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# Students' Development of Geometric Reasoning About the Derivative of Complex-Valued Functions

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# STUDENTS' DEVELOPMENT OF GEOMETRIC REASONING ABOUT THE DERIVATIVE OF COMPLEX-VALUED FUNCTIONS

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## ABSTRACT

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The purpose of this study was to explore the nature of students' reasoning about the derivative of a complex-valued function, and to study ways in which they developed this reasoning while working with *Geometer's Sketchpad* (*GSP*). The participants in this study were four students from one undergraduate complex analysis class. The development of participants' reasoning about the derivative of a complex-valued function was captured via video-recording and screen-capture software in a four-day interview sequence consisting of a two-hour-long interview each day. This reasoning was interpreted through the theoretical perspective of embodied cognition. The findings indicated that students manifested embodied reasoning through gesture and speech, through algebraic and geometric inscriptions, and through interaction with the physical environment and the virtual environment provided by *GSP*. The findings further indicated that students needed to advance their geometric reasoning about the derivative of a complex-valued function in three essential ways in order to reason geometrically about the derivative as a local linear approximation. First, with help from gesture and speech, they recognized that they did not know how to characterize a linear complex-valued function. Second, with help from algebraic and geometric inscriptions, they reasoned that a linear complex-valued function $f(z)$ rotates and dilates every circle by the same

amounts $Arg\big(f'(z)\big)$ and $|f'(z)|$, respectively. Finally, through embodied reasoning in both the virtual and physical environments, students recognized the need to focus on how a complex-valued function rotates and dilates small circles only.

These findings suggest that one approach to improving student learning about the derivative of a complex-valued function is to highlight these three geometric aspects of the derivative, and to offer students opportunities to reason about this geometry in embodied ways listed above.

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# CHAPTER I

# INTRODUCTION

Imagine you could only think in terms of mathematical formulae to solve a mathematics problem. You could not think about shapes or color. You would have no access to such familiar concepts as "up" or "down." Geometry would no longer be visual; it would be no less abstract than any other form of mathematics. In short, you could only calculate. Nonetheless, you could still perceive certain visual properties, even if you could not visualize them. You could calculate the volume of a sphere, because you know that a sphere is a *thing* with the volume formula $\frac{4}{3}\pi r^3$. You could derive the slope of a line knowing that slope is $\frac{\Delta y}{\Delta x}$. You could even describe a tangent line to a function $f$ at $x_0$ by calculating its slope via the derivative value $f'(x_0)$. However, you would have no idea what a sphere or a tangent line looked like, or how a steeper slope compares graphically to a more gradual one. You would not be able to "see" the mathematics; it might even appear just to be a collection of arbitrary rules.

While undergraduate students might be capable of reasoning geometrically, they seem to use it disconnected from the algebraic formulae that describes the geometry (Danenhower, 2006; Panaoura, Elias, Gagatsis, & Giatilis, 2006; Presmeg, 2006; Sfard, 1992). Without this connection, students may lack the mathematical guidance in approaching mathematical concepts that geometric reasoning seems to provide (Katz &

Barton, 2007; Núñez, 2004, Sfard, 1993). In contrast with students, experts regularly appear to rely upon geometric reasoning to guide their intuition, while they utilize algebra to establish precision (Kuo, Hull, Gupta, & Elby, 2013; Núñez, 2004; Sierpinska, 2000; Sfard, 1993; Szydlik, 2000). As such, the National Council of Teachers of Mathematics (NCTM, 2009) standards list the abilities to reason both algebraically and geometrically and to integrate different mathematical representations as essential facets of mathematical reasoning.

Prior research indicates that certain forms of technology known as dynamic geometric environments (DGEs) might help students develop these forms of reasoning and connections between them by grounding students' reasoning in the physical environment (Hollebrands, 2007; Olive, 2000). They may help students "see" the mathematics more clearly by emphasizing visual representations (Arcavi & Hadas, 2003; Barrera-Mora & Reyes-Rodríguez, 2013), or by making ideas expressed by abstract algebraic formulae more concrete (Hollebrands, 2007; Olive, 2000). Thus, such technology may help students refine their mathematical intuition and in justifying logical mathematical arguments, such as that required for developing or critiquing formal proofs (Arcavi & Hadas, 2000; Battista, 2007; Laborde, 1998; Mariotti, 2001, 2002; Marrades & Gutiérrez, 2000).

Technology is not the only method past researchers have found to help students connect algebraic and geometric reasoning. The usage of inscriptions also appears to have aided students in this respect (Châtelet, 2000; de Freitas & Sinclair, 2012; Gibson, 1998; Samkoff, Lai, & Weber, 2012; Zazkis, Dubinsky, & Dautermann, 1996). Dörfler (2001) describes three stages of diagrammatic reasoning: construction, experimentation, and observation. Construction refers to the actual creation of an inscription. Experimentation involves manipulating pieces of the inscription, perhaps to explore intuitions or test conjectures. This manipulation can occur either literally within the context of the inscription or through the usage of other tools such as motion-based gestures or DGEs. Observation involves reflecting on properties of the inscription itself or on past experimentation. Movement through these stages may encourage a unification not only of algebraic and geometric reasoning, but also of inscriptions, gestures, and speech as they support algebraic and geometric reasoning through diagrammatic reasoning (Châtelet, 2000; Chen & Herbst, 2013; Roth & McGinn, 1998; Soto-Johnson & Troup, 2014; Zazkis et al., 1996).

The nature of many of the connections between these aspects suggests that people learn in a way congruent with their experience with the physical environment. For example, Châtelet (2000) notes that diagrams (defined later in this chapter) can inform gestures and that gestures inform diagrams. Through real-world motion via gesture, new possibilities unfold for usages of the diagram. Through these discoveries regarding the nature of the diagram, perception of the diagram itself may change, thereby encouraging new gestures. He furthermore suggests that novel usages of a diagram to investigate an existing mathematical concept can only arise via gesture. This statement that learning is necessarily grounded in the experience of the learner within the physical environment is part of a learning theory known as embodied cognition.

A learner might be encouraged to connect abstract mathematical concepts back to his more concrete real-world experiences through reasoning tools other than diagrams. Through a teaching experiment involving a physical representation of the complex plane that used floor tiles as the coordinate system, dot stickers to represent points, and string to represent vectors, Nemirovsky, Rasmussen, Sweeney, and Wawro (2012) observed in-service teachers learn that multiplying a point by a complex number rotates that point 90°. Furthermore, this particular representation seemed to encourage their participants to notice discrepancies between their algebraic and geometric approaches, which past research suggests is non-trivial (Danenhower, 2006; Panaoura et al., 2006; Sfard, 1992; Tall & Vinner, 1981).

Finally, while research on topics involving the teaching and learning of complex numbers also suggests that students tend to compartmentalize algebraic and geometric approaches (Danenhower, 2006; Panaoura et al., 2006), this research is sparse. Danenhower (2006), Panaoura et al. (2006), and Soto-Johnson and Troup (2014) administered clinical interviews to investigate their participants' reasoning regarding complex numbers, while Nemirovsky et al. (2012) and Harel (2013) conducted teaching experiments to actively encourage their participants' reasoning about complex numbers. Given that research is limited in the field related to the teaching and learning of complex numbers and variables, I wanted to investigate students' algebraic and geometric reasoning about the derivative of a complex-valued function.

Furthermore, I wanted to investigate the possibility that mathematics students' usage of inscriptions created by *GSP* might help them develop geometric reasoning about the derivative of a complex-valued function. I designed my study to investigate the potential relationships between participants' usage of this technology, their algebraic and geometric reasoning, and their usage of gesture. In particular, this dissertation study built on my proposal study to answer the following research questions:

Q1 What is the nature of students' reasoning about the derivative of complex-valued functions?

Q2 What is the nature of the development of students' reasoning about the derivative of complex-valued functions while utilizing *Geometer's Sketchpad* (*GSP*)?

To answer these questions, I developed a sequence of *GSP*-based tasks involving the derivative of a complex-valued function. I had four participants, which I placed into two groups of two. Using these tasks, I conducted a four-day interview sequence consisting of two-hour-long interviews each day for each of these two groups of participants. I video-recorded these interviews, then transcribed all gesture, speech, and technological data. Finally, I qualitatively analyzed these data to produce my findings.

Through *GSP*, I hoped to help my students connect the algebraic definition of the derivative that they already knew to more geometric reasoning about the derivative. This connection could have helped students to "see" the mathematics involved in the derivative of a complex-valued function. This research was intended to discover new ways in which students could come to view the derivative of a complex-valued function as more than just an algebraically calculated limit. In this way, students were encouraged to see this particular mathematical concept as an actual meaningful entity, rather than just an arbitrary "rule."

Findings from this study indicated that my participants reasoned in three distinct embodied ways and required three critical advancements in geometric reasoning to reason about the derivative of a complex-valued function as an amplitwist. That is, they grounded their reasoning in gesture and speech, they integrated their reasoning via algebraic and geometric inscriptions, and they embodied their reasoning in both the virtual and the physical environments. In the process, they reasoned that they needed to characterize the geometry of a line in $\mathbb{C}$, they needed to realize that a linear complex-valued function $f(z)$ rotates and dilates every circle by $Arg\left(f'(z)\right)$ and $|f'(z)|$, respectively, and they needed to reason that this reasoning about rotation and dilation only applies to small circles for a general complex-valued function.

To aid the reader in following chapters, I now discuss terms and definitions relevant to this research study. Furthermore, I provide a brief overview of mathematical concepts related to the derivative of a complex-valued function.

## Definitions

This study aimed to capture participants' development of a geometric reasoning of the derivative of a complex-valued function. "Reasoning" is the word I use to describe a participant's particular way of explaining a concept. Merriam-Webster defines the verb "reason" as follows:

1. To think in a logical way
2. To form (a conclusion or judgment) by thinking logically.

As I could not directly observe the nature of my participants' thoughts, instead of utilizing the word "reasoning" in a literal way in accordance with the definitions presented above, I use it to mean the chain of arguments my participants used to convince themselves or others of truth. This usage is in accordance with *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009), which defined "reasoning" as "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4). For my research, I was particularly interested in my participants' algebraic and geometric reasoning. I use algebraic reasoning to mean a "process involved in solving problems that mathematicians can easily express using algebraic notation" (Carraher &

Schliemann, 2007, p. 670), and geometric reasoning to entail "the invention and use of formal *conceptual systems* to investigate shape and space" (Battista, 2007, p. 843). Note that these definitions of algebraic and geometric reasoning complement NCTM's discussion of these terms, wherein algebraic reasoning involves algebraic manipulating equations, reasoning purposefully with formulae to solve problems, using symbols in a meaningful way, and integrating algebra with geometry. The National Council of Teachers of Mathematics (NCTM) states geometric reasoning is comprised of spatial elements of reasoning such as constructing geometric objects, modeling geometry with algebra, and conjecturing and critiquing geometric arguments.

While I focused on students' "reasoning" methods or approaches, other authors use different words since they are discussing slightly different—though related—concepts. I use the authors' wordings in the literature review to remain true to their original intentions. I use the word "representation" when discussing Panoura et al. (2006), and the word "form" while reviewing Danenhower (2006). Thus, "form" refers to the particular symbolic nature a written mathematical number takes, and "representations" are the means by which participants convey mathematical ideas. For example, Danenhower describes the Cartesian form $a + bi$ or the polar form $re^{i\theta}$ of a complex number, and Panaoura et al. suggest that geometric diagrams or algebraic equations could function as mathematical representations.

A diagram, as defined by Dörfler (2001), is not necessarily geometric in nature. Rather, a diagram is composed of elements spatially arranged in a specific, structural way in some physical medium. As such, learners can utilize a diagram to discover spatial relationships between these elements based on the diagram's structure. Dörfler clarifies,

> What is important is the spatial structure of a diagram, the spatial relationship of its parts to one another and the operations and transformations of, and with, diagrams. The constituent parts of a diagram can be any kind of inscription like letters, numerals, special signs, or geometric figures (Dörfler, 2001, p. 39).

While Dörfler further notes, "diagrams are of such a wide variety that a generic definition appears both impossible and impractical" (p. 41), he does list several examples of diagrams. These include Cartesian graphs, geometric figures, arrows, fractions, matrices, systems of linear equations, arithmetic terms, algebraic terms, and function terms. Thus, it appears that a diagram is an inscription where the arrangement of its elements carries some meaning.

This sort of arrangement can be seen in a fraction where the uppermost position signifies the numerator, and the lower position signifies the denominator, or in a circle, which is a very specific arrangement of points. Without this separation of numerator and denominator, there is no fraction; if I take a set of points which are not equidistant from the center, it is not a circle. Furthermore, a diagram can be either algebraic or geometric in nature, as evidenced by the previous list of examples. Students may develop connections between algebraic and geometric inscriptions due to attendance to their respective spatial structures, the meanings inherently contained in their arrangement, and correspondences in meaning between the inscriptions. For example, a student might find new meaning in an algebraic formula by constructing, observing, and manipulating vectors within a graph that signify various components of the given formula (Soto-Johnson & Troup, 2014). Students' usage of diagrams or inscriptions to discover or reason about mathematical concepts through construction, observation, and experimentation of the diagrams or inscriptions is a process known as *diagrammatic reasoning* (Dörfler, 2001).

Given these definitions, forms, representations, diagrams, or methods of reasoning can reasonably be algebraic or geometric nature. In my paper, I often refer to this nature as primarily algebraic, primarily geometric, both algebraic and geometric, or neither algebraic nor geometric. Algebraic forms, representations, and reasoning methods often correspond to the usage of symbolic mathematical equations, while geometric forms, representations, and reasoning frequently utilize more geometric diagrams. Forms and representations are encompassed by the idea of an "inscription," which Roth and McGinn (1998) define as "signs that are materially embodied in some medium … and because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects" (p. 37). Thus, inscriptions are things which are externally accessible, whether through some written representation or through a virtual environment such as a dynamic geometric environment (DGE).

According to this definition, algebraic equations and diagrams are both examples of an inscription, regardless of whether they were handwritten or computer-generated. Thus, in the context of my research, I consider a "form" in Danenhower's (2006) sense as a type of algebraic inscription, and that an inscription in Panaoura et al.'s (2006) sense could function as a "representation." Their previously referenced examples of "representations"—namely geometric diagrams and algebraic equations—are both inscriptions; such a diagram is a geometric inscription and a written equation is an algebraic inscription. My study aimed to investigate how the usage of inscriptions produced with *GSP* influences my participants' geometric reasoning about the derivative of a complex-valued function. Thus, the mathematical concept of the derivative of a complex-valued function merits a more in-depth discussion.

### Derivative of a Complex-Valued Function as a Local Linearization

Within the context of real-valued functions, the derivative function has a well-recognized geometric interpretation. Namely, at a point $x$, $f'(x)$ is the slope of the line tangent to the graph of the function at the point $(x, f(x))$. However, as the graph of a complex-valued function is four-dimensional in nature, the generalization of this concept is not straightforward. To begin overcoming this problem, one can represent the graph of a complex-valued function with two separate planes. On one plane, we can plot a point $z$ or a set of points. I refer to this plane as the input plane. The points we plot on the other plane—the output plane—is controlled by the points we chose to plot on the input plane: if we plotted a point $z$ in the input plane, then we plot the corresponding point $f(z)$ in the output plane. Note that this scheme is a generalization of the typical representation of a two-dimensional real-valued transformation $g: \mathbb{R} \to \mathbb{R}$ (an input line and an output line), rather than the more common representation of the graph of a function $g(x)$ on a single real plane (the plotted points $(x, g(x))$).

Now, as Needham (1997) points out in his book *Visual Complex Analysis*, we may consider an infinitesimal complex number emanating from a complex point $z$, and the effect of a complex-valued function on this infinitesimal. Informally, consider an extremely small circle centered on $z$ in the input plane. More formally, we may consider the effect of a complex-valued function $f$ on a circle of radius $\epsilon$ centered around the point $z$ in the input plane, where $\epsilon > 0$ but is also arbitrarily small (i.e., smaller than every positive real number, so $\epsilon$, being an infinitesimal, is not a real number). The derivative evaluated at $z$ describes the image of this circle of infinitesimal radius. In particular, "The length of $f'(z)$ must be the magnification factor, and the argument of $f'(z)$ must be the angle of rotation," (p.197) a concept that Needham refers to as an "amplitwist." That is, if we transform the infinitesimal circle around $z$ by dilating it by a factor of $|f'(z)|$ and rotating it by the argument of $f'(z)$, we should obtain the proper shape of the image of this circle under the function $f(z)$. Thus, the value of the derivative of a complex-valued function at a given point geometrically describes how a small circle around a point will be expanded and rotated. Therefore, the derivative of a complex-valued function provides a linearization that locally approximates the function, as "'expand and rotate' is precisely what multiplication by a complex number means" (p.196).

With this understanding, I now briefly discuss the mathematical meanings of each item in a list (included in Chapter III) of relevant concepts related to the derivative that were captured in the interview tasks. This list is as follows:

1. The behavior of a given function (e.g., how points, lines, or circles are transformed)
2. $\epsilon-$neighborhoods around a given point
3. Local versus global properties
4. The relationship between magnitude and dilation
5. The relationship between argument and rotation
6. The meaning of "linearization" or "linear" in the complex plane
7. Conformality (circles are mapped to circles)
8. Approximate conformality

The first item in this list largely refers to the geometric objects I asked my participants to construct (described in Chapter III), though it also encompasses conjectures my participants made about how these objects were transformed under the functions I asked them to investigate. This may include investigations such as whether lines and circles are preserved, what sort of shapes lines and circles do map to, how the function transforms a single point, or how the function transforms the entire plane. By $\epsilon-$neighborhoods around a point $z$ I mean any reference my participants made to a small circle of radius $\epsilon$, which surrounds the point $z$, though formally an $\epsilon-$neighborhood of $z$ should be centered at $z$. Item 3 notes that I paid attention to my participants' application of their reasoning about the derivative of a complex-valued function.

Item 3 was particularly relevant in my participants' attempts to generalize their reasoning about the derivative of a linear complex-valued function to the derivative of a non-linear complex-valued function. In the case of the linear function, it is possible to reason about the derivative in a global way. Namely, since the derivative is constant, a linear transformation will rotate and dilate every circle in exactly the same way regardless of its location or size. In particular, if the derivative is $a+bi$, then every circle will map to an image with a radius that is a factor of $|a+bi|$ larger, and rotated $Arg(a+bi)$ counterclockwise with respect to the pre-image. Since the derivative is variable in a non-linear function, this global reasoning strategy no longer applies. For a non-linear function, proper reasoning about the derivative is necessarily local. In particular, the derivative evaluated at a point $z$ now describes only how a sufficiently small circle around $z$ rotates and dilates under the transformation; it no longer describes how *every* circle around $z$ is rotated and dilated. This distinction is encapsulated in Needham's (1997) description of the derivative as a *local* linear approximation of the function.

Items 4, 5, and 6 in the aforementioned list are also accounted for by Needham's (1997) explanation of the derivative. The magnitude of the derivative evaluated at a point describes the dilation of a sufficiently small circle around that point, while the argument of the derivative describes its rotation, as previously discussed. A linearization, or linear approximation, of a complex-valued function $f(z)$ at a point $z_0$ is a linear function $g(z) = mz + b$ which behaves similarly (though not identically; $g$ only *approximates* $f$) to $f(z)$ near the point $z_0$. In the context of this project, the relevant point was that $f'(z_0) = g'(z_0) = m$, which means that $f(z)$ transforms a sufficiently small circle around $z_0$ in nearly the same way that $g(z)$ would transform this same circle. Since $g$ is only an approximation of $f$, this circle may transform to a slightly distorted circle under $f$ instead of another perfect circle, much as a tangent line at $x_0$ does not perfectly describe the nature of the curve it approximates near the point $x_0$.

In this light, items 7 and 8 are also relevant points; in linear functions my participants observe that circles always map to circles, and in $f(z) = \frac{1}{z}$ that circles map to circles unless they intersect the origin, in which case the circle "breaks." This seems to be a relevant observation in reasoning about the derivative of a linear complex-valued function. Item 8, approximate conformality, is my summary of my participants' attempts to generalize to the derivative of a non-linear complex-valued functions, where if a small enough circle is considered, it maps to another shape which resembles a circle, but may be slightly distorted as previously discussed. Thus, the ideas of conformality and approximate conformality are strongly related to the meaning of a “linear approximation” to a complex-valued function.

## Organization of the Dissertation

Chapter II further explores concepts related to the derivative of a complex-valued function by reviewing literature related to forms of reasoning, the teaching and learning of complex numbers, dynamic geometric environments, and gesture. Chapter II concludes with a discussion of a theoretical framework motivated by the literature. This theoretical framework guided this study. Chapter III presents data collection and analysis methods, and findings from my pilot study related to my participants’ patterns of development of reasoning about the derivative of a complex-valued function. Chapter IV provides results from my study. Particularly, it summarizes the ways in which students developed reasoning about the derivative of a complex-valued function as an amplitwist, as well as ways in which they developed reasoning about requisite mathematical concepts. Finally, Chapter V provides a discussion of my results, implications for research and teaching, limitations of my dissertation study, and directions for future research.

CHAPTER II

# LITERATURE REVIEW

The purpose of my research was to extend the literature on students' reasoning with inscriptions, specifically those inscriptions created with a dynamic geometric environment (DGE) in the field of complex numbers. In particular, I was interested in the nature of students' reasoning as they explored the derivative of various complex-valued functions via a sequence of *Geometer's Sketchpad (GSP)* labs designed to emphasize dynamic properties of the derivative. I was primarily interested in how undergraduate students' reasoning about the derivative of a complex-valued function progresses throughout the interview, and in particular how the geometric and algebraic reasoning of the participants develop. I was additionally interested in tracking which aspects of the tasks and interview process potentially contribute to the development of this reasoning. In particular, the research questions that I addressed in this study were:

Q1 What is the nature of students' reasoning about the derivative of complex-valued functions?

