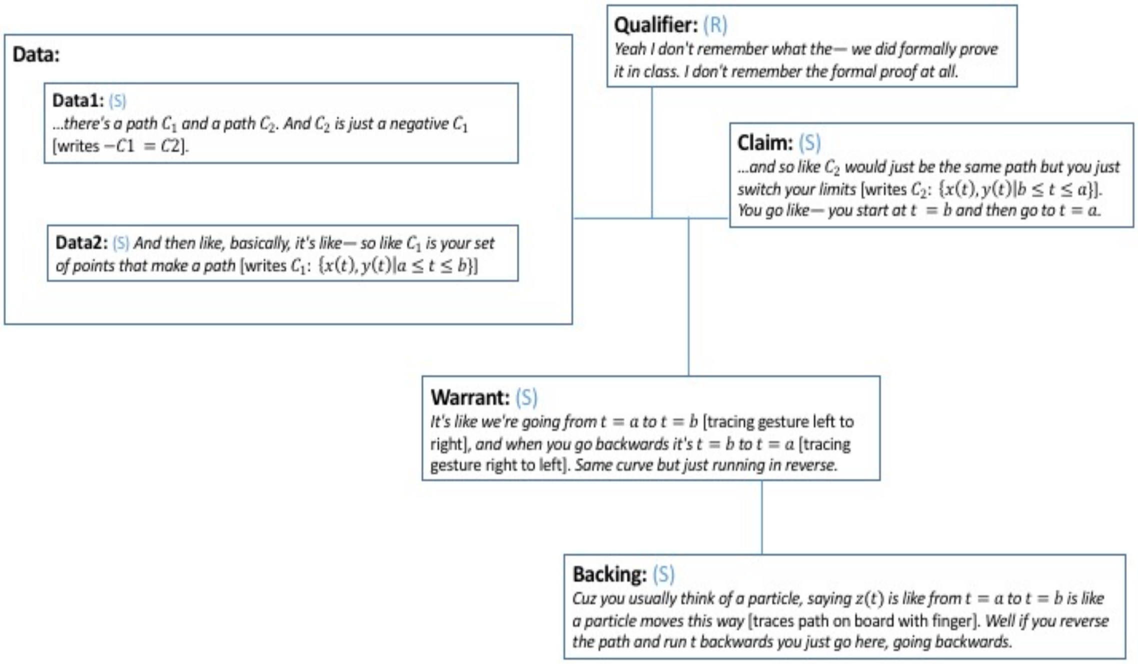


Figure 99. Toulmin diagram for Riley and Sean, Argument 2, Task 8.

S: Just the opposite.

R: Yeah, so I mean it's— so we take this and we integrate it right? [draws bracket emphasizing $i\int_{2\pi}^{0} d\theta$ inscription]. And we're going to be sort of using the Fundamental Theorem of Calculus. And so we do like— it's going to end up being $0 - 2\pi$. Whereas if we had done it, um to be an integral like this [rewrites $i\int_{0}^{2\pi} d\theta$ inscription] , it would've just ended up being 0 to 2π so it just comes out to— I mean it comes out to the same exact thing but instead of being like, $f(a) - f(b)$, it's $f(b) - f(a)$. I don't know if that really answers your question.

Int: Yeah totally. It just basically boils down to the Calc II, I guess, sort of result.

Int: Ok great. Um, so I think we're good with that sort of example. So feel free to erase that if you'd like. We can move onto something a little bit different.

In response to my aforementioned follow-up question, Sean once again claimed that switching the limits yields "the opposite" result (line 27). As spokesman, Riley emphasized the simplified version of the symbolic integral (lines 28-29). Employing formal-symbolic reasoning, she clarified that "we're going to be sort of using the Fundamental Theorem of Calculus" to evaluate the integral (line 29). She concluded that doing so yields an answer of $0 - 2\pi$, aside from the remaining i factor (line 30). Riley then compared this result to the integral $i\int_{0}^{2\pi} d\theta$, in which case she argued that "it comes out to the same exact thing but instead of being like, $f(a) - f(b)$, it's $f(b) - f(a)$" (lines 31-32). She qualified this argument with the verbiage, "I don't know if that really answers your question" (line 33), but I assured her that she and Sean adequately discussed the task (lines 35-36). Argument 3 is summarized below in Figure 100.

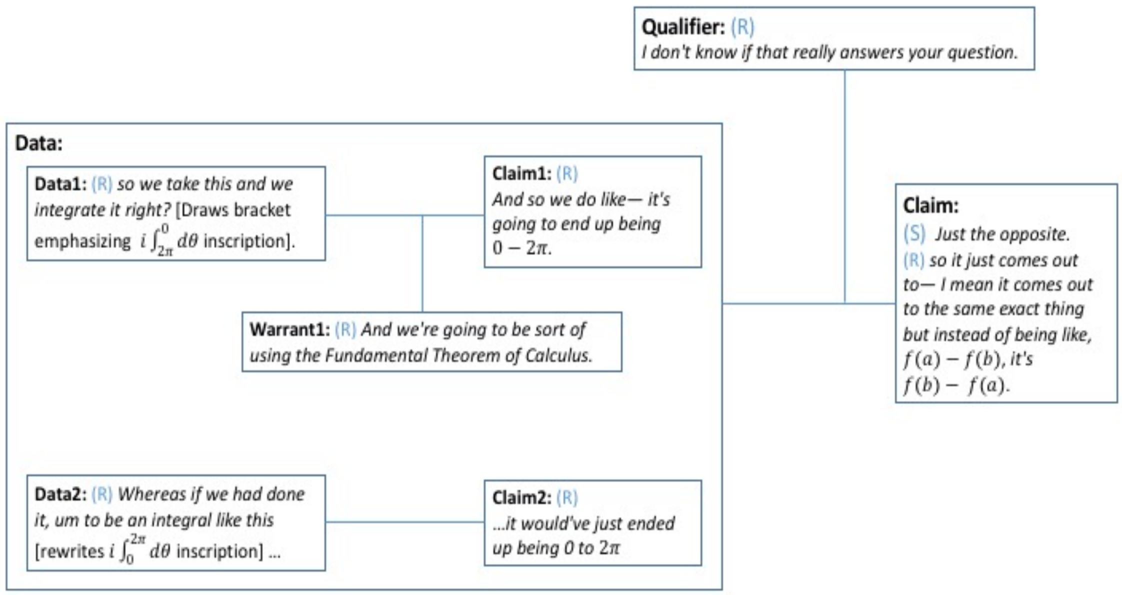


Figure 100. Toulmin diagram for Riley and Sean, Argument 3, Task 8.

Task 9a—Dan and Frank

Task 9a (see Appendix C) involved integrating the function $f(z) = \frac{z+2}{z}$ over the semicircular path $z = 2e^{i\theta}$, $0 \leq \theta \leq \pi$. I read this information aloud to Dan and Frank, and Frank wrote the corresponding symbolic inscriptions for this function and path (lines 1-7). Dan and Frank's first argument, Argument 1, centered around which method(s) they could use to evaluate this integral.

Dan initially authored a suggestion that working with a parametrization would be easiest, and conjectured that this might be the only approach, given that the semicircular path is not closed (line 8). As relayer, Frank reiterated Dan's claim that this path is not closed, but qualified this statement with the word "right," indicating a lack of certainty.

Dan confirmed that the path is not closed, stating that the path is a semicircle as support for this assertion (line 10). Because Dan and Frank discussed a geometric property of the semicircular path, this exchange suggests embodied reasoning.

Int: So moving to something slightly different— So I'll kind of give you this setup and if you want me to repeat anything that's totally fine. So we're now going to consider some path *C* that's given by the semicircle $z = 2e^{i\theta}$, where theta ranges from 0 to π. And we want to find the integral around this path *C*, of the function $\frac{z+2}{z}\,dz$. [Frank writes everything].

F: So I think I understood you, right? [Points to his writing]

Int: Yeah so we have that path there. We're going to find this integral around the path *C* of that function $\frac{z+2}{z}$.

D: Ok. Wouldn't it be easiest to parametrize this? Isn't that the only option since it's not a closed path?

F: It's not a closed path right?

D: No, it's a semicircle. Um, and— oh you could also use the, um [pauses] um, integration theorem that says if your function is continuous and there is a contour that connects those two points in some domain, then you can just find the antiderivative and evaluate at the endpoints.

F: Right, because this is just [draws Argand plane]—

D: Yeah.

F: Where that's 2 [labels 2] and that's minus 2 [labels -2] and that's $2i$ [labels $2i$] so it's just that [draws semicircular path through 2, $2i$, and -2].

F: So we could find some domain [draws dotted domain/region around the path] where— cuz I mean, obviously, there's the discontinuity at zero, but that's irrelevant for this. So we could go and find an antiderivative for this, um, keeping in mind that we'd have some kind of logarithm, um, which means, because we have a point on the negative real axis, we'd have to choose a branch cut that doesn't include the negative real axis.

D: Yeah.

Subsequently, Dan authored a claim that they could evaluate an antiderivative at the endpoints of the contour (lines 10-12). This claim represents formal-symbolic reasoning because it refers to a formal theorem involving the symbolic evaluation of an antiderivative. As support for this assertion, Frank authored a warrant describing why such an antiderivative exists (lines 15-21). As part of this warrant, Frank drew a geometric diagram depicting the semicircular path passing through the points 2, $2i$, and -2, implementing embodied reasoning (lines 15-16; see Fig. 101). In describing a domain wherein an antiderivative exists, Frank once again exemplified embodied reasoning by authoring the statement, “because we have a point on the negative real axis, we'd have to choose a branch cut that doesn't include the negative real axis" (lines 20-21). Argument 1 is summarized in Figure 102.

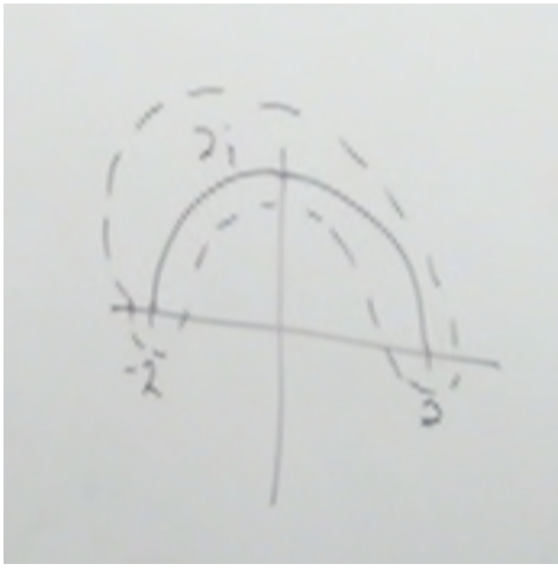


Figure 101. Frank's sketch of the semicircular path and a domain enclosing the path.

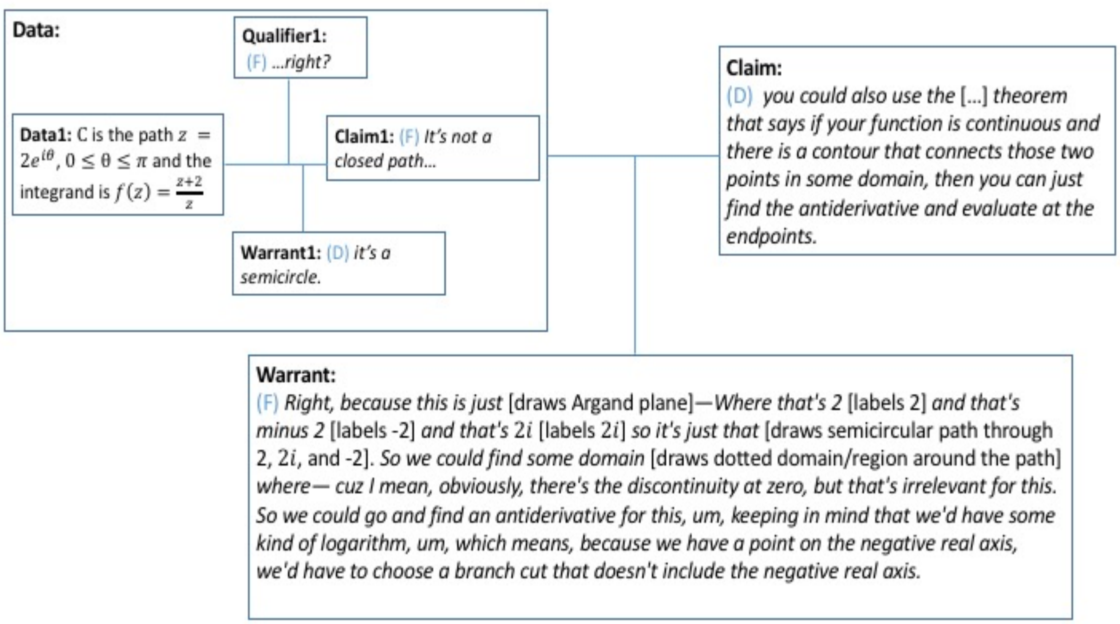


Figure 102. Toulmin diagram for Dan and Frank's Argument 1 for Task 9a.

Following this first argument, Frank acknowledged as author that they could alternatively use parametrization methods to evaluate the integral in question (line 23). This acknowledgement led to two distinct but related arguments about the task, which I refer to below as Argument 2a and Argument 2b, corresponding to the parametrization and Fundamental Theorem approaches, respectively. Arguments 2a and 2b initially occurred simultaneously as Dan and Frank silently wrote symbolic inscriptions on the board. However, later in each argument, the other participant interjected either to challenge the other student's assertion or to verify the correctness of various statements, as I detail below. It should be noted that although each participant initially pursued a separate approach to the task, Dan and Frank's resulting arguments were still collective in the sense alluded to previously.

23 *F:* It's either we do that, or we parametrize it.
24 *D:* I feel like—
25 *F:* What would you prefer?
26 *D:* [Both laugh] Um—
27 *F:* I mean, the parametrization should be simple.
28 *D:* Cuz you could just split that one up into two.
29 *F:* Yeah.
30 *D:* Cuz you have one plus 2 over z.
31 *F:* Yeah cuz this is [writes dz =] equivalent to 2i [writes $2ie^{i\theta}$]
32 *D:* Let's just do the branch cut one.
33 *F:* You wanna do the branch cut?
34 *D:* Yeah it's easier. And then we could do the—
35 *F:* We can do the parametrization after.
36 *Int:* Yeah you might as well do both just to see that you get the same thing.
37 *D:* Ok I'll do the branch cut one.
38 *F:* You'll do the branch cut?
39 *D:* Yeah I'll attempt that.
40 *F:* Ok. You give that a shot. I would use the branch cut with—
41 *D:* The negative—
42 *F:* Yeah the negative imaginary axis. That shouldn't have any issues.
43 *D:* Ok sweet.
44 *F:* Ok I'll start working on the parametrization. I really hope we get the same answer.
45 *D:* I might make a mistake and you'll catch it.

After Frank acknowledged parametrization as an alternative method for approaching this task, Dan and Frank discussed which method they wished to pursue (lines 24-36). Frank appeared to prefer the parametrization approach, as indicated by his assertion that it "should be simple" (line 27). As author, Dan added that they could split up the integrand into the sum $1 + \frac{2}{z}$ (lines 28-30), a symbolic claim. Frank continued by authoring the symbolic claim that $dz = 2ie^{i\theta}$ (line 31), implicitly using the datum that the path can be represented by $z = 2e^{i\theta}$ with $0 \leq \theta \leq \pi$. However, Dan suggested that they "just do the branch cut one" (line 32), indicating his preference for the method they initially discussed in Argument 1. He additionally remarked that he finds this method easier than parametrization (line 34). Frank conceded but suggested that they also try the parametrization approach afterwards (line 35).

Curiously, however, Dan chose to attempt this "branch cut" method himself (line 37) and Frank offered to work on the parametrization (line 44). In hindsight, it seems Dan interpreted my agreement with Frank as a direction to break up the task in this way. While this is not what I intended, I did not stop them from taking this course of action because I wanted their response to the task to unfold as naturally as possible. Frank and Dan decided that Dan should use a branch cut along the negative imaginary axis (lines 40-43), which they discussed further at a later time. Frank mentioned that he hoped he and Dan "get the same answer" (line 44). Notably, Dan responded, "I might make a mistake and you'll catch it" (line 45), which turned out to be an apt characterization of Argument 2b.

Subsequently, Dan and Frank began silently writing symbolic inscriptions supporting their respective preferred approaches to the task. In particular, Dan chose to split up the integrand as he alluded to previously in lines 28-30 (line 46); thus he took on the role of relayer to voice his symbolic claim. Meanwhile, Frank substituted $z = 2e^{i\theta}$

into the given function $\frac{z+2}{z}$ and utilized his prior claim that $dz = 2ie^{i\theta}$ to conclude that $\int_C f(z) = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} (2ie^{i\theta})\, d\theta$, which he then simplified to $\int_0^{\pi} \left(1 + e^{(-i\theta)}\right) \left(2ie^{i\theta}\right) d\theta$, using symbolic reasoning to "cancel those 2's" (lines 47-48; line 53). While Frank wrote these inscriptions, Dan wrote a symbolic claim that his integral $\int \left(1 + \frac{2}{z}\right) dz$ equals $z\Big|_0^{2\pi} + 2Log\, z\Big|_0^{2\pi}$ (line 49), but realized that his limits of integration differed from Frank's (lines 50-52).

46	*D:*	[Writes $\int \frac{z+2}{z} = \int \left(1 + \frac{2}{z}\right) dz$]
47	*F:*	[Writes $\int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} (2ie^{i\theta})\, d\theta$]. We can cancel those 2's [crosses out the 2's in the first factor with the
48		quotient].
49	*D:*	[Writes = $z\Big|_0^{2\pi} + 2Log\, z\Big|_0^{2\pi}$]
50	*D:*	[Glances at Frank's work and sees his bounds are different] Oh we're going from 0 to pi, my bad.
51		[Erases 2π and writes π in his evaluations in the last line].
52	*F:*	Oh did you go to 2π? [Looks at D's work] That looks a lot prettier than what I'm doing.
53	*F:*	[Writes $= \int_0^{\pi} \left(1 + e^{(-i\theta)}\right) \left(2ie^{i\theta}\right) d\theta = 2i \int_0^{\pi}$]
54	*F:*	Um, so we've pulled this out [points to 2i], $e^{i\theta} + 1,\, d\theta$. [writes this expression in as the integrand].
55	*D:*	[Writes = $\pi + 2(Log\, \pi - Log\, 0)$]
56	*F:*	Which actually works out nicely [writes = 2i(]

After acknowledging Dan's error, Frank continued with his symbolic calculation by pulling out the constant $2i$ in front of the integral, leaving him with the expression $2i \int_0^{\pi} (e^{i\theta} + 1)\, d\theta$ (lines 53-54). During this time, Dan silently evaluated his previous expression at 0 and π to obtain the new expression $\pi + 2(Log\, \pi - Log\, 0)$ (line 55). Frank commented that his calculations "work[ed] out nicely" (line 56) and was about to finish evaluating the integral when he glanced over towards Dan and noticed that Dan had stopped writing. In an effort to help, Frank asked Dan if he was "having a problem" (line 57).

F: You having a problem?
D: Um yeah, can we take a log of—
F: Yeah your branch cut is here, right? [Points to negative imaginary axis] So go from $\pi/2$ to $3\pi/2$.
D: But you still have a problem, cuz your log is— let's call that log(z) [writes $log(z) = ln|z| + i\,Arg(z)$].
F: Plus i Arg, yeah.
D: But if my z is zero, or if I input a zero, then this is still undefined, right?
F: It's not though, because your branch cut is on the negative real [I think he meant imaginary] axis, so the argument is a little a, arg [erases Arg and writes arg].
D: But I'm talking about the real component, right?
F: Oh hang on. What real component?
D: Zero. So what I'm saying is like— so you're splitting this into real and imaginary right?
F: Yeah.
D: But your real is the magnitude of z. But if I'm plugging in enpoints, and one of them is a zero—
F: But it's not.
D: Oh you're saying that because of the—
F: Because of your branch cut, your endpoints are— well, your starting point is $\pi/2$ and $3\pi/2$.
D: Oh ok, gotcha.
F: Cool?
D: Yep. [erases the bounds for evaluation of the log in his last two lines].

Dan responded by starting to express concern about evaluating a logarithm with their chosen branch cut (line 58), but Frank reassured him that there would be no issues (line 59). However, Dan persisted that there is still a problem when taking the Argument of the value 0, and he and Frank continued to disagree because Dan interpreted the path as starting at an argument of 0 (lines 60-70). But Frank ultimately pointed out that because they adjusted their chosen branch cut to take place along the negative imaginary axis, the arguments of the start and end points of the path are instead at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, respectively (lines 71-73). Dan realized his error and adjusted his inscriptions accordingly (lines 74-76).

Int: So I know different books kind of have different notations here. When you have the arg with the little a as opposed to the capital A, which one are we talking about?
D: So the capital A is from $-\pi$ to π.
F: Right.
D: So the standard one.
Int: The principal one?
D: Yeah the principal argument! [points to me]
F: The little 'a' is basically just any argument, based on whatever system we chose to define. So in this case Dan is using a branch cut along the negative imaginary axis. So because of that, his arguments for the points are $\pi/2$ and $3\pi/2$ [traces with pen along the positive real and negative real axes, resp. when saying $\pi/2$ and $3\pi/2$].
Int: Ok great. There's not necessarily a consensus among the textbooks and stuff.

Because Frank and Dan previously disagreed about whether the argument should be written as 'arg' or 'Arg' (lines 64-65), I asked them to comment on the difference between these two definitions (lines 77-78). Dan responded that "the capital A is from $-\pi$ to π," and Frank agreed (lines 79-80). As spokesman, Dan then referred to this type of argument as "the standard one" (line 81), and I asked if this was the same as the principal argument (line 82). Dan excitedly agreed and pointed at me, indicating that he had forgotten the particular name (line 83). Frank elaborated on this distinction, and recapitulated the agreement he and Dan reached about their chosen branch cut and the resulting values of the argument along the provided path (lines 84-87). In attributing the values $\pi/2$ and $3\pi/2$ of the argument to their choice of branch cut, Frank used this choice of branch cut as a warrant in Argument 2b that built off Dan's datum regarding the complex logarithmic definition. This portion of Argument 2b relied upon embodied-symbolic reasoning because Dan and Frank symbolically labeled the argument of various complex points based on their geometric position relative to their chosen branch cut. Moreover, Frank's tracing gestures along the axes in their diagram (see Fig. 103) substantiate the embodied aspect of this portion of the argument.

Figure 103. Frank's gestures tracing the positive (left) and negative (right) real axis.

F: [Finishes writing his last inscription: $= 2i\,[\frac{1}{i}\, e^{i\theta} + \theta]\,\big|_0^{\pi}$]
F: [Writes $= 2i\,[(-ie^{i\pi} + \pi) - (-i\ +\ 0)]$]. So that's just— [writes $= 2i($], e to the i pi is negative 1, so that's just [finishes writing $= 2i\ [\ i + \pi + i\ ]$] which is 2i times 2i + pi [writes this]. So that's 2 pi i minus 4 [writes $2\pi i - 4$ and boxes this answer].
D: [Replaces evaluation bounds so that inscription now reads $z\big|_2^{-2} + 2 \text{ Log } z\big|_2^{-2}$]
F: I feel like I screwed something up, but we'll see what you get.
Int: What did you feel like you screw up, like some algebraic thing or?
F: No, it's just we haven't really done too many functions in class where um, we have much on the numerator. So it's just weird getting a result of this form [points to his answer of $2\pi i - 4$.]
Int: I see.
F: I mean um, I *feel* like I went through it properly, but I'm just not sure if I made a silly mistake somewhere.
Int: The answer just doesn't seem like a clean number?
F: Exactly. It's not quite clean enough for me, but still, $2\pi i - 4$ isn't bad.

After this resolution to their disagreement, Frank returned to writing inscriptions for Argument 2a (line 89). In particular, he used symbolic reasoning to evaluate his last expression, concluding that $\int_0^{\pi}(1 + e^{(-i\theta)})\,(2ie^{i\theta})d\theta = 2i\int_0^{\pi}\ (e^{i\theta} + 1)\ d\theta =$ $2i\,[\frac{1}{i}\ e^{i\theta} + \theta]\,\big|_0^{\pi} = 2i\,[(-ie^{i\pi} + \pi) - (-i\ +\ 0)]$ (lines 89-90). He then authored the symbolic warrant $e^{i\pi} = -1$ to finish simplifying his previous expression, allowing him to claim that the result was $2\pi i - 4$ (lines 90-92). During this time, Dan updated the bounds in his previous inscription in Argument 2b to indicate evaluation of each term from $z = 2$ to $z = -2$ (line 93).

Once Frank obtained his answer of $2\pi i - 4$, he qualified Argument 2a with the statement, “I feel like I screwed something up, but we'll see what you get” (line 94). Because his argument was complete, I asked about the nature of Frank’s uncertainty (line 95), and Frank discussed how the integrand in this task appeared different symbolically than what he was used to in class (lines 96-97). Consequently, he felt uneasy when he obtained an answer that was “not quite clean enough” compared to what he was familiar with (lines 99-102). Argument 2a is summarized in Figure 104.

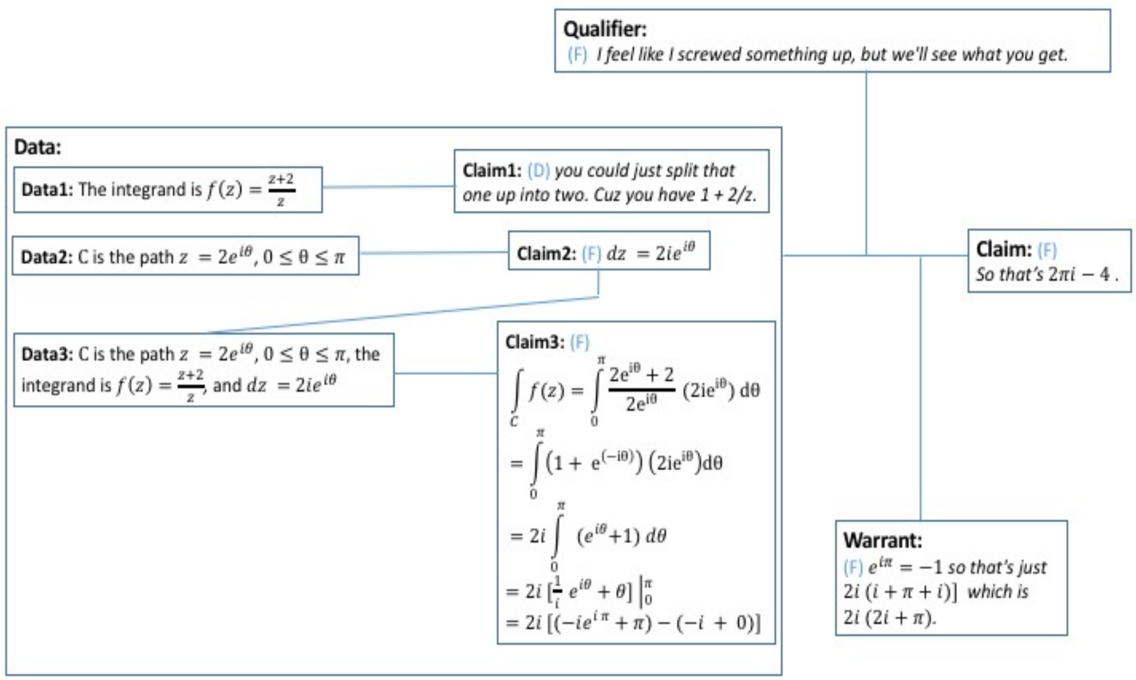


Figure 104. Toulmin diagram for Dan and Frank’s Argument 2a for Task 9a.

Int: Yeah it could be worse [laughs]. So let's see how Dan is doing [F walks over to D].
D: [Laughs] I was making a really stupid mistake. So I have 2 plus— I have i in the first argument, and, um— [in the mean time, he has gone from the last documented line, $z\big|_{2}^{-2} + 2\ \mathrm{Log}\ z\big|_{2}^{-2}$, to rewriting this as $4 + 2\ [\ln 2 + \ i\pi/2 - \ln 2 - \ i\ 3\pi/2]$.]
F: Hang on, you're going from 2 to -2 right? [points to his evaluation bounds] So it'll be minus 4 [adds minus sign in front of 4].
D: Yeah whoops. Cool, so um— so minus.
F: Do you have the same issue in terms of what you're subtracting from what? Did you go from -2 to 2? I'm not sure.
D: Yeah cuz $\pi/2$ is the argument for 2, right? So then minus 2—
F: Hang on. Cuz you should have— um, $ln|-2|\ +\ i\ arg(-2)$ [writes this below Dan's last step]
D: Yeah but that's the same as this right?
F: ln(2) and arg(-2)
D: Oh! I'm so stupid.
F: ...is $3\pi/2$.
D: Ok. I see
F: Yeah I just wanted to make sure you didn't have the signs backwards.
D: I did. So $\frac{i3\pi}{2}$, $-\frac{i\pi}{2}$ [changes the two argument calculations]. Ok so that just goes to $-4 + 2\pi i$.

With Argument 2a completed, I suggested we check in with Dan on the other approach to the task, and Frank walked over to Dan's portion of the whiteboard (line 103). In the time following Dan and Frank's previous discussion regarding the choice of branch cut and its implications on evaluating logarithms, Dan symbolically evaluated the expression $z\big|_{2}^{-2} + 2\ \mathrm{Log}\ z\big|_{2}^{-2}$ as $4 + 2\ [\ln 2 + \ i\pi/2 - \ln 2 - i\ 3\pi/2$ (lines 104-106). However, Frank relayed the datum that their path starts at $z = 2$ and ends at $z = -2$, and used symbolic reasoning to author a claim that the 4 in Dan's expression should actually be negative (lines 107-108). After Dan agreed to this revision, Frank continued to question whether Dan switched the starting and ending point in evaluating the rest of the expression (lines 110-111). As support, he elaborated that Dan's symbolic evaluation should include $ln|-2|\ +\ i\ arg(-2)$, which he simplified to $\ln 2 + \ i\ 3\pi/2$ (lines 113 and 115). In response, Dan maintained that this expression was equivalent to what he previously wrote, but quickly realized his error (line 116-118). After revising his argument calculations, Dan simplified his expression using symbolic reasoning and obtained an answer of $-4 + 2\pi i$ (line 120).

F: Yeah, $2\pi i$ minus 4. Yep, so we got it two different ways: using parametrization and using the fundamental integral theorem, assuming a branch cut that doesn't deal with any discontinuities.
Int: So you're pretty confident that those are right, then?
F: Pretty confident, yeah.
Int: Yeah I mean, it's a good sign when you get the same answer in two different ways.
D: That's the simpler way for sure [his way].
Int: Yeah so I guess now that you've done both, which do you sort of prefer?
D: Definitely this one [his, the antiderivative way]
F: Yeah!
D: I was screwing up the endpoints-- I was plugging in the angles rather than the actual points themselves. But yeah, once I figured out what I was doing wrong, yeah this was a lot easier.
F: I mean, I don't think parametrization is really going to steer you wrong, unless you make an algebra error which is easy considering how much algebra there is. But um, it's kind of the brute force "let's bring a sledge hammer to it" type of thing, versus *that's* a lot more elegant [points to D's approach].
Int: Ok great. So yeah they both have their advantages and disadvantages, certainly.

As spokesman, Frank reiterated Dan's claim and stated that he and Dan both got the same answer, also explicitly providing a warrant for Dan's approach in Argument 2b (lines 121-122). Because he and Dan obtained the same answer in two different ways, Frank expressed that they were confident about the correctness of this answer (lines 123-125). Dan added that his approach (in Argument 2b) was "the simpler way for sure," despite his mistakes and the disagreements he and Frank had (lines 126-131). Frank agreed (line 129), and elaborated that while parametrization is rarely "going to steer you wrong, […] it's kind of the brute force, 'let's bring a sledge hammer to it' type of thing, versus *that's* a lot more elegant [points to Dan's approach]" (lines 132-134). I was surprised to hear about their mutual preference for the method in Argument 2b, particularly Frank's characterization of it as "a lot more elegant," given how lengthy Argument 2b was. Argument 2b is summarized below in Figure 105.

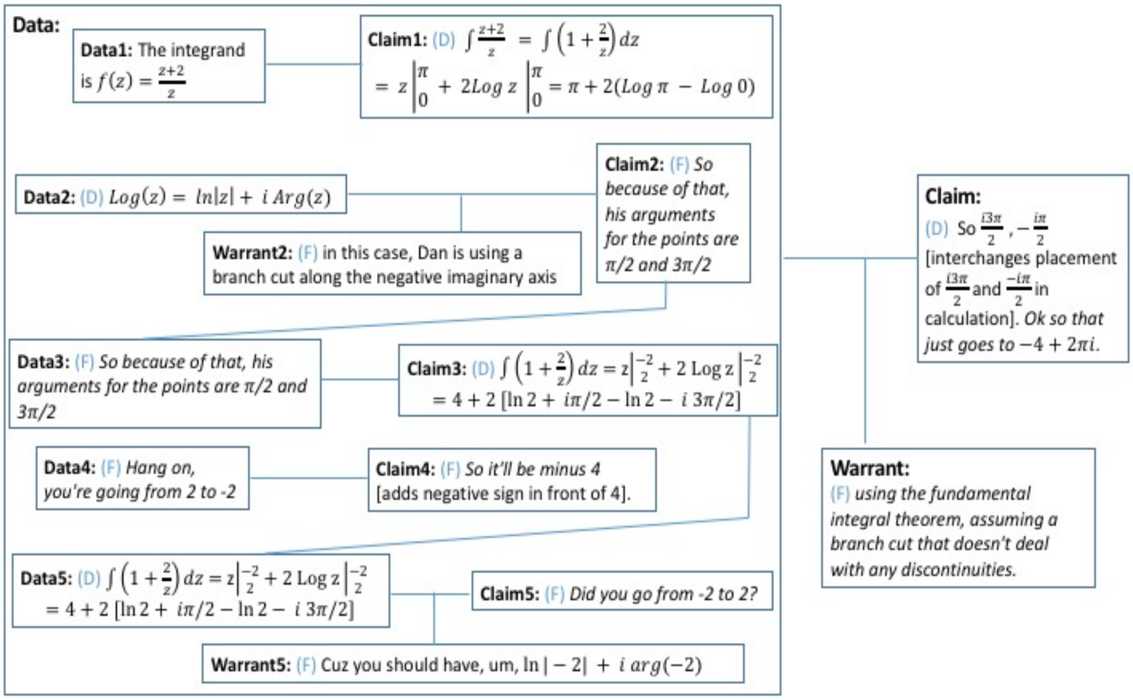


Figure 105. Toulmin diagram for Dan and Frank's Argument 2b for Task 9a.

Task 9b – Dan and Frank

In the second portion of Task 9, participants evaluated the same integral as in Task 9a, except the semicircular path C now ranged from π to 2π (lines 1-2). As spokesman, Frank commenced Argument 1 by replacing the bounds for theta in the inscriptions from the last task, which remained on the whiteboard (line 3). Given that Frank and Dan pursued two different methods in Task 9a, I clarified that they could choose either method this time (line 4). In accordance with their comments towards the end of Task 9a, Dan and Frank both quickly chose to use Dan's antiderivative method from Argument 2b in Task 9a (line 5).

Int: Ok so what if we now modify C to be a similar semicircle, but now this time theta going from π to 2π? We're going to look at the same integral.
F: [Erases range for theta and replaces with π to 2π.]
Int: And so you can choose either method.
D&F: Let's use that one [point to D's branch cut method from Task 9a]
F: So in this case we can just do the branch cut at the positive imaginary axis.
D: Yep, so pi over 2, mhm.
F: Math was easier without branch cuts [we all laugh].
D: [Changes the evaluation bounds from previous inscription to -2 to 2].
F: So I'm going to draw this out just for my own visualization. [Starts drawing the plane, the points 2, -2, and the semicircular path].
D: Can I do this? [doing the computation]
F: Yeah you can go ahead and start.
F: So there's our path. So yeah if we just our branch cut along— I'll just do the imaginary axis [draws in the branch cut along positive imaginary axis] then I'm sure we won't run into any issues.

With the new path in mind, Frank authored a claim that he and Dan "can just do the branch cut at the positive imaginary axis" (line 6). As spokesman, Dan agreed and characterized this branch cut by its complex argument of $\pi/2$ (line 7). Both portions of this claim represent embodied reasoning in that Frank presumably visualized the semicircular path to decide on a branch cut which avoided the path, and Dan's reformulation relied on describing the positive imaginary axis using a different geometric attribute. While Frank joked that math was easier without branch cuts (line 8), Dan updated the symbolic inscriptions from the last task to evaluate their antiderivative from $z = -2$ to $z = 2$ (line 9). Using embodied reasoning, Frank sketched the path C "for [his] own visualization" (see Fig. 106) while Dan started working on the symbolic computation (lines 10-13). As spokesman, Frank re-voiced his first claim that a branch cut along the positive imaginary axis would avoid "any issues" (line 15).

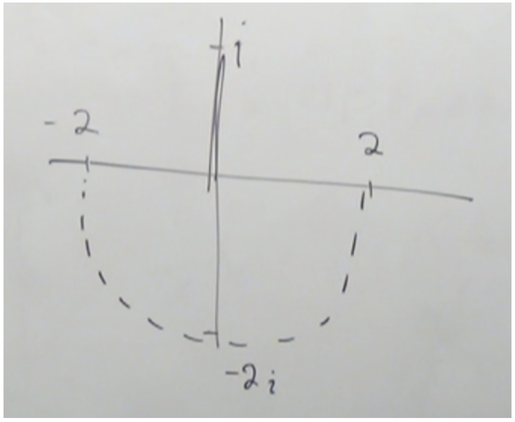


Figure 106. Frank's sketch of the semicircular path in Task 9b.

D: [Writes/ has been writing $2(-2)\ +2[\ln 2+\ i\frac{3\pi}{2} - \ln 2 -\ i\frac{\pi}{2}]$.] Does that look okay so far?
F: I just want to make sure that the arguments, um—[D writes simplified answer of $4\ +\ 2\pi i$ and boxes it] So we're going -2 to 2. So the argument of -2 is $\pi/2$ in this case?
D: Um, so, yes. Well we're going from 2 to -2, right?
F: Are we?
D: I thought we were. Oh we're not! We're going from -2 to 2. Are we going from 2 to -2 or -2 to 2? [directed at me. They both look at me].
Int: So theta's going from pi to 2 pi.
D: Ok so it's -2 to 2. Ok so 2— so yeah this is right. Hang on, because we're going 2 first.
F: No we're going from pi to 2 pi, in the normal system, which is from here to here [uses pen to rotate along path from -2 to 2].
D: Minus 2 to 2, so you evaluate it at 2 first.
F: Oh God yeah, I'm an idiot. Yeah.
D: [Laughs] So ok.
F: My bad. Yeah you had it completely right. $4 + 2\pi i$.

Dan finished altering the symbolic inscriptions from Task 9a, and claimed that $\int\left(1+\frac{2}{z}\right)dz = 2(-2)+2\left[\ln 2+\ i\frac{3\pi}{2} - \ln 2 -\ i\frac{\pi}{2}\right] = 4+2\pi i$ (line 16-18). He qualified this assertion by asking Frank if his inscriptions appeared to be accurate (line 16). As spokesman, Frank revoiced the datum from line 9 with the phrase "we're going from -2 to 2" and authored an embodied claim that "the argument of -2 is $\pi/2$ in this case" (line 18). Dan then challenged Frank's datum and the pair disagreed about the start- and endpoints of the path for this task, eventually asking me (lines 19-22). Rather than answering Dan's question directly with the starting and ending *z*-values, I repeated the corresponding values of theta that I initially provided to them (line 23). This allowed Dan and Frank to confirm that Dan's inscriptions were correct (lines 24-30). In particular, Dan provided an embodied-symbolic warrant that they evaluated the expression at $z = 2$ first because their path ran from -2 to 2 (lines 24 and 27). Argument 1 is summarized below in Figure 107.

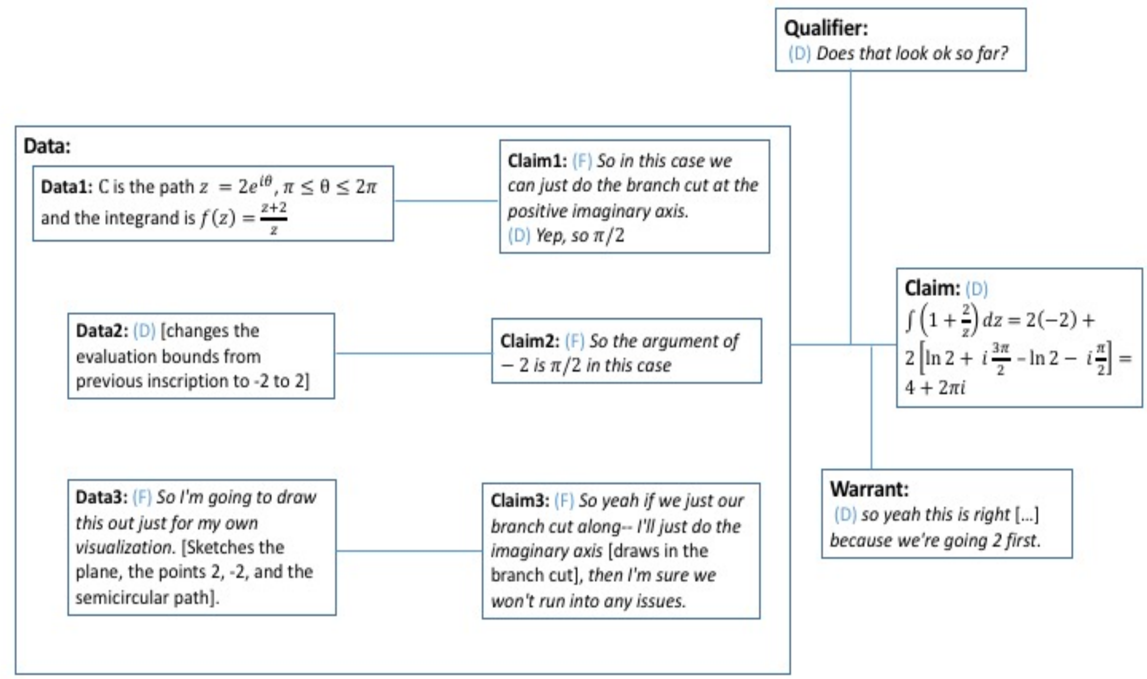


Figure 107. Toulmin diagram for Dan and Frank's Argument 1 for Task 9b.

Because Dan and Frank disagreed several times about the values of theta in Argument 1, I asked them to briefly recap how they decided on $3\pi/2$ and $\pi/2$ (line 31). This catalyzed a second argument, Argument 2. Dan began by relaying the definition of a complex logarithm as well as their choice of branch cut (lines 32-33). Using these data, he claimed that the complex argument θ must satisfy $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$ along the path C (lines 33-34). However, he expressed some uncertainty about this claim via the modal qualifier, "Is this right, Frank?" (lines 34-35).

Frank agreed with Dan's claim, and Dan reiterated as spokesman that this choice of branch cut produced a "new argument definition" (lines 36-37). But shortly after, Frank reneged on his agreement with Dan and sought to verify that Dan represented a full circle (line 38). Frank's question appeared to confuse Dan, but Dan initially went along with this change, and the pair concluded that θ should in fact range from $\pi/2$ to $5\pi/2$ (lines 39-40). However, Dan then expressed uncertainty about Frank's decision to represent a full circle, tracing a whole circle in the air with his pen (see Fig. 108). He followed this embodied reasoning with an explicit qualifier, "Is that right?" (line 41). Meanwhile, Frank changed Dan's prior symbolic inscription $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$ to indicate the new values of θ for the full circle (line 42). But Dan insisted that this was incorrect (line 43), and he and Frank disagreed once more about this (lines 44-45).

Int: Maybe just explain real briefly where the $3\pi/2$ and the $-\pi/2$ come from?
D: So um, the definition of Log(z) [writes $Log(z) = ln|z| + i\ arg(z)$], um and then we're choosing a branch cut of the log, so we're doing the log— the branch cut at $\pi/2$. So our argument is from [writes $\frac{\pi}{2} \leq$ and then changes to <] actually it should be that [<]. [finishes writing $\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$]. Is this right Frank?
F: Yeah.
D: Ok so this is our new argument definition.
F: Or actually, make [erases the 3]. Are you representing a whole circle?
D: Sorry, $\pi/2$ to—
F: $5\pi/2$?
D: Oh [Traces whole circle in the air with his pen] Ok yeah. Is that right?
F: [Writes in $5\pi/2$ instead of $3\pi/2$]
D: No.
F: [Looks surprised] Yeah.

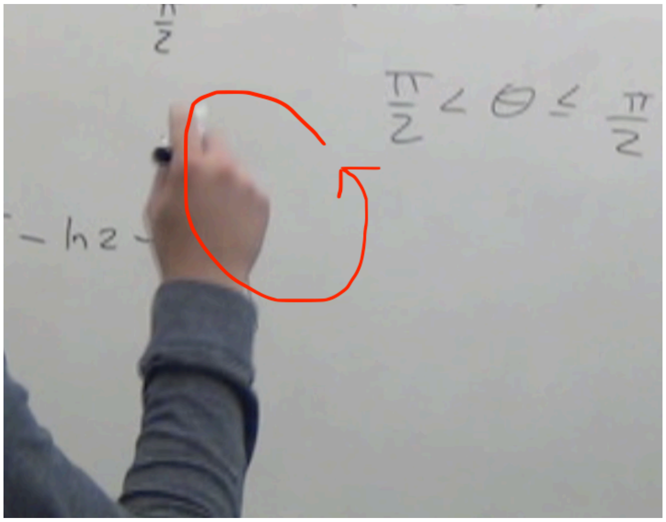


Figure 108. Dan gestures the path of a full circle in Task 9b, Argument 2.

Frank finally conceded that θ should not go from $\pi/2$ to $5\pi/2$, and supplied a symbolic warrant that such a range for θ would present a problem with the terms involving $\pi/2$, pointing to the $-\frac{i\pi}{2}$ term in their prior inscription (line 46). Accordingly, Frank changed the inscription so that it read $-\frac{3\pi}{2} < \theta \leq \frac{\pi}{2}$ (lines 46-47). But this proposed fix was short-lived, as he concluded that the $\frac{3\pi}{2}$ term would be "too big" (lines 47-48). Because I did not completely understand the reasoning behind Frank's last claim, I asked him to clarify what was too big (line 49). Instead of directly answering this question, Frank appeared to start re-explaining the contents of Argument 2 from the beginning. In particular, he relayed their choice of branch cut and the fact that this influenced the respective values of the complex argument (lines 50-51). As he continued to summarize, he claimed that the "new" arguments for -2 and 2 were $\pi/2$ and $3\pi/2$, respectively (lines 51-53). This claim reflected embodied reasoning, in that he rotated his pen on the diagram from their branch cut towards the point $z = -2$ (see Fig. 109).

D: Yes. So that is our new branch cut, right?
F: Actually hang on, no. Because that presents an issue here [points to the $\pi/2$ in their work]. So it would be pi over 2 in the upper bound [writes $-\frac{3\pi}{2} < \theta \leq \frac{\pi}{2}$]. No but then that's too big [points to $\frac{3\pi}{2}$ in work].
Int: What's too big?
F: So basically the branch cut is along the positive imaginary axis, so um, we're defining the arguments with respect to that now. [Erases his range of thetas]. So -2 is at an argument, from our new perspective, at $\pi/2$ [rotates pen from $\pi/2$ to the point -2] and 2 is at an argument, from our new perspective, of $\frac{3\pi}{2}$. So yeah, that's where those numbers are coming from. That's our $\frac{3\pi}{2}$ [points to the $\frac{3\pi}{2}$ inscription] and this is actually positive $\frac{\pi}{2}$ [points to $-i\frac{\pi}{2}$ term] but because of the integral and we're subtracting, we have to throw in a negative sign.

Ultimately, Frank answered my initial question which began Argument 2, and he clarified, "So yeah, that's where those numbers are coming from" (line 53). While pointing out the terms in their symbolic inscriptions that corresponded to $\theta = \frac{3\pi}{2}$ and $\theta = \frac{\pi}{2}$, Frank also authored a symbolic warrant that explained why they ended up with a negative sign in front of the term corresponding to $\theta = \frac{\pi}{2}$ (lines 53-55). Argument 2 is summarized in Figure 110 below.

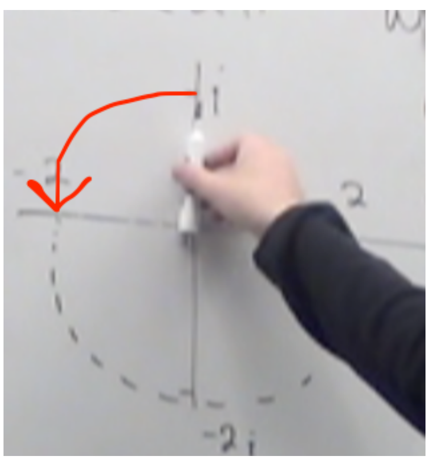


Figure 109. Frank gestures a rotation from the branch cut towards $z = -2$ in Task 9b.

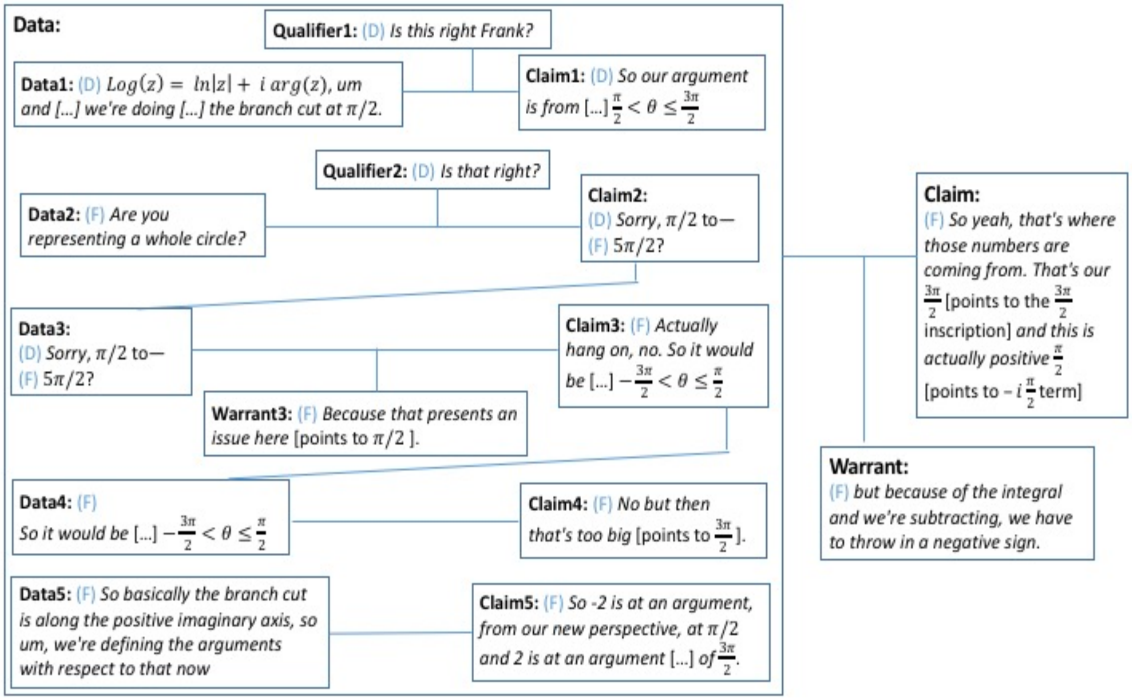


Figure 110. Toulmin diagram for Dan and Frank's Argument 2 for Task 9b.

Task 9c—Dan and Frank

Task 9 required participants to evaluate the same integral as in the last two parts, but now over a full circular path (lines 1-4). Before Dan and Frank provided their response to this task, I also reminded them of their answers from 9a and 9b (line 1). Frank rather quickly authored a symbolic claim that the answer is $2\pi i$ (line 5). Dan began to author a warrant for this claim (line 6), but I accidentally started speaking at the same time and asked Frank why he said $2\pi i$ (line 7). Accordingly, Frank authored a warrant that "you can just add up the two, uh, integrals along the semicircles" (line 8). This warrant represented embodied-symbolic reasoning because it characterized the new circular path as the concatenation of the two pervious semicircular paths and related this geometric description to the sum of two symbolic inscriptions.

Int: Ok great. So you got $4 + 2\pi i$ then. And so the other semicircle was $-4 + 2\pi i$ right?
D: Mhm.
Int: Ok then what if we make C the entire circle then? So theta going from 0 to 2π. What's our integral going to be?
F: Well, $2\pi i$ [looks at Dan with delight on his face].
D: Because you have one—
Int: So why do you say that?
F: So there's 2 ways to look at it. One-- you can just add up the two, uh, integrals along the semicircles. The first one was—
D: *Can* you? We used two different branch cuts.
F: Oh maybe not. Oh then it's not two, yeah it is— I think it is still $2\pi i$ though. You can't *add* them.
Int: You can't add what?
D: I don't know if we can just assume that we can add the two paths, because the two different branch cuts—
F: No right, I agree with you on that. But here, look at it this way.

As Frank continued to elaborate on this warrant (line 9), Dan interrupted Frank with a challenge to this warrant. In particular, he expressed doubt because he and Frank utilized two different branch cuts in Tasks 9a and 9b (line 10). Frank tentatively agreed with Dan's concern, but still maintained, as spokesman, that the integral would come out to $2\pi i$ (line 11). In doing so, he contradicted his previous warrant with the statement, "You can't *add* them" (line 11). When I asked him to clarify what he meant by this, Dan stepped in as spokesman and reiterated his concern about the two different branch cuts (lines 13-14). Frank agreed with Dan more definitively this time, and suggested a second way of looking at the task (line 15). This began a second argument pertinent to this task, Argument 2. Argument 1 is summarized in Figure 111.

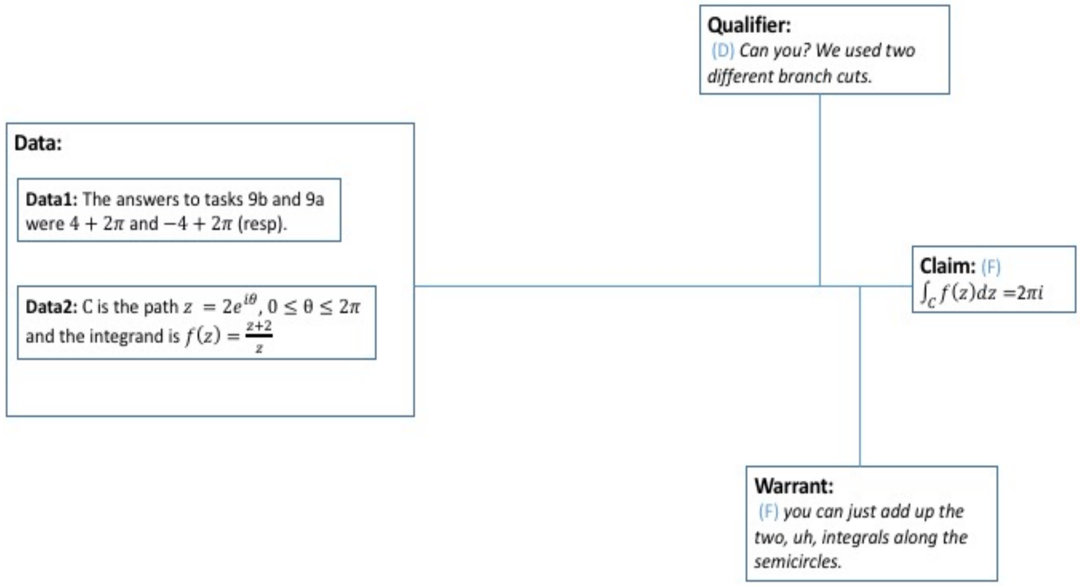


Figure 111. Toulmin diagram for Dan and Frank's Argument 1 for Task 9c.

Frank's second way to consider the problem was to rewrite $f(z)$ as $1 + \frac{2}{z}$ using symbolic reasoning, so that $\int_C \frac{z+2}{z} dz = \int_C 1\, dz + \int_C \frac{2}{z}\, dz$ (lines 16-18). After writing the corresponding symbolic inscriptions on the board, Frank authored a new claim that he thought the answer was instead $4\pi i$ (line 19). As support for this assertion, he elaborated that the first term $\int_C 1\, dz$ is the integral of an analytic function over a closed curve (lines 19-20). Using this as a datum, he authored an embodied claim that there exists a domain *D* that contains the curve (line 20). Using formal-symbolic reasoning, Frank discussed how applying the Cauchy-Goursat Theorem as a warrant yielded a value of 0 for this first term (lines 20-21).

F: This is just this integral [writes $\int_C 1\,dz$]
Int: And you can name the other things like C_1 or C_2 or something.
F: Yeah I shouldn't even need to [continues writing: $+\int_C \frac{2}{z}\,dz$ so inscription reads $\int_C 1\,dz + \int_C \frac{2}{z}\,dz$]. Umm, actually no I think it is $4\pi i$. Because this [points to $\int_C 1\,dz$] is just the integral along a closed curve, um that's [points to 1] an analytic function. There is a domain D that contains the curve. So by the Cauchy-Goursat Theorem, this is going to go to 0 [crosses off $\int_C 1\,dz$ and writes 0].
F: *This* [points to $\int_C \frac{2}{z}\,dz$] has a discontinuity at 0, um, so we have— if we use CIF, um this is the integral on some circle of radius 2 [writes $\int_{C_2^+}$] positively oriented, centered at 0, of 2 over— this is effectively just z minus 0, [writes integrand as $\frac{2}{z-0}$] dz.
F: So we know from Cauchy's Integral Formula that's $2\pi i$ times this function [points to 2] evaluated at zero, which is just 2—
D: …which is just $4\pi i$.
F: [Writes $= 4\pi i$] So $4\pi i$. So this total is just $4\pi i$ [points to his original sum of two integrals]
D: …which was the sum of our two previous—
F: Yes.
D: Ah, that's interesting.

Moving on to the second term, Frank used symbolic reasoning to author a datum that $\frac{2}{z}$ has a discontinuity at zero (line 22). As spokesman, he rewrote the path C as C_2^+ to align with the conventional notation from their class, and described the curve as positively oriented (line 23). Continuing as spokesman, Frank used formal-symbolic reasoning to rewrite $\frac{2}{z}$ as $\frac{2}{z-0}$, a symbolic form that aligned with his subsequent formal warrant, Cauchy's Integral Formula (lines 23-25). In particular, he used this warrant to author a symbolic claim that $\int_{C_2^+} \frac{2}{z-0}\,dz = 2\pi i(2)$, which Dan simplified as $4\pi i$ (lines 25-27). As relayer and then spokesman, Frank repeated the answer of $4\pi i$ and described it as a "total" of the two integrals $\int_C 1\,dz$ and $\int_C \frac{2}{z}\,dz$ (line 28). Dan added that this was also the sum of the answers from Tasks 9a and 9b, and Frank agreed (lines 29-31). A summary of Argument 2 is provided in Figure 112 below for reference.

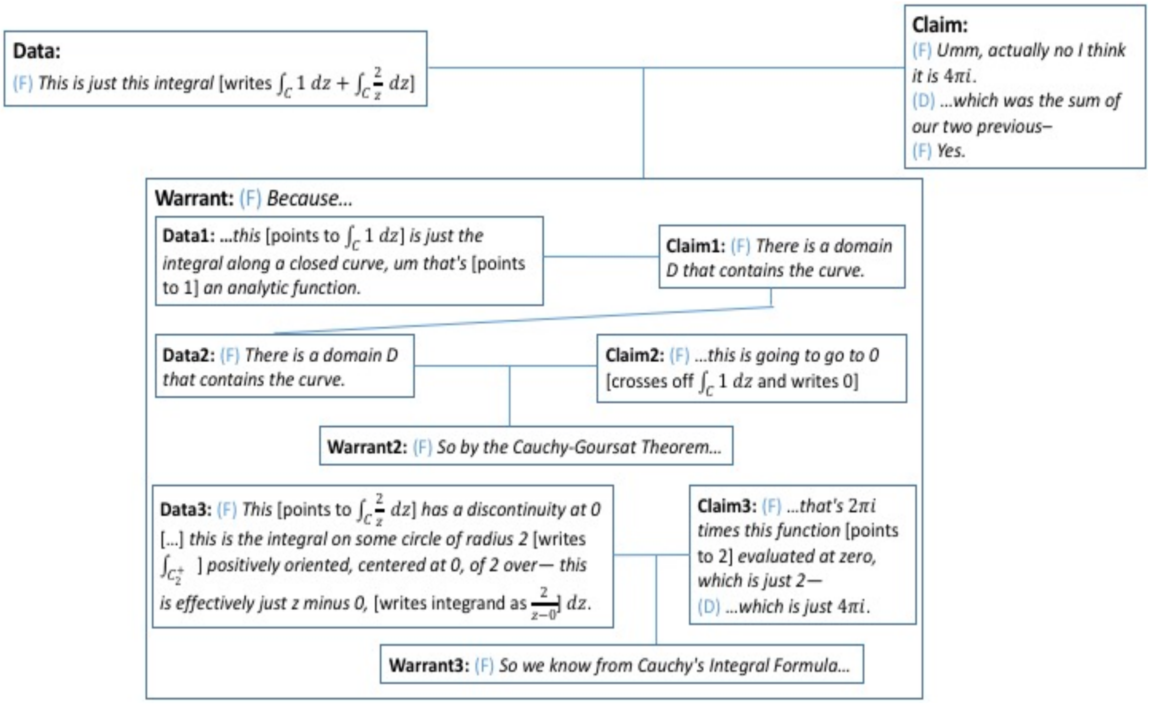


Figure 112. Toulmin diagram for Dan and Frank's Argument 2 for Task 9c.

Because Dan stated "Ah, that's interesting" (line 31), I asked him and Frank if they thought Dan's last observation was a coincidence (line 32). Frank did not think so, and Dan tried to reconcile this finding with his previous concern about using two branch cuts. Specifically, he suggested, "You could just use different branch cuts" (line 34). Frank agreed, and added that their parametrization still holds (lines 35-37). I also mentioned that Frank's parametrization method in Task 9a did not involve any branch cuts, yet he and Dan obtained the same answer.

Int: So is that a coincidence, do you think, then?
F: No [shakes head] No I don't think so.
D: You could just use different branch cuts right?
F: Yeah, cuz I mean, fundamentally—
D: It's the same
F: Yeah, you're parametrizing it just as you need to.
D: Ok yeah, I'd agree with Frank that it's always like that.
Int: Ok yeah. And also recall that you did the— uh, your methods too Frank, that didn't involve any branch cuts, too right? Where it's just straight brute force parametrization. So that might be another way to think about it too.
F: Right.
Int: Ok great. So something somewhat related— but we can erase that if you'd like.

Task 9a—Riley and Sean

As with Dan and Frank, Riley and Sean began their response to Task 9a by writing inscriptions corresponding to the setting I provided. Specifically, Riley drew the semicircular path on the board as an embodied datum (line 3; see Fig. 113). Meanwhile, Sean provided a symbolic description of the path and the integral of interest, as spokesman (lines 4-6). He qualified their argument by conjecturing, "I think we'll have to straight parametrize this; I don't see there's another way" (line 6). Sean continued by authoring a symbolic claim that they could write the integrand as $1 + \frac{2}{z}$, and qualified this claim with "I mean, maybe" (lines 8-9).

Int: Ok so now we're going to look at a path C, um, that's the semicircle $z = 2e^{i\theta}$, where theta is going from 0 to π.
R: [Draws the path on board as I read off the task]
S: [Writes $C: z = 2e^{i\theta}, 0 \leq \theta \leq \pi$].
Int: Ok yeah cool. And um, how would you find the integral of the function $\frac{z+2}{z}\,dz$ for this curve C?
S: [Writes $\int_C \frac{z+2}{z}\,dz$] Well I think we'll have to straight parametrize this, I don't see there's another way.
R: Um—
S: I mean, maybe it makes sense to split up this integral of 1 dz plus integral of 2/z dz. [writes $\int_C 1\,dz + \int_C \frac{2}{z}\,dz$]
R: Right, okay.
S: And then we just parametrize it and you end up getting, let's see—
R: Well you don't have to necessarily parametrize it right? Can't we just choose a branch for Log that makes it work? Cuz it's not a full circle; it's a semicircle.
S: You could.
R: I mean, it might not save that much work, but like—
S: Yeah.