Q2 What is the nature of the development of students' reasoning about the derivative of complex-valued functions while utilizing *Geometer's Sketchpad* (GSP)?

Since my participants utilized *GSP* to help in developing a geometric reasoning about the derivative of a complex-valued function, this chapter centers on literature related to previous publications on the nature of algebraic, geometric, and diagrammatic reasoning, the derivative of complex-valued functions, and finally the usage of DGEs to provide inscriptions which support reasoning. Comparatively little research has been conducted regarding complex-valued functions and pedagogy within the field of complex numbers in general, so it was natural to attempt to extend the extensive findings about reasoning and DGE usage into this domain.

In this chapter, I first discuss research on algebraic, geometric, and diagrammatic reasoning, as well as connections between these approaches. I additionally outline the nature of connections between inscriptions, speech, and gesture in the context of diagrammatic reasoning. I continue by summarizing the mathematics education literature on complex numbers, which currently appears sparse. After summarizing the literature on complex numbers, I transition to an overview of the usage of dynamic geometric environments (DGEs) and related computer software to create inscriptions and help students develop their reasoning methods. I additionally discuss gesture as a means of promoting learning, and as part of an integrated system together with speech. Finally, I overview various interpretations of the framework of embodied cognition, and outline my own view.

## Forms of Reasoning

In this section, I focus on algebraic, geometric, and diagrammatic reasoning, as well as connections between each type of reasoning. I discuss research on students' experiences with each form of reasoning, as well as their associated potential benefits and pitfalls. The National Council of Teachers of Mathematics (NCTM, 2009), defined reasoning and sense making together as "accurately carry[ing] out mathematical procedures, understand[ing] why those procedures work, and know[ing] how they might be used and their results interpreted" (p. 3). They clarified that reasoning is "the process of drawing conclusions on the basis of evidence or stated assumptions" (p. 4) and that sense making is "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (p. 4). This includes the ability to reason algebraically, the ability to reason geometrically, and the ability to integrate different representations[1]. If students can connect representations to each other and to their own existing knowledge, they may be more able to remember what they have learned (Hiebert, 2003, as cited by NCTM, 2009). This integration is important in that algebraic representations can inform geometric reasoning, and geometric representations can inform algebraic reasoning (Katz & Barton, 2007, as cited by NCTM, 2009).

NCTM (2009) considered algebraic reasoning to involve mindful manipulation, reasoned solving, and connecting algebra with geometry. Mindful manipulation involves mentally visualizing calculations, understanding the arithmetic properties underlying algebraic manipulations, and purposefully selecting algorithms based on context. Reasoned solving consists of seeing solutions both in context and as a sequence of formal logical deductions. Finally, connecting algebra with geometry involves both the ability to translate algebraic and geometric inscriptions from one form to the other and the ability to integrate algebraic and geometric reasoning. Geometric reasoning includes forming conjectures about properties of geometric objects, construction and evaluation of these conjectures and associated arguments, and the ability to utilize multiple geometric approaches. Geometric reasoning was defined as "the invention and use of formal *conceptual systems* to investigate shape and space" (p. 843, emphasis in original). Spatial

[1] Even though my interest was in reasoning, while discussing related literature I use the authors' terminology to remain true to the intention of their original work. For the purposes of this discussing this study, I consider an inscription to be a form of representation.

reasoning, a component of geometric reasoning, was further described as “the ability to observe geometric objects or processes, form conjectures about them, and use them to perform other operations” (p. 843). Goldin and Kaput (1996) defined an object as a thing on which a learner operates while reasoning, and a representation as something that stands in for some concept, object, or process. Thus in geometric reasoning “one reasons *about* objects; one reasons *with* representations” (NCTM, 2009, p. 844).

Both algebraic and geometric reasoning can involve identifying representations, attending to structure and patterns, utilizing previously learned algorithms, forming conjectures, or reflection on a previously developed solution. NCTM (2009) claimed that reasoning progresses through three stages: empirical, preformal, and formal. In the empirical stage, claims are supported by cases; in the preformal stage, claims are supported by intuition and insight; in the final formal stage, claims are rigorously justified through logical proof. To stimulate this transition, NCTM suggested implementing tasks that “require students to figure things out for themselves,” “ask[ing] students questions that will prompt their thinking—for example, ‘Why does this work?’ or ‘How do you know?’” (p. 11), and “encouraging students to ask probing questions of themselves and one another” (p. 11).

Carraher and Schliemann (2007) identified algebraic reasoning as “processes involved in solving problems that mathematicians can easily express using algebraic notation” (p. 670). This definition of algebraic reasoning complements NCTM’s aforementioned description. This particular system of reasoning may help students at all levels to make algebraic generalizations (Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2000, 2005; Lee, 1996; Mason, 1996; Radford, 1996a/1996b;

Schliemann, Carraher, & Brizuela, 2006) and visualize novel structures of spatial objects (Boester & Lehrer, 2007, as cited by NCTM). That is, algebraic reasoning may inform geometric reasoning, and thus one can view an algebraic concept as both a process and as an object.

However, many research studies (Danenhower, 2006; Dubinsky & Harel, 1992; Gray & Tall, 1994; Otte, 1993; Panaoura, Elias, Gagatsis, & Giatilis, 2006; Sfard, 1991/1992/1995; Sfard & Linchevsky, 1994) suggested that students both compartmentalize algebraic and geometric approaches and tend to view mathematical concepts as either processes or objects, but not both. That is, students may believe that algebraic expressions are necessarily processes and thus not objects (David, Tomaz, & Ferreira, 2014). Carraher and Schliemann (2007) claimed, “some authors treat procedural approaches as inherently more primitive and thus in need of replacement by an object orientation; others treat procedural interpretations as different, yet nonetheless desirable even in advanced mathematical thinking” (p. 672). Sfard (1991, 1992) suggested that a learner must develop a procedural view before they can obtain an object view and that connecting these two views of a mathematical concept is inherently difficult. This difficulty may help explain why students sometimes do not recognize logical contradictions between different representations of the same mathematical concept (David et al., 2014; Lee, 1996; Harel & Sowder, 2005; Tall & Vinner, 1981).

Tall and Vinner (1981) further reported that recognizing a contradiction between two distinct beliefs is inherently difficult. For example, while investigating how students reasoned about limits, they found that students tended to dismiss counterexamples as exceptions to a given statement rather than a disproof of it, a finding later corroborated by

Harel and Sowder (2005). Tall and Vinner theorized that a student would likely not even recognize the contradiction unless both contradictory notions are evoked into the student's conscious mind simultaneously, causing a cognitive conflict. A similar phenomenon seems to have occurred in Danenhower's (2006) and Panaoura et al.'s (2006) research (discussed later in this chapter), where students failed to notice a contradiction between disparate results simply because the results were expressed in different representations, or because one solution was obtained via geometric reasoning while the other was algebraically calculated. Furthermore, Tall and Vinner reported that their participants might have formed incorrect beliefs about the limit concept because they seemed to favor a dynamic intuitive notion of limit over the formal algebraic definition. For instance, Tall and Vinner found that some of their students felt strongly that a limit can never be attained by its function and that a limit bounds the function. Those participants that did appeal to the formal definition were more often able to answer questions correctly.

Experts also appeared to rely on dynamic or geometric ideas to guide their reasoning, but were able to shift between types of reasoning more appropriately (Arcavi, 1994; Kuo, Hull, Gupta, & Elby, 2013; Lithner, 2008; Núñez, 2004; Redish & Smith, 2008; Sfard, 1993; Szydlik, 2000; Wertheimer, 1959). However, unlike Tall and Vinner's (1981) participants, experts did not utilize these ideas exclusively. Instead, they used algebraic reasoning to reason about atypical cases and to keep their visualizations or geometric reasoning accurate with respect to formal mathematical definitions. In this way, algebraic reasoning appears to lend precision to geometric reasoning (Sierpinska, 2000; Tall & Vinner, 1981; Williams, 1991) and thus helps prevent unwarranted assumptions that may appear intuitive given a certain visualization or geometric model.

Despite these potential problems, dynamic or geometric reasoning can still be helpful to help students reason strategically. Tall and Vinner (1981) found that even in the absence of a well understood formal definition of limit, students' dynamic ideas still helped them guide their thoughts in beneficial ways, though not always entirely correct. Therefore, Tall and Vinner's study provided some additional motivation for helping students connect formal definitions to their pre-existing intuitive ideas, often dynamic or geometric in nature. This finding is in keeping with other researchers' (Danenhower, 2006; Hiebert, 2003; Katz & Barton, 2007; Kuo et al., 2013; Panaoura et al., 2006; Sfard, 1992; Sherin, 2001) claims that the ability to both translate between representations and meaningfully connect them is essential. Without this integration, a student may have relatively little to guide their mathematical reasoning. As Sierpinska (2000) notes, "It is not enough to just make the structural content more concrete through working in low dimensions and using visualization. In fact, visualizations themselves are problematic in that they may lead to irrelevant interpretations which make the understanding more, not less difficult" (p. 244). While Sierpinska's observations applied specifically to linear algebra content, it is possible that similar difficulties with visualization may arise in other contexts. In particular, because complex numbers are often visualized as vectors, her results on vectors in the context of linear algebra seem applicable to complex analysis.

While Tall and Vinner (1981) found that most of their participants did not appeal to the formal algebraic definition of limit to guide their reasoning, Williams (1991) reported that 10 students he interviewed professed a surprising amount of faith in graphs and function formulas. He commented that “It is as though problems of continuity, topological properties of the real line, and a myriad of other difficulties that they realize might arise in taking limits are magically taken care of for students in the process of drawing a graph” (p. 234). So, it would seem that Lagrange’s (n.d.) concern that students would be deprived of mathematical learning opportunities because of dynamic tools was indeed valid. However, Lagrange thought in particular that computer programs would deprive students of the opportunity to interpret symbolic forms (Sherin, 2001), while in William’s study, it appeared that the students’ reliance on graphical tools and those same symbolic forms was depriving them of an opportunity to develop their algebraic reasoning about limits. The effects of computer programs on learning are discussed later in this chapter.

Rather than emphasize any one form of representation, physics education researchers Kuo, Hall, Gupta, and Elby (2013) argued that the ability to blend formal and conceptual mathematical reasoning is essential in problem-solving, that this blending can be described via symbolic forms, and that teaching students how to integrate these styles of reasoning is a feasible teaching goal. According to Kuo et al., conceptual analysis is the three-step process of qualitatively analyzing the context of the problem, selecting an appropriate solution strategy, and interpreting the answer obtained in context to check for validity. By formal mathematical reasoning they meant the manipulation of algebraic expressions. They noted that experts began with conceptual reasoning, while novices skipped this first step and immediately began manipulating equations without much thought for context (Larkin, McDermott, Simon, & Simon, 1980; Simon & Simon, 1978), much as David et al.’s (2014) participants applied the distributive property without much regard for the actual structure of the presented algebraic statement. Two of their participants reflected this distinction; one attacked the problem directly with equations and algebra, while the other reasoned through the problem in context first. In this respect, the former participant's problem-solving behavior more closely aligned with "standard problem-solving procedures advocated by researchers and taught to students" (p. 55), and the latter's demonstrated a more complete grasp on the material.

Investigating students' reasoning methods in solving a physics problem involving velocity and acceleration, Kuo et. al (2013) found that out of their 13 students, six students utilized either symbolic forms-based reasoning or blended processing. Sherin (2001) had previously reported that symbolic forms help students integrate algebraic equations with some physical situation. Kuo et al. reaffirmed Sherin's results by reporting that symbolic forms and blended processing led to solutions which could be generalized more easily, and thus appeared to make their students' reasoning more suited for a greater variety of tasks. However, they add a note of caution, stating,"constraining students to expert behavior may not be the road to expertise" (p. 53). In particular, they worry that conceptual blending or usage of symbolic forms may simply become another step in our already overly procedural solution strategies.

Kuo et al.'s (2013) novice participants' tendency to ignore context and skip straight to solving problems algebraically may have been due to the way they thought about and approach mathematical representations. In particular, Sierpinska (2000) suggested that students tend to think in practical rather than theoretical ways. She drew a parallel in this classification of practical and theoretical ways of thinking to Vygotsky's (1962) distinction between spontaneous and scientific concepts, respectively.

Furthermore, she noted that scientific concepts, or theoretical thinking, is integrated into systems, and may thus make contradictions more apparent between distinct representations. Again, Sierpinska's results concerned vectors in linear algebra, but as complex numbers are often positioned as vectors, these results seemed relevant for this study. For experts, this distinction is less clear-cut, since scientific reasoning is more familiar to them than to a novice, and is thus not entirely separate from their spontaneous reasoning[2]. Just as seen in previous research on algebraic reasoning (Tall & Vinner, 1981; Williams, 1991), this theoretical or scientific reasoning is utilized by experts when they are confronted with an unfamiliar case or contradiction, or when they need to justify their chosen strategy in approaching a problem.

Sierpinska (2000) additionally observed the phenomenon of compartmentalization between types of reasoning, dubbing this issue "the obstacle of formalism" (p. 210). That is, her students seemed to view formal representations of a mathematical object as though the representation *was* the object it was representing. As such, they could not perceive the structure of the linear transformations they investigated with *Cabri* or integrate distinct representations. Some students even described what they saw with *Cabri* only in terms of the computer environment, and did not connect these rather literal observations back to any mathematical principles.

Despite these issues, Sierpinska (2000) suggested that inscriptions were able to help students advance their reasoning. She stated that "semiotic representation systems become themselves an object of reflection and analysis in theoretical thinking because they constitute the only medium through which theoretical thinking may prove its

[2] Compare to Zazkis, Dubinsky, and Dautermann (1996), who report that visualization and analysis become progressively more integrated as expertise is gained. This research is discussed more later in this chapter.

existence and convey its meanings" (p. 212). Furthermore, she found that students were able to overcome the obstacle of formalism when they were placed in pairs and assigned their own individual tutor. While a tutor may have helped students overcome this difficulty on a small scale, it remained an issue in the context of a classroom setting with a single instructor responsible for all students. The course content was specifically designed with an awareness of this problem in mind, and the course itself took place in a computer lab. However, even with these special arrangements, the obstacle of formalism remained.

David et. al (2014) observed a similar problem in their students' treatment of the distributive property. In particular, they created a "shower" visualization to help them remember how to multiply through parentheses when distributing. However, this visualization seemed to encourage students to associate *all* parentheses with the distributive property, causing them to distribute at inappropriate times during an algebraic procedure, or in contexts where the distributive property does not apply. Just as Sierpinska's (2000) students failed to grasp the structure of linear transformations, so David et al.'s students could not understand the structure of the distributive property. Instead, they seemed to view a parenthesis as a command to multiply, even though David et al. reported that the students' teacher had presented the distributive property in a structural way. Thus, visualizations such as the shower inscription can cause students to overgeneralize mathematical properties. Even so, David et al. suggested that these same visualizations could still help students recognize contradictions between the representations of a mathematical concept and the concept itself. In other words, visualizations or geometric inscriptions might help students overcome the obstacle of formalism discussed by Sierpinska (2000).

Lee (1996) offered another example of overgeneralization in algebraic reasoning. When asked to produce an algebraic proof that two consecutive numbers always sum to an odd number, 49 of 113 high school students represented the sum algebraically either as $x + (x + 1)$ or $2x + 1$. This first inscription led students to overgeneralize by assuming that $x$ represented an even number and that $x + 1$ represented an odd number, perhaps due to the "$+ 1$" contained within this inscription. While 49 students may have produced this inscription, only 8 students in the entire sample created solutions considered correct. Other difficulties the students experienced involved giving distinct names to the two consecutive numbers (i.e., $x$ and $y$), forming an incorrect algebraic inscription for two consecutive numbers (e.g. $1x$ and $2x$), or conflating even and odd with negative and positive.

This behavior reflects other researcher's (David et al., 2014; Kuo et al., 2013; Larkin et al., 1980; Simon & Simon, 1978) observation that students may ignore the structure or context of an inscription in favor of beginning algebraic manipulations immediately. It also serves as evidence for students' tendency to over-generalize—they showed $2x + 1$ was odd for a limited number of examples (Harel & Sowder, 2005; Mason, 1996; Radford, 1996a; Radford & Berges, 1988), thereby implying that these few examples substantiate the claim that $2x + 1$ is *always* odd. Lee (1996) further noted that while students may have been able to see patterns, they had difficulty communicating the patterns they did see, utilizing inscriptions to represent these patterns, or identifying a mathematically useful pattern. Students' ability to generalize may change greatly with context. When Lee asked his participants to find the number of dots in a certain figure in a sequence of increasing rectangles, most students were able to perceive a pattern, though perhaps not a mathematically useful pattern. Even so, most participants provided a correct answer. Lee concludes that algebraic inscriptions appear to facilitate the ability to generalize.

Mason (1996) suggested that students are generally not aware of the depth of meaning contained in the question "Does it *always* work?" Furthermore, he implied that this may be because teachers and students both focus on techniques to manipulate equations and symbols rather than ways to generalize mathematical concepts. Finally, he suggested that computer software might help students and teachers alike refocus on to the meaning of and justifications for generalizations, as this software can quickly solve most traditional rote school problems immediately. It might also help students form conjectures, for which Mason outlined three approaches.

First, students can manipulate a representation in such a way as to make the claim apparent from the representation. Given computer software's flexibility in generating representations, it may help the student find a representation that fits the claim. Second, students can find a local rule to build the next term from the previous. Again, the availability of computer software makes testing rules easier, and can accommodate testing rules in larger number. Third, students can find a pattern and extract a formula, mirroring Kuo et al.'s (2013) discussion of physics experts' tendencies to first analyze a situation and then select an appropriate strategy. Computer software provides a greater abundance of examples and has a capacity to make abstract mathematical concepts more concrete, and thus may help students generalize patterns as well. Later in this chapter, such computer software—particularly dynamic geometric environments (DGEs)—are discussed further.

Both Mason (1996) and Lee (1996) implied that generalization lies at the center of algebra based on their historical analyses, but Radford (1996a) contested this point. Rather than agree that the sole point of algebra is generalization, he included problem solving as another purpose that functions in tandem with algebra. To clarify, he stated that problem solving serves “as a primary need for knowledge” (p. 108), while generalization drives this need.  He was also aware of the potential pitfalls on which Mason (1996) and Lee (1996) commented. In particular, he noted that students often attempt to justify a universal claim with a single example, or attempt to establish a functional rule for a sequence by showing that it provides the correct value for a single “special” term. Despite his difference of opinion, he agreed that generalization is context-dependent and that algebraic representations help students generalize their mathematical observations.

Algebraic inscriptions alone may not be able to facilitate students’ attempts to generalize in all contexts, but geometric inscriptions may help in this regard (Battista, 2007). On the other hand, they may actually blockade students’ ability to generalize based on a diagram’s properties (Clements & Battista, 1992; Yerushalmy & Chazan, 1993), since such pictures or diagrams typically only represent one single unique case (Presmeg, 1997). However, a learner may nonetheless appropriately classify them via perceptual abstraction as described by von Glasersfeld (1991, 1995), even without visualization. In perceptual abstraction, some experience is viewed as its own isolated entity. In the initial stage, the experience cannot be visualized. Once visualization becomes possible, the entity is described as having been internalized (Steffe, Cobb, & von Glasersfeld, 1988). At this point, a learner can identify “structure, pattern, and operations from experiential things and activities” (p. 859) by reflecting upon a visualization for the purpose of determining its structure. At this level, the concept is separable from its context. Finally, at the second level of interiorization, “operations can be performed on the material without re-presenting [visualizing] it and symbols, acting as ‘pointers’ to the originally abstracted material, can be used to substitute for it” (p. 860).

Students’ mathematical learning could additionally be positively influenced via DGEs (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale, Swart, & Black, 2014). Mariotti (2001, 2002) claimed DGEs may help students approach geometric exploration and form meaningful conjectures. They may emphasize geometric properties by requiring some explicit specification of their properties to properly construct a figure. Jones (2000) distinguished between drawings and figures constructed in a DGE via a “drag test”. In particular, a figure retains its properties if moved within the DGE, and remains within the same classification. For example, a square would remain a square if dragged because the user constructed a shape utilizing properties of a square. A drawing has no such immutability. A square might be drawn free-hand within the DGE without any reference to properties of a square. Such a construction would likely cease to be a square the moment a single vertex was moved. In addition to emphasizing geometric properties and classifications, they also easily allow for creation of several objects of the same class with a single construction in a continuous and dynamic way. For example, if a circle were to be constructed in *Geometer's Sketchpad* (*GSP*), it could then be moved around the graph to various locations, and its radius could be increased or decreased, thereby representing a large variety of distinct circles.

Marrades and Gutiérrez (2000) claimed that DGEs are primarily advantageous because with a DGE, students can easily construct complex geometric figures, perform a large variety of operations on these figures, and generate a large number of examples of a certain kind of figure. Battista (2007) and Laborde (1998) suggested that draggable figures help with geometric analysis and conceptualization by providing a representation that can be manipulated subject to certain movement constraints. Dörfler (1993) suggested that DGEs do not help students do something faster or better, but are in fact environments that require an entirely new "cognitive view than has been taken in traditional approaches to geometry instruction" (p. 883). Battista further cautioned that DGEs may de-emphasize formal proofs such as those found in high school geometry. The potential effects of DGEs on students' mathematical learning are discussed in more detail later in this chapter.

In addition to DGEs, diagrammatic reasoning as described by Dörfler (2001) may further encourage students to integrate their algebraic and geometric reasoning. According to Dörfler, there are three phases of diagrammatic reasoning: *construction*, *experimentation*, and *observation*. Diagrammatic reasoning can involve either algebraic or geometric inscriptions, as well as inscriptions created within DGEs. Learners can switch between and return to any of these phases; they do not have to occur in strict order.

In the construction phase, the learner creates or adds to an inscription. This could be an algebraic equation, geometric shapes, a graph, or any other written representation of a mathematical concept. In the experimentation phase, the learner manipulates parts of the inscription, potentially to test a conjecture or to prepare for a transition to the observation phase. This could involve, for example, applying algebraic operations to an equation or moving a vertex around within a graph. Finally, the observation phase involves reflecting on the inscription itself or a prior experiment with it for the purpose of discovering mathematical truths inherent in the inscription, or in their manipulation of the inscription.

This manipulation of diagrams may positively influence mathematical reasoning. For example, both Samkoff, Lai, and Weber (2012), and Gibson (1998) found that diagrams helped their participants create proofs to justify mathematical statements. Gibson (1998) investigated students' creation of diagrams to construct real analysis proofs. He reported that his students utilized diagrams for at least four different reasons: to help them comprehend the information available to them, to reason about whether a statement was true or false, to discover new ideas, and to help them communicate or write down their own ideas. Because Gibson was primarily interested in discovering why students might utilize diagrams, he did not explicitly ask them to create one, instead opting to investigate instances when the construction of diagrams might arise naturally in the course of students' thinking about mathematical concepts. Unlike Gibson, I did explicitly ask my students to create several inscriptions with *GSP*.

Samkoff et al. (2012) investigated eight mathematicians' approaches to a proof involving the sine function and their usage of diagrams in order to classify and explore ways in which these diagrams might be beneficial for developing proofs. One of the more common ways in which diagrams were used was to observe and investigate properties that might be true. While Samkoff et al. believed that a complete classification scheme for the ways in which mathematicians interact with diagrams may not exist; they did identify four ways in which diagrams *might* be used. This included: noticing properties and generating conjectures, estimating the truth of an assertion, suggesting a proof approach, and instantiating an idea or assertion.

One mathematician drew more conviction from his diagram than from a more formal argument that he developed. In fact, his diagram convinced him that his formal proof was, in fact, incorrect. Rather than allowing an abstract formal proof to alter his reasoning, this mathematician found himself more convinced by his diagram, and perhaps a bit by prior experience as well. He further convinced himself of the truth of his diagram and the existence of a flaw in his proof by producing specific counterexamples to his proof.  Thus, the mathematician relied more on the reasoning produced via the creation of the diagram and its geometric properties than on any algebraic argument. Diagrammatic reasoning, in this case, took precedence over the formal mathematical argument. Furthermore, as discussed later, Châtelet (2000) argued that diagrams and gestures are inextricably linked, while Goldin-Meadow (2003) noted that gesture and speech are also inseparable. Diagrammatic reasoning and inscriptions may thus serve to unify not only students' algebraic and geometric reasoning, but also their gesture and speech. The usage of gesture as a means of inferring the nature of students' reasoning is discussed further later in this chapter. Here I limit my discussion to relationships between each of these aspects and diagrammatic reasoning or inscriptions.

Inspired by Châtelet (2000), de Freitas and Sinclair (2012) reported that diagrams and gestures are indeed linked. For these researchers as well as for Châtelet, diagrams were thought to “lock” or “capture” gesture:

> A diagram can transfix a gesture, bring it to rest, long before it curls up into a sign, which is why modern geometers and cosmologers like diagrams with their peremptory power of evocation. They capture gestures mid-flight; for those capable of attention, they are the moments where being is glimpsed smiling (Châtelet, 2000, p.10).

Thus, diagrams can act as tools to depict gesture in a deeper way than mere representation. In addition to describing a gesture, a diagram may include both virtual and real aspects, both explicit and implicit facets, and kinematic features, all within the same diagram. Furthermore, de Freitas and Sinclair (2012) stated that simply “by adding a dotted line to the paper, a new dimension can be brought into being; an arrow might forge out new temporal relationships between objects. These excavations enable the virtual and the real to become coupled anew” (p. 138).

According to Châtelet (2000), gestures and diagrams work together to open up new avenues of exploration and to make novel gestures and diagrams accessible. Furthermore, as pointed out by de Freitas and Sinclair (2012), this process never terminates, as gesture is too varied in form to be described by some finite set of algorithmic rules or classification scheme. Therefore, gestures and diagrams should not be considered separately. Châtelet claimed, “extracting one from the other is awkward and possibly misleading” (de Freitas & Sinclair, 2012, p. 137). Furthermore, Châtelet observed that the ability to apply a diagram in a novel way to an existing mathematical concept has led to many historical mathematical breakthroughs.

For example, he described one such diagram utilized by Cauchy as part of his work in developing a method for integrating over singular points. The diagram depicts a line through a singular point that bends into an upper semicircle to circumvent the singular point. Châtelet seemed to view this diagram in a very physical way, even writing "the plane is made flesh, as it were" (p. 34). Rather than view the point as abstract, Châtelet inferred that the diagram brings the point closer to a more real, more concrete existence. In de Freitas and Sinclair's (2012) words, "…the diagram constitutes the point as a material bump on the surface of the page.…Taking a very material, physical point of view, Châtelet reads [the point] as being made flesh by a 'cut out' in the complex plane in which the point is now enveloped" (p. 139).

According to Châtelet, gesture must be somehow utilized, implicitly or explicitly, to create this kind of diagram. Thus, it seems that the creation of a diagram is inextricably linked to an embodied experience: "not a tentative deictic pointing at something on the surface, but an actual physical creasing or cutting out which marks up the surface and conjures its virtual folds" (de Freitas & Sinclair, 2012, p. 139). Other researchers (Chen & Herbst, 2013; Roth & McGinn, 1998) added that speech works together with gesture to give life to a diagram, and that in fact inscriptions provide a link between speech and gesture.

Since algebra appears to be connected to verbal ideas, and geometry to visual ones (Barrera-Mora & Reyes-Rodríguez, 2013; Battista, 2007; Boester & Lehrer, 2007; Carraher & Schliemann, 2007; Cory & Garofolo, 2010; Goldin-Meadow, 2003; Sierpinska, 2000; Sfard, 1992; Zazkis, Dubinsky, & Dautermann, 1996), it seems natural to assume that speech could be predisposed toward algebraic ideas and gesture might be more adept at representing geometric ideas (Goldin-Meadow, 2003). Diagrammatic reasoning may thus also help students connect their algebraic and geometric reasoning.

Similarly, Zazkis et al. (1996) observed that oscillating between visualization and analysis helped their students answer abstract algebra questions about the symmetry group of a square. They reported that most of their participants combined visual and analytic approaches into a single strategy. Based on their observations, they suggested a model for how learners may use these approaches. Initially, an individual may view visualization and analysis as two distinct strategies. They may use both approaches, but treat them as disconnected from one another. They may even be able to transition from one to the other within the same mathematical activity, albeit with immense mental effort. As time goes on and the individual moves more frequently between visualization and analysis, the individual may see these two approaches as progressively more integrated, and movement between strategies becomes more and more trivial. Finally, visual and analytic strategies may become so integrated that neither the individual nor any outside observer can reliably separate the two.

Students' difficulties integrating algebraic and geometric reasoning and generalizing mathematical concepts properly could be addressed via diagrammatic reasoning with algebraic and geometric inscriptions and dynamic geometric environments (DGEs) (Barrera-Mora & Reyes-Rodríguez, 2013; Châtelet, 2000; Chen & Herbst, 2013; Danenhower, 2006; David et al., 2014; de Freitas and Sinclair, 2012; Hollebrands, 2007; Lee, 1996; Olive, 2000; Panaoura et al., 2006; Sfard, 1995; Tabaghi & Sinclair, 2013; Tall, 2003; Tall & Vinner, 1981; Vitale et al., 2014; Zazkis et al., 1996). Since reasoning necessarily involves the integration of algebraic and geometric representations, students should be encouraged to connect these approaches. While students typically experience difficulty in connecting algebraic and geometric reasoning, experts appear to do so regularly to beneficial effect (Arcavi, 1994; Kuo et al., 2013; Lithner, 2008; Núñez, 2004; Redish & Smith, 2008; Sfard, 1993; Sierpinska, 2000; Szydlik, 2000; Wertheimer, 1959; Williams, 1991). In particular, geometric reasoning appears to provide guidance, while algebraic reasoning seems to encourage precision (Sierpinska, 2000; Tall & Vinner, 1981; Williams, 1991). Providing students with opportunities to transition between different forms of reasoning, thereby integrating these two approaches (Zazkis et al., 1996) may thus help them increase in mathematical proficiency and comprehension.

As this section demonstrated, students' difficulties in integrating algebraic and geometric approaches occur across a wide variety of contexts and mathematical domains. Since I researched students' reasoning about the derivative of complex-valued functions, in the next section I review the math education literature on complex numbers.

**The Teaching and Learning of Complex Numbers**

There have not been a large number of educational studies that focus on complex numbers (Danenhower, 2006; Harel, 2013; Nemirovsky et al., 2012, Panaoura et al., 2006; Soto-Johnson, 2014; Soto-Johnson & Troup, 2015). The studies involving clinical interviews appeared to suggest that students favor algebraic representations of complex numbers over geometric representations (Panaoura et al., 2006) and their Cartesian form, and have difficulty utilizing alternate mathematical forms of them efficiently (Danenhower, 2006). Furthermore, though most students do not seem to prefer geometric representations, research involving teaching experiments suggests that such reasoning may be encouraged through the use of a model complex plane (Nemirovsky et al., 2012) or other embodied activities such as the usage of inscriptions and gestures (Soto-Johnson & Troup, 2014).

In a study designed to identify problems that students in an introductory complex analysis course were likely to have, Danenhower (2006) observed that his participants had the ability to shift only between algebraic or Cartesian forms and polar form of complex numbers. A form is a particular way of representing a complex number. For example, the symbolic form uses the letter $z$ to refer to a single complex number, while Cartesian form represents complex numbers as $x + iy$. Other forms include polar form ($re^{i\theta}$) and exponential form ($r \cos\theta + r \sin\theta$). Most students compartmentalized these forms rather than viewing each piece as part of a larger coherent whole.

Danenhower (2006) investigated students' usages of Cartesian, polar, vector, and symbolic forms by asking students to convert various instances of $\frac{a+ib}{c+id}$ into either the form $x + iy$ or the form $re^{i\theta}$, whichever the students preferred. He thought that students might recognize that some forms were more useful for certain operations than others. For instance, he expected that students would convert to polar form to divide two complex numbers, as division tends to be easier with polar form than with Cartesian form. Finally, Danenhower observed that while students were generally able to navigate between forms, they were still more proficient within each form than translating between them. Additionally, "nearly half did not have good judgment about when to shift to another form" (2006, p. 151). That is, contrary to expectations, students were not generally able to choose a form that simplified the problem, such as using Cartesian form for addition or polar form for multiplication. Instead, students seemed to choose an initial form and then switch between forms somewhat arbitrarily.

Danenhower's (2006) claims were substantiated by Panaoura et al. (2006), who conducted a study that explored the ways in which high school students translated algebraic statements to geometric pictures and vice versa. For example, one of the tasks was set up to determine whether these students would recognize the algebraic equation $|z - 1 + i| = \sqrt{2}$ as a semicircle. Another corresponding task asked students to find the algebraic equation that defined a given semicircle. Panaoura et al. reported that "the geometric approach was used more frequently, while the pupils used the algebraic approach more consistently and in a more persistent way" (p. 681). This result suggests that the students were more often correct within their algebraic attempts at these conversion problems, or at least wrong in a more consistent way. Despite their apparent familiarity and comfort with their algebraic representations, they still attempted geometric lines of reasoning more frequently. Considering that students seem to favor algebra (Kuo et al., 2013; Panaoura et al., 2006), it is possible that students consider geometry to be an exploratory method, and thus appear to use geometric representations in exactly this fashion.

This assumption regarding the purpose of geometry may not even be harmful or incorrect. Several researchers (Arcavi, 1994; Kuo et al., 2013; Lithner, 2008; Redish & Smith, 2008; Sfard, 1993; Sierpinska, 2000; Wertheimer, 1959) noted that mathematical experts relied on visual or geometric pictures to drive their intuition and overarching thought processes of a given mathematical topic or concept. It is therefore reasonable to suggest that students utilizing geometric thought processes in a discovery-oriented setting could have a similar effect. Szydlik (2000) also suggested that both experts and students with internal sources of conviction used algebra primarily as clarification for atypical cases or disambiguation of these same cases. That is, geometry or metaphor drove their intuition and initial thought processes, and algebra served to increase precision of thought. Under this frame of reference, it seems entirely natural to begin mathematical investigations with geometric representations and then transition back to algebraic reasoning over time. This benefit cannot be achieved, however, if students remain unable to integrate algebraic and geometric reasoning due to their tendency to compartmentalize these two modes of reasoning as separate from one another.

Panaoura et al. (2006) found that "The phenomenon of compartmentalization indicating a fragmental understanding of complex numbers was revealed among pupils who implemented the geometric approach" (2006, p. 681). That is, their participants tended to treat geometric and algebraic representations as relatively separate systems, though they could occasionally extend a representation to a different form. For example, even after some students converted a geometric representation to an algebraic one, they still could not reverse this process to convert from algebraic back to geometric, even for the same complex number. Furthermore, like Danenhower's (2006) students, Panaoura et al.'s participants did not seem to have a good idea of which representation they should have been working with at any given time, as they did not always utilize geometric reasoning productively.

In light of Danenhower's (2006) and Panaoura et al.'s (2006) work, I was interested in the development of my participants' geometric and algebraic reasoning about the derivative of a complex-valued function. It has been noted, or at least implied, that geometric reasoning is often visual in nature, and that algebraic reasoning is more closely associated with verbal aspects (Barrera-Mora & Reyes-Rodríguez, 2013; Battista, 2007; Boester & Lehrer, 2007; Carraher & Schliemann, 2007; Cory & Garofolo, 2010; Goldin-Meadow, 2003; Sierpinska, 2000; Zazkis et al., 1996). Furthermore, Sfard suggested that thinking about a mathematical concept in an operational context is often verbal in nature, while thinking about this same concept as a structure rather than an operation is often visual. Sfard (1991) additionally stated that once a mathematical object is properly reified in a student's mind, that student should be able to switch adeptly between operational and structural modes of thought, viewing each line of thinking as two sides of the same coin rather than two entirely different, disconnected, isolated ideas.

However, Sfard (1992) pointed out that this reification of operational and structural viewpoints into a single mathematical entity seems to be inherently difficult for students to accomplish. Furthermore, she posited an association between algebraic and operational modes of thought, as well as between geometric and structural lines of reasoning. This conclusion was supported by Danenhower's (2006) finding that students have difficulty translating between algebraic and geometric representations, and Panaoura et al.'s (2006) research suggesting that students compartmentalize geometric and algebraic representations rather than using them together effectively as a single system.

Taking a historical perspective, Harel (2013) demonstrated via a sequence of teaching experiments that in-service teachers and pre-service mathematics education sophomores had difficulty viewing algebraic inscriptions such as $x^3$ as a single number in certain contexts. However, through a sequence of activities based on the DNR (duality, necessity, repeated reasoning) framework and the historical development of complex numbers, his participants were able to learn how to extract a mathematical representation from a given context and to reason about the meaning of these representations. The principle of duality refers to the interplay between the ways students reason mathematically, and the tools they use to understand mathematics such as theorems, conjectures, proofs, definitions, and problem solutions. Necessity refers to the idea that for students to learn a concept, they must have an intellectual need for it, much as Sfard (1992) suggested that for a student to transition from viewing a mathematical concept as a process to viewing it as an object (such as seeing a function as either a process or an object), students must have a reason to view the concept as more than a process. Finally, repeated reasoning is the principle that students must practice reasoning to learn and retain these concepts.

Using these principles, Harel (2013) conducted a teaching experiment that consisted of work in small groups, discussions in which the whole-class took part, and lectures. In solving systems of cubic equations, most students adopted an approach based on trial and error, while some used algebra to reduce the equations to a more familiar quadratic form. However, Harel reported that students who used algebraic approaches abandoned their methods in favor of the trial-and-error approach introduced by other students in their working groups. Furthermore, students could not generalize the quadratic form to higher powers—they could not perceive that $(x^3)^2 + Ax^3 + B$ is quadratic with respect to $x^3$. This suggests that they were unable to view $x^3$ as a single entity, instead considering the representation as descriptive of the "cubing" operation. This observation is reflective of students' difficulties in generalizing algebraic inscriptions observed in studies from other fields of mathematics, as discussed in a previous section in this chapter.

In keeping with these other studies, students also did not always seem aware of the meaning of certain algebraic inscriptions. In sharing a historically-based cubic formula, Harel (2013) expected students would be surprised that this formula did not yield all the roots of a cubic equation. However, he observed that the students did not show the expected surprise, and later determined that this was because they were not clear on certain logical principles such as the difference between necessary and sufficient conditions and the meaning of quantifiers. They were also not able to answer completely why numbers of the form $a + b\sqrt{-1}$ could be considered a meaningful expression. The students simply stated that $i = \sqrt{-1}$ to solve $x^2 + 1 = 0$ "because that's what we were told in school" (p. 30). Following a teacher prompt, they reflected that they were unsure why such a number was invented to solve an equation like $x^2 + 1 = 0$ and no such number was invented to solve other equations with no solution such as $x = x + 1$. This further reflects students' difficulties generalizing algebraic inscriptions and deriving structural meaning from them.

In addition to these obstacles, students found it difficult to attach geometric meaning to operations such as addition and multiplication on complex numbers, and even to the assignment of a complex number $a + b\sqrt{-1}$ to an ordered pair $(a, b)$. Harel (2013) determined that much of this difficulty was caused by students' lack of understanding of the function concept, and the meaning of "one-to-one" and "onto". Once this difficulty was addressed, students were able to produce a parallelogram rule for the addition of complex numbers and the proper way to rotate and dilate a vector representing a complex number under multiplication.

In another teaching experiment, Nemirovsky et al. (2012) demonstrated that embodied cognition helped students learn that multiplying by $i$ corresponds to a rigid $90°$ rotation of the entire complex plane. Their students used a "floor tile" as a representation of the complex plane in which students could physically move either themselves or stick-on dots and string around to represent complex numbers. With this model, the students reasoned and tested ideas about multiplication by $i$, and eventually found that this operation corresponds to a $90°$ rotation of the entire complex plane. In addition to utilizing the "embodied" complex plane, students calculated algebraic equations corresponding to their embodied actions to test and corroborate their results. The students' usage of embodied reasoning may have facilitated their cognitive development, as Nemirovsky et al. found that their students noticed when their corresponding algebraic reasoning and embodied actions disagreed. In contrast, previous literature suggests that students tend to have difficulty noticing when different styles of reasoning yield contradictory results.

Similarly, Tall and Vinner (1981) stated that students can possess contradictory concept images of a single mathematical concept, and furthermore that it may often be difficult to make the contradiction apparent to the students. Danenhower (2006) found that students have difficulty translating between algebraic and geometric representations, possibly further obscuring the students' ability to recognize a contradiction that might occur across the different types of reasoning two representations might naturally suggest. In the same vein, Panaoura et al. (2006) discovered that their students tended to view algebra and geometry as entirely distinct ways of thinking, suggesting that even if a student did notice a contradiction, it might be attributed to a change in the inscription rather than any flaw in thinking (Soto-Johnson & Troup, 2014). However, none of these projects utilized embodied reasoning in the style of Nemirovsky et al. (2012). Thus, it is possible that the students' usage of the embodied complex plane itself helped them to recognize the discrepancy in reasoning between two representations. As a result of recognizing this inconsistency, these students managed to formulate a more correct understanding of multiplication by $i$ as a rotation rather than a reflection.

Since implementation of embodied cognition activities was helpful to Nemirovsky et al.'s (2012) students, I believed that allowing my participants to utilize a dynamic geometric environment (DGE) could help them connect algebraic and geometric reasoning more readily themselves. While I did not utilize a physical embodied complex plane, I did use computer software that has the capability to model transformations on the complex plane in real time, which allowed my participants to at least simulate potential embodied actions within a virtual environment. That is, they were able to interact with a virtual copy of the complex plane by moving the mouse with their hand, and thus in some sense manipulate a geometric inscription of an abstract environment through physical activity. While this environment may not exactly be "embodied" in the proper sense of allowing for manipulation of an actual physical layout, it nonetheless allowed for an interactive, explorable environment via actual physical movement of the mouse. Furthermore, my findings suggest that this embodiment of their reasoning did help them integrate their reasoning in this way.

As the DGE was not "embodied" as much as Nemirovsky et al.'s (2012) complex plane, it was possible that students may not have benefited from it as directly. That is, whereas embodied actions take place within the physical environment, interaction with a

DGE requires two levels of abstraction. First, a student needs to view their virtual motion within the DGE in tandem with their real world embodied action of moving the mouse. Second, the student needs to recognize a parallel between action they take regarding mouse movement and the corresponding actions they would take within the real world. Due to these extra layers of abstraction, it was possible that the benefits my participants derived from a DGE might not have paralleled the benefits experienced by Nemirovsky et al.'s students in using a physical representation of the complex plane. However, it seemed that my participants did benefit by grounding their reasoning via gesture, speech and inscriptions produced with the aid of *GSP*.

On the other hand, I at least considered a DGE a few steps closer to an opportunity for an "embodied" experience, if still somewhat abstractly represented. It was considered a way to concretize certain "simulated" embodied actions (Alibali & Nathan, 2012; Bazzini, 2001; Botzer & Yerushalmy, 2008; Tall, 2003), to be discussed more in a later section. A DGE allowed for testing of mathematical hypotheses and provides observable feedback as a direct result of the participants' interaction with it. The nature of these tests was much more observable than any similar mental simulations the participants might otherwise have performed, and likely more reliable as well. Thus, Nemirovsky et al's (2012) research functioned as further support for having included opportunities for my participants to actively manipulate a dynamic technological environment themselves during my formal interviews. Within this environment, my participants were guided toward developing reasoning about the derivative of a complex-valued function. DGEs are discussed in greater detail in the following section.