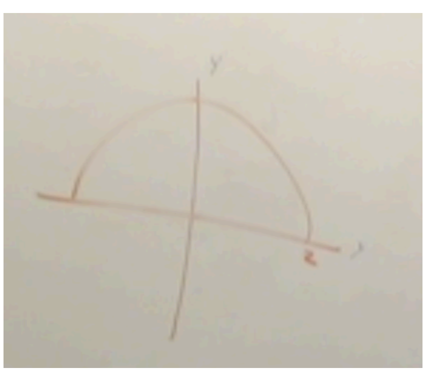


Figure 113. Riley's initial diagram for the semicircular path in Argument 1, Task 9a.

Riley agreed (line 10), and Sean began to author another claim regarding what parametrizing would yield, however Riley interrupted him (line 11). She proceeded to author a rebuttal proposing an alternate approach to the task not involving parametrization (lines 12-13). In particular, she employed formal-embodied reasoning to consider choosing a branch of the Log function that "makes it work," and clarified that they could do so because the path is a semicircle rather than a full circle. Sean agreed that this was an alternate possibility, and Riley acknowledged that her approach would not necessarily save computational effort (lines 14-16). This first argument, Argument 1, is summarized in Figure 114.

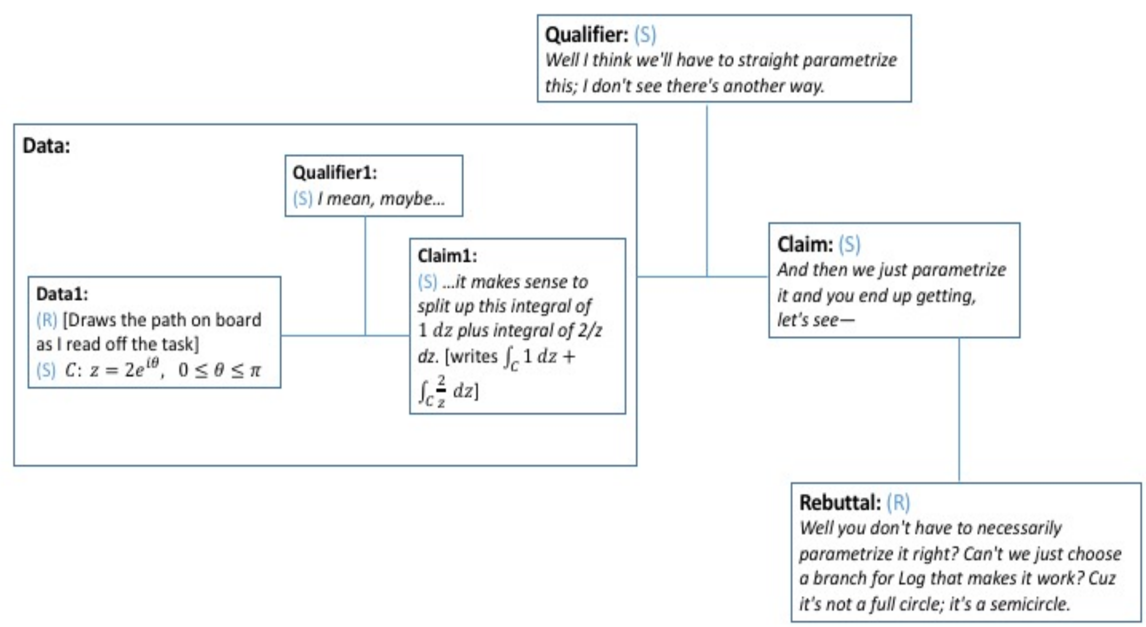


Figure 114. Toulmin diagram for Riley and Sean's Argument 1, Task 9a.

Commencing a second brief argument, Riley continued to describe her alternate method avoiding parametrization. She began by authoring a formal-embodied warrant dictating a choice of branch cut that avoids the semicircular path (lines 17-18). While articulating this warrant, Riley produced an embodied tracing gesture back and forth along their path to highlight her phrase “anywhere on this semicircle” (line 18; see Fig. 115). She then clarified that she considered a branch cut at an angle of $-\pi/4$ to satisfy the aforementioned requirement (lines 18-19; see Fig. 116).

R: But like, if you choose a branch, you just have to choose one so that you have your ray where it's not analytic, doesn't fall anywhere on this semicircle [traces back and forth along path with hand]. So when you choose Log, branch at $-\pi/4$, [draws branch cut at $-\pi/4$] um—

S: This is just [underlines $\int_C 1\, dz$ and writes 2π underneath], yeah just 2π times 2, over 2 [writes $\frac{2\pi(2)}{2}$].

R: Well it's not over 2π, it's just over π right? Cuz you're not integrating over the full semicircle.

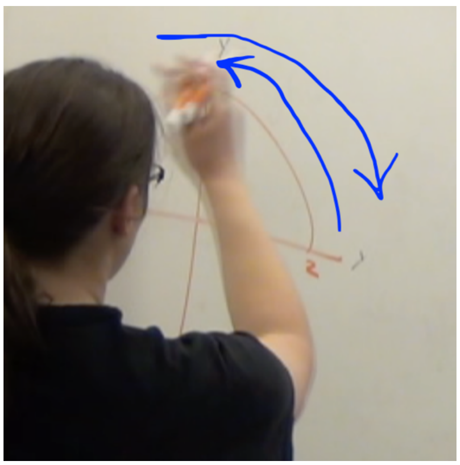


Figure 115. Riley’s gesture representing “anywhere on this semicircle” in Argument 2.

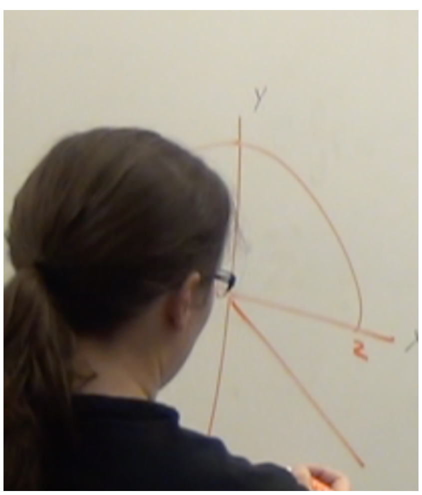


Figure 116. Riley’s chosen branch cut in Argument 2, Task 9a.

Subsequently, Sean interjected and finished describing his solution that incorporated parametrization. Using symbolic reasoning, he underlined the first integral

$\int_C 1\, dz$ and authored a claim that this first result is 2π. As spokesman, he clarified that this result came from evaluating the expression $\frac{2\pi(2)}{2}$ (line 20). Riley once again challenged Sean's conclusion, implementing embodied-symbolic reasoning to claim that the "2π" should instead read "π" because "you're not integrating over the full circle" (line 21). This challenge catalyzed a third argument, as I discuss below. A summary of Argument 2 is depicted in Figure 117.

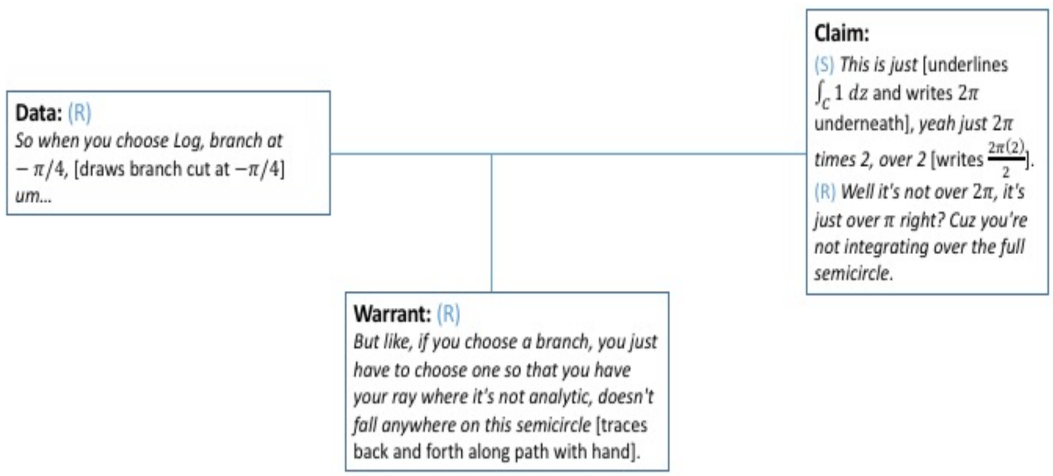


Figure 117. Toulmin diagram for Riley and Sean's Argument 2, Task 9a.

S: Well it's 2π radius, then half that [writes $\frac{2\pi r}{2}$], so it would be 2π. For circumference. Take half of circumference. So a circle of radius 2, you put a 2 in there. So you have $2\pi r$, so $2\pi r$, the full circumference is 4π. Then you have to divide by 2 to get 2π. Cuz we're just integrating the arc length of this point [points to 1 dz].

Following Riley's challenge, Sean clarified his previous symbolic inscriptions in a third argument. Specifically, as spokesman, he explained that he used the symbolic formula $2\pi r$ as a datum (line 22). He then authored a symbolic warrant elucidating that he took "half that" (line 22). As backing for this warrant's correctness, Sean identified his $2\pi r$ formula as "for circumference" (line 22), and described how "the full circumference is 4π" (lines 23-24). According to Sean, dividing by 2 then yielded the 2π result, and summarized that he essentially calculated arc length of their curve (lines 24-25). Thus, this backing incorporated embodied-symbolic reasoning. Argument 3 is summarized in Figure 118.

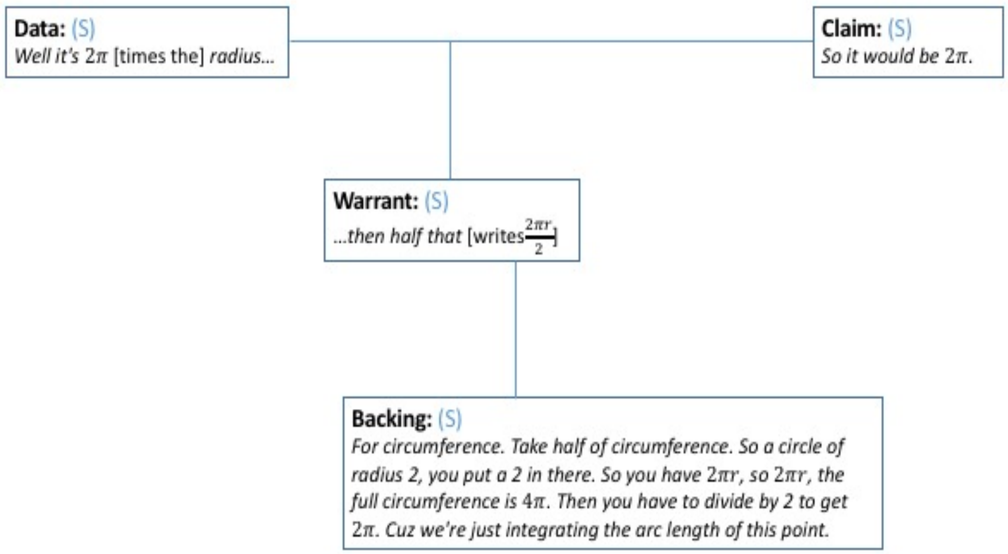


Figure 118. Toulmin diagram for Riley and Sean's Argument 3, Task 9a.

R: Yeah but you're not— cuz um, wait. Cuz, ok so to do this, you just do [draws arrow from $\int_C 1\, dz$] um, $f(a) - f(b)$ or whatever, right? Um, and but it matters where you're integrating to and from – you're just integrating from 0 to π [writes in limits of 0 and π to the first integral].

Once again, Riley challenged Sean's argument, beginning a fourth argument pertinent to Task 9a. She began Argument 4 with a formal-symbolic warrant as she drew an arrow from the symbolic inscription $\int_C 1\, dz$ and explained that "you just do […] $f(a) - f(b)$" (lines 26-27). She qualified this warrant with the phrase, "or whatever, right?" (line 27). Notice that she transposed the order of subtraction in the statement of the Fundamental Theorem, but perhaps recognized this in her qualifier. In any case, she claimed that "it matters where you're integrating to and from" (line 27). This claim appeared to involve primarily symbolic reasoning, as she indicated the limits of integration as 0 to π as and added these to the integral inscription as symbolic data (line 28).

By this moment in Riley and Sean's response to Task 9a, it became clear to me that Riley and Sean were conflating aspects of their two different approaches to the task, and this was what was causing their disagreements. In particular, Sean initially chose to parametrize their path, while Riley wanted to take an antiderivative and employ the Fundamental Theorem. However, in Argument 4, Riley attempted to apply limits of integration for θ to evaluate $\int_C 1\,dz$, yet the integrand was not an expression of θ. This conflation of approaches continued in Argument 5, as detailed below. Argument 4 is summarized in Figure 119.

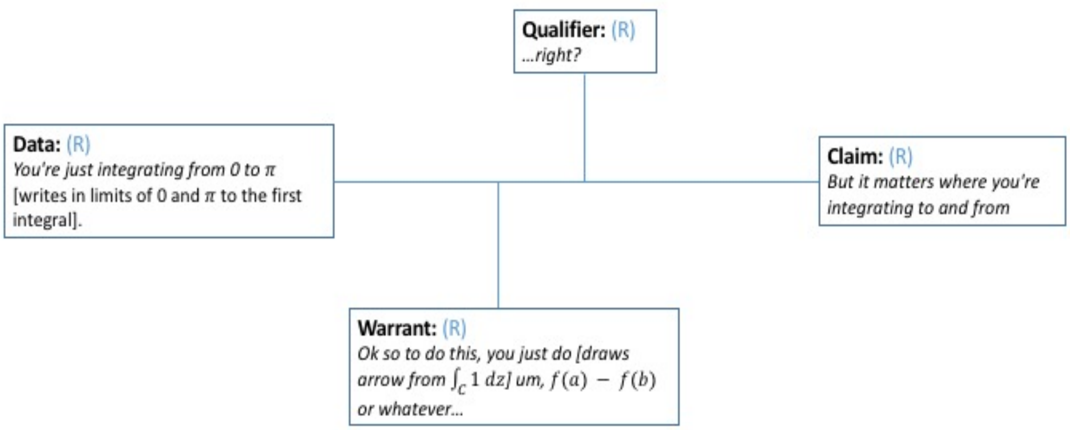


Figure 119. Toulmin diagram for Riley and Sean's Argument 4, Task 9a.

S: But when you do that, though [draws his own arrow from $\int_C 1\,dz$], you have this here, is z [points to $z = 2e^{i\theta}$]
R: Yeah but the curve—
S: ...so dz is $2i$ [writes $\int_0^\pi i2e^{i\theta}\,d\theta$]. So it should be 2π— wait, sorry [erases most recent inscriptions]. So same thing, same curve. It's just this [points to $z = 2e^{i\theta}$] from there [points to range of theta]. So I just say if integral of 1 dz is $2ie^{i\theta}\,d\theta$, 0 to π [writes $\int_0^\pi 2ie^{i\theta}\,d\theta$]. So $\frac{2i}{i}$ [times], $e^{i\theta}$, from π to 0 [writes from 0 to π but misspeaks] which is 2 [gets interrupted]—
R: Wait no this isn't right. It should be— It shouldn't be π there should it? It should be negative—two and negative 2, right [for limits of integration]?
S: Well we're doing this parametrization [points to $z = 2e^{i\theta}$].

Sean responded to Riley's prior challenge by pointing out that her limits of integration corresponded with the symbolic characterization of z as $2e^{i\theta}$ (lines 29-30). In doing so, he drew his own arrow from the $\int_C 1\, dz$ inscription in a manner similar to Riley, and symbolically claimed that the integral should be expressed as $\int_0^{\pi} i2e^{i\theta}\, d\theta$ (line 32). As a symbolic warrant, he began to elucidate what dz became in this new characterization, but did not finish verbalizing this statement (line 32). Sean authored a claim that this symbolic representation of the integral yielded a value of 2π, but retracted this claim with the qualifier, "wait, sorry," and erased his recent inscriptions (line 32).

Sean relayed the prior data, "so same thing, same curve," and that $z = 2e^{i\theta}$ and theta ranges from 0 to π (line 33). As spokesman, Sean set up an integral equivalent to his prior one as a symbolic warrant, merely transposing the 2 and i compared to the last version (lines 33-35). However, once again, Riley challenged Sean. This time, she called his warrant into question via the qualifier "it shouldn't be π there should it? […] right?" (lines 36-37). Due to the aforementioned conflation of their two methods, Riley claimed that instead the limits of integration should be 2 and -2 (lines 36-37). However, note that these two values correspond to the starting and ending values of z, rather than θ. As spokesman, Sean clarified that he was using a parametrization (line 38). He complemented this symbolic rebuttal by once again pointing to the inscription $z = 2e^{i\theta}$. A summary of Argument 5 is depicted in Figure 120.

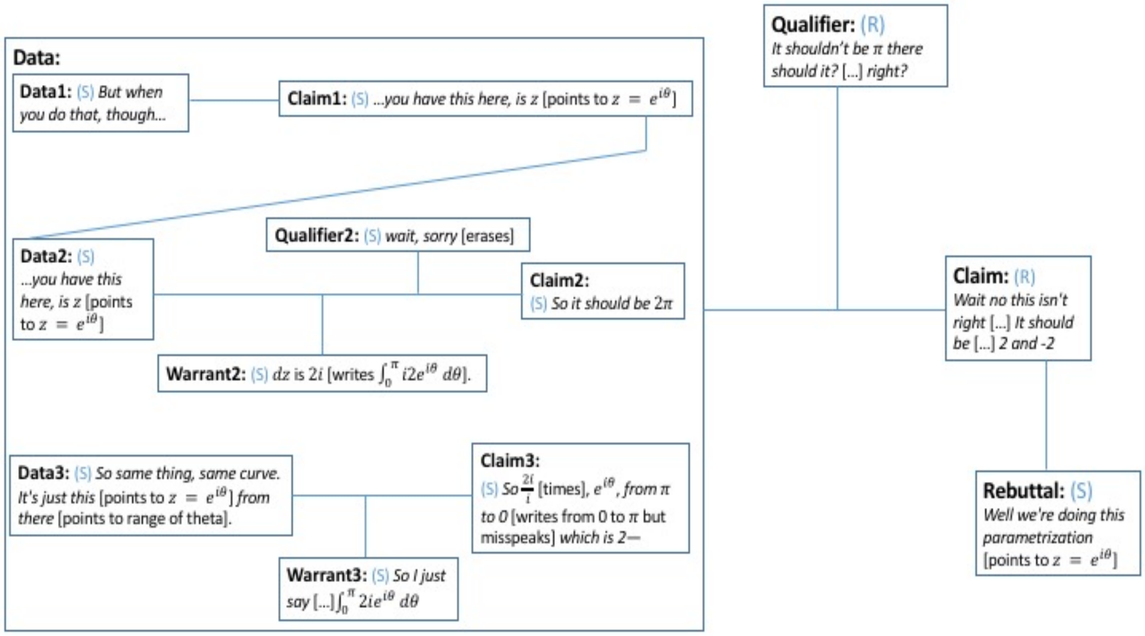


Figure 120. Toulmin diagram for Riley and Sean's Argument 5, Task 9a.

R: Okay, you're parametrizing it—
S: So it should be the arc length. So you get 2 times—[writes $2[cos\,\pi + i\,sin\,\pi - cos(0) - i\,sin(0)]$], then erases this last step.
S: Ok i, so [cosine of] π is negative 1 [writes 2(-1-1) =]
R: It should come to 4, right?
S: [Writes $= -4$ and looks at Riley with look of surprise].
R: Well, so—
S: Yeah?

Riley began Argument 6 by acknowledging Sean's parametrization method (line 39). With this datum in mind, Sean began to author a warrant, relaying that the integral "should be the arc length" (line 40). Implementing formal-symbolic reasoning, he then began to evaluate $e^{i\theta}$ using Euler's formula, but erased his corresponding inscriptions when finished (lines 40-41). He attempted this step once more, clarifying that $\cos\pi = -1$ (line 42). Once she saw Sean write $2(-1-1)$, Riley challenged Sean by authoring a claim that the answer should be 4, but once again qualified this challenge with the word "right?" (line 43). However, Sean concluded his calculation with an answer of -4 and glanced at Riley with a puzzled facial expression (line 44), and qualified his answer with the word “Yeah?” (line 46). Argument 6 is depicted below in Figure 121.

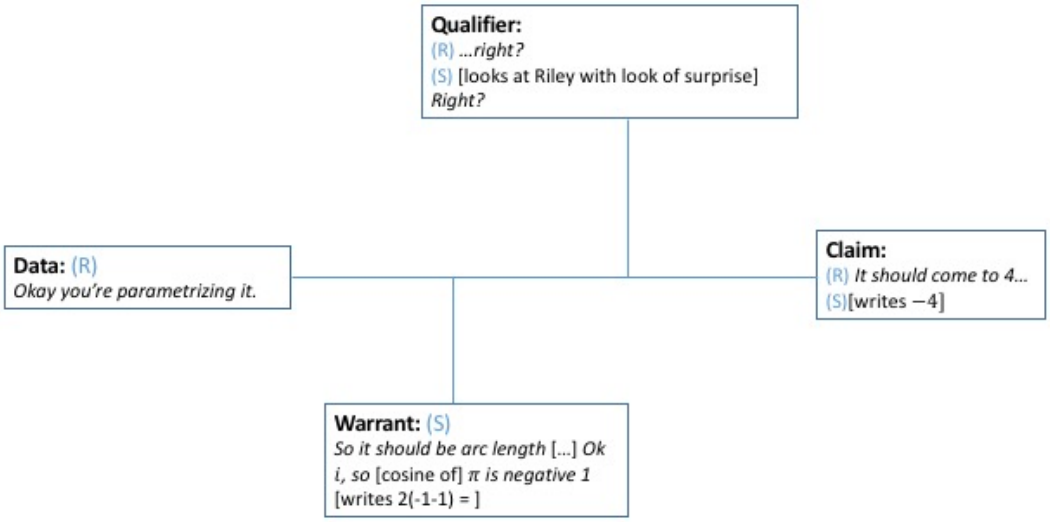


Figure 121. Toulmin diagram for Riley and Sean’s Argument 6, Task 9a.

R: Umm, because you take here its end paths between these two points right? [Points to orange diagram] So here we're going from 2 to -2 [writes $2 \rightarrow -2$].
S: Mhm.
R: Umm, because this is still analytic right?
S: It's what?
R: Analytic.
S: Mhm.
R: So we can say just z is the antiderivative from our point a to our point b. [writes $z \,\Big|_{2}^{-2}$]
S: Yes.
R: So we just we just do $-2 - 2 = -4$, right? No crying involved. Uh, because it doesn't matter what the path is.
S: Ok. It seemed arc length for some reason.
R: Uhh, yeah I don't really know why you're getting that. Sorry.
Int: Well so our path here is this semicircle here. You're saying it wouldn't matter which path we're taking to get from—?
S: For this first integral.
Int: For the first part?
R: Well yeah but it doesn't matter because it's analytic everywhere.

Riley began Argument 7 by elaborating on her prior disagreement in Argument 6, this time using her preferred antiderivative method. Employing embodied-symbolic reasoning, she first pointed to the endpoints of their orange semicircular path, and wrote the corresponding inscription “$2 \rightarrow -2$” to indicate these points symbolically (lines 47-

48). Riley then authored a formal-symbolic warrant that the integrand, 1, is analytic (lines 50-53), and Sean agreed (line 54). Because this integrand is analytic, Riley symbolically claimed that the antiderivative for 1 is z, and that they should evaluate this antiderivative at the two endpoints (lines 54-56). In particular, she relayed her prior symbolic claim from Argument 6 that the value of the integral should be -4 (line 56), once again qualifying this assertion with the word "right?"

Riley bolstered this claim by authoring the formal-embodied warrant, "it doesn't matter what the path is" (lines 56-57). Sean conceded but also mentioned, "It seemed [like] arc length for some reason," seemingly unconvinced that their two methods would obtain the same answer. Riley qualified this discrepancy by telling Sean, "I don't really know why you're getting that. Sorry" (line 59). Because Riley and Sean were unable to reconcile their two approaches, I asked a follow-up question about Riley's statement regarding path choice. She attributed this path-independence to her previously established analyticity, invoking this formal-embodied reasoning as backing for her warrant's validity (line 64). Argument 7 is summarized in Figure 122.

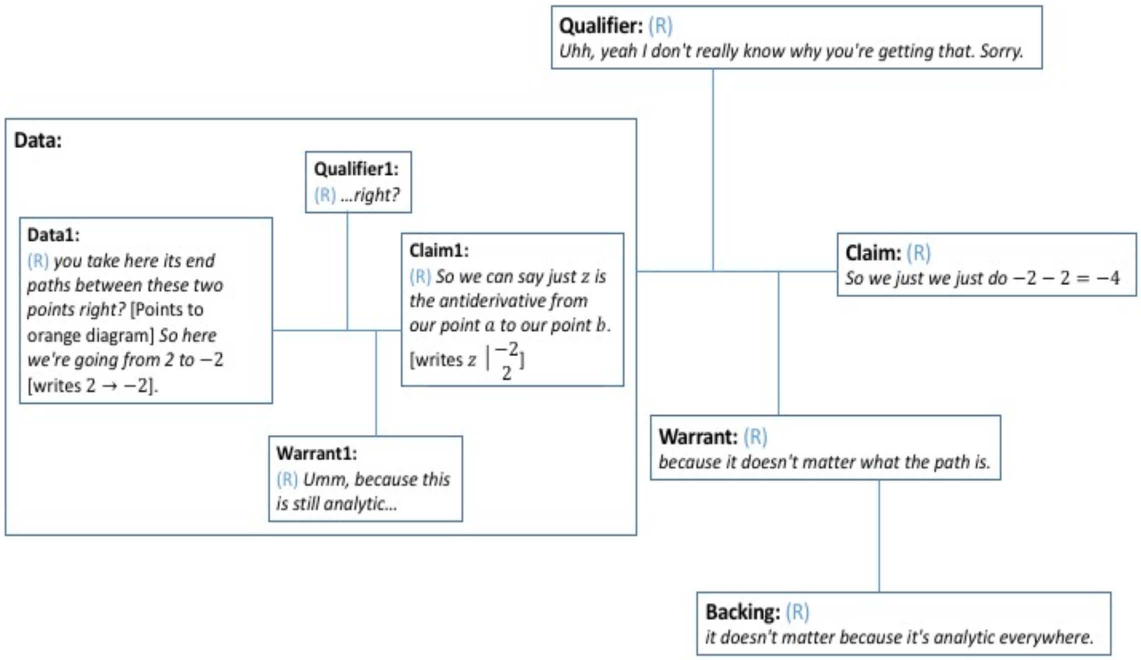


Figure 122. Toulmin diagram for Riley and Sean's Argument 7, Task 9a.

R: Uh, for the second integral, you can parameterize it or you can pick the branch of the log so that it works. Um, so, so if we want to say— Ok so this second integral is $2/z$, right? [Writes $\int_C \frac{2}{z}$] Um, dz. So then, let's see— our z is going to be equal to $2e^{i\theta}$ [writes this], theta is greater than 0 and is less than or equal to π, right? [Writes $0 \leq \theta \leq \pi$]

S: Mhm.

R: Uhh, $f(z)$ is $2/z$, so we just do— um, so we say, "um remember that formula we had earlier?" We just say it's going to be 2, over, uh $2e^{i\theta}$, uh, times $2ie^{i\theta}\, d\theta$ [writes this]. Uh and then you go from 0 to π, right? So these [$e^{i\theta}$] cancel, these [2's] cancel, and so it becomes $2i$, which makes sense because you could cancel it this way right? And you'd just retain that 2, um, and so then that becomes $2i$ [integral of] 0 to π of theta, is equal to $2\pi i$.

R: And sort of conceptually, that makes sense because— it seems like it should make sense that it'd be, the semicircle should be half of $2\pi i$ [traces semicircle in air with finger], which it would be what it was if it was the full circle [traces full circle counterclockwise], but then we double it because it's $2/z$ [writes 2/z on board]. Um, so it sort of makes sense that it comes to $2\pi i$.

R: So then we add those together, and so it's $2\pi i$ minus 4. [Sean writes $2\pi i - 4$ next to the original integral inscription/setup].

Having satisfactorily resolved their discrepancy about the first integral, Riley turned to the second integral in their sum, $\int_C \frac{2}{z}\, dz$, beginning Argument 8. She acknowledged that they could evaluate this integral using either of their previous methods, and relayed the symbolic data that $z = 2e^{i\theta}$ and $0 \leq \theta \leq \pi$ (lines 65-68). Sean agreed (line 69), and Riley appealed to their prior parametrized work as a symbolic warrant. She used this warrant to author a symbolic claim, rewriting the integral as $\int_0^{\pi} \frac{2}{2e^{i\theta}} 2ie^{i\theta} d\theta$ (lines 70-72).

Next, Riley relayed the limits of integration, qualifying this data with the word "right?" (lines 71-72). She authored a symbolic claim that several of the factors "cancel" so that the integral simplifies to $2i \int_0^{\pi} \theta\, d\theta = 2\pi i$ (lines 72-74). Riley qualified this assertion with the phrase, "you could cancel it in this way, right?" (line 73). She authored an embodied-symbolic warrant confirming the reasonableness of their result, in that the integral of $\frac{1}{z}$ over a full circular path should be $2\pi i$. Therefore, according to Riley, integrating over half a circle should yield half of $2\pi i$, but because the integrand is $\frac{2}{z}$, the extra factor of 2 should double that result so that they obtain $2\pi i$ (lines 75-78). While she verbally articulated this warrant, she provided two tracing gestures corresponding to the semicircular and circular paths, respectively (see Fig. 123).

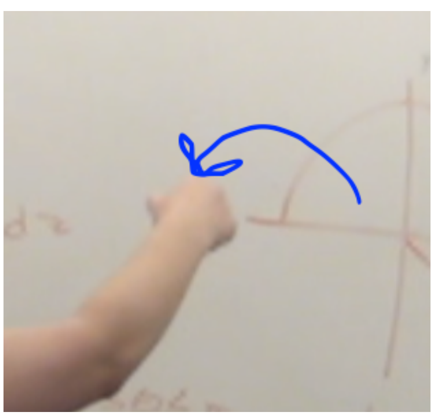
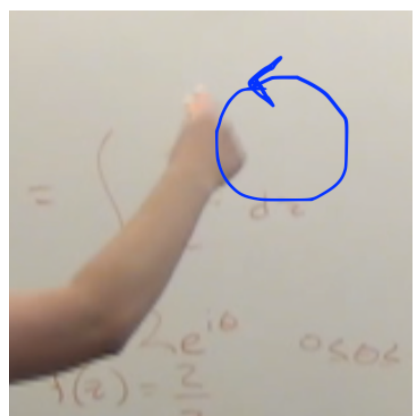



Figure 123. Riley's semicircular (left) and circular (right) gesture in Argument 8.

Returning to the original task, Riley authored a symbolic warrant that they needed to add the values of the two integrals together to obtain the overall result (line 79). She symbolically concluded that this yields a result of $2\pi i - 4$ (line 79), and Sean relayed this conclusion in the form of a symbolic inscription on the board (lines 79-80). Argument 8 is depicted in Figure 124.

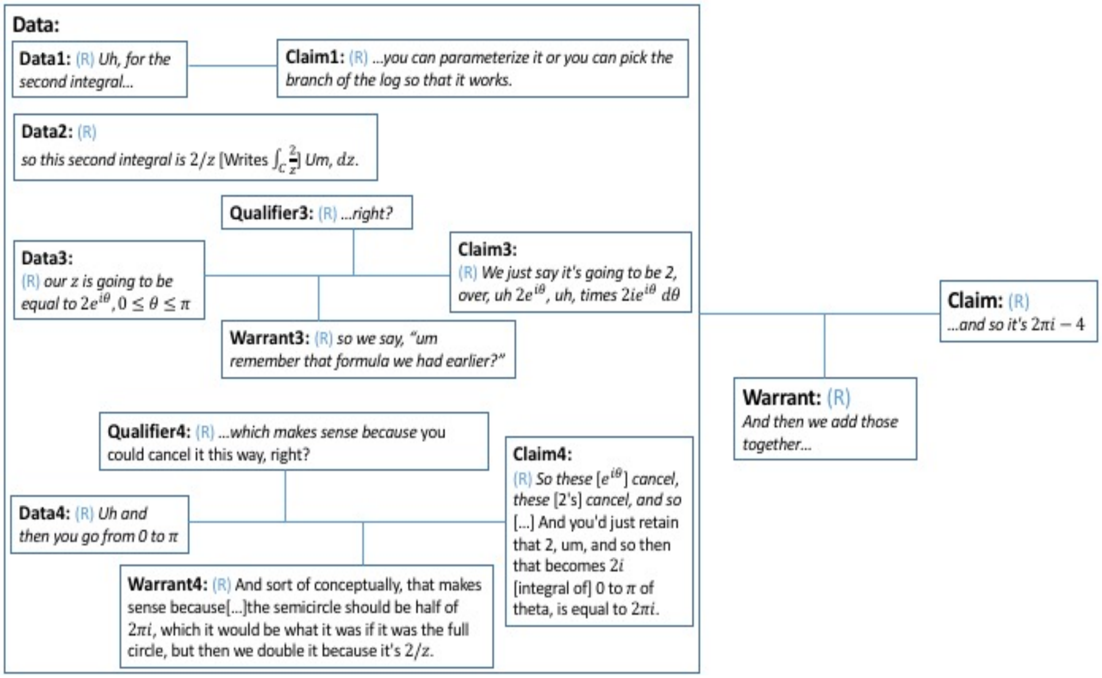


Figure 124. Toulmin diagram for Riley and Sean's Argument 8, Task 9a.

Tasks 9b and 9c – Riley and Sean

In responding to Task 9b, Riley and Sean simultaneously answered Task 9c before I asked them about it. Accordingly, I present Riley and Sean's arguments pertaining to Tasks 9b and 9c as one section. Sean began his and Riley's first argument as spokesman, as he sketched the new semicircular path C_2 (see Fig. 125) and wrote a corresponding symbolic description of this path (lines 1-4). Riley authored a symbolic claim that the value of the integral should be $4\pi i$, and qualified her claim with the word, "right?" (line 5). Because she did not immediately elaborate, I asked her about this assertion (line 6). Riley responded by authoring an embodied-symbolic datum that she and Sean already evaluated the integral of $1/z$ over a full circle in Task 6; she relayed their prior answer of $2\pi i$ (lines 7-8). She once again qualified this statement with the word, “right?” (line 8). While speaking the words “full curve” (line 7), Riley also provided an embodied gesture as she traced the full circle with her whiteboard marker (see Fig. 126).

Int: Ok cool. So what if we now have, maybe call it C_2 or something, be the semicircle, now kind of going the rest of the way. So from π to 2π. So same function, $z = 2\,e^{i\theta}$, um but now theta is going from π to 2π. And so let's look at the same integral over C_2 now.
S: [Draws new path on previous diagram and labels C_2; writes $C_2: 2e^{i\theta}, \pi \leq \theta \leq 2\pi$]
R: Ok, so I think that eventually this is going to come to $4\pi i$. Right?
Int: What makes you say that?
R: Uh because, ok-- So basically, we already did the integral for the full curve for $1/z$ right [traces circle with marker]? Uh let's call this $C_1 + C_2$ right? It's like this full curve. And it comes to $2\pi i$, right? [writes $\int_{C_1+C_2} \frac{1}{z}\,dz = 2\pi i$]. And so the curve for twice that should just be $1/z$, $C_1 + C_2$, is $4\pi i$ [writes $\int_{C_1+C_2} \frac{2}{z}\,dz = 4\pi i$]. So that's cool. And then basically what we're doing is we're adding that to the integral for $1\,dz$, right, over the full curve [writes $+ \int_{C_1+C_2} 1\,dz$].
R: And this [points to the integral of 1] is, uh— it's going to come to equal 0 [circular tracing gesture] because it's a closed curve and because, uh, it's analytic— which was something, what was it, the Cauchy—
S: It was Cauchy's Theorem.
R: Right, um, his name is in every single theorem.
All: [Laughs].

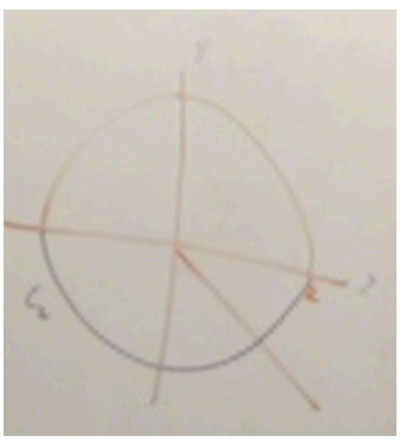


Figure 125. Sean’s drawn path C_2 in Argument 1, Task 9b/c.

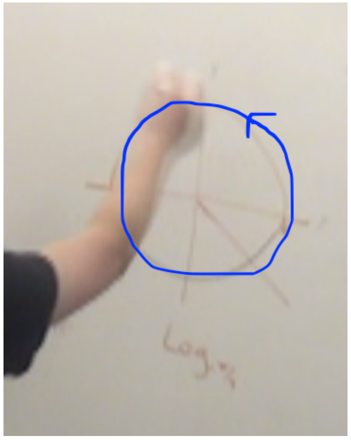


Figure 126. Riley's tracing gesture "for the full curve" in Argument 1, Task 9b/c.

Riley symbolically expressed this full circular path as the concatenation $C_1 + C_2$ of the two semicircular paths C_1 and C_2, and authored a claim that twice the aforementioned integral is $4\pi i$ (lines 9-10). Next, she authored a symbolic datum that they were adding this integral of $2/z$ to the integral of 1 over the path $C_1 + C_2$ (lines 10-11). Riley authored an embodied-symbolic claim that the integral of 1 over this full circular path vanishes (line 12). The embodied aspect of this claim consisted of a tracing gesture Riley produced while saying "it's going to come to equal 0" (line 12; see Fig. 127). She supported her assertion with a formal-embodied warrant, Cauchy's Theorem (lines 13-14). However, she could not fully remember the name of the theorem, as evidenced by her qualifier, "what was it?" (line 13), and Sean stepped in as spokesman to name the warrant (line 15).

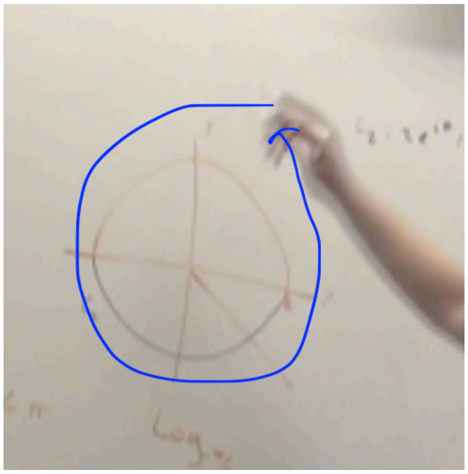


Figure 127. Riley's tracing gesture for "it's going to come to equal 0" in Argument 1.

R: But yeah, it just— If this is a closed curve [circular tracing gesture], it doesn't really matter how it's oriented; It might have to be simple—and it's an analytic function that you're integrating [points to integrand of 1], then it's always going to come to equal 0. So, um— so I mean we can work out, um, the second curve and add them together, but it should come to, just $4\pi i$.

S: Like all we're doing is just changing the bounds from like 0 to π, to now π to 2π, so like here [points to Task 9a inscriptions for the integral of 2/z] you get positive 4, and there [points to Task 9a inscriptions for integral of 1] you get just another $2\pi i$, so you add them and altogether you get $4\pi i$. So that's with the nitty gritty details, but this [points to Riley's inscriptions] reasoning is more elegant.

Int: So you're saying the integral over $C_1 + C_2$ is $4\pi i$?

S: [Nods head in agreement]

R: Yeah.

Subsequently, as spokesman, Riley supplied formal-embodied-symbolic backing for their warrant's validity. In particular, she listed the embodied aspects of the curve that allowed her to apply the formal theorem that yielded the symbolic answer of 0 (lines 18-20). While articulating this backing, she gestured a closed path beginning and ending at the point $z = 2$ (line 18; see Fig. 128). Next, Riley authored a symbolic warrant that once they had the value of the integral $\int_{C_1+C_2} 1\, dz$, they could add that to the integral $\int_{C_1+C_2} \frac{2}{z}\, dz$ to obtain the value of $\int_{C_1+C_2} \frac{2+z}{z}\, dz$ (lines 20-21). She also relayed her previous conjecture that this last integral would be $4\pi i$ (line 21). Accordingly, Riley answered what I intended to be Task 9c in response to Task 9b. While communicating her warrant, Riley traced along the path C_2 as she discussed "work[ing] out the second curve" (lines 20-21; see Fig. 129).

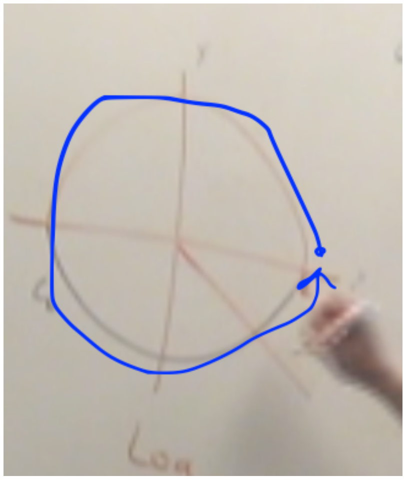


Figure 128. Riley's tracing gesture for "if this is a closed curve" in Argument 1.

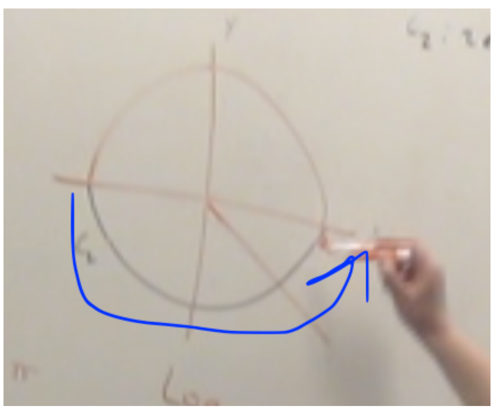


Figure 129. Riley's tracing gesture for "work out the second curve" in Argument 1.

Afterwards, Sean stepped in and authored a warrant for Task 9b rather than 9c. Using symbolic reasoning, he discussed changing the limits of integration to reflect the new range of theta for the curve C_2 (line 22). He then pointed to previous symbolic inscriptions on the board for calculating $\int_{C_1} 1\, dz$ and $\int_{C_1} \frac{2}{z}\, dz$ in Task 9a as he identified their counterparts using the curve C_2. Specifically, he supplied values of 4 and $2\pi i$ for the integrals $\int_{C_2} \frac{2}{z}\, dz$ and $\int_{C_2} 1\, dz$, respectively, and relayed Riley's statement that they could add these results together (lines 22-24). However, he acted as ghostee when describing this sum, for he referred to adding $(2\pi i + 4)$ and $(2\pi i - 4)$ as the values of $\int_{C_2} \frac{z+2}{z}\, dz$ and $\int_{C_1} \frac{z+2}{z}\, dz$, respectively. Ultimately, however, both Riley and Sean obtained the same result that $\int_{C_1+C_2} \frac{z+2}{z}\, dz = 4\pi i$, and Sean commented that he viewed Riley's solution as "more elegant" (lines 25-28). Argument 1 is summarized in Figure 130.

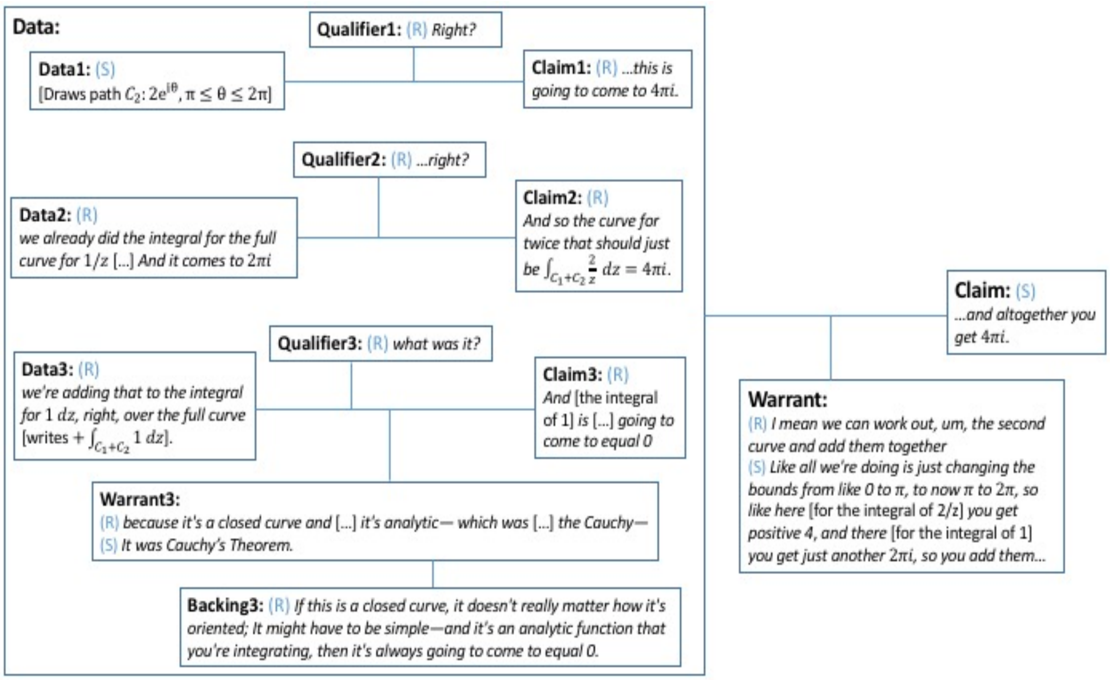


Figure 130. Toulmin diagram for Riley and Sean's Argument 1, Task 9b/c.

Int: I guess that's sort of jumping a little bit ahead, because that's what I was going to ask next. But just for the C_2 part— that bottom semicircle, what's that integral come out to be?
R: Um, that should also be $2\pi i$, right?
S: Plus 4.
R: Ok yeah, sorry. Yeah I was just thinking the $2/z$ [points to her inscription for the integral of $2/z$], but yeah. But plus 4.
S: So we know from the Cauchy Theorem, Cauchy-Goursat [circular tracing gesture], that this will just be—
R: [As Sean is finishing his sentence] I guess at this point we're just working backwards, but you can do the same exact process as far as— now you're going from—
S: It's like a dramatic argument— we know we know the ending. [Writes C_1: $2\pi i - 4$ *and* C_2: $2\pi i + 4$, then a horizontal line, and $4\pi i$ underneath to represent a total].

Because Riley and Sean ended up discussing Task 9c prior to answering 9b, I acknowledged this change in ordering and redirected them to the integral along just C_2 (lines 29-30). This initiated a second argument, which Riley began by authoring a claim that this integral "should also be $2\pi i$" (line 31). Once again, she qualified this claim with the word, "right?" (line 31). Implementing symbolic reasoning, Sean challenged Riley's assertion and claimed that they needed to add 4 to Riley's result (line 32). Riley agreed with Sean's addendum, and symbolically explained that she inadvertently thought about only the integral of $2/z$ (lines 33-34).

Afterwards, Sean and Riley articulated three warrants in short succession, not all of which were completed. Specifically, Sean mentioned the Cauchy-Goursat Theorem as a formal-embodied warrant, the embodied aspect of which was a circular tracing gesture around the path $C_1 + C_2$ (lines 35-36; see Fig. 131). Riley interrupted the end of Sean's warrant with an incomplete warrant of her own, wherein she generally stated that "we're just working backwards, but you can do the same exact process" (lines 37-38). I could not definitively discern which world or worlds she invoked when making this statement, because the "same exact process" from before involved all three worlds at various times. Before Riley was able to elaborate on her warrant, Sean interrupted Riley and added that "It's like a dramatic argument—we know we know the ending" (line 39). As spokesman, he elaborated on this seemingly metacognitive statement with symbolic inscriptions summarizing the values of the integrals around paths C_1 and C_2, as well as the sum of these integrals (lines 39-40). This concluded Argument 2, which is depicted in Figure 132.

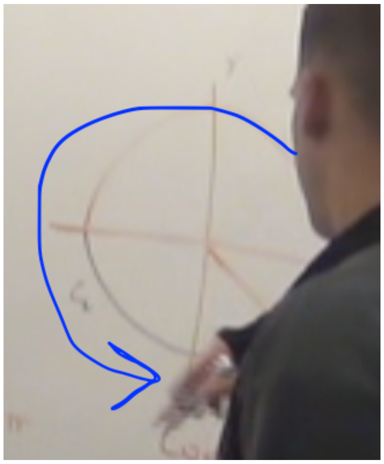


Figure 131. Sean's circular tracing gesture for "Cauchy-Goursat" in Argument 2.

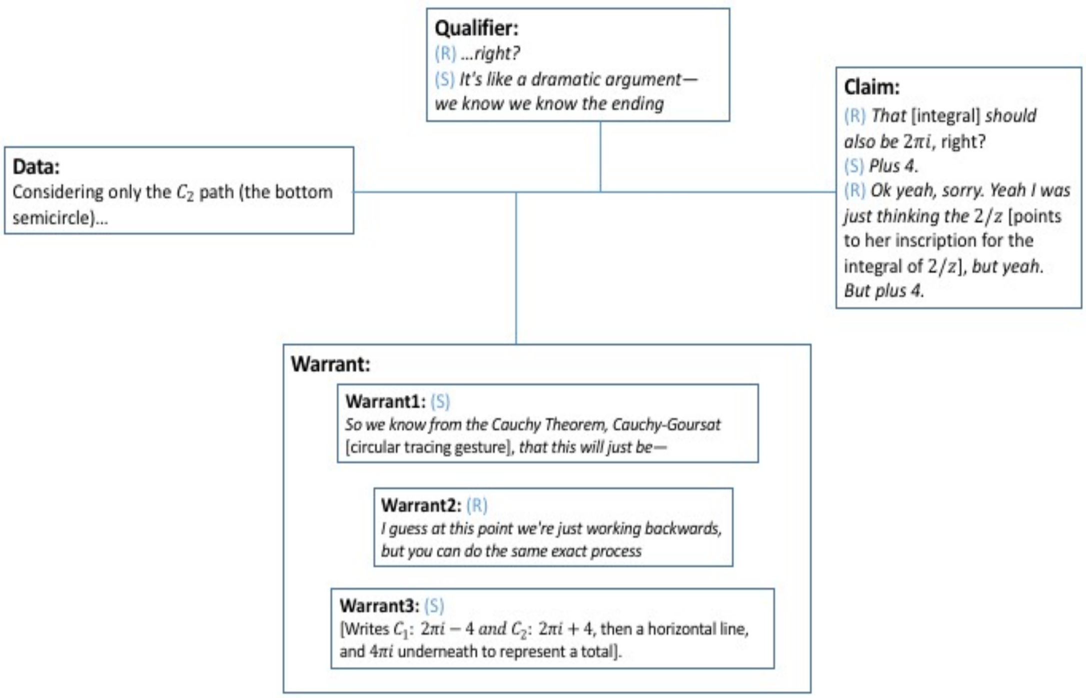


Figure 132. Toulmin diagram for Riley and Sean's Argument 2, Task 9b/c.

R: [Writes $\int_{C_2} \frac{z+2}{z}\, dz = 2\int_{C_2} \frac{1}{z}\, dz + \int_{C_2} 1\, dz$] . So again we do this. Curve 2 is just— well we can choose any curve we like, so we just go [integral from] -2 to 2 of dz [writes $\int_{-2}^{2} dz$]. I should probably stick to the same variable— it's going to equal 4 [writes = 4]. So that's this guy [draws arrow from $\int_{C_2} 1\, dz$ inscription to her $\int_{-2}^{2} dz$ inscription] and then here [points to $2\int_{C_2} \frac{1}{z}\, dz$] we'd parametrize in exactly the same way. Um, and it's just our bounds would be again flipped [makes crossing gesture with index finger on one hand, and marker in the other hand], just like last time right? Um which is going to make it come out to, um—

R: Well, ok the curve— the curve itself is different, yes or no? Because now it's going from $-\pi$ to π.

S: From π to 2π. We're going this way [counterclockwise].

R: Ok sure. So it's going to be π to 2π of $\frac{2}{2e^{i\theta}} 2e^{i\theta}$ [writes $\int_{\pi}^{2\pi} \frac{2}{2e^{i\theta}} 2e^{i\theta}\, d\theta$. Again, these cancel [the $e^{i\theta}$ factors] and these cancel [the 2s] and then it comes to [writes $2i(2\pi - \pi) = 2i\pi$]. I mean, but it's pretty much the same thing we had done before.

Next, Riley began Argument 3 by elaborating on Warrant2 from their second argument, as she signified with the phrase "so again we do this" (line 41). As spokeswoman, she wrote the statement $\int_{C_2} \frac{z+2}{z}\, dz = 2\int_{C_2} \frac{1}{z}\, dz + \int_{C_2} 1\, dz$ as a symbolic datum (line 41). Calling upon the path independence discussed in Argument 1, Riley continued as spokeswoman to articulate a formal-embodied-symbolic warrant that they could write $\int_{C_2} 1\, dz$ as $\int_{-2}^{2} dz$ (lines 41-42). Recall from Argument 1 that Riley previously attributed the formal analyticity of the integrand 1 as backing allowing her to conclude that they could invoke any embodied path between the endpoints -2 and 2 when symbolically evaluating such an integral. Riley relayed the symbolic value of this integral from Argument 1 as 4 (lines 43-44).

Riley then turned her attention to the other integral $2\int_{C_2} \frac{1}{z}\, dz$, and authored a warrant that they could "parametrize in exactly the same way" (lines 44-45). She authored an embodied-symbolic warrant that their "bounds would be again flipped" (lines 45-46), producing a two-handed crossing gesture to indicate flipping (see Fig. 133). Riley also qualified this warrant: "just like last time, right? […] Well, okay the curve—the curve itself is different, yes or no?" (lines 46-48). Using embodied-symbolic reasoning, Sean clarified that their limits of integration should be from π to 2π because their path traveled counterclockwise (line 49).

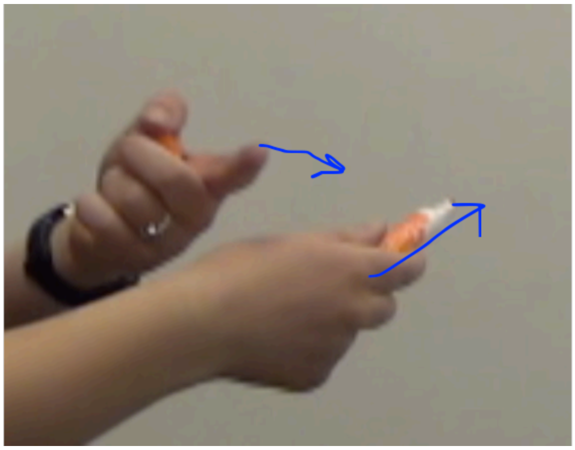


Figure 133. Riley's gesture for "flipped" in Argument 3, Task 9b/c.

Riley agreed with Sean's response, and symbolically set up the integral parametrized by θ (line 50). She relayed the cancellation of certain factors in the integrand as a symbolic warrant (lines 50-51). Riley symbolically evaluated the integral to obtain a claimed answer of $2i\pi$ (line 51), and qualified her assertion by reminding us that "it's pretty much the same thing we had done before" (line 52). Argument 3 is summarized in Figure 134.

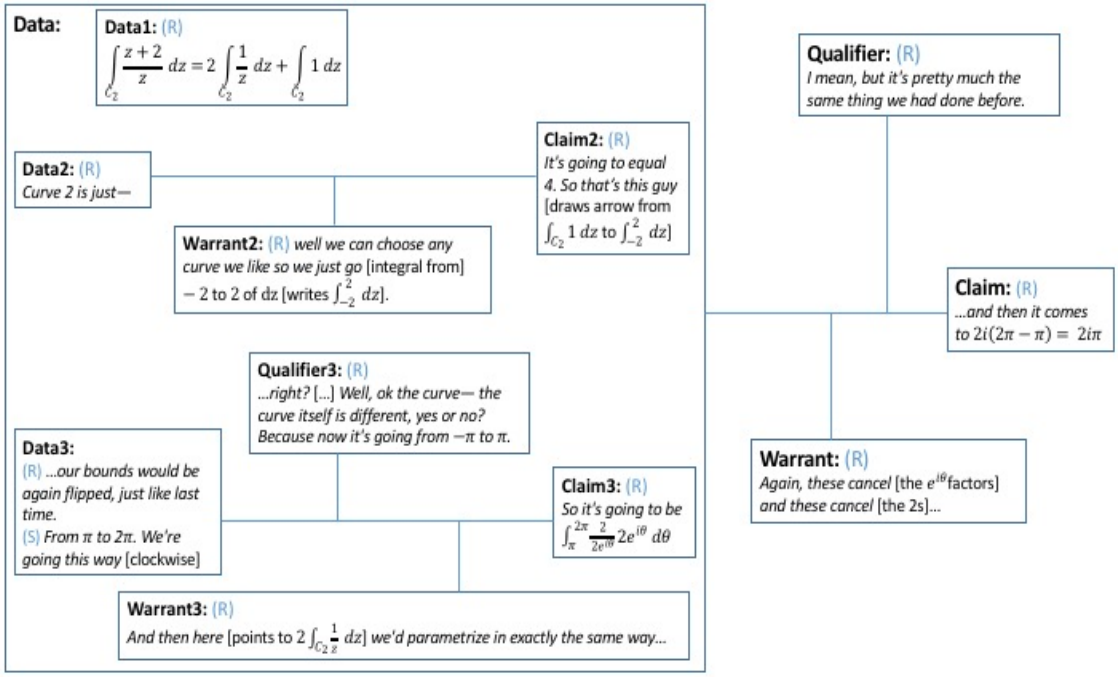


Figure 134. Toulmin diagram for Riley and Sean's Argument 3, Task 9b/c.

Int: Ok, and so over the whole circle, $C_1 + C_2$ then— um so it looks like you're claiming then that's just the sum of the individual integrals over C_1 and C_2 respectively?
R: Yes.
Int: So any sort of intuition about why you can do that?
R: Uhh, again he [Prof. X] proved why you can do that, and I don't remember what the proof was. But uh it just kind of makes—
S: Because when you add the paths together it's piecewise smooth. So it's like one giant path, pretty much. So really we could've just parametrized the integral from 0 to 2π if we wanted to, at this point. But since the path's connected very well [gesture bringing both hands together to represent connectedness], you [pauses and doesn't finish sentence]—
Int: So when you're saying "adding the paths together," you mean like literally just add $C_1 + C_2$?
R: You just continue along— yeah you just continue along the path.
S: I mean like, yeah, so there's like a different— if there's a different kind of path [gestures tracing a squiggly path between $z = -2$ and $z = 2$ with the pen in hand], we'd still have to do a separate integral for this path, but the integral of the whole path is just the sums [points to C_1 path then C_2 path]. In this case, since it's a part of the same circle we could just change the boundaries from 0 to 2π. But in general, you'd still have the integral over path C_1 [tracing gesture over path C_1] plus the integral over path C_2 [tracing gesture over his prior visualized path from $z = -2$ to $z = 2$, not the semicircular C_2]; as long as those paths are piecewise continuous then I think you're good.
Int: Okay.

Implicit to Riley and Sean's previous arguments was the claim that the integral over the full circle $C_1 + C_2$ was equal to the sum of the integrals over C_1 and C_2. Thus, I asked them about this facet as a follow-up question (lines 53-54), which began a fourth argument. Riley agreed that she used this fact (line 55), but did not elaborate on why it is true. Accordingly, I asked her and Sean if they had any intuition behind why this result holds (line 56). As she had done previously in Task 8, Riley noted that Professor X had proved this result in class, but she could not remember the details (lines 57-58). However, Sean stepped in to provide some insight.

Sean authored a formal-embodied warrant that “when you add the paths together it’s piecewise smooth,” and as spokesman, continued, “So it’s like one giant path, pretty much” (lines 59-60). Because $C_1 + C_2$ can be thought of as one path, Sean claimed that “we could’ve just parametrized the integral from 0 to 2π” (line 60). Sean began to articulate an alternate way of thinking about this problem in terms of connectedness, but paused and did not finish his sentence (lines 61-62). However, when describing the path as “very well connected,” Sean produced a gesture bringing both his fists together to illustrate this connectedness (see Fig. 135). Thus, his incomplete thought incorporated embodied reasoning.

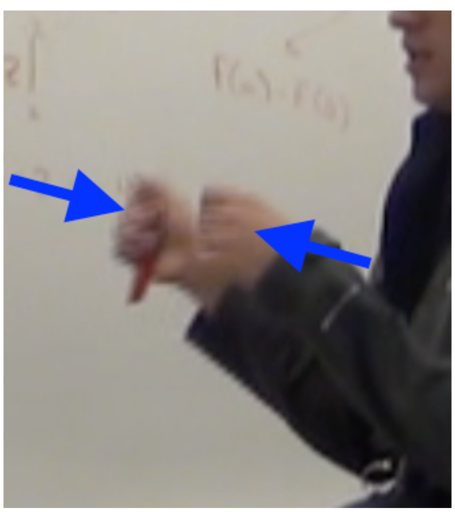


Figure 135. Sean’s gesture for “very well connected” in Argument 4, Task 9b/c.

Because I assumed throughout Riley and Sean's response to Task 9 that by "adding the paths together" they meant concatenation of paths, I asked a deliberately "naïve" follow-up question to elicit a more precise description of what they meant by this (line 63). In response, Riley spoke of $C_1 + C_2$ as "just continu[ing] along the path" (line 64), an embodied addendum to Sean's previous warrant. Sean also added embodied specificity to the warrant by illustrating "a different kind of path" as he traced a meandering path between $z = -2$ and $z = 2$ as an alternate choice for C_2 (lines 65-67; see Fig. 136). But even in this case, Sean maintained that "the integral of the whole path is just […] the integral over path C_1 plus the integral over path C_2" (lines 67 & 69-71). While authoring this embodied-symbolic claim, Sean provided tracing gestures for the paths C_1 and C_2 while discussing these two integral summands, using his visualized and hypothetical C_2 instead of the original semicircle (lines 70-71; see Fig. 137).

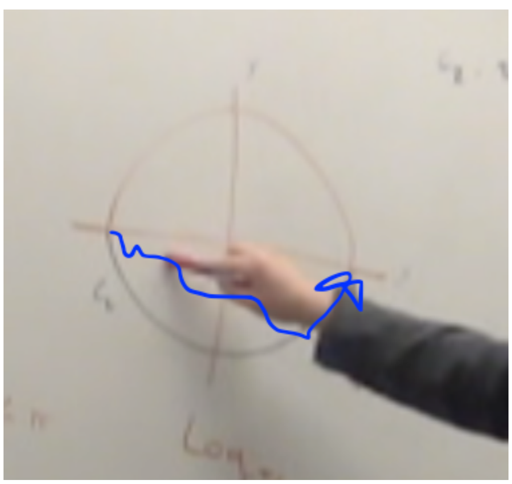
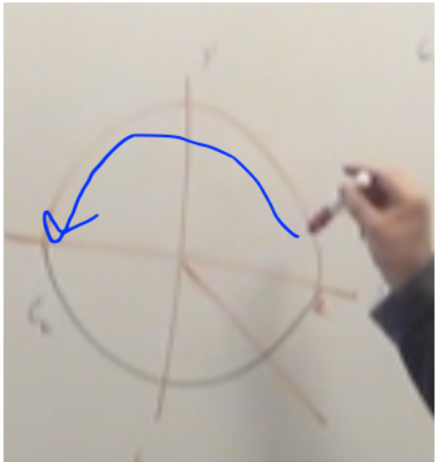


Figure 136. Sean's tracing gesture for "a different kind of path" in Argument 4.