## Dynamic Geometric Environments

In this section, I review research on the benefits of technology—DGEs such as *Cabri*, *GeoGebra*, or *Geometer's SketchPad (GSP)*— in connecting differing representations and styles of reasoning as well as potential pitfalls to avoid in using this same technology. While students seem to have formed numerous misconceptions regarding foundational mathematical topics such as functions and limits, technology has been utilized with some success to curb these issues (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale et al., 2014). Some authors suggested that computer programs that provide dynamic construction tools seem to help these same students either correct previous misunderstandings or avoid them altogether (Arcavi & Hadas, 2000; Cory & Garofolo, 2010). Others warned of potential pitfalls of overemphasizing the usage of these programs, suggesting that computer over-usage may detract from the learning that naturally occurs within a traditional paper-and pencil environment (Kieran, 2007; Lagrange, n.d.; Olive, 2000). Much of the research included a caution that regardless of the choice of technology, the effects of the technology is dependent on how it is implemented in the classroom (Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Vitale et al., 2014). At the very least, utilizing computers for pedagogical purposes does appear to change the nature of what is being learned. DGEs seem to emphasize different topics altogether than the physical environment, for better or for worse, but can still greatly benefit students if used appropriately (Arcavi & Hadas, 2000; Barerra-Mora &

Reyes-Rodríguez, 2013; Heid & Blume, 2008; Hollebrands, 2007; Lagrange, n.d.; Olive, 2000; Pea, 1985; Salomon, 1990; Tabaghi & Sinclair, 2013; Vitale et al., 2014).

Pea (1987) continued in this vein by describing various ways in which the existence of computer technology makes new mathematical teaching goals possible and gives students novel methods to explore concepts related to previously existing teaching goals. In particular, students manipulating computer programs may become more fluent in reasoning about mathematics, exploring mathematics, learning problem-solving methods, learning how to learn, and "integrating different mathematical representations" (p. 106) such as algebraic equations and diagrams. Pea claimed that "manipulable dynamically linked, and simultaneously displayed representations from different symbol systems are likely to be of value for learning translation skills between different representational systems" (p.110). The introduction of mathematical computer environments thus may provide rich opportunities for exploring the intuitive dynamic notions experts are already using to guide their mathematical thought (Fey, 1984; Sfard, 1993). Along these same lines, Salomon (1990) suggested that DGEs might help students by providing interactivity, intelligent guidance, dynamic feedback, and multiple representations of mathematical objects.

Pea (1987) further stated that the usage of dynamic computer programs makes cognitive processes more visible within a clinical research setting. The problem that has always plagued any study of cognition still exists: thought processes are still invisible and can only be inferred. Fortunately, DGEs provide a new data source that contributes to the study of cognition. Within a dynamic environment set up with the express purpose of mathematical investigation, it was possible to observe participants' activity within the environment. Reasons for the ways in which they were exploring may have been inferred more reliably than within a paper-and-pencil setting, as the computer program encouraged input where a pencil could have easily been held unused.

In addition, just as Vygotsky (1962) said that the zone of proximal development must involve progression of thought even with the removal of outside assistance, Salomon (1990) claimed that true cognitive change occurs when students' thought processes are changed even outside the context of the computer program. Otherwise a risk of "deskilling" is present. That is, it is possible that a student could become dependent on the technology itself to drive his or her thought, and eventually may become altogether incapable of developing similar mental models without the technology upon which he or she has grown so reliant. Perhaps the positive effects a student experiences may simply fade with time. Still, computer technology provides more opportunities to expand a student's zone of proximal development in areas where the technology allows the computer to function as a "more capable peer" who assists the learner.

There were, however, concerns about what aspects of thought we might be in danger of losing due to the usage of technology in the teaching and learning of mathematics (Heid & Blume, 2008; Lagrange, n.d.; Kieran, 2007). For instance, Kieran (2007) worried that the shift in focus technology causes may come at the expense of more precise symbolic static forms. Heid (1984) found that students of a technology-intensive introductory calculus course failed to properly develop handwritten symbolic procedures over twelve weeks of working with *muMath*, graphing functions, and generating tables. These same students learned these procedures in three weeks with instruction focused particularly on the procedures the students needed to learn. Heid and Blume (2008) suggested that technology such as spreadsheets may reduce students' chances to manipulate symbolic forms, even while providing a new perspective on algebra.

Nonetheless, technology may help overcome these same obstacles. While Heid's (1984) students could not carry out the necessary symbolic manipulations by hand, they could still interpret and reason about the theoretical results. Given multiple representations of functions to consider, Yerushalmy's (1991) eighth grade students could apply graphing techniques and solve related traditional problems, even though their instruction did not focus as much explicitly on graphing itself (as cited in Heid & Blume, 2008). Technology expands students' opportunities to investigate multiple representations, and offers a new perspective on mathematical conceptualization, representation, generalization, symbolic manipulation, and modeling. Even actions as simple as zooming, scaling, or scrolling can positively influence students' reasoning.

Used in the proper way, technology could present results in a way believable to the students and could help students discover firsthand in a dynamic way exactly why their intuitive notions can often be wrong. Students may be able to generalize more appropriately from a large number and variety of computer-generated examples (Heid & Blume, 2008). Technology provides students with the invaluable opportunity to see mathematics and discover it for themselves. Technological programs thus carry with them the hope of reducing students' apparent reliance on teachers, textbooks, or other mathematical authority (Harel & Sowder, 2005). Instead of just being told by teachers how math works, they can investigate their theories and intuitions themselves, and maybe even discover *why* math works the way it does. Care must be taken, however, not to let technology deprive students of other forms of reasoning (Heid & Blume, 2008; Kieran, 2007; Lagrange, n.d.).

Due to these concerns, much of the research made recommendations about *how* to use this technology in the proper way. For example, Arcavi & Hadas (2000) particularly recommended asking for predictions prior to events that are expected to feel counter-intuitive to the students. In addition, they cautioned that students may not initially conduct worthwhile investigations with the provided tools without some guidance, and suggested asking the students questions that require predictions to prevent this problem. They added that under this scheme, dynamic geometric environments (DGEs) may help students build up or correct their own ideas, and refine their intuitions regarding the correctness or potential proofs of formal mathematical statements.

Therefore, I gave my research participants opportunities to explore aspects of the derivative of a complex-valued function before interviewing them more thoroughly. This progression may have helped my participants to develop their reasoning regarding this topic and potentially allowed them to make more progress during the following task-based interviews. Since the task-based interviews main intent was to observe how the students refine their reasoning about the derivative of complex functions over time, it seemed helpful to suggest potentially useful ideas or ask questions to encourage exploration of these ideas. This small amount of probing may have helped them avoid the potentially fruitless investigations against which Arcavi and Hadas (2000) warned, and possibly may have streamlined later interviews.

Lagrange (n.d.) continued the cautionary theme by warning that computer programs may deprive students of learning how to interpret symbolic forms, as the programs could lead the students to interpret symbols graphically rather than understand their original meaning. That is, he worried that computer programs could actually increase the gap between students' algebraic and geometric reasoning, rather than help bridge it. He built a conceptual framework which assumes that students approach functions at three different levels of dependencies: "sensually-experienced" dependencies in a physical system, magnitudes, and various representations. Despite his previously noted concerns, Lagrange (n.d.) stated that technology can help students grasp the inherently difficult concepts of covariational reasoning provided that references to bodily activity are utilized, as he believed these are crucial to a proper understanding of functions. He did not resolve his and Kieran's (2007) concern that symbolic forms would be interpreted graphically rather than dealt with, choosing simply to say that this is a valid concern worth keeping in mind when utilizing dynamic geometric environments.

Thus, by grounding participants' experience with DGEs via their real-world experiences, these DGEs may be able to render certain abstract mathematical ideas more concrete. This grounding seems to encourage a more vivid conceptual understanding of those abstract concepts. Tall (2003) noted that DGEs may encourage embodied reasoning about mathematical concepts. He connected features of DGEs to Bruner's (1966) three modes of representations (as cited in Tall, 2003): symbolic, iconic, and enactive. Symbolic representations involve numerical, logical, or linguistic concepts; iconic representations provide visual or sensory information; enactive representations involve actions themselves as representation.

Tall (2003) noted that a DGE provides all levels of representation. The user's experience with the interface itself provides an enactive representation, images representative of selectable options constitute iconic representations, and keyboard input and the program's internal processing are symbolic representations. He thus argued that such technology could support embodied reasoning through enactive and visual experiences such as "allow[ing] the user to interact in a physical way by pointing, selecting, and dragging objects onscreen to extend the embodied context of real-world calculus" (p. 9). However, Tall was also aware that there is a sense in which students' experience with DGEs are not embodied, realizing that "applications have a largely symbolic interface, producing graphic *output* on the screen, but with little embodied *input*" (p.9, emphasis in original).

It may further help students connect this form of reasoning with symbolic-proceptual reasoning (the ability to see mathematical symbols as both a process and an object as in Sfard (1991)) and formal-axiomatic reasoning (the ability to argue logically towards a theorem from a set of axioms). Indeed, DGEs may be configured to support any form of reasoning (Tall, 2003), and provide warrants of truth for any form of reasoning. According to Tall, such warrants are established in embodied reasoning if behavior of an object is as expected, in symbolic-proceptual reasoning if a property of an object can be calculated to be as claimed, and in formal-axiomatic reasoning if a claim can be logically proved from the given set of axioms. Such reasoning can be supported via generic organizer programs or cognitive roots. Generic organizers allow students to manipulate and investigate examples and non-examples of some mathematical property or object. Cognitive roots are concepts salient to the student that also promote further development toward formal reasoning or embodied concepts. For example, local straightness is a cognitive root for differentiation (Tall, 2003). Thus, Tall's research again demonstrated that DGEs may support embodied reasoning.

Some researchers (Hollebrands, 2007; Olive, 2000) further suggested that for pedagogical purposes, *Geometer's Sketchpad* (*GSP*) in particular could help students ground their reasoning in the physical environment. Olive warned that *GSP* requires some basic knowledge of geometry to be used well, since it is possible to construct shapes, which either preserve or do not preserve its initial properties. The user must therefore possess the ability to decide which properties she wants her object to preserve before actually constructing it. He further noted that a computer makes it possible to test a large number of examples at once. This instantaneous feedback could be a major advantage of a dynamic geometric environment (DGE), possibly allowing students to test a previously overwhelming number of conjectures. This includes the ability to construct lines and circles and some proficiency in reading and using a two-dimensional graph. They should also be able to compare slopes of lines or areas and radii of circles. Overall, a typical high school geometry class should easily provide sufficient background to beneficially utilize *GSP*. My participants were all students who took an undergraduate complex analysis course, so it was expected that all of them had taken high school geometry at some point.

Hollebrands (2007) catalogued high school honors geometry students' usage of *GSP* and their associated strategies while they investigated geometric transformations. He reported that students used *GSP* measures to explore relationships, create and verify conjectures, and check the correctness of their constructions. He additionally found that strategies could be either reactive or proactive, and that *GSP*'s effects were related to the types of strategies employed. Reactive strategies developed one step at a time, as students perform an initial action, then perform subsequent actions in reaction to what they see occur on screen. If a student expects something particular to occur for a given action, or makes predictions associated with a given action, he or she was said to be utilizing a proactive strategy. *GSP* appeared to hamper the reasoning of students who employed reactive strategies, while it supported the reasoning of students who employed proactive strategies.

Thus, Hollebrands (2007) concluded that “the ability to measure and drag coupled with carefully crafted tasks and questions posed by the instructor is not enough to assist students in learning new mathematical concepts” (p. 190). He suggested directing students to reflect on the relevant relationships between these concepts and the technology they are using and make connections and distinctions between the mathematical and technological realms. He alluded to the importance of this distinction by noting that the *GSP* interface led students to believe that points can move, and are not simply locations in space, due to the fact that these constructed “points” can be labeled and then dragged around without changing their names.

Jones (2000) agreed that to make progress with *Cabri*, his 12-year-old participants needed to be able to separate features of the software from mathematical geometric properties. Thus, he constructed three phases in his interviews to help his students form this distinction. In the first phase, he helped students gain familiarity with *Cabri*. In the second phase, he asked students to create a rhombus, a square, and a kite. Finally, he asked them to discover relationships between these quadrilaterals. He discovered that the DGE functioned as a link between their spontaneous reasoning (as in Vygotsky, 1962) and their formal mathematical reasoning. That is, they began primarily with descriptions that did not involve any formal mathematical language, transitioned to explanations that utilized terminology that directly referenced the DGE, and finally arrived at a formal mathematical explanation that involved terminology independent of the DGE. This sequence mirrors the progression Tall (2003) detailed through the enactive, iconic, and symbolic modes of reasoning.

A slightly different progression was demonstrated in Vitale et al.'s (2014) work, which reported on third- and fourth-graders usage of a dynamic geometric environment (DGE) to investigate their intuitions about geometric shapes. They began their interview by asking students assigned the "grounded integration" condition about "intuitive" concepts through familiar actions or problems. Following this, they attempted to help students ground their reasoning through embodied concepts. Finally, they presented the students with problems specifically designed to challenge students to distinguish between settings where it is or is not appropriate to apply their developed reasoning.  I followed a similar format for my dissertation study interviews. First, my students investigated familiar functions with *GSP*. They followed this with explanations of how the derivative is "rotated" and "dilated" with accompanying gestures. Finally, I challenged my participants to identify non-differentiable points, determine derivative values for a rational function, and reconstruct an algebraic formula for this rational function.

In Vitale et al.'s (2014) research, students in the "numerical integration" condition were provided with symbolic measurements rather than embodied concepts. Vitale et al. noted that children tend to categorize objects based on characteristics which are noticeable, but do not distinguish them well from objects of a different type. In contrast, adults or experts tend to classify the same objects based on less noticeable abstract properties that more properly distinguish between different types of objects. Heid and Blume (2008) observed a similar phenomenon in their students, who "frequently use[d] linear functions as prototypes for functions" (p. 79). They concluded that students could place overly restrictive criteria on a certain class of object by viewing certain properties of the presented example as though they were properties of the entire class.

Vitale et al. (2014) found that students who were assigned the grounded integration condition were more likely to correctly identify the presented shapes than those assigned the numerical integration condition. They further felt that one of the most critical tasks given to the students in the grounded integration condition was the validation step, where students guided virtual hands into place to check that certain conditions were met. These hands simulated real-world gestures. For example, a student might place the two virtual hands at right angles to each other to check that a corner of a shape really was a 90° angle. Given that DGEs are two leaps of abstraction away from truly embodied actions, it is noteworthy that by the end of the study, most of the participants in this study actually performed the gestures represented by the virtual hands on the screen. Vitale et al. (2014) concluded that students need activities with salient intuitive ideas that nonetheless require formal reasoning.

Furthermore, DGEs might help students bridge the apparent gap between the intuitive motion-oriented visual models of limit with the algebraic, motionless, formal definition. While Cory and Garofalo (2010) called the visual model naïve, even experts familiar with the formal algebraic definition often employ dynamic imagery when asked to describe how they think about limits (Presmeg, 2006). Cory and Garofalo (2010) used a series of interactive, dynamic sketches "somewhat successfully" through the five stages of covariational reasoning about limits defined by Cottrill et al. (1996). While the students made progress through the stages of covariational reasoning, it remained unclear whether they strengthened the connections between their visual and verbal modes of thought.

It is also possible that DGEs may emphasize geometric or visual representations and de-emphasize algebraic or verbal representations. Barrera-Mora and Reyes-Rodríguez (2013) found that teachers working on problems in mathematical, hypothetical, and real-world contexts formed conjectures strongly based on visual representations (Arcavi & Hadas, 2003) they constructed in *Cabri*. However, they reported that these teachers, who were well-versed in either mathematics or engineering, did use *Cabri* in a beneficial way. The DGE in this case appeared to act as a reorganizer, essentially allowing teachers to formulate conjectures and create procedures with data and tools not available in most other non-technological contexts.

Still, Barrera-Mora and Reyes-Rodríguez (2013) additionally found that not all teachers verified their constructions appropriately, contrasting Olive's (2000) assertion that integration of algebraic and geometric representations emerges from a clash between conjecture and observation. Perhaps this integration did not emerge because the teachers were using different representations for different purposes, and thus did not realize contradictions that may have arisen. This possibility was substantiated by Dennis & Confrey's (1996) assertion (as cited in Arcavi & Hadas, p. 40) that in the "coordination and contrast of multiple forms of representation…often one sees a particular form of representation as primary for the exploration, whereas another may form the basis of comparison for deciding if the outcome is correct."

Heid and Blume (2008) added that while such technology could indeed help reify objects and processes as defined by Breidenbach, Dubinsky, Hawks, and Nichols (1992) by making multiple representations more accessible, representations developed by technology may advance students' reasoning differently than representations developed by hand. However, they also noted that even physical representations can be unhelpful. They describe how Meira (1998) found that students do not grasp linear relationships any more easily when presented with a modeling winch or spring mechanism than when dealing with purely symbolic inscriptions.

Heid and Blume (2008) further suggested that potential benefits a learner could enjoy from a dynamic geometric environment (DGE) may be dependent on the level at which the student allows the program to make decisions for him or her (Zbiek, 1998, as cited by Heid & Blume). For example, giving a student direct control over the parameters of a mathematical entity appears to allow him or her to identify invariant geometric properties. However, if a student does not consider the relationship between the actions he or she takes and the outcomes of those actions when dealing with multiple representations, those representations may remain compartmentalized (Schoenfeld, Smith, & Arcavi, 1993, as cited by Heid & Blume). In essence, the mathematical activity that students accomplish with a DGE, not the DGE itself, determines learning.

Building on previous work (Sierpinska, 2000; Sinclair & Tabaghi, 2010), Tabaghi and Sinclair (2013) found that while interacting with an eigenvector sketch in *Sketchpad*, four undergraduate students and one graduate student were able to integrate the synthetic-geometric mode of thinking with the analytic-arithmetic mode. Synthetic-geometric reasoning corresponds roughly to reasoning about geometric figures or graphical representations of spatial objects, and analytic-arithmetic reasoning corresponds to reasoning via formulas. That is, synthetic-geometric reasoning is related to geometric reasoning as described by Carraher and Schliemann (2007) in that both involve reasoning about geometric properties of mathematical objects, while analytic-arithmetic reasoning is related to Battista's (2007) algebraic reasoning due to the reliance of both on manipulation of algebraic symbols (Soto-Johnson & Troup, 2014).

In summary, previous research suggested overall that technology can certainly be used to help students and teachers alike refine their mathematical ideas (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Marrades & Gutiérrez, 2000; Mason,1996; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale et al., 2014). With the help of *Cabri,* both teachers and students could more easily posit and test mathematical conjectures (Barrera-Mora & Reyes-Rodríguez, 2013; Jones, 2000). *GSP* helps students form a more vivid conceptual understanding of abstract ideas by making them appear more concrete (Hollebrands, 2007; Olive, 2000). Similar computer environments have additionally been found to aid students in determining whether their current mathematical intuitions are correct, as well as in evaluating or creating formal proofs (Arcavi & Hadas, 2000; Battista, 2007; Laborde, 1998; Mariotti, 2001, 2002; Marrades & Gutiérrez, 2000). Cory and Garofolo (2010) found that a series of dynamic sketches helped students advance their level of geometric reasoning, and Pea (1987) claimed that technology could naturally emphasize dynamic ideas more than static ones.

While it has been suggested that technology could damage students' ability to learn mathematical concepts (Kieran, 2007; Lagrange, n.d.), it appears that many more researchers reported that technology could in fact be used to help students refine their intuitions of these concepts (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Marrades & Gutiérrez, 2000; Mason,1996; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale et al., 2014). Finally, allowing students to use technology may help reduce students' reliance on mathematical authorities such as teachers or textbooks (Pea, 1987).

Not all effects of technology are necessarily positive, though. Computers can easily test large numbers of examples at once, so they could potentially exacerbate students' existing issues with improper generalizations (Clements & Battista, 1992; David et al., 2014; Lee, 1996; Olive, 2000; Radford, 1996a; Yerushalmy & Chazan, 1993). Furthermore, students may become reliant on the technology used to teach them to the extent that they cannot reason mathematically without that same technology (Salomon, 1990). Thus, technology should be used carefully, and students should be guided in its use. Used carefully, students can investigate aspects of mathematical thought that did not previously occur to them, and refine or correct their existing mathematical interpretations. Allowing my students to use technology may have even elucidated some aspects of their cognition that would not otherwise have been noticeable. In addition to utilizing technology to enhance my inferences about my participants', I carefully attended to their produced gestures. This usage of gesture and technological action gave me a greater ability to determine whether my participants were utilizing algebraic or geometric reasoning at certain points in time, for example. Literature on gestures and how they connect to thought and communication is discussed in the following section.

## Gesture

In this section, I discuss research on the relationship between gesture and inscription usage, the effects of gesture on cognition, and the possibility of using gesture to help students bridge the conceptual gap between algebraic and geometric reasoning. Studies in gesture suggested various ways I could have complemented my planned study, such as connecting algebra to geometry via diagrammatic reasoning (Châtelet, 2000; Chen & Herbst, 2013; de Freitas & Sinclair, 2012; Dörfler, 2001; Roth & McGinn, 1998; Samkoff et al., 2012; Zazkis et al., 1996), or simply drawing stronger conclusions from the additional data that gesture provides. Gesture itself was considered by some to naturally arise due to thinking, and therefore has the capability of physically manifesting otherwise invisible cognitive processes, or at least providing more information than speech conveys alone (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene, Rasmussen, & Stephan, 2012; Roth, 2001). As Keene et al. claimed, "gestures can be used as a window into what students in a classroom are thinking" (p.367).

According to much of the previous gesture research, gestures can also inform inscriptions. Châtelet (2000), for example, suggested that certain gestures may be inherent in particular diagrams. A dotted line might convey motion that could then be expressed in a corresponding iconic hand gesture in the direction of the line. One vector could represent the way another vector rotates and dilates when the two are multiplied together; this motion could be expressed with a flick of the wrist to convey rotation

(Soto-Johnson & Troup, 2014). In this way diagrams can "capture" (Châtelet, p.10) or "transfix" (Châtelet, p. 10) gesture. Similarly, Sinclair and Tabaghi (2010) suggested that diagrams provide a link between gestures and speech for mathematicians reasoning about eigenvectors. Chen and Herbst (2013) further stated that diagrams and gestures can work together to bring a diagram to life, and Roth and McGinn (1998) reported that inscriptions can help coordinate speech and gesture.