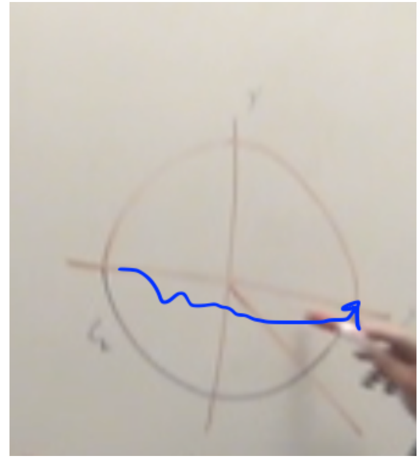


Figure 137. Sean's tracing gestures for "integral over path C_1"(at left) and "integral over path C_2" (at right) in Argument 4, Task 9b/c.

Again appealing to embodied-symbolic reasoning, Sean returned to his initial claim that "we could've just parametrized the integral from 0 to 2π" (line 60). Specifically, he authored a warrant that such a parametrization is possible in the case where C_2 is a semicircle because $C_1 + C_2$ comprises one full circular path (lines 68-69). Sean concluded Argument 4 by qualifying his and Riley's more general claim that the sum of the integrals along C_1 and C_2 yields the integral along $C_1 + C_2$: "as long as those paths are piecewise continuous then I think you're good" (line 71). Argument 4 is summarized in Figure 138.

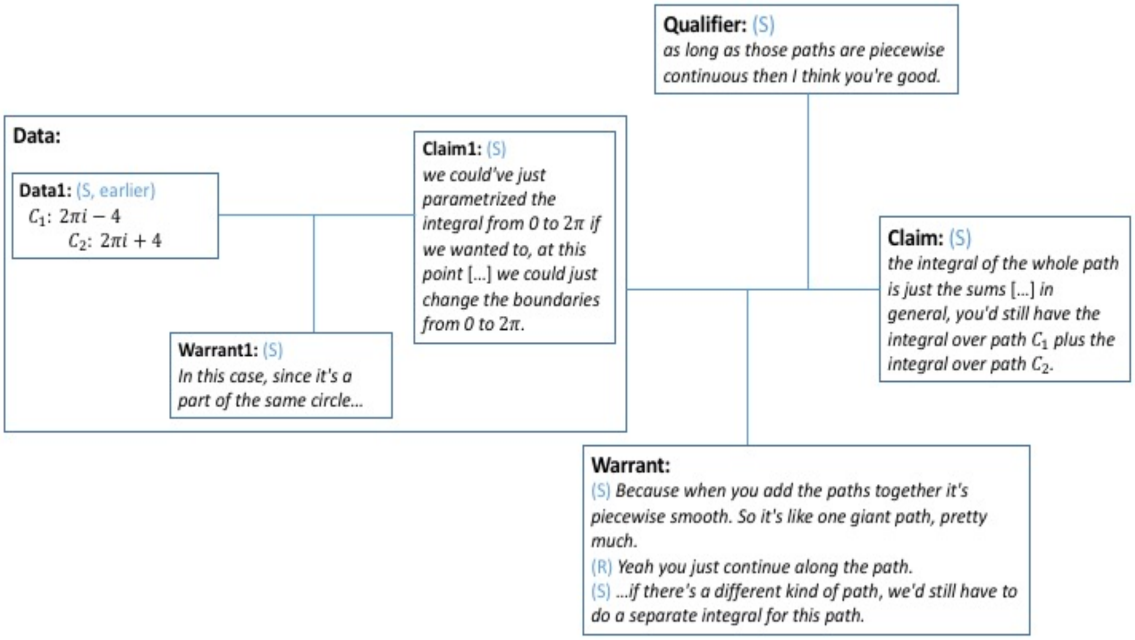


Figure 138. Toulmin diagram for Riley and Sean's Argument 4, Task 9b/c.

R: Um, plus like I just flew back to Calc 2 again, right, and it's like you had like an integral, say an integral over this region [draws region under a negatively sloped line from $x = 0$ to another unlabeled positive number] plus the integral over this region [shades under a positive constant function from an unlabeled negative x-value to $x = 0$] then the integral over like, the full, from here to here [points to the start and endpoint x-values] , or here to here [points to the same two points in opposite order] or whatever, um, is just going to be the sum of them [simultaneously points to both regions using two-finger gesture]. Um, here [points to her Calc 2 diagram] you'd have to split it up because it's not smooth. But if you had like a smooth curve, and you added these two together [shades two areas under a downward-facing parabola] you can do that because just physically it's the area, so you just add the area. Here, [looks towards the Task 9 inscriptions] it's not physically the area, but it still sort of intuitively makes sense that you can add them together.
Int: Yeah. Ok great. Um, alright. So you kind of answered what I wanted to talk about in a slightly different order, but that's actually great.
R: Nice that's good!
Int: Ok, so maybe let's kind of get rid of all this, then. Moving on.
R&S: [Erasing board].

As a follow-up, Riley supplied a fifth argument appealing to integration in real-variable calculus: "I just flew back to Calc 2 again" (line 73). She authored an embodied datum considering the integral of a piecewise function comprised of two linear pieces joined at $x = 0$ (lines 74-76 see Fig. 139). With this function in mind, Riley authored an embodied-symbolic claim that the integral of this function over a closed interval is equal to the sum of the areas of the respective regions between each piece of the function and the x-axis (lines 76-79). Using embodied reasoning, she indicated the sum of the two areas via a two-fingered pointing gesture (see Fig. 140). Riley also authored an embodied-symbolic warrant that the integral of this piecewise function needed to be "split up" over each piece because her function was "not smooth" (lines 79-80). Although Riley did not define what she meant by "smooth," she appeared to use this word to indicate a piecewise function that she could not readily describe using a single function formula.

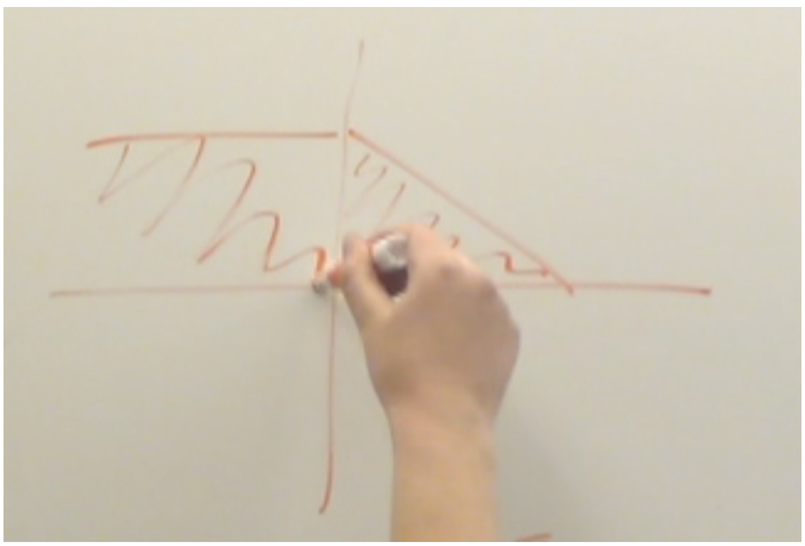


Figure 139. Riley's "Calc 2" diagram with two "not smooth" regions in Argument 5.

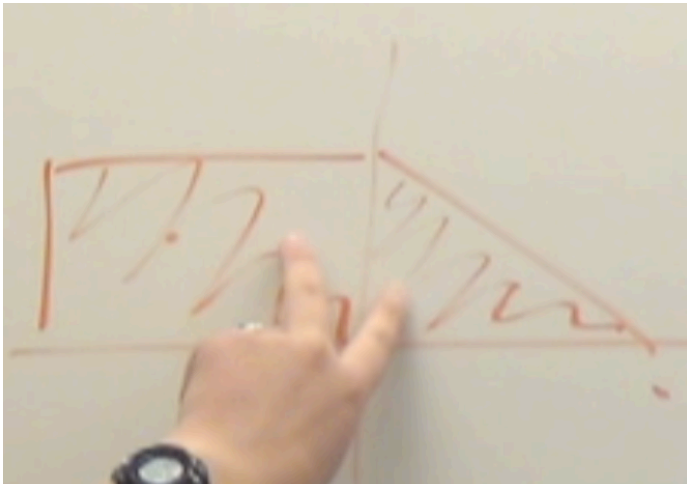


Figure 140. Riley's pointing gesture illustrating "the sum of them" in Argument 5.

Afterwards, Riley authored a second embodied datum by drawing a second curve that she identified as "smooth" (line 81). She once again considered the area of the region bounded between this function and the x-axis, and compared this full area to the areas of two sub-regions (see Fig. 141). In particular, she authored an embodied-symbolic claim that in order to evaluate the integral over the full region, "you just add the area" (line 82). Riley bolstered this claim with an embodied warrant: "you can do that because just physically it's the area" (line 81). Although this statement is also true of her first "non-smooth" function, Riley's argument seemed to implicitly indicate that this second "smooth" function could be identified by a single formula. For example, evaluating the integral $\int_{-2}^{2} 3 - x^2\, dx$ could be accomplished by adding $\int_{-2}^{0} 3 - x^2\, dx$ and $\int_{0}^{2} 3 - x^2\, dx$, but the same could not be done with the first function without using two separate integrand formulas.

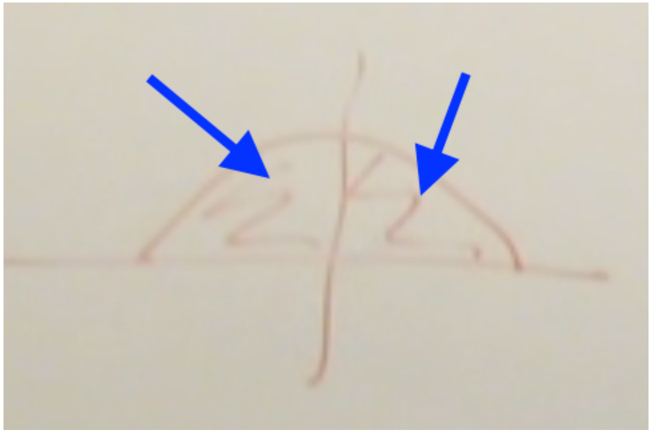


Figure 141. Riley's diagram of 2 regions under a "smooth" curve in Argument 5.

Returning to the present context of the complex plane, Riley authored an embodied warrant that although integration no longer generally represents area under a curve, "it still sort of intuitively makes sense" to claim that $\int_{C_1+C_2} \frac{z+2}{z}\, dz = \int_{C_1} \frac{z+2}{z}\, dz + \int_{C_2} \frac{z+2}{z}\, dz$ (lines 82-83). Hence, Riley explicitly instantiated *thinking real, doing complex* (Danenhower, 2000) in the sense that her experience with adding areas in Calculus 2 informed her embodied intuition in this task. Argument 5 is summarized below in Figure 142.

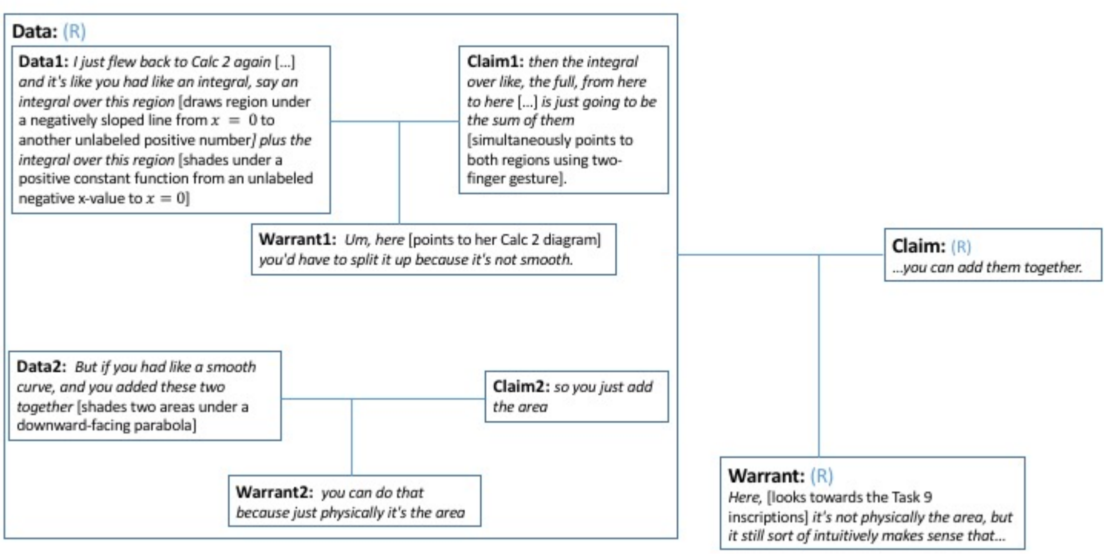


Figure 142. Toulmin diagram for Riley and Sean's Argument 5, Task 9b/c.

Task 9 Summary

In task 9, we continue to observe a comparative abundance of embodied reasoning from Riley and Sean versus Dan and Frank. However, much of Dan and Frank's embodied reasoning mirrored Riley and Sean's, in the sense that both pairs drew diagrams of the various paths involved, and both pairs gestured a full circular path in response to Task 9b. This latter observation is particularly notable because Task 9b only explicitly considered the bottom semicircle, which Riley and Sean denoted C_2. In fact, Riley provided whole-circle gestures even in Task 9a before I asked about C_2. Additionally, Sean displayed more embodied reasoning in Task 9 than in previous tasks, particularly throughout Argument 4 of Task 9b/c.

Another distinguishing factor between the pairs in Task 9 was that Riley and Sean responded to Task 9c before 9b, and Riley demonstrated a clever way of finding the integral of $2/z$ as twice the integral of $1/z$, which they computed in Task 6. In task 9, both pairs exhibited the property that individuals within the pair pursued different solution approaches. In Riley and Sean's case, they were not as explicit in articulating these choices of approach to each other, and this led to several disagreements and challenges that culminated in additional follow-up arguments. On the other hand, Dan and Frank more clearly delineated their division of labor; they clearly verbalized the decision for Dan to pursue a logarithmic approach while Frank used a parametrization. Nevertheless, both pairs of participants ran into some difficulty agreeing on their limits of integration in Task 9, which created some confusion between them when they needed to reconcile their respective methods. Finally, notice that of the two pairs, only Sean and Riley consciously verbalized *thinking real, doing complex* (Danenhower, 2000).

Task 10—Dan and Frank

Task 10 required participants to consider the implications of traversing a circular path twice on the complex integral along such a path (lines 1-2). As spokesman, Dan re-voiced the datum of traversing a circular path twice by gesturing the motion of this path in the air, using the tip of his whiteboard marker (line 3; see Fig. 143). After a moment of silence, I clarified the intent of the task by reminding Dan and Frank that the last task involved a circular path traversed once, and I asked them what would happen if this path was traversed twice (line 4). As spokesman, Frank reworded the latter portion of my question and produced a gesture similar to Dan's but oriented in the opposite direction (line 5; see Fig. 143). Both Dan and Frank's gestures exemplify embodied reasoning because they represent motion along a visualized path. Having clarified the setting for this task, I finished asking them whether traversing the path twice would affect the value of the integral (line 6), and Dan and Frank gathered their thoughts (lines 7-8).

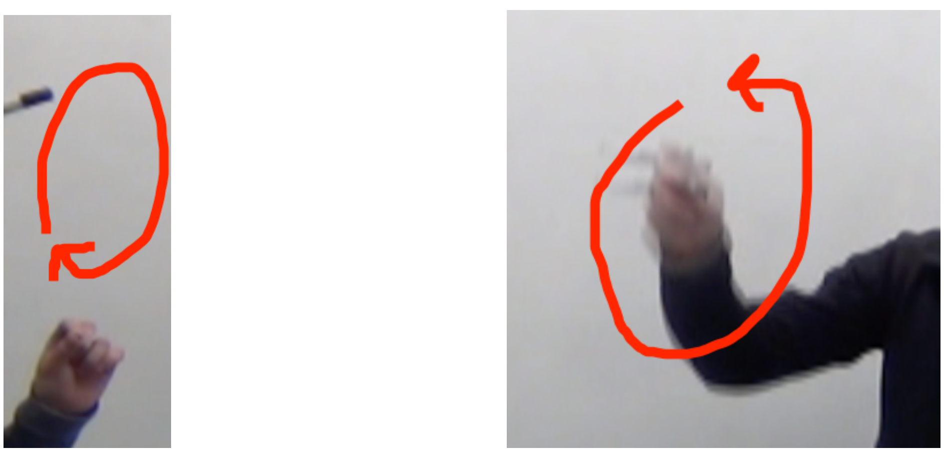


Figure 143. Dan's (left) and Frank's (right) gesture a circular path traversed twice.

Int: So just something a bit more general. Is it permissible to travel over a circular path twice? And if so, how does that affect the value of a complex path integral?
D: [Traces hypothetical path in the air twice with pen]
Int: So for instance, the circle that we did, we were traversing once. So if we were traversing twice—
F: Yeah so travel over it again [Traces circular path in the air twice].
Int: Is that going to affect the value of the integral? And if so, how?
D: Umm, hmm.
F: That's a good question. I've never even thought about that.
D: So if you think of it like a vector field—
F: Yeah
D: Then it shouldn't affect— if it was conservative—
F: Yeah, it depends on if it's analytic or not, I guess [looks at me].
D: It's just double the value of whatever—
F: Yeah it'll just double the value, as far as I'm concerned. Because if the function is analytic, then it's basically a "conservative vector field" [air quotes gesture] in which case the result of the first time around is zero so [D starts talking] so 2 times zero is zero.
D: But even if it's not, and so it gave you some type of— let's say -5 is your answer. Well—
F: Or like more commonly, if we did something like this [writes $\int \frac{1}{z}$ and starts to write the bounds, but erases]. Or if our function was, let's say, 1 over [writes $f(z) = 1/z$] and our curve was the circle of radius 1, oriented positively, and centered at 0. And we're going from say $0 \leq \theta \leq 4\pi$ [writes this]. So basically we're travelling along the circle twice [draws the path twice, where the second time around looks like a larger circle, presumably since he wanted to see the second time around]. I shouldn't have drawn it bigger, but—
F: Then if we went around it once, then it'd just be $2\pi i$ [writes $2\pi i$]. So if we went around it twice, it should just be two times that, I imagine. [Writes $2(2\pi i)$ and accidentally simplifies as $2\pi i$,then erases] I should've drawn a 4. [Writes $4\pi i$]. That makes sense to me.

Afterwards, Dan began to author a datum considering the integrand as a conservative vector field (lines 9-11) and Frank added that the effect of traversing the path twice depends on whether the function is analytic (12). Frank qualified this assertion with the phrase “I guess” and looked at me for validation (line 12). Dan began to author a symbolic claim that the integral of such a path has a doubling effect (line 13), and Frank relayed that it will “just double the value” (line 14). As spokesman, Frank clarified Dan’s datum that an analytic function represents a “conservative vector field,” and concluded that in this case, the integral around the circular path traversed once is zero. Therefore, according to Frank, traversing the path twice would double the value of this integral and thus yield an answer of zero (lines 14-16).

Dan began to author a symbolic rebuttal to Frank’s claim, and considered a situation wherein the value of the integral was nonzero, but Frank interjected (line 17).

Frank went on to author data regarding a specific function $f(z) = 1/z$ and a circular path traversed counterclockwise around the origin (lines 18-20). Using embodied-symbolic reasoning, he related the doubly traversed path to a symbolic statement about the corresponding values of theta (line 20). Transitioning to embodied reasoning, he then drew a diagram of the circular path, and accidentally increased the radius slightly as he traced the path a second time (lines 21-23). With this data in mind, Frank authored an embodied-symbolic claim that traversing the path once would yield an integral of $2\pi i$ (line 24). Therefore, Frank argued, traversing the path twice should double this value, and he concluded that the answer should be $2(2\pi i) = 4\pi i$ (lines 24-26). He qualified this claim with the phrase, "I imagine," conveying at least some level of uncertainty (line 25). This first argument is summarized in Figure 144 below.

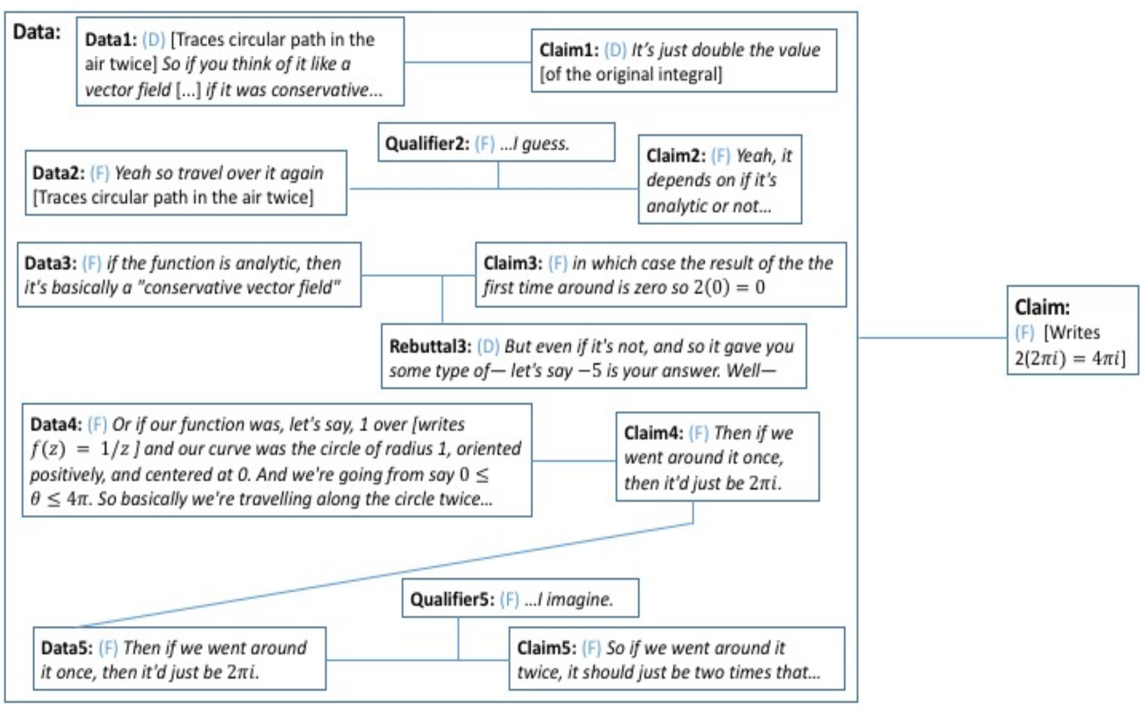


Figure 144. Toulmin diagram for Dan and Frank's Argument 1 for Task 10.

Although Dan and Frank articulated a claim for this task in Argument 1 and illustrated this claim via an example, they did not provide a warrant for this claim. Accordingly, I asked a follow-up question about whether they saw a connection between this task and the last (lines 27-31). This follow-up question prompted a second argument pertinent to Task 10, which began as Dan relayed a datum about parametrizing the path (lines 32-33). In particular, he reiterated that $0 \leq \theta \leq 4\pi$ when the circle is traversed twice. Dan then used this datum to author an unfinished symbolic claim relating the integral from 0 to 4π to two integrals: one from 0 to 2π and another from 2π to 4π (lines 33-34).

Likely as a result of their prior discussion in Task 9c concerning adjusting branch cuts when adding two integrals, Dan began to articulate how this issue manifested itself in the present context (lines 34-35). However, Frank interrupted and finished this warrant as spokesman, conveying the reason they could combine the two integrals in Dan's previous claim (line 36). Frank altered Dan's symbolic inscription from line 34 to explicitly indicate that they were adding the two integrals (line 36). As spokesman, Dan agreed and reworded Frank's warrant (line 37), and he and Frank qualified their argument by expressing a moderate degree of confidence (lines 38-39). This second argument is summarized below in Figure 145.

Int: So just kind of intuitively. What about based on what you guys found in the last question, where instead of— it eventually amounts to just adding the two semicircles together and that gets you the whole circle?
F: Right
Int: So if you're travelling over the same circle twice, do you see that as being the same?
D: Yeah I do. Because you could think of it like if you have— So let's say we're going to parametrize it right? So you have it from 0 to 4pi, circle traversed twice. Well that's the same as if you were to go from 0 to 2pi [writes $\int_0^{4\pi} = \int_0^{2\pi} \quad \int_{2\pi}^{4\pi}$] and then if you are adjusting for arguments or whatever, then you're just saying—
F: You adjust the branch cuts as required, and just add them [writes a + in between D's two integrals].
D: Yeah if you were to add anything involving arguments, it would be the same thing, so—
F: Yeah, yeah that seems reasonable to me.
D: I mean, that's how I would imagine it.

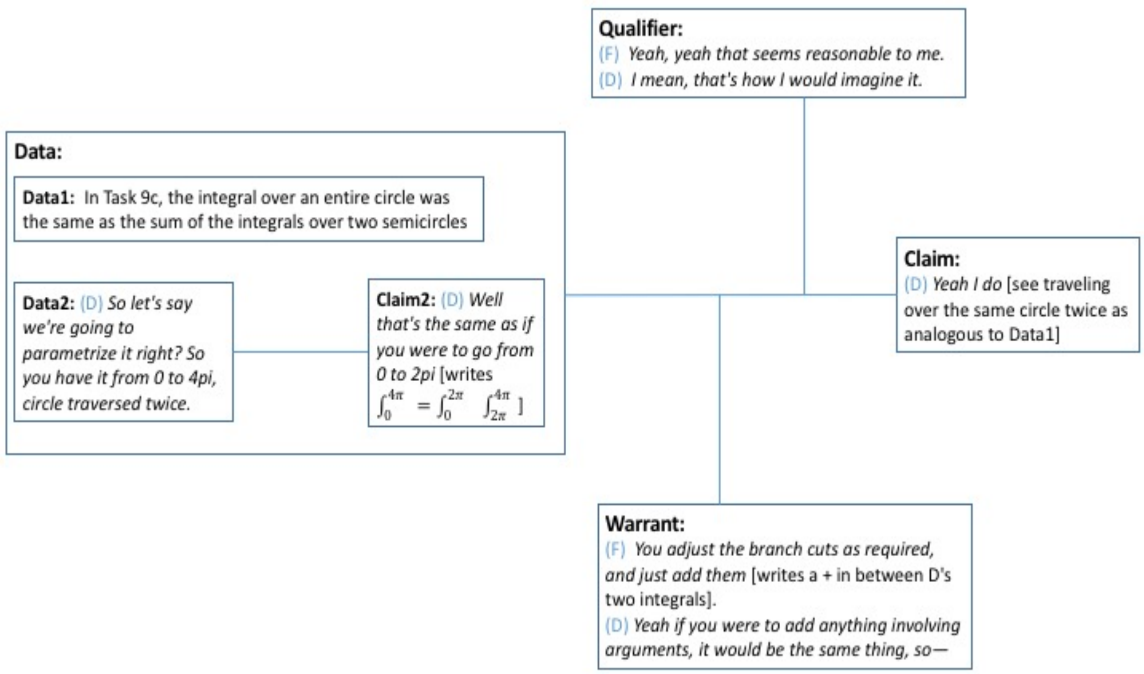


Figure 145. Toulmin diagram for Dan and Frank's Argument 2 for Task 10.

Subsequently, I asked Dan and Frank to conjecture about what happens if the circular path is traversed *n* times (line 40). This catalyzed a very brief third argument wherein both participants articulated symbolic claims that the integral would be "*n* times" the value of the integral involving the path traversed only once (lines 41-42). Specifically, Dan brought up the multiplication by *n*, and Frank relayed the phrase "*n* times" but clarified the quantity that is multiplied. This succinct Argument 3 is depicted in Figure 146 below. As another follow-up, I asked the pair whether they had discussed ideas related to Task 10 in class (line 43). Consistent with Frank's earlier comment that he had never thought about this idea before (line 8), Dan also denied having seen it in class (line 44). Frank agreed, and clarified that they "usually only went around things once" (line 45).

Int: So generalizing, if we traversed that circle n times instead, what would your guess be?
D: It would be n times—
F: n times the value around it once.
Int: Cool. So that's not something you guys discussed in the class though?
D: No.
F: No, we usually only went around things once. I think the only time that we saw things wrapping multiple times was the winding numbers, where—
D: What was that called again?
F: Rouche's theorem. Where if you um—
D: Or was that just the argument principle? I think it was the Argument Principle.

Figure 146. Toulmin diagram for Dan and Frank's Argument 3 for Task 10.

Frank did recall considering "things wrapping multiple times" when learning about winding numbers (line 46), and he and Dan attempted to recall what theorem involved this concept (lines 47-49). Frank wrote symbolic inscriptions representing the Argument Principle (line 50), and then explained the meaning of the various symbols within the equation (lines 50-53). Frank then related the symbolic statement of this result to a geometric interpretation, namely the "number of times that the image of $f(C)$ winds around the origin" (lines 53-60). I asked him and Dan if they saw this idea as related to Task 10, and they both replied that they did not. Rather, Frank explained, "that was the only time we had ever really seen things going around points multiple times" (lines 63-64).

50 *F:* What was it again? It was just [writes $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = z_f - p_f$]. So if you evaluate this integral on some
51 curve that's closed [points to the path C in the integral] of— and you have the derivative of some
52 analytic function over the function itself, then that's equivalent to the zeros of the function minus the
53 number of poles in the function [points to z_f and p_f, resp.]. But a way to geometrically visualize that,
54 that we talked about, was the number of times that it— basically the result that you get is the number
55 of times that—
56 *D:* $f(C)$ yeah
57 *F:* …that the image of $f(C)$ winds around the origin. I think that was really the only time that we saw
58 things wrap around multiple times. But we never really talked about going over a path multiple times.
59 *Int:* And $f(C)$, C is the same curve—
60 *F:* C is the curve. $f(C)$ is just the image of that curve under that function.
61 *Int:* Ok. So do you see that as related to what you were just talking about though?
62 *D:* Nah.
63 *F:* Not particularly. I just— that was the only time we had ever really seen things going around points
64 multiple times.
65 *Int:* Ok cool. Let's see so I have one more kind of general— well I guess it's not too general. It follows a
66 specific function but there's some different things to consider.

Task 10—Riley and Sean

After I read Task 10 to Riley and Sean (lines 1-4), Sean began Argument 1 by authoring a symbolic claim that "you should get twice your value" (line 6) when traversing the circular path twice. Riley qualified Sean's claim with the statement, "Yeah that would make sense," and elaborated by authoring a formal datum considering an analytic function (line 7). Instantiating embodied-symbolic reasoning, she claimed that the integral of such a function vanishes due to an embodied warrant that the circular path is closed (lines 7-8). As she had done many times before, she qualified this assertion as a question, using the word "right?" (line 8).

Afterwards, Sean relayed the symbolic inscription $e^{i\theta}\Big|_0^{2\pi}$ from task 9 as a datum (lines 9-10). He explained that he and Riley obtained this result for "one of the values" in the last task. Recall that Riley and Sean expressed $\frac{z+2}{z}$ as $1 + \frac{2}{z}$ and integrated each term along one full circle. However, notice that the integral of neither term produces this exact symbolism; Sean appeared to notice this, as he soon erased his symbolic inscription as an unspoken qualifier (line 12). Meanwhile, Riley authored an embodied datum by drawing the twice-traversed circular path and considering values of theta up to 4π (lines 11-13; see Fig. 147). Sean stared at Riley's diagram and authored a symbolic claim that "eventually things cancel out" (line 14). Using formal-symbolic reasoning, he elaborated that "the analytic parts of the function go to 0" (line 14). I interpreted this to mean that if they rewrote a rational function as a sum of two or more terms, as they did in Task 9, then the integral of each analytic term would vanish. Sean qualified this assertion with the phrase, "of course," expressing a high degree of confidence (line 14), and Riley agreed with his conclusion (line 16). Argument 1 is summarized in Figure 148.

Int: Ok and this is still somewhat related, since we were talking about circular paths and whatnot. So keeping in mind a circular path, is it permissible to travel over a circular path twice? And if so, does that affect the value of a complex path integral? So in this previous example we were thinking of just traversing the circle once. What if we do that twice?
R: Ok, um—
S: You should get twice your value.
R: Yeah that would make sense. I mean, so I mean if it's an analytic function, then you're still going to get 0, since it's still a closed path, right?
S: But— probably shouldn't have erased this part— so one of the integrals we got, like we got to this value, say from 2π to 0 [writes $e^{i\theta}\Big|_{0}^{2\pi}$].
R: [Draws Argand plane.] So if we do to 4π—
S: [Erases what he just wrote]
R: [Draws circular path twice]
S: Right, so eventually things cancel out. So like, the analytic parts of the function go to 0 of course, but—
R: Right.

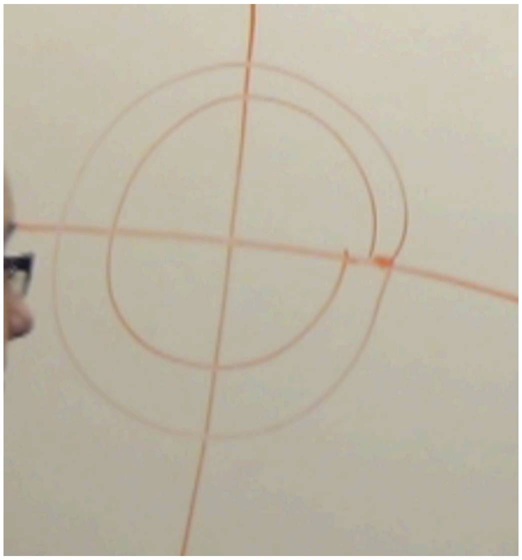


Figure 147. Riley's diagram for a circular path traversed twice in Argument 1, Task 10.

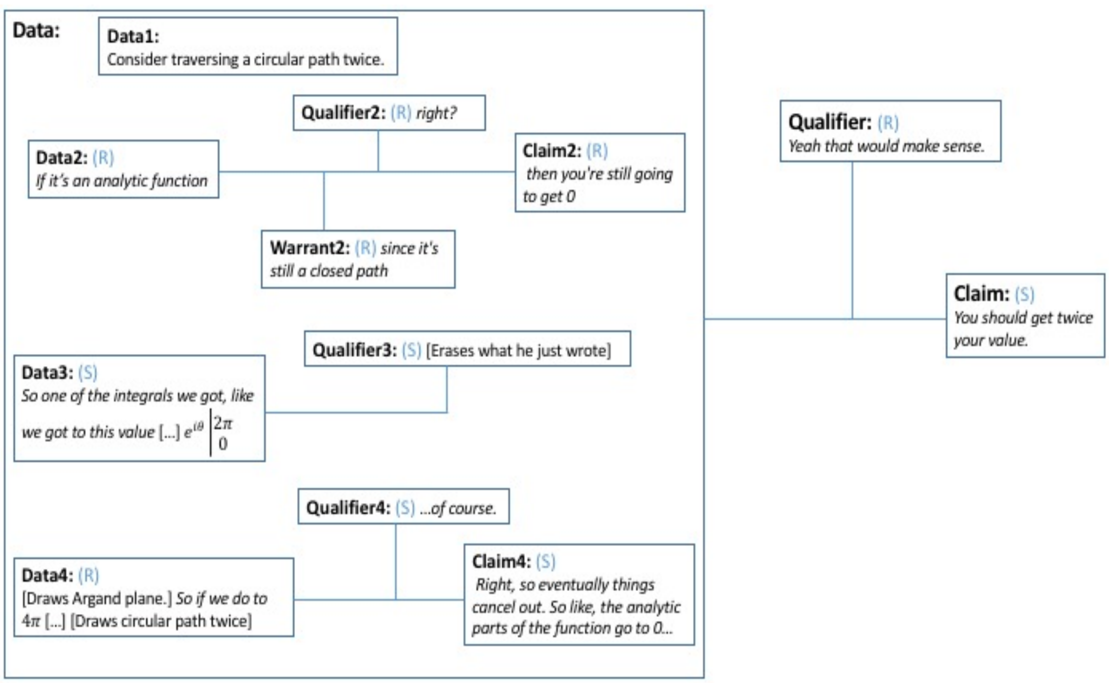


Figure 148. Toulmin diagram for Riley and Sean's Argument 1 for Task 10.

S: But for this [points to Riley's diagram] it would just wind around twice [circular tracing gesture], so it would be, yeah—
R: So it should just be double, cuz I mean, eventually— so for like the 1/z [writes $\int \frac{1}{z} dz$]
S: So we go over like this now [writes $2i \int_0^{4\pi} d\theta$].
R: …we ended up with the integral from 0 to 2π of um, so it's dt right, [integral] of 1 dt right? [writes $= i \int_0^{2\pi} 1\ dt$] So instead of at 2π [circular path gesture], if we wound twice it would be [same circular path gesture]—
S: $2i$.
R: Let's call curve two cuz twice winding around [writes $\int_{C_2} \frac{1}{z}\ dz$]. It's going to end up at 0 to 4π, right? [writes $= i \int_0^{4\pi} 1\ dt$] And so it's just going to end up at $4\pi i$. Yeah?
S: Mhm.
R: So it's just double.

As spokesman, Sean returned to the present task by pointing to Riley's diagram as an embodied datum. With this datum in mind, he began a second argument with an embodied warrant that the circular path "would just wind around twice" (line 17). While speaking these words, he provided a circular gesture illustrating one full circle of the path rather than two (lines 17-18; see Fig. 149). As spokeswoman, Riley re-voiced their prior claim from Argument 1: "So it should just be double" (line 19). Continuing as spokeswoman, she repeated their finding from Task 9 that the integral of $1/z$ over one full circle became $i \int_0^{2\pi} 1\ dt$, though previously they expressed this result using θ rather than t (lines 19 & 21-22). As done previously, Riley qualified this symbolic warrant with the word, "right?" (line 21). Riley then specified an embodied-symbolic datum considering how the upper limit of integration would change to 4π "if we wound twice" (lines 22-23). While speaking these quoted words, she produced the same circular gesture that Sean did previously (see Fig. 149).

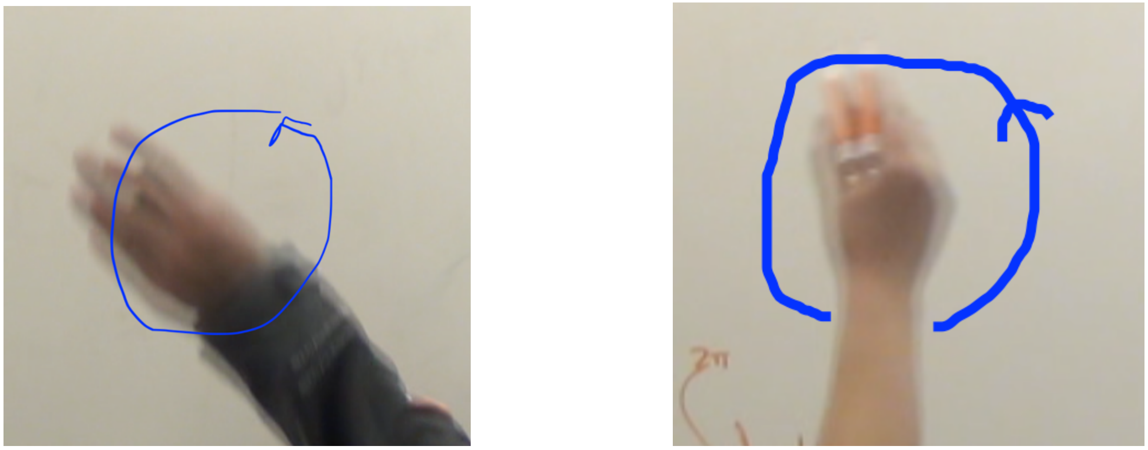


Figure 149. Sean's (left) and Riley's (right) gesture for winding twice, Argument 2.

As Riley finished articulating her symbolic warrant, Sean wrote the symbolic inscription $2i\int_0^{4\pi} d\theta$ (line 20). Accordingly, when Riley later began to author her corresponding symbolic claim about the new path's integral (lines 22-23), Sean was eager to interrupt and say "$2i$" (line 24). Notice that Sean's inscription is actually incorrect because he doubled Riley's integral $i\int_0^{2\pi} 1\, dt$ but also changed the upper limit of integration to 4π, which would have an extra doubling effect. Unaffected by the interruption, Riley continued to articulate her claim. As spokeswoman, she created the symbolic notation C_2 to represent the new twice-traversed circular path. She then authored a symbolic claim that $\int_{C_2} \frac{1}{z}\, dz = i\int_0^{4\pi} 1\, dt$ (lines 25-26). Riley qualified this assertion by asking Sean, "It's going to end up at 0 to 4π, right?" (line 25). Finally, Riley authored a symbolic claim that they should obtain an answer of $4\pi i$, and relayed her previous claim that "it's just double" (lines 26-28). She qualified this assertion by asking Sean, "Yeah?" (line 26). Argument 2 is summarized below in Figure 150.

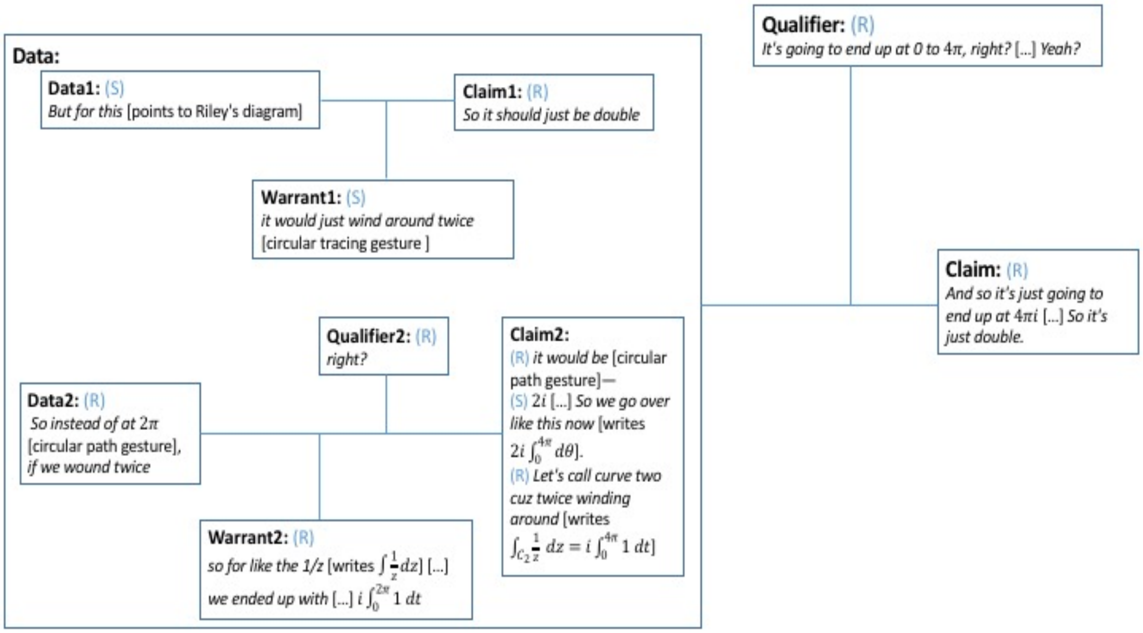


Figure 150. Toulmin diagram for Riley and Sean's Argument 2 for Task 10.

Int: Ok, and so in general, do you think you could find a way to generalize that if you're traversing, you know, n times?
R: Um yeah, so if you're traversing the same path n times then it would probably— it would make sense that um, that it would just be n—
S: It would be n times the integral of one path.
R: Yeah. Of whatever your function is [writes $n\int_{C_1} f(z)\ dz$]. Um I guess to represent that mathematically, it would have to be like— n times C_1 seems like it would be really bad, like, notation—
S: $C_1 + C_2 + \ ... + C_n$, where $C_1, C_2, \dots, C_n$ are all C_1.
R: Let's just call it C_n [writes $\int_{C_n} f(z)\ dz$].
Int: Yeah, without getting too hung up on notation, um so you're thinking it's just going to multiply like that. Is this something you talked about in class, kind of towards the end?
S: No. I don't think so.
R: No I don't think we talked about multiple winding around.
Int: Ah, ok ok. Cool. Yeah so there's kind of more general versions of like, Cauchy Integral Formula, for instance, that incorporate the idea of a winding number.
R&S: [Both suddenly have "aha moment" looks on their faces and remember].

Following Argument 2, I asked Riley and Sean if they could generalize this argument to discern what would happen if the circular path was traversed n times (lines 29-30). This began Argument 3, as Riley relayed the embodied datum that the curve is traversed n times (line 31). She used this datum to author a symbolic claim about the resulting value of the integral along such a path (lines 31-32), but did not finish her statement. Sean interrupted and finished the claim: "It would be n times the integral of one path" (line 33). As spokeswoman, Riley added to this claim to specify an integrand, and wrote corresponding symbolic inscriptions (line 34). However, she qualified her choice of symbolism as unmathematical, acknowledging that "n times C_1 seems like it would be really bad, like, notation" (lines 35-36).

In response to Riley's hesitation, Sean proposed an alternate symbolization as spokesman. Specifically, he suggested they think of such a path as "$C_1 + C_2 + \ldots + C_n$, where $C_1, C_2, \ldots, C_n$ are all C_1" (line 37). As spokeswoman, Riley altered this notation even further, and claimed they could write the integral as $\int_{C_n} f(z)\ dz$ (line 38).

Argument 3 is summarized below in Figure 151. After this argument, I asked Riley and Sean if they had seen something similar in their class, and they both answered that they did not recognize "multiple winding around" (lines 39-42). I told them that there are more general versions of Cauchy's Integral Formula that involve the concept of a winding number, and they both had looks on their faces that suggested they had indeed talked about such a concept in their course (lines 43-45). Sean recalled that they did discuss winding numbers at the very end of the course (line 46). This realization sparked one final argument pertinent to Task 10, as I detail next.

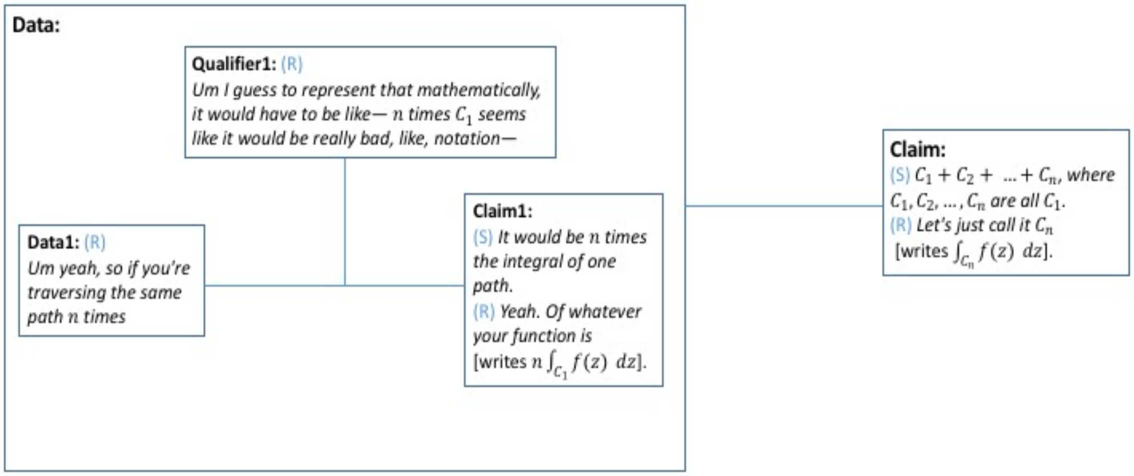


Figure 151. Toulmin diagram for Riley and Sean's Argument 3 for Task 10.

S: We did cover winding number at the very end. It's like the *last,* last class—
R: How many times like the function $f(z)$ goes around the origin or something?
S: So if my memory's correct, it was [writes $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)-a}dz$] was like the number of times that $f(z)$ winds around the point a.
R: Right.
S: So if you do like, just no a, then it's the number of—
R&S ...times you wind around the origin.
R: Again, I think that makes sense, right? Because if we— I mean, if we take this [points to $\int_{C_2}\frac{1}{z}\,dz$] um, and we divide like $4\pi i$, divide by $2\pi i$, then that's just going to be 2 [writes $\frac{4\pi i}{2\pi i}$], so it makes sense that it runs twice around the origin [circular gesture running twice around], right? So we hadn't connected it in that way, but I get— yeah we had talked about that like the last couple of days.
S: Very briefly though.
Int: Well, cool. Yeah there's a lot of interesting stuff there. Unfortunately, with just one semester you don't really get to everything. Ok yeah cool.
Int: Ok I have one more big task, um, to look over. So why don't we erase this.

Riley began Argument 4 with the embodied-symbolic datum that she remembered the winding number as representing "how many times like the function $f(z)$ goes around the origin," but qualified this datum as a question that ended with "or something?" (line 47). As spokesman, Sean elaborated on this datum, but also qualified his remark with "if my memory's correct" (line 48). He remembered the winding number as the symbolic formula $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)-a}dz$, which represents "the number of times that $f(z)$ winds around the point a (lines 48-49). Sean authored a symbolic claim that if $a = 0$ then this formula describes the number of "times you wind around the origin," which he and Riley uttered in unison (lines 51-52).

Riley reflected on their argument thus far and qualified it as "mak[ing] sense, right?" (line 53). She elaborated by authoring a symbolic warrant that related their answer from Argument 3 to Sean's formula for the winding number (lines 53-54). Accordingly, she claimed that this formula for winding number corroborated their findings from Argument 3 (lines 54-56). When describing the circular path as "run[ning] twice around the origin" (line 55), Riley gestured the motion of the circular path, this time with two full circles (see Fig. 152). This concluded Argument 4, which is depicted in Figure 153.

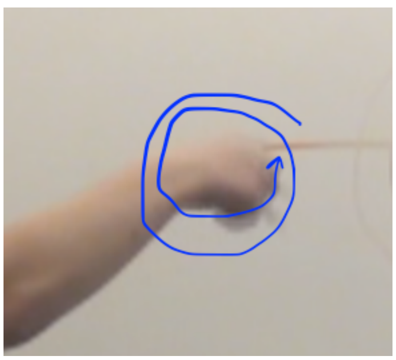


Figure 152. Riley's gesture for a circular path traversed twice in Argument 4, Task 10.

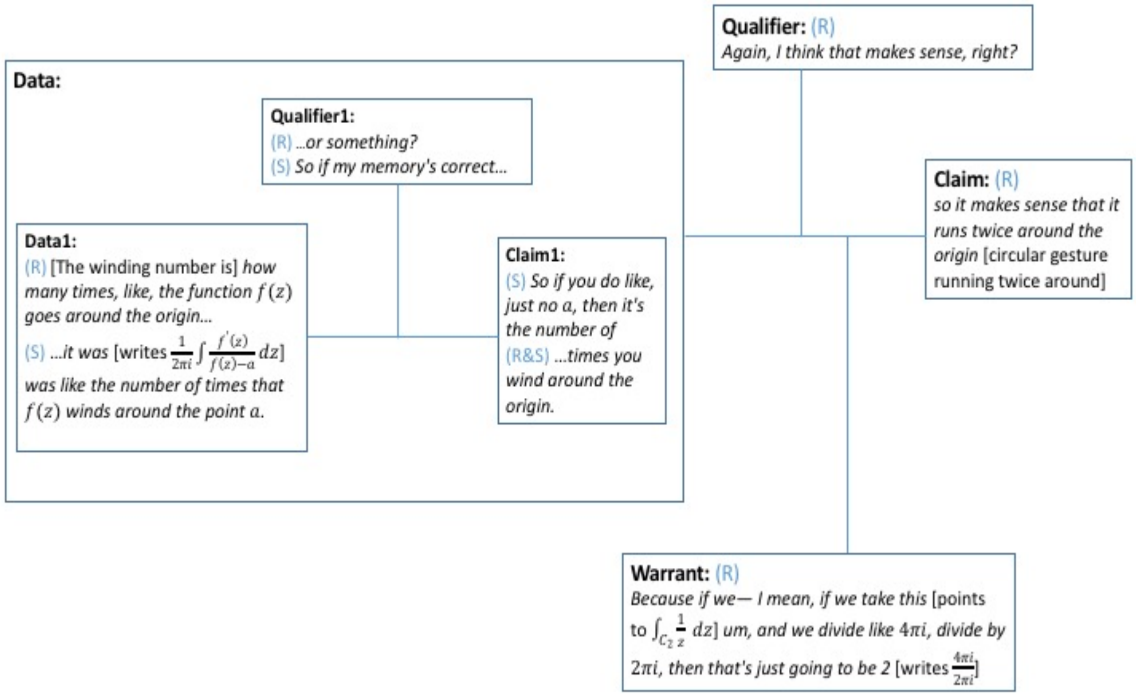


Figure 153. Toulmin diagram for Riley and Sean's Argument 4 for Task 10.

Task 10 Summary

All four participants individually produced the same circular tracing gesture in the air representing a circular path traversed twice. Sean and Riley both used very similar language ("wind around" and "wound it twice," respectively) while producing this gesture, and until the end of Argument 4, their gestures corresponding to a multiply-traversed path only explicitly illustrated one full circular motion, rather than several. One key difference between Dan and Frank's response as compared to Riley and Sean's was that Dan and Frank chose to incorporate the language of a "conservative vector field" to justify the portions of the integral that vanished, while Riley and Sean attributed this to the analyticity of the function and the fact that the path was closed. Moreover, Dan and Frank discussed having to adjust branch cuts when traversing the circular path a second time, whereas Riley and Sean incorporated parametrization techniques that did not need to account for branch cuts. Ultimately, both pairs arrived at the same conjectures and eventual claims for Task 10 with respect to traversing the path twice and n times, and both pairs stated the Argument Principle when I asked if this task reminded them of anything they had done in class.

Task 11 – Dan and Frank

Int: So let's consider the function $f(z)$ equals— it's going to be a fraction. On the top it's just 1, so 1 over— yeah on the bottom we have $z\ (z^2 - 1)$. Yeah, like that [F writing the function down as I say it.] And we're going to let L be a simple closed positively oriented curve, so the SICOPOC abbreviation for that, such that $f(z)$ is continuous on L.
F: Hold on, let me write this out properly. L is— I guess I'll use the abbreviation [writes SICOPOC]. I never actually used it. Uh, [writes 'such that'] you said $f(z)$ is continuous on L?
Int: Uh huh, yeah.
F: [writes f is continuous on L] . Ok
Int: And so I'd like you guys to find all possible values of the integral of that function over L.
F: So we want this [writes $\int_L f(z)\, dz$]
Int: So we can define L to be different curves, but find all the different values you could get for the integral of that thing over L.
F: Ok. Um—

Task 11 required participants to consider the possible values of the integral of the function $f(z) = \frac{1}{z(z^2-1)}$ along a simple, closed, positively oriented curve L such that f is continuous on L (lines 1-4). Recall from Chapter III that the course instructor used the abbreviation SICOPOC for a simple, closed, positively oriented curve. Surprisingly, Frank noted that he never personally used this abbreviation during the course, but wrote down inscriptions summarizing the information provided (lines 5-13). Consistent with his previous preference for using an antiderivative in the last few tasks, Dan authored the suggestion, "Can't you just put it into the antiderivative?" (line 14).

Frank decided to pursue an alternate method, and relayed the given attributes of the curve as data (line 15). Using the fact that the curve was simple, closed, and positively oriented, he authored a claim that there are four possible values for the integral, and suggested they "draw it geometrically" (lines 15-17). He also qualified this assertion with the phrase, "I think," expressing some uncertainty that he and Dan would revisit later in the task. Using embodied-symbolic reasoning, Frank drew an Argand plane and plotted points corresponding to poles he discerned from rewriting the function f as $\frac{1}{z(z+1)(z-1)}$ (lines 18-21).

D: Can't you just put it into the antiderivative?
F: Well, hmmm. I would just think of it— it's a simple closed positively oriented curve. So if we draw it geometrically, because there's basically only like 4 options, I think, for the different values. [Draws Argand plane.]
F: The poles of this function are at zero [plots the origin point], and this is just $(z+1)(z-1)$ [writes factored form of denominator.]. So we've got the pole there [points at the origin] of order one, a pole here [draws point at $z = 1$] of order 1 at the point 1, and a pole here [draws point at $z = -1$] of order 1 at the point -1.
D: So you could do the Residue Theorem, right?
F: You could. Or, um, if we think of it in terms of the Cauchy-Goursat Theorem, L could be, like, something like that [draws circular path around the point -1] around one of the points. It could be around two of the points [draws path around both 0 and -1]. It could be around all three of the points [draws a circular path around all three]. Or it could be around none of the points [draws a circular path containing none of the points, then looks at me for approval]. Uh, geometrically speaking, I think those are the only options.
D: So if L is around none of them, it [the integral] is zero.
F: So yeah. If we have the integral on L of $f(z)$ such that L contains none of the points, it'll just be 0 [writes $\int_L f(z)\,dz = 0$].
F: If it contains— hang on, we should probably write this nicely.

Using these data, Dan authored a formal claim that they could implement the Residue Theorem, and qualified the claim with the question, "right?" (line 22). Frank agreed but once again preferred a different approach, supplying the Cauchy-Goursat Theorem as a formal warrant for a subsequent claim. In particular, Dan used this theorem to author a formal-embodied claim that the integral along a path containing none of the poles is zero (line 29). Frank contributed backing for this warrant's validity by sketching possible paths L surrounding none, one, two, and all three of the poles, thus articulating why the warrant applies to at least one of these cases (lines 23-27; see Fig. 154). Note that Frank did not consider several other potential paths L, namely the two other possible ways to include two of the poles. For instance, L could also surround either the points 0 and 1, or -1 and 1, and these different choices of paths can affect the value of the integral. However, Dan and Frank did eventually notice this issue, as I discuss later.

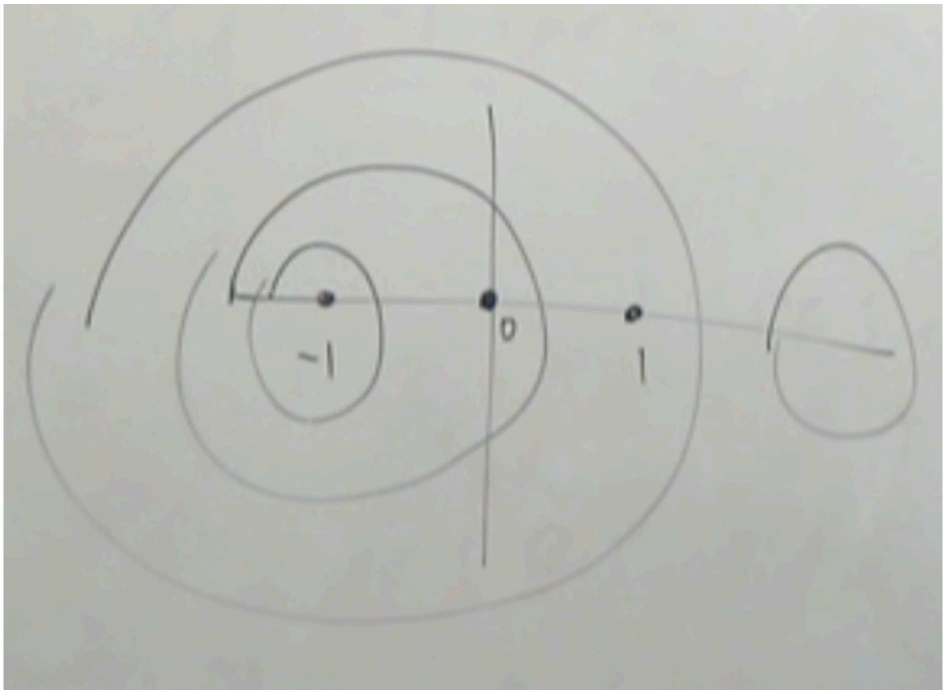


Figure 154. Frank's possible paths for L in Task 11.

After authoring these embodied examples, Frank looked at me for approval, and qualified his backing with the statement, "Uh, geometrically speaking, I think those are the only options" (lines 27-28). After Dan authored the aforementioned claim from line 29, Frank re-voiced this claim as spokesman, and wrote the supporting symbolic inscription $\int_L f(z)\,dz = 0$ (lines 30-31). Frank started to consider the next case wherein *L* contains a pole, but qualified this first argument with the phrase, "hang on, we should probably write this nicely" (line 32). This modal qualifier signaled the start of a second argument, Argument 2. Argument 1 is summarized with the Toulmin model depicted in Figure 155 below.

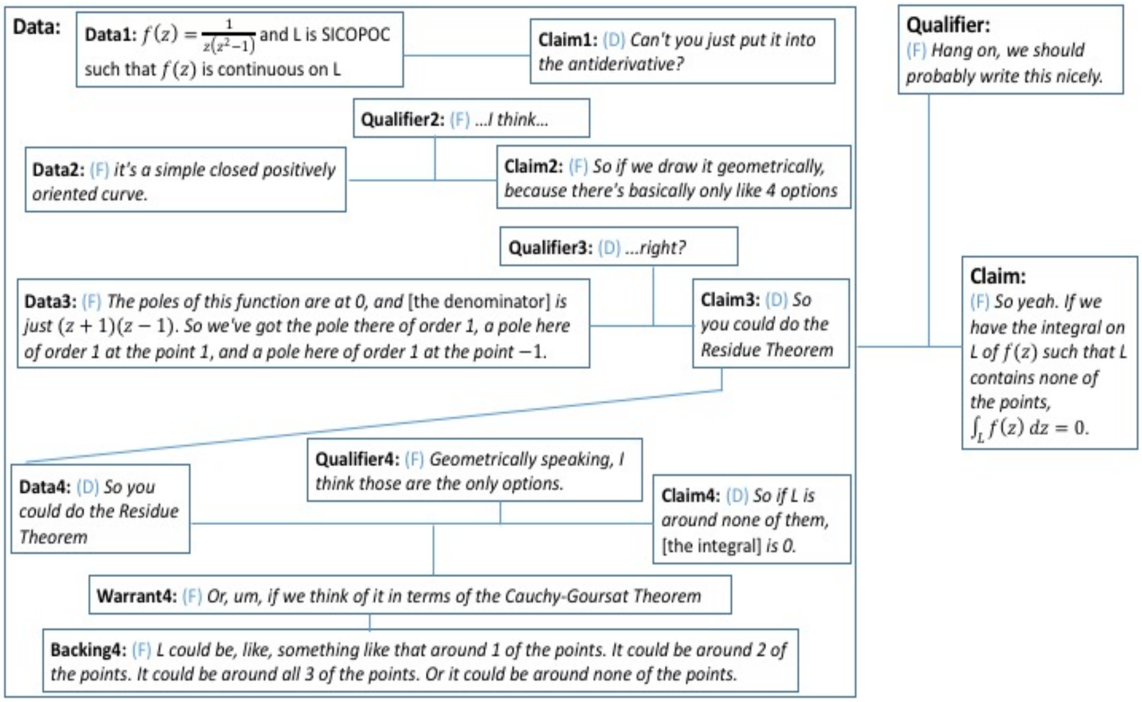


Figure 155. Dan and Frank's Toulmin diagram for Argument 1 for Task 11.

In accordance with Frank's qualifier commencing Argument 2, I suggested that it might be helpful to label the potential paths as L_1, L_2, and so on (line 33). Frank agreed, and started labeling his drawn paths (lines 34-35). However, he quickly changed his mind regarding how he wanted to label these paths, as evidenced by the qualifier, "Wait, you know what? Let's be smart about this" (lines 35-36). He erased his recent labels and authored a claim that, given one of his paths contained no poles, he could label this path L_0 (lines 36-37). Dan agreed (line 38), and Frank similarly concluded that the paths he had drawn surrounding k poles could be labeled L_k for $k = 1, 2, 3$ (lines 39-40). As such, he implemented embodied-symbolic reasoning, corresponding his symbolic labels with the number of poles each path surrounded in the diagram he drew. Relaying their previous claim that the integral using the path L_0, Frank additionally claimed that the integral along L_1 is $2\pi i$ but expressed this claim as more of a question. Rather than look at me for approval this time, Frank looked over at Dan, who began to verify Frank's claim but qualified this verification with the phrase, "umm, wait" (line 43). Because Dan's verification comprised a separate complete argument, I present his response as Argument 3; Argument 2 is summarized in Figure 156.

Int: And you can maybe make an L_1, L_2, etc.
F: Sure, yeah. We'll kind of correspond these [erases his recent inscriptions]. So this is L_1 [labels the path around none of the singularities L_1], this is L_2 [labels the path around just -1], this is L_3. Wait, you know what? Let's be smart about this [erases the labels just drawn]. This one contains no points so this one can be L_0 [labels the path around none of the singularities]
D: Ah, yes.
F: This one can be L_1 [the one around -1] and this one can be L_2 [the one around -1 and 0], and this one can be L_3 [the path around all 3] since it contains all three.
F: So the integral on L_0 of $f(z)$ is zero [writes this], by the Cauchy-Goursat Theorem.
F: The integral on L_1 of $f(z)$ is [pauses] $2\pi i$? [Looks over at D]
D: Umm, wait. Cuz you'd have, for L_1, so you'd have your $f(z)$ is $\frac{1}{z(z-1)}$, and then you'd evaluate that at uh, -1.
F: Sorry, say that again?
D: So, okay. So it's just right off the— so you have like this formula [writes $f(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{z-z_0}$]. Right?
F: Right.
D: So $f(z)$ in the L_1 case is [writes $f(z)$ =] 1 over—
D: z times $(z-1)$ [writes $f(z) = \frac{1}{z(z-1)}$]. So you have to evaluate this [his new "$f(z)$"] at z_0 which is -1. So you'd have 1 over -1(-2), which is 1/2, times $2\pi i$, which is just πi.
F: Wait hang on. But does that mean we can get different results based on different poles?

Dan proceeded to author a formal-symbolic datum that implicitly used Cauchy's Integral Formula to dictate how to symbolically represent the integrand (line 43). With this manifestation of the integrand in mind, he concluded that they should evaluate the expression $\frac{1}{z(z-1)}$, which he referred to as $f(z)$, at the point $z = -1$ (lines 43-44). Because Dan used the symbolism $f(z)$ to denote a function different from the given integrand, he acted as ghostee. Perhaps due to the fact that $f(z)$ now represented two distinct formulas, Frank asked Dan to repeat himself (line 45). Dan responded with formal-symbolic reasoning, stating an incomplete version of Cauchy's Integral Formula for Derivatives: $f(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{z-z_0}$ (line 46). In particular, his equation included the variable n only outside the integral, but not as part of the integrand. For reference, the complete equation should read $f(z_0) = \frac{n!}{2\pi i} \int \frac{f^{(n)}(z)}{(z-z_0)^{n+1}}$. Accordingly, it appeared that Dan meant to use the case where $n = 0$.

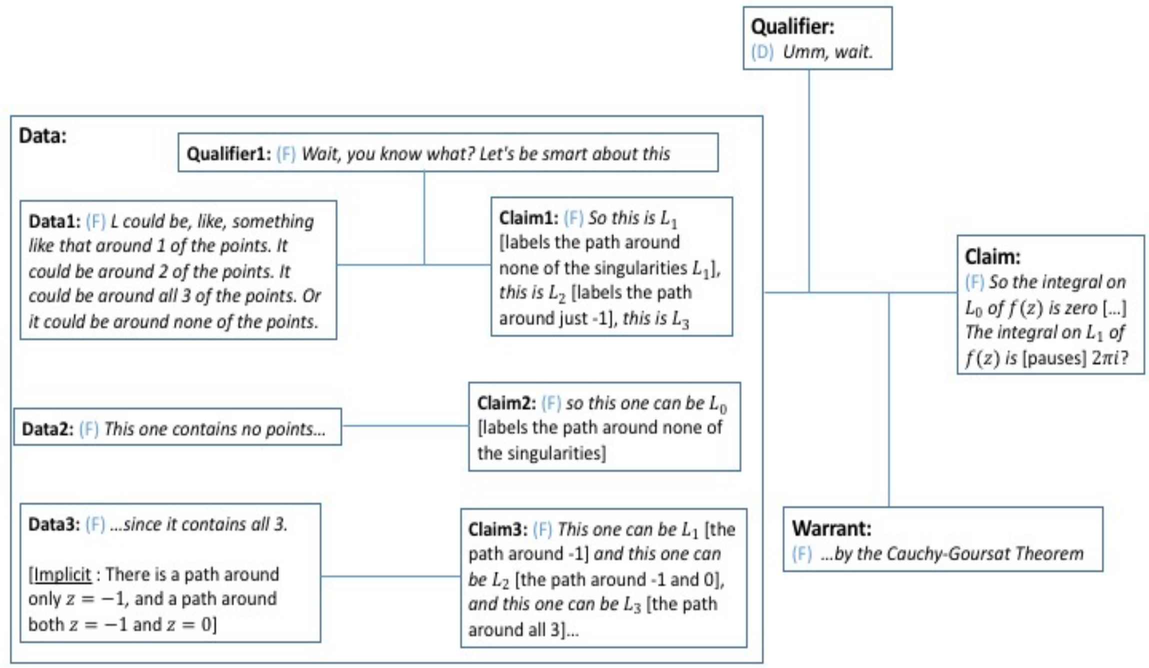


Figure 156. Dan and Frank's Toulmin diagram for Argument 2 for Task 11.

Nevertheless, Frank agreed with Dan's symbolic inscription (line 47). As spokesman, Dan re-voiced his prior claim from lines 43-44 that "you'd evaluate [his new f] at, uh, -1" clarifying that here $z_0 = -1$ (lines 49-50). This time, Dan additionally provided a formal-symbolic warrant that this function is $f(z) = \frac{1}{z(z-1)}$ in the context of the path L_1 (lines 48-49). As such, he symbolically concluded that the integral should be $\frac{1}{-1(-2)} 2\pi i = \pi i$ (lines 49-50). However, after Dan came to this conclusion, Frank realized the aforementioned issue regarding obtaining different answers for the integral depending on which poles the path surrounds. He articulated this realization via the qualifier, “Wait hang on. But does that mean we can get different results based on different poles?” (line 51). This insight opened a new fourth argument, as I detail following Figure 157, which summarizes Argument 3.

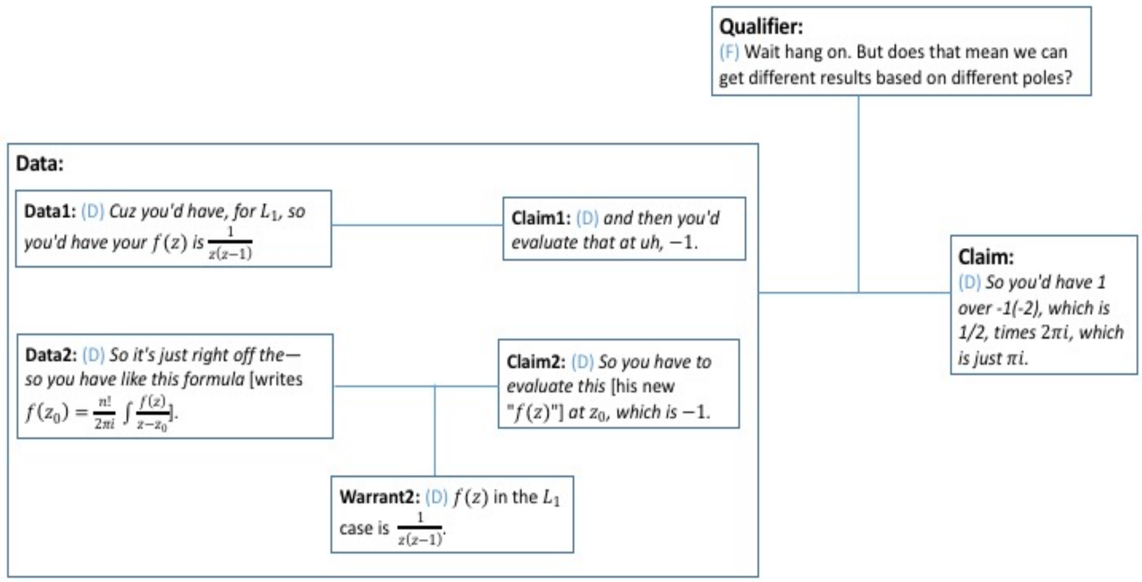


Figure 157. Dan and Frank’s Toulmin diagram for Argument 3 for Task 11.

Frank commenced Argument 4 by elaborating on his concern expressed at the end of Argument 3. First, as spokesman, he recapitulated their original choice of L_1 as well as the resulting value of the integral, πi (line 52). But then he introduced an alternate description of L_1, such that this curve only encloses the origin (lines 52-53). Using this embodied datum, he authored a formal-symbolic claim that symbolically identified “$f(z)$” to be $\frac{1}{(z+1)(z-1)}$ based on the formal Cauchy Integral Theorem (lines 53-54). Consequently, Frank implemented this theorem to author a warrant involving evaluation of this new function f at the point $z = 0$ (lines 54-55). Multiplying this result of -1 by $2\pi i$ as before, Frank used symbolic reasoning to conclude that the value of the integral along this new path is $-2\pi i$ (lines 55-56). He then named this new path L_1^* to distinguish it from the previous path L_1(line 57).

F: So based on our definition of L_1, we're getting πi [writes $\int_{L_1} f(z) = \pi i$]. But let's say L_1 is the curve that just contains the point 0. Then this $f(z)$ [the one D renamed] is actually [erases the denominator of D's new $f(z)$ for Cauchy Integral Formula] $(z-1)(z+1)$. And we want that [points to his new $f(z)$] evaluated at the point 0, so that's just 1 over $-1(1)$ which is just -1, which is -1. In which case it'd be $-2\pi i$ [writes $-2\pi i$].

F: So let's say L_1^*. The integral along that of $f(z)$ is actually $-2\pi i$? Yeah, so I guess this is getting a little more complicated than I anticipated, because um, if, if L_1 is a curve that contains solely the point -1, then as Dan showed, well it's going to be πi. If L_1 is a curve containing solely the point 0, it's going to be $-2\pi i$ [points to the answer he calculated in line 56]. If L_1 is a curve containing solely the point 1, then this becomes— [erases the denominator in "new f(z)" and writes $z(z+1)$]. In which case this is um—

D: Oh wait I should just correct this, because it's still there but n is zero [changes denominator in original CIF inscription to be raised to (n+1)].

F: …in which case this become $1(1+1)$. Or no, I'm sorry, it contains, no yeah yeah. It contains the point 1. Which is 1/2 [writes $\frac{1}{1(1+1)} = \frac{1}{2}$]. So we have that times $2\pi i$. So that's just πi [writes $= \pi i$].

D: Yeah. So we have $-\pi i$, πi, and— what was the other one?

F: Um we never got $-\pi i$. The other one was positive I think.

D: Oh yeah you're right

F: So it was either πi, or $-2\pi i$ [writes the other option for an answer of $\int_{L_1} f(z) = -2\pi i$]. Where this one is the case where L_1 contains the point 0 [draws arrow pointing toward the $-2\pi i$ and labels as 0], and this is the case where it contains either -1 or 1 [points to the other answer of πi]. Umm—

Seemingly surprised by the result of this last integral, Frank checked his result and uttered the qualifier "so I guess this is getting a little more complicated than I anticipated" (lines 57-58). As spokesman, he reiterated the two different answers they obtained by using the respective paths L_1 and L_1^* (lines 58-60). Using a process analogous to the calculations for the other two paths, Frank argued that $f(z) = \frac{1}{z(z+1)}$ in the case where the path encloses solely the pole at $z = 1$ (lines 60-62). At this time, Dan suddenly recognized his omission of n from the symbolic inscriptions pertaining to Cauchy's Integral Formula for derivatives (lines 63-64). Meanwhile, Frank continued his calculation by authoring a formal-symbolic warrant, evaluating $f(1) = \frac{1}{1(1+1)}$, but briefly doubted whether he was plugging in the correct value of z (line 65).

To resolve this doubt, Frank provided backing for his warrant's correctness, citing the fact that the path "contains the point 1" (lines 65-66). This backing was embodied-symbolic in nature because it coupled a geometric property of the path with symbolic inscriptions evaluating the function at a particular value. With his concern abated, he finished stating his warrant by arguing that the value $\frac{1}{1+1} = \frac{1}{2}$ needed to be multiplied by $2\pi i$, and employed symbolic reasoning to author a claimed result of πi (line 66). Dan relayed the previous answers obtained from integrating along a path containing one pole, but mistakenly mentioned $-\pi i$ as part of the list, and Frank reminded Dan that they had not obtained such an answer. As spokesman, Frank recapitulated the two answers of $-2\pi i$ and πi, and reiterated the corresponding paths used to obtain these answers (lines 70-72). This fourth argument is summarized in Figure 158 below.

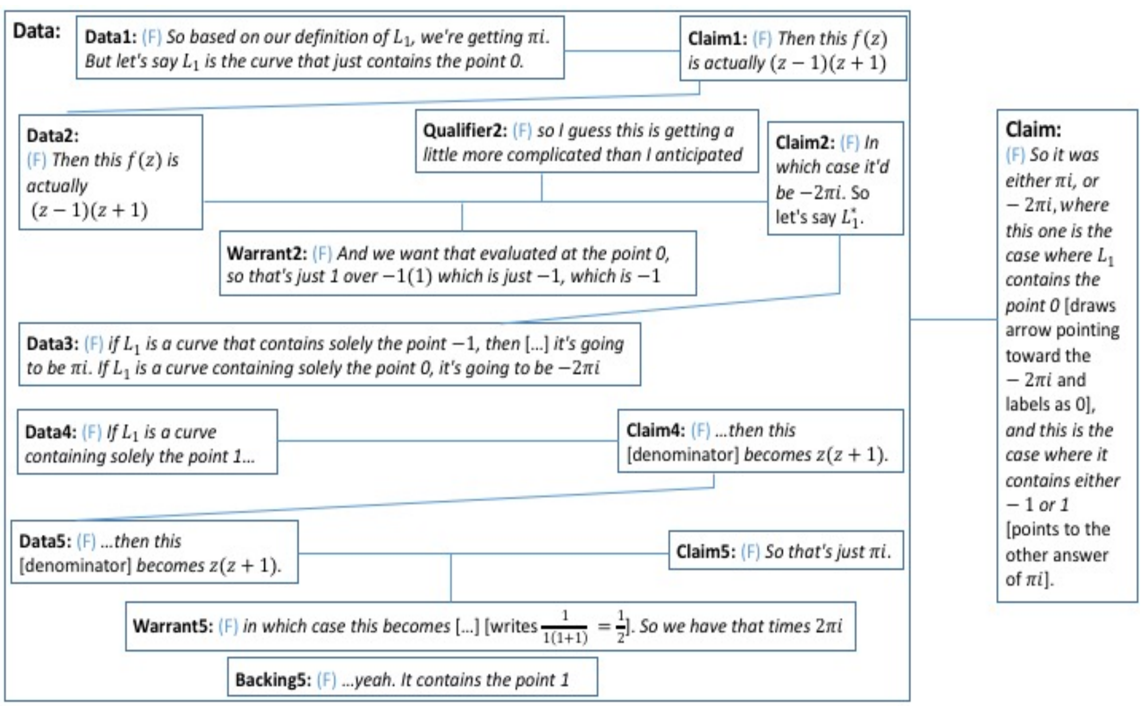


Figure 158. Dan and Frank's Toulmin diagram for Argument 4 for Task 11.

Next, Dan began a fifth argument by considering what results were possible if the path of integration contains two of the aforementioned poles (line 73). Frank authored an embodied claim that there were three possible paths for L_2, and provided an embodied warrant describing the three potential pairs of poles that the path could enclose (lines 74-75). With these three possible paths in mind, Dan authored a formal claim that "it'd probably be easier to do [the] Residue Theorem" (line 76). Frank agreed, and Dan authored the rebuttal that "otherwise you'd have to do partial fraction decomposition" (lines 78-79). Dan also clarified his claim with the warrant that this theorem makes the symbolic calculations easier when considering paths that surround two or more poles (lines 79-80). A summary of Argument 5 is depicted in Figure 159.

D: Now if it contains two—
F: Well then we have three possibilities again for L_2, because either it contains -1 and 0, 0 and 1, or -1 and 1.
D: Yeah and it'd probably be easier to do Residue Theorem?
F: Yeah.
D: Cuz otherwise you'd have to do partial fraction decomposition, or you'd have to do— yeah you'd have to do partial fraction decomposition. So to do Residue Theorem is easier when you're doing two or more [singularities within the path].
F: Yeah so I guess we could use the Residue Theorem for that.
Int: Well so I have a couple quick questions here. So the $f(z)$ that you defined to be, like, $\frac{1}{z(z+1)}$
F: Yes. I guess it's simpler if we call it g, um, just so it doesn't get confused for the definition of f that you gave us.
Int: Ok yeah that was kind of— [F replaces "new f" with g throughout inscriptions].

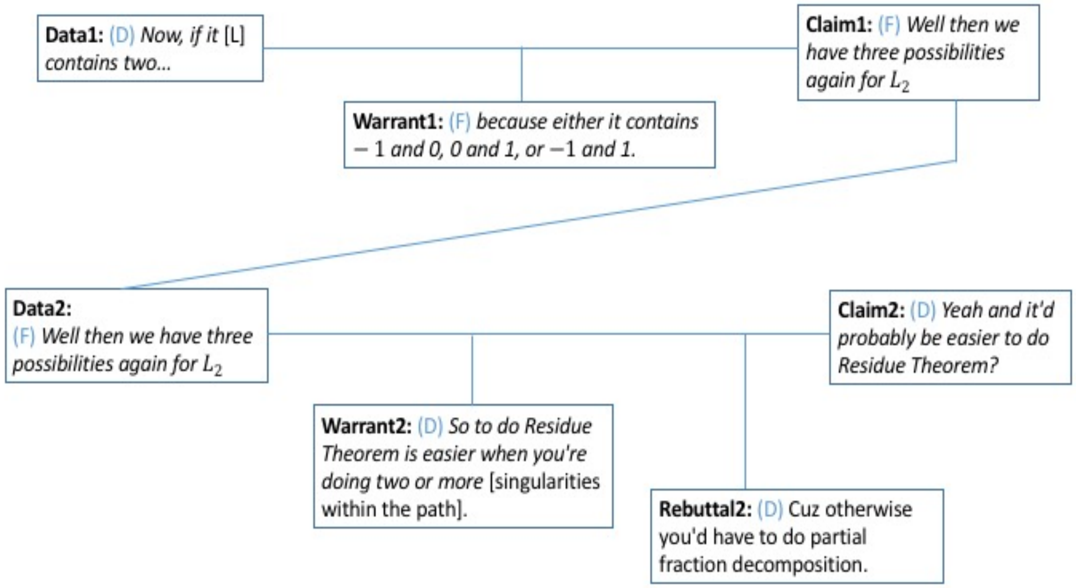


Figure 159. Dan and Frank's Toulmin diagram for Argument 5 for Task 11.

At this time, I asked a couple follow-up questions about Dan and Frank's recent arguments. Specifically, I first asked about the function they labeled $f(z)$, given that I initially provided them with the integrand introduced as $f(z)$ (line 82). Apparently having recognized this potential ambiguity beforehand, Frank interjected and conceded that they should have labeled their other function g so as to avoid this issue (lines 83-84). He quickly adjusted their previous inscriptions on the board to reflect this change in notation (line 85). This catalyzed a short sixth argument in which Frank recapitulated the role of this function now denoted g.

F: Basically, we're just finding some function g that you can put in the numerator such that the definition provided for f [points at the $f(z)$ formula given in task] corresponds with the form required of the integral [circles the CIF inscription in the air with his pen]. So if um— if we're supposing at z_0 where the discontinuity occurs at positive 1, then this is going to be $z-1$ on the bottom there [points to denominator of CIF]. So to complete that form [points at board, presumably to original $f(z)$], all we need the g to be is $\frac{1}{z(z+1)}$. So that's where that's coming from.

Int: So both of those functions are g then? Under the integral and on the left side?

F: Uh yeah [writes in an n^{th} derivative of the g on the left hand side of CIF equation]. In the sense that the integral over the curve C [writes a C as the path in the integral in CIF], or L in our case— uh so I'll write L [erases C and writes L]. The integral over the curve L, um, of g over $(z-z_0)$ to the $n+1$, where z_0 is some discontinuity of g, times $\frac{n!}{2\pi i}$, is equivalent to the n^{th} derivative of g, evaluated at that discontinuity. Cauchy's Integral Formula.

F: And basically all we did for the L_1 case is we put the function f into various forms that made it suitable to work with for Cauchy's Integral Formula. But you're saying we should do residues for that? [Turns to Dan]

As spokesman, Frank proffered a warrant clarifying that this function g allowed him and Dan to put the integrand in a form amenable to the application of Cauchy's Integral Formula (lines 86-88). He then illustrated this in the specific case where their path surrounded the point $z=1$, in which case Frank concluded that they symbolically altered the integrand so that it had a denominator of $z-1$ (lines 89-90). From this claim, he additionally surmised that this choice of denominator yielded a choice of $g(z)=\frac{1}{z(z+1)}$ (lines 90-91). Frank explained, via a formal-symbolic warrant, that this choice of g served to "complete that form," namely the form dictated by the theorem (line 90). This sixth argument is summarized in Figure 160.

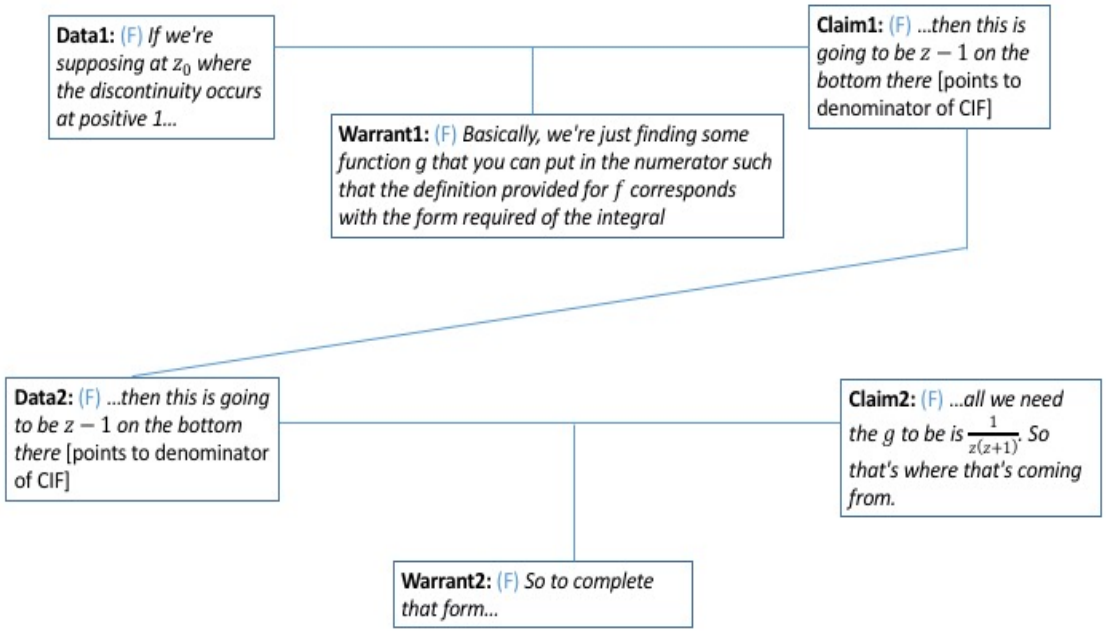


Figure 160. Dan and Frank's Toulmin diagram for Argument 6 for Task 11.

To ensure that I understood how Frank and Dan used this function g, I asked Frank if both of the functions in the revised inscriptions for the Cauchy Integral Formula, $g(z_0) = \frac{n!}{2\pi i} \int \frac{g^{(n)}(z)}{(z-z_0)^{n+1}}$, where actually supposed to be g (line 92). Frank responded by essentially reciting the various symbolic pieces in the theorem (lines 93-97), and as spokesman, changed the name of the curve from C to L in his inscriptions to align with the notation I initially provided. Continuing as spokesman, Frank re-voiced his first warrant from Argument 6 to underscore the role of g in their work once again. When finished, Frank inquired about Dan's claim in Argument 5 regarding the relative ease of employing the Residue Theorem for this task (lines 99-100). This also happened to be relevant to the other clarification question I intended to ask Dan before moving on (lines 101-102).

Int: Well, so that brings me to my other question. So you were saying Dan, that if we didn't use residue theory that we'd have to use a partial fractions decomposition?
D: Yeah [but gives me a look that suggests he's unsure].
Int: So why do you say that?
D: Um, because we still want to use this formula, right? [Points to CIF]. Um, and if you have multiple discontinuities— it's like the extended Cauchy-Goursat Theorem, right?
F: Yeah.
D: So, if you have one— one simply, um— or one positively oriented uh, closed contour, then the, um, integral is the same as the sum of the—
F: ...of the smaller ones, yeah.
F: So we could just use the extended Cauchy-Goursat Theorem then, in the sense that [draws new plane] here's minus 1, here's 0, and here's 1 [plots these points]. Let's say our curve contains 0 and 1 [draws path containing 0 and 1]. Then we can choose small curves that contain 0 and 1 separately [draws little curves around 0 and 1], evaluate the integral about them, which we already did [points to the answers for L_1] right here, and then add them to each other.
D: Oh yeah you're right. Yeah sweet.

When I asked Dan about his rebuttal to the claim about residues (from line 78), he verbally maintained that a partial fraction decomposition would be necessary, but his facial expression indicated some uncertainty (line 103). In response to my inquiry for an explanation (line 104), Dan began a seventh argument by authoring a formal datum that they "still want to use [Cauchy's Integral] formula" (line 105). As such, he concluded that when the path of integration contains multiple discontinuities, the formula becomes similar to that in the extended Cauchy-Goursat Theorem (lines 105-106). Dan qualified this assertion with the word "right?" (line 106), and Frank agreed with Dan's claim (line 107).

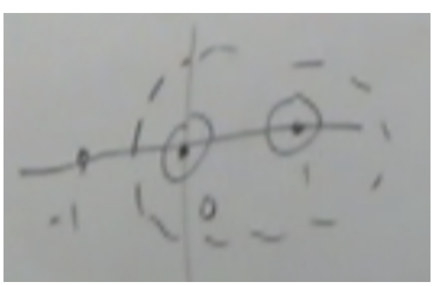


Figure 161. Frank's example paths illustrating the Extended Cauchy-Goursat Theorem.

Dan continued to explain this connection by arguing that if the contour is "simpl[e], um—or one positively oriented uh, closed" then the value of the integral over this contour is equivalent to a sum (lines 108-109). Frank interjected as spokesman and characterized this value as the sum of "the smaller ones" (line 110). As relayer, he affirmed Dan's suggestion to use the extended Cauchy-Goursat Theorem, and illustrated the theorem's applicability using an example. Instantiating embodied reasoning, Frank authored a datum by drawing a new plane as well as a path surrounding the points 0 and 1 (lines 111-113; see Fig. 161).

As spokesman, Frank articulated a warrant that explained the embodied and symbolic connections between this example path and the theorem. Using formal-embodied reasoning, Frank argued that the theorem allowed them to draw two small paths surrounding the points 0 and 1, respectively (lines 113-114; see Fig. 161). Then, according to Frank, they could evaluate the integrals around each of these smaller paths, and pointed to their existing symbolic inscriptions corresponding to these two values (lines 114-115). Using formal-symbolic reasoning, Frank concluded that the theorem allows them to add these two integrals to obtain the value of the integral along the larger original path (line 115). A summary of Argument 7 is depicted in Figure 162.

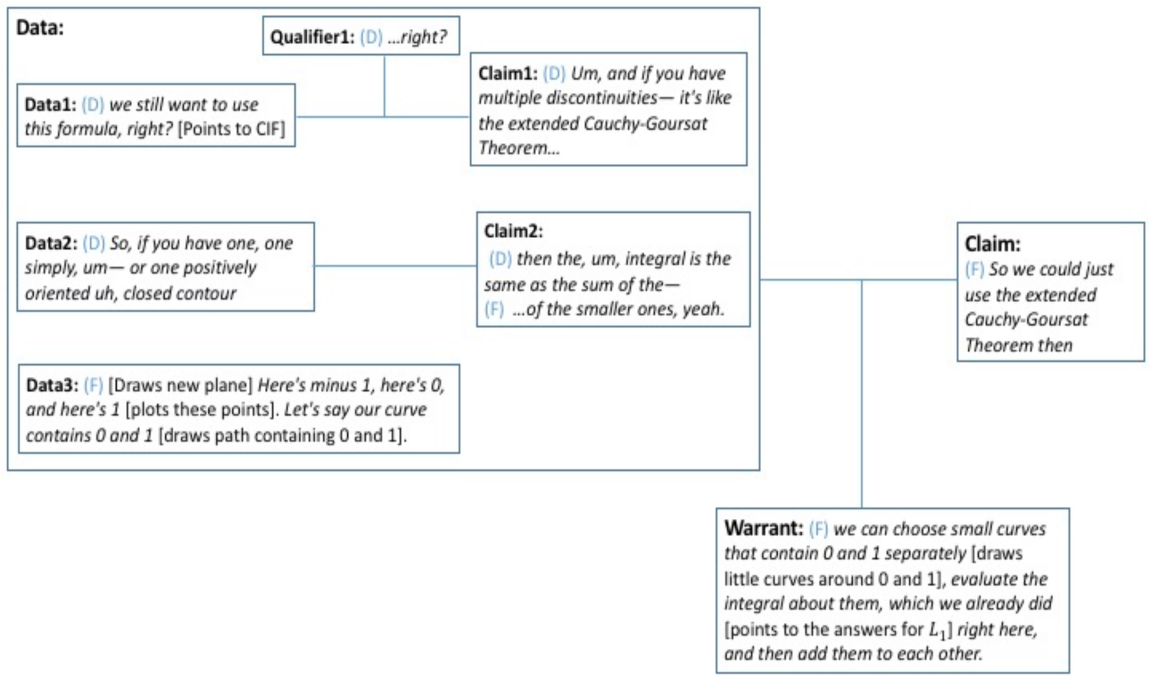


Figure 162. Dan and Frank's Toulmin diagram for Argument 7 for Task 11.

F: So for L_2— I mean we can do the same thing for L_3. We can just keep using the extended Cauchy-Goursat Theorem.

D: Oh, yeah add up the ones that we know. Oh yeah.

F: So yeah. So if L_2 includes -1 and 0, then the— yeah, so, if this is L_2 [writes $\int_{L_2}$] where L_2 includes— here, we'll do it exactly like this [labels L_2 as the curve containing 0 and 1 in his previous picture] 0 and 1, then [writes $f(z)$ in the integrand] the value is simply um, $\pi i + (-2\pi i)$, which is $-\pi i$ [writes = $-\pi i$].

F: If it includes -1 and 0 [points to picture], we have this these same values [points to the answers for the L_1 case] so it's going to be $-\pi i$ again.

F: If it includes -1 and 1 [points to diagram], it'll be $2\pi i$. So those are the three possibilities for that.

F: For all three— uh, if it contains all three of them— so the options are either $-\pi i$, or $2\pi i$?

D: Yeah. Isn't that backwards? I thought you said πi and $-2\pi i$.

F: Those were for the L_1 's [points to the list of answers for L_1]. So if L_2 contains 0 and 1, where this is the one about 0 [draws label for $-2\pi i$] and this is the one about plus or minus 1 [labels the πi answer], then if L_2 contains 0 and 1, then it's $-2\pi i$.

D: So L_3 should be 0.

F: L_3 should just be— yeah, $2\pi i - 2\pi i$. So the integral around L_3 of $f(z)$ is just 0 [writes $\int_{L_3} f(z) = 0$].

Int: So how do we get the 0, then?

F: Because we again use the extended Cauchy-Goursat Theorem. And if L_3 contains all the points, then for -1 we have πi, for 0 we have $-2\pi i$, and for 1 we have πi [points to the respective answers from the L_1 case as he's listing them verbally]. And if you add up all those values you get 0.

Next, Frank suggested that he and Dan continue to apply the extended Cauchy-Goursat Theorem in the L_3 case (lines 117-118). As spokesman, Dan agreed, and clarified that doing so would allow them to express the integral as the sum of values they already calculated (line 119). Because they had only outlined a method to calculate the integral along an "L_2" path, Frank began an eighth argument by considering the exact values of the integral along each possible L_2 path. He first authored an embodied datum considering the case they ended Argument 7 with, namely a curve enclosing the points 0 and 1 (lines 120-122). Using this particular L_2, Frank authored a symbolic claim that the integral is $\pi i + (-2\pi i) = -\pi i$ (lines 122-123). Note that the two terms in this sum corresponded to the two values of their previous integrals around L_1 when the path enclosed the points $z = 1$ and $z = 0$, respectively.

Similarly, Frank concluded that if L_2 contains the points -1 and 0, then the integral around this path is also $-\pi i$. To support this assertion, he authored a symbolic warrant clarifying that the corresponding values for the integral around L_1 surrounding each of these two points happened to also be πi and $-2\pi i$ (lines 124-125). Continuing in this manner, Frank claimed that the integral around a path containing the points -1 and 1 is $2\pi i$, concluding the possible values for the integral around L_2 (line 126). He then moved on to the L_3 case, and relayed the two possible distinct values for the integral around L_2 (line 127). However, Dan challenged Frank's statement because he remembered the two answers as πi and $-2\pi i$ rather than $-\pi i$ and $2\pi i$ (line 128). In response, Frank reminded Dan that the πi and $-2\pi i$ answers corresponded to integrals around L_1, and recapitulated his answers for the L_2 cases (lines 129-131).

With this disagreement resolved, Dan authored a symbolic claim that the integral around L_3 "should be 0" (line 132). As spokesman, Frank agreed and clarified that they should obtain $2\pi i - 2\pi i$ (line 133). I subsequently asked Dan and Frank for a little more detail about how they obtained an answer of 0 (line 134). Frank responded by authoring backing for his previous warrant's correctness, relaying the three answers obtained from integrating along the possible L_1 paths (lines 135-137). Frank reiterated that they used the Extended Cauchy-Goursat Theorem to add up these three values and obtain their answer (line 137). A summary of Argument 8 is depicted in Figure 163 below.

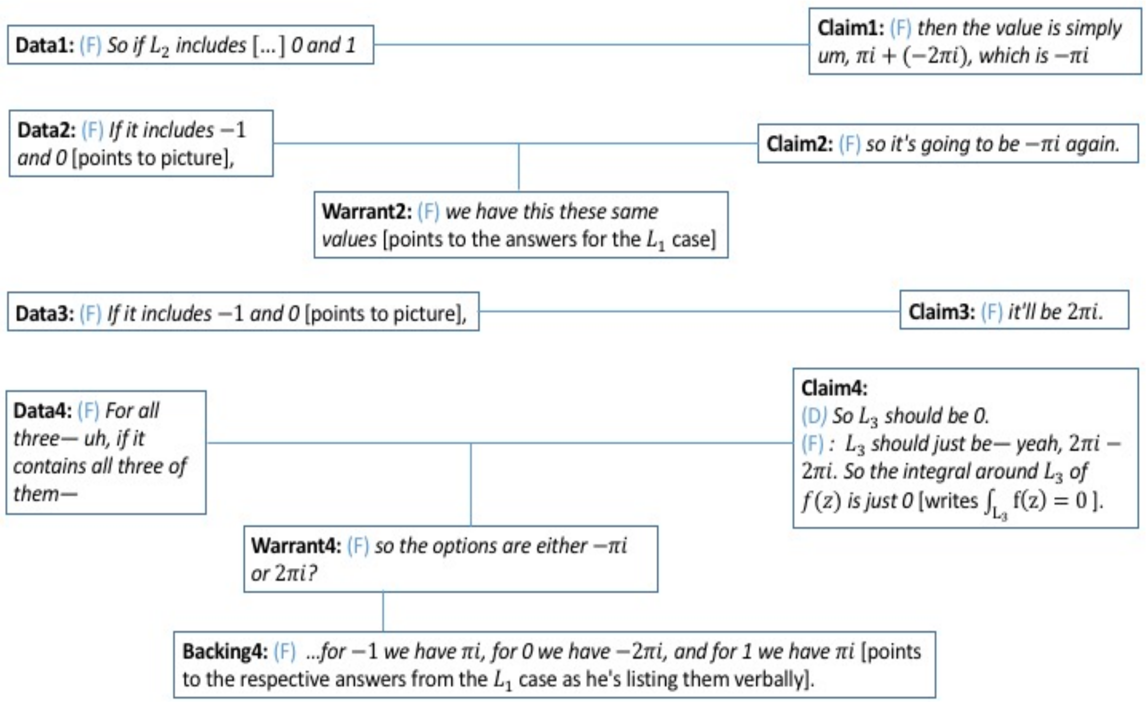


Figure 163. Dan and Frank's Toulmin diagram for Argument 8 for Task 11.

Int: Ok perfect. Um so, you found for L_0 that that integral was 0 also, and L_3 turned out to be 0. Does that seem curious to you guys? Do you think that's just a coincidence or do you think that's something—
D: I think that's probably coincidental [seems unsure, and looks at F]. I think it's specific to this function.
F: I guess, I think it only— I think it works because—why does it work? Ok, it works for this function because the uh— the values basically cancel each other out. I'm trying to think if there's like a greater explanation for why that happens.
D: Well cuz imagine if your function is this instead. $f(z) = \frac{1}{z(z+1)}$ [writes this]. Then if you have your um, curve containing all the singularities then it's still not going to be 0, like we just showed [points over to recent inscriptions].
F: Right.
D: So it's not— you can't generalize that to any function. It's coincidental I think.
F: Right. It can only be generalized if the curve contains singularities where the integral about each singularity is the exact same. So I guess if all the singularities are symmetrical in a way? Then it works if your curve contains all of them?

Once Dan and Frank finished providing the different possible values of the integral, I asked a follow-up question about the fact that they obtained the same answer of zero for integrals around different paths (lines 138-139). In response, Dan claimed that this was "probably coincidental" but looked unsure as he glanced at Frank (line 140). Frank tentatively agreed, and qualified Dan's claim with the phrase, "I guess" (line 141). Using symbolic reasoning, Frank authored a warrant explaining that "the values basically cancel each other out," presumably referring to the summands in the L_3 calculation (lines 141-143). Dan authored a second datum in the form of a symbolic hypothetical function $f(z) = \frac{1}{z(z+1)}$ (line 144).

Using embodied-symbolic reasoning, Dan claimed that the integral around a path enclosing both $z = 0$ and $z = 1$ is "still not going to be 0" (line 145). He supported this assertion with a warrant that referenced their recent symbolic inscriptions (lines 145-146). Frank agreed (line 147), and Dan used his previous claim to conclude more definitively that "you can't generalize that to any function. It's coincidental I think" (line 148). Frank agreed, and authored an embodied-symbolic warrant that supported Dan's assertion. Specifically, he argued that the integral vanishes when "the curve contains singularities where the integral about each singularity is the exact same" (lines 149-150). This led Frank to hypothesize that symmetry played a role in obtaining an integral of zero (lines 150-151). Argument 9 is summarized in Figure 164.

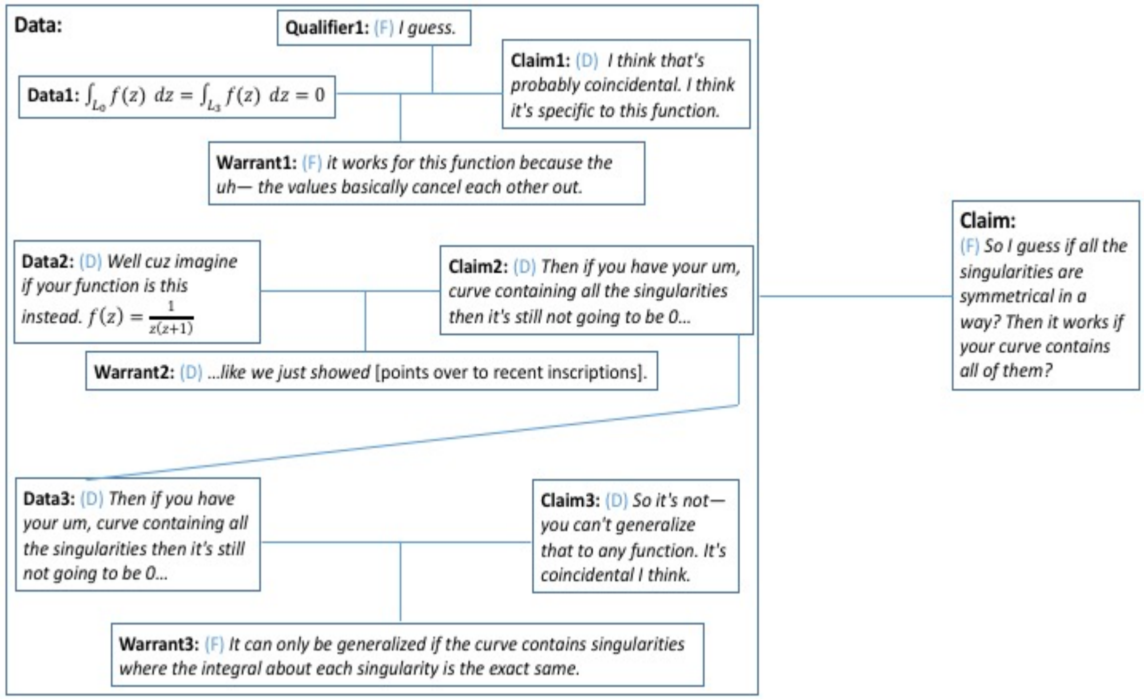


Figure 164. Dan and Frank's Toulmin diagram for Argument 9 for Task 11.

Int: So what made you think this symmetry is at play here?
F: Umm [pauses for several seconds]
Int: Like that they cancel each other out somehow, the two individual integrals?
F: Yeah, in the sense that um, if we put another discontinuity out at 2 and -2 [labels points 2 and -2 on graph] so if we had like a $z^2 - 4$ term here as well [writes in this term in the denominator of $f(z)$] um, then I'm under the impression that if the curve contained all the points, I think it would work out to 0 again [draws path around all 5 points]. Mainly just because the integral about 2 and -2 would end up just cancelling with each other, similarly to what happened here. I think.
Int: Maybe want to just try it real quickly to verify that?
F: Ooh, actually no!
D: I'm going to say no too, because the values wouldn't be the same [points to $\frac{1}{z(z+1)}$ inscription].
F: Hang on, no I don't think that'll work.
Int: What makes you think that won't work now?
F: Because— no wait, hang on. Hang on— do you mind if I erase this, Dan?
D: Yeah.
F: [Erases most of their prior work] Let's clean this up a little bit.
D: My counterexample is not a good counterexample though. Because you'd have different values for the— if we plugged this into the Cauchy-Goursat.

Subsequently, I asked Frank a follow-up question about why he thought symmetry was at play (line 152). This catalyzed a tenth argument, which Frank began with a long pause (line 153). In order to elicit more detail, I relayed their previous observation that the two individual integrals "cancel[ed] each other out somehow" (line l54). Frank responded by altering the given function $f(x)$ to include additional poles at $z = 2$ and $z = -2$. Using embodied-symbolic reasoning, he plotted these two additional points on their diagram and appended a corresponding $(z^2 - 4)$ factor to their symbolic inscriptions (lines 155-156). As author, Frank concluded that the integral around all poles of this new function "would work out to 0 again," and drew such a path in their diagram (lines 157-158). In support of this claim, he authored a warrant that "the integral about 2 and -2 would end up cancelling with each other," as when they integrated the original function (lines 158-159). He also provided a qualifier, "I think," conveying a degree of uncertainty (line 159).

Because of Frank's hesitation, I asked him and Dan if they wanted to verify Frank's claim regarding this altered version of the function f (line 160). Shortly after, however, Frank changed his mind about the value of this new integral (line 161), and Dan agreed (line 162). As author, Dan followed with a symbolic warrant that "the values wouldn't be the same," and pointed to the $z(z + 1)$ portion of the denominator (line 162). As spokesman, Frank again reiterated that he did not think the integral vanished (line 163). I again asked why he thought this, and he erased his previous inscriptions in an effort to "clean this up a bit" (line 167). Dan also authored a qualifier to clarify that his counterexample was not ideal to illustrate their argument (line 168-169). Because Frank erased the majority of the inscriptions and they indicated a desire to pursue a different strategy, I treat this portion as the end of Argument 10, which is summarized in Figure 165.

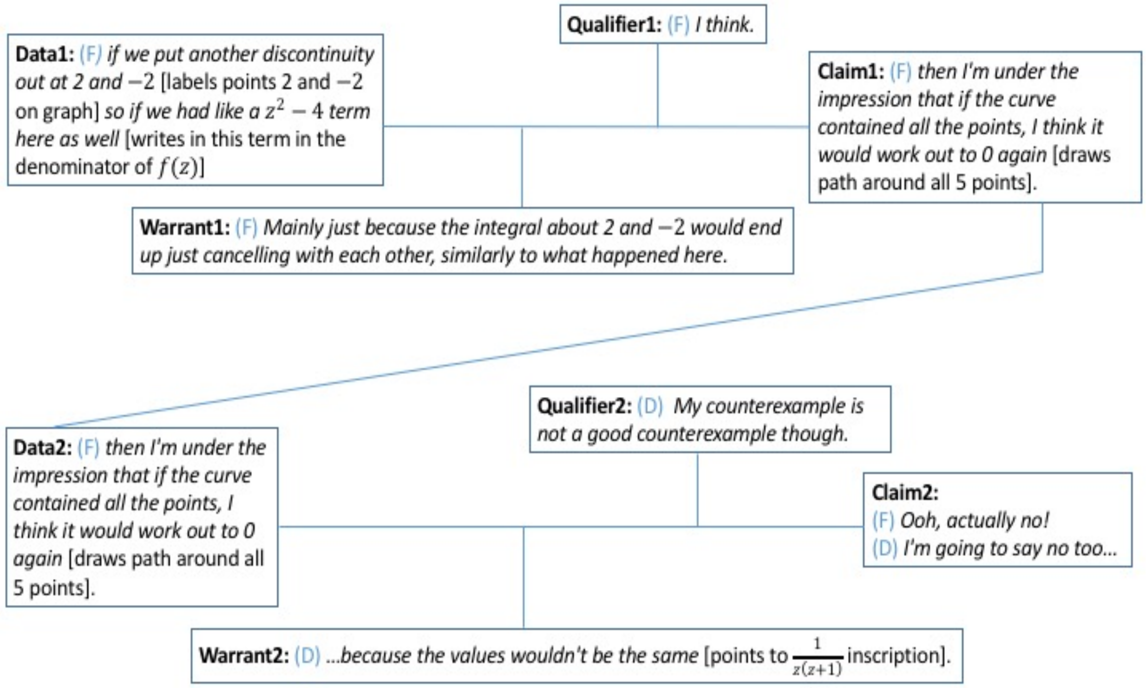


Figure 165. Dan and Frank's Toulmin diagram for Argument 10 for Task 11.

F: Right. Ok so let's suppose that $f(z)$ is actually $\frac{1}{z(z^2-1)(z^2-4)}$ [writes $f(z) = \frac{1}{z(z^2-1)(z^2-4)}$]. Then we know that this is going to be just, again expand to [writes $\frac{1}{z(z+1)(z-1)(z+2)(z-2)}$]. In which case [draws new plane] the poles are going to occur at 0, plus and minus 1, and plus and minus 2 [plots these points]. Ok so, um, we know that the integral on L of, uh, in this case our g function, if we choose to evaluate—
Int: So what's your L here? You're considering the path around all of those?
F: Right. I'm actually trying to figure out exactly how I want to break this up. Do you want to go through the whole thing again with this? [Laughs]
D: No.
F: I don't want to either. Let's just do the 0 case and see what happens. So let's assume that L is just some closed path [draws small path around 0] about the point 0, that doesn't include any of the other zeros, or poles I should say. Then the integral on L— we need a function g of the form [writes $\int \frac{\frac{1}{(z+1)(z-1)(z+2)(z-2)}}{z-0}\,dz$].
F: So that's just going to be this [points to the numerator, i.e. "g"] evaluated at 0. Which is $\frac{1}{1(-1)(2)(-2)}$ [writes this]. Which is just 1/4, in which case the integral over L of $f(z)\,dz$ is just $\frac{\pi i}{2}$ [writes $= \frac{\pi i}{2}$]. Um actually yeah I think it would end up working. Cuz—
D: Cuz you'd just have negatives of— each time you do the formula, you should have the negative of—
F: Yeah. So it should work in the same way. So if we had a curve about all the points, it should— it should end up giving us 0 again.
Int: Okay.
D: If there is symmetry.

Argument 11 opened as Frank altered $f(z)$ again, this time providing the symbolic datum that $f(z) = \frac{1}{z(z^2-1)(z^2-4)}$. As spokesman, he employed symbolic reasoning to rewrite this function as $\frac{1}{z(z-1)(z+1)(z+2)(z-2)}$ (lines 170-171). After drawing a new Argand plane, Frank used this rewritten form of the function as a warrant to conclude that the poles occur at $0, \pm 1,$ and ± 2, and plotted these points on his newly drawn plane (lines 171-173). Frank began to discuss the impact of this change on the integral of this function, but I interjected to ask which path they were integrating along (line 175). Frank replied that he was trying to decide on that, and he asked Dan if they should calculate all possible values of the integral of this function (lines 176-177). They laughed, and Dan replied with a definitive "no" (line 178).

Instead, Frank suggested they focus on the integral along a path enclosing just the origin (line 179). He authored a corresponding embodied datum describing such a path, and drew the path on their diagram (lines 179-181). Subsequently, Frank authored a symbolic warrant that such a path required a choice of "g" that yielded the integrand $\frac{\frac{1}{(z+1)(z-1)(z+2)(z-2)}}{z-0}$ (lines 181-182). He used this warrant to justify a symbolic claim that they should evaluate the numerator of this expression at $z = 0$, and multiply by $2\pi i$ to obtain an answer of $\frac{\pi i}{2}$ (lines 183-184). Frank then authored a qualifier that he thought "it would end up working," referring to achieving an integral of zero using a path surrounding all the poles of this function (line 185). He made this more explicit when he claimed, as spokesman, that "if we had a curve about all the points, it should […] end up giving us 0 again" (lines 187-188). Dan supported this assertion with an incomplete warrant concerning the presence of negatives when evaluating the integrals around subsets of the collection of poles (line 186). Argument 11 is summarized in Figure 166 below.

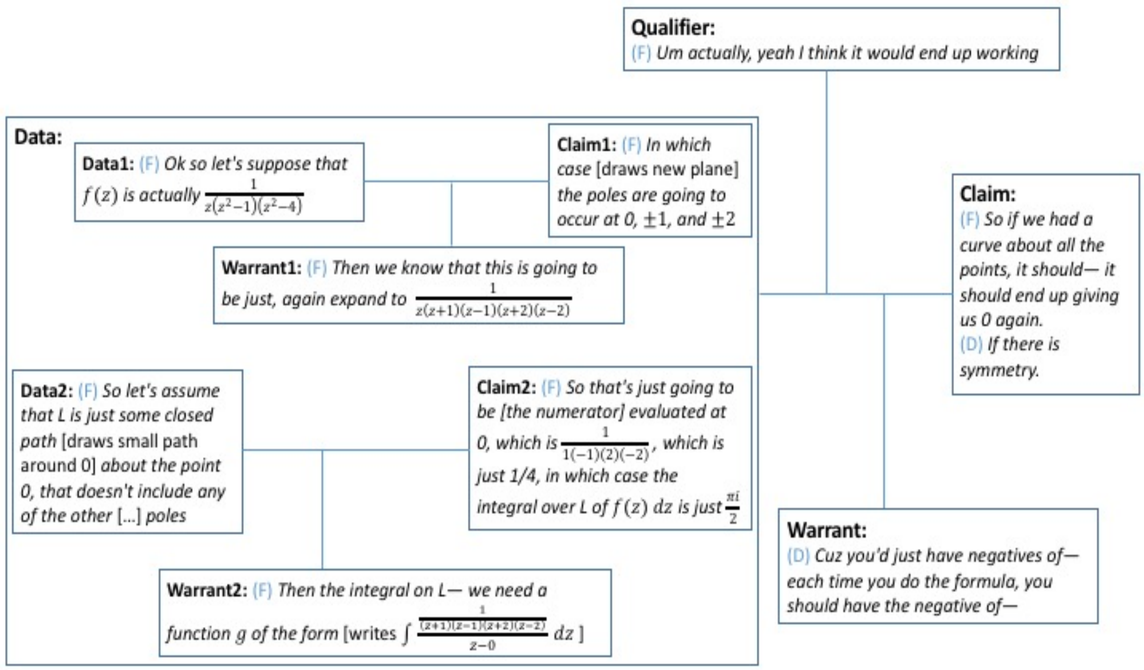


Figure 166. Dan and Frank's Toulmin diagram for Argument 11 for Task 11.

Int: Ok so other than the symmetry, you don't see anything more general at play here, that you would just guess it's automatically zero?
D&F: [Both stare at the board in silence]
Int: I'm not saying that there is, necessarily, but—
D: I don't think so. I mean, if that were true, then— I'm just thinking of it like this. If that were true and any curve that contained all the singularities of some function were to be 0, then why would we even have the extended Cauchy-Goursat Theorem? Because then we would just say, "Oh it's always zero!"
F: Yeah cuz if we chose to make this singularity on the far right 3 instead of 2, um, then it wouldn't be 0 right? It wouldn't just cancel out. I think this is only working because um, yeah, because fundamentally they are negatives of each other [points at the plotted points on the plane] it ends up, uh we end up getting matching values. I think that wouldn't necessarily be the case if the zeros were chosen arbitrarily, or the poles, I should say.
F: Um yeah, so I feel confident in saying that the symmetry is certainly like, pretty important as to why it's always working out to 0 when we include all of them. I'm sure there are geometric arrangements where it would work out to 0 that aren't symmetrical. But I'd have to spend some time to come up with one.
Int: Ok yeah, sounds good. Ok I just have a couple sort of questions that are more general and don't really deal with specific functions any more. But kind of given what we've talked about in the interview here.

Subsequently, I decided to ask one more follow-up question about this observation regarding vanishing integrals, which began a twelfth argument. In particular, I inquired about whether they thought anything other than symmetry caused the integrals over different paths to be zero (lines 191-192). Once again, Dan and Frank stared at the board in silence for quite some time, so I clarified that I was not necessarily implying that there was another obvious explanation (lines 193-194). Dan claimed that he did not think there was anything else at play. He explained by authoring a rebuttal considering what would happen if the integral around a curve containing all poles of the integrand was always zero (lines 195-197).

Frank agreed, and articulated another example wherein the pole at $z = 2$ is replaced with one at $z = 3$ (line 198). He concluded that the integral around a path containing all poles would no longer be zero, but qualified this claim with the word, "right?" Frank also authored a symbolic warrant that "it wouldn't just cancel out" (lines 198-199). Ultimately, Frank confidently claimed that symmetry was the reason the integral vanished when the path enclosed all poles (lines 203-204). As spokesman, he reiterated with a symbolic warrant that "this is only working because […] fundamentally they are negatives of each other […] we end up getting matching values" (lines 199-201). Finally, Frank qualified this argument with a statement that other "geometric arrangements" of asymmetric poles could still yield a vanishing integral, but he would "have to spend some time to come up with one" (lines 204-206). Argument 12 is summarized below in Figure 167.

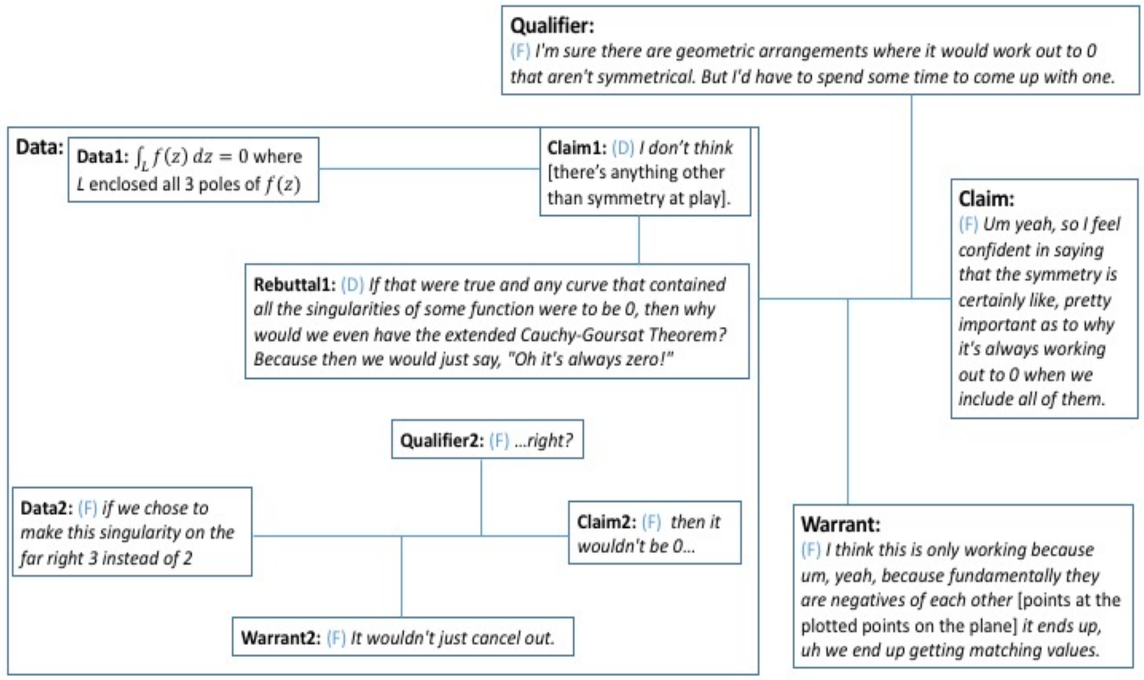


Figure 167. Dan and Frank's Toulmin diagram for Argument 12 for Task 11.

Task 11 – Riley and Sean

Sean began the pair's response to Task 11 as spokesman, symbolically rewriting the provided function in factored form (lines 1-2). After I provided the additional data related to the path *L* (lines 3-4), Riley asked about the meaning of the Jordan curve portion of the hypothesis (line 5). Sean clarified by authoring the formal claim, "we just said it was— simply connected positively oriented curve" (line 7). Seemingly unsure about Sean's claim, Riley questioned whether these two descriptions were actually equivalent (line 8). Because her qualifier was directed at me, I started to respond that they had the SICOPOC acronym (line 9). Riley once again asked if this is equivalent to the Jordan condition (line 10), and this time Sean claimed the affirmative (line 11). As spokeswoman, Riley summarized this condition as "a nice curve" (line 12), and given that neither Riley nor Sean mentioned the usual definition of a Jordan curve as having an interior and exterior, I provided this clarification (line 13).

Int: Ok so let's consider the function $f(z) = \frac{1}{z(z^2-1)}$ so all in the denominator there.
S: [Writes $f(z) = \frac{1}{z(z^2-1)} = \frac{1}{z(z+1)(z-1)}$]
Int: Yeah like that. And let's let L be a simple closed Jordan curve. [S writes L: simple closed Jordan curve]
R: What's a Jordan curve?
Int: Do you remember what that is? [Directed at Sean]
S: We didn't call it Jordan, we just said it was— simply connected positively oriented curve
R: Is that the same thing?
Int: So you had that SICOPOC or whatever
R: So that's all a Jordan curve is?
S: Yeah.
R: So a nice curve.
Int: Well, so the Jordan— that part is really just saying that the curve has an interior and an exterior.
R: Ok.
Int: So I believe when I was there you talked about it briefly.
S: Yeah.
Int: The proof of the theorem is really difficult in general so you definitely didn't talk about that but—
S: There is a theorem in the first chapter of the book about this, like, in that really really, like, real-analysis-type section like 1.6—
R: Ok so it just has to have an inside and an outside.
S: Yeah the Jordan's Theorem was closely related to an interior and exterior, so Jordan curve.
Int: Yeah exactly. So we're dealing with that nice type of curve here, but yeah. So yeah think of your SICOPOC definition. [Sean smiles]
R: Then yeah that would probably be a more slightly specific version of a Jordan curve.

I also reminded Riley and Sean that this idea was briefly discussed in class, which I observed, though they did not prove the Jordan Curve Theorem (lines 15-17). This seemed to trigger Sean's memory, as he recalled a theorem from the "real-analysis-type section" of the book that mentioned this property (lines 18-19). Riley recapitulated the definition as spokeswoman in the form of a claim (line 20), and Sean submitted a formal warrant that the theorem implies a SICOPOC is Jordan (line 21). Riley qualified this by hypothesizing that a SICOPOC "would probably be a more slightly specific version of a Jordan curve" (line 24). This first argument is summarized in Figure 168.

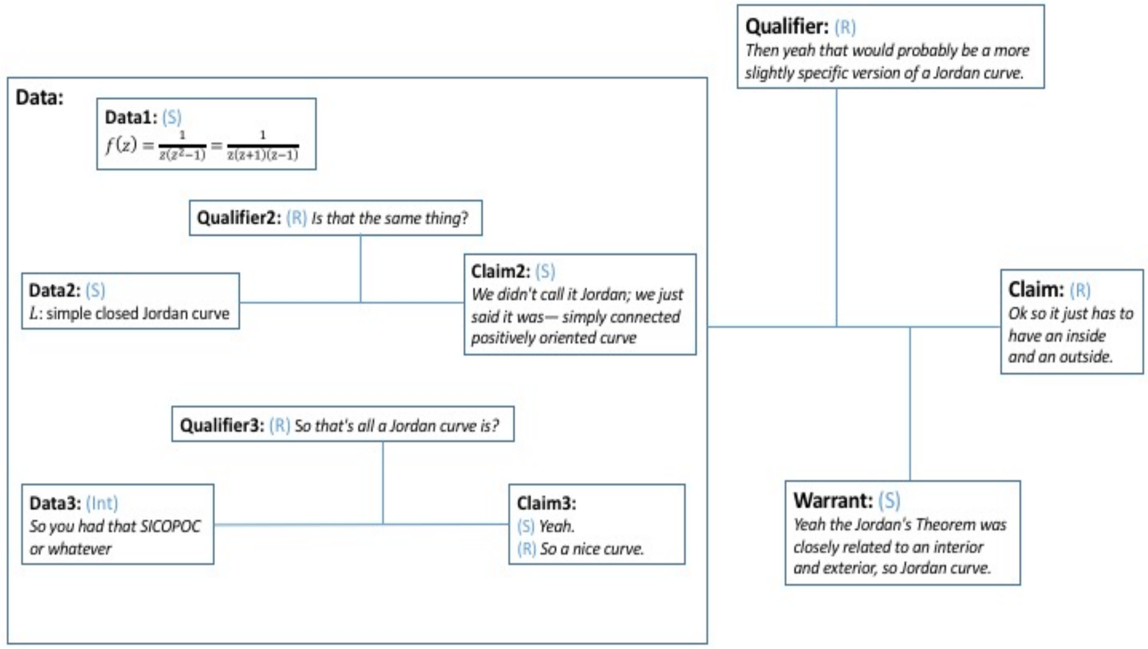


Figure 168. Riley and Sean's Toulmin diagram for Argument 1 for Task 11.

With the clarification about Jordan curves settled, I continued to articulate the directions for Task 11 (line 25-29). This began a second argument, starting with Sean relaying the provided data (line 26). As author, Riley considered the three "points of discontinuity" at $z = -1, 0,$ and 1 as an embodied-symbolic datum (lines 30-31) and qualified her datum with "right?" Sean agreed (line 32), and Riley then drew a path containing none of these points as an embodied datum (line 33; see Fig. 169). She began to author a warrant that the function is analytic over this region (line 34), but Sean interrupted as spokesman to name the region enclosed by this path as *region 1* (line 35). Riley continued her line of reasoning as she authored a symbolic claim that the integral around this path is zero, but qualified this assertion with the word "right?" (line 36). Sean agreed with her claim (line 37) and Riley continued to discuss and sketch other potential paths surrounding the various poles (lines 38-40; see Fig. 170).

Int: And so we'll stick with it being positively oriented and everything. And we'll also say that $f(z)$ is continuous on L. [Sean writes this] Ok so why don't you walk me through all the different possible values that you could think of for the integral of this function over L. So we're integrating this function over L, where, you know, other than having these properties, we're kind of free to vary what L would be.

R: So, let's see, we have three points of discontinuity right? At $1, 0$, and -1? [Draws Argand plane and plots these three points]

S: Mhm.

R: Um, so we can have a region that includes none of them [sketches such a region], in which case this is going to be analytic—

S: Region 1 [labels the inside of her drawn curve].

R: ...over the entire region and so [the integral of] that will equal 0 right? [writes $= 0$ next to region 1]

S: Mhm.