In contrast, Soto-Johnson and Troup (2014) stated that gesture served as the link between inscriptions and speech. Their participants initially reasoned just with words, then began building a diagram when this was no longer sufficient. Their algebraic inscriptions and gestures both served to support their construction of a geometric inscription of a mathematical equation involving complex numbers. As they constructed this diagram, they appeared to act out the gestures their diagrams may have captured. While Soto-Johnson and Troup's (2014) participants ultimately integrated algebraic and geometric reasoning, students' difficulties with viewing both types of reasoning as part of a single system is long-standing, well-documented, and exists across a variety of mathematical topics (Danenhower, 2006; David et al., 2014; Dubinsky & Harel, 1992; Gray & Tall, 1994; Kuo et al., 2013; Larkin et al., 1980; Otte, 1993; Panaoura et al., 2006; Sfard, 1991, 1992, 1995; Sfard & Linchevsky, 1994; Sierpinska, 2000; Simon & Simon, 1978; Tall & Vinner, 1981; Vygotsky, 1962) as discussed earlier in this chapter.

Gesture itself could be utilized to discover and describe the ways participants reason, both algebraically and geometrically (Alibali & Nathan, 2012; Goldin-Meadow, 2003; Keene, Rasmussen, & Stephan, 2012; Roth, 2001). While Clement (2000) defended clinical interviews as a legitimate means to collect data on otherwise invisible cognitive processes, Goldin-Meadow's (2003) and others' work (Alibali & Nathan, 2012; McNeill, 1992, 2005; Roth, 2001) made the case for including gesture along with speech in the analysis of these interviews. Much of this research strongly suggested that gesture and speech form a single, integrated system.

In one study, (Goldin-Meadow, 2003) participants were asked to describe a picture or a short animated clip under two conditions: gesture permitted and gesture not permitted. Overall, those who were allowed to gesture were more easily able to recall and describe the picture or short they were shown, whereas those who were not allowed to gesture exhibited stilted, less natural speech patterns and apparently more difficulty remembering the information in some cases. That is, not allowing speakers to gesture negatively impacted their ability to speak. This phenomenon suggests that gesture and speech are essentially tied to the same system. Each facet of this system of communication depends on the other. This is not to say that gesture and speech always convey the same information. In fact, discrepancies between the information expressed in speech and the information communicated by co-occurring gesture arise fairly frequently. Goldin-Meadow dubbed these occurrences as gesture-speech mismatches.

In another of her experiments, Goldin-Meadow (2003) observed teachers' and students' gestures as they interact with each other, paying particularly close attention to the production of gesture-speech mismatches. She found that both teachers and children produce these mismatches, and that children who mismatch more frequently seem to be more primed to learn new ideas than those students who reliably match gesture with speech. Furthermore, teachers were seen to modify their explanations based on the children's produced gestures, often entirely unconsciously. It was not uncommon for teachers to express one potential solution strategy in speech, and present an entirely disparate solution strategy in gesture for the exact same mathematical task. Sometimes the strategy presented in speech was correct and the strategy presented in gesture was not. The children would on occasion fixate exclusively on the strategy presented in gesture and ignore the speech-presented strategy altogether, regardless of whether these strategies were correct. Overall, however, the gesture utilized by both teacher and student seemed beneficial to the communication between them.

While gesture seems to be primarily used for communication, Goldin-Meadow (2003) theorized that perhaps this is not in fact its reason for existence. She noted that while speakers gesture less frequently when alone than when talking to others, the difference between these situations in numbers of gestures produced is small (though statistically significant). She explained this by suggesting that gesture is primarily caused by thought, though its main function appears to be communication. Others might have said that there is no real difference between thought and communication, defining invisible thought merely as discourse with oneself (Sfard, 2008). Regardless of how we define communication, discourse, and thinking as a social construct, the fact remains that the occurrence of gesture certainly appears to reduce cognitive load.

In yet another experiment on gesture, Goldin-Meadow (2003) asked participants to remember a random string of letters (or words for younger participants) under the same two conditions as before: gesture permitted and no gesture permitted. The participants were even given an unrelated task to perform between memorization and recall to prevent certain confounding variables. For example, requiring the participants to do something in between prevented them from storing physical information in some overly concrete way, such as through a sustained hand shape or held gesture. Furthermore, it allowed for sufficient time lag for the experiment to test actual short-term memory. Without the time lag, participants might be able just to parrot back the proper words without ever really registering those words in any sort of memory, consciously or unconsciously.

With these confounding variables accounted for, Goldin-Meadow's (2003) results are compelling. Those who were allowed to gesture could more easily recall the string of letters or words they had memorized than those who were denied the ability to gesture. According to similar experiments (Keene, Rasmussen, & Stephan, 2012), gesture is capable of lightening cognitive load for both verbal and visual information. For example, taking the perspective that thinking and discourse are equivalent as Sfard (2001) outlined, Keene et al. detailed how taken-as-shared gestures affected their participants' understanding of concepts related to differential equations. Within the context of my study, this suggested that allowing the participants the ability to gesture and create diagrams (which Châtelet (2000) refers to as "gesture captured mid-flight") would reducee the cognitive strain, thereby allowing them to more easily explore and think about novel mathematical ideas. Interviewing my participants in pairs may have further increased the frequency with which they gestured, however slightly. Even if the increase was small, it seemed worthwhile to give my participants the opportunity to leverage the natural benefits that arise from gesture. Indeed, my participants seemed to benefit from producing iconic gestures of rotation and dilation when reasoning about how various complex-valued functions map circles, for example.

As students become more familiar with a certain mathematical procedure, they may not need to gesture as much to support their reasoning (Alibali & DiRusso, 1999;

Marrongelle, 2007; Soto-Johnson & Troup, 2014). Both Marrongelle and Alibali and DiRusso investigated students' usage of gesture, but reported contradictory findings. Alibali and DiRusso saw that students gestured less as they became more familiar with a counting task, while Marrongelle observed no such reduction in gesture while her participants investigated differential equations. However, Marrongelle did observe a shifting in the purpose of gesture. In particular, she reported that her participants appeared to utilize graphs and gesture mainly to support their reasoning while they attempted to develop an algorithm, but primarily to clarify their ideas when applying an algorithm they had already developed previously. In the latter situation, her participants appeared to use the algorithm itself to reason.

In addition to noting a reduction in gesture as their undergraduate calculus students progressed through a series of tasks involving related rates, Garcia and Engelke (2012) also observed that their participants gestured more frequently when they were stuck on a problem. Soto-Johnson and Troup (2014) found that undergraduate complex variables students' gestures did not reduce in frequency overall, but did change in character from predominantly iconic (representative gestures) to primarily deictic (pointing gestures). Vitale et al. (2014) reported that the purpose of gesture began primarily as a way to remind themselves of a geometric concept and that as this need faded, gesture's primary purpose transitioned into a tool for validation. This suggested that Marrongelle's (2007) and Alibali and DiRusso's (1999) findings may not be entirely contradictory, as Soto-Johnson and Troup's participants' iconic gestures did reduce as they progressed through the tasks, but gestures overall did not. Furthermore, the purpose of the gestures in Soto-Johnson and Troup's research appears to corroborate

Marrongelle's findings regarding their purpose: the participants utilized iconic gestures to help them reason through novel tasks, while later they applied deictic gestures to better communicate their ideas to one another.

Alibali and Nathan's (2012) research provided further evidence that analysis of gestures naturally fits within an overarching framework of embodied cognition. For example, pointing gestures suggest that thought is grounded in the physical environment in some capacity, especially when Wilson's (2002) moderate interpretation of embodied cognition is considered in conjunction with Alibali and Nathan's writings. In particular, Wilson wrote:

> by doing actual, physical manipulation, rather than computing a solution in our heads, we save cognitive work. However, there is also a sense in which these activities are not situated. They are performed in the service of cognitive activity about something else, something not present in the immediate environment (p. 629, 2002).

Similarly, Alibali and Nathan (2012) argued that environments and actions may be simulated rather than directly experienced, such as when one may imagine walking down an aisle in an organized roomful of chairs, without actually being present in such a room. Drawing upon previous real-life experience, one may even imbue these imagined chairs with shape, feel, and texture, and perhaps even count the number of rows of chairs in this imaginary room. One could easily imagine a person actually pointing at various locations in space as he or she counts the chairs that exist only in his or her mind. In this way, representational gestures could arise from mentally simulated embodied actions. Metaphoric gestures and language, such as one feels of speaking "up," "down," or "blue," could develop from environmentally based conceptual metaphors in a similar way. For example, the emotion of sadness could reasonably convey the idea of someone in a posture which is not altogether upright, but rather somewhat slouched or literally "down".

So, if an imaginary mental picture triggers a gesture, why would that gesture not also be imaginary? Should the individual not simply imagine pointing at rows of chairs within his or her mind rather than actually pointing at areas of empty space entirely unrelated to the purpose of the gesture? Alibali and Nathan (2012) felt that when a thinker is simulating or "reliving" an event, cognitive load may increase in response, to the point that the speaker produces a gesture to lessen this burden. This line of reasoning was backed by Goldin-Meadow's (2003) claim that gesture can indeed lessen cognitive strain, and can therefore be produced as the result of thinking, rather than merely utilized for the purpose of facilitating communication between individuals. Like Alibali and Nathan, Goldin-Meadow also claimed that gesture can provide a window into the mind, for similar reasons. In particular, because gesture is often produced as a result of thinking, it could provide context clues to aid in inferences regarding the nature of the thought that triggered that gesture.

Alibali and Nathan (2012) further stated that a gesture triggered by a simulated event or mental image typically seems to be representational in nature, and that many of the produced representational and metaphorical gestures they observed implicitly utilized the linguistic phenomenon known as fictive motion. There may therefore be a natural connection between the use of gesture and the tendency to rely on language employing fictive motion, which also seems to be metaphoric or representational in nature. This relationship begins to feel stronger as one imagines an experienced hiker trying to describe to his or her friend the shape of the trail as it runs up the mountainside. It is easy to imagine that his or her finger may really move upward in a somewhat winding trajectory as he or she recalls his previous experiences along the trail mentally. In producing this gesture, the hiker has converted the merely metaphoric phenomenon of fictive motion into the very real motion of his finger.

Even within the context of such a commonplace example we see a completely natural dynamic description of a mountainside trail that is entirely static and unmoving. However, when a hiker speaks of an unmoving trail and moves his or her finger to describe it, no one seems to have any trouble reconciling the fact that the described entity is static and the way in which it is described is dynamic. In this light it seems strange that students would so regularly have difficulty pairing dynamic geometric mathematical entities with their static algebraic definitions. Therefore, Alibali and Nathan's findings on gesture serve to suggest that student-produced gesture itself could help bridge the conceptual gap between geometric and algebraic representations that seems to plague students so commonly.

Núñez (2004) noted that there is a parallel conceptual gap between experts' ideas of standard $\varepsilon - \delta$ definitions of continuity and students' ways of thinking about the same topic. As discussed previously in this chapter, mathematical experts typically seem to view these very algebraically presented definitions in geometric, dynamic, or generally visual ways. In contrast, students appear to favor algebraic representations over geometric representations (Kuo et al., 2013; Panaoura et al., 2006). At first glance, it may feel somewhat surprising that experts would so commonly take a view that does not directly reflect the formal definition. That is, if the formal definition is so algebraic in nature, why do experts consistently pursue geometric modes of thought? Núñez (2004) compared this phenomenon to the linguistic notion of "fictive motion."

Fictive motion (Talmy, 1988, 2003) occurs when a speaker refers to an unmoving object as though it were moving, such as when a native English speaker might say quite naturally that "the fence runs along the road," or "the trails winds up the mountain." Neither the fence nor the trail is actually moving, yet they are attached to active verbs, which inarguably convey a sense of motion. In the context of mathematics, a professor might speak of a graph that "approaches" or "gets closer to" an asymptote. Just as in the context of everyday speech, the professor does not mean to say that the points of the graphs themselves are jostling up and down or streaming steadily toward the limit value defined by the asymptote. Rather, the everyday English speaker is conveying that as one moves along the road, the traveler will find more of the fence, or that as a hiker winds up the mountain, they will be on some part of the same trail the entire way.

Similarly, the mathematics professor is attempting to communicate the idea that for any error bound, one can always take a domain value large enough so that the associated range value is within that selected error bound. Formally, for every $\varepsilon > 0$ there exists a $\delta$ such that if $x \geq \delta$ we have $|f(x) - M| < \varepsilon$, where $M$ is the range value of the asymptote. In attempting to avoid any occurrence of fictive motion, we have algebraically defined what it means for a function to "approach" $M$ as $x$ approaches $\infty$. So, it would seem that algebraic and formal definitions could simply be the natural consequence of avoiding any reference to fictive motion. By purposefully attempting to be formal and precise and removing the intuitive, immediately understandable language that involves fictive motion, we arrive at a formal, precise, and much less intuitive or dynamic algebraic definition.

It could even be argued that this less intuitive definition is more mathematically correct; Núñez (2004) went so far as to say that points cannot actually move since points are not actually real. It could also be said that since points are defined as locations in space, once one moves (at all), he or she stops occupying the space where they once were and begin occupying a new space, and are thus at an entirely new and different point. Different locations means different points, so points, being defined by their location alone, do not and cannot move. However, Núñez also claimed that there is a metaphorical notion of *something* moving as successively larger values of $x$ are taken. Perhaps experts' geometric lines of thought simply arise from their drive to restore the intuitive sense of the definition that was lost when fictive motion was removed from the language. The algebraic formal definition may supply precision and clarify ambiguous or pathological cases, but geometric ideas and the associated fictive motion could provide a powerful source of intuition and suggest natural ways of thinking about associated mathematical problems.

Thus, while there seems to be a conceptual gap between students' use of algebraic and geometric reasoning, Núñez (2004) felt that fictive motion might help explain this chasm. He noted that while algebraic ideas and formal definitions provide precision and guidance, geometric ideas in conjunction with fictive motion provides a powerful source of intuition. Goldin-Meadow (2003) suggested that since gesture and speech form a single integrated system used for both thought and communication, gesture can be analyzed in addition to speech as a second source for inferring student's thought processes. Garcia and Engelke (2013), Alibali and DiRusso (1999), Marrongelle (2007), and Soto-Johnson and Troup (2014) all suggested that gestures can help support mathematical reasoning, particularly in novel contexts. Alibali and Nathan (2012) added that thought appears to be grounded in the physical environment or embodied experiences within it, even though these actions and the environment could sometimes merely be mentally simulated. Additionally, they noted that gesture occurring in tandem with a mental simulation tends to be representational in nature, echoing Núñez's thoughts on fictive motion. Finally, many researchers (Châtelet, 2000, Chen & Herbst, 2013; de Freitas & Sinclair, 2012; Roth & McGinn, 1998; Sinclair & Tabaghi, 2010) suggested that inscriptions link words and gestures, while Soto-Johnson and Troup (2014) posited that gestures provide the connection between words and inscriptions. I chose to employ embodied cognition as my theoretical perspective for this project because it allows me to interpret both algebraic and geometric reasoning made apparent by gesture and actions taken within a dynamic technological environment such as *Geometer's Sketchpad (GSP)*.

## Theoretical Framework

Due to the nature of my research questions, it was important for me to choose a framework that allowed me to connect reasoning and inscriptions. In this section, I detail the nature of each aspect of my framework as well as the connections between them. In particular, the framework of embodied cognition both served as the lens through which I interpreted my data and potentially aided my participants in developing geometric reasoning for the derivative of complex-valued functions. Furthermore, *GSP* provided inscriptions through which my participants investigated the derivative of these complex-valued functions, and both their technological actions and their gesturing helped me infer the nature of their reasoning and their usage of inscriptions. Finally, both gesture and inscriptions may have helped my participants bridge the gap so many students seem to experience (Danenhower, 2006; Panaoura, et al., 2006; Sfard, 1992) between their algebraic and geometric reasoning strategies (Alibali & DiRusso, 1999; Châtelet, 2000; Chen & Herbst, 2013; de Freitas and Sinclair, 2012; Gibson, 1998; Goldin-Meadow, 2003; Roth & McGinn, 1998; Samkoff et al., 2012; Zazkis et al., 1996). In this section, I supply a brief summary of embodied cognition and connect this theoretical perspective to diagrammatic reasoning with dynamic geometric environments (DGEs). In each subsection, I additionally detail the utility of each within the context of my project.

**Embodied Cognition**

In essence, research under an embodied cognition perspective is fundamentally concerned with some relationship between reasoning and actions taken within the physical environment (Anderson, 2003). However, the nature of this relationship differs between researchers; thus these alternative views must be carefully considered (Anderson, 2003; Wilson, 2002). Particularly, some researchers suggested that an organism's mental models are directly influenced by the organism's experience with the physical environment (Alibali & Nathan, 2012; Lakoff & Nuñez, 2000; Wilson, 2002). In reviewing several claims made within the context of embodied cognition, Wilson noted that even cognition that occurs solely within the mind is still body-based. She referenced as evidence the quality of "reliving" (p. 633) certain memories, the progression of skills from deliberately applied to automatic, and the usage of mental imagery in problem-solving tasks. She further stated that these "domains of cognition listed above are all well-established and non-controversial examples of offline-embodiment" (p. 634), and additionally claimed the assumption that cognition must be analyzed as an inextricable part of the environment is “deeply problematic” (p. 1). Lakoff and Nuñez suggested the cognitive mechanism of metaphorical projection—doing mathematics with metaphors of mathematics that are based on our bodily interactions with our world, such as conceptualizing an abstract set as having physical existence, or envisioning a point moving along some path in space.

To further describe embodied cognition, Soto-Johnson and Troup (2014) summarized how

> …other researchers (Châtelet, 2000; de Freitas & Sinclair; 2012; Nemirovsky et al., 2012) avoid referencing mental models in favor of discussing only the physical experience of the learner, choosing to view learning and experience within the environment as inherently inseparable. In other words this latter group of researchers view knowing as doing, which is observable unlike mental models. Nemirovsky et al. elaborate on this phenomenological view by describing how a learner might project some “realm of possibilities” (p. 291) onto some perceived environment. This realm of possibilities is fluid, potentially changing in real-time in response to interactions with the environment. The creation of inscriptions, interaction with inscriptions, and produced gestures could influence these realms of possibilities (p. 112).

Under the view of cognition that relates mental models to an organism’s experience with the environment, DGEs, gestures, and inscriptions all function as tools to help a thinker lighten his or her cognitive load by manipulating the environment in beneficial ways. The existence of inscriptions saves work by allowing a learner to avoid storing certain information in short-term memory, while produced gesture may itself reduce cognitive load in a similar way (Goldin-Meadow, 2003). Alibali and Nathan (2012) added that mental simulations (such as reliving an event as described by Wilson (2002)) may increase cognitive load to the point a gesture is produced to lessen it. Gesture can additionally provide a window into the mind, as gesture seems to be produced as a result of thinking (Goldin-Meadow, 2003). In other words, gesture may serve as an external representation of an internal representation. Alibali and Nathan (2012) further stated that a gesture produced in this fashion typically seems to be representational in nature. Thus, student-produced gestures could help students bond the gap between geometric and algebraic representations that seems to exist in students' minds.

DGEs, gestures, and inscriptions do not merely serve to reduce cognitive load. Wilson (2002) noted that one of the most powerful and under-utilized claims that is made within the perspective of embodied cognition is that even "offline cognition is body based" (p. 632). That is, thought that occurs invisibly, outside the realm of the physical environment, is still strongly driven by bodily experience. Thus, while many cognitive theories positioned the body as a servant of the mind where mental schema are adapted based on perceived experience (Piaget & Cook, 1952; Sfard, 1992; von Glasersfeld, 1995), this view of embodied cognition reverses these roles and places the mind as subservient to the body. Instead of mental schemas determining bodily actions, mentally simulated bodily actions are crucial to the functioning of the mind.

Under the latter view, instead of functioning to reduce cognitive load or prime mental mechanisms, bodily experience is itself an inextricable part of the learning process. Because embodied reasoning is seen as fundamentally phenomenological, research under this view is primarily concerned with participants' personal experience with mathematics, rather than the details of any particular cognitive mechanism. The focus of research on diagrams, gestures, and inscriptions might seek to discover the nature of participants' personal experience with mathematics, utilizing their embodied actions as a means of inference.

Furthermore, diagrams, gestures, and inscriptions are interrelated and could act as a driving force in refining a learner's projected realm of possibilities. The creation of a diagram could "capture" gesture (Châtelet, 2000), and alteration of a diagram could create new possibilities for further actions and previously unnoticed relationships between mathematical objects (de Freitas & Sinclair; 2012). Nemirovsky et al.'s (2012) work suggested that embodied actions might help students integrate algebraic and geometric reasoning. For example, an embodied complex plane such as the one used by Nemirovsky et al. may assist students in making connections between algebraic and geometric inscriptions. The usage of diagrams may aid students in accomplishing this same goal (Soto-Johnson & Troup, 2014).

The difference in these views appears to be one of focus: one view is concerned with developing a formal theory of mental models and cognitive mechanism as influenced by embodied action, while the other, more phenomenological view focuses exclusively on lived, personal experience. Furthermore, the two views do not seem entirely incompatible: Nemirovsky et al. (2012) noted the existence of an "explanatory gap" (p. 2) between formal, more traditionally scientific accounts of a phenomenon and the subjective experience of the same phenomenon. An impersonal description of pain as "the firing of C-fibers" (p. 2) does not invalidate or even take precedence over someone's personal experience with pain, nor vice versa. Similarly, postulations about the cognitive way the mind responds to bodily actions should not be seen as contradictory to inferences made about a learner's personal experience with his mind and environment. However, embodied cognition itself appears to be defined differently within each view. While all perspectives of embodied cognition appear to assert that a learner's experience with the physical environment is essential to the learning process, they differ regarding how this cognition takes place. Do embodied actions drive mathematical thought located within the mind, or do the embodied actions themselves constitute the mathematics?