R: We could have one that includes this one [draws path enclosing $z = 0$], one that includes these two [draws path enclosing $z = 1$ and $z = 0$] , these 2 [draws path enclosing -1 and 0], all three [draws path enclosing all three], or I guess one like this [draws path enclosing -1 and 1 but not 0].

Int: Sorry just real quickly— so for the one that included none of the poles there, so you said that integral would be 0?

R: Yeah because then the function would be analytic over the entire region, and so we go back to Cauchy's Theorem that just says, um, we are— we go with a closed curve and it equals 0 if it's analytic over the curve and the entire region inside the curve.

Int: Ok great. Sorry continue.

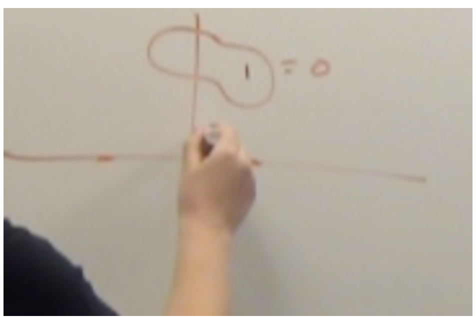


Figure 169. Riley's diagram with region 1 containing no poles, Argument 2, Task 11.

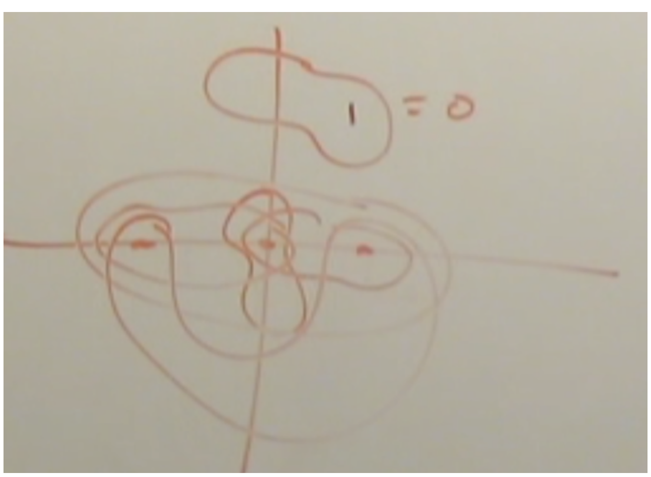


Figure 170. Riley's various possible paths L enclosing the poles in Argument 2, Task 11.

Before Riley continued to discuss these other options, I asked her and Sean to clarify how they knew the integral over “region 1” was zero (lines 41-42). In response, Riley relayed her previous warrant and additionally provided embodied-formal backing for this warrant’s validity by mentioning the applicability of Cauchy’s Theorem in light of the function’s analyticity (lines 43-46). The embodied aspect of this backing came from a circular tracing gesture she produced in the air to instantiate “a closed curve” (line 44; see Fig. 171). This clarification signaled the end of Argument 2, which is depicted in Figure 172.

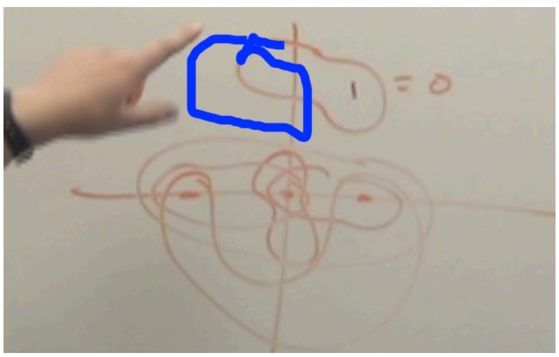


Figure 171. Riley’s gesture for "closed curve" mimicking curve 1 in Argument 2.

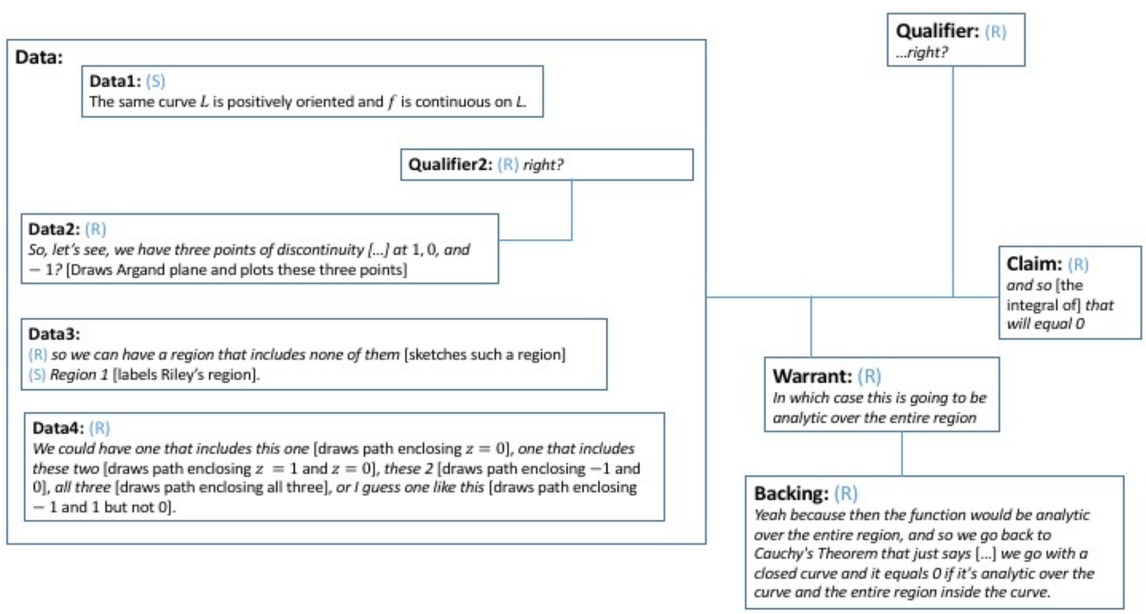


Figure 172. Riley and Sean’s Toulmin diagram for Argument 2 for Task 11.

Sean began Argument 3 by previewing the invocation of the Extended Cauchy-Goursat Theorem as a formal warrant (line 47). But first he clarified his datum, embodied by a dotted elliptical curve surrounding the poles 0 and 1 (lines 47-48; see Fig. 173). Combining symbolic, formal, and embodied reasoning, Sean authored a claim that the integral over his elliptical path is equal to the sum of the integrals around “small circles” surrounding each of the two poles, which he drew in red (lines 48-51; see Fig. 174). Sean began to discuss the values of the integrals along these small circles, but Riley interjected that she would prefer to use residue theory (line 52). Sean authored a formal claim that both methods are equivalent (line 53), and supported this assertion with a symbolic warrant detailing the two respective solutions (lines 53-57).

S: And then, per the extended Cauchy-Goursat theorem, if you're like saying— say we have a reasonable closed— like our L is like this [draws in elliptical dotted curve surrounding 0 and 1]. We know we can really just do— it was— we can really just— this integral over this one curve is now just the integral over two small circles enclosing these poles [draws smaller circles around each of the two poles]. And we know each of the integrals—
R: I mean I'd just use the Residue Theorem, honestly.
S: It's the same thing. So the extended Cauchy-Goursat theorem sort of says like, basically these [integrals] constrained to each of the small circles around the poles are each going to be $2\pi i$, so $4\pi i$ [writes $4\pi i$ next to his red dotted region]. Or you could do the Residue Theorem, which was $2\pi i$ times the residue for each point [Writes $\int_L f(z)dz = 2\pi i \sum_i Res[f(z), z_i]$; Riley points to the poles at $-1, 0, 1$] right?
R: But it's not going to be $4\pi i$ here, is it? [Points to Sean's answer] Because it's not— such a simple f. Like it's not just $1/z$. Cuz then don't you have to, um, didn't you have to divide it up like partial fractions and stuff?
S: I think so. Yes. Yes, you're right. [Erases answer of $4\pi i$]
R: So, so it'll come to something, but it's—
S: So partial fractions, or— So this one's a little better [points to the sum of the residues].
R: Right and so the residue—

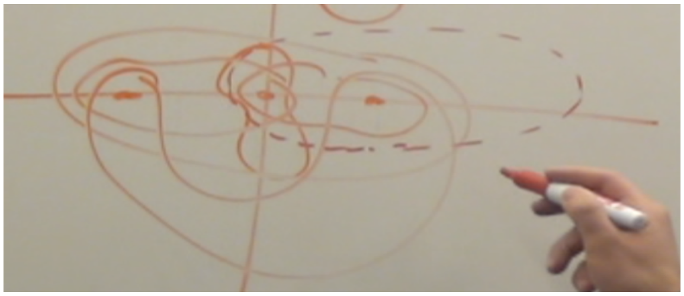


Figure 173. Sean's new dotted region for Extended Cauchy-Goursat, Argument 3.

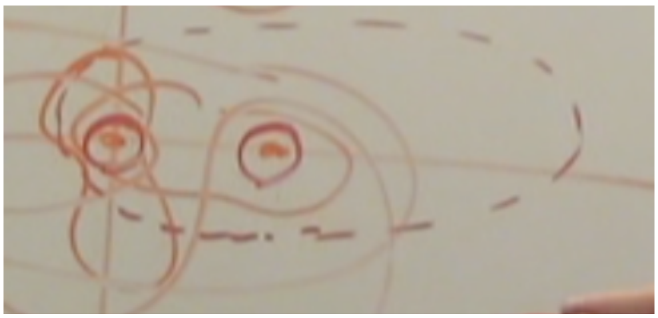


Figure 174. Sean's red "small circles" in Argument 3, Task 11.

However, Riley challenged Sean's warrant with the qualifier, "But it's not going to be $4\pi i$ here, is it?" and cited the fact that f was not "simple" (in the colloquial sense) like $\frac{1}{z}$ (lines 58-59). Accordingly, Riley authored a symbolic claim that in such instances, they would have to proceed via partial fractions (lines 59-60). Sean qualified Riley's claim by hesitantly agreeing with her and erasing his previous answer of $4\pi i$ (line 61). He also agreed with Riley by identifying the residue method as "a little better" (line 62). Argument 3 is summarized below in Figure 175.

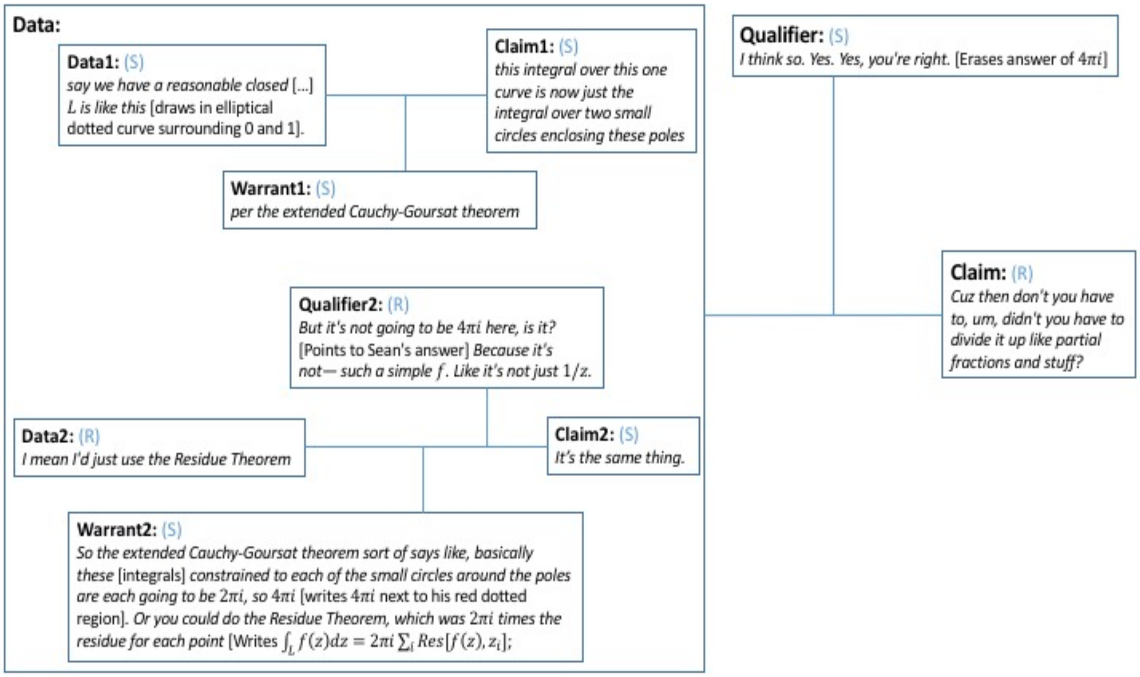


Figure 175. Riley and Sean's Toulmin diagram for Argument 3 for Task 11.

S: Yeah let me get hyper-specific. So there's 0 and 1 [points over towards the diagram], so you say that [writes $2\pi i \lim_{z\to 0} zf(z) + 2\pi i \lim_{z\to 1}(z-1)f(z)$] And then that equals your integral.
Int: So what's that going to be in this case?
R: So in this case, you take—
S: [Writes $= \frac{2\pi i}{(-1)(1)} + 2\pi i\left(\frac{1}{1(2)}\right) = -2\pi i + \pi i = -\pi i$].

After reaching this agreement, Sean got "hyper-specific" and determined the value of the integral using the residue approach (lines 65-66). This segment incorporated embodied-symbolic reasoning, in that Sean pointed to the points 0 and 1 in order to decide which limits to evaluate in the symbolic residue calculation. Sean did not proceed to evaluate the specific value of his symbolic expression, so I asked him to do so (line 67), and he symbolically obtained $-\pi i$ (line 69). This brief exchange comprised Argument 4, which is depicted in Figure 176 below.

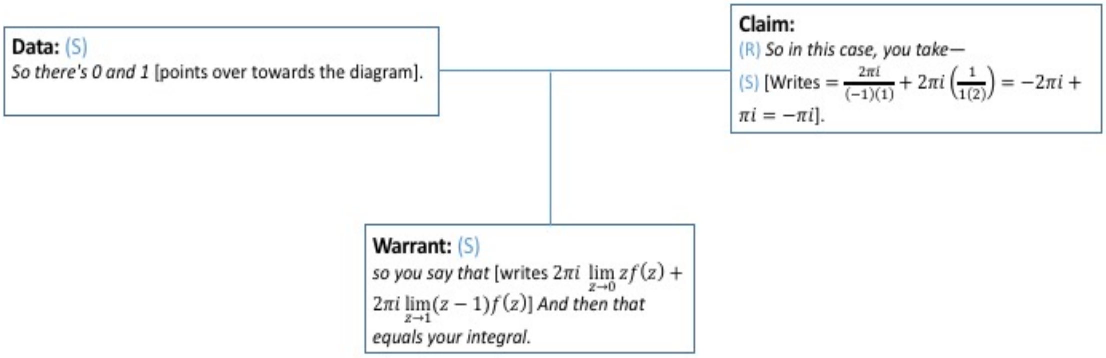


Figure 176. Riley and Sean's Toulmin diagram for Argument 4 for Task 11.

R: Ok so that's just like for those two points. But you can—
Int: So this is that red dotted curve around the two—
R: Right yeah in this case
S: But this generalizes to like—
R: Like but you can do it for, um, whichever points you like. So like the residue— here let's just— The residue for like um, $f(z)$ at 0 was -1 [writes $Res[f(z), 0] = -1$] was it? Right? [looks at Sean's previous inscriptions]
S: Yes.
R: Then the residue at $f(z)$, for $f(z)$ for— uh, sorry that was at 1, right?
S: Mhm.
R: ...is 1/2 [writes $Res[f(z), 1] = 1/2$]. And the residue, um, of $f(z)$ at -1, should be like $-1/2$ or something? [writes $Res[f(z), -1] =$]
S: Also 1/2.
R: Oh it's just 1/2. Ok. [Finishes writing $= 1/2$]
Int: Could you just briefly walk through how you're getting those computations for the residues?

Riley began a fifth argument by authoring a symbolic datum that Sean's approach in Argument 4 only used two specific poles (line 70). However, she and Sean pointed out that this approach could be generalized to other poles (lines 73-74). As spokeswoman, Riley relayed Sean's previous findings about the values of the residues at $z = 0$ and $z = 1$, then applied their symbolic warrant to evaluate the residue at the pole $z = -1$ (lines 75-83). She qualified her claim with the questions "was it?" and "Right?" as she glanced back at Sean's symbolic inscriptions (lines 75-76, 78). Because Riley was not providing detail about how she obtained her computations, and because she was uncertain and initially incorrect about the last value she obtained, I asked her to elaborate on these calculations (line 84). This catalyzed a new argument; Argument 5 is summarized in Figure 177.

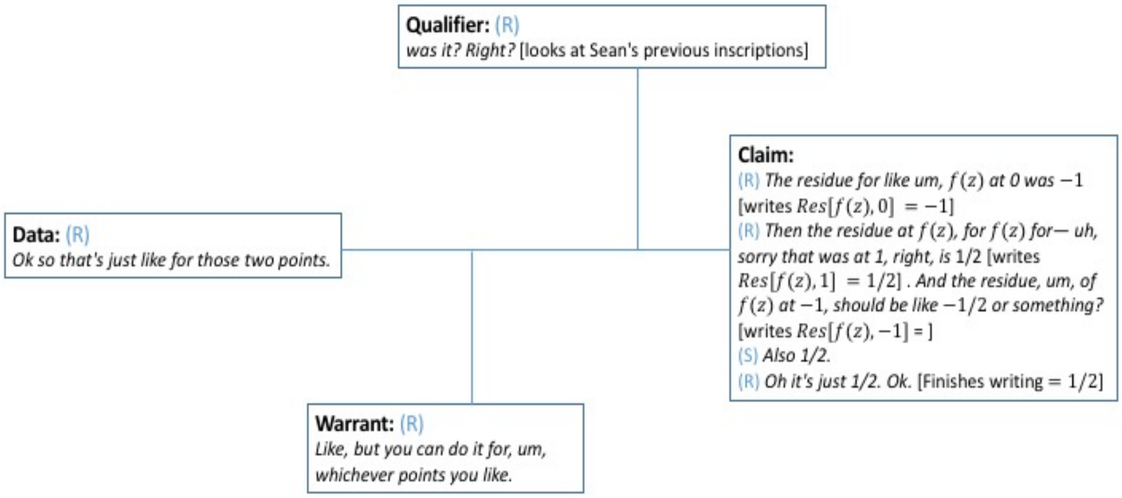


Figure 177. Riley and Sean's Toulmin diagram for Argument 5 for Task 11.

Riley began Argument 6 by implementing symbolic reasoning to identify $z = -1, 0$, and 1 as first-order poles (lines 85-86). She provided the definition of the residue of a function $f(z)$ at a generic point z_0, though she forgot to mention a limit (lines 86-88). Thus, Sean stepped in as spokesman and mentioned taking the limit of Riley's symbolic expression as z approaches z_0 (line 89). Next, Sean authored a datum considering the case where one's pole is of order k, and Riley claimed "Then you have to do derivatives and stuff" (line 91). Using formal-symbolic reasoning and as spokesman, Sean wrote out corresponding inscriptions for Riley's claim (line 92) and Riley labeled Sean's symbolism with "pole of order k" (line 93).

R: Yeah so the residue for— so you figure out what um— so these are all poles, so you look at what order the pole is. In this case, it's nice; they're all first-order. So the residue, um, of a function at a point z_0 say, [writes $Res[f(z), z_0] =$] is just going to be, uh, let's see, $(z - z_0)f(z)$, evaluated at z_0. [writes $= (z - z_0)f(z)$ eval @ z_0]
S: Yeah so the limit of that function as z goes to z_0, times that. And of course, for like z^2 or for like a pole of order k, then you say, ok—
R: Then you have to do derivatives and stuff.
S: [Writes $Res[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)]$. Yeah so a pole of order k [Riley writes "pole of order k" underneath Sean's symbolism].
S: Which of course works here because the 0^{th} derivative is just the function itself, and k is 1 in all these cases [points to his recent inscriptions].
R: Did that answer the question?
Int: Yeah definitely. Um, and so, in order to use like the Residue sort of theory, is there anything that you're using in particular about the function f or the domain that we're looking at?

After explaining this general form of the residue calculation, Sean authored a symbolic claim that this form also "works here" in the $k = 1$ case, and provided a symbolic warrant that the function f corresponds to the zeroth derivative in this case (lines 94-95). Afterwards, Riley qualified this clarification by asking, "Did that answer the question?" (line 96). I responded that they did, but asked another follow-up question regarding any assumptions they were implicitly imposing on the function or domain in order to use the residue theorem (lines 97-98). This began Argument 7, which is detailed below; Argument 6 is summarized in Figure 178.

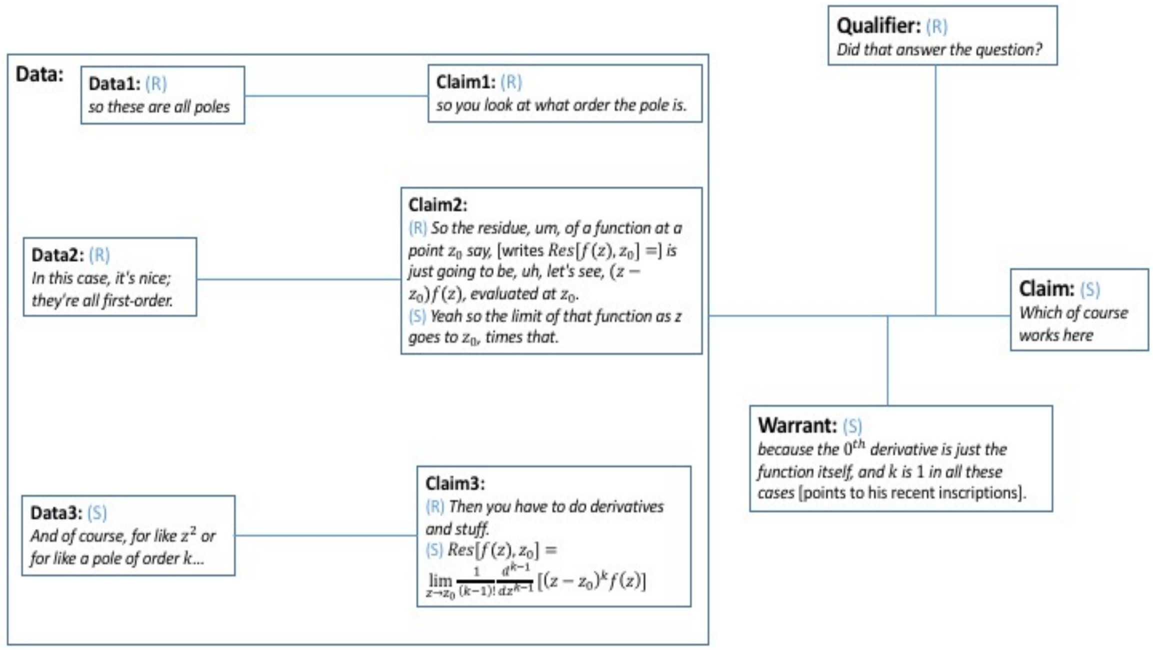


Figure 178. Riley and Sean’s Toulmin diagram for Argument 6 for Task 11.

R: So the function— it’s important that it be—
S: Expressed as a Laurent series, for residues? [smiles at Riley]
R: Yeah I think that's part of it. But like f, uh, has to be analytic all along L, [tracing gesture mimicking some closed path L] uh, so it definitely has to be analytic everywhere on L. So like you couldn't have L actually going through one of the points. And it has to be analytic everywhere inside of L, except for those specific points. Like they have to be isolated points. Um [writes f analytic on L and in L except @ isolated z_i]. Yeah so isolated points. Um, let's see what else.
Int: So what about the domain itself? Like this region that's enclosed here—

Riley began Argument 7 by considering the function f; she started to author a claim about properties of f allowing the use of residue theory, but Sean interrupted with the formal claim that the function needs to be “expressed as a Laurent series” (line 100). Riley qualified Sean’s claim by commenting, “Yeah I think that’s part of it” (line 101), but added that f needs to be analytic along L (line 101). This formal verbiage in her addendum also accompanied embodied reasoning, in that Riley traced out a visualized closed path L while verbalizing her statement (lines 101-102; see Fig. 179).

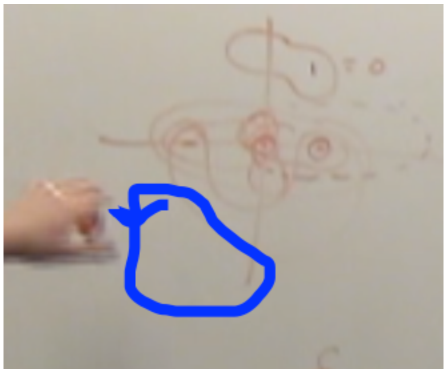


Figure 179. Riley's tracing gesture for some path L, Argument 7, Task 11.

Using her clarification as a datum, Riley authored a formal-embodied claim that *L* could not pass through one of the poles as a result of this analyticity (lines 102-103). She continued to provide another property that the function has to be analytic "everywhere inside of *L*, except for those specific points" (lines 103-104), a property she used as a datum for the claim that such points must be "isolated" (lines 104-105). As spokeswoman, Riley summarized her requirements for f as brief inscriptions on the whiteboard (lines 104-105). She paused for some time, and because she had not mentioned requirements about the domain, I asked a follow-up question about this (line 106). Argument 7 is summarized in Figure 180 below.

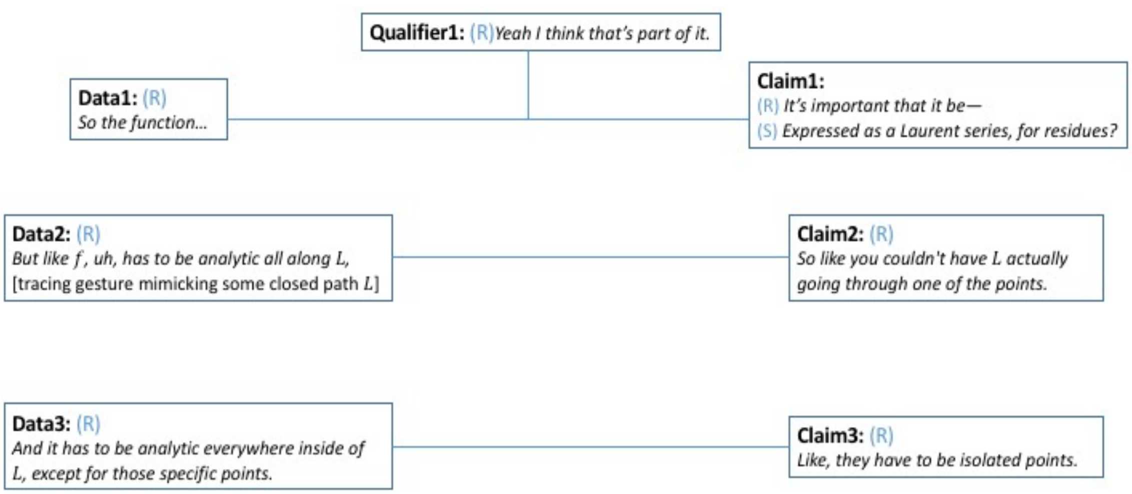


Figure 180. Riley and Sean's Toulmin diagram for Argument 7 for Task 11.

In response to my follow-up question, Sean began Argument 8 by authoring a formal-embodied warrant that "the only poles you can consider are the poles in the region" (line 107). He used this warrant to claim that this was the reason they did not calculate the residue at $z_0 = -1$ inside their red curve previously (lines 107-108). He paused for a while after authoring this claim, so I asked Sean and Riley about the given assumption that L was a simple closed curve (line 109). In response, Riley provided an embodied example of a non-simple curve that "loops around itself" (lines 110-111; see Fig. 181).

S: The only poles you can consider are the poles in the region. Hence why we didn't do the residue for -1 in that red curve [points over to diagram], cuz the poles aren't in that domain. So—
Int: So when we say that L was a simple closed curve up here, the fact that it's simple here—
R: Yeah so like—let's think— so like a non-simple curve would be like one that like, that loops around itself or something like that, so like maybe a figure eight. Um, and—
Int: So what would go wrong there?
R: So if you had a singularity there [draws a point inside the upper portion of figure eight] for instance, um—
S: It would go wrong there.
R: Yeah, um— It does— I mean, I think it does have to be simple. But— [extended pause]
S: I think so, yeah. I think so too.
R: Maybe. I mean— [extended pause]

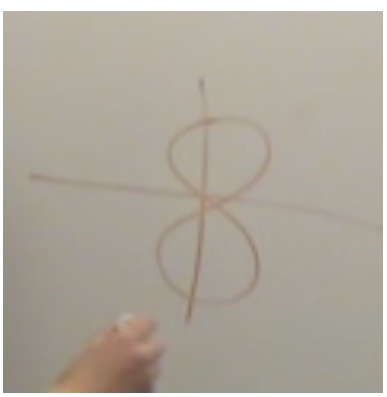


Figure 181. Riley's example of a non-simple curve in Argument 8, Task 11.

I asked her what would go wrong in the residue theorem if they used such a curve (line 112), and she responded by authoring an embodied datum considering a singularity in the upper portion of her figure eight (lines 113-114). Sean claimed that "it would go wrong there," but did not provide a reason as to why (line 115). Riley concluded that the curve does have to be simple, but both she and Sean qualified this conclusion with verbiage and pauses that suggested some uncertainty (lines 116-118). Argument 8 is depicted in Figure 182.

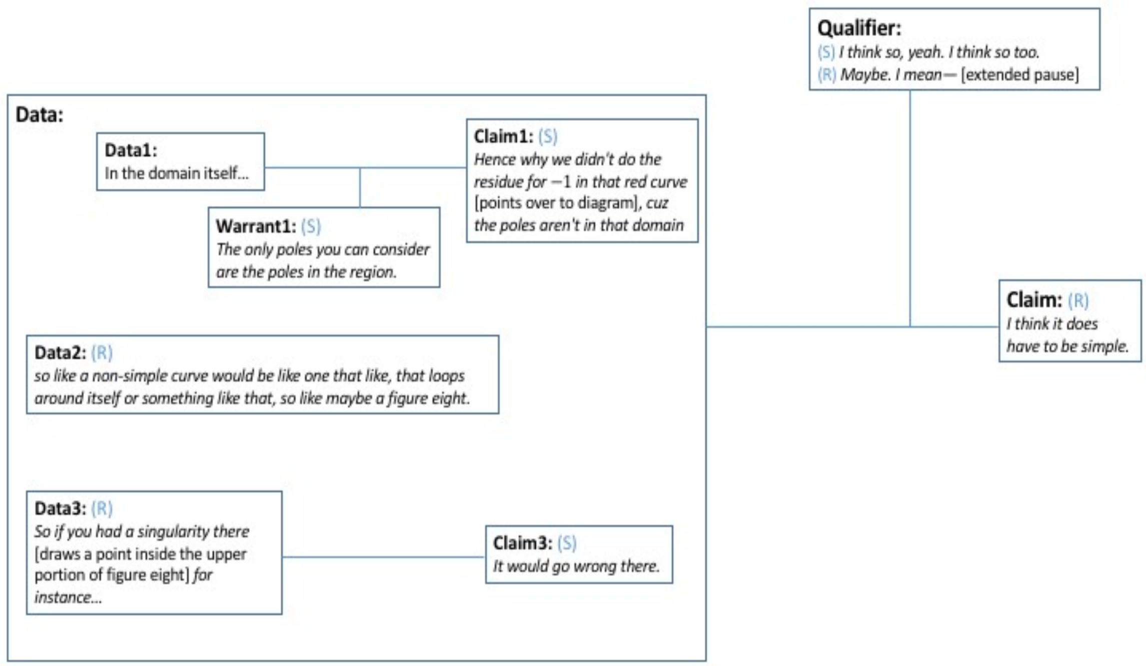


Figure 182. Riley and Sean's Toulmin diagram for Argument 8 for Task 11.

Given that Riley and Sean reached an impasse regarding the simplicity assumption for *L*, I asked if Riley's example curve was a Jordan curve (line 119), and she indicated that this was what was confusing her (line 120). She began a new ninth argument by remarking that her figure-eight shape appeared to have "an internal and external point" (line 121). Riley generated another example curve as an embodied datum (lines 121-122; see Fig. 183), and claimed that this limacon with inner loop would still not be simple (line 122). However, Riley drew a point within the inner loop (see Fig. 183) and claimed that is was not clear whether this point would be considered as an interior or exterior point with respect to the entire limacon (lines 122-123). This led her to qualify this argument by questioning whether Jordan curves can be non-simple (line 124).

Int: Does that curve— is that still a Jordan curve? Does it still have an interior and an exterior?
R: That's what I'm confused about. Cuz it seems like— I mean *this* one I think seems like— kind of like it has an internal and external point. Like if you had a curve like this [draws a limacon with inner loop] then it wouldn't be simple either but it's not clear whether this point [draws a point in the "inside" of the inner loop] like this region would be on the inside or the outside. Um, but the fact that it's simple and a Jordan curve— Can you have a Jordan curve that's not simple?
Int: What do you think?
S: I'm not sure.
R: I don't know the definition very well. So I don't know if, um, that incorporates the, sort of idea of being a simple curve.
Int: Well so— so you tackled this using Residue Theory. Is there a different, um— I know a lot of these things are very similar, but is there like a different theorem that you've talked about that maybe would also be relevant that would get you the same result?

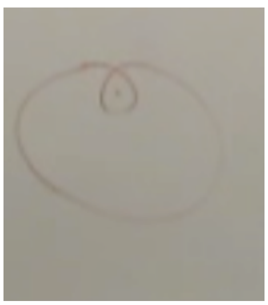


Figure 183. Riley's limacon with inner loop and point within inner loop, Argument 9.

I redirected this question back to Riley and Sean, and Sean replied "I'm not sure" (lines 125-126). Riley added to her qualifier by stating that she does not "know the definition very well" (lines 127-128). Because the pair had once again reached an impasse about the assumptions needed for residues, I suggested that maybe they could pursue a different approach (lines 129-131). This catalyzed a new argument, as detailed following Figure 184, which summarizes Argument 9.

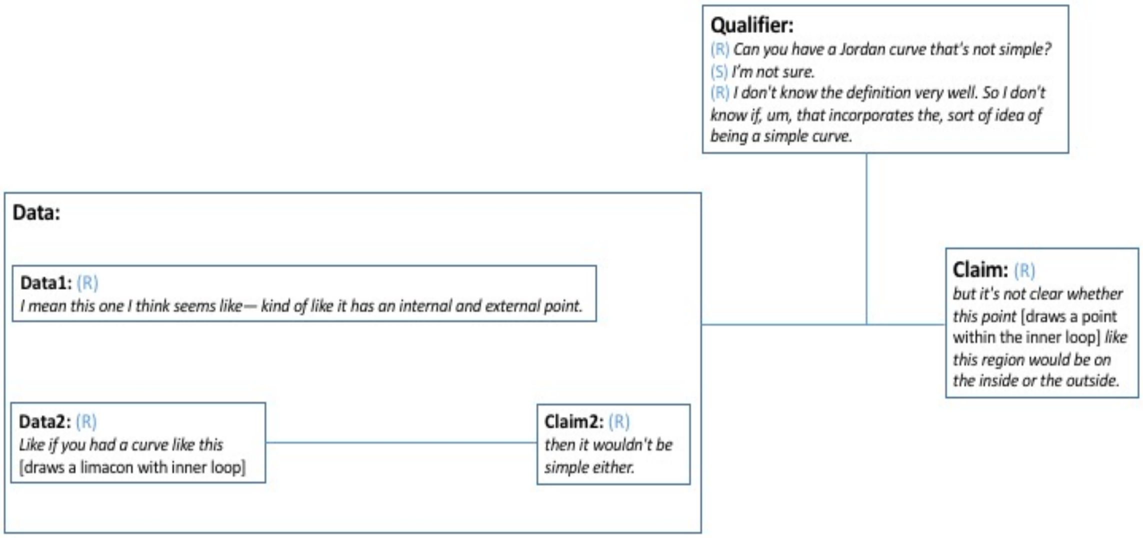


Figure 184. Riley and Sean's Toulmin diagram for Argument 9 for Task 11.

As an alternative approach, Riley suggested that she and Sean employ the "extended Cauchy Integral Formula" as a formal warrant (line 132). Implementing embodied reasoning, she drew a region and plotted several points inside the region representing "points of discontinuity" (lines 135-137; see Fig. 185). She clarified that, in their case, these discontinuities were located at $z = -1, 0$, and 1, and qualified her datum with the word "right?" (line 137).

R: Yeah like you could use the extended Cauchy Integral formula, which would be—
S: Mhm
Int: So what does that look like?
R: So that one— so say we had a region [draws a blob region] and you have like, however many points of discontinuity [plots several arbitrary points inside curve]. So in this case, we have $-1, 0,$ and 1, right?
Int: Mhm.
R: Um, basically what it says is the integral around this whole curve [draws an orientation arrow pointing counterclockwise], assuming you have to be analytic along the whole curve for $f(z)$, um, so the integral along the curve, uh, is equal to the— like the integrals around each of these [draws in small circles around each of the singularities] summed.
Int: Ok.
R: Um, so if we had, say, $f(z)$ [writes $f(z) = 1/z$] one easy way to do it would be $\frac{1}{z} + \frac{1}{z-1}$ [writes $+\frac{1}{z-1}$], or something like that. Um, because like those— well I don't know if that's a good example, but like you end up splitting like into curve 1—
Int: Well why don't we work with the one that I gave you.
R: Yeah let's work— yeah it'll come out the same so—

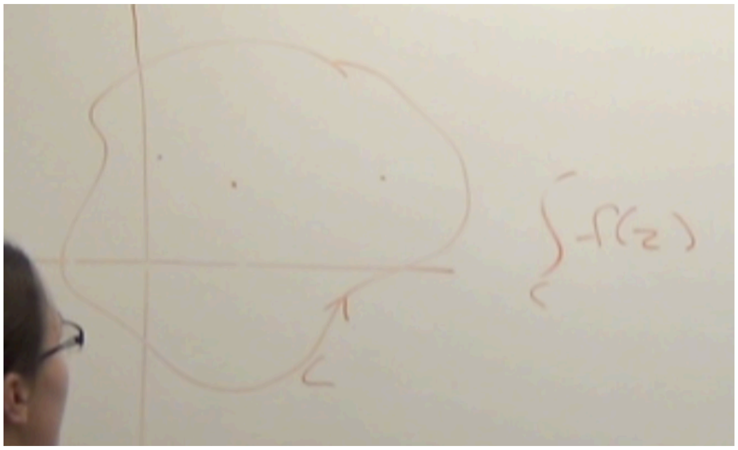


Figure 185. Riley’s sample region and points of discontinuity, Argument 10, Task 11.

Returning to her warrant, she added that “you have to be analytic along the whole curve” (line 140). As author, she invoked this warrant to surmise that the integral along this curve is equal to the sum of the integrals around small circles about the discontinuities (lines 139-142). This embodied-symbolic reasoning incorporated an updated diagram in which Riley drew in the aforementioned small circles (see Fig. 186). Afterwards, she began to exemplify her argument with a different function, but I redirected her to work with the provided function from Task 11 (lines 144-148). Argument 10 is summarized in Figure 187.

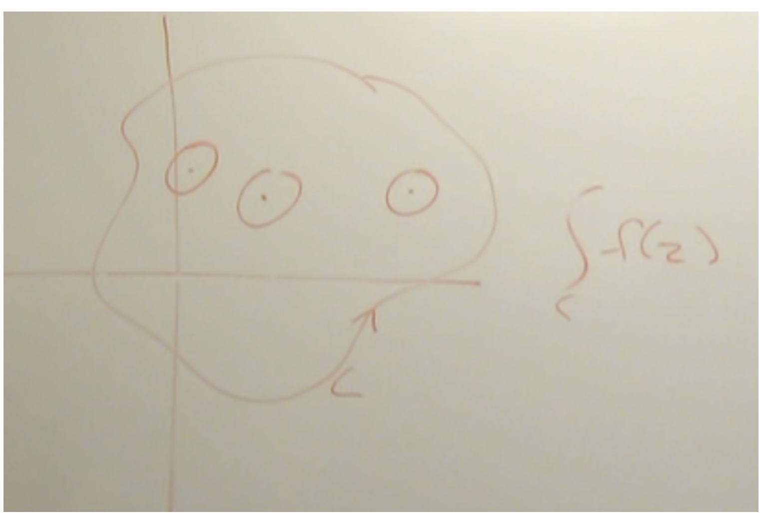


Figure 186. Riley’s "integral around each of these summed," Argument 10, Task 11.

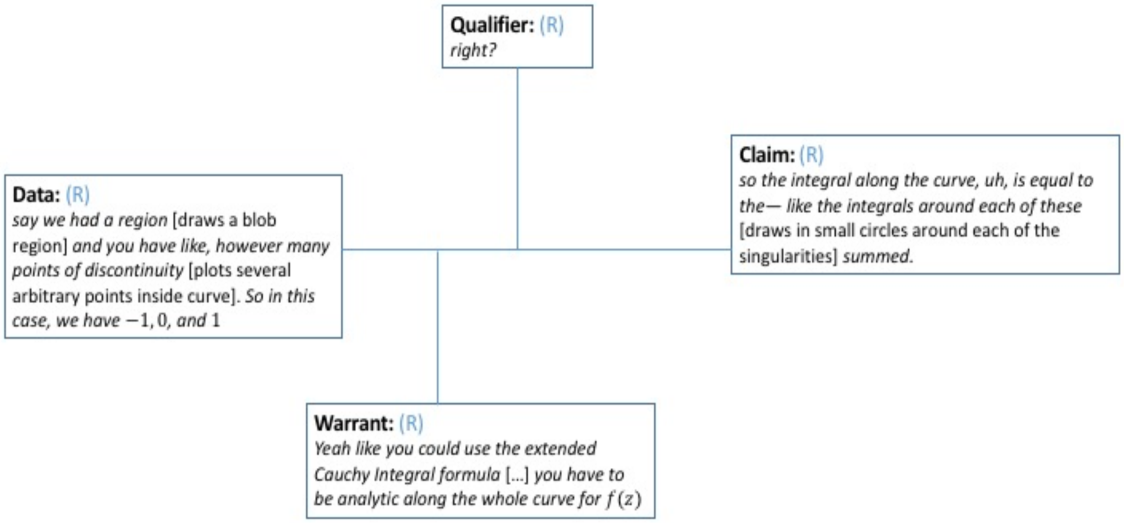


Figure 187. Riley and Sean's Toulmin diagram for Argument 10 for Task 11.

Following my redirection back to the task at hand, Sean began Argument 11 by authoring a symbolic datum that considered a partial fractions decomposition, and Riley qualified this suggestion with the question, "What is it going to be? Um" (lines 149-152). Sean symbolically set up the partial fractions decomposition, and employed the usual technique of evaluating the resulting equation at conveniently-chosen values of z to obtain $A = -1$ and $B = \frac{1}{2} = C$ (lines 153-156). He used these resulting values of A, B, C as a symbolic warrant to claim that $\frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1} = -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{z+1}\right)$ (line 157), thereby completing Argument 11, which is depicted in Figure 188.

S: So what you first do is you split it into partial fractions.
R: Yeah so let's split that um—
S: So you have like,
R: What is it going to be? Um—
S: So this would be— [writes on board $\frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$ next to original $f(z)$ inscription. Then writes $A(z-1)(z+1) + Bz(z+1) + Cz(z-1) = 1$].
S: Alright. [Writes $z = 1,\ B(2) = 1.\ z = 0, A(-1)(1) = 1.\ z = -1,\ C(-1)(-2) = 1.\ B = 1/2.\ A = -1.\ C = 1/2$.]
S: [Goes back up to original decomposition and writes $= -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{z+1}\right)$.]
Int: So there's your partial fraction decomposition.
R&S: Yes/Mhm [simultaneously].
Int: And so what do we do from there?

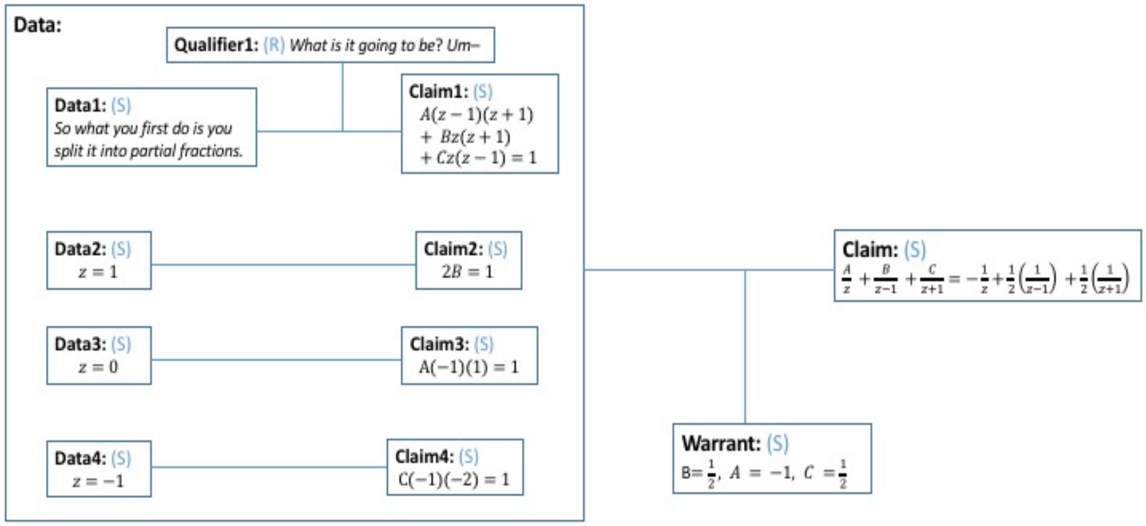


Figure 188. Riley and Sean's Toulmin diagram for Argument 11 for Task 11.

Riley began Argument 12 by symbolically replacing the integrand $f(z)$ by its partial fractions decomposition (lines 161-162). She employed embodied-symbolic reasoning as she authored a claim that "we choose three curves" based on the warrant that "we have 3 points of discontinuity" (lines 162-163). She somewhat arbitrarily designated these curves to be circles of radius $1/2$ (line 163), and claimed that the integral over the entire curve is equal to the sum of the integrals about each of these three small circles (lines 163-168). She qualified this assertion with the question, "Am I doing this right?" (line 165). She also expressed some hesitation about her observation that the function is not analytic over these three circles, but hypothesized that they could still parametrize (lines 169-170). This brief argument is summarized in Figure 189.

R: Ok so we're still taking the integral over the curve C, um— and, or the curve L, I guess. Uh, [writes $\int_L -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{z+1}\right)\, dz$]. And basically we say— we have 3 points of discontinuity, so we take it, um— we choose three curves, so let's say curves of radius 1/2, um and we can split— this integral should be the same as the integral of the function around like, a curve of 1/2 [note she means circle] centered at 0 of the same function. Am I doing this right?

S: Yep.

R: Plus [the integral around a circle centered at] 1/2 centered at -1 [writes $\int_{C_{\frac{1}{2}}(-1)} f(z)\, dz$] plus a curve of radius 1/2 centered at 1 [Writes $= \int_{C_{\frac{1}{2}}(0)} f(z)\, dz + \int_{C_{\frac{1}{2}}(-1)} f(z)\, dz + \int_{C_{\frac{1}{2}}(1)} f(z)\, dz$]. Um, so, uh this comes out and it's going to— oh we're going to— it's still not analytic but we can parameterize, right?

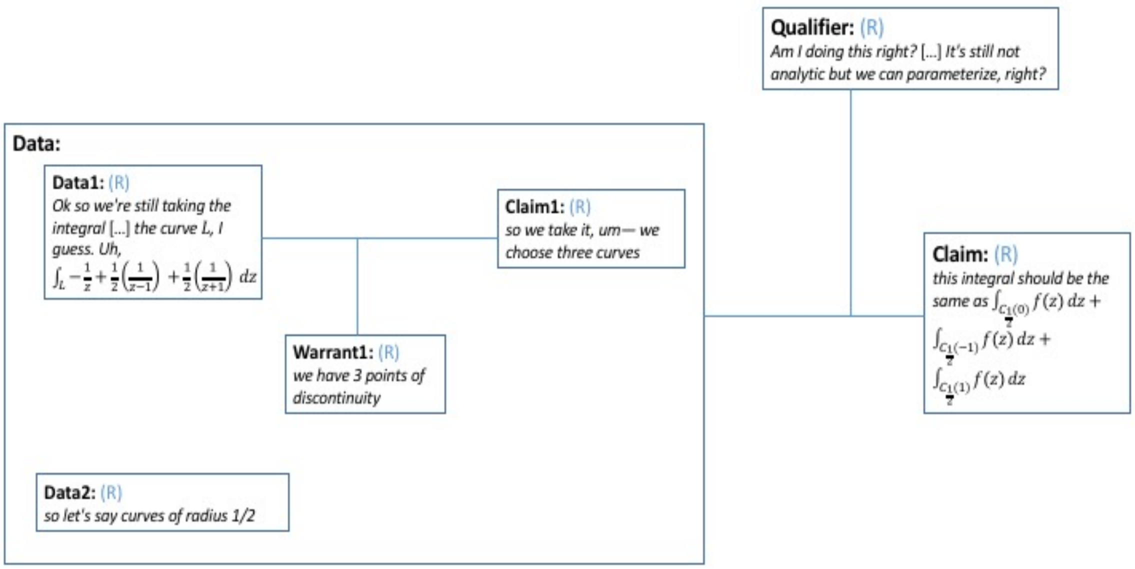


Figure 189. Riley and Sean's Toulmin diagram for Argument 12 for Task 11.

Riley discussed the details of evaluating each of the three integrals during Argument 13, which she began as spokeswoman, once again drawing an arrow from her symbolic inscription to signify its evaluation (lines 171-172). Using formal-symbolic reasoning, she authored a warrant resting on the analyticity of the terms $\frac{1}{z-1}$ and $\frac{1}{z+1}$, and supported this warrant with backing for its validity: "we can split up the integral" (lines 172-173). Riley then authored a claim that the integrals of both these terms vanish (lines

173-174). Accordingly, as spokeswoman, Riley symbolically rewrote the integral $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz$ as $\int_{C_{\frac{1}{2}}(0)} -\frac{1}{z}\,dz$ (lines 175-176).

R: So if we do centered at 0 [draws arrow from $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz$], then we're going to say ok so [the integral of] the curve of radius 1/2 centered at 0 of $f(z)$. Uh, these are both going to be analytic [points to $\frac{1}{z-1}$ and $\frac{1}{z+1}$ terms in original integrand]. And again, we can split up the integral. So um, so these will both go to 0 [points again to the $\frac{1}{z-1}$ and $\frac{1}{z+1}$ terms in integrand], these two aspects of it. So, but we still have to do the integral of $-1/z\,dz$ over— well not over L, over [circle of radius] 1/2 centered at 0 [writes $\int_{C_{\frac{1}{2}}(0)} -\frac{1}{z}\,dz$ on the other side of her arrow].

R: Um, and then I guess technically you want that right now [writes + $\int$] that's going to be like $\frac{1}{2}\frac{1}{z-1} + \frac{1}{z+1}$ right? [Writes integrand $\frac{1}{2}\frac{1}{z-1} + \frac{1}{z+1}\,dz$] It just so happens that this one [draws arrow from $\frac{1}{2}\frac{1}{z-1} + \frac{1}{z+1}$] is 0 [writes 0 next to the tip of her arrow] because it's analytic over that specific region [points to the $C_{\frac{1}{2}}(0)$ at the bottom of the first integral/term].

R: This, [points to the integral of $-1/z$] I mean— again, we've gone over this enough times, like this is going to equal $-2\pi i$ because it's [the integral of] $1/z$ over this closed circle [circular tracing gesture in air] uh, and then you just take the negative right? [writes $= -2\pi i$ next to her most recent symbolic inscriptions]

S: Mhm.

R: Yeah, ok.

Int: The curve L you're considering here is the one going around all three points then? Or which L are you considering?

R: Yes, I'm considering L that goes around all three points, yeah.

Int: Ok.

With the first term simplified significantly, Riley turned her attention to the integral of the other two terms, $\int \frac{1}{2}\frac{1}{z-1} + \frac{1}{z+1}\,dz$ (lines 177-178). She again qualified this symbolic datum with the word “right?” (line 178). Riley symbolically claimed that this portion of the integral vanishes, based on a formal warrant appealing to the function’s analyticity in the region (lines 178-180). Finally, she returned to the integral $\int_{C_{\frac{1}{2}}(0)} -\frac{1}{z}\,dz$ and claimed that its value is $-2\pi i$ (lines 181-182). As a supporting warrant, Riley explained that one only needs to modify the familiar result of the integral of $1/z$ over a closed circle by negating the answer (lines 182-183). While she discussed this familiar symbolic result, she produced an embodied tracing gesture to represent the circular path (see Fig. 190). She also qualified this assertion with the rhetorical question, “we’ve gone over this enough times […] right?” (lines 181 & 183). Given we had discussed the integral over one particular path for some time, I asked Riley to clarify which path she was integrating over in this argument as a way to segue into the rest of the possible values of the integral in Task 11 (lines 187-190). Argument 13 is summarized in Fig. 191.

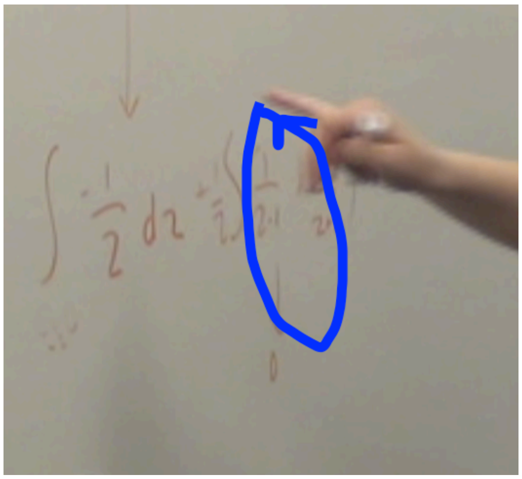


Figure 190. Riley’s tracing gesture for "integral over this closed circle,” Argument 13.

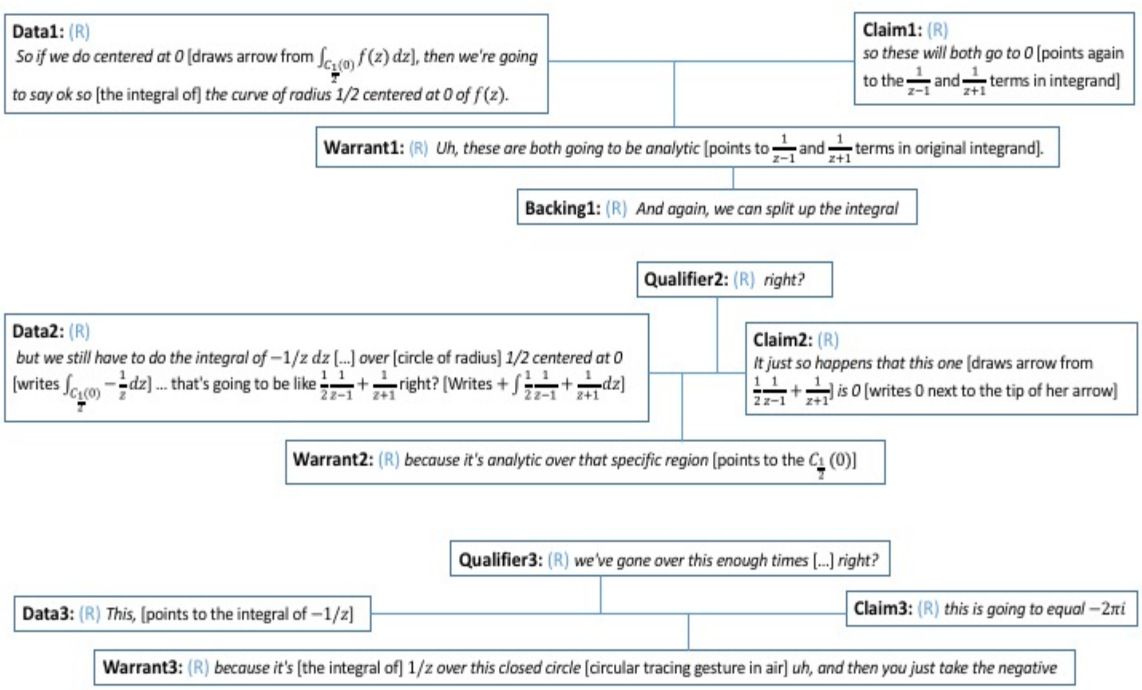


Figure 191. Riley and Sean’s Toulmin diagram for Argument 13 for Task 11.

Next, Riley discussed how to alter her approach from Argument 13 to obtain the integral over other possible paths L. Specifically, she began Argument 14 by considering the embodied datum of a path L "that wasn't around all three points" (line 191), and claimed that in such a case, one would just omit whichever terms in the expression $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz + \int_{C_{\frac{1}{2}}(-1)} f(z)\,dz + \int_{C_{\frac{1}{2}}(1)} f(z)\,dz$ coincided with poles not surrounded by the path (lines 191-193). After this aside, Riley returned to the expression $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz + \int_{C_{\frac{1}{2}}(-1)} f(z)\,dz + \int_{C_{\frac{1}{2}}(1)} f(z)\,dz$ in order to evaluate the two remaining terms; recall that up to this moment, she had evaluated $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz$.

Turning her attention to the second integral $\int_{C_{\frac{1}{2}}(-1)} f(z)\,dz$, Riley mentioned that one could evaluate this integral in a similar fashion to $\int_{C_{\frac{1}{2}}(0)} f(z)\,dz$, "except for now, you're centered at -1 instead" (lines 193-194). In particular, she authored a datum that $\frac{1}{2}\frac{1}{z+1}$ was now the portion of the partial fractions decomposition that was not analytic, and that the other two terms vanished under the integral (lines 195-197). After a lengthy pause, Riley authored a symbolic claim that $\int_{C_{\frac{1}{2}}(-1)} f(z)\,dz = \pi i$, and qualified this claim with "right?" (line 198). As spokesman, Sean represented this answer as $\frac{1}{2}(2\pi i)$, which Riley acknowledged as equivalent to her claim (lines 199-200). Sean authored a formal-symbolic-embodied warrant for their claim, namely that "you'll get 1 over (z minus your pole), integrate over a circle, centered at your pole, which always goes to $2\pi i$. Because that's the only pole in your circle" (lines 201-202).

R: Um, you could do it— if you did it for an L that wasn't around all three points, then you would just omit whichever of these three integrals that it didn't go around [points to the 3 terms in previous argument]. Um, so and then you do the same thing like over here [draws arrow from the integral over $\int_{C_{\frac{1}{2}}(-1)} f(z)\,dz$ inscription], except for now, you're centered at -1 instead.
R: So like that would be— now this one's the one that's not analytic [writes bracket under the last term $\frac{1}{2}\frac{1}{z+1}$ in partial fractions decomposition)]. These two both go to zero [puts brackets under the other two terms in the partial fractions decomposition]. And [pauses]—
R: It still comes to πi, right? [writes πi next to arrow from $\int_{C_{\frac{1}{2}}(-1)} f(z)\,dz$ inscription]
S: 1/2 times $2\pi i$.
R: Well ok, yeah sure. $\frac{1}{2}(2\pi i)$ [writes $\frac{1}{2}(2\pi i) = \pi i$].
S: Cuz eventually you'll get 1 over (z minus your pole), integrate over a circle, centered at your pole, which always goes to $2\pi i$. Because that's the only pole in your circle.
R: Yeah and so finally, you know, over here [draws arrow from third integral term $\int_{C_{\frac{1}{2}}(1)} f(z)\,dz$] you're going to get the exact same thing, $\frac{1}{2}(2\pi i)$, um, once you've parametrized that [points to $\frac{1}{2}\frac{1}{z+1}$ term] because like, the one will cancel— um, it really doesn't matter where you're centered. It comes to $2\pi i$. So then you get—
S: Zero.
R: Zero.
R: Is that the same thing we get if we do all three residues? Cuz it should come to the same thing.
S: Yep. $\frac{1}{2} + 1 - \frac{1}{2} = 0$.
Int: So you're adding those ones [residue inscriptions] in orange, yeah. So you have the $-1, 1/2$ and $1/2$. Ok. Yeah, so you get the same thing. Ok cool.

Finally, Riley considered the third integral $\int_{C_{\frac{1}{2}}(1)} f(z)\,dz$ and authored a symbolic claim that the result would still be $\frac{1}{2}(2\pi i)$, citing the symbolic warrant that “the one will cancel” and that “it really doesn’t matter where you’re centered” (lines 203-205). In total, then, Riley and Sean added their three integrals to obtain zero (lines 206-208), and Riley pondered whether they would get the same result by adding the three residues from before (line 209). Sean affirmed Riley’s suspicion symbolically by adding $\frac{1}{2} + 1 - \frac{1}{2}$ to obtain zero (line 210). Argument 14 is summarized in Figure 192.

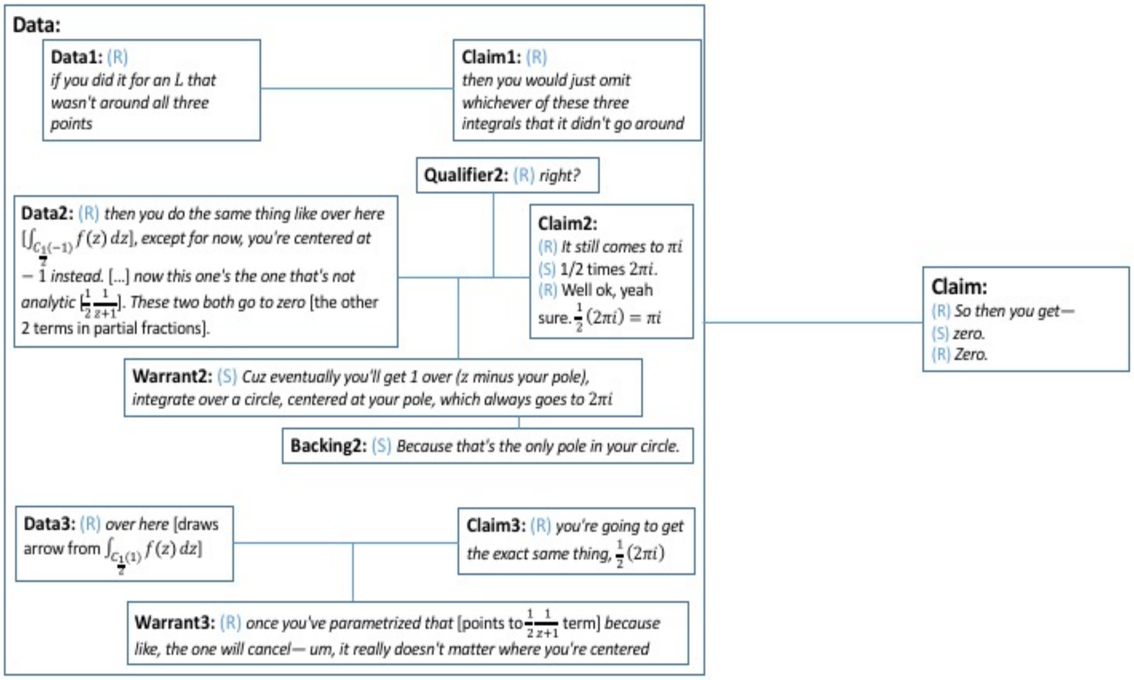


Figure 192. Riley and Sean's Toulmin diagram for Argument 14 for Task 11.

Int: Are there any other possible paths that would get an integral of 0 that you could see, without kind of going through all the details? Like so this one, maybe it wasn't completely clear that it would come out to 0, right? You had the paths around all three. And you had the original path that you talked about that didn't enclose any of the singularities; that came out to be an integral of 0. Um, do you think any of those other paths would have an integral of 0?
R: I don't think so. Right? Because um— I mean you can do it that way [points to the partial fractions work] but it's easier if you think about the residues. When you add up all three residues— because you're just multiplying by $2\pi i$, so to make the full integral equal to 0, you have to make the sum of the residues equal 0.
S: So it either includes all three points or it includes none of them.
R: Yeah so I guess— all I can think of would be a non-simple curve that winds around one of the points multiple times. But—
Int: But we're only restricting to simple curves.
R: …but we're not sure that would work, so—
Int: Yeah in this setup, we're considering simple curves—
R: So yeah, it would just be a simple curve.
R: I can't think of any [others] that would make it equal 0, cuz there's no other way to add them [points to the residues on board].