In either case, diagrams, gestures, and inscriptions seem to benefit students of mathematics, whether by “greas[ing] the wheels of the thought process” (Wilson, 2002, p. 629) or by helping students experience otherwise abstract mathematics within the physical environment. For my research, I viewed embodied cognition to include both bodily actions taken within the physical environment for the purpose of doing mathematics and reasoning about mathematics, which I assumed was based on our participants’ subjective experience with their world. I identified with Lakoff and Nuñez’s (2000) argument that students’ understandings of mathematics are based on their personal experience with the world. I additionally adopted Wilson’s (2002) view that reasoning is body-based, including that which does not take place within the immediate physical environment. This view seems to suggest that actions taken within a virtual environment, such as a dynamic geometric environment (DGE), could reasonably be abstracted back by students to actions taken within the real world.

I further believed both that perceptuo-motor activity can influence reasoning and that this reasoning influences bodily actions. Thus, diagrammatic reasoning, produced gesture, inscriptions, and algebraic and geometric reasoning were all directly relevant under my views of embodied cognition, and their usage allowed me to make inferences regarding the nature of my participants’ reasoning methods. I acknowledged Nemirovsky et al.’s (2012) position that there is a “mechanistic” side and a phenomenological side to embodied cognition, and agreed that these two accounts are not necessarily contradictory.

I did not postulate any cognitive mechanisms driven by my participant's embodied actions. Rather, I suggested a relationship between how my participants utilize DGEs, gestures, and inscriptions. As the view of embodied cognition that knowing is doing assists research projects intended to investigate the ways in which students utilize reasoning, DGEs, speech, gestures, and inscriptions, I chose to adopt this perspective for my research. To this end, I focused my analysis on my participants' observable diagrammatic actions and produced gestures, particularly those actions which appeared to help them create diagrams justifying certain algebraic statements involving complex numbers.

Recent research in the field of complex numbers suggested that this perspective possesses practical teaching applications. Nemirovsky et al.'s (2012) study demonstrated that allowing students to utilize a physical representation of the complex plane helped these students successfully reason geometrically about multiplication by the complex number $i$. Furthermore, while these students did not stop reasoning about this problem algebraically as well, they noticed when an inconsistency arose between the answer they obtained via algebra and the one obtained via embodied geometric reasoning. Other studies on both complex numbers (Danenhower, 2006; Panoura et al., 2006) and real numbers (Sfard, 1992; Tall &Vinner, 1981) have suggested that students typically have difficulty recognizing when a contradiction occurs, particularly when they utilized two different representations. Other researchers such as Cottrill et al. (1996) further claimed that much of previous literature on the limits of real-valued functions suggests that students tend to have significant problems learning associated concepts (Artigue, 1992; Cornu, 1981, Sierpinska, 1992), most commonly because students frequently seem to believe that the limit of a function is never actually attained by the function (Cottrill et al., 1996; Tall & Vinner, 1981).

However, none of these prior studies encouraged embodied reasoning in their participants as Nemirovsky et al. (2012) seem to have accomplished with their embodied complex plane. Therefore, it is possible that students reasoning in such an embodied way may have an easier time noticing inconsistencies between different representations or approaches to a given problem. Indeed, my participants seemed to more easily reason geometrically about the derivative of a complex-valued function because they were encouraged to think in some similarly embodied way. This embodied reasoning appeared to help them connect their algebraic and geometric reasoning about the derivative of complex-valued functions.

### Diagrammatic Reasoning and Dynamic Geometric Environments

For my project, I took both algebraic and geometric representations produced by *Geometer's Sketchpad* (*GSP*) to be inscriptions, and was thus able to consider my participants' work with *GSP* a form of diagrammatic reasoning. As discussed previously, this form of reasoning appeared to help students integrate their algebraic and geometric reasoning. Furthermore, this usage of technology fit under my theoretical perspective of embodied cognition. The dynamic environment that *GSP* provides can be considered more concrete than the simulated actions and environments described by Alibali and Nathan (2012), but less concrete than Nemirovky et al.'s (2006) embodied complex plane. As my participants' usage of *GSP* seemed to form some sort of middle ground between these two extremes, and both extremes fit under the embodied cognition perspective, my participants' usage of *GSP* reasonably fit under this same perspective.

It is thus reasonable to suggest that my participants' experiences with *GSP* may have allowed them to connect their algebraic and geometric reasoning more easily and recognize inconsistencies between these two methods of reasoning due to an unidentified error in one, as embodied cognition did for Nemirovsky et al's (2006) students. On the other end of the spectrum, it is similarly possible that the usage of this technology could have helped reduce cognitive load, just as Alibali and Nathan (2012) described that gestures produced during simulated actions within a simulated environment can reduce the cognitive strain these simulations create. A produced gesture (e.g. pointing at rows of imaginary chairs) represents simulated action and thus reduces cognitive load by introducing some aspect previously only imagined into the real physical environment. Similarly, an action taken in *GSP* could serve to represent a thought or action that was previously only abstract, and thus reduce cognitive load in a related way. In fact, in my study I found that *GSP* did support my students' reasoning through its facilitation of mathematical investigation and the creation of inscriptions.

**Summary**

Overall, previous research suggested that students may be able to reason more effectively about mathematical concepts if they enhance their algebraic and geometric reasoning and develop connections between them (Carraher & Schliemann, 2007; Danenhower, 2006; Hiebert, 2003; Kaput, 1995, 1998; Katz & Barton, 2007; Kuo et al., 2013; Panaoura et al., 2006; Sfard, 1992; Sherin, 2001; Sierpinska, 2000; Tall & Vinner, 1981; Williams, 1991). Since experts appear to integrate different forms of reasoning into a single approach in a beneficial way (Arcavi, 1994; Kuo et al., 2013; Lithner, 2008; Núñez, 2004; Redish & Smith, 2008; Sierpinska, 2000; Sfard, 1993; Szydlik, 2000;

Wertheimer, 1959; Williams, 1991), it seems advantageous to find productive ways to encourage student to use both forms of reasoning together. With these two approaches working in concert, students may develop the ability to reason about mathematical concepts both intuitively and precisely. Unfortunately, students appear to have difficulty integrating different forms of reasoning and representations (David et al., 2014; Danenhower, 2006; Dubinsky & Harel, 1992; Gray & Tall, 1994; Harel & Sowder, 2005; Kuo et al., 2013; Lee, 1996; Otte, 1993; Panaoura et al., 2006; Sierpinska, 2000; Sfard, 1995; Sfard & Linchevsky, 1994; Tall & Vinner, 1981). In my study, I focused on my participants' usage of algebraic and geometric reasoning to explore the development of these reasoning methods.

There is some evidence to suggest that diagrammatic reasoning with both algebraic and geometric inscriptions may help students overcome this phenomenon of compartmentalization (Battista, 2007; Châtelet, 2000; Chen & Herbst, 2013; de Freitas and Sinclair, 2012; Gibson, 1998; Lee, 1996; Radford, 1996b; Roth & McGinn, 1998; Samkoff et al., 2012; Zazkis et al., 1996). Dynamic geometric environments such as *GSP* may also help in this regard (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Marrades & Gutiérrez, 2000; Mason,1996; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale et al., 2014). Within the context of complex-valued numbers, researchers have discovered via clinical interviews that students strongly compartmentalize different lines of thought, particularly separating algebraic and geometric approaches (Danenhower, 2006; Panaoura et al., 2006). Through a teaching experiment, Harel (2013) reaffirmed students' difficulties with generalization of algebraic inscriptions and the successful integration of algebraic and geometric reasoning strategies. He additionally found that teachers were able to reduce novel algebraic expressions to a familiar form, though this approach did not always help them move toward a solution. Nemirovsky et al. (2006) found that allowing "a class of prospective secondary school teachers" (p.6) the ability to move around on a physically represented complex plane on his classroom floor, they were able to more proficiently connect their algebraic calculations with their graphical "embodied" explorations.

Dynamic geometric environments (DGEs) can be used as well to help correct students' flawed mathematical ideas, and perhaps to connect students' algebraic and geometric reasoning (Arcavi & Hadas, 2000; Barrera-Mora & Reyes-Rodríguez, 2013; Cory & Garofalo, 2010; Heid & Blume, 2008; Hollebrands, 2007; Jones, 2000; Lagrange, n.d.; Marrades & Gutiérrez, 2000; Mason,1996; Olive, 2000; Tabaghi & Sinclair, 2013; Tall, 2003; Vitale et al., 2014). Programs like *Cabri* and *GSP* can help students and teachers make sense of some of the more abstract ideas, evaluate formal proofs, or even just check their own intuitions for correctness (Arcavi & Hadas, 2000; Barrera-Mora and Reyes-Rodríguez, 2013; Battista, 2007; Hollebrands, 2007; Jones, 2000; Laborde, 1998; Olive, 2000). Computers have additionally been found to help students refine their ability to reason geometrically and naturally emphasize dynamic or geometric ideas (Cory & Garofolo, 2010; Mariotti, 2001/2002; Marrades & Gutiérrez, 2000; Pea, 1987). As a result, some authors have worried that students will lose some of their ability to grasp the meaning of symbols and static concepts (Heid & Blume, 2008; Kieran, 2007, Lagrange, n.d.; Schoenfeld, Smith, & Arcavi, 1993).

Other potential problems include further confusing students about whether large numbers of examples constitute a complete formal proof, or risking students becoming too reliant on computers in their mathematical thinking. However, the potential gains to learning with technology appear to outweigh these inherent risks, provided such technology is used purposefully and carefully. Finally, the ways in which students use technology provide data regarding how they might be thinking about the problem on which they are working. In my study, I capitalized on the use of technology to explore my participants' development of inscriptions in the context of complex-valued functions.

In addition to the nature of students' technological explorations, it appears that gesture can also be used to infer student thought processes (Alibali & DiRusso, 1999; Garcia & Engelke, 2013; Marrongelle, 2007). Furthermore, gesture appears to be connected to the notion in language known as fictive motion, and that both gesture and fictive motion could provide a means to help students connect the gap between their algebraic and geometric representations (Goldin-Meadow, 2003; Núñez, 2004). This power in inference comes from the observation that gesture and speech form a single integrated system, and that gesture seems closely tied to geometric ideas while speech seems related to algebraic ones. (Goldin-Meadow, 2003). Furthermore, gesture, speech, and inscriptions all appear linked within the context of diagrammatic reasoning (Châtelet, 2000, Chen & Herbst, 2013; de Freitas & Sinclair, 2012; Roth & McGinn, 1998; Sinclair & Tabaghi, 2010). Finally, thought appears to be grounded in the physical environment (Alibali & Nathan, 2012). Thus, gesture as a data source informs researchers about student cognition under the framework of embodied cognition. Therefore, I catalogued my participants' produced gestures as an extra data source to help me code my participants' reasoning method as either algebraic or geometric.

In the next chapter, I discuss the methods utilized and procedures followed in my study, in addition to research leading me to select these particular protocols. In Chapter IV, I present findings from my dissertation study, organized by task and group. Finally, in Chapter V I provide answers to my research questions and their implications for teaching and research.

# CHAPTER III

# METHODOLOGY

The purpose of this study is to contribute to the literature on algebraic and geometric reasoning about complex analysis, specifically as both kinds of reasoning relate to diagrammatic reasoning, inscriptions, and gestures. The relationship between reasoning, inscriptions, gestures, and speech is detailed in the previous chapter. This study seeks to answer the research questions:

Q1 What is the nature of students' reasoning about the derivative of complex-valued functions?

Q2 What is the nature of the development of students' reasoning about the derivative of complex-valued functions while utilizing *Geometer's Sketchpad* (GSP)?

In this chapter, I detail the nature of the methodology I chose to employ for this study, including discussions regarding the type of study I chose to conduct, a description of my participants, the development of the *GSP* tasks, the structure of the interviews, and the nature of my data collection and analysis. I follow this chapter with results in chapter IV. Note that prior to conducting this study, I obtained approval from the Institutional Review Board (IRB) for the methods detailed below. My IRB form can be found in Appendix A.

## Design Experiment

As my research questions focus primarily on the nature of students' reasoning, and not on any easily measurable quantity, a qualitative methodology was more appropriate for my study. Furthermore, I investigated the ways in which students advance their geometric reasoning about the derivative of a complex-valued function via tasks developed for this study. Therefore, a design experiment structure was chosen to capture the nature of my exploration, where this dissertation study constitutes iteration one of the design experiment. Before iteration one, I conducted a pilot study where I developed tasks intended to help participants explore potential meanings of the derivative of complex-valued functions. I referred to this pilot study as iteration zero of my design experiment. Iteration zero is thus the name for the first iteration of my design experiment, while my second iteration is called iteration one. In both iterations, I focused my analysis on isolating participants' reasoning strategies within the context of these tasks, in accordance with my research questions stated at the beginning of this chapter.

All design experiments share five common themes, according to Cobb, Confrey, diSessa, Lehrer, and Schauble (2003). First, their primary purpose is to answer questions about how students learn and possible ways to support this learning. My research question involves both the nature of students' reasoning regarding the derivative of a complex-valued function and the ways in which the tasks I created support this reasoning. I developed the pilot study tasks particularly to provide my participants with the opportunity to develop geometric reasoning about the derivative of a complex-valued function as an *amplitwist* (Needham, 1997). Furthermore, the improvements I made to the task flow and task design for my dissertation study were in large part motivated by my desire to find a way to address common difficulties my participants experienced in my pilot study. Thus, the study design itself was motivated in large part by observing and reflecting on how my pilot study and current participants learned and finding possible ways to support this learning, just as Cobb et al. described.

Second, design experiments build on prior research to investigate potentially new educational opportunities and teaching methods. For my pilot study, the design and implementation of the tasks I utilized in my interviews were largely motivated by findings suggested in previous research studies. Namely, students may exhibit sophisticated reasoning about a particular mathematical object if they can view it both as a process and an object (Sfard, 1992), or via a synthesis of algebraic and geometric methods of reasoning (Danenhower, 2006; Panaoura et al., 2006). In my dissertation study, my pilot study results and the tasks I employed shared these motivations. In addition, I included one of Sfard's (1991) observations regarding reification. In particular, she claimed that in cases of true reification of some mathematical process/object pair, the student should exhibit the ability to reason about the process in reverse. Thus, I included a fifth task in my dissertation study which requires students to produce a derivative value at a point of their choosing given geometric information garnered with the aid of *GSP* and produce a matching algebraic function formula. Thus, this new task reversed the process required by my other tasks, which provided a function formula and required students to relate the function's geometric behavior to the derivative value at specified points. This last task additionally provided information which I did not obtain in iteration zero. A more detailed description of the purpose of each of these tasks follows in the Interview Structure section of this chapter.

Third, a design experiment must contain both prospective and reflective aspects to its methodology. Thus, before detailing my findings, I describe which aspects of reasoning about the derivative of complex-valued function I expected my participants to develop as a result of the tasks I placed before them. Furthermore, I improved the task design based on observations from my pilot study. I discuss the development of these tasks in the Task Development section of this chapter.

The fourth aspect of a design experiment is iterative design. Thus, for my dissertation study, I conducted another iteration on this project, again interviewing students in four two-hour blocks, and asking them to complete similar but improved tasks. Based on the prospective and reflective aspects of the design experiment and the suggestions made to me by my dissertation committee, several weaknesses of the pilot study were identified, and improvements were made to the design of the study for this next iteration. These weaknesses included inconsistent group sizes, a relative lack of preparation in task flow and design, and a lack of data to differentiate reasoning developed in my participants' complex analysis class from reasoning developed in my interview sequence.

To address these identified weaknesses for my dissertation study, I ensured that all students were interviewed in pairs, improved the interview task flow and task design, and included observations from my participants' complex analysis classes. I discuss each of these improvements in the relevant sections in this chapter. That is, the student pairing process and observations from the complex analysis classes are both included in Setting and Participants. A description of the methods involved in acquiring data from the classroom is provided under Data Collection, and a summary of the data collected from the classroom is provided under the subheading Classroom Observations in the Setting and Participants section. Finally, I discuss improvements to the ordering and administration of the tasks in the Interview Structure section, while improvements to the tasks themselves are discussed under a heading of Task Development.

The fifth aspect of a design experiment as outlined by Cobb et al. (2003) is that results from the study “informs prospective design” (p. 11). That is, these results must suggest a particular way of implementing the means used in the design experiment as part of some potential educational instruction. As such, in the summary of my findings at the end of Chapter IV, I include the mathematical concepts I felt each task emphasized, as evidenced by the nature of my participants’ reasoning within each task. I additionally detail some suggestions for the implementation of these tasks within the classroom.

Within a qualitative study, the researcher should examine the data in a way that allows him or her to provide extensive and detailed information regarding that case (Patton, 2001). For a design experiment, data should “support the systematic analysis of the phenomenon under investigation” (Cobb et al., 2003, p.12), including data regarding how the students learned and what tools they utilized in facilitating this learning. Cobb et al. additionally list gesture, tasks, and the nature of the social interaction between participants as potential data sources. Thus, I collected data in a way that allows for such extensive investigation. In accordance with the IRB, I obtained signed permission from all participants affected by these data-gathering steps, as well as permission from the complex analysis professor to video-tape and attend his class. I asked the professor to identify the complex analysis classroom sessions that were related to the derivative complex-valued functions, and video-recorded these sessions. In addition, I kept notes of concepts the professor taught in class, gestures and chalkboard drawings he employed, and questions and conversations that occurred between the students and the professor.

After the conclusion of the course, I conducted a four-day task-based interview sequence intended to guide participants through the development of geometric reasoning about the derivative of a complex-valued function. In accordance with the IRB, I again obtained signed permission from my participants to record them and report on their work in the interview. While detailing these findings in this dissertation paper, I utilize gender-preserving pseudonyms. I additionally compensated participants for their time with their choice of a $25 Starbucks or iTunes gift card once they completed all the interviews. This compensation was also approved by the IRB.

In accordance with the IRB, I recorded all interviews, utilizing a video recorder to collect audio and visual data, as well as screen-capture software to record the technological actions my participants took with *GSP*. I developed a sequence of tasks for my participants to complete during a four day sequence of two-hour-long interviews, as well as task worksheets that supplied instructions and questions for the first two tasks. These tasks are discussed in detail later in the Methods section. I only developed worksheets to pair with the first two tasks; I supplied the instructions verbally for later tasks. The reader must keep in mind that participants completed Task 1 and Task 2 according to pre-written instructions, while they completed all subsequent tasks according to verbal instructions, which I supplied.

All participants were paired to reduce potential mathematically induced stress, in accordance with the IRB. All participants completed the same sequence of tasks, so Task 4, for example, for one participant pair was the same Task 4 completed by all other pairs.

For Tasks 1 and 2, I created worksheets for participants to follow as they learned how to use *Geometer's Sketchpad* (*GSP*). However, later tasks had no worksheets associated, so that the participants could conduct a free-form exploration of the derivative of a complex-valued function; the lack of specific direction theoretically allowed the students to investigate the phenomena on which they wished to focus.

I purposefully selected participants to maximize potential progress through the tasks and development of geometric reasoning about the derivative of a complex-valued function. I selected students that the complex analysis professor recommended; the professor told me the students he suggested demonstrated geometric reasoning about complex numbers within his class. I placed students in a single pair when they knew each other well or the course professor felt they would work well together. In particular, I avoided pairing students who had rarely or never worked with each other before.

## Methods

The method for iteration one of my design experiment, my dissertation study, is discussed below. I first describe the setting and participants for my experiment, the manner in which I developed and improved the *GSP* tasks, and the structure of the interviews. I conclude this section with a discussion of my data collection and analysis techniques.

### Setting and Participants

In my pilot study, I collected data only from the interviews I administered. As such, I had no way of differentiating the advancements in reasoning my participants developed in their complex analysis class from the progression they experienced as a result of the interview sequence of my pilot study. Therefore, in an effort to document this development better, I observed the parts of the participants' complex analysis class in which the professor discussed the geometry and algebra of the derivative of a complex-valued function. These records were intended to help me connect students' experiences in my interview with their experiences in the classroom, and to give me a greater chance of determining in which setting various aspects of their reasoning about the derivative of a complex-valued function developed. The methods used to collect these data follow later under Data Collection, and the classroom observations are summarized in the following section.

**Classroom Setting**

Before selecting participants for my dissertation study, I attended part of a complex analysis course. I obtained permission for attending and recording these classes from both the professor and the students in class, in accordance with the IRB. I enlisted the professor's help in identifying which class sessions most related to the derivative of a complex-valued function, and attended those days. Thus, I did not attend every day of the complex analysis class, but only those days the professor felt were relevant to my topic, as well as a few days leading up to the commencement of the topic of the derivative of a complex-valued function. While attending, I video-recorded the entire class session and kept handwritten notes of the concepts the professor discussed, the marks he made on the chalkboard, and the gestures he employed to explain these concepts. In addition to these, I took notes on questions students asked, and conversations that occurred between student and professor. Based on the professor's recommendations, I attended all classes from the beginning of the semester up to the class the professor had informed me marked the end of their discussion of the derivative of a complex-valued function.

As the course was lecture-based, students listened and took notes, asking questions occasionally for clarity. Broadly, the content of the course was algebraically motivated, though the professor related the algebra to graphical phenomena at least some of the time, as discussed in the rest of this section. In discussing the occurrences of the course below, I refer to some of my participants by their pseudonyms, which are Christine, Zane, Edward, and Melody. My participants are described in more detail later in this section.

In the course sessions leading up to the professor's construction of the derivative of a complex-valued function, the professor defined $i = \sqrt{-1}$ algebraically as the number such that $i^2 = -1$, and graphically as a unit vector along the vertical imaginary axis. Following this introduction, the professor pointed left, then right while stating that $i$ and $-i$ are differing values, and defined the Cartesian form of a complex number via the equation $z = a + ib = (a, b)$. He then drew coordinate axes and a vector $(a, b)$ in the first quadrant roughly at a $45°$ angle. Following this presentation, Zane asked if multiplying $3 + 2i$ by $i$ is equivalent to rotating the vector $3 + 2i$ by an angle of $90°$, and the professor confirmed and clarified that the rotation must be counterclockwise.

This class observation was particularly important when Zane's development was explored. In this class session, Zane showed at least cursory geometric reasoning in class about how multiplication by a complex number can rotate and dilate a vector. Similarly, this class session's observation notes provide a possible reason for some errors seen in my interviews: in the example $(7 + 2i)z = 7z + 2iz$ the professor notes that the $7z$ "stretches" the vector by 7 and that the $2iz$ "turns 90°" and "doubles it." This example may have motivated students' reasoning that the real part of a complex-valued derivative is the factor by which the input circle stretches, and the imaginary part is the factor by which the input circle rotates. A recollection of this classroom example may also explain Christine's tendency to attempt geometric reasoning about the derivative of a complex-valued function via vector addition.

After covering multiplication as a stretch and rotation, the professor noted that conjugation corresponds to a reflection across the real axis. He also covered polar representation $z = Re^{i\theta}$ and that $R$ is a "stretching factor." In this same discussion, he noted that given $z_i = r_i e^{i\theta}$, we have $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$, which is the algebraic statement indicating that if two complex numbers are multiplied, the resulting vector is obtained by multiplying the vector's magnitudes and adding their arguments. Finally, the professor provided algebraically motivated presentations on DeMoivre's formula, roots of unity, the triangle inequality, and how to represent complex numbers as two-dimensional vectors, and a geometric description that "complex multiplication is rotation followed by dilation."

At this point, the professor provided a short symbolic example of the derivative of a complex-valued function by stating that the derivative of $e^{\phi+i\theta}$ is $(1+i)e^{\phi+i\theta}$, and noted again that multiplication by $z = Re^{i\theta}$ is geometrically equivalent to rotation by $\theta$ and dilation by $R$. The professor drew a graph of a hand on a pair of planes to demonstrate. He placed the "input" drawing of a hand on the left plane, and its rotated and dilated image on the right plane. He further leveraged this example to explain that we "can't really graph functions" when we consider complex-valued inputs and outputs, so a "split-screen view" must be utilized instead. In this discussion, the professor further clarified that "dilate" and "stretch" have equivalent graphical meanings.

At the beginning of the second week of the course, the professor began algebraically reasoning about limits of rational functions by considering the coefficients on the dominating terms. In this class, $\epsilon-$neighborhoods were first referenced geometrically as a "little tiny ball," and utilized to formally define a limit $L$ via a convergence of the sequence $z_n \to L$. The professor further clarified that $z_n \to L$ means that for every ball centered at $L$ of positive radius there is an $M$ such that if $n \geq M$, then $z_n$ lies in that ball. In the professor's words, every disk centered at $L$ eventually "ensnares the flea." The professor further noted that polynomials exhibit "good behavior" inside a disk, or more precisely, there exists $M$ such that $|p(z) - p(z_0)| \leq M|z - z_0|$, and showed an example of "bad behavior" with $f(z) = \frac{\bar{z}}{z}$. While he did also draw a graph of a $\delta-$neighborhood mapping to a corresponding $\epsilon-$neighborhood and noted that the circle mapped to a circle, precise amounts of rotation and dilation were not discussed at this time. Rather, algebraic verifications of limit rules followed. This included the fact that limits preserve products, sums, and reciprocals, and that given $f(z) \to L$ for a continuous function $f$, if $z$ is close to $z_0$, then $f(z)$ is always close to $L$. That is, if $z$ is in the input ball, $f(z_0)$ is in the output ball.

After discussing complex multiplication and constructing limits, the professor algebraically constructed the derivative in class, paralleling the real-valued definition. In particular, he defined the difference quotients $\frac{f(x+h)-f(x)}{h}$ and $\frac{\Delta w}{\Delta z} = \frac{f(z)-f(z_0)}{z-z_0}$, and then defined $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = L$, in the case where $L$ exists. Initially, the professor discussed algebraic verifications of derivative rules familiar from the real-valued case such as the product rule, quotient rule, and preservation of addition and scalar multiples.

The professor additionally lectured about the derivative according to Caratheodory's definition, culminating in an algebraic definition of the derivative as a local linearization. More precisely, the professor noted the derivative exists if and only if $f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\sigma$ , where $\sigma \to 0$ as $z \to z_0$. The geometry of linear complex-valued functions was not discussed at this time, though the professor did mention rotation and dilation again when discussing composition of functions. In particular, the instructor stated that the first function of the composition could be considered a rotation and dilation, and that the second function of the composition could be characterized as a second rotation and dilation that occurs after the first. That is, the first function rotates and dilates an input circle, and the second function rotates and dilates the resulting image.

The class day after constructing the derivative formally, the professor reminded the class that $f(z)$ is differentiable at $z_0$ if and only if a local linear approximation exists at $z_0$. The professor leveraged this fact to derive the Cauchy-Riemann equations algebraically via matrices. He additionally mentioned that in conformal maps, that "global distortion can be weird, even though local behavior can be fine." In the same lecture, the professor started with the real and imaginary parts $u = x + 2xy$ and $v = 7x + 3y$ to obtain the derivative $f'(z) = 3 + 7i$. He noted that the dilation factor is $|3 + 7i| = \sqrt{9 + 49}$ and the rotation is $\theta = \tan^{-1}\frac{7}{3}$. Zane asked if it was true that if the dilation factor was less than 1, then the image would shrink, and the professor confirmed this was true. The professor continued by defining holomorphic functions as those which are differentiable on an open region, and algebraically calculated that for $f(z) = z^2 + 2z$ near $i$, the dilation factor is $2\sqrt{2}$ and the angle of rotation is 45°.

Later, the professor promised to talk about how to visualize analytic complex-valued maps. To begin, he reminded the class of rotation and dilation on a vector, and lectured about the multiple-valued nature of $Arg(z)$. He related $\theta$ to the physics concept of "phase," and drew a contour map to demonstrate a phase portrait. He further mentioned that when $f'(z) = 0$, dilation "destroys" and conformal geometry breaks down. The complex-valued function $f(z) = 3iz + z^2$ demonstrates this behavior, so the professor noted that this function is quadratic and non-linear, the angles double, the radii square, and the map collapses.

On the remaining class days before the professor informed me his instruction of the derivative was complete, he surveyed a few more uses for the derivative, such as deriving Laplace's equation $u_{xx} + u_{yy} = 0$, checking differentiability via the Cauchy-Riemann equations, and investigating harmonic conjugates. In the first case, the professor related Laplace's equations to the geometric reasoning that convexity in the $x$ and $y$ directions "cancel exactly," so the graph must resemble a "saddle" or "soap film." This relationship allowed the professor to elaborate that "every soap film is related to a conformal map," and that this relationship is a local property that involves rotation and dilation. Finally, he continued to use the "soap film" aspect of geometric reasoning to demonstrate stereographic projection. The final few topics covered involved linear fractional transformations. The professor first proved that if $f$ is holomorphic in region $G$ and real-valued, then $f$ is constant in $G$. He additionally noted a relationship existed between linear algebra and linear fractional transformations of the form $\frac{Az+B}{Cz+D}$, where $AD - BC \neq 0$.

On the last class day I attended, the professor proved the chain rule and noted that $\frac{1}{0} = \infty$, and that linear fractional transformations map circles and lines to circles and lines. He also listed the linear fractional transformations as dilation, translation, and reciprocation. Finally, he noted that in stereographic projection, reciprocation of the planes corresponds to a $180°$ rotation of the sphere about the $x$ −axis, which interchanges the north and south poles. At the end of this class day, the professor informed me his discussion of derivatives was complete. The events described in this section occurred over a total of ten class days, of which three days were spent introducing various forms of complex numbers and basic operations involving complex numbers, one day was devoted to constructing a definition of limit, and six days involved the derivative directly.

**Participant Selection**

In my pilot study, I selected four students, but one of them could not be scheduled for interviews due to logistical difficulties. Thus, I interviewed two together and one alone. This created a potential conflating factor and led to finding differences across groups which may or may not have resulted from the difference in group size alone. Therefore, one of the largest improvements for my dissertation study was the requirement that all students interviewed were placed in pairs. This prevented any one student from working alone, thereby mitigating both a potential conflating factor and possible frustration with the tasks themselves.

Thus, I selected four undergraduate students who had recently completed an undergraduate course in complex analysis to participate in the interviews to complete the tasks in pairs and explored further questions that I posed. While I invited the whole complex analysis course students to participate in these interviews, only four students agreed to be interviewed due to logistical constraints. The first group was interviewed after their final exam at the conclusion of the Spring 2014 semester, which was the same semester they took their undergraduate complex analysis course. One of these students identified as a physics major, while the other stated he was an applied math major with a focus on computer science. The second pair consisted of two applied math majors, one of which stated he had a focus on computer science and statistics. I interviewed this second group at the beginning of Fall 2014.

Regarding the participants in the first pair, in accordance with the IRB, I will refer to the physics major as Christine, and the applied math major with a computer science focus as Zane. When discussing the second group, I refer to the applied math major with a focus on computer science and statistics as Edward, and the other applied math major as Melody. All four students had taken the same complex analysis course I had observed immediately prior to these interviews in the Spring 2014 semester.

Due to the lecture-based nature of the course, all participants were relatively quiet in the complex analysis course itself, though some did occasionally ask questions. Zane in particular asked a question clarifying the nature of rotation and dilation involved in the mapping of a $\delta -$neighborhood to an $\epsilon -$neighborhood, as discussed previously in this section. Aside from this small clarification however, I did not observe any of my participants ask any questions about rotation, dilation, or the derivative of a complex-valued function as it relates to local linearization. I did not observe them ask questions about the geometry of local linearization either, as the presentation of local linear approximations remained largely algebraic in nature. Rather, they appeared attentive and took notes as the professor lectured as previously described in this section.

**Task Development**

Task development was an ongoing process that spans the entirety of both my pilot study and dissertation study. That is, the development of the tasks used for my pilot study began before the onset of my dissertation study. Therefore, I begin my discussion at the initial creation of the tasks in the pilot study, include improvements to the tasks during the pilot study, and end with improvements to the tasks between the pilot study and dissertation study.

The overall goal of the tasks I developed for my interview sequences was to encourage reasoning about the derivative of a complex-valued function as described by Needham's (1997) concept of an amplitwist. Needham describes an amplitwist as one possible way of reasoning geometrically about such a derivative. The construction of the amplitwist is as follows. Given a complex-valued function $f(z)$, first consider an $\epsilon$ −neighborhood $\mathcal{N}$ around a point $z_0 = (x_0, y_0)$ in the complex plane. Next, calculate the magnitude of the derivative $M = |f'(z_0)|$ and the counterclockwise angle between the derivative and the positive real axis $\theta = Arg\big(f'(z_0)\big)$. Finally, observe that the image $f(\mathcal{N})$ of the $\epsilon$ −neighborhood is dilated by a factor of $M$ and rotated by a factor of $\theta$. This rotation and dilation resulting from the derivative of the complex-valued function forms the amplitwist.

My primary intent in both conducting the interviews and creating the interview tasks was guiding my participants toward reasoning geometrically about the derivative of a complex-valued function as a local linearization as characterized by Needham's (1997) description of the amplitwist. That is, I wanted them to develop the following ideas:

1. The derivative describes how a given function transforms a small circle around a given point; in particular, small circles are mapped to shapes that are approximately circles.
2. The magnitude of the derivative is the factor by which the function dilates the image of the small circle with respect to its pre-image.
3. The argument of the derivative is the angle by which the function rotates the image of the small circle with respect to its pre-image.

In order to develop these tasks, I adopted questions from Soto-Johnson (2014). Additionally, before my pilot study interviews commenced, I asked the participants' complex analysis course professor for advice regarding the progression of the tasks and the content he felt might be desirable. He suggested that I direct the students first toward simple polynomial functions such as $f(z) = z^2$, and advance to more complicated functions such as $f(z) = e^z$, perhaps even by building this latter function up through Taylor series and utilizing the knowledge gained from the polynomial functions.

Based on the complex analysis professor's advice, the first task I developed was an exploration of the function $f(z) = z^2$ (see Appendix B) and the second was an exploration of the function $f(z) = e^z$ (see Appendix C). The task worksheet I wrote for each function contained similar questions regarding how points and various circles were transformed by the given function.

Since I intended the interviews to last four days for two hours each day, before the pilot study, the complex analysis course professor suggested investigating the additional function $f(z) = \frac{1}{z}$ if time allowed, on the basis that this function preserves circles. He suggested that students might reason about this function geometrically more easily in some ways than even simple polynomial functions. I did not write an additional worksheet for this function, but chose to include it in the pilot study interviews if there was time after the students explored the first two functions, $f(z) = z^2$ and $f(z) = e^z$. The worksheets associated with these first two functions are located in Appendix B and Appendix C, respectively. I placed the function $f(z) = \frac{1}{z}$ after the first two in my interview plan because it was the only function I had prepared for which the derivative did not exist everywhere. My participants of the pilot study finished all prepared tasks before the 8-hour interview sequence concluded, so I included an additional task involving $f(z) = |z|$ to help participants investigate reasoning about a function with no derivative.

The overall goals of the tasks remained unchanged between the pilot study and dissertation study, though some improvements were made. First, due to a lack of rich findings and its tangential relationship to the amplitwist concept, I omitted the task $f(z) = |z|$ from my dissertation study's interview sequence entirely. Second, some wording previously contained in the Task 1 and Task 2 worksheets caused some confusion in all my pilot study participants. These slightly modified worksheets are included in Appendix B and C, respectively. In particular, the question "What do you think the output will look like if the input is a circle that contains the point $1 + i$?" led every student group in my pilot study to ask for clarification regarding whether I meant that the point must be on the circle itself, or within the region enclosed by the circle. Thus, for my dissertation study I changed some related wording on the Task 1 and Task 2 worksheets to minimize confusion. For example, I changed the aforementioned question to "What do you think the output will look like if the input is a circle where $1 + i$ is within the area enclosed by the circle?" All wording changes were directly related to this particular source of confusion.

As an additional result of findings from the pilot study interviews, I included a task involving complex-valued linear functions to help students develop geometric reasoning about the derivative of a complex-valued function. This final task involved an unknown complex-valued rational function and was included to help participants develop reasoning about the derivative as a local property of a complex-valued function. For this task I used the function $f(z) = \frac{2z+1}{(z+i)(1-z)}$, and its goals are more carefully discussed in the following section.

**Interview Structure**

I interviewed four students total, grouped into two pairs in accordance with the IRB, with each pair progressing through the same four-day interview sequence, though because of my participants' schedules, these four days were not consecutive for any participant pair. Each interview lasted approximately two hours. Although I had a planned schedule for each of the four days, each group did not necessarily complete all the same tasks on the same day, instead progressing at slightly different rates on different tasks. However, by the end of the four-day sequence both participant groups had completed all the same tasks in the same order, with the exception of the task involving the rational function, which only the second pair had time to fully address. Because of their different rates of progress through this itinerary, it became convenient to organize their interview around tasks rather than days to reduce the number of potentially conflating factors between the two groups. Thus, my following discussion of the interview structure is organized by task and describes how the groups of my dissertation study actually progressed through the interviews.

In the first task, students followed instructions on a lab worksheet to construct the function $f(z) = z^2$ with the aid of *Geometer's Sketchpad* (*GSP*) and predict how the function maps points, lines, circles, and the entire complex plane (see Appendix B). The intended goals of this task were for participants to use these investigations help students establish proficiency with *GSP* and determine the mapping behavior of a complex-valued function, particularly as it relates to circles. Both the objectives and the instructions for the second task were similar to the first task, except that I asked students to construct another function $f(z) = e^z$ (see Appendix C). Investigating another function allowed them to continue gaining proficiency with *GSP* as well as the opportunity to compare and contrast this new function with the behavior of the function of the previous task.

For Task 3 in the pilot study, I had only planned on allowing for free exploration with and without *GSP*. However, during this free exploration in my pilot study, I found that both my groups required an investigation of linear complex-valued functions to continue developing their reasoning about the derivative of a complex-valued function. Thus, for my dissertation study I purposefully prepared a linear complex-valued function for my participants to investigate during this task in my dissertation study. For this task in my dissertation study I prepared a linear complex-valued function with a complex-valued derivative. I chose to construct a function with a complex-valued derivative because I found from my pilot study that participants incorrectly generalized from functions such as $f(z) = 2z$ and $f(z) = iz$ with purely real or purely imaginary derivatives. In particular, such functions appeared to lead them to reason that the real part of the derivative is a dilation factor and the imaginary part of the derivative is a rotation factor. Planning for a linear function with a complex-valued derivative allowed me to keep some of the free form feel of the pilot study while also more meticulously preparing for potential reoccurrences of these same previously unexpected developments of the pilot study.

In my dissertation study, I administered Task 3 by removing access to *GSP* and asking participants to describe their current geometric reasoning about the derivative of a complex-valued function. After they suggested that the derivative gives the slope of the tangent line or provided other such reasoning, I suggested they apply their reasoning to the linear function I had previously prepared, namely $f(z) = (3 + 2i)z$.

I re-introduced *GSP* later at a point selected by me based on the nature of the participants' discussion. For example, if the participants appeared to have seriously considered several aspects of their reasoning, made conjectures about what might happen in various situations regarding some particular function, and wanted to test their ideas with the aid of *GSP*, I might reasonably have reintroduced *GSP* at this point to allow them to verify or overturn these hypotheses. Alternatively, if no new advancements in reasoning seemed forthcoming and participant responses degenerated into a circle of previously voiced reasoning patterns, I re-introduced *GSP* at this point to encourage novel thought. In the remainder of Task 3 after *GSP* was reintroduced, I intended to allow participants to test their previously developed conjectures and continue to explore their geometric reasoning about the derivative of a complex-valued function with the aid of *GSP*. The intended purpose of this task was to help participants develop geometric reasoning about the analog of a line in the case of complex-valued functions, and to relate the magnitude and argument of the derivative to the way the linear function dilates and rotates a circle.

In Task 4, I asked participants to generalize their geometric reasoning about the derivative of a complex-valued linear function in order to reason geometrically about the derivative of a general complex-valued function. To accommodate this goal, I suggested to participants that they use their previous *GSP* labs to investigate and test conjectures. In order to further substantiate the generality of the claims they produced, I suggested that they verify their geometric reasoning with the new function $f(z) = \frac{1}{z}$. Thus, in Task 4, participants primarily generalized and tested their reasoning with the aid of *GSP* via the functions $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$. The intended goal of this exercise was to support students' efforts to generalize the geometric reasoning they developed in Task 3 to the general case.

Note that for the pilot study, I initially utilized $f(z) = \frac{1}{z}$ as a standalone task in the style of Tasks 1 and 2. However, I found in the pilot study that students needed to investigate linear complex-valued functions before they could adequately reason geometrically about the derivative of a non-linear complex-valued function. Thus, while I kept $f(z) = z^2$ and $f(z) = e^z$ as the first two tasks in my dissertation study to introduce participants to *GSP* and expose them to a non-linear complex-valued function early in the interview sequence, I reserved $f(z) = \frac{1}{z}$ as a third non-linear function until after they had investigated a linear complex-valued function. After this investigation, I introduced $f(z) = \frac{1}{z}$ in order to allow them to generalize their reasoning about complex-valued linear functions to both familiar and unfamiliar complex-valued non-linear functions, as previously described.

I developed a new task, Task 5 for the dissertation study and replaced the pilot study task involving investigation of the non-differentiable function $f(z) = |z|$. I included this task specifically to address a common weakness in geometric reasoning displayed in the pilot study. In particular, my pilot study participants experienced difficulty reasoning geometrically about the derivative of a complex-valued function as a local property. Reasoning geometrically with linear complex-valued functions with the aid of *GSP* appeared to allow them to reason geometrically about the derivative of a complex-valued function as an amplitwist in part, but did not appear to help them grasp that this reasoning applies only to small circles in the general case. As such, all my participants of the pilot study experienced difficulty generalizing that the amplitwist related geometric reasoning to the case of non-linear complex-valued functions such as $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$. In order to meet this goal, I created a new task, which required participants to determine the value of a derivative at a particular point in a pre-constructed unknown rational transformation. The rational transformation I utilized for this task was $f(z) = \frac{(2z+1)}{(z+i)(1-z)}$.

Given findings from my pilot study, the intended goal of Task 5 in my dissertation study was to encourage students to develop reasoning about the derivative as a local property and to develop reasoning about points of non-differentiability as they relate to the amplitwist concept. Note that for the other tasks in the interview sequence, I provided a known complex-valued function and asked my students to construct and investigate it with the aid of *Geometer's Sketchpad* (*GSP*) in the hopes that they developed geometric reasoning about how the derivative of the function relates to the way the function maps circles. In this task, I asked students to undertake this process in the reverse direction. That is, prior to the interviews, I constructed the rational function $f(z) = \frac{(2z+1)}{(z+i)(1-z)}$ with the aid of *GSP*. I did not provide this function formula to my participants, though I did tell them the function I constructed is of the form $f(z) = \frac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are polynomials. I gave them access to *GSP* and asked them to use it to choose a point and determine the value of the derivative of the function I had constructed at that point. I additionally asked them to reconstruct the function formula. In other tasks I gave my participants access to an algebraic function formula and asked them to discover geometric information. In contrast, in this final task I provided my students with the means to discover geometric information on their own with the aid of *GSP* and asked them to use this information to determine algebraic information.

My hope was that this task would give them an opportunity to investigate non-differentiability in a complex-valued function as well as an opportunity to develop reasoning about the derivative of a complex-valued function as a local property in general. I replaced $f(z) = |z|$ due to the fact that $f(z) = |z|$ is real-valued and the fact that this function does not map any circle to another circle. My reasoning in creating the new task was that by asking them to determine both non-differentiable points and derivative values at particular points, they would naturally consider smaller circles as a means of focusing on these particular points. In turn, the consideration of these smaller circles might lead them to develop geometric reasoning about why small circles were necessary to complete such a task, and thus why the derivative necessarily describes a local property in the general case.