As a follow-up question, I asked Riley and Sean if they could think of any other paths that would yield an integral of zero (lines 213-217). This commenced Argument 15, which Riley began with the claim "I don't think so," though she provided her usual qualifier of "right?" to signal some uncertainty (line 218). She mentioned that she preferred to think in terms of residues, and she considered situations in which adding "all three residues" yields a result of zero (lines 218-219). In particular, she incorporated symbolic reasoning to author a claim that the "full integral" vanishes precisely when the sum of the residues is zero (lines 220-221). To support this assertion, she authored a symbolic warrant that the residue calculation amounts to multiplying $2\pi i$ by the sum of the residues, i.e. "you're just multiplying by $2\pi i$" (line 220).

Accordingly, Sean symbolically surmised that the only way one could obtain a sum of zero from these three terms is if the curve contains either all three poles or none of them (line 222). Riley hypothesized that perhaps a non-simple curve might also yield a vanishing integral, but I reminded her that in this task we were only considering simple curves (lines 226-228). In response, she concluded that she could not think of any other ways to obtain zero, once again citing that "there's no other way to add [the three residues]" (lines 229-230). Argument 15 is summarized in Figure 193.

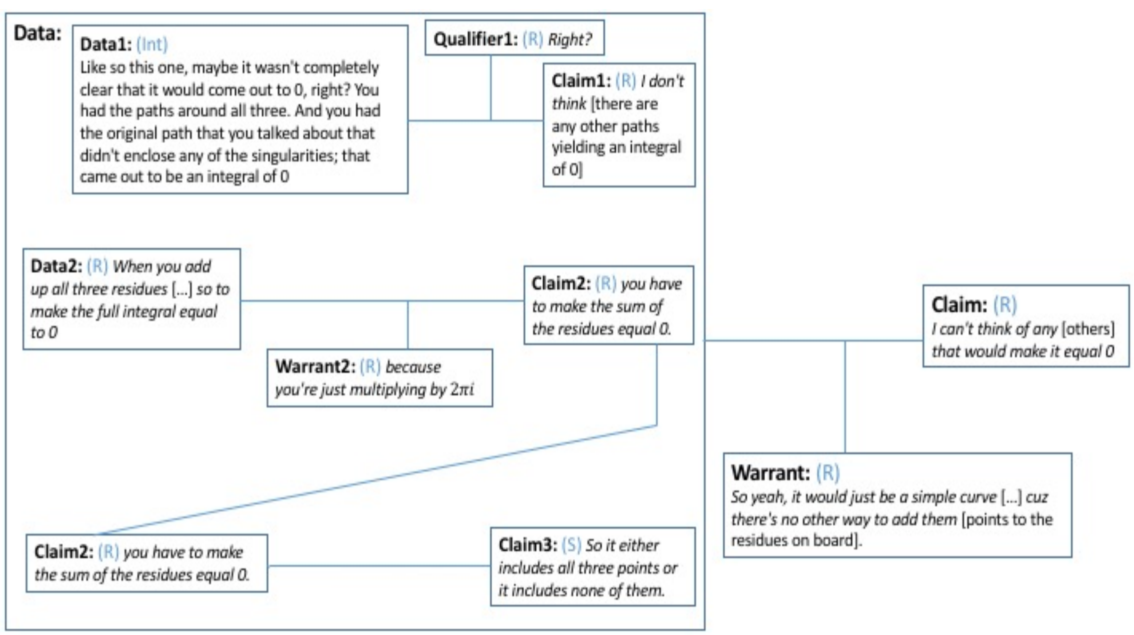


Figure 193. Riley and Sean's Toulmin diagram for Argument 15 for Task 11.

Because Riley mentioned a preference for Residue Theory, I asked one final follow-up question about this (lines 231-232). This catalyzed one final argument related to Task 11, as follows. Riley claimed that she finds this method “easiest” (line 233), and Sean authored a symbolic warrant that this ease comes from not having to “do partial fractions” (line 234). Riley conceded that even though partial fractions are “not that bad” (line 235) the decomposition “takes more time” (line 237). Sean provided symbolic backing for this warrant’s correctness by discussing a situation in which the denominator contains higher-order poles and hence “the partial fractions would take forever” (lines 239-241). Argument 16 is depicted in Figure 194.

Int: Sounds good. So Riley, you said something about how the Residue Theory, you think, is kind of the easiest way to consider these sort of integrals?
R: Yeah so I think it's definitely the easiest, um—
S: Cuz you don't have to do partial fractions
R: You don't have to do partial fractions. I mean, they're not that bad. But—
S: [Shrugs] Yeah.
R: But it takes more time, whereas for this [points to residue inscriptions/calculations]— for those poles they'll cancel.
S: Say there's like, z^2 (points to denominator in $f(z)$ inscription] or something way more complicated, like we've seen before, like z^2, $(z+i)^3$— the partial fractions would take like forever. Whereas to do the residues it's much easier.
R: Yeah.

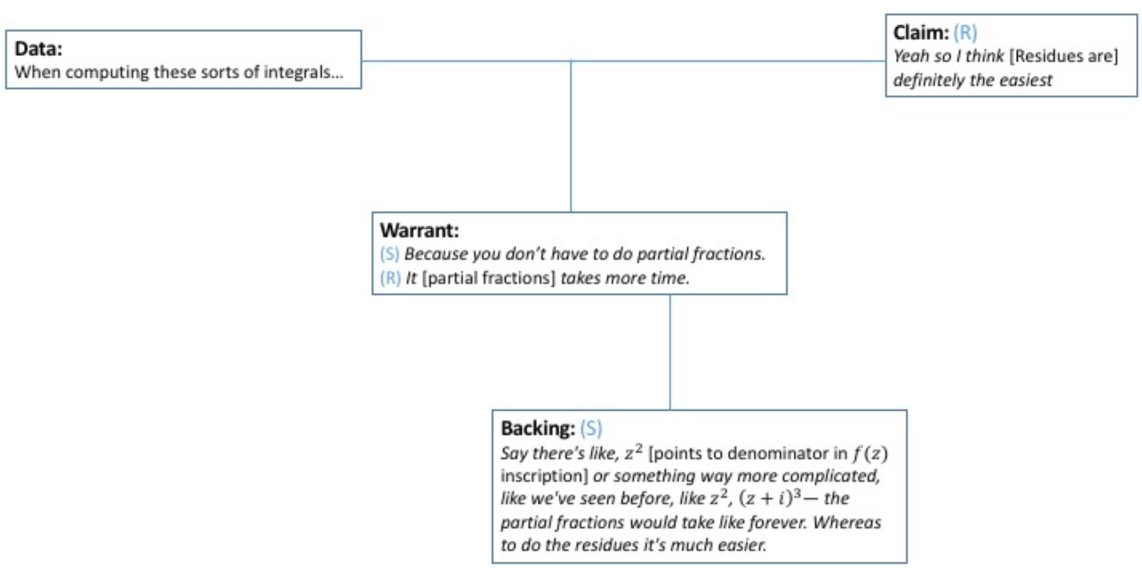


Figure 194. Riley and Sean’s Toulmin diagram for Argument 16 for Task 11.

Task 11 Summary

All four participants insisted at some point during this task that to find the integral of the function along a path containing multiple poles, they would have to either use residue theory or decompose $f(z)$ into partial fractions. In such instances, both pairs also indicated a preference for residues, although when I asked Dan about why he felt partial fractions might be necessary, he and Frank ended up changing their minds and used the Extended Cauchy-Goursat Theorem instead. Moreover, when I asked Riley and Sean what assumptions were needed to use the Residue Theorem, they struggled, particularly when discussing why they thought L needs to be simple and/or Jordan, and if you can have a Jordan curve that is not simple. In Task 11, Riley and Sean continued to evidence more embodied reasoning than Dan and Frank. However, the majority of Dan and Frank's embodied reasoning alluded to small circular paths around poles contained inside the path L while applying the Extended Cauchy-Goursat Theorem, and this was also a common source of embodied reasoning for Riley and Sean as well. As in previous tasks, Riley produced many tracing gestures embodying a closed path as she discussed the integral around such a path, though she tended to be the only participant to do so in Task 11.

Task 12 – Dan and Frank

The twelfth task required participants to provide a general personal characterization of the integral of a complex function, as well as to compare this description to that of a real-valued integral (lines 1-4). Because Dan and Frank's initial response did not contain an explicit argument, I focus the discussion more on the participants' instantiation of Tall's (2013) three worlds. Dan responded first, admitting that he did not attribute any particular meaning to a complex integral (line 5). Subsequently, Frank recalled Professor X's first lesson on integration in their complex variables course, and provided an embodied description of integration as "adding up a bunch of vectors […] along the curve" (lines 6-8). Dan agreed (line 9), and Frank added that such a characterization makes sense because "fundamentally, vectors are all these complex numbers are" (lines 10-11). This statement suggests an invocation of the formal-embodied world, in the sense that the formal identification of complex numbers as vectors lent credence to the embodied drawing that he remembered from class.

Int: What would you say— so what does the integral of a complex valued function represent, just in general, to you? And how is this different from a real-valued function? How is it the same? So just kind of keeping in mind what we've talked about or maybe just how've you thought during the semester. What do you think of with a complex valued integral?

D: I don't know if I actually *have* any type of understanding of what that can represent.

F: I mean, I'm thinking back to like the first lesson that we had, in which case the way I guess I would think about it, which would especially make sense in terms of parametrization, would be adding up a bunch of vectors, uh, along the curve. I think that's what [Dr. X] drew.

D: Yeah.

F: I mean, that's kind of the only thing that makes sense to me. Because fundamentally, vectors are all these complex numbers are. Um, but when we talk about things, kind of like the Cauchy-Goursat Theorem or we're just evaluating it about circles, I don't really take the time to think about it in terms of vectors. I just think about the formulas and the theorems that we regularly deal with. But I guess in the purest form, I think it's just adding up a bunch of vectors.

Frank followed the aforementioned observation with an illuminating remark regarding the way he thinks about integration of complex functions. In particular, he explained that he does not "really take the time to think about [integration] in terms of vectors" when integrating specific functions. Rather, he "just think[s] about the formulas and the theorems that we regularly deal with" (lines 12-13). Accordingly, he compartmentalizes the characterization of integration as adding up vectors, acknowledging it as "the purest form," but one that he need not invoke when computing integrals or using the Cauchy-Goursat Theorem (lines 13-14).

D: Just kind of different from— like if you think about Calc 1, doing calculus in real variables, there's all sorts of ways you can visualize it. You can visualize it—
F: Area under the curve.
D: Yeah area under the curve, or distance that you're travelling, or work, you know. There's so many ways that you can understand it. For this, I honestly don't know.
F: Yeah I guess adding up a bunch of vectors is the simplest way I can think of it, um, yeah in this context.
Int: Can you think of a context in the complex variables where maybe the integral does relate to area somehow, given that you have that Calc 1 analog in the back of your mind? When you think of an integral in the complex situation, can you think of maybe a case where maybe it does represent area?
D: I mean maybe if $f(z)$ is like 1. And—
F: Yeah.

Afterwards, Dan contrasted Frank's formal-embodied vector description against integration of real-valued functions. He pointed out that "doing calculus in real variables, there's all sorts of ways you can visualize [integration]" (lines 15-16). Frank proffered the archetypal embodied portrayal of "Area under the curve" (line 17), and Dan listed additional interpretations as distance travelled and work (line 18). However, in the complex setting, Dan conceded, "For this, I honestly don't know" (line 19). Frank agreed and relayed his previous description involving vector addition (lines 20-21). Because I wanted to hear more of their thoughts about the relationship between integration in the real and complex settings, I asked Dan and Frank a follow-up question about complex contexts wherein integration still represents area (lines 22-24). This question initialized a Toulmin (2003) argument, as I detail below.

Dan began this argument by authoring a symbolic datum considering the function $f(z) = 1$ (line 25). He qualified this contribution with the phrase "I mean maybe," suggesting some uncertainty. Because neither he nor Frank expanded on why this might represent area, I asked Dan why integrating such a function might yield an area, and also asked what path of integration he had in mind (lines 27-29). Frank responded by asking Dan if he was thinking about double integrals, as when integrating a density function over a region in multivariable calculus (lines 31-32). Accordingly, he suggested an embodied-symbolic warrant relating the symbolism of a double-integral to the area of a region via a function embodied as a density.

Int: So why do you think— so with the function 1, what does that give you?
D: Um, hmm—
Int: Like what path are you considering, maybe?
F: Are you thinking like double integrals maybe, from Calc 3? Where like if you integrate 1, as like a density function over some region? Yeah.
D: Umm, I don't know.
Int: Well so is there a way to relate one of these types of integrals to a double integral?
F: I mean, yeah in terms of Green's Theorem. In the sense that a lot of the integrals that we've been doing are about like, closed curves. Uh, in which case you can represent the line integral as a Green's Theorem double integral. So I guess, in that way, we could come up with a way to solve for the area of something.
F: Um, and I mean, that also makes sense because it's a closed curve, and like it's kind of irrelevant trying to find the area of a shape that's not closed [laughs]. Um, so yeah that's really the only connection I can bring. Do you see anything other than that?
D: I don't think so.

Although Dan did not confirm that he necessarily thought about the situation in this way (line 32), I asked him and Frank how they might relate such an integral to a double integral (line 33). As author, Frank answered that Green's Theorem could provide such a connection, given that most integrals they dealt with involved closed paths. Hence, he provided formal-symbolic backing for his prior warrant's validity by describing why a double integral applied to the situation at hand. This backing also contained a formal-embodied aspect, in the sense that he perceived the theorem to be applicable based on the closed attribute of the path. He further elaborated, "that also makes sense because […] it's kind of irrelevant trying to find the area of a shape that's not closed" (lines 38-39). Frank concluded, with some hesitation, that "in that way, we could come up with a way to solve for the area of something" (lines 36-37). This sole argument about Task 12 is summarized in Figure 195 below.

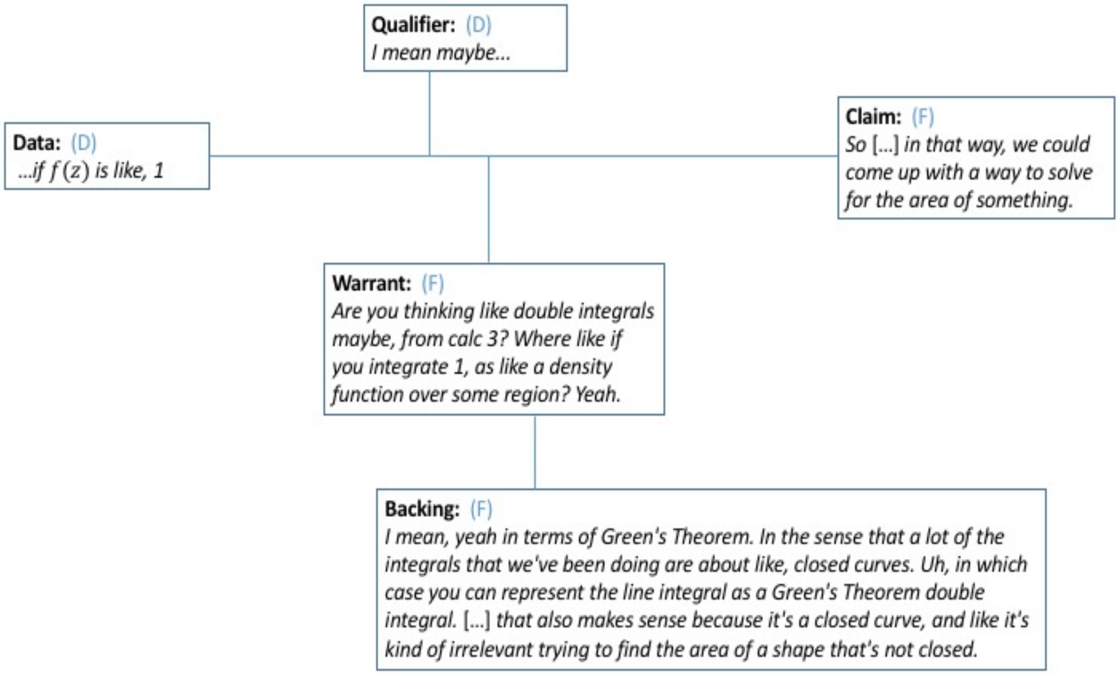


Figure 195. Toulmin diagram for Dan and Frank's Argument 1, Task 12.

Task 12 – Riley and Sean

Although I articulated Task 12 to Riley and Sean in such a way that did not favor any one of Tall's (2013) three worlds (lines 1-5), Riley responded by discussing her lack of a "good *geometric* interpretation" for integration in the complex setting (line 6). She provided the usual interpretation of real integrals as "area under the curve," but deliberately avoided *thinking real, doing complex* (Danenhower, 2000) as she clarified that "that analogy doesn't apply […] for complex functions" (lines 6-10). As such, she reiterated her difficulty with embodying integration of complex functions (lines 10-11). Sean suggested that he and Riley attempt to come up with such an embodied interpretation (line 12). In response, Riley recalled discussing "displacement and stuff" from earlier in the interview, but conceded that she does not naturally think of such physical interpretations, but rather "algebra" (lines 13-14). Regardless, Sean suggested they try something to "see where it goes" (line 15).

Int: Ok. Sounds good. Ok so just a couple kind of general questions— uh, these were kind of specific with examples and things like that. But kind of, with all this in mind, um— What do you think the integral of a complex-valued function represents, and how is this different, say, from a real-valued function and how is it the same? Just generally speaking, what do you think of when you see a complex integral?
R: Um, yeah I don't really have a good geometric interpretation of that. I mean when I think about a real-valued integral, I'll always think area under the curve.
Int: Right.
R: But that doesn't— that analogy doesn't apply for complex variables, and um— for complex functions, rather. Um, so yeah I don't really have a good like, um, way I can think of it concretely, in terms of like a picture or something.
S: We could try to inspect it.
R: And we did— we did a little bit of that earlier with like displacement and stuff, but I don't immediately think like, 'Oh! Displacement!' I just think, 'Oh! Algebra! [laughs]'
S: I kind of think of a similar idea. Let's see where it goes.
R: So I don't know. How do you think of it?

Rather than continuing to discuss their thoughts and tendencies in generality, Sean began an argument connecting the complex setting "back to Calc 3" (lines 17-18). He authored a symbolic datum considering a complex function $f(z)$ as "an ordered pair of points," $u(x, y), v(x, y)$ (line 17). Continuing with symbolic reasoning, Sean relayed his previous penchant towards Calc 3 and wrote corresponding inscriptions for integration in that setting as $\int_C \vec{F} \cdot dr$ (lines 20-21). Riley authored a hesitant claim that such inscriptions represent work, and qualified this statement by questioning, "right? Or Something?" (line 22).

After relaying his previous datum about an ordered pair, Sean re-voiced this as spokesman in the language of vectors (lines 23-24). He then authored a new embodied-symbolic data considering an arbitrary point z expressed as a vector, the function $f(z) = z$, and a semicircular, positively oriented path passing through his point (lines 24-27; see Fig. 196). Continuing with embodied-symbolic reasoning, Sean authored a claim that dz represents an "incremental path," which he illustrated by sketching a small vector between nearby points along the curve (lines 27-29; see Fig. 197). Consequently, Sean concluded as spokesman that the original integrand can be rewritten as a dot product between the vector $f(z) = z$ and the differential vector dz (lines 29-31).

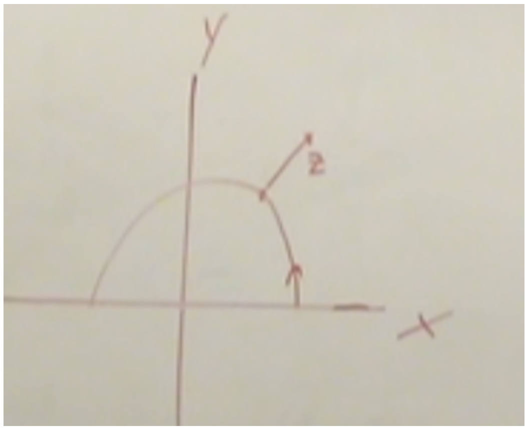


Figure 196. Sean’s initial setup for Argument 1, Task 12.

S: [Writes $\int_L f(z)dz$]. Ok so $f(z)$ is an ordered pair of points [writes = $u(x,y)\,,v(x,y)$]. So I always think of it as like— kind of coming back to Calc 3.

R: You need to close your parentheses.

S: So I always— I love thinking back to Calc 3. I love thinking back to [writes $\int \vec{F}$] function— it was like F dotted with dr, over your cuve C [finishes writing $\int_C^{\square} \vec{F}\cdot dr$]. And so—

R: That ended up being a work function right? Or something?

S: Yeah. So— so think of this [points to $f(z)$] as an ordered pair of points. So you can think of this as some kind of vector, I guess. So say we have this point [plots a point] and say your function is just z. There's your vector right here [draws a vector and labels z]. And say you're going on a semicircle [draws in semicircle passing through the point z]. Then dz is just your— oh we're going positive orientation [draws arrow on semicircular path indicating positive orientation]. Then dz is just your incremental path, so if we're going like here to here [draws points near the point z] so there's a little differential element there [draws tiny vector between the two local points on curve]. So it'd be like this [points to z vector] multiplied by that [points to differential vector], which in the language of Calc 3, we'd say this dotted with that [points to $\vec{F}\cdot dr$]. And then we'd just add it up over the whole thing [gestures with little segments along the semicircular path].

S: So it's like $\vec{F}$ is kind of like your two-dimensional vector field [puts “vector” in air quotes] that you're taking like the dot product of with your path.

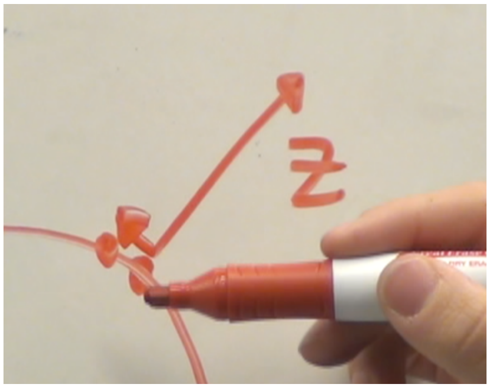


Figure 197. Sean's vector dz representing an "incremental path," Argument 1, Task 12.

As he articulated this embodied-symbolic claim, Sean pointed to the embodied vectors on the whiteboard to convey which objects to "multiply," and also pointed to the symbolic inscription $\int_C \vec{F} \cdot dr$ to highlight how the "language of Calc 3" influenced his complex reasoning (lines 30-31 & 33-34). Finally, he clarified that the integral symbol meant adding up the results given by his dot products, and he provided an embodied gesture to illustrate this summing of vectors along the semicircular path (lines 31-32; see Fig. 198). As I discuss further in the Task 12 Summary and in Chapter V, note that Sean's application of the dot product to complex numbers is incorrect, given that complex numbers have their own well-defined product. Nonetheless, Argument 1 is summarized in Figure 199.

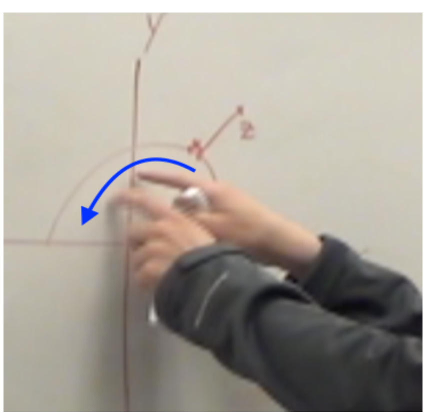


Figure 198. Sean's gesture for "Just add it up" in Argument 1, Task 12.

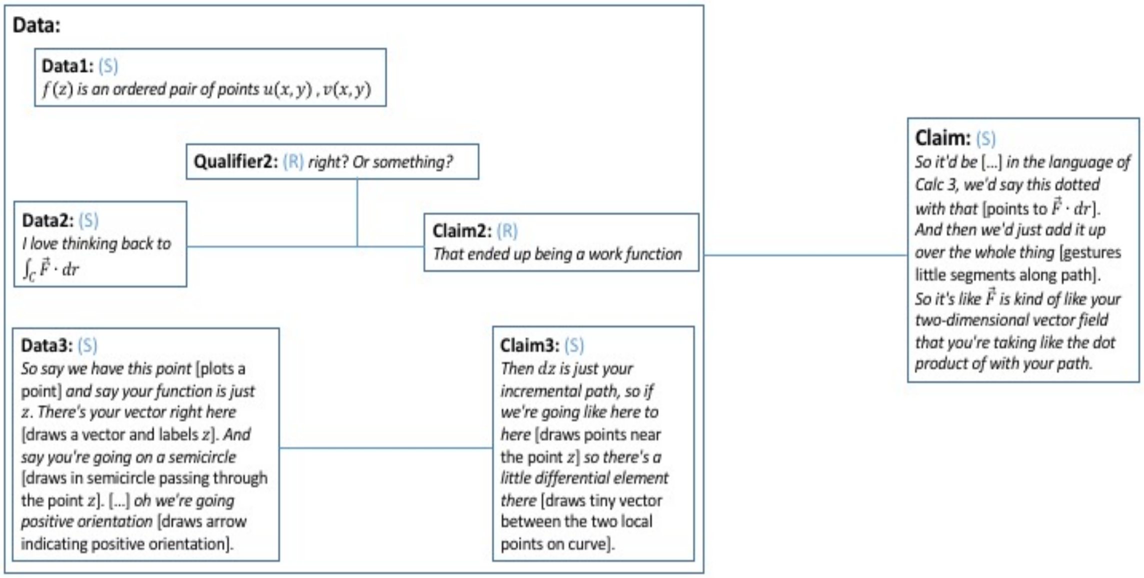


Figure 199. Toulmin diagram for Riley and Sean's Argument 1, Task 12.

Following up on her previous claim in Argument 1, Riley began Argument 2 by revisiting her connection to a work application (line 35). In particular, she authored an embodied-symbolic datum characterizing the function $F(z)$ as "the force being applied at z" (line 36). Citing an embodied warrant of "force applied over distance," Riley re-voiced her claim as spokeswoman and concluded "you could think of it in terms of work" (lines 38-39), though she qualified this claim with "I guess" (line 38).

R: Yeah, can you uh— right ok. Cuz I'm pretty sure I always thought of those as like a work function. $F(z)$ is like the force being applied at z.
S: Mhm [shakes head yes].
R: So I guess if you take that parallel then um, force applied over distance, so it's— yeah I guess you could think of it in terms of work.
S: Yeah so this gives you zero cuz you're always perpendicular [points to the two vectors z and dz]. But yeah, a calc 3 perspective. But um, that's what I think of. But here's your components. It's kind of like F [writes $\vec{F} = \mathrm{P(x,y), Q(x,y)}$], $P(x,y)$ and $Q(x,y)$ as your x and y component. This is kind of like your x and y component [points to $u(x,y), v(x,y)$]. That's how I think about it; I think about it like vectorially.

Sean agreed and authored a symbolic claim that this integral would be zero; he provided a supporting embodied warrant that z and dz are "always perpendicular" (line

40). Sean additionally provided backing for this warrant's field by emphasizing the connection to Calc 3 as he drew symbolic parallels between the real component functions $P(x, y)$ and $Q(x, y)$, and the complex component functions $u(x, y)$ and $v(x, y)$ (lines 41-44). As spokesman, Sean summarized that he thinks of complex integration "vectorially" (line 44). Argument 2 is depicted in Figure 200 below.

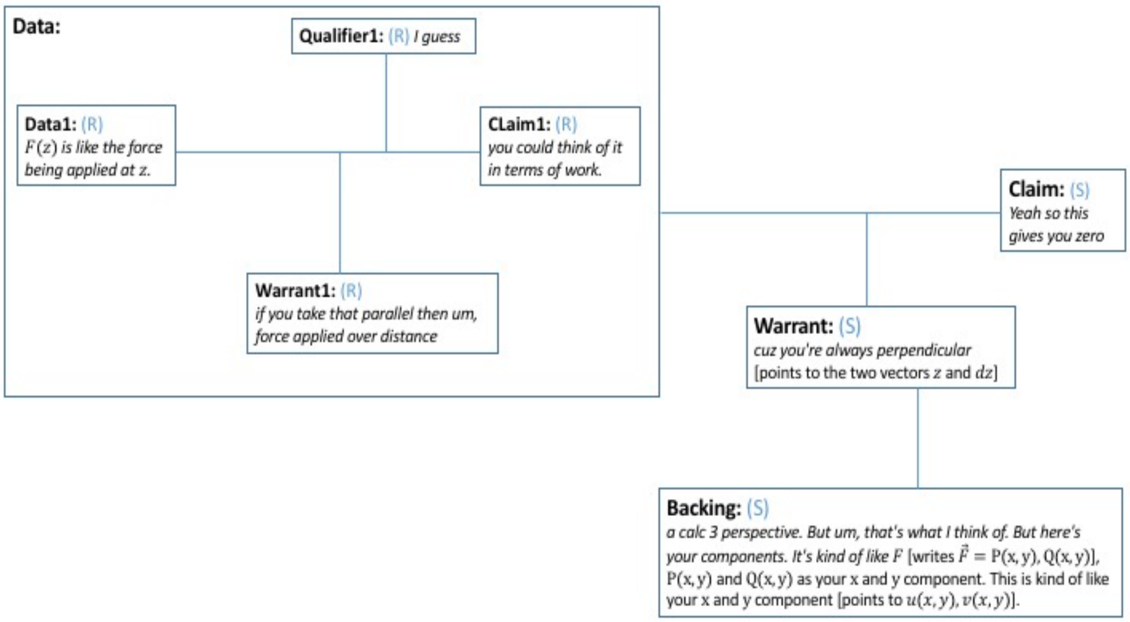


Figure 200. Toulmin diagram for Riley and Sean's Argument 2, Task 12.

Int: Ok cool. And do you remember like, ways to relate like a single integral like that $\vec{F} \cdot dr$ to maybe like a double integral or anything like that?
S: If it's a closed region, then you can do um, Green's Theorem. So if, so if you were in Calc 3 [points to $\int_C^{\square} \vec{F} \cdot dr$] then you could say, use the definition [points to the P and Q components from before] then you could say this is really [writes double integral symbols]. So say C is really the boundary of a region D, so D is some type of region, then this is really just [writes integrand of $Q_x - P_y$].
R: You remember so many more equations than I do [laughs]. But we did also brush over very briefly that that works for uh, complex as well.
S: Yeah so I think it was used in the theorem of the— the Cauchy Theorem of integrals.

Subsequently, I asked Riley and Sean if they remembered a way to relate an integral such as $\int_C \vec{F} \cdot dr$ to a double integral (lines 45-46). In response, Sean began a third argument by authoring an embodied datum considering a closed region D and its boundary C (lines 47, 49-50). He clarified his formal warrant, Green's Theorem (line 47), and symbolically concluded that one can write the previous single integral as the double integral $\iint Q_x - P_y$ (line 50). Moreover, Sean and Riley provided backing for their warrant's field, in that Sean identified Green's Theorem as a Calc 3 property, but Riley argued that this result "works for uh, complex as well" (lines 51-52). Sean additionally qualified Riley's contribution to the backing by remarking, "I think it was used in […] the Cauchy Theorem of integrals" (line 53). Argument 3 is summarized in Figure 201 below.

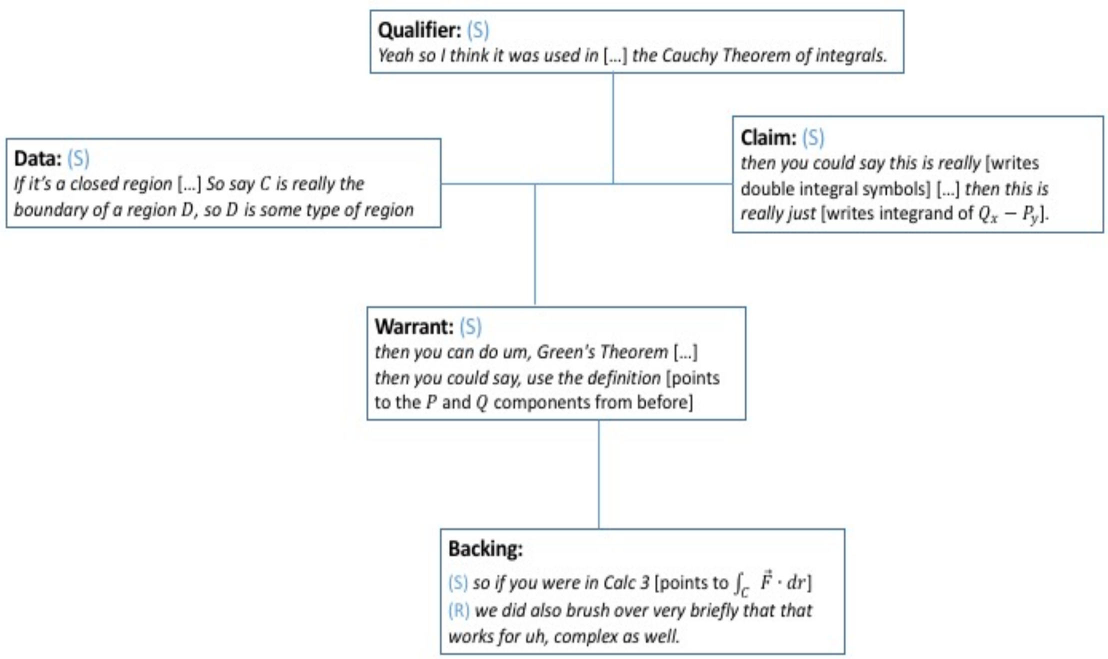


Figure 201. Toulmin diagram for Riley and Sean's Argument 3, Task 12.

Afterwards, I asked another follow-up question about how they interpret double integrals, this time probing more specifically about Riley and Sean's embodied perceptions (lines 54-55). Riley answered that she thinks about "an area bounded by two curves" (line 56), and Sean agreed (line 57). Riley then elaborated on her thoughts via

Argument 4, as follows. She began by authoring an embodied datum considering a region D and an embodied claim about connecting two curves within this region (lines 58-60; see Fig. 202). She also provided her usual qualifier of "right?" to obtain affirmation (line 60). Employing embodied-symbolic reasoning, Riley authored a datum detailing the limits of integration and specifying the integrand $f(z) = 1$. In this case, Riley concluded that the double integral of this function over the region D yields the area of D (line 62).

Int: So when you have a double integral like that then, what does that usually tell you, like physically or just— what do you think of when you see a double integral as opposed to like a single integral?
R: Um, I think of an area bounded by two curves.
S: [Nods head in agreement]
R: Yeah so you can choose any region you like. But we're just saying that it's over the region *D* right? So if *D* is like this [draws a region *D*] then you can choose a couple curves and so you go one curve [unintelligible] right? And so you start by integrating the two curves and you integrate from this point to that point [points to right-most point then left-most point on curve] and you end up with, over that region— so if it was 1 [the function] then you'd end up with the area of that region. If you're integrating over a function, um, then, depending on what the function represents, that could be like, the mass of the region or something like that. Like, depending, like if you had a density function then that would be the mass, right?
S: Mhm.
R: Um, but yeah. When I see a double integral, I always think area.
S: [Shakes head in agreement]
Int: And when you see a Calc 1 integral, you think of area under the curve, you said?
R: Yeah. And I guess that's—

On the other hand, Riley authored another datum considering a more involved integrand (lines 62-63) and articulated an embodied warrant indicating that the physical interpretation of this integral depends on what $f(z)$ represents (line 63). She ultimately decided to consider a density function, and concluded that the integral of such a function yields a mass (lines 64-65). She once again qualified this assertion with the word "right?" (line 65), and Sean affirmed her claim (line 66). As spokeswoman, Riley returned to her previous statement that she thinks of areas when she comes across double integrals, and Sean agreed (line 67-68). Riley's Argument 4 is summarized in Figure 203.

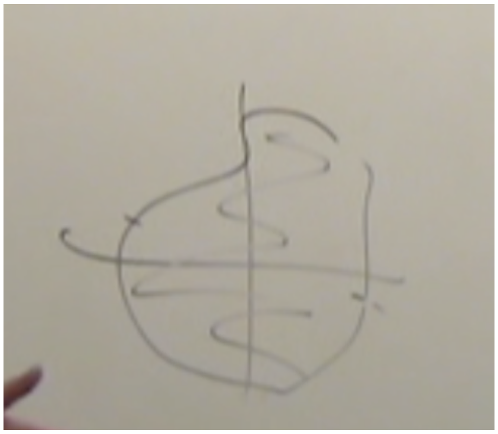


Figure 202. Riley's diagram illustrating the region D in Argument 4, Task 12.

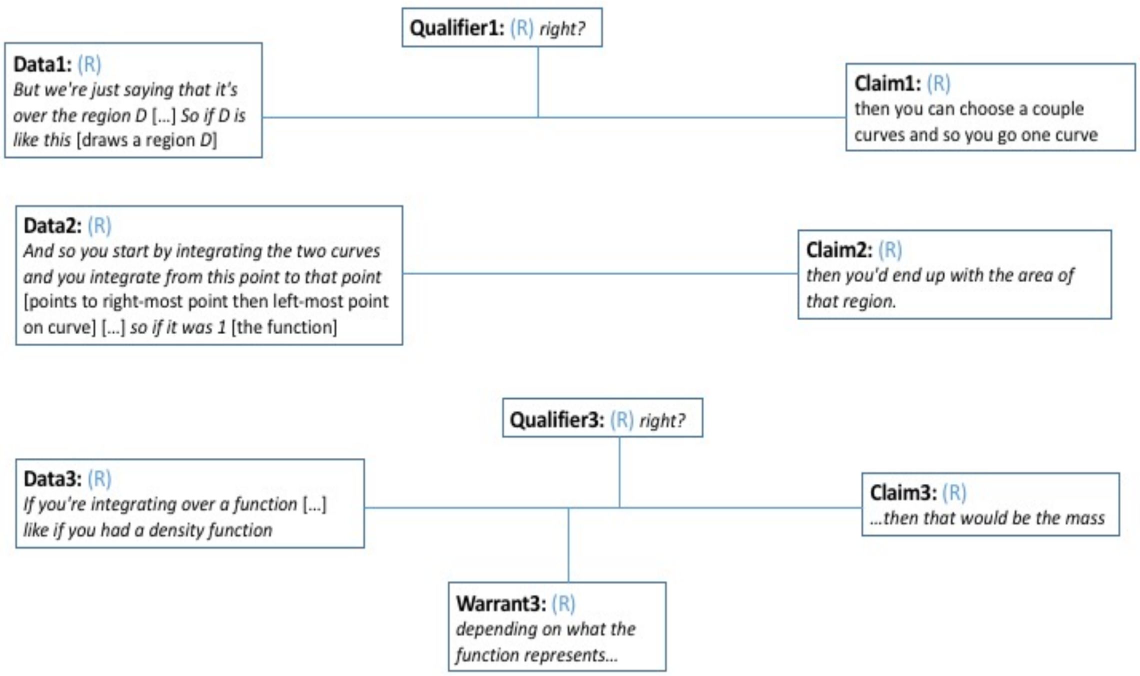


Figure 203. Toulmin diagram for Riley's Argument 4, Task 12.

S: And so like— sometimes double integrals you can think of as like $f(x, y)$ is like a little surface over the *x-y* plane, so it's like a volume kind of, of that surface over the *x-y* plane up there. That's always positive. Those are the applications, just integrating over two variables, really.
Int: Cool. Well that's one of the nice things about this area, is there are tons of applications, so—

In the brief embodied-symbolic Argument 5, Sean authored a warrant that one can think of $f(x, y)$ as a "surface over the *x-y* plane" (lines 71-72). Thus, he concluded that a double integral in such a context represents the volume of that surface (lines 72-73). Argument 5 is depicted in Figure 204 below.

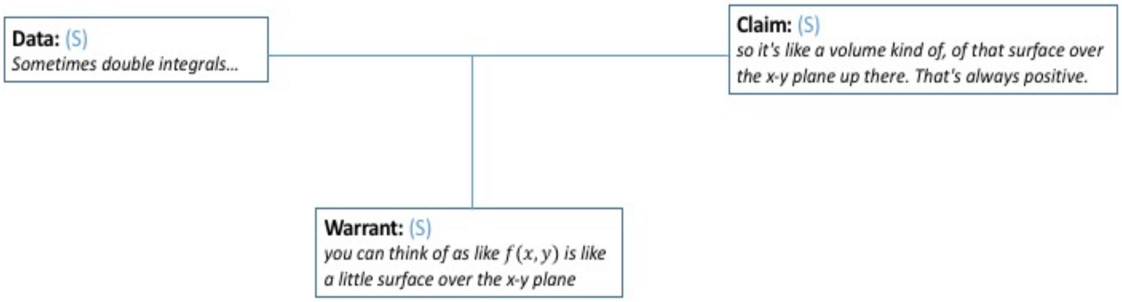


Figure 204. Toulmin diagram for Riley and Sean's Argument 5, Task 12.

S: Of course then I always, I always go to like— like a more general version, saying like this is not just a simple flat curve. It's like an actual surface. It's like, ok well this really just turns into the double integral of the like, um, the curl of F dotted with the surface [writes $(D \times F) \cdot dS$], which reduces to this [points to Green's Thm] for the *x-y* plane, but it goes to this more general version, I think it was Stoke's Theorem at this point.

S: But it makes me question, like, could this like generalize with complex numbers? But kind of not, because it's like you have a third dimension so it's kind of weird. Maybe there is but I don't really know. But Complex, you see like a— cuz some of the— I know some of the applications of complex numbers are like, electric fields and certain things with like flow fields and arrows which are all two-dimensional, and since complex is two-dimensional you can use like all these results [points to Green's/Stokes] but three-dimensional, I don't know whether it goes there or not.

Int: Well, so what makes uh, complex valued functions harder to visualize just in general?

Subsequently, Sean discussed a "more general version" (line 75) of his and Riley's thoughts in Argument 6. He authored an embodied datum considering a setting comprised of a three-dimensional surface rather than just a curve (lines 75-76). Sean then authored a symbolic warrant detailing how the integral setup changes and incorporates the curl of the function in this context (lines 76-77). Incorporating formal-symbolic reasoning, he claimed that this revised symbolism reduces to Green's Theorem "for the *x-y* plane, but it goes to this more general version, I think it was Stokes' Theorem at this point" (lines 77-79).

With these connections in mind, Sean proffered a tentative claim about extending this type of argument to the complex setting (line 80). He immediately followed this claim with a rebuttal reflecting his interpretation that “it’s like you have a third dimension so it’s kind of weird” (lines 80-81). Sean also qualified his claim by conceding, “Maybe there is but I don’t really know” (lines 81-82). In support of his connection, he authored a formal-embodied warrant listing various physical interpretations of complex applications, all of which he identified as two-dimensional and thus compatible with Green’s Theorem (lines 82-85). As such, he pondered whether these applications could generalize to three dimensions (line 85). Sean’s Argument 6 is depicted in Figure 205.

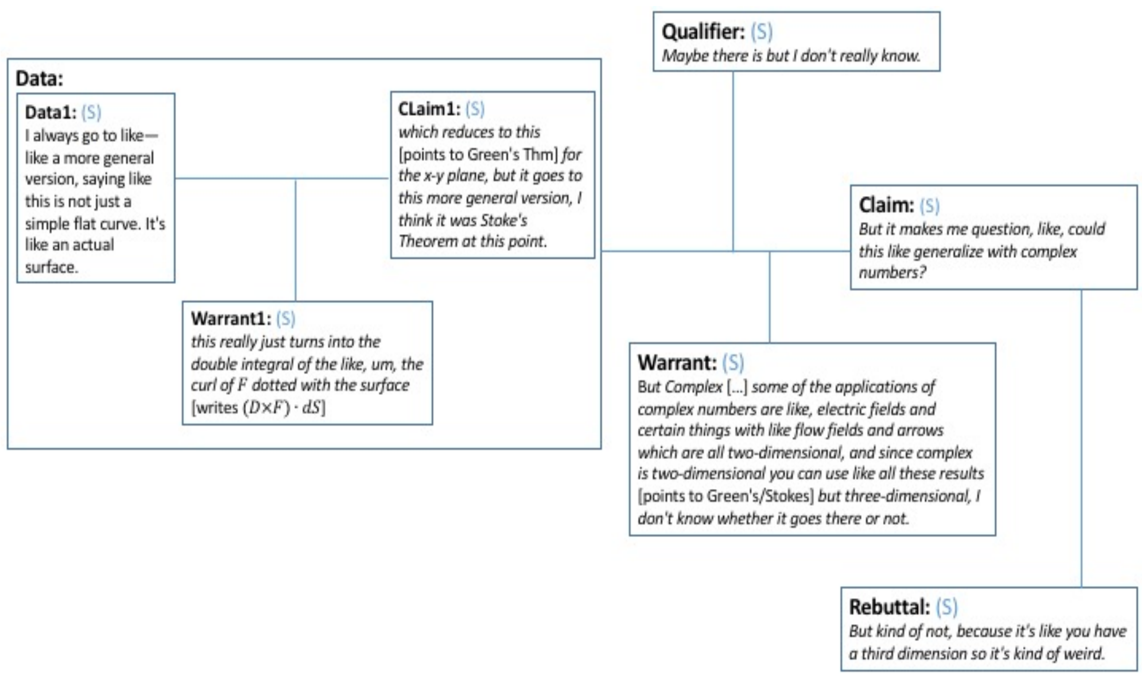


Figure 205. Toulmin diagram for Sean’s Argument 6, Task 12.

87 *R:* Well you're going from an x-y plane to an x-y plane. So, so when you're going from like the x to y
88 right, and you're graphing just a normal function, you sort of graph an independent variable and a
89 dependent variable. And it's easier to graph in two dimensions cuz that's how we draw things, right?
90 *S:* Mhm.
91 *R:* Um, when you're going from already two dimensions and graphing to another two dimensions, that
92 becomes four-dimensional, uh, graph, if you want to do independent and dependent variables, which
93 we don't graph very well because— three dimensions is hard enough and four, you have to have color
94 or time or something and it's not very fun.
95 *S:* And you have things like cuz like $\mathbb{R}^2$ is just the x-y plane it's just two real axes. But then you have like
96 $\mathbb{C}^2$ is like four axes and you can't really visualize that at all. So it's much harder to visualize having an
97 extra axis. So you lose your visuals, but you still have all the math! You just generalize it to n
98 dimensions for $\mathbb{C}^n$ but generally past 3-D our brains are lost.

As one final related follow-up question, I asked Riley and Sean what makes complex functions difficult to visualize (line 86), which prompted Argument 7 as follows. Riley responded by authoring an embodied datum that "you're going from an *x-y* plane to an *x-y* plane" (line 87), and contrasted this against a "normal function" in which case "you're going from like the *x* to *y*" (line 87). In the latter setting, she authored the embodied claim that "it's easier to graph in two dimensions" and offered the embodied warrant "cuz that's how we draw things" (line 89). She returned to the complex setting and claimed that graphing "becomes four-dimensional […] which we don't graph very well" (line 92), and cited the embodied warrant that "you have to have color or time or something" as the fourth dimension (lines 93-94). As spokesman, Sean added to this warrant by articulating that there are essentially "4 axes and you can't really visualize that at all" (lines 95-96). Accordingly, he claimed, "So you lose your visuals, but you still have all the math!" (line 97). Argument 7 is summarized in Figure 206.

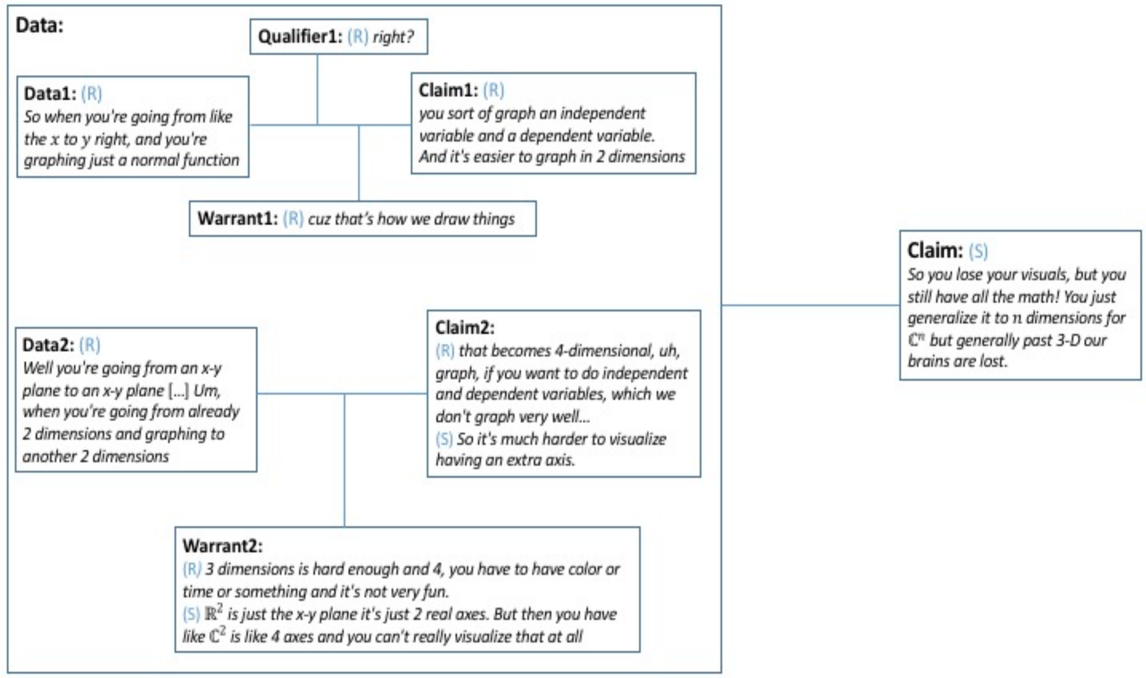


Figure 206. Toulmin diagram for Riley and Sean’s Argument 7, Task 12.

Task 12 Summary

Both pairs of participants discussed an embodied interpretation for complex integration as summing vectors, however neither completely fleshed out the geometric details of this description. Although Sean’s “Calc 3” approach from Argument 1 was on the right track, he incorrectly applied the dot product to the complex setting, as the complex field $\mathbb{C}$ is endowed with its own multiplication operation on vectors. Compared to Dan and Frank, Riley and Sean formed more actual arguments in response to Task 12, though to be fair, several of these stemmed from follow-up questions I asked. During these extra arguments, Riley and Sean instantiated backing for their warrants’ field when alluding to Calculus 3 ideas for integration; this is notable because this type of backing was used only sparingly throughout most of the other tasks.

Additionally, in this task, Riley and Sean demonstrated two types of reasoning related to *thinking real, doing complex*. First, Riley deliberately attempted to avoid thinking of complex integration as area under the curve, thus avoiding *inappropriate* invocation of thinking real, doing complex. On the other hand, Sean made connections back to Calculus 3 that supported thinking of complex numbers and functions as vectors. As discussed previously, some of these connections were beneficial while other details were problematic. Although both pairs of participants exhibited these two types of reasoning individually in other tasks, this task was relatively unique in that they alluded to both in the same task. That being said, this task seems like a natural candidate for both types of reasoning to show up, in that it called for a general overview of participants' thoughts on the meaning behind complex integration.

Task 13 – Dan and Frank

The thirteenth and final task required participants to discuss the conditions under which a complex function has an antiderivative (lines 1-3). Frank began Argument 1 by commenting that this question was on their recent exam (line 4). He authored formal-embodied data considering the existence of a simply-connected domain that contains the path of integration (lines 4-6). Frank also stated a qualifier that "if f is analytic everywhere, then it's really easy" (line 6). As relayer, he reiterated the aforementioned data and concluded that under these circumstances, f has an antiderivative (lines 7-8). Using symbolic reasoning, he also claimed that computing such an antiderivative would entail application of "our Calc 2 techniques—just treating z as our variable instead of x" (lines 8-9). Argument 1 is summarized in Figure 207 below.

Int: Ok great, yeah that's good. Um and then, just one last thing, given that we've kind of brought up at several points, antiderivative ideas. So when does a complex valued function have an antiderivative? And why would that be useful to know when you have an antiderivative?

F: This was on the test [points at D and laughs]. Um, if f is an analytic function in an, um— let's assume that we're integrating it over some,uh, some curve and there is a simply connected domain that contains that curve. Um, then— or really, if f is analytic everywhere, then it's really easy. But as long as we're integrating it in some region or on some curve and there is a simply-connected domain that contains that curve, then uh, f has an antiderivative. And we can solve for it the same way that we would uh, using our calc 2 techniques— just treating z as our variable instead of x.

Int: So thinking back, maybe, on the first few tasks, where I think sometimes your initial preference would be to use an antiderivative, but then you know, maybe you were a little hesitant at some points to see if you could actually use that or not. Um, how does that relate to your answer about the question I just asked? So you now kind of remember when you have an antiderivative. So thinking about the earlier tasks then, does that dictate how you might approach a certain integral? Like whether you have an antiderivative or not? Or like when you might go to something like a straight parametrization or something like that?

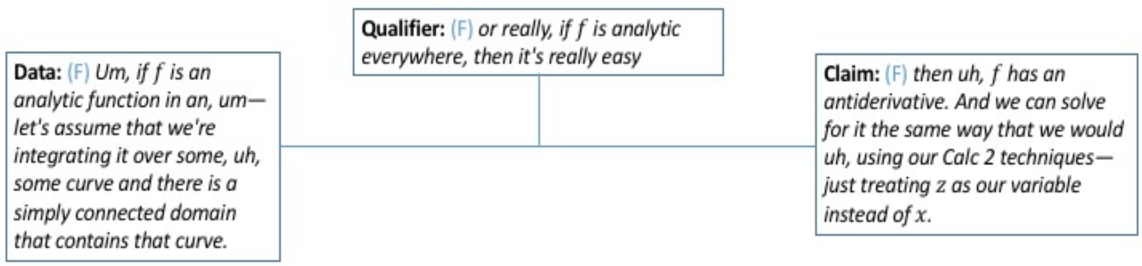


Figure 207. Toulmin diagram for Dan and Frank's Argument 1, Task 13.

With this brief response to my initial question in mind, I asked Dan and Frank to reflect on their preference for applying antiderivatives as opposed to parametrization techniques in previous tasks (lines 10-16). This initiated a second argument relevant to Task 13, which Dan alone provided. As spokesman, he considered a formal datum wherein the integrand is entire, and concluded, "you'd want to use an antiderivative" (lines 17-18). He also qualified this assertion with the word "clearly," expressing a high degree of certainty.

D: I think, um, if there— like if the function is analytic everywhere, so you have an entire function, then clearly you'd want to use an antiderivative. But it got messy having to deal with branch cuts. Um, especially if you're thinking of like, trying to program a computer, for instance, to evaluate these types of integrals. It's a lot easier to just parametrize it, rather than deal with breaking up your function, making some different branch cuts that'll make it work— to make sure that your function is analytic on the contour between those two points.
D: So I don't know. I think like, in general, it's just easier to think about parametrizing rather than having to deal with branch cuts and trying to make a function be analytic on some contour.
Int: Are there going to be cases where you can't parametrize to find the integral? Or is that always a solution and it just might get kind of messy?

Subsequently, he authored a symbolic datum considering programming a computer to evaluate such integrals (lines 19-20). According to Dan, in this situation, “It's a lot easier to just parametrize it, rather than deal with breaking up your function, making some different branch cuts that'll make it work— to make sure that your function is analytic on the contour” (lines 20-22). Dan’s claim represents embodied-symbolic reasoning, in that his hypothetical method relates symbolically “breaking up your function” to the embodied process of choosing a proper branch cut. Curiously, Dan proceeded to explain that “in general, it’s just easier to think about parametrizing rather than having to deal with branch cuts and trying to make a function be analytic on some contour” (lines 23-24). This statement seemed inconsistent with Dan’s preference for applying antiderivatives on earlier tasks such as Tasks 9a and 9b, so I asked whether they thought there were situations in which one cannot parametrize to find a complex integral (lines 25-26). Argument 2 is depicted in Figure 208 below.

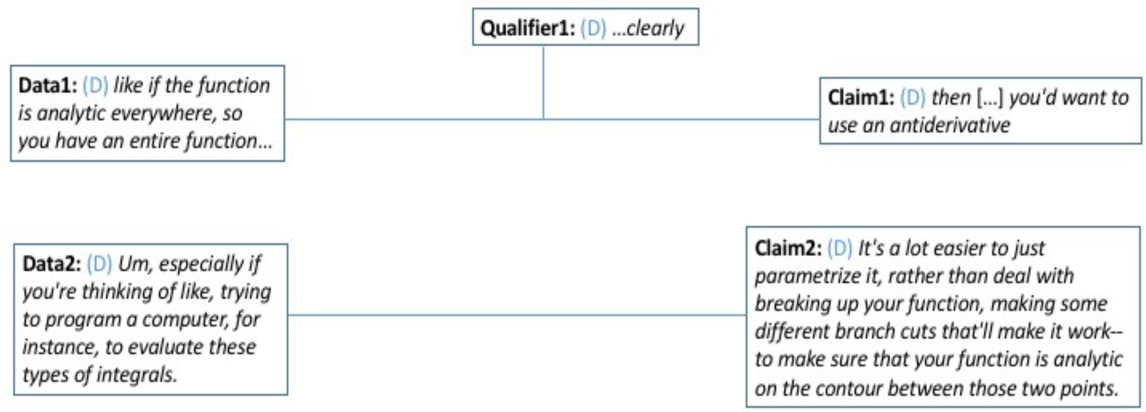


Figure 208. Toulmin diagram for Dan and Frank's Argument 2, Task 13.

D&F: [Both make slanty faces, suggesting unsure]
F: Yeah I think—
D: You can always do that, pretty much.
F: Yeah I mean, like—
D: Cuz all the formulas that we used later on, in our course, I mean— we were always just like, "Yeah you could parametrize it! But it just might take a long time."
F: Right. So there's always a way to like brute force it, in that sense, um, through a parametrization. But yeah as Dan pointed out, there are really— the issue that comes up as to whether or not that simply-connected domain exists, and if so, where— how do we define it? Like where are its discontinuities now and how do we then adjust the arguments at play? Um, so yeah. I think yeah in the same context that you [points at D] put it. Like if you were dealing with computing, then um, computers don't care how much algebra they have to do, so. I mean forcing it to do a uh a parametrized curve really wouldn't be the end of the world. But uh, yeah I guess that would be my only yeah—the only reason why I would use parametrization, is it's simpler in that sense. But I guess the machinery behind it is more complicated.
Int: Ok great. Well that's all I have for you guys. Thanks so much for taking the time to chat with me.
D: We're happy to help.

In response to my follow-up question, Dan and Frank stood in silence with confused looks on their faces for several seconds (lines 27-28). After this pause, Dan authored a claim that "you can always do that," and qualified this assertion with the phrase "pretty much" (line 29). He additionally authored a formal-symbolic warrant, explaining that towards the end of their course, the class discussed parametrization as a viable but potentially tedious alternative to "all the formulas that we used later on" (lines 31-32). As spokesman, Frank agreed with Dan's prior claim that parametrization is always possible, though he recognized it as a "brute force" symbolic method (line 33).

Continuing as spokesman, Frank recapitulated Dan's prior embodied comments about the importance of finding a simply-connected domain containing the path of integration, and the need to adjust arguments relative to the choice of branch cut (lines 33-36). He also re-voiced Dan's datum from Argument 2 about using computers to perform tedious computations in the context of parametrization, clarifying that "computers don't care how much algebra they have to do" (lines 36-39). He summarized, using symbolic-formal reasoning, that he would only use parametrization in the sense that it is "simpler," yet "the machinery behind it is more complicated" (lines 39-41). Argument 3 is depicted in Figure 209 below.

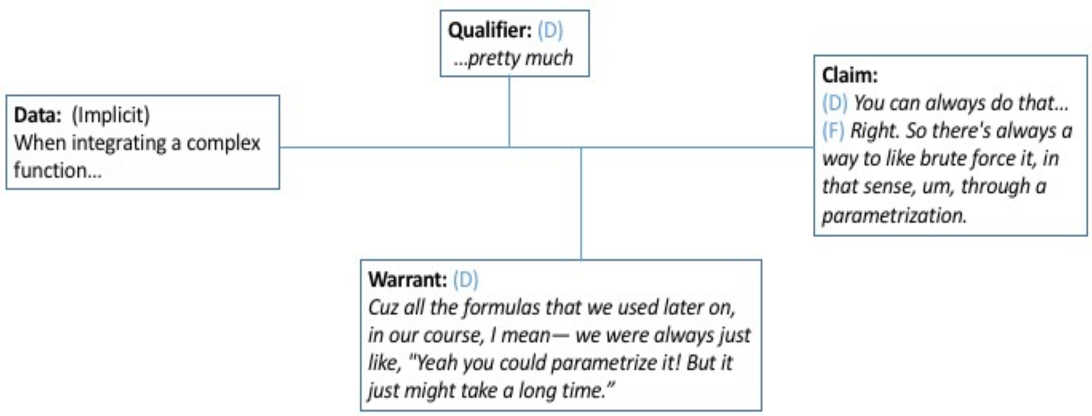


Figure 209. Toulmin diagram for Dan and Frank's Argument 3, Task 13.

Task 13 – Riley and Sean

Riley first responded to Task 13 by authoring a formal claim that whenever the function in question is analytic, it has an antiderivative, and Sean agreed (lines 5-6). She further added that such analyticity implies the existence of harmonic component functions u and v, but qualified this second assertion with "I guess [...] right?" (line 7). However, Sean challenged her and authored an alternate datum of differentiability as the requirement needed for the harmonic property (line 8). As spokesman, Sean implemented formal reasoning to clarify that “if it’s differentiable over a whole region then it becomes analytic” (line 10). Riley relayed this statement (line 11) and Sean authored a supporting warrant that differentiability merely along a line does not guarantee analyticity (line 12). Argument 1 is summarized in Figure 210.

Int: Cool. Ok great. So I just have one more question for you guys. And it deals kind of nicely with something that we talked about really towards the beginning of the interview. And this is about antiderivatives. So I want to know, when does a complex-valued function have an antiderivative, and why would this be useful to know when you have an antiderivative?
R: Um, so when it's analytic, you always have an antiderivative.
S: Mhm.
R: And I guess, if it's analytic, then you have a harmonic function for u and v right?
S: That's differentiable.
R: But if it's—
S: What I'm saying is like if it's differentiable over a whole region then it becomes analytic.
R: If it's differentiable on a region, then it's analytic.
S: Yeah cuz if it's just a line then it's not analytic.

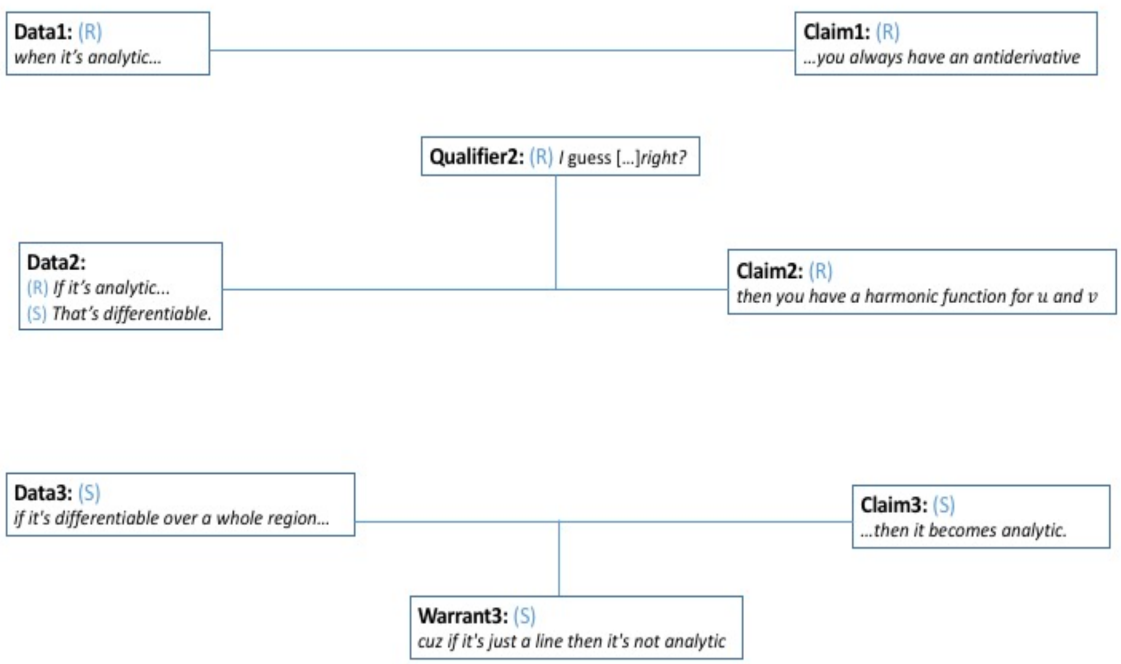


Figure 210. Toulmin diagram for Riley and Sean’s Argument 1, Task 13.

Afterwards, Sean provided a second argument expanding upon his warrant in Argument 1. He authored a symbolic datum considering the integral setup in Calculus 1, which he qualified as “pretty easy” and “simple” because there is a “one-dimensional axis" (lines 13-14). He then authored a formal-symbolic claim that $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$, provided that f "has no jump discontinuities or anything like that" (lines 13-15). Sean turned to the complex setting and authored an embodied-symbolic datum considering the integral $\int_A^B f(z)\,dz$, where A and B are endpoints along some curve C (lines 16-17). He claimed that, so long as the assumptions are met, using the Fundamental Theorem is "much easier" than parametrizing the curve C, and authored a symbolic warrant that one can simply "pick endpoints and subtract" (lines 17-20). At this time, Riley began writing the necessary requirements on f in order to invoke the Fundamental Theorem (line 20); Sean stepped in, agreed, and added to them (line 21). In doing so, Riley challenged Sean's claim about a simple closed curve, and instead claimed that they should be working within a simply connected domain (lines 21-22), which Sean acknowledged and finished articulating (lines 22-23). Finally, he added that one can use "any paths through the domain" (line 23). Figure 211 summarizes Argument 2.

S: So in Calc 1 [writes $\int_a^b f(x)\,dx$] we'd say, pretty easy, one-dimensional axis and this is $F(b) - F(a)$ for $F'(x) = f(x)$ [writes $= F(b) - F(a)$ and $F'(x) = f(x)$], the antiderivative. Simple, assuming this [points to $f(x)$] has no jump discontinuities or anything like that.
S: But for this one, [writes $\int_C f(z)$ then erases C and writes limits of A and B] so I'll say C is like point A to point B on a curve C. So if this works, it's much easier because instead of using the whole parametrization, like z equals some function of t, parametrizing $f(z(t))$, $z'(t)$, dz/dt, integrate all the way through, you know, hard. But if you use this [points to FTC inscriptions] then you pick endpoints and subtract. [Riley starts writing her assumptions for FTC: "if f analytic on..."]
S: Yep, if f is analytic on your simple closed—
R: Simply connected.
S: Yep, on your simply connected domain D. And it holds for any paths through the domain.

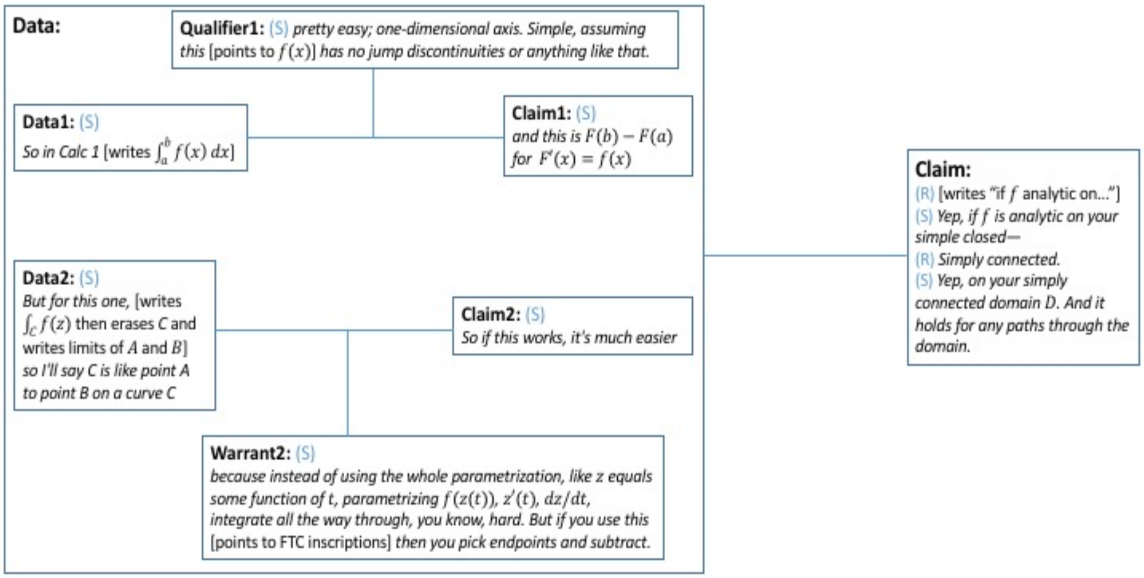


Figure 211. Toulmin diagram for Riley and Sean's Argument 2, Task 13.

S: So like— so kinda like in our example, we had like $1/z = f(z)$. We were using— ok you use this little curve here from like i to $-i$, going that way [draws semicircular path from i to $-i$]. So you parametrize it; it'll work, but it took a lot longer, probably especially the more complicated ones. But if you can find an antiderivative [writes $Log(z)$] then that'd work. Except you have to find another branch. So like if you choose Log of like, $-\frac{\pi}{4}$, so the branch is like there [draws a segment from origin at angle of $-\frac{\pi}{4}$].
R: Yeah so then you would do— you'd have to—
S: Yeah so in this case, you know f, it's 1/z and our domain— you just pick like, a domain that avoids the origin [draws in dotted domain around the path] and it's analytic there, so we can always find an antiderivative. Then we have a nuance of this being well defined in the region [points to Log(z)]. But in general you just pick some nice little region that has our endpoint and beginning point, such that this [points to $1/z$] is analytic, find the antiderivative, and evaluate at both points and subtract.

Subsequently, Sean began a third argument by alluding to a particular example. He authored an embodied-symbolic datum considering $f(z) = 1/z$ and a semicircular path from $z = i$ to $z = -i$ (lines 24-25; see Fig. 212). Sean claimed that pursuing a parametrization approach to integrating this function takes longer than applying an antiderivative, and qualified this claim with "probably especially the more complicated ones," though it is unclear whether Sean meant paths or functions by the word "ones" (lines 25-26). On the other hand, Sean authored a claim that the antiderivative would "work," and cited an embodied warrant that they would have to adjust the branch cut for the Log function so as to avoid intersecting their chosen path (lines 27-28). Accordingly, he chose a branch cut at an argument of $-\pi/4$ radians and drew this on their diagram (lines 28-29; see Fig. 213).

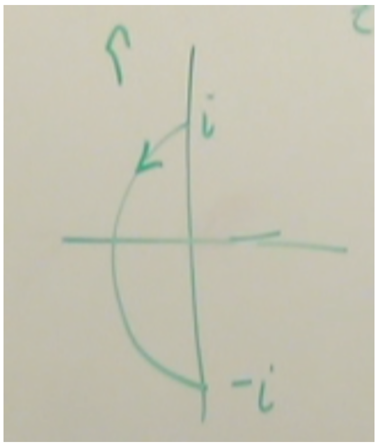


Figure 212. Sean's path gamma in Argument 3, Task 13.

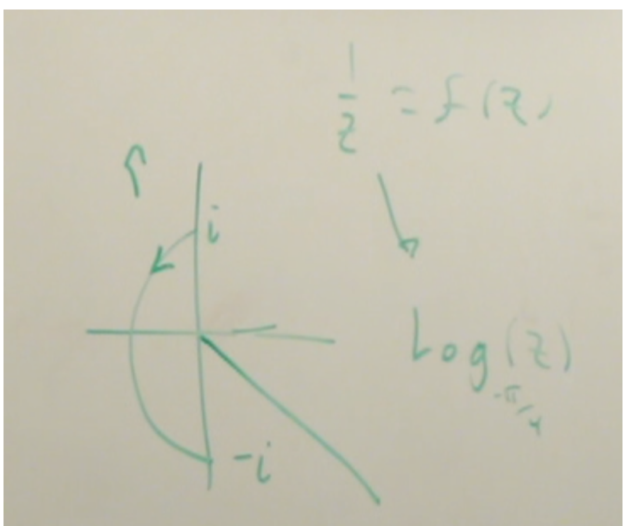


Figure 213. Sean's chosen branch cut in Argument 3, Task 13.

Next, Sean relayed their choice of function and authored an embodied datum by choosing a domain that "avoids the origin" (lines 31-32; see Fig. 214). He claimed that this choice of domain allows them to find an antiderivative, and cited the embodied-formal warrant that the function is analytic there (lines 32-33). Sean also cautioned, in the form of a rebuttal, that the logarithm function needs to be well-defined in the chosen region (line 33), but ultimately concluded that "in general, you just pick some nice little region that has our endpoint and beginning point, such that $[1/z]$ is analytic, find the antiderivative, and evaluate at both points and subtract" (lines 34-35). Argument 3 is summarized in Figure 215 below.

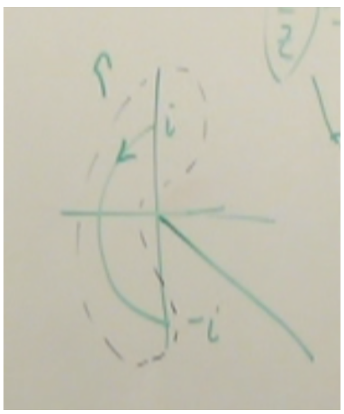


Figure 214. Sean's dotted domain in Argument 3, Task 13.

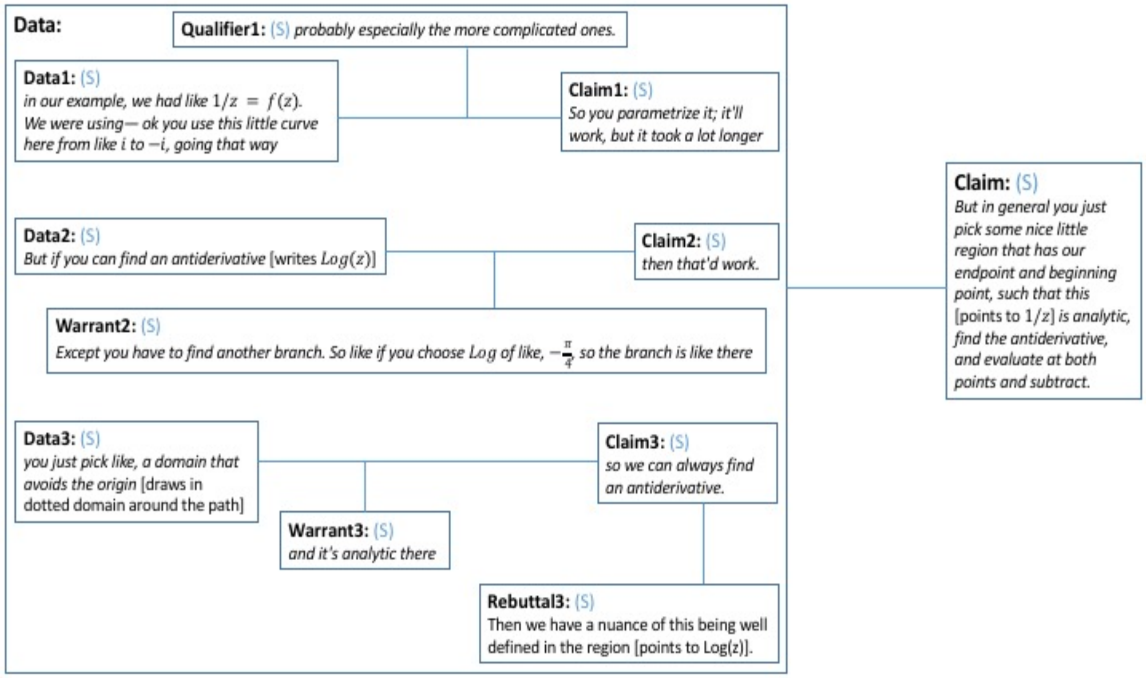


Figure 215. Toulmin diagram for Riley and Sean's Argument 3, Task 13.

Subsequently, I asked a follow-up question about whether Riley and Sean could think of a situation in which they would not be able to parametrize nicely (lines 36-40). In response, Sean began a fourth argument by discussing the opposite scenario, namely one in which the Fundamental Theorem did not apply but parametrization would. He authored an embodied datum extending their previous semicircular path to a full circle (line 41; see Fig. 216). In such a case, he authored a formal-embodied claim that they cannot concoct a domain in which the function is "analytic everywhere" (lines 41-42), and cited an embodied warrant that "that pole in the middle messes everything up" (line 42).

Int: So that could certainly make our lives easier, then. Can you think of maybe a situation or a function where, um, you can't necessarily parametrize nicely? Cuz so far, you know a lot of the examples that we work through, for instance, have nice circles or semicircles, where maybe things cancel when you parametrize. Can you maybe think of examples that you did, or just— or not necessarily things that you did but where maybe if you try to parametrize it doesn't turn out so nicely?

S: Yep, so if we extend our half-circle to a full circle [completes path to become full circle], we cannot get a domain that makes it analytic everywhere because that pole in the middle messes everything up [erases previous dotted domain].

R: And then you're forced to parametrize. But—

S: The only way to exclude would be, like ok my domain is this annulus [draws extra dotted region around origin] but that's not simp— that's no longer simply connected. You have to get— the only way to get a simply connected domain with this curve is to also include the non-analytic point [points to origin]. So you can't do that [points to FTC] and you have to use the methods we've done before.

R: But what would be like a curve— so in this case, you're going over a circle so it's easy to parametrize. So but what would be like a curve that's not easy to parametrize, but it is analytic over that curve? So that you'd have to use the antiderivative?

S: Hard to parametrize? I mean—

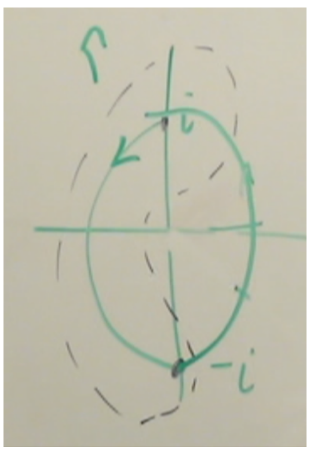


Figure 216. Sean's revised path as a full circle, Argument 4, Task 13.

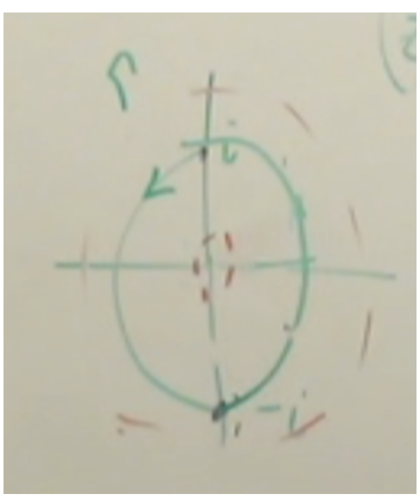


Figure 217. Sean's dotted annular domain, Argument 4, Task 13.

As such, Sean erased their previous domain, and Riley further concluded, “then you’re forced to parametrize” (line 44). Sean then discussed a potential new annular domain that avoided the origin (see Fig. 217), however he claimed that choosing such a domain would not allow them to employ the Fundamental Theorem, due to the embodied-formal warrant that this domain is no longer simply connected (lines 45-48). At this time, Riley acknowledged that Sean had not fully answered my follow-up question, and re-voiced my question as spokeswoman (lines 49-51), which catalyzed a fifth argument. Argument 4 is depicted in Figure 218.

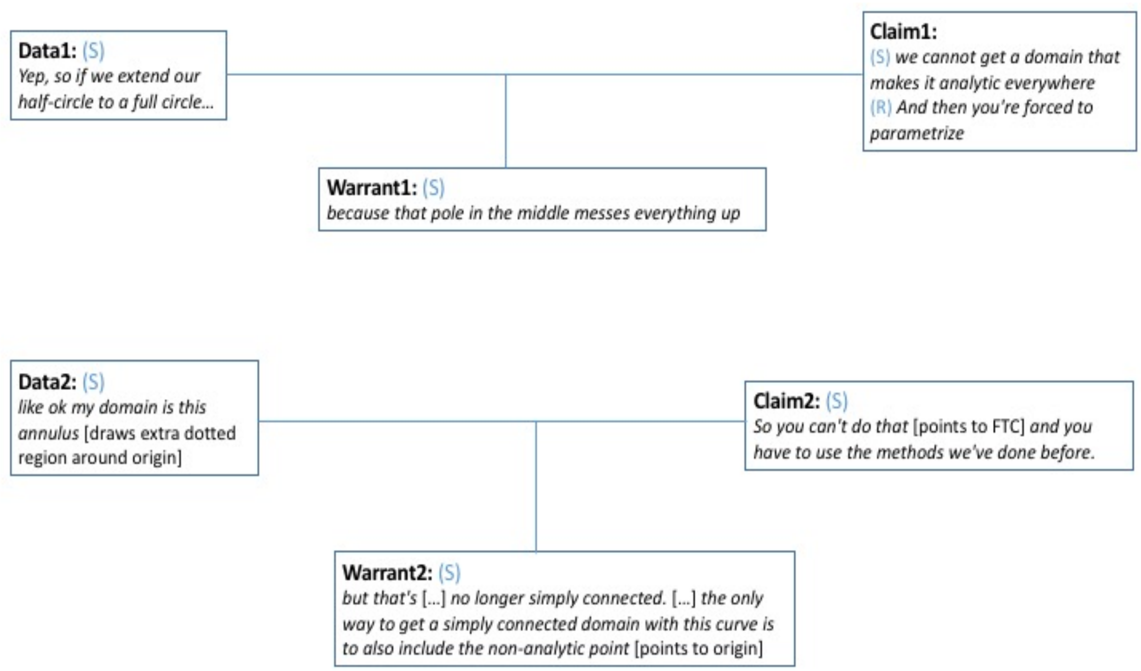


Figure 218. Toulmin diagram for Riley and Sean’s Argument 4, Task 13.

After Sean paused for a moment following Riley’s re-phrased question, she decided to answer the question herself in Argument 5. She began by authoring an embodied datum considering a path with “a lot of sharp edges” (lines 53-54; see Fig. 219). She concluded that one could still parametrize this path, but “it’s a pain” (line 54), and cited an embodied-symbolic warrant that one would have to parametrize each linear piece separately (line 55). On the other hand, she authored an embodied-symbolic claim that one need only know the "value of the antiderivative" at the two endpoints of her jagged path (lines 55-57). As a formal-embodied warrant, she indicated that path-independence allows one to draw a "smooth path" connecting the two endpoints instead (lines 57-58; see Fig. 219). Argument 5 is summarized in Figure 220.

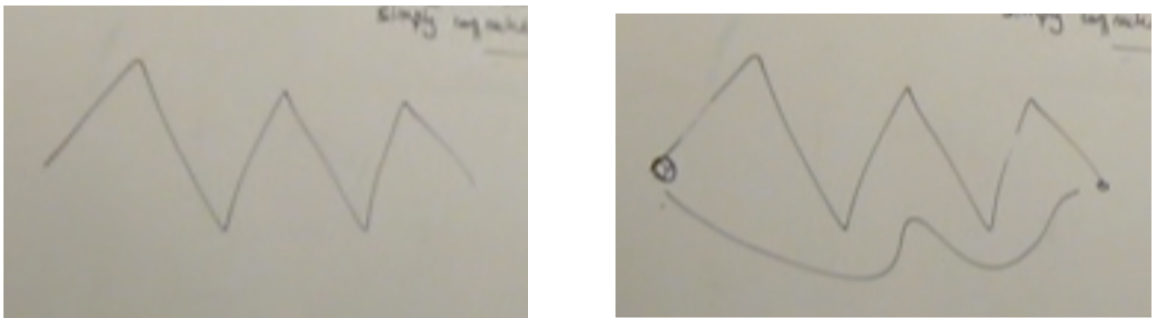


Figure 219. Riley's jagged path (left) and smooth alternative (right) in Argument 5.

R: I mean, you could have like a really uh, like a uh, not a smooth curve, right? Um so I think anything where you'd have like a lot of sharp edges [draws such a path]. You can parametrize it but it's a pain, because you'd have to do separate functions for each of these regions. And so with an antiderivative all you have to do is know the value of the antiderivative here and here [draws in the end points to her jagged curve]. Because it works for any point. So you could've just as easily chosen like a smooth function, or a smooth path, rather [draws in smooth path between the two endpoints of jagged curve].

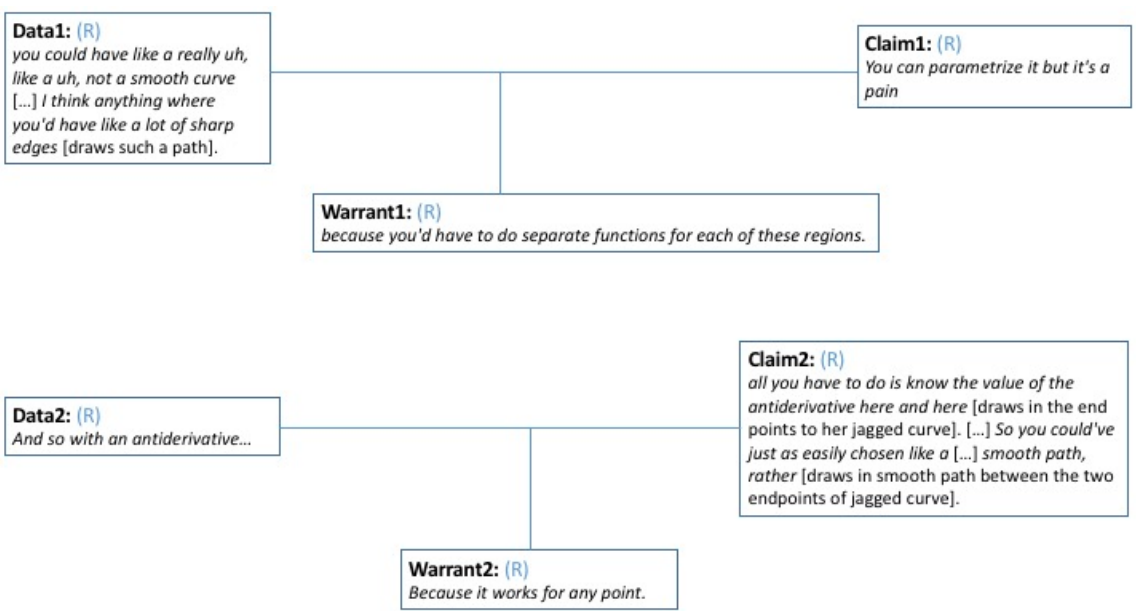


Figure 220. Toulmin diagram for Riley and Sean's Argument 5, Task 13.

Due to the difficulty of parametrizing certain paths, Sean articulated one final argument in which he described the benefits of working with analytic functions in physics applications. He began Argument 6 by authoring an embodied datum comprised of a complicated path, which he described as a “strange blob function” (line 59; see Fig. 221). Sean authored a qualifier expressing his doubt over whether such a path could even be parametrized (lines 59-60), but Riley claimed, “Piecewise you could parametrize just about anything” (line 61). Sean conceded in the form of an embodied warrant that they could “just chop it up” but insisted it would be very difficult (line 62).

S: So say we have this strange blob function [draws in blob shape]. Maybe you could parametrize this, I mean I'm not sure, but I doubt it. It would be hard. So that's kind of why in physics we love—
R: Piecewise you could parametrize just about anything.
S: Yeah just chop it up but it would be really, really hard to do.
S: Hence why in physics we're saying, ok we love conservative work functions because it's path-independent, and you just do *this* minus *that* really [points to start and end point].
S: Whereas if it's a non-conservative work field, you then have to do like the whole path and friction forces [tracing gesture along path] and it's really annoying. So that's much easier to make sure that you have a function that's analytic, so that you can do things like that, in case you have a path that's horrendous to parametrize.
Ok great, yeah. I think we're all set then.

Figure 221. Sean’s “strange blob” path in Argument 6, Task 13.

As such, Sean expressed physicists' preference for "conservative work functions," in that they afford one the favorable property of path independence as discussed in Argument 5 (lines 63-64). In contrast, Sean considered the datum of a "non-conservative work field," in which case he authored an embodied claim that one must deal with numerous friction forces, which he alluded to using a tracing gesture along his path (lines 65-66; see Fig. 222). Therefore, as spokesman, Sean reiterated that it is "much easier to make sure you have a function that's analytic" (lines 66-68). Argument 6 is summarized in Figure 223.

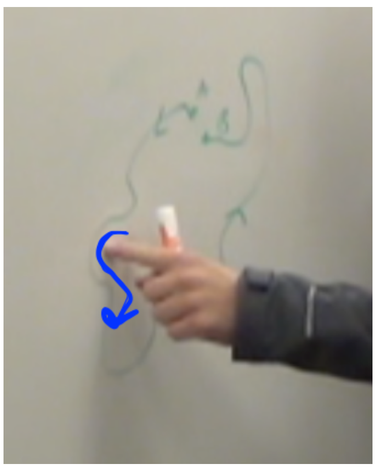


Figure 222. Sean's tracing gesture, Argument 6, Task 13.

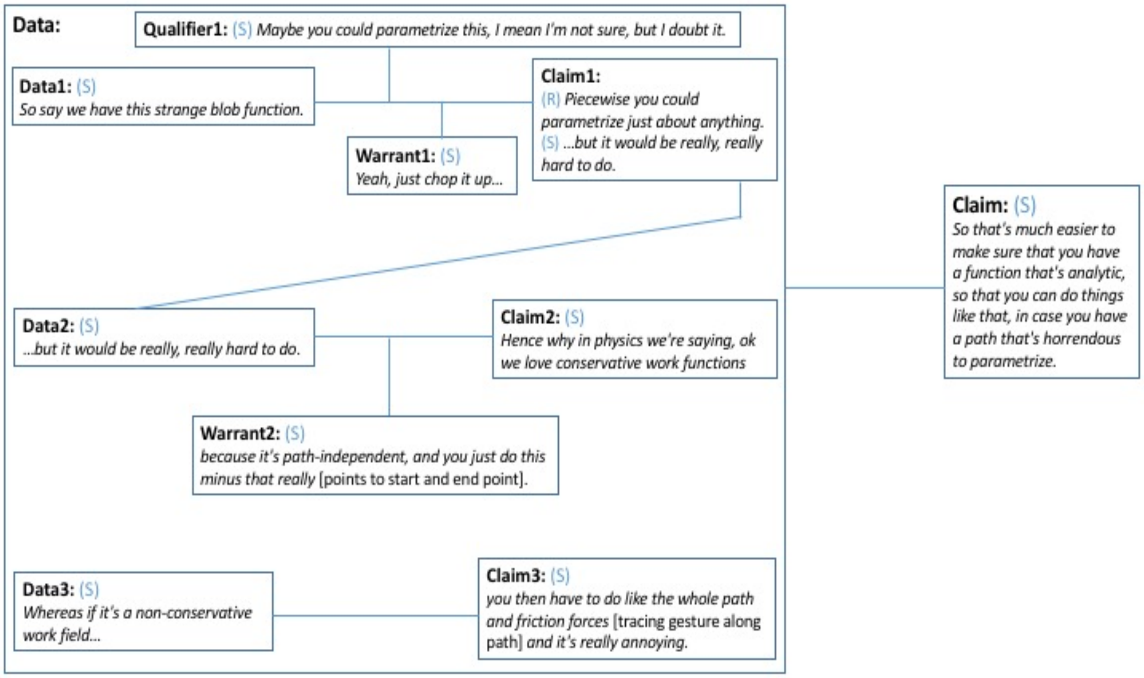


Figure 223. Toulmin diagram for Riley and Sean's Argument 6, Task 13.

Task 13 Summary

Unsurprisingly, both pairs of participants discussed similar requirements needed to invoke the Fundamental Theorem or calculate antiderivatives, and everyone seemed to be in agreement about their general preference for using this theorem instead of parametrizing. However, the pairs expressed different reasons for their preferences, and in how they would abate the tedium of parametrizing more exotic paths. For instance, Riley and Sean appealed to physics applications to discuss why analytic functions and path independence are appealing, and suggested dealing with parametrizing messy functions by breaking them up piecewise. On the other hand, Dan and Frank suggested implementing technology to ease the burden of parametrizing more complicated paths, though this likely would also entail some sort of piecewise approach implicitly.

Another interesting finding in Task 13 was that Sean expressed more modal qualifiers than usual in this task, perhaps because this task especially pushed him to think about the idiosyncratic hypotheses present in the theorems they invoked throughout. Sean's abundance of qualifiers also happened to coincide with fewer qualifiers from Riley, and she challenged more of Sean's claims than in previous tasks. Perhaps this is because she felt comforted by Sean's aforementioned uncertainty and did not need to highlight her own. Finally, Sean instantiated more embodied reasoning than in other tasks, particularly when describing physics applications, and Riley and Sean collectively exhibited more embodied reasoning than Dan and Frank. In the next chapter, I discuss the results presented in Chapter IV address my research questions. I frame this discussion within the context of the literature presented in Chapter II, and provide teaching and research implications of my findings. I also discuss future directions of my research, and acknowledge the limitations present in my study.

CHAPTER V

DISCUSSION

In the previous chapter, I detailed the nature of my four participants' nuanced collective argumentation as these undergraduate pairs responded to the thirteen integration tasks listed in Appendix C. These results served to address my aforementioned guiding research questions:

Q1 How do pairs of undergraduate students attend to the idiosyncratic assumptions present in integration theorems, when evaluating specific integrals?

Q2 How do pairs of undergraduate students invoke the embodied, symbolic, and formal worlds during collective argumentation regarding integration of complex functions?

In this final chapter, I summarize key findings related to these two research questions and situate these results within the existing literature discussed in Chapter II. Afterwards, I discuss theoretical implications of my dissertation with respect to framing collective argumentation in mathematics education research. I additionally proffer pedagogical implications arising from my results, and delineate the limitations of my study. Finally, I outline potential directions for future research in collective argumentation in light of my results and proposed theoretical addendums.

Summary of Key Findings

Treatment of Theorem Premises

My first research question regarded the manner in which undergraduate student pairs attended to the assumptions pertaining to integration theorems. Generally speaking, neither pair of participants initially appeared confident nor certain about the premises needed for employing certain tools, approaches, or theorems. Participants repeatedly expressed such uncertainty through explicit verbal modal qualifiers, as well as nonverbal qualifiers involving indicators such as facial expressions. In this section, I recapitulate several examples from the interviews to illustrate both this initial uncertainty and the manners in which participants were able to eventually reach consensus, or at least make significant progress in the task, following such qualifiers. I also discuss the significance of these results relative to the existing mathematics education literature incorporating Toulmin's (2003) scheme. Next, I briefly refer to established embodied cognition research to substantiate my contention that the types of *nonverbal* qualifiers exhibited by my participants can play a vital role in shaping collective argumentation. Finally, I highlight the manners in which participants instantiated Danenhower's (2000) phenomenon of *thinking real, doing complex* while attending to the nuanced hypotheses of integration theorems.

From qualification to consensus. Although participants expressed uncertainty via their qualifiers in many of the tasks, Task 11 appeared to elicit some of the most consequential modal qualifiers from both student pairs. In particular, Riley and Sean shared many discussions during Task 11 about whether two formal methods, approaches, theorems, or definitions were equivalent. For instance, in Argument 1, Riley and Sean discussed whether the SICOPOC conditions (simple, closed, positively oriented contour) were equivalent to a Jordan curve. Riley expressed her uncertainty about Sean's statements regarding the equivalence of these properties by repeatedly voicing qualifiers such as, "Is that the same thing?" and "So that's all a Jordan curve is?" Afterwards, in Argument 3, Riley expressed a preference for using the Residue Theorem instead of the Extended Cauchy-Goursat Theorem, but Sean claimed that these two results represent "the same thing." Following each of Riley's qualifiers, Sean either provided additional support for his assertions, or revised a previous assertion based on Riley's feedback. Hence, these explicit modal qualifiers shaped the trajectory of the pair's argumentation.

Due to the multifaceted nature of Task 11, which required participants to find all possible values of a particular integral by incorporating different paths of integration, the student pairs also had to come up with careful symbolic notation to keep track of these various paths. The ensuing conversations about such notation allowed participants to reflect on important features of integration. For instance, in Arguments 1 and 2, Frank initially believed there were only four possible paths of integration that would yield distinct answers; these four paths corresponded to the *number of poles* enclosed by a path. However, at the end of Argument 2, Dan expressed uncertainty in Frank's approach via the qualifier, "Umm, wait," which led to a third argument in which Dan examined the dependency in Cauchy's Integral Formula on which pole was enclosed by the path of integration. Consequently, Frank reflected on their work thus far and articulated the qualifier, "Wait hang on. Does that mean we can get different results based on different poles?" As such, these explicit modal qualifiers paved the way for subsequent arguments in which Dan and Frank came to realize that the task was more complicated than they initially anticipated. In the arguments that followed, they were able to revise their notation to talk about the different path choices, and they ultimately exhausted the different solutions.

Another example underscoring the importance of explicit modal qualifiers was when Riley and Sean questioned themselves in Task 5b about whether the integrand function needs to be differentiable in order to employ parametrization. By explicitly qualifying such arguments, and in conjunction with my follow-up questioning, they eventually reached a consensus that the function only needs to be continuous. However, because they did not spend significant time in their course carefully justifying continuity arguments, the students exhibited substantial difficulty justifying whether the function $\bar{z}$ is continuous or not. In particular, they pursued limit calculations to try to show this function was not continuous, but muddled their symbolic limit inscriptions. More generally, when I pushed participants to justify *why* given functions were continuous, they primarily relied on backing for their warrants' field. For instance, Dan mentioned in Task 5b that during their complex variables course, "we just kind of looked at something and said, 'look, it's clearly continuous' or 'it's discontinuous at this point.'" Similarly, Sean mentioned in Task 5b that their professor identified continuity arguments as more germane to a complex *analysis* course in which students are already familiar with continuity proofs in the real-valued context. Both student pairs also mentioned that the professor emphasized differentiability more than continuity in the course.

Although Dan and Frank exhibited more confidence and decisiveness when deciding a function's continuity, they faltered a bit when justifying their application of Cauchy's Integral Formula in Task 6. In particular, when Dan claimed they could produce a simply-connected domain containing the path L, Frank questioned the existence of such a domain, and his attempt at drawing one resulted in a domain that was not simply-connected. However, as with the above examples, Dan and Frank's eventual consensus resulted from an explicit modal qualifier. Summarily, the importance of such explicit qualifiers across the interviews was that they often led to follow-up arguments wherein the participants discussed assumptions in greater detail, including their applicability to the integral at hand. As such, my findings corroborate previous researchers' (Alcock & Weber, 2005; Inglis, Mejia-Ramos, & Simpson, 2007; Simpson, 2015; Troudt, 2015) contention that one should consider the full Toulmin (2003) model when analyzing undergraduate level mathematical arguments.

Nonverbal qualifiers. Another theme that I observed related to modal qualifiers was that participants employed *nonverbal* qualifiers that shaped the flow of their argumentation. For example, participants would look at me or each other for validation after voicing a claim, and this led to follow-up arguments or clarifying remarks. In such instances, I was purposeful about not attending to these looks directly during the interview in order to not interrupt participants' reasoning process. Frank instantiated this phenomenon most often, including during Task 3 (Argument 1), Task 5a (Argument 1), Task 5b (Argument 1), Task 7 (Argument 1), and Task 10 (Argument 1). Notice that this behavior took place in Argument 1 of each of the aforementioned tasks. As mentioned previously, I did not address Frank directly, and this meant that it was up to the students to sort out their uncertainty. Consequently, these nonverbal qualifiers led to clarifications or follow-up arguments about nuanced integration hypotheses or questionable integral results.

Moreover, because I deliberately avoided providing validation for Frank's hesitant statements, he eventually turned to Dan instead of me. This turning point occurred in Task 11, during which Frank first looked to me for approval in Argument 1 after listing possible path choices, but then looked to Dan in Argument 2 to verify his claim about the value of a particular integral. Subsequently, in Argument 9 Dan also looked to Frank for validation after answering a follow-up question that I posed for them. I noticed far fewer of these nonverbal glances as qualifiers between Riley and Sean, though they did exist sporadically. For instance, Sean glanced over at Riley inquisitively with a look of incredulity after obtaining a surprising value for an integral in Task 9a Argument 6. I discuss potential teaching implications associated with such findings in a later section.

This theme is significant in that it points to a way in which the existing collective argumentation framework can be extended to account for important embodied and social considerations. In particular, research on embodied cognition posits that, "Through interaction with others, utterances become collective or group-phenomena" (Nemirovsky & Ferrara, 2009). This stance treats *utterances* as multimodal rather than just verbal, encompassing both overt and covert aspects of communication such as facial expression, gestures, tone of voice, eye motion, gaze, and body poise, among others (see also Arzarello, 2006).

Moreover, much of participants' embodied reasoning incorporated gestures that alluded to visualized processes or conveyed other geometric information. I detail this finding in the section addressing my second research question, but note this theme here to underscore the multimodality of participants' collective argumentation in the sense described above. Ultimately, these findings compel me to contend that collective argumentation analysis should classify participants' multimodal *utterances* into Toulmin components and speaker roles, as opposed to just their *verbiage*.

Thinking real, doing complex. Despite participants' occasional struggles with formal hypotheses, both student-pairs were regularly cognizant of the *thinking real, doing complex* (Danenhower, 2000) phenomenon, and expressed their desire to avoid inappropriate applications of it. For instance, Frank cautioned in Task 3 Argument 1 that "you can't really say *velocity*, I guess, in the context of complex numbers, would be my understanding," when discussing a physical interpretation of $\frac{dz}{dt}$. Subsequently, in Argument 2 Frank hypothesized that they could "still" visualize this quantity as a tangent vector: "I mean, wouldn't it still be tangential?" Moreover, in Task 4 Argument 1, Dan and Frank initially had trouble describing a geometric interpretation of $\int_a^b \frac{dz}{dt}\, dt$ because they did not know how to reconcile the fact that the horizontal axis in their diagram did not represent time, yet the dt and limits of integration corresponded to times. This perceived conflict between *thinking real* and *doing complex* led Frank to comment, "I'm tempted to think of this in terms of real numbers, but I know the analogy doesn't work." However, following this impasse, he and Dan agreed that they could still borrow some intuition from the notion of a line integral, as Dan claimed "So it's just like a line integral." Finally, during Task 5b Argument 2, Frank expressed uncertainty about how the limit definition for continuity might transfer from the real case to the complex setting: "I just don't know what the analog is necessarily in terms of transferring that to complex numbers, or if it's really different."

Riley and Sean made similar statements regarding avoiding *thinking real, doing complex* during their interview as well, however not as often as Dan and (especially) Frank. For instance, in Task 4 Argument 3, Riley recited the common geometric interpretation of integration of real functions as "area below the curve," but immediately acknowledged that "that's not the case for um, like with complex variables." Later, during Task 9b Argument 5, Riley once again verbalized her deliberate avoidance of this interpretation of integration of complex functions, this time while discussing "splitting up" one original integral into a sum of two integrals. In particular, she discussed how she "flew back to Calc 2 again," wherein the area interpretation for integration in the real-valued case allows one to break up one large area under a "smooth curve" into two smaller areas. However, in the context of complex functions, she cautioned that, "Here, it's not physically the area." Summarily, the above examples illustrate ways in which both pairs of participants explicitly articulated a desire to avoid inappropriately extending properties of real-valued integration to the complex context.

However, this is not to say that the students always successfully avoided such pitfalls. One notable example of *thinking real, doing complex* that was ultimately unproductive occurred when Sean described his general interpretation of complex integration in Task 12. He began his response by correctly noting that complex numbers can be represented as vectors, and that he therefore tends to "think of it as like— kind of coming back to Calc 3." However, he ultimately conflated multiplication of complex numbers with a dot product of vectors when explaining his perceived connection between the integrals $\int_L f(z)\,dz$ and $\int_C \vec{F} \cdot dr$. In particular, his argument appealed to "the language of Calc 3" to equate the product of complex numbers with a dot product. But note that these two vector operations are not equivalent because the latter yields a real-valued scalar, while the former yields another complex number/vector. That said, if Sean had interpreted complex multiplication as a rotation and dilation instead of dot product, he would essentially have a correct embodied interpretation of complex integration, for he finished his description by articulating summing such products repeatedly over the entire path. As such, I found Sean's embodied description quite impressive, considering several mathematicians from a recent study (Oehrtman, Soto-Johnson, & Hancock, 2018) did not successfully provide as complete of a geometric description of complex integration.

Another important theme related to *thinking real, doing complex* pertains to instances in which participants instantiated this phenomenon in *productive* ways. This is noteworthy in the sense that the existing literature on the teaching and learning of complex variables tends to mostly refer to *thinking real, doing complex* in a pejorative light. For instance, Danenhower (2000) found that one student concluded that $f(z) = (2z - x)^2$ was a polynomial and thus differentiable everywhere. Similarly, Troup (2015) noticed that his participants initially wanted to characterize the derivative of a complex function as the slope of a tangent line, but did not "know what slope means in complex world" (p. 178).

Alternatively, my dissertation illuminates ways in which this type of thinking might actually support productive reasoning in complex analysis. For instance, in Task 2, Riley and Sean invoked a warrant characterizing the *x* and *y* coordinates of the unit circle in $\mathbb{R}^2$ using cosine and sine in order to describe the corresponding real and imaginary coordinates in the Argand plane. Subsequently, in Task 4, Sean argued that the task identity was procedurally the "same exact thing" as the "Calc 1 version" of the Fundamental Theorem of Calculus, in the sense that "we can find the antiderivative […] then just plug in the endpoints and subtract." Similarly, during Task 13 Frank claimed that computing a complex antiderivative would entail application of "our Calc 2 techniques—just treating z as our variable instead of x."

In summary, there were three manners in which my participants attended to the *thinking real, doing complex* phenomenon in the present study: (1) purposefully avoiding inappropriate applications of it; (2) extending real intuition to the complex setting erroneously; and (3) extending real intuition to the complex setting in productive ways. With some exceptions, participants were mostly cognizant about avoiding the unproductive versions of *thinking real, doing complex* but implementing the productive ones. As such, I discuss potential teaching implications arising from these findings in a subsequent section.

Invoking Tall's Three Worlds

My second research question inquired about the nature of students' invocation of Tall's (2013) three worlds during collective argumentation about complex integration. Quite unsurprisingly, my participants' formal reasoning dealt primarily with Cauchy's Integral Formula, the Cauchy-Goursat Theorem, the Cauchy-Riemann equations, and related results when evaluating specific integrals. However, more illuminating were the ways in which participants invoked formal-symbolic, formal-embodied, or embodied-symbolic reasoning to justify the implementation of such theorems.

For instance, Riley (and eventually Sean) explicitly illustrated their embodied-symbolic reasoning by drawing arrows on the whiteboard between symbolic inscriptions and embodied paths of integration which they had sketched on the board. This type of embodied-symbolic reasoning was most prevalent in tasks which required participants to discuss multiple integral results within the same argument or sequence of arguments, especially when employing partial fractions decompositions in Task 9a. This type of reasoning was also prevalent in Task 11 when participants invoked the residue theorem or extended Cauchy-Goursat Theorem. For example, Dan and Frank also created symbolism in Task 11 to allude to embodied paths of integration enclosing a given number of poles. That said, Riley demonstrated this type of reasoning as early as Task 4, when she drew an arrow from Sean's symbolic integral inscription representing arc length to her drawn path.

Additionally, when discussing limits and paths, all four participants produced symbolic inscriptions but also conveyed corresponding dynamic gestures embodying their chosen paths of approach or paths of integration. For example, in Task 2 both Dan and Riley provided similar tracing gestures accompanying their symbolic inscriptions pertaining to the real and imaginary axes in the Argand Plane. Subsequently, in Task 3 Riley gestured in reference to "a little vector pointing off" as she discussed her interpretation of $\frac{dz}{dt}$ as a tangent vector. When discussing symbolic limit inscriptions in Task 5a regarding the definition of the derivative $f'(z)$, Dan also gestured horizontal and vertical lines with the palm of his hand to illustrate their two chosen paths of approaching a generic point z_0, and Sean provided very similar gestures in Task 5b Argument 2. Moreover, in Task 5c Frank produced a circular tracing gesture as he discussed the symbolism for parametrizing such a circular path. These examples serve to highlight the ways in which my participants provided additional embodied support for their symbolic inscriptions, thus instantiating one way in which the symbolic and embodied worlds can intersect in the context of complex integration.

There were also instances in which the two student-pairs displayed notable differences in their predilections towards certain worlds when implementing various integration approaches. For instance, Dan and Frank incorporated a comparative lack of embodied reasoning versus Riley and Sean, especially in Tasks 4 and 5. On the other hand, in Task 5b, Riley and Sean chose to incorporate a mostly embodied method for integrating the complex conjugate $\bar{z}$. The pair plotted tangent vectors along the circular path of integration as well as conjugates resulting from reflection transformations, and Riley and Sean also enacted visual vector addition.

Another noteworthy distinction between Dan and Frank's versus Riley and Sean's invocation of the three worlds was that Dan and Frank tended to prefer calling upon formal theorems, while Riley and Sean incorporated more parametrization and partial fractions decompositions. This comparative preference for formal reasoning was especially prominent in Task 6, during which Dan and Frank invoked Cauchy's Integral Formula. This choice of approach resulted in several follow-up arguments in which Dan and Frank provided backing statements and supporting embodied diagrams regarding simply-connected domains. In contrast, Riley and Sean provided only one argument about parametrization, containing no backing statements. Finally, a minor difference in the two pairs' symbolic inscriptions was that Sean incorporated a Newtonian "dot" notation when discussing time derivatives in Task 4, while Dan and Frank did not incorporate such notation.

A surprising result related to Task 12 was that all participants interpreted the statement of the task as a request for an *embodied* meaning of integration of complex functions, even though I did not specify or encourage any particular world. Specifically, I read the task statement aloud, which asked, “What do you think the integral of a complex-valued function represents, and how is this different from a real-valued function and how is it the same?” In response, participants immediately responded in terms of a geometric interpretation. In particular, Riley responded, “I don’t really have a good geometric interpretation of that.” Similarly, Dan replied, “I don’t know if I actually *have* any type of what that can represent,” but Frank added that he recalled something that Dr. X drew about “adding up a bunch of vectors, uh, along the curve.”

Afterwards, Frank also stated that while he recognizes complex numbers as vectors, “when we talk about things, kind of like the Cauchy-Goursat Theorem or we’re just evaluating [the integral] about circles, I don’t really take the time to think about it in terms of vectors. I just think about the formulas and the theorems that we regularly deal with.” The fact that my participants generally did not think of a purely embodied interpretation of integration while evaluating specific integrals is not particularly surprising, given that the majority of mathematicians from a recent study (Oehrtman et al., 2018) could not produce such a description when explicitly asked to do so. Nevertheless, I found it interesting that the students interpreted my question as a request for a geometric interpretation specifically, despite my intentionally open-ended phrasing of the question that did not favor any particular representation.

The aforementioned instantiations of *thinking real, doing complex* in the previous section also have important connections to Tall’s (2013) Three Worlds lens. Specifically, recall that Tall emphasizes the role of prior mathematical knowledge in shaping an individual's cognitive structure, using the *met-before* construct: "a structure we have in our brains now as a result of experiences we have met before" (p. 23, italics in original). Through the innate *set-befores* of recognition, repetition, and language, individuals enact three corresponding forms of knowledge compression: categorization, encapsulation, and definition. As such, the *thinking real, doing complex* phenomenon can be characterized according to the met-before of the structure of the real numbers that is imposed (sometimes inappropriately) onto new mathematical concepts in $\mathbb{C}$ such as the complex integral. This process is ostensibly enabled by the recognition set-before and manifested in the definitions of complex objects using the "language of Calc 3," as Sean put it in Task 12. In particular, when participants explicitly concluded that the structure of the real numbers was inapplicable to the present complex context, I contend such instances revealed a glimpse into the students' mental categorization of interpretations of calculus concepts into those that align with their real-valued counterparts, and those that cannot. In other instances, we witnessed episodes in which participants' categorizations were unsuitable, such as when Sean equated a dot product of two vectors with the product of two complex numbers.

Recall that Wawro (2015) found that her participant's argumentative successes were primarily due to the fact that he was "flexible in his use of symbolic representations, proficient in navigating the various interpretations of matrix equations, and explicit in referencing concept definitions within his justifications" (p. 336). Similarly, in the setting of the complex numbers, researchers argued for the importance of students' ability to recognize when certain forms of a complex number are most convenient, as well as the ability to switch between forms (Danenhower, 2006; Karakok et al., 2014; Panaoura et al., 2006; Soto-Johnson & Troup, 2014). Accordingly, this literature mutually suggests a potentially strong connection between representational fluency and effective mathematical argumentation, particularly within the setting of complex analysis. As such, the results of my study corroborate this link by illustrating how students' embodiment, symbolism, and formalism collectively inform their argumentation about integration. In particular, my participants were most successful and confident in supporting their assertions pertaining to integration when they could (1) proficiently alternate between or merge the embodied, symbolic, and formal worlds; and (2) properly reconcile *thinking real* with *doing complex.*

Implications for Framing Collective Argumentation

In this section, I discuss how my study complements and extends the mathematics education literature regarding students' mathematical argumentation, particularly regarding how Toulmin's (2003) model is adopted to the context of collective argumentation. For instance, not only did my participants' explicit qualifiers catalyze new arguments, but follow-up arguments also ensued when individuals challenged each other's assertions or changed their own mind. In the following subsections, I provide examples of such challenges from my results and discuss theoretical implications of these challenges in framing collective argumentation.

According to Krummheuer (2007), individuals participate in collective argumentation in two ways: (1) the production of statements categorized according to Toulmin's model, and (2) an individual's speaker role (author, relayer, etc.). Notice that both of these forms of participation primarily serve to either introduce new ideas or support/re-voice existing ideas. However, they do not account for disagreement between parties or changing one's own mind following internal reflection. Accordingly, I contend that a third type of participation can drive collective argumentation, namely *challenging*.

Previously, I illustrated how both my results and existing mathematics education literature support embodied addendums to analyzing Toulmin components and speaker roles. Specifically, I argued that nonverbal qualifiers and the general multimodality of utterances should be accounted for when analyzing these two forms of participation in collective argumentation. My study also incorporated increased specificity to analysis of Toulmin components by classifying backing statements according to Simpson's (2015) three types. This allowed me to notice important themes related to my research questions, such as the aforementioned prevalence of backing for a warrant's field when participants justified the continuity of specific functions. Moreover, I noticed an overall abundance of backing for the validity of participants' warrants as opposed to backing for their correctness. Below, I propose an augmented theoretical framing of collective argumentation in which all three forms of participation operate in tandem and influence one another in multimodal manners (see Fig. 224).

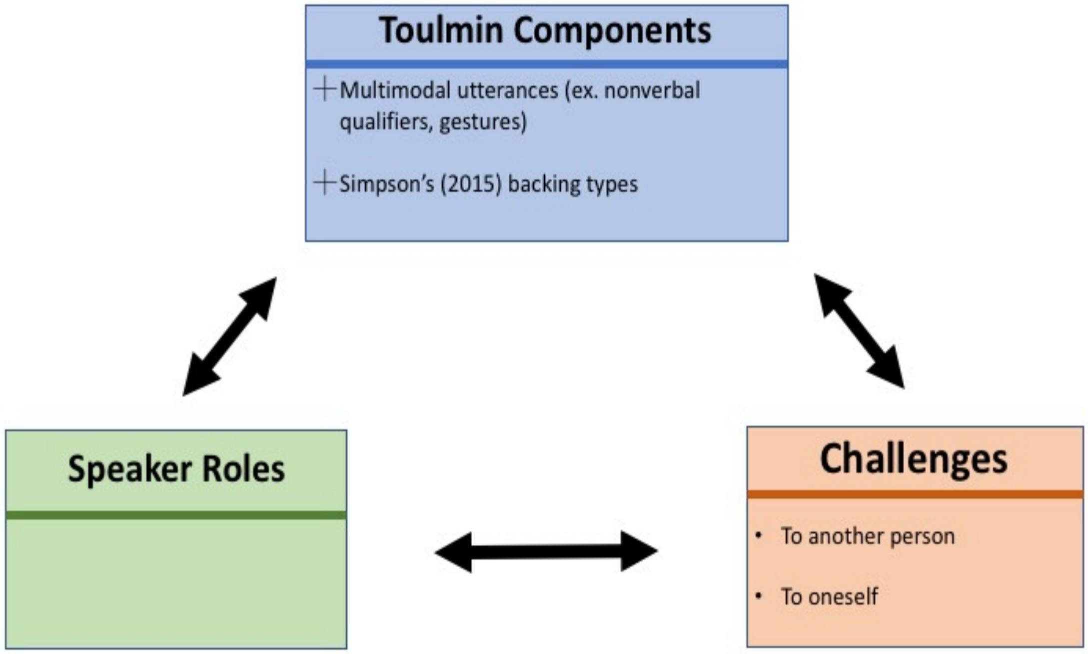


Figure 224. Proposed augmented collective argumentation framework.

Toulmin Components and Challenges

As mentioned previously, participants occasionally disagreed with either the other student's contributions or with a Toulmin component that they themselves previously stated. In other words, a participant would proffer a Toulmin component (datum, claim, etc.) and then either participant would challenge that component. Such challenges then resulted in new arguments, sub-arguments, or further clarification/support for the statement in question. For example, after labeling some paths of integration as L_1, L_2, and L_3 in Task 11, Frank challenged his own symbolic notation in Argument 2 when he decided, "wait, let's be smart about this." He then erased his previous labeling, and revised his symbolism to instead denote the number of poles enclosed by the path. This revision ended up giving Dan and Frank less ambiguous notation to talk about the different possibilities for their integral, which they outlined in subsequent arguments.

Hence, Frank's self-imposed challenge was a catalyst for future Toulmin components and arguments.

Later in Task 11, during Argument 8, Frank summarized the list of possible integral values obtained from paths of integration enclosing two poles. During this summary, Dan challenged Frank's warrant about some of the answers obtained, and this led Frank to re-voice and provide additional clarification for how the previous answers were obtained. This exchange allowed the pair to subsequently reach the consensus that the integral around all three poles should vanish. Accordingly, sometimes both types of challenges occurred within the same task, each shaping the trajectory of the pair's collective argumentation in different ways. In particular, while Frank's challenge to his own symbolism in Argument 2 impacted how future arguments transpired, Dan's challenge to Frank's summary in Argument 8 resulted in Frank providing additional support for existing statements within the present argument.

Riley and Sean also instantiated similar connections between challenges and new or revised Toulmin components. For example, during Task 4 Argument 3, Riley argued that the integral $\int_a^b \frac{dz}{dt}\, dt$ represented arc length. But Sean challenged her assertion, insisting that this integral actually represented a "change in position." This challenge affected the pair's subsequent Toulmin components in both of the aforementioned ways. First, Sean provided embodied-symbolic backing for his warrant's correctness by incorporating two new vectors $\vec{r}_1$ and $\vec{r}_2$ into their existing diagram. His and Riley's disagreement then catalyzed a follow-up argument, Argument 4, about how they could obtain arc length by integrating the *magnitude* of the original integrand, which Sean chose to express as $v(t)$. Hence, Sean's challenge resulted in both additional support to an existing argument, as well as the creation of a new argument whose purpose was to address their disagreement. Accordingly, my results corroborate Weber et al.'s (2008) contention that "Challenges […] from the student's classmates invite the student to be explicit about the warrant being employed and provide backing to support the warrant's legitimacy" (p. 249).

Challenges and Speaker Roles

In the present study, participants' challenges and speaker roles were intimately connected, suggesting a bidirectional relationship between these two components of collective argumentation. Specifically, sometimes challenges induced specific speaker roles; in other instances, certain speaker roles resulted in challenges. In this section, I discuss some specific examples of each, beginning with the former. I also theorize hypothetical ways in which the latter relationship might naturally manifest itself in collective argumentation.

There were several ways in which challenges evoked certain speaker roles in response to, or in further support of, the challenge. The first way in which I noticed such relationships was that challenges caused participants to re-voice a previous statement as spokesman. This happened either when one participant responded to a challenge, or when the student who articulated the challenge wished to clarify the aspects of a statement with which he or she did not agree. Sean instantiated the latter type in the following exchange from Argument 1 of Task 13. First, Riley claimed that whenever the function in question is analytic, it has an antiderivative. She further added that such analyticity implies the existence of harmonic component functions u and v. However, Sean challenged her statement, insisting that *differentiability* was the requirement needed for the harmonic property. As spokesman, Sean clarified, “What I’m saying is, like, if it’s differentiable over a whole region then it becomes analytic.” He continued to assert that differentiability merely along a line does not guarantee analyticity.

A second respect in which participants’ challenges influenced their speaker roles was that, following a challenge, participants authored additional support for a statement in the form of new backing, an additional warrant, or clarifying the data used. For instance, in Task 9a, Sean authored a claim that the integral $\int_C 1\,dz$ results in a value of 2π. However, Riley challenged Sean’s assertion, instead claiming that the “2π” should instead read “π” because “you’re not integrating over the full circle.” Following Riley’s challenge, Sean clarified his previous symbolic inscriptions by explicitly identifying the symbolic formula $2\pi r$ as a datum. He then authored a new warrant elucidating that he took “half that.” As backing for this warrant’s correctness, Sean identified his $2\pi r$ formula as “for circumference,” and described how “the full circumference is 4π.” According to Sean, dividing by 2 then yielded the 2π result, and he summarized as spokesman that he essentially calculated arc length of their curve.

Finally, participants responded to challenges in a third respect by reminding someone of something already said (as relayer). For example, in Task 11, Frank listed the two possible distinct values for the integral around L_2 (their notation for paths enclosing two poles). However, Dan challenged Frank’s statement because he remembered the two answers as πi and $-2\pi i$ rather than $-\pi i$ and $2\pi i$. In response, Frank reminded Dan that the πi and $-2\pi i$ answers corresponded to integrals around L_1, and recapitulated his answers for the L_2 cases as relayer. As evidenced in the previous example, sometimes challenges catalyzed clarifications or addendums via multiple speaker roles.

Conversely, individuals' choices of speaker roles can influence the appearance of a challenge in several respects. A relatively obvious example is that *authoring* new claims, warrants, or backing can induce a challenge if another person does not agree with the statement put forth. This particular sequencing happened often in the present study, including in many of the aforementioned examples from the previous section. Alternatively, one might also encounter instances in which an individual challenges someone's re-voicing of a statement as *spokes(wo)man* during a collective argument. In particular, a student might make a claim or author a datum, which another student may then re-voice; but the original student might disagree about whether the re-voiced syntax matches the semantic intent of the original claim. For example, one student might verbally introduce the datum of a path of integration as a circle of radius 3, traversed counterclockwise and centered about the origin. As spokes(wo)man, another student might then express this path symbolically as $C_0^+(3)$, but the first student might challenge this re-voiced symbolism and instead wish to use the symbolism $C_3^+(0)$. Although I did not incorporate this latter scenario into my data analysis in the present study, further research could tease out the exact nature of this relationship between challenges and speaker roles in collective argumentation.

Speaker Roles and Toulmin Components

Ostensibly, the relationship between speaker roles and Toulmin components is fairly straightforward in the context of collective argumentation, in that each Toulmin component is contributed via one of the four speaker roles. As such, I focus here on some implications for researchers' treatment of the speaker roles themselves. For instance, something that I noted in the present study was when individuals incorporate speaker roles in nonverbal or embodied manners. For instance, there were instances in which one participant made a statement and then corroborated her or his verbiage with an embodied gesture as spokes(wo)man, such as the aforementioned examples wherein participants produced circular tracing gestures while discussing paths of integration. This nonverbal re-voicing can also manifest as one individual produces a gesture to capture what another individual previously stated in words. In the present study, this occurred when one participant traced the real or imaginary axes in the air while the other student discussed paths of approach with regard to limits in continuity or differentiability calculations.

Alternately, individuals or groups of individuals can instantiate a discordance between their speech and gesture content. This phenomenon, which is commonly referred to as *gesture/speech mismatch* in existing gesture research, can illuminate important features of students' cognition. For instance, Alibali and Goldin-Meadow (1993) found that such mismatch actually "appears to be a stepping-stone on the way toward mastery of a task" (Goldin-Meadow, 2003, p. 51). Moreover, Goldin-Meadow, Alibali, and Church (1993) found that fourth grade students who exhibited three or more mismatches during mathematical equivalence tasks conveyed significantly more problem solving strategies, using gestures alone, than students who produced less than three mismatches. Thus, the researchers argued that students who mismatch speech and gesture not only have more strategies at their disposal than students whose speech and gesture match, but that these extra strategies lie in students' gestures themselves.

Analogously, I hypothesize that this phenomenon might be captured in the ghostee speaker role, which occurs when an individual attributes a different or new semantic meaning to existing syntactic content. Given that gestures can act as "a window into what students in a classroom are thinking" (Keene, Rasmussen, & Stephan, 2012), an individual could repeat an existing statement with respect to his or her verbiage, yet the individual's gesture could signify a different intended semantic meaning. In the present study, the ghostee speaker role was the least prevalent for both student pairs, however the phenomenon that I am describing arose for Riley and Sean in Argument 1 of Task 9bc. When determining the value of $\int_{C_1+C_2} \frac{2+z}{z}\,dz$, where C_1 and C_2 were semicircular paths in the upper-half and lower-half planes (respectively), Riley argued that this integral is equivalent to summing $\int_{C_1+C_2} 1\,dz$ and $\int_{C_1+C_2} \frac{2}{z}\,dz$. Afterwards, Sean agreed and repeated Riley's verbiage of "adding them together," but his pointing gestures semantically referred to summing $\int_{C_1} \frac{2+z}{z}\,dz$ and $\int_{C_2} \frac{2+z}{z}\,dz$.

Accordingly, such situations suggest a potential way in which gesture-speech mismatch might align with the ghostee speaker role, as well as how gesture-speech mismatches might be extended to more social situations such as a collective argument. Other examples of this phenomenon might have occurred in other tasks during my interviews, however I did not explicitly code for gesture-speech mismatches in the present study. Thus, future work could further investigate this potential relationship between mismatches and the ghostee speaker role. I hypothesize that there might also be a relationship between gesture-speech mismatches and students' navigation of Tall's three worlds, in the sense that one's speech and gesture might respectively attend to two different worlds. Again, these relationships could be explored via future research. Nevertheless, my participants' use of gestures to instantiate the spokes(wo)man and ghostee speaker roles suggest that a more comprehensive framing of speaker roles, especially one including individuals' gestures, might benefit collective argumentation analysis.

Implications for Instruction

As discussed in the previous section, I contend that there are at least three distinct manners in which individuals can participate in collective argumentation, and I have illustrated several ways in which these components can work in tandem. Accordingly, when the individuals in question are students, and when classroom interactions include students' collective argumentation, this necessitates that instructors consider (1) how to attend to each of these three pieces, and (2) how the pieces can intertwine. Such considerations are especially important in classrooms driven by inquiry-oriented practices, in which students:

> learn new mathematics through *inquiry* by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems. Thus, the first function that student inquiry serves is to enable students to learn new mathematics through engagement in genuine argumentation (Rasmussen & Kwon, 2007, italics in original).

Along these lines, it is also essential for instructors to keep in mind that they can directly shape students' argumentation in subtle ways. Indeed, even as interviewer wherein my intended role was not to provide instruction, some of my follow-up questions initiated additional arguments or challenges, especially when such questions asked for clarification about a participant's previous statement. Other times, my interjections induced particular speaker roles in my participants, such as when I would ask someone to recapitulate a prior statement and they would respond as relayer. Moreover, there were instances in which participants expected me to add to the conversation, as evidenced via particular eye gaze and facial expressions directed at me, but I deliberately did not.

Ultimately, my findings suggest that verbal and nonverbal qualifiers can significantly shape collective argumentation, in the sense that they can catalyze follow-up arguments or stimulate additional clarifications in an existing argument. As such, instructors must be attuned to both types of qualifiers.

For instance, instructors might use students' qualifiers to glean important information about how students view authority in the classroom, and perhaps to subsequently shape these views. As discussed previously, in Task 11 Frank's nonverbal qualifiers transformed from looking to *me* for approval, to looking to *Dan* for approval. I was pleased to observe this because I found that Dan subsequently stepped in and tried to verify Frank's claims himself rather than wait to see if I would validate Frank's statements. Accordingly, instructors might wish to redirect students' qualifying looks for approval back towards the students in order to shift the perceived source of authority from teacher to student(s) during collective argumentation. And more generally, instructors must be mindful of how their questioning and scaffolding (or lack thereof) can shape students' argumentation in the above ways. Indeed, such implications also align with Krummheuer's (2007) findings, in which elementary students appealed to their teacher's presence or absence of intervention following students' claims as a warrant to support or refute these claims.

Moreover, considerations about students' perceived sense of authority in the classroom could shape the prevalence of *challenges* in students' collective argumentation. According to Weber et al. (2008), in classroom environments where the teacher is perceived as the sole arbiter of students' reasoning, "we believe students will be unlikely to challenge their classmates' arguments, believing that it is the teacher's job to do so” (p. 259). As such, and given the demonstrated importance of challenges to collective argumentation in the present study, instructors should cultivate learning environments in which “warrants become explicit and the subject of debate […] warrants become the claim to be justified, engaging students in a higher level of mathematical reasoning” (Weber et al., 2008, p. 258). For the reasons conveyed previously, it is a nontrivial task as an instructor to mediate all the various aspects of collective argumentation, and the idiosyncrasies of how this can be done should be the object of further research, as I discuss in a later section.

Additionally, the manners in which my participants joined embodied reasoning with symbolic and formal reasoning highlight the potential roles of visualization and geometry in the study of complex integration. Although complex variables courses tend to focus on symbolic computations and applications involving integration, my results point to an important consideration for teaching such a course. Specifically, they suggest that instructors might want to more explicitly highlight how the symbolism that abounds during the integration unit of a complex variables course can intertwine with the embodied and formal worlds. For instance, after providing a formal definition for a simply-connected domain or a simple curve, students could benefit from drawing numerous examples and counterexamples with one another. At times, my participants conflated some of these formal requirements, suggesting that additional care should be taken to produce examples that satisfy one requirement but not another. Moreover, given Riley and Sean’s difficulty with justifying the continuity of certain complex functions, instructors might wish to review this topic prior to beginning the integration unit, as continuity is an assumption needed for many of the integration theorems.

A related pedagogical implication that arises from the present study involves educators' emphasis of geometric interpretations of foundational arithmetic at the beginning of a complex analysis course. In particular, my participants occasionally reached impasses during embodied reasoning about integration due to errors in their geometric characterizations of complex arithmetic. A notable example of this was Sean's conflation of the dot product with complex multiplication, as discussed previously. Moreover, in Task 5b, Sean mis-plotted several complex conjugates and this confused him and Riley during his embodied description of an integral. In Task 4, Riley and Sean also ended up with the wrong resultant vector when performing vector subtraction visually. Such difficulties with complex arithmetic were not limited to Riley and Sean. In particular, Dan and Frank conflated the complex reciprocal $\frac{1}{z}$ with the complex conjugate $\bar{z}$ in Task 6, wherein both participants claimed that the two operations were equivalent. They attempted to justify this claim by writing the function $\frac{1}{z}$ in several forms, including z^{-1} and $cos\theta - isin\theta$, the latter of which is only accurate when the point z lies on the unit circle.

The aforementioned findings suggest that complex variables instructors might need to further emphasize the geometry of complex arithmetic at the onset of the course if they wish for their students to develop a geometric interpretation of integration. In particular, instructors might especially stress that although complex numbers can be represented as vectors graphically, they are equipped with a multiplication operation that is structurally different than the dot- and cross-products studied in multivariable calculus. Moreover, a focus on fluency between the various forms of a complex number, especially those involving the exponential form, might prevent instances such as Dan and Frank's false equivalence of $\frac{1}{z}$ and $\bar{z}$, and is advocated for by other researchers in this domain as well (Danenhower, 2006; Karakok et al., 2014). Finally, Soto-Johnson and Troup (2014) found successes in participants' proficiency between algebraic and geometric reasoning about complex arithmetic following a complex variables course incorporating dynamic geometry software. As such, I underscore their contention that the effective implementation of technology such as *GeoGebra* or *Geometer's Sketchpad* could be effective in cultivating students' geometric conceptions of complex arithmetic, which can be employed when discussing embodied interpretations of complex integration.

As discussed in a previous section, there were three ways in which my participants explicitly addressed *thinking real, doing complex* (Danenhower, 2000): (1) purposefully avoiding inappropriate applications of it; (2) extending real intuition to the complex setting erroneously; and (3) extending real intuition to the complex setting in productive ways. Aside from a few notable exceptions, my participants were mostly cognizant about avoiding the unproductive versions of *thinking real, doing complex*, but implementing the productive ones; an important question is *why*? While this was not an explicit research question in the present study, I note here that it is possible that my participants' attention to *thinking real, doing complex* might be partly attributed to Professor X's explicit statements about building upon the intuition from real-valued functions in order to define analogous complex structure. He made such statements in class regularly throughout the integration unit, and I speculate that he made similar remarks during previous units as well based on how he introduced integration of complex functions.

In particular, on Day 1 of the integration unit, Professor X reminded the class that they had previously defined complex derivatives by mimicking the limit definition of the real-valued derivative. Analogously, he introduced complex integration by stating, “It can’t be the same as in real analysis, for reasons you’ll see in a minute; but let’s take the definition from real variables, and as best we can, mimic what we get to define something in the complex case.” Accordingly, I suspect that instructors might be able to instill in students an awareness of *thinking real, doing complex* via continual and explicit conversations regarding (1) how the intuition from real-valued functions plays a part in defining the complex-valued counterparts and (2) where that intuition needs adjustment in the complex setting. In other words, instructors might benefit from making explicit any productive versus unproductive met-befores using the set-before of “the language of Calc 3.” That said, such matters can and should be explored through future research.

Finally, my results suggest potential teaching implications regarding the use of language with respect to the formal assumptions in integration theorems. In particular, care should be taken when assigning acronyms like SICOPOC or ASCODOD. Ostensibly, these abbreviations are a convenient way to express several conditions or hypotheses succinctly in a proof or when writing out the statement of a theorem. However, in doing so, there is the potential danger of lumping assumptions together in such a way that students do not have to think carefully about each of the separate statements or when they are using each particular assumption during integration computations. For instance, this might be why Riley and Sean got confused about the “Jordan” phrasing of Task 11 and related considerations during Arguments 1-3. On the other hand, Dan and Frank did not seem to require as much clarification about the statement of Task 11. As I read this task aloud to them, Frank immediately expressed a desire to "write this [the assumptions] out properly," and when I added that he and Dan could use the SICOPOC abbreviation from class, he responded, "I guess I'll use the abbreviation. I never actually used it [in class]." To be clear, I am not suggesting that such abbreviations are always harmful, but rather that instructors should be careful to explicitly highlight each assumption as it comes up in the problem or proof.

Limitations and Future Research

Although the results of my study suggest potential teaching implications and extensions for theoretically framing collective argumentation, I acknowledge several limitations preventing further interpretation of my findings. In this section, I disclose known limitations related to data collection at the chosen institution, as well as unexpected circumstances that arose during data collection. Afterwards, I suggest possible avenues for future research based on the observations discussed in this chapter and the results detailed in the previous chapter.

Limitations

Unfortunately, there were several unavoidable difficulties that I encountered during my study that arose from the logistics of collecting data at the particular institution in which my participants resided. In particular, internal policies at this institution regarding the conducting of interviews mandated that I submit the exact interview tasks to the institution early on in process of designing my study. This meant that I could not adjust my tasks after conducting classroom observations, and I was not able to significantly deviate from the submitted list of questions. Consequently, the ensuing classroom observations did not inform how I conducted the interviews to the extent that I had originally intended, as I could not add new tasks based on what I observed in class. Along these lines, Professor X also had to slightly alter his original schedule for the integration unit mid-semester due to pacing considerations, so my scheduled observations did not completely cover the entire integration unit. Hence, I was not able to observe how the class discussed several of the later integration topics such as residue theory and integral applications.

A related consequence of the aforementioned issue was that some of the tasks I asked participants happened to be strikingly similar to exercises they had completed in class or on a previous exam. For instance, after I asked Dan and Frank about the conditions under which a complex function has an antiderivative in Task 13, Frank remarked, “This was on the test” as he pointed at Dan. In such cases, participants’ collective argumentation might have included fewer explicit warrants and backing if they did not need to think as deeply about supporting their assertions or convince one another of their claims. Ultimately, I do not find this to be problematic, in that students can certainly encounter similar situations in authentic classroom sessions wherein they recall a problem they have interacted with previously. I mention this finding here only to acknowledge the fact that students’ prior exposure to certain tasks likely affected their ensuing collective argumentation in response to such tasks, for better or for worse.

Another similar outcome arising from the predetermined rigidity of my tasks manifested during the students’ interviews as I read Task 11 aloud to the participants. In particular, my use of the word “Jordan” confused Riley and Sean, and this property become the object of follow-up arguments that required additional clarification. During my classroom observations, the class did not meaningfully discuss this property when articulating the various integration theorems, but I did not change the wording of this task out of consideration of the aforementioned interview requirements. Accordingly, the assumptions I listed in the setup of Task 11 (and potentially others) were not exactly how they would have been phrased in class, and this might have caused undue confusion about what I was asking in the tasks. At times, I attempted to adjust for this scenario by additionally re-voicing some of the hypotheses using Professor X's acronyms such as SICOPOC. However, as mentioned previously, Frank admitted that he never really used those abbreviations on his own.

An unfortunate limitation also arose from an accidental omission on my part during Dan and Frank's interview. Specifically, after an interesting follow-up conversation after Task 7, I inadvertently skipped over Task 8. As such, I did not have enough data from that task to identify any similarities or differences in how the two student pairs reasoned about the effect of reversing a path's orientation on the resulting integral across such a path. More generally, I witnessed instances during the interview in which one participant had more to say about a topic, but was interrupted by the other participant, leaving some arguments initially incomplete. I tried not to intervene in these cases, so as not to disturb the natural flow of the pair's collective reasoning, but nevertheless this meant that sometimes one participant's reasoning overshadowed another's.

As mentioned previously, participants occasionally recognized some aspects from the interview tasks from previous exam questions from their class. However, I did not request copies of these exams as artifacts, so I was unable to directly compare the two scenarios or verify that the tasks were indeed similar. While this was not an explicit goal of the present study, future work could incorporate such considerations for the purposes of studying their transfer of knowledge as captured by their Toulmin argumentation schemes. In particular, it might be illuminating to compare how individual students initially reason about an exam problem on their own, compared with how they collectively support their assertions when working through a similar problem at a later time. In particular, such an investigation could capture how students' mathematical justifications shift over time and in individual versus social contexts. In the next section, I detail other potential directions for pertinent future research.

Future Research

With the aforementioned results from a controlled interview setting in mind, I am excited to ascertain in future research how my proposed theoretical addendums to collective argumentation play out in more authentic classroom interactions, as well as in other mathematical contexts. Specifically, I would like to collect data wherein I can carefully analyze the collective argumentation of larger groups of students rather than just pairs. Moreover, I am particularly interested in how teachers dynamically mediate the three proposed types of participation in collective argumentation during an actual class session, as doing so is certainly not a trivial task.

Although previous studies (Krummheuer, 1995, 2007; Rasmussen, Stephan, & Allen, 2004; Stephan & Rasmussen, 2002) have investigated students' real-time argumentation in classrooms, they have primarily done so via a truncated Toulmin model lacking qualifiers and rebuttals, and researchers rarely found evidence of explicit backing. These researchers' justification for omitting such components from their analysis has largely been that such components were non-existent in their K-12 students' arguments. However, while it is perhaps unsurprising that younger students would not explicitly articulate backing or rebuttals in the traditional sense, I believe my proposed amendments to analyzing argumentation would indeed account for more subtle or embodied versions of qualifiers, backing, and rebuttals. It is my contention that features such as gestures and facial expressions can capture this type of important information when analyzing students' mathematical justifications. As such, future work could shed immense light on how we view collective argumentation within authentic classroom settings, especially those that involve younger students who may not always be able to verbalize some of the nuances of their mathematical reasoning.

I would additionally like to investigate such matters in courses for pre-service teachers via future research. I hypothesize that this particular population of students might argue in a relatively unique manner due to their dual roles as students and prospective teachers. Specifically, pre-service teachers might call upon specialized support for their mathematical assertions that stems from how they might teach a hypothetical student. Indeed, in previous work, mathematics professors appealed to pedagogical explanations when articulating their reasoning about continuity of complex functions (Soto-Johnson, Hancock, & Oehrtman, 2016). Moreover, I hope to find ways to further refine how we can view collective argumentation in order to inform subsequent research and ultimately improve the practice of teaching. Eventually, my aim is to provide research-based professional development for in-service teachers so that they may carefully attend to the interrelated factors shaping students' collective argumentation. Attention to students' argumentation is particularly important in modern standards-based curricula. In particular, one of the Common Core State Standards Initiative's eight

Standards for Mathematical Practice is to “construct viable arguments and critique the reasoning of others” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Future work could also more intricately link classroom observations and student interviews. Specifically, it might be of interest to investigate how the establishment of norms regarding what counts as justification influences or shapes students’ collective argumentation about integration. This could illuminate, for instance, how a complex analysis professor’s justifications about continuity of complex functions during class inform students’ warrants related to continuity hypotheses when applying integration theorems in an interview setting or during future classes. Such research could complement work conducted by Fukawa-Connelly (2011) in abstract algebra, in which he conducted Toulmin analyses to conclude that students appropriated key features of their professor’s argumentation structure in their own subsequent proofs. Along these lines, I hypothesize that Professor X in my study influenced my participants’ attention to *thinking real, doing complex* (Danenhower, 2000) via explicit statements about building upon the intuition from real-valued functions. As such, future work could illuminate more definitively how instructors might influence students’ invocation of this phenomenon in complex variables courses by instilling and nurturing specific norms to this end.

Although complex analysis has been a rich setting to explore the boundary between intuition and formality, I also plan to investigate students’ argumentation within more foundational settings such as calculus. Part of my rationale for this avenue of research is that I want to investigate the contexts in which some of students’ intuition and embodiment are formed in the first place. In particular, I intend to study how students’ reconciliation of everyday notions such as steepness and accumulation with the symbolism and theorems in calculus manifests in students' collective argumentation. Such work could, in turn, further inform research on the teaching and learning of complex analysis, in that I could gain pertinent insight into the *thinking real* aspect of *thinking real, doing complex.*

Previous research on integration of real-valued functions has illuminated a few aspects that might be helpful in such an investigation, but unfortunately the majority of these studies have primarily focused on students' misconceptions static products rather than the processes by which students reach these faulty conclusions. For instance, Orton (1983) and Grundmeier et al. (2006) reported students' difficulties relating the concept of the definite integral to a limit, as well as incomplete or completely incorrect definitions of the definite integral. Mahir (2009) also found that students had trouble identifying when areas should be treated as negative contributions to the definite integral. While such findings suggest important potential misconceptions, future work could illuminate *why* such confusion might occur in calculus courses by detailing the processes of students' (collective) argumentation as they make and support similar claims.

Another context in which I could test my proposed amendments to framing collective argumentation is linear algebra. Like complex analysis, undergraduate linear algebra courses tend to be comprised of a healthy mixture of symbolic calculation and formal proof, with theoretical results such as the Invertible Matrix Theorem complementing symbolic manipulations such as row-reducing matrices. Moreover, topics such as linear transformations and eigenvectors are often visually embodied by studying the manners in which particular vectors or shapes are mapped under certain transformations. Accordingly, the mathematical context of linear algebra is amenable to the theoretical lens of Tall's (2013) Three Worlds.

For instance, Thomas and Stewart (2011) investigated how undergraduate students attended to the embodied, symbolic, and formal worlds with respect to eigenvalues and eigenvectors. They found that explicitly highlighting embodied interpretations of linear maps throughout the course allowed some students to proficiently describe eigenvectors and eigenvalues using the language of "change of direction" and "steepness" (p. 283), though ultimately many participants still preferred symbolic representations. Moreover, although she did not incorporate the Three Worlds framework, Wawro (2015) attributed her participant's successes in the observed linear algebra course to his representational fluency and explicit reference to definitions in his mathematical justifications. Future work should investigate the potential for similar successes in *collective* argumentation, particularly in a classroom where students are regularly given opportunity to "reflect on the different symbolic forms and translations between them," as Wawro encouraged.

Finally, it is worth noting that several important components of my study and proposed theoretical additions to framing collective argumentation concern explicit statements that resulted from reflecting on one's own thoughts or prior statements. Specifically, both *challenges to oneself* and avoiding inappropriate invocations of *thinking real, doing complex* appear to instantiate aspects of *metacognition*, broadly described as thinking about one's own thinking (Schoenfeld, 1987). Accordingly, future research could further illuminate these and other potential relationships between collective argumentation and metacognition. Such work could also complement and/or extend recent efforts to reformulate metacognition as a social process related to a *collaborative zone of proximal development* (Goos, Galbraith, & Renshaw, 2002). Indeed, Goos et al. found that secondary students'

> challenges eliciting clarification and justification of strategies stimulated further monitoring that led to errors being noticed or fruitful strategies being endorsed. On the other hand, causes of metacognitive failures could be traced to the absence of such challenges (p. 218).

Hence, future studies could help discern how the types of challenges that I witnessed in my dissertation, both to oneself and to another student, shape undergraduate students' metacognition on the individual and collective levels.

REFERENCES

Aberdein, A. (2005). The uses of argument in mathematics. *Argumentation*, *19*, 287-301.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior, 24,* 125-134.

Alibali, M. W., & Goldin-Meadow, S. (1993). Gesture-speech mismatch and mechanisms of learning: What the hands reveal about a child's state of mind. *Cognitive Psychology*, *25*, 468-523.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa, 9* (Special issue on semiotics, culture and mathematical thinking), 267–299.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247-285.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3/4), 175-190.

Clement, J. (2000) Analysis of clinical interviews: Foundations and model viability. In Lesh, R. and Kelly, A., *Handbook of research methodologies for science and mathematics education* (pp. 341-385). Hillsdale, NJ: Lawrence Erlbaum.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches.* Thousand Oaks, CA: Sage.

Committee on the Undergraduate Program in Mathematics. (2015). Complex analysis. *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide, 2015*. Mathematical Association of America. 1529 Eighteenth Street NW, Washington, DC. Retrieved June 11, 2016 from http://www2.kenyon.edu/Depts/ Math/schumacherc/public_html/Professional/CUPM/2015Guide/Course%20Groups/Complex%20Analysis.pdf

Danenhower, P. (2000). Teaching and learning complex analysis at two British Columbia universities. *Dissertation Abstract International 62(09).* (Publication Number 304667901). Retrieved March 5, 2011 from ProQuest Dissertations and Theses database.

Danenhower, P. (2006). Introductory complex analysis at two British Columbia Universities: The first week–complex numbers. *CBMS Issues in Mathematics Education*, *13*, 139-170.

Evens, H., & Houssart, J. (2004). Categorizing pupils' written answers to a mathematics test question: 'I know but I can't explain'. *Educational Research*, *46*(3), 269-282.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). “You're going to want to find out which and prove it”: Collective argumentation in a mathematics classroom. *Learning and instruction*, *8*(6), 527-548.

Fukawa-Connelly, T. (2011). Toulmin Analysis: A tool for analyzing teaching and predicting student performance in proof-based classes. *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education*, Portland, Oregon.

Goffman, E. (1981). Footing. In *Forms of talk. Ders*. Philadelphia: University of Philadelphia Press.

Goldin-Meadow, S. (2003). *Hearing Gestures: How our hands help us think*. Chicago: University of Chicago Press.

Goldin-Meadow, S., Alibali, M. W., & Church, R. B. (1993). Transitions in concept acquisition: Using the hand to read the mind. *Psychological Review*, *100*(2), 279-297.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics, 49*, 193-223.

Grundmeier, T. A., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *16*(2), 178-191.

Hawker, C. (1986). The effects of replacing some manual skills with computer algebra manipulations on student performance in business calculus. *Dissertation Abstracts International, 47A,* 2934.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19,* 3-25.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*, *17*(1), 123-134.

Hollebrands, K. F., Conner, A., & Smith, R. C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. *Journal for Research in Mathematics Education*, 324-350.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modeling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Judson, P. (1988). Effects of modified sequencing of skills and applications in introductory calculus (Doctoral dissertation, University of Texas at Austin, 1988). *Dissertation Abstracts International, 49A*, 1397.

Judson, T. W., & Nishimori, T. (2005). Concepts and skills in high school calculus: An examination of a special case in Japan and the United States. *Journal for Research in Mathematics Education*, 24-43.

Karakok, G., Soto-Johnson, H., & Anderson-Dyben, S. (2014). Secondary teachers' conception of various forms of complex numbers. *Journal of Mathematics Teacher Education, 18*(4), 327-351.

Keene, K., Rasmussen, C., & Stephan, M. (2012). Gestures and a chain of signification: the case of equilibrium solutions. *Mathematics Education Research Journal, 24*, 347-369.

Krummheuer, G. (1995). The ethnology of argumentation, In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229-269). Hillsdale, NJ: Erlbaum.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *The Journal of Mathematical Behavior*, *26*(1), 60-82.

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind.* Chicago, IL: University of Chicago Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by.* Chicago: University of Chicago Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Levinson, S. C. (1988). Putting linguistic on a proper footing: Explorations in Goffman's concepts of participation. In P. Drew, A. Wootton, & G. Ervin (Eds.), *Exploring the interaction* (pp. 161–227). Cambridge: Polity Press.

Lobato, J. (2006). Alternative perspective on the transfer of learning: History, issues, and challenges for future research. *The Journal of the Learning Sciences,* 15(4), 431-449.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior*, 21, 87-116.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, *40*(2), 201-211.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. University of Chicago Press.

Mejia-Ramos, J. P. (2008). The construction and evaluation of arguments in undergraduate mathematics: A theoretical and a longitudinal multiple case study. Unpublished PhD Thesis, University of Warwick, UK.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation.* San Francisco, CA: Jossey-Bass.

Miller, P. (2009). Theories of developmental psychology. New York: Worth.

National Council of Teachers of Mathematics. (2009). *Focus in High School Mathematics: Reasoning and Sense Making*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common Core State Standards for mathematics: Standards for mathematical practice. Retrieved November 4, 2017 from http://www.corestandards.org/Math/Practice/.

Nemirovsky, R., & Ferrera, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, 70, 159-174.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, *21*(2), 287-323

Oehrtman, M., Soto-Johnson, H., & Hancock, B. (2018, January). Experts' construction of mathematical meaning for derivatives and integrals of complex-valued functions. Paper presented at the MAA Invited Paper Session on Research in Undergraduate Mathematics Education: Highlights from the Annual SIGMAA on RUME Conference. Joint Mathematics Meetings, San Diego, CA.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1-18.

Palmiter, J. R. (1991). Effects of computer algebra systems on concept and skill acquisition in calculus. *Journal for Research in Mathematics Education*, 151-156.

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G. P. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International Journal of Mathematical Education in Science and Technology*, *37*(6), 681-706.

Patton, M. Q. (2002). *Qualitative research and evaluation methods.* Thousand Oaks, CA: Sage.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. *Handbook of research on the psychology of mathematics education*, 205-235.

Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *The Journal of Mathematical Behavior*, *26*(3), 189-194.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *The Journal of Mathematical Behavior*, *23*(3), 301-323.

Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. In *PME CONFERENCE* (Vol. 4, pp. 4-089).

Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *The Journal of the Learning Sciences*, *11*(2-3), 275-300.

Schiralli, M. & Sinclair, N. (2003). A constructivist response to 'Where mathematics comes from.' *Educational Studies in Mathematics, 52 (1)*, 79-91.

Schoenfeld, A. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 189-215). Hillsdale, NJ: Lawrence Erlbaum.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, *22*(1), 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44-55.

Simpson, A. (2015). The anatomy of a mathematical proof: Implications for analyses with Toulmin's scheme. *Educational Studies in Mathematics*, *90*(1), 1-17.

Soto-Johnson, H., Hancock, B., & Oehrtman, M. (2016). The interplay between mathematicians' conceptual and ideational mathematics about continuity of complex-valued functions. *International Journal of Research in Undergraduate Mathematics Education*, *2*(3), 362-389.

Soto-Johnson, H., & Troup, J. (2014). Reasoning on the complex plane via inscriptions and gesture. *The Journal of Mathematical Behavior*, *36*, 109-125.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459-490.

Tall, D. O. (2004a). Building theories: the three worlds of mathematics. *For the Learning of Mathematics, 24*(1), 29-32.

Tall, D. O. (2004b). Thinking through three worlds of mathematics. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway, 4, 281-288.

Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), 5-24.

Tall, D. O. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM*, *41*(4), 481-492.

Tall, D. O. (2013). *How humans learn to think mathematically: exploring the three worlds of mathematics*. Cambridge University Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.

Thomas, M. & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal, 23*, 275-296.

Toulmin, S.E., (1958, 2003). *The uses of argument.* Cambridge, UK: University Press.

Troudt, M. (2015). *Mathematicians' evolving personal arguments: Ideas that move proof constructions forward*. Published Doctor of Philosophy dissertation, University of Northern Colorado.

Troup, J. D. S. (2015). *Students' development of geometric reasoning about the derivative of complex-valued functions*. Published Doctor of Philosophy dissertation, University of Northern Colorado.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Wagner, J. F. (2006). Transfer in pieces. *Cognition and instruction*, *24*(1), 1-71.

Wawro, M. (2015). Reasoning about solutions in linear algebra: the case of Abraham and the Invertible Matrix Theorem. *International Journal of Research in Undergraduate Mathematics Education*, *1*(3), 315-338.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247-261.

Weinstein, M. (1990). Towards an account of argumentation in science. *Argumentation, 4*(3), 269-298.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic bulletin & review*, *9*(4), 625-636.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4. *Journal for Research in Mathematics Education, 27(4)*, 435-457.

APPENDIX A

INTEGRATION THEOREMS AND RELATED RESULTS

1. *M-L Inequality*: If f is a complex-valued, continuous function on the contour C and if $|f(z)| \leq M$ for some constant M and for all z on C, then $\left|\int_C f(z)dz\right| \leq ML$ where L is the arc length of C.

2. *Cauchy-Goursat Theorem*: If a function f is analytic at all points interior to and on a simple closed contour C then $\int_C f(z)dz = 0$.

3. *Green's Theorem*: Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If P and Q are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then $\oint_C (P\ dx + Q\ dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy$.

APPENDIX B

IRB MATERIALS

UNIVERSITY of NORTHERN COLORADO

Institutional Review Board

DATE:	April 8, 2015
TO:	Brent Hancock
FROM:	University of Northern Colorado (UNCO) IRB
PROJECT TITLE:	[732996-2] Undergraduate Mathematics Majors’ Geometric and Algebraic Reasoning about Integration of Complex-valued Functions (REVISED)
SUBMISSION TYPE:	Amendment/Modification
ACTION:	APPROVED
APPROVAL DATE:	April 8, 2015
EXPIRATION DATE:	April 8, 2016
REVIEW TYPE:	Expedited Review

Thank you for your submission of Amendment/Modification materials for this project. The University of Northern Colorado (UNCO) IRB has APPROVED your submission. All research must be conducted in accordance with this approved submission.

This submission has received Expedited Review based on applicable federal regulations.

Please remember that informed consent is a process beginning with a description of the project and insurance of participant understanding. Informed consent must continue throughout the project via a dialogue between the researcher and research participant. Federal regulations require that each participant receives a copy of the consent document.

Please note that any revision to previously approved materials must be approved by this committee prior to initiation. Please use the appropriate revision forms for this procedure.

All UNANTICIPATED PROBLEMS involving risks to subjects or others and SERIOUS and UNEXPECTED adverse events must be reported promptly to this office.

All NON-COMPLIANCE issues or COMPLAINTS regarding this project must be reported promptly to this office.

Based on the risks, this project requires continuing review by this committee on an annual basis. Please use the appropriate forms for this procedure. Your documentation for continuing review must be received with sufficient time for review and continued approval before the expiration date of April 8, 2016.

Please note that all research records must be retained for a minimum of three years after the completion of the project.

If you have any questions, please contact Sherry May at 970-351-1910 or Sherry.May@unco.edu. Please include your project title and reference number in all correspondence with this committee.

Brent -

Hello and thank you very much for your patience with the UNC IRB process.

The first reviewer, Dr. Collins, has provided approval based on the clear and thorough revisions/ clarifications submitted. It is now more clear how you will accomodate any students who do not wish to participate, and/or not wish to have their recordings on audio and/or video used in your study.

I've subsequently reviewed both your original and revised materials and have no further requests for revisions or additional materials. Please be sure to use all revised/amended protocols in your participant recruitment and data collection. Best wishes for a successful study and don't hestiate to contact me with any IRB-related questions or concerns.

Sincerely,

Dr. Megan Stellino, UNC IRB Co-Chair

<u>Narrative: UNC IRB Application</u> -- Undergraduate Mathematics Majors' Geometric and Algebraic Reasoning about Integration of Complex-valued Functions

A. **Purpose**
   1. **Research Questions**

The study of complex numbers and variables is one of the undergraduate mathematical domains that have not received much attention from mathematics education researchers. The few studies that do exist in the domain of complex variables have focused primarily on complex arithmetic and forms of a complex number (Danenhower, 2006; Karakok, Soto-Johnson, & Anderson-Dyben, 2014; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012; Panaoura, Elia, Gagatsis, & Giatilis, 2006; Soto-Johnson & Troup, 2014). While there are several ongoing studies investigating more advanced topics such as continuity and differentiation, there is no existing literature regarding integration of complex-valued functions, despite this being a central topic of any complex analysis course for undergraduates. In particular, it is unclear as of yet how undergraduate students reason algebraically and geometrically with the notion of integration of complex-valued functions. Fortunately, there exists research within other mathematical domains (such as linear algebra) regarding how students reason algebraically and geometrically about mathematics (Sierpinska, 2000; Tabaghi & Sinclair, 2013), and I intend to utilize the associated reasoning framework to assist my data analysis.

Due to the absence of any mathematics education research in this field, my[1] study is designed to contribute to the literature on teaching, learning, and understanding undergraduate mathematics, particularly in the area of complex analysis. Specifically, the purpose of this qualitative research project is to explore undergraduate mathematics majors' geometric and algebraic reasoning about integration of complex-valued functions. My research questions are:

1. What is the nature of undergraduate mathematics majors' reasoning with respect to integration of complex-valued functions?
2. What relationships exist between undergraduate mathematics majors' algebraic and geometric reasoning when integrating complex-valued functions?
3. What types of reasoning do undergraduate mathematics majors invoke when they apply powerful integration theorems such as Cauchy's Integral Formula to compute integrals?
4. How do undergraduate mathematics majors attend to the assumptions present in powerful integration theorems, when reasoning about integrals in practice?

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[1] "My" and "I" refer to the principal researcher (B. Hanock). The "research advisor" refers to H. Soto-Johnson.

Anecdotally, undergraduate students in complex analysis courses tend to be able to proficiently compute complex-valued integrals by applying powerful results such as Cauchy's Integral Formula, but it is often unclear (from looking at traditional written student work) how students are actually reasoning about their computations or why they can even apply the theorems that they do. As such, my research aims to provide teaching implications for educators as to how they can draw students' attention to any assumptions implicitly being used when computing integrals in complex analysis.

2. **Review Category**
   This research falls under the expedited review category because the research activities present no more than minimal risk to human participants (see section C for details) and data collection will be in the form of video-recordings and student-work artifacts. Furthermore, my research is designed to describe group characteristics from a population who is not vulnerable. There is no appreciable, foreseeable risk associated with completion of the tasks or questions beyond the risk typically associated with solving math problems and all information regarding the participants will be kept strictly confidential. Details regarding this confidentiality are provided in the Methods section below.

B. Methods

1. **Participants**
   Participants will be selected from students at the United States Air Force Academy enrolled in the *Math 451: Complex Variables* course, which is offered in the spring of 2015. This is generally a small-enrolled course with approximately 17 students composed of primarily juniors or seniors. The course will already be recorded by the institution for instructional purposes. For this reason, I will ask everyone 18 and older for permission to utilize pre-recorded video-taped classroom observations (see description below) for the purpose of describing group characteristics and the general classroom environment. I will also select a subset of four (two pairs) of these students to take part in two 90-minute task-based interviews per student pair. This subset of four will be purposeful (Patton, 2002) because I hope to interview students who can easily articulate their thoughts and work well together. In order to ensure such a selection I will talk to the instructor of the course to get an idea of which students might reason similarly or work well together.

   I will contact the four students selected to participate in the interviews via email. **Document A** is a sample letter that will be used in the email. **Document B** is a consent form for these four participants. Prior to the study's commencement, I will visit the Math 451 class and request class participation from the entire class. **Document C** is a copy of the consent form that I will distribute after describing the purpose of the research and debriefing the class. A detailed description of the debriefing for the classroom observations is included in the next section, and a copy of the debriefing for interview participants is included in **Document G**.

2. **Data Collection Procedures**
In this section I describe the data that I hope to collect, the purpose of these data, and debriefing protocols. There will be three sources of data: video-taped classroom observations, student-work artifacts, and video-taped task-based interviews. Conducting classroom observations and collecting student-work artifacts will not require any extra time from the students.

I will sit in on the course during the integration unit of the course (approximately five to eight class sessions), but will not be an active class participant. The purpose of the classroom observations is to document what information is covered by the instructor, and to establish a "base-line" for what students know about integration theory before taking part in the subsequent interview. The course will already be recorded by the institution for instructional purposes. Video-taping will result in stronger research because it allows me to "retain a rich record of behavior that can be reexamined again and again" (Clement, 2000, p. 577). It will also allow me to document field notes on the spot as I observe the class. Although the instructor is not the focus of my research, his sequencing of events and how he teaches the content will most likely influence students' reasoning to some degree. Accordingly, I would like to have this information as part of my research. Thus, I will ask everyone 18 and older for permission to utilize pre-recorded video-taped classroom observations (see description below) for the purpose of describing group characteristics and the general classroom environment. Those who do not give permission will have identifying information edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted.

In addition to observing the complex variables class several times, I would also like to collect copies of select student work, such as homework exercises and quizzes/exams. These artifacts will help triangulate my classroom observations and task-based interview findings. In particular, they will help me to assess what students know before going into the interview, as well as how they typically solve problems in complex analysis.

The third component of the data I wish to collect is in the form of two 90-minute, task-based, semi-structured interviews. I will conduct two such interviews per pair of student participants, and these interviews will be scheduled to take place near the end of the spring 2015 semester. During these interviews, I will ask the pair of students to work together to solve some tasks related to integration of complex-valued functions. Participants will be asked to communicate with one another aloud and write down their thoughts on an accompanying whiteboard. Appendices D and E contain some sample interview questions, though I will also be asking follow-up and clarification questions throughout.

While the students work on the tasks, I will encourage the students to elaborate on their discoveries, theories, ideas, reasoning, conjectures, etc. Such probing will allow me to encourage the students to think aloud and to request clarification about their remarks. A research assistant will be responsible for video-recording the teaching experiment, but will not take part in any subsequent analysis of the data. The interviews will be conducted at the United States Air Force Academy in a room familiar to the students. Summarily, a timeline of the various data collection is shown below.

Time	Activity	Participants
Late March - Early April	Class observations during unit on integration (5-8 classes)	1 researcher All students
Late April/Early May	Conduct task-based interview	1 researcher 2 pairs of consenting students

The debriefing process will occur when I invite the class to partake in the video-taped classroom observations. I will inform the students of the purpose of the research, and let them know that the observations will be videotaped over the course of 5-8 class sessions. I will disclose that I am interested in their geometric and algebraic understanding of complex-valued integration, including their use of gestures, and thus request that the participants allow me to utilize recorded video-taped classroom sessions. I will inform them of the importance of viewing video-recorded classroom sessions (i.e. gather rich data that can be observed multiple times) and that the student-artifacts will help substantiate my conjectures. Further, the video-recorded classroom sessions will be used for the purpose of describing group characteristics and the general classroom environment. Only video data artifacts (e.g. screenshots and quotes) for those students who agree to participate in interviews will be used in data analysis. Those who do not give permission will have identifying information edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted. Finally, I will inform them about the opportunity to take part in the aforementioned task-based interviews, if they are contacted by me at a later date

All the students will be informed orally and through the consent forms that they are not required to participate in the research and that their course grade will not be affected if they choose to not participate in the research. Moreover, those who do not give permission will have identifying information edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio.

Furthermore, I will inform the participants that for dissemination purposes I request that I be allowed to use video with their images, especially where I want to illustrate their use of gestures or diagrams to convey their understanding. I will honor students' request to not use their images; such students will have the option to participate in the research, but I will only use their remarks and describe their gestures. More detail is provided in section B4.

3. **Data Analysis Procedures**
The following data analysis procedure are relevant only for those students who will be interview participants. Video data for students in the recorded class sessions who are not interview participants will not be used in data analysis. Given my research is qualitative in nature, I will use qualitative methods to analyze the data, and potentially use software such as *Elan* to organize my data. The data analysis for the student interviews will begin with me transcribing participants' exact verbiage word-by-word in Microsoft Excel or a program such as *Elan*, and noting and describing any important gestures made by the participants. Each segment of the participants' response will then be coded for types of reasoning (c.f. Sierpinska, 2000). Through many viewings of the video data, as well as reviewing the coded reasoning data, I will then use *generative* analysis techniques to develop a theoretical model of the observed data. Generative analysis entails open interpretation of large episodes. I hope to inductively determine themes that emerge from the data, and use the other types of data collected (classroom observation field notes/video, student artifacts) to triangulate my findings, as discussed below.

Relevant episodes of the classroom observation video data and field notes will also be analyzed to substantiate or negate findings from the aforementioned student interview analysis. These classroom observations will be used in this way only to triangulate observations for interview participants, not for all students who allow me to view classroom observation data. Any phrases or gestures used by interview participants or the professor during the class observations will be compared to those used by interview participants during the interviews, and similar gestures will be documented by taking screenshots of the video data and labeling the corresponding gestures. The student-work artifacts will be used to triangulate my findings by offering a way to compare what students have previously written down to what they write as inscriptions during the interview tasks. The analysis of these three sources of data will help me answer my three research questions.

4. **Data Handling Procedures**
As the lead researcher, I (B. Hancock) will have access to the data, and my research advisor will also have access to the data as needed. But all data (video as well as PDF copies of student-work artifacts) will be stored on my password-protected computer. Any hard copies of the written work will be scanned and saved as an electronic copy on the external drives, and the hard copies will be shredded. In case my research advisor needs to view any data, I

will back-up all the data on a USB flash drive and have her store this drive in a locked file-cabinet in her UNC office. As this is a pilot study for a larger dissertation, all data will be stored for up to 4 years or until data analysis and dissertation are completed. All video data will then be destroyed.

Some of the data will be synthesized and portrayed as group results, but student excerpts will be used to substantiate my hypotheses and theoretical models. Furthermore, images of interview participants' gestures will be included as part of the results – this is standard reporting for such research questions. As mentioned above, I will request to use images of the interview participants from the video, and thus the identity of the participants will only be protected if they choose to abstain from sharing their images. In such a case, these participants will be assigned a pseudonym to use with their remarks, and will be guaranteed that I will not use their images in any dissemination materials. In an effort to convey their gestures, I will include rich descriptions rather than images of these participants. All participants in the interview component of my research, regardless of whether they want their images used in dissemination materials, will be assigned a pseudonym to help protect their identity to the greatest extent possible under the aforementioned conditions. The video-recorded classroom sessions will be used for the purpose of describing group characteristics and the general classroom environment. Only video data artifacts (e.g. screenshots and quotes) for those students who agree to participate in interviews will be used in data analysis. Those who do not give permission will have identifying information edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted.

C. Risks, Discomforts and Benefits

The risks inherent in this study are no greater than those normally encountered during regular classroom participation. Such minimal risks include participants being embarrassed about their responses, insecure about sharing their work, or worried that they will say something incorrect. I will attempt to mitigate these risks by assuring the students that I am not concerned about whether their responses/work/remarks are incorrect, and rather I am interested in how they reason algebraically and geometrically about complex variables concepts.

A benefit of participating in this research is that the students may gain a deeper understanding of the geometry behind the arithmetic and analysis of complex numbers and variables by simply discussing the interview tasks with another student from the class. Indirect benefits include contributing to the knowledge base of teaching and learning complex variables, which could result in an improved course for future students.

D. Costs and Compensations

Participation is voluntary. Because observations and interviews will be conducted on-base at the US Air Force Academy, there should not be any cost incurred by the participants exceeding normal transportation to school. The only other foreseen costs are the time costs associated with the interview, which will last 180 minutes at most (two 90-minute sessions). Compensation will not be provided for consenting participants but snacks will be available during the interview periods.

E. Grant Information

I have recieved travel funds via University of Northern Colorado's College of Natural and Health Sciences' 2014-2015 Student Research Fund, in the amount of $200.

Documentation:

Document A: Email invitation to participate in interviews
Document B: Consent form for interview participants
Document C: Consent form for non-interview participants
Document D: Consent form for course instructor
Document E: Sample Day 1 Interview Questions/Tasks
Document F: Sample Day 2 Interview Questions/Tasks
Document G: Debriefing for Interview Participants

Document A:
Email Invitation to Participate in Interviews

Dear ________________,

My name is Brent Hancock and I am a graduate student in educational mathematics at the University of Northern Colorado. I am conducting a study to investigate undergraduates' algebraic and geometric reasoning about complex-valued functions. The purpose of this letter is to invite you to participate in a task-based interview along with one of your peers in Math 451.

I am interested in exploring how undergraduate mathematics majors, such as you, view ideas such as integration of complex valued functions. As part of this research I intend to gather data on how undergraduates communicate their ideas using diagrams, gestures, metaphor, etc. In order to explore this phenomenon I am inviting you to participate in an end of semester interview, spread out over two sessions. Here are the important facts regarding the interview:

- The interview will last a maximum of 180 minutes (90 minutes at most per session), and you will work with a classmate to discuss the tasks with one another.
- This interview will take place on campus and will be video-taped so I can analyze the data at a later time.
- We can find a time that works for you and your classmate during the second half of April or early May, after you have completed the integration unit.

Given the purpose of my research, I would like to share portions of your video-clips during presentations and it is possible that I may want to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. In such a case, I will only describe your responses rather than use pictures. In any case, I will assign you a pseudonym – care will be taken to protect your identity.

I hope you will be willing to participate in this study especially since the results of this study could inform improved teaching methods of complex variables and other mathematical domains. This interview would also be a great opportunity to review and discuss Math 451 course material with a fellow classmate. Please do not hesitate to contact me if you have any questions regarding the study or the protocol for the study. You may contact me at brent.hancock@unco.edu .

Sincerely,

Brent Hancock

Document B:
Consent Form for Interview Participants

UNIVERSITY of NORTHERN COLORADO

Consent Form for
Human Participants in Research

Project Title: Undergraduate Mathematics Majors' Geometric and Algebraic Reasoning about Integration of Complex-valued Functions
Researcher: Brent Hancock, Graduate Student, School of Mathematical Sciences, University of Northern Colorado
Phone Number: (818) 730-9615
E-mail: brent.hancock@unco.edu

Research Supervisor: Dr. Hortensia Soto-Johnson, Department of Mathematical Sciences, University of Northern Colorado
Phone Number: (970) 351-2425
E-mail: hortensia.soto@unco.edu

I am investigating how undergraduate mathematics majors perceive integration of complex-valued functions. I am interested in how students such as yourself communicate your understanding of these mathematical concepts through diagrams, gestures, body movements, and metaphors. In order to explore this phenomenon I invite you to participate in a video-recorded interview, spread over two 90-minute sessions. The purpose of the interview is to ask you and a Math 451 classmate to discuss some problems and concepts related to integration of complex valued functions. The results of this study could inform improved teaching methods of complex variables and other mathematical domains, and participation in this study would be a great opportunity to review and discuss Math 451 course material with a fellow classmate in a low-pressure environment.

The interview is designed to allow me, as a researcher, more time to observe you interact with complex variables content. During this interview you will engage in some integration problems with a fellow classmate, where I ask that you articulate your thoughts. The two of you should converse with one another, share ideas, and question each other's conjectures, if applicable. I may ask probing questions simply to get a better understanding of what you are attempting to convey. There is no need to worry if you say something that is incorrect because I am interested in how you might use geometric and algebraic reasoning to answer questions, or how such reasoning is developing throughout the activities. Recall from my invitation letter that:

- We can find a time that works for you and your classmate during the second half of April or early May, after you have completed the integration unit.
- The interview will last a maximum of 180 minutes (two sessions lasting 90 minutes at most), and you will work with a classmate to discuss the tasks with one another.
- This interview will take place on campus and will be video-taped so I can analyze the data at a later time.

Given the purpose of my research, I would like to share portions of your video-clips in both interviews and relevant class sessions during presentations and it is possible that I may want to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. In such a case, I will only describe your responses rather than use pictures. In any case, I will assign you a pseudonym when reporting any results – care will be taken to protect your identity.

All data will be stored on my (Brent Hancock's) personal computer, which is password protected; thus no one will have access to this data other than potentially my research advisor.

There are no foreseeable risks to participating in this study other than some discomfort if you do not feel comfortable answering a question. You may also benefit from participating in this research if reflecting on these activities and questions allows you to gain a new perspective of topics in complex variables.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910.

Please feel free to contact me via phone or email if you have any questions and retain one copy of this letter for your records. Thank you for assisting me with this research.

 ________________ (Participant Initials)

If willing to participate in the interview and <u>willing to disclose your identity</u> i.e., agreeing to have your video shared with others at conference presentations, publications, etc. please complete the following.

Name (please print)	Signature	Date

Researcher's Name	Researcher's Signature	Date

If willing to participate in the interview but <u>prefer to have identity protected</u>, please complete the following.

Name (please print)	Signature	Date

Researcher's Name	Researcher's Signature	Date

Document C:
Consent Form for Non-Interview Participants

UNIVERSITY of NORTHERN COLORADO

Consent Form for
Human Participants in Research

Project Title: Undergraduate Mathematics Majors' Geometric and Algebraic Reasoning about Integration of Complex-valued Functions
Researcher: Brent Hancock, Graduate Student, School of Mathematical Sciences, University of Northern Colorado
Phone Number: (818) 730-9615
E-mail: brent.hancock@unco.edu

Research Supervisor: Dr. Hortensia Soto-Johnson, Department of Mathematical Sciences, University of Northern Colorado
Phone Number: (970) 351-2425
E-mail: hortensia.soto@unco.edu

I am investigating how undergraduate mathematics majors perceive integration of complex-valued functions. I am interested in how students such as yourself communicate your understanding of these mathematical concepts through diagrams, gestures, body movements, and metaphors. In order to explore this phenomenon I request that you allow me to utilize 5 to 8 of the video-taped class sessions already being recorded by the mathematics department while you are in the complex-variables class (Math 451).
By viewing video of the class I will be able to observe the different ways in which interview participants convey their understanding of complex variables and the recording will allow me to watch the episodes on multiple occasions. During class sessions I attend, I will not be an active participant during the class. I will simply take notes of my observations. There is no need to worry if you say something that is incorrect because I am only interested in how interview participants use geometric and algebraic reasoning to answer questions or how such reasoning is developing through the semester.

Given the purpose of my research, I would like to share portions of video-clips involving interview participants during presentations and it is possible that I may want to incorporate photos that illustrate their gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. The video-recorded classroom sessions will be used for the purpose of describing group characteristics and the general classroom environment in order to triangulate observations for interview participants. Only video data artifacts (e.g. screenshots and quotes) for those students who agree to participate in interviews will be used in data analysis. Those who do not give permission will have identifying information edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted.

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your Math 451 course grade. You also have the option to participate in different aspects of the research. You may choose to:

a. participate in the video-taping where we are allowed to use episodes showing your face,
b. participate in the video-taping where we are NOT allowed to use episodes showing your face but where we are allowed to use your remarks,
c. not participate in the research at all.

All data will be stored on Brent Hancock's computer, which is password protected, thus no-one will have access to this data other than those involved in the study (B. Hancock and Hortensia Soto-Johnson).

There are no foreseeable risks to participating in this study other than some discomfort if you do not feel comfortable being video-taped or are embarrassed by your work. It is possible that some video footage might accidentally capture your face or actions/words, especially if you are working closely with someone who has agreed to be video-taped. However, the video-recorded classroom sessions will be used for the purpose of describing group characteristics and the general classroom environment. Only video data artifacts (e.g. screenshots and quotes) for those students who agree to participate in interviews will be used in data analysis. If you do not give permission to have your image/audio captured on video, all identifying information will be edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted so there will be no record of your involvement.
You may benefit from participating in this research in reflecting on your work, hence gaining a new perspective of complex numbers and complex variables.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910. Please feel free to contact me via phone or email if you have any questions and retain one copy of this letter for your records. Thank you for assisting me with this research.

Page 2 of 3 ________________ (Participant Initials)

If willing to participate in classroom video-taping and willing to disclose your identity i.e., agreeing to have your video shared with others at conference presentations, classes, publications, etc. please complete the following.

____

Name (please print) Signature Date

____

Researcher's Name Research's Signature Date

If willing to participate classroom video-taping but prefer to have identity protected, please complete the following.

____

Name (please print) Signature Date

____

Researcher's Name Research's Signature Date

If not willing to participate in the research, please complete the following.

____

Name (please print) Signature Date

____

Researcher's Name Research's Signature Date

Page 3 of 3 ________________ (Participant Initials)

Document D:

Consent Form for Course Instructor

UNIVERSITY of NORTHERN COLORADO

Consent Form for
Human Participants in Research

Project Title: Undergraduate Mathematics Majors' Geometric and Algebraic Reasoning about Integration of Complex-valued Functions
Researcher: Brent Hancock, Graduate Student, School of Mathematical Sciences, University of Northern Colorado
Phone Number: (818) 730-9615
E-mail: brent.hancock@unco.edu

Research Supervisor: Dr. Hortensia Soto-Johnson, Department of Mathematical Sciences, University of Northern Colorado
Phone Number: (970) 351-2425
E-mail: hortensia.soto@unco.edu

I am investigating how undergraduate mathematics majors perceive integration of complex-valued functions. I am interested in how students communicate their understanding of these mathematical concepts through diagrams, gestures, body movements, and metaphors. In order to explore this phenomenon I request that you allow me access to 5 to 8 class sessions already recorded for instruction purposes through your institution while you teach the complex-variables class (Math 451) and that you allow me access to some of your students' completed homework assignments, exams, including any physical models that they create, and quizzes or misc. class work.

By using video-taped class sessions I will be able to observe the different ways in which you and the students convey an understanding of complex variables and the recording will allow me to watch the episodes on multiple occasions. Students' class work will be used to substantiate my interpretations of the classroom observations. During class sessions I attend, I will not be an active participant during the class. I will simply take notes of my observations. There is no need to worry if you say something that is incorrect because I am only interested in how your students use geometric and algebraic reasoning to answer questions or how such reasoning is developing through the semester.

Page 1 of 3 ________________ (Participant Initials)

Given the purpose of my research, I would like to share portions of your video-clips during presentations and it is possible that I may want to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. In such a case, I will only describe your responses rather than use pictures. In any case, I will assign you a pseudonym when reporting any results – care will be taken to protect your identity.

Please note that you are not under any obligation to participate in this research. You also have the option to participate in different aspects of the research. You may choose to:

a. participate in the video-taping where I am allowed to use episodes showing your face and where I am allowed to use your students' work,
b. participate in the video-taping where I am NOT allowed to use episodes showing your face but where I am allowed to use your remarks and your students' work, or
c. not participate in the research at all.

All data will be stored on Brent Hancock's computer, which is password protected, thus no-one will have access to this data other than those involved in the study (B. Hancock and Hortensia Soto-Johnson).

There are no foreseeable risks to participating in this study other than some discomfort if you do not feel comfortable being video-taped or are embarrassed by something you might say during class. It is possible that I may accidentally video-tape you, especially if you are working closely with someone who has agreed to be video-taped. The video-recorded classroom sessions will be used for the purpose of describing group characteristics and the general classroom environment. Only video data artifacts (e.g. screenshots and quotes) for those students who agree to participate in interviews will be used in data analysis. . If you do not give permission to have your image/audio analyzed from video data, all identifying information will be edited out of the video-recordings (both visual and audio) using the video editing software Camtasia Studio. The original identifying video will then be deleted so there will be no record of your involvement. You may benefit from participating in this research if reflecting on your work, hence gaining a new perspective on the teaching complex variables.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact Sherry May, IRB Administrator, Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-1910. Please feel free to contact me via phone or email if you have any questions and retain one copy of this letter for your records. Thank you for assisting me with this research.

If willing to participate in classroom video-taping and to provide student work and willing to disclose your identity i.e., agreeing to have your video shared with others at conference presentations, classes, publications, etc. please complete the following.

Name (please print)	Signature	Date

Researcher's Name	Research's Signature	Date

If willing to participate classroom video-taping and to provide student work but prefer to have identity protected, please complete the following.

Name (please print)	Signature	Date

Researcher's Name	Research's Signature	Date

If not willing to participate in the research, please complete the following.

Name (please print)	Signature	Date

Researcher's Name	Research's Signature	Date

Document E:
Sample Day 1 Interview Questions/Tasks

1. What do you believe the word "parametrization" signifies when describing a complex path integral?

2. Explain how you would represent $z(t) = e^{it}$ as a position vector of a moving point in the complex plane, using vector component notation.
   a. What are the two components?

3. Consider the paths defined by $z_1(t) = e^{i\pi t}$ and $z_2(t) = 1 + t$.
   a. How would you sketch $z_1(t) + z_2(t)$ on the board?
   b. Is this the same as joining the end of one path to the beginning of the other?
   c. If not, what function would you construct to describe a parametrization of such a path?

4. If $z = f(t)$ is a parametrized curve, what does $\frac{dz}{dt}$ represent physically at each point?
   a. How would you draw this?

5. If $z = f(t)$ is a parametrized curve described as a complex-valued function of *t*, how would you provide a geometric interpretation of the identity $\int_a^b \frac{dz}{dt} dt = f(b) - f(a)$?
   a. What is this identity commonly called?

6. Consider the function $f(z) = \bar{z}$
   a. Is this function analytic?
      i. If so, where is it analytic?
      ii. If not, how do you know?
   b. Is it possible to find $\oint_L \bar{z} dz$ where *L* is a circle of radius *r* traversed counterclockwise?
      i. If so, what is its value?
      ii. If not, how do you know?
   c. Does the value of the integral depend upon the radius of the circle?
      i. If so, how?
      ii. If not, how do you know?

Document F:
Sample Day 2 Interview Questions/Tasks

1. How would you find $\oint_L \frac{1}{z}dz$ if L represents the unit circle $|z| = 1$ traversed counterclockwise? Explain.

2. What if L (from problem 1) is now a circle of radius 2, centered about the origin, traversed counterclockwise? Explain.

3. What if L (from problem 2) is now traversed clockwise? Explain.
   a. How does parametrizing a path in the reverse direction affect the value of a complex path integral? Explain.

4.
   a. Let C be the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$. How would you find $\oint_C \frac{z+2}{z}dz$? Explain.

   b. Now let C be the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$. How would you find $\oint_C \frac{z+2}{z}dz$? Explain.

   c. Finally, let C be the whole circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$. How would you find $\oint_C \frac{z+2}{z}dz$? Explain.

   d. How did your answer (in part c) compare to your previous answers (in parts a, b)?

5. Is it permissible to travel over a circular path twice, and if so how does that affect the value of a complex path integral?

6. Let $f(z) = \frac{1}{z(z^2-1)}$ and let L be a closed rectifiable Jordan curve on the complex plane such that $f(z)$ is continuous on L. Find all possible values of $\oint_L f(z)dz$. For each possibility sketch the curve L that results in that value. Explain your answer.

   a. Which curves L (from problem 5) resulted in an answer of 0 for our integral? Why did this happen?
   b. What assumptions about $f(z)$ did you use to find each value of the integral? Why?
   c. What assumptions about L did you use? Why?

7. What does the integral of a complex valued function represent? How is this different than the integral of a real valued function? How is it the same?

8. When does a complex valued function have an antiderivative? Why would this be useful to know?

Document G:
Debriefing for Interview Participants

Thank you for taking the time to participate in these teaching experiments. As I mentioned in the invitation letter, the purpose of my research is to explore undergraduate mathematics majors' algebraic and geometric understanding of complex variables topics related to integration. This interview process is designed to allow me, the researcher, more time to observe you interact with complex variables content related to integration. During these two interview sessions you will engage in some integration problems with a fellow classmate, where I ask that you articulate your thoughts. The two of you should converse with one another, share ideas, and question each other's conjectures, if applicable. I may ask probing questions simply to get a better understanding of what you are attempting to convey. There is no need to worry if you say something that is incorrect because I am interested in how you might use geometric and algebraic reasoning to answer questions, or how such reasoning is developing throughout the activities. Recall from my invitation letter that:

- The interview will last a maximum of 180 minutes (two sessions lasting 90 minutes at most), and you will work with a classmate to discuss the tasks with one another.
- This interview will be video-taped so I can analyze the data at a later time.

Given the purpose of my research, I would like to share portions of your video-clips during presentations and it is possible that I may want to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, I am requesting permission to do so, but if you would prefer that I protect your identity, then I will honor your request. In such a case, I will only describe your responses rather than use pictures. In any case, I will assign you a pseudonym when reporting any results – care will be taken to protect your identity.

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your course grade. Please complete the consent form and then we will begin with the first interview session.

APPENDIX C

INTERVIEW TASKS

Part 1 Interview Questions/Tasks

1. What do you believe the word "parametrization" signifies when describing a complex path integral?

2. Explain how you would represent $z(t) = e^{it}$ as a position vector of a moving point in the complex plane, using vector component notation.
    a. What are the two components?

3. If $z = f(t)$ is a parametrized curve, what does $\frac{dz}{dt}$ represent physically at each point?
    a. How would you draw this?

4. If $z = f(t)$ is a parametrized curve described as a complex-valued function of *t*, how would you provide a geometric interpretation of the identity $\int_a^b \frac{dz}{dt} dt = f(b) - f(a)$?
    a. What is this identity commonly called?

5. Consider the function $f(z) = \bar{z}$
    a. Is this function analytic?
        i. If so, where is it analytic?
        ii. If not, how do you know?
    b. Is it possible to find $\oint_L \bar{z} dz$ where *L* is a circle of radius *r* traversed counterclockwise?
        i. If so, what is its value?
        ii. If not, how do you know?
    c. Does the value of the integral depend upon the radius of the circle?
        i. If so, how?
        ii. If not, how do you know?

Part 2 Interview Questions/Tasks

6. How would you find $\oint_L \frac{1}{z} dz$ if L represents the unit circle $|z| = 1$ traversed counterclockwise? Explain.

7. What if L (from problem 1) is now a circle of radius 2, centered about the origin, traversed counterclockwise? Explain.

8. What if L (from problem 2) is now traversed clockwise? Explain.
    a. How does parametrizing a path in the reverse direction affect the value of a complex path integral? Explain.

9.
    a. Let C be the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$. How would you find $\oint_C \frac{z+2}{z} dz$? Explain.

    b. Now let C be the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$. How would you find $\oint_C \frac{z+2}{z} dz$? Explain.

    c. Finally, let C be the whole circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$. How would you find $\oint_C \frac{z+2}{z} dz$? Explain.

10. Is it permissible to travel over a circular path twice, and if so how does that affect the value of a complex path integral?

11. Let $f(z) = \dfrac{1}{z(z^2 - 1)}$ and let L be a closed rectifiable Jordan curve on the complex plane such that $f(z)$ is continuous on L. Find all possible values of $\oint_L f(z)dz$. For each possibility sketch the curve L that results in that value. Explain your answer.

12. What does the integral of a complex valued function represent? How is this different than the integral of a real valued function? How is it the same?

13. When does a complex valued function have an antiderivative? Why would this be useful to know?

APPENDIX D

SAMPLE EXCERPT FROM CODEBOOK

Time	Who?	Speaker Role	Verbiage	Misc. notes	Toulmin	Worlds	Backing
Task 3							
Argument 1							
3:46	I	R	Ok sounds good. Um, ok so next, if z = f(t) is a parametrized curve [Sean writes z = f(t)], what does dz/dt represent physically [Sean writes dz/dt] at each point, and how would you draw this?		D1	S	
4:05	R	R, A, A	So let's see. So you have, like, z is a [draws curve on board]-- it's like some curve, right? It's parametrized. Um so like, dz/dt is sort of breaking it into little chunks. If we're at this point [draws a point on the curve], I guess [pauses] um, it would just be a little directional kind of infintessimal um, pointer, [draws in tangent vector at this same point] that says where we're going along this curve. **So if your curve is oriented this way--**	Until this point, Sean did all the writing on the board.	D1 (contd), C1, D2, Q2, C2, **D3**	E-S (relate symbolic z = f(t) to drawing of curve), E,	
4:39	S	A	Tangent vector.		C2(contd.)	E	
4:39	R	S	**[draws arrows on curve]** then dz/dt would look like a little vector pointing off [gesture with hand of a tangent vector] to where the next, uh, z is. It's not like actually a tangible concept, because it's infinitely small, but that's how I think of it.		**D3 (contd),** C, Q	E	
Argument 2							
4:53	S	A	Yeah if you think about tangent vectors-- so, for this [points at diagram from Task 2] we would get, like, z'(t) = -sin t + i cos t [writes this], and then our unit vector is, like, in this direction [draws in green unit vector on circle]. And we could call our tangent vector T, I guess.		D1, C1=D2, C2	E, S, E, S	
5:16	R	A	Yeah, it's going to be, like-- it should be parallel to the slope of the line at that point [traces finger along her path for this task].		W2	E	
5:22	I		And so if you-- if this is representing, like, the physical path of an object or something, does that tangent vector tell you anything physically about what's going on then?				
5:32	S	A	Velocity.		C	E	
5:33	I		So it's your veloctiy vector?				
5:34	S & R		Yeah.				

Time	Who?	Role	Verbiage	Other notes	Toulmin	Worlds	Backing
Task 4							
Argument 2							
7:56	I		Ok perfect, so yeah I saw you had the capital F notation before, so I was glad that you explained what that is and everything. So ok, we'll come back a little later, um, to talk a little more about antiderivatives and things of that nature. But, ok cool, so anything you wanted to add to that, Riley?				
8:11	R	A	Um, let's see-- so it's going to be-- I think this has to be true [FTC] for any path between these two points?		Q	E-S-F (formal statement about the relationship between the identity and the path)	
8:18	S		Mhm.				
8:19	R	S	It has to always be true.	D = (D from Arg 1)	C	F	
8:22	I		Yeah so we're really talking about any path from a to b there.				
8:27	R	A	Yeah so it's nice because it makes it more flexible, since like if we're working here [in the Calc 1 case -- points to circled Calc 1 version on board] if you're only in one dimension, there's only one way to get between, between the two points.		C (contd), W	E	
8:36	S		Yeah.				
8:37	R	A	But in 2 dimensions, you can take any path you'd like [path gesture], cuz it works for any path.		W (contd)	E	
8:40	S	A	And there are like, fairly technical things. Like you have to assume this [f] is continuous on the interval here [a to b] and you have to assume that the path here is piecewise smooth, or something like that. So there's no special-- the antiderivative is defined so there's no like, breaks, or anything. So you have to make sure it's a "well-behaved" path.	Good that he brought up the conditions/assumptions to use FTC. Riley didn't really mention these before	B1	F, E	Correctness
8:55	R	A	But generally those are the ones we're working with, so--		B1 (contd)		Field
8:59	S	A	Yeah, then of course, you have to distinguish between this thing [point to FTC] like, the integral of a complex function, versus the integral of, not a real variable [points to int of f(t)] but actually a complex variable. And that's when it gets a little more involved, but I guess that's later [in the interview].		B2	F-S	Validity (attention to when it applies vs. not)