**Data Collection**

To collect data, I attended and video-recorded some of my participants' complex analysis classes prior to their interviews, in accordance with the IRB. I asked their professor which classes he believed would be relevant to my dissertation study on the derivative of complex-valued functions, and he suggested I attend from the beginning of the semester to the point where he finished his discussion of the derivative of a complex-valued function. During these classes, I first set up a camera at the back of the class to capture the professor, his board work, and student activity. I additionally attended these classes and took handwritten notes of the lecture material, gestures the professor employed, and student behavior. I utilized these notes and videos to provide the description of the setting earlier in this chapter, and to triangulate the reasoning students purportedly learned in class with the reasoning they displayed in the interview sequence I administered.

I additionally video-recorded all interviews with one camera placed in front of the participants and utilized screen-capture software to record the work the participants performed with the aid of *GSP*. As such, the camera captured the participants' gestures and boardwork, while the screen-capture software recorded the actions the participants carried out with the aid of *GSP*. This recording was approved by both the IRB and the participants I interviewed. I pointed the camera toward the computer and the participants while they worked on the computer, and I rotated the camera toward the chalkboard when they stood up to write something there. I collected all handwritten notes that participants created on paper, though only Zane and Christine produced such notes.

For both groups of students, I conducted four interviews, each of which lasted roughly two hours. For both pairs, I introduced them briefly to *Geometer's Sketchpad (GSP)*, and instructed them on how to build the transformation $z \rightarrow z^2$ (mathematically equivalent to the complex-valued function $f(z) = z^2$) with the aid of *GSP*. The task worksheet contained many of the intermediary questions I wanted the participants to answer, so throughout the first two tasks I primarily provided technical support with *GSP*. This support included helping them find buttons, walking them through the construction of a transformation, and teaching them how to construct and move mathematical objects such as points, vectors, and circles. As only one mouse was available to each pair of participants, I ensured that all students had an opportunity to control the mouse, and thus the precise actions taken with the aid of *GSP*.

I helped them frequently in these ways near the beginning of the interviews. However, their need for my assistance diminished as the interviews continued, as they began establishing proficiency with *GSP* and trading control of the mouse without prompting. I also probed the students to say more about the reasoning they offered about mathematical concepts such as the derivative of a complex-valued function or the geometric behavior of a particular function, to clarify something they had just said, or to justify their reasoning and explanations. I additionally sometimes asked one of the participants to explain what they thought the other participant was saying or to re-voice the other participant's explanations.

**Data Analysis**

My analysis began by watching the videos in conjunction with the screen-captured *GSP* recordings to determine which segments of the interview provided the most relevant content related to my research purpose. In particular, I looked for places where the participants appeared to be making progress toward a conception of the derivative of a complex-valued function as a local linearization. This included times when the participants discussed prerequisite or related ideas, which include, but are not necessarily limited to:

1. The behavior of a given function (e.g., how points, lines, or circles are transformed)
2. $\epsilon -$neighborhoods around a given point
3. Local vs. global properties
4. The relationship between magnitude and dilation
5. The relationship between argument and rotation
6. The meaning of "linearization" or "linear" in the complex plane
7. Conformality (circles are mapped to circles)
8. Approximate conformality

I developed the tasks with these developments in reasoning in mind. Therefore, I looked particularly for the ideas listed above. For example, I particularly consider the first three items on this list to be important prerequisite concepts for a complete development of reasoning geometrically about the derivative as an amplitwist. The first few questions of each task are questions about function behavior, followed by questions about how circles of various sizes around a particular point in the domain were mapped into the image. Familiarity with a particular function's behavior might help participants more easily connect their reasoning about its derivative to the function.

As the derivative is often described as a local linearization, the behavior of $\epsilon-$neighborhoods around particular points is essential to the development of reasoning geometrically about the derivative. Items 4, 5, and 7 are exactly the ideas I wanted my participants to develop as part of the four-day interview sequence. These items detail the particular characterization of the derivative of a complex-valued function towards which I guided my participants. That is, an $\epsilon-$neighborhood around a point $z$ in the domain is mapped into the co-domain as follows:

1. The associated image of the $\epsilon-$neighborhood is approximately a circle (item 7).
2. The function rotates the image by the argument of the derivative of the function at $z$ (item 5).
3. The function dilates the image by the magnitude of the derivative of the function at $z$ (item 4).

I did not explicitly plan for items 6 or 8 in the pilot study, but they seemed to arise naturally during the progression of all participants through the interview tasks. Thus, I purposefully prepared for these ideas to arise once again in the dissertation study.

For this iteration of the design experiment, I transcribed all recorded gesture, speech, and usage of inscriptions. These documented inscriptions include the drawings and equations students created on paper with a pencil, the drawings and equations students created on a chalkboard, and the constructions created with the aid of *Geometer's Sketchpad* (*GSP*). For this research, I was particularly interested in segments where participants described the way in which they reasoned about the derivative of the complex-valued function, segments where participants struggled with some related conceptual difficulty, segments where participants appeared to resolve a previous difficulty, and segments where participants suggested a new idea for the first time. After completing the transcription, I imported the *ELAN* file into an Excel spreadsheet for coding. I coded lines as algebraic based on speech and concurrent gesture if the speaking participant appeared to be reasoning about formulas or other formal symbolic representations of mathematical concepts.

If a participant referred to the real part and imaginary part of a complex number while appearing to point successively at two pieces of an imagined complex number in addition to speaking the words “real” and “imaginary,” I coded this speech as algebraic. I coded the same speech as geometric if the participant said the same phrase while moving his or her hand horizontally (along the real axis) and then vertically (along the imaginary axis) or vice versa, in conjunction with speaking the words “real” and “imaginary,” or making these gestures while speaking these words. I coded lines as geometric based on speech and concurrent gesture if the speaking participant appeared to be reasoning about graphs, shapes, spatial transformations, or other similar abstract entities that a learner could potentially think of as hypothetically existing within some real space. (See Table 1 for an example of this analysis.)

Table 1
*Coding Examples*

| Verbiage | Alg/ Geo | Gestures | Technology | Task Progress |
|---|---|---|---|---|
| Z: So when you get closer, keeps on curving, then bends in on itself, and it's still never technically inside. | Geo | Points at screen with right index finger, traces screen in large clockwise loop. | | |
| Z: Cause this function is doing the $z^2$ of the $z^2$, right? Don't we have the first function to find the circle $z$ to $z^2$ and now we're doing $z$ to $z^2$ again from that other point | Alg | No produced gesture | | |
| M: I don't know, that's what, so $z$, this one we want to look at just $z$, and then I want to map $2z$ | | While moving arm right, right index finger flicks to the left, then twists clockwise as hand transforms to a C shape, with index finger held above and thumb below, pointing at 1st quadrant of right graph. Moves index finger up and down as hand drifts left  | | |
| M: So the $i$ is what's doing the rotating. Which makes sense cause if we had 2 $i$'s multiplied together that should rotate it 90 degrees, I mean rotate it 180 degrees, which is just the same as multiplying by negative 1 | Alg | Bounces right hand inward twice. Left index finger rotates in counterclockwise horizontal circle around right hand. Rotates left index finger counterclockwise again pointing down, and bounces right hand to the right then the left. Moves right hand in clockwise vertical arc | | Still believes the $i$ is what is doing the rotating. Understands that multiplying by -1 corresponds to a rotation of 180 degrees |
| E: Yeah, it goes that way because the imaginary axis is the rotation, and so that's why it, it wraps around, because 2 pi would be a full rotation | | Points at screen | Spirals mouse counterclockwise around origin up to top of blue circle, then moves mouse down between origin and output twist | Explains rotation with imaginary axis |

*Note.* Alg stands for algebraic, and Geo stands for geometric.

Usage of gestures or technology could help clarify the nature of the participant's reasoning, as described in my theoretical framework. There were several lines I did not code as either algebraic or geometric due to a lack of compelling evidence in either direction for the nature of participants' reasoning. For example, if a participant referred to the real and imaginary parts of a complex number, it is possible that they could have been referring either to the $x-$ and $y-$ values attached to the Cartesian form $x + iy$ or to the horizontal and vertical components of a graphical representation of a complex number. In some cases, gestures could be used to clarify the nature of participants' reasoning, as described earlier in this chapter. However, in the absence of gesture this ambiguity between algebraic and geometric reasoning remained.

In addition to categorizing exchanges as algebraic or geometric, I looked particularly for places where the participants appeared to be making progress toward an idea of a derivative as a local linearization. This aspect of coding was similar to my prior description of particularly relevant segments. In this case, however, I additionally looked for ideas that specifically seemed to help the participants overcome previous misconceptions or appeared to lead them toward a completely developed ability to reason geometrically about the derivative as a local linearization. I described these events briefly in a column on an Excel spreadsheet labeled "Task Progress." Finally, I wrote remarks for any miscellaneous comments the participants made or things they did that appeared to be interesting. For example, I often commented on whether I felt the participants' stated reasoning was correct, and if not, I documented possible reasons for why the participant might have utilized their specific form of incorrect reasoning. Once the data were coded, I wrote summaries of each day for each set of participants. In each summary, I detailed the events that had been selected as significant in the previous stage and any conversations that either appeared to lead up to these events or were somehow prerequisites to understanding the concepts contained in the selected events.

Finally, I grouped my summaries by task and looked for common occurrences both within a single group's interviews as well as similarities and differences between the two groups' interviews. That is, I performed both a cross-case and within-case (Merriam, 2009; Patton, 2001) analysis on the written summaries, referencing both the Excel spreadsheet and the actual raw data to obtain supporting evidence as necessary. For the cross-case analysis, I looked for aspects of progress toward reasoning about the derivative as a local linearization common to both interviewed groups, as well as aspects unique to a certain group or participant. For the within-case analyses, I looked for ideas that appeared essential in the participants' progress, particularly those ideas which recurred within a group or were actually a conceptual part of reasoning about the derivative as a local linearization as described above. I also considered advancements in reasoning essential if they appeared to highlight or resolve a previous difficulty in reasoning. I summarize these developments in Chapter IV, where I discuss the nature of the development of geometric reasoning about the derivative as a local linearization for each set of participants.

# CHAPTER IV

# FINDINGS

In this chapter, I detail the progress of my four participants' reasoning through a sequence of tasks modified as a result of previous analyses as part of my pilot study to address the research questions.

Q1 What is the nature of students' reasoning about the derivative of complex-valued functions?

Q2 What is the nature of the development of students' reasoning about the derivative of complex-valued functions while utilizing *Geometer's Sketchpad* (*GSP*)?

The results of analyses of data are grouped by task and analyzed across both pairs of participants. Some basic observations throughout this chapter are made regarding the participants' usage of speech, gesture, and *Geometer's Sketchpad* (*GSP*). The reader will note that to answer the questions posed in the task worksheets (see Appendix B and Appendix C) and the questions I asked, Melody and Edward utilized geometric reasoning and gesture more than Zane and Christine, who appeared to prefer algebraic reasoning and speech. That is, I coded a large amount of Edward's and Melody's reasoning as geometric, while I coded much of Christine and Zane's reasoning as algebraic. This coding process was described in Chapter III. Furthermore, Melody and Edward produced more gestures than Zane and Christine. While this chapter details my observations of the participants' behaviors throughout the tasks, a comparative analysis with a summary of recurring themes throughout the interview can be found in Chapter V.

Each pair of participants began by exploring the behavior of the functions $f(z) = z^2$ and $f(z) = e^z$. Each groups' work with the first function is reported under Task 1 below, and their work with the second is reported under Task 2. In Task 3, I removed *GSP* and asked the groups to describe their reasoning about the derivative of the complex-valued function. At this time I asked probing questions to clarify the participants' responses, such as directing them to describe what the derivative of $f(z) = z^2$ tells them about the function near the point $1 + i$. Later, in this same task I provided access to *GSP* and asked them again to describe their reasoning about the derivative, and again asked similar probing questions. At some point during this task, the groups decided they could not reason geometrically about a line in the complex plane, so I introduced linear functions organically as they arose in the participants' discussion. Once the groups decided they could reason about a constant derivative, I asked them to generalize their reasoning to the non-linear functions $f(z) = z^2$, $f(z) = e^z$ and $f(z) = \frac{1}{z}$. I report these results under Task 4. Finally, I asked one of the groups to determine where an unknown rational function is differentiable, to estimate the derivative at a point of their choosing, and to construct an algebraic inscription for this rational function. Progression through this task is detailed under Task 5, which completed the interview sequence.

## Task 1: Investigating $\mathbf{f(z) = z^2}$

### Zane and Christine

While Christine and Zane primarily utilized reasoning coded as algebraic, they used reasoning coded as geometric as well (see Table 2). At the beginning of Task 1, I helped Zane and Christine construct the function $f(z) = z^2$ with the aid of *GSP*. In particular, I showed them where to find the menus referenced by the task worksheet (see

Appendix B). This included helping them find the calculator, the button used to define a new transform, and the button used to construct a point based on algebraically calculated coordinates.

Table 2
*Task 1 Codes for Zane and Christine*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 103 | C: Okay so we had $a$ plus $ib$ and we should have squared the whole thing, but we didn't, we squared | x | | | writes on paper (calculates $(a+ib)^2$) |
| 216 | Z: Well, you're squaring it, so | x | | | |
| 217 | Z: That means there's two | x | | | |
| 218 | C: So if we cubed it would go around three times? | x | | | |
| 396 | C: Does it move the circle, or does it deform it? Probably deforms it a little bit | | | | |
| 407 | C: Sucked in, looping in some way | | x | hitchhiker thumb on left hand, pointing left | |
| 425 | Z: Okay so it''s kind of like the whole, when we did the first one around the unit circle it looped around twice | | x | points at screen while moving hand in a clockwise circle. points at screen while moving hand in two small circular motions | |

*Note.* Alg stands for algebraic and Geo stands for geometric.

I additionally reminded them to switch between mouse tools during construction of $f(z) = z^2$. This included reminding them to click the arrow tool before trying to drag a circle around the graph, or reminding them to click the circle construction tool to construct a circle. Furthermore, each time Zane and Christine opened a new *GSP* file to start a new task (see Appendix B and Appendix C), I requested that they construct a green unit circle around the origin for reference. I let Christine and Zane learn how to use *GSP* on their own primarily as they followed the worksheets (see Appendix B and Appendix C) but provided assistance if they seemed to become frustrated. This mostly involved reminding them to switch from the construct circle tool to the arrow tool when they had difficulty dragging their circle around the plane. I also occasionally reminded them to unselect certain objects. For example, a circle cannot be dragged in the *GSP* environment unless only the circle is selected, so sometimes I showed my participants that they had more selected than they realized. As I provided this kind of help, they constructed their first *GSP* function: $f(z) = z^2$.

After constructing this function, Christine dragged $z$ around the plane and Zane noted the point labeled $z^2$ moved as a result. Afterward, Christine read the first warm-up question: "Where do you expect the point labeled $z^2$ to go if you put $z$ on $1 + i$?" Zane and Christine answered this question by independently calculating that $(1 + i)^2 = (1 + i)(1 + i) = 1 + 2i - 1 = 2i$. This and the following discussion was coded as algebraic due to their apparent reliance on symbolic calculation to answer the given questions. However, when they moved the point $z$ to $1 + i$, they noted that the point $z^2$ was also at $1 + i$, which led Christine to inspect her reasoning more closely: "Okay, it didn't go anywhere. Did I do my math wrong?" While I pointed out that the function's real and imaginary parts were $x^2$ and $y^2$, respectively, Christine and Zane calculated the correct real and imaginary parts themselves by calculating algebraically that $(a + bi)^2 = a^2 - b^2 + i2ab$, though not without some apparent recollection of a similar calculation:

> Christine: So what we want is a squared plus, isn't that minus cause of the i squared or does that go away? It's a minus.
>
> Zane: plus 2iab.
>
> Christine: That's weird, cause I could swear that in Algebra 2 it was something else. So it's a$^2$ plus 2iab plus b$^2$.

I coded this reasoning as algebraic due to the fact that Christine and Zane were currently engaged in calculating $(a + bi)^2$ on paper through symbolic manipulation. When Christine asked me whether this is correct, I told her it was and she responded, "Okay, so ignore whatever we learned in Algebra 2." This algebraic reasoning through $(a + bi)^2$ appeared to remind her of a previous class discussion, but somehow also seemed to persuade her that what she learned in Algebra 2 was incorrect. It is not clear whether she decided that her prior reasoning was completely incorrect or merely did not apply in this current particular context. Regardless, once the participants algebraically calculated the real and imaginary parts they successfully reconstructed the function $f(z) = z^2$ with relatively little guidance.

For the second question, "Where should you put $z$ to send $z^2$ to $i$, Christine attempted to solve for $a$ and $b$ in the equation $a^2 - b^2 + 2iab = 0 + i$. As before, this and the following calculations were coded as algebraic for similar reasons. Christine noted, "We should let the real part be $0$, and the other part should be $1$." Thus, they broke this equation into real and imaginary parts, from which they obtained the two equations $b = \frac{1}{2a}$ and $a^2 = b^2$. Zane started trying to reason about fractions algebraically, which seemed to allow Christine to solve this system of equations:

> Christine: Square root of one over square root of 2 which is the square root of 2 over 2 which is on our unit circle I think like, is that one ninety?
>
> Zane: That one's $45$.
>
> Christine: Yeah, that's what I meant. Half of $90$.

So, once they found $a = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$, Christine and Zane identified the correct geometric point as the point on the unit circle rotated $45°$ counterclockwise from the positive $x$ −axis.

The next question they addressed utilized the green unit circle they had just constructed with the aid of *Geometer's Sketchpad* (*GSP*). In particular, the question read "What do you think happens to $z^2$ when you move around the green [unit] circle once? Test your theory." Christine and Zane's initial predictions were that $z$ would move "back and forth through the circle", or "to the origin and back", though they did not explicitly clarify what they meant by this. Christine waved her index finger up and down through the center of the unit circle as she spoke "back and forth," so perhaps she believed the output point would move up and down along the $y$-axis within the unit circle. However, they discovered through *GSP* that in reality the point $z^2$ moves around the green circle twice. The following discussion was coded as primarily geometric, due to the fact that in this discussion Christine and Zane referred to the movement of a circle constructed with the aid of *GSP* through various points on the graph.

> Christine: Do you think it just bounces back and forth?
>
> Zane:…Wait are you talking about, like through the circle (moves finger up), or around it (drops arm)?
>
> Christine: Like, through it, if this was our circle it would just go like 'tchu, tchu, tchu, tchu.'(moves finger back in forth in front of screen) Well okay, not that many times, but, no wait, hold on, cause if it starts here (point $(1,0)$) while we're here (point $(1,0)$) and we move over here (point $(-1,0)$, it should end up back here (point $(1,0)$), right?...Do you know what I'm saying though? Move our little z to here (point $(0,1)$). It should be like at the origin.
>
> Zane:…Wait why is it supposed to go to the origin when it's up top?

As seen in the above exchange, Christine did not voice any particular reasoning for why the image point should go through the circle, or to the origin. As such, Zane questioned Christine's reasoning about why the image point should go to the origin at all. Christine and Zane briefly attempted to reason algebraically but did not advance another hypothesis. At this point Zane suggested, "If we say it moves around the circle we can just do it." Despite the lack of reasoning about why any of Zane's or Christine's predictions might make sense, Zane uttered a nearly correct theory, which omitted only the fact that the image point should travel *twice* around the unit circle. Perhaps he realized their reasoning had stopped progressing and he may have been anxious to see the true behavior of the image point. Having suggested a theory, Zane proceeded to experiment with the aid of *GSP* by dragging the point $z$ around the circle, and observed, "So there it goes around once (at point$(1,0)$), wait because yeah, it looped around twice, didn't it?" Christine did not seem to notice this fact until Zane pointed it out, after which she commented, "Did it? Yeah, you're right!"

Because neither Zane nor Christine voiced much reasoning before their experimentation, I asked them to explain why the output point moved twice around the circle.

> Interviewer: Okay tell me why you think that happened?
>
> Christine: Because Zane moved his mouse?
>
> Zane: Why did it move around twice? Well, you're squaring it, so that means there's two.
>
> Christine: So if we cubed it, it would go around three times?
>
> Zane:…Coordinates grow faster and they shrink faster, was it the specific (trails off)?
>
> Christine: It made it halfway.
>
> Zane: Yeah, when it turned exactly once it's halfway. Okay so, so, so as we're moving around. Trying to think of the rate of change and that kind of stuff.

This exchange suggests that their geometric observation that an image point moves around the input circle twice for every time the corresponding pre-image point allowed them to connect to the algebraic inscription $f(z) = z^2$. However, for Zane and Christine, who told me later that they preferred to use algebra over geometry, it almost seemed enough to say that the image point moves around the origin twice because the function is $z^2$, which has an exponent of 2 in it. Christine correctly extended this reasoning to $z^3$, whose image point really would move three times around the unit circle if the pre-image point moved around the unit circle once. That is, while they made a pertinent geometric observation, they did not really appear to connect this observation to any sort of geometric reasoning, preferring instead to connect it to algebraic reasoning about the formula $f(z) = z^2$. Still, without the geometric observation they made with the aid of *GSP*, they may not have made this particular connection to the algebraic formulae. This question also appeared to motivate Zane to start reasoning about rates of change as he reasoned about the rate at which the coordinates increase or decrease as the pre-image point moves. So, the participants' reasoning here may suggest that these questions about function behavior really might help prime students' reasoning about the derivative later, which describes a rate of change.

After answering these questions about point behavior, they began constructing and transforming circles. They colored the input circle blue and the output circle red, which they referred to as pink. Zane, who was manipulating *GSP* at this point, deferred to Christine for guidance:

> Zane: Okay, what are we supposed to do, move the circle?
>
> Christine: And watch how it moves the pink one.
>
> Zane: Don't you think it's interesting how that part is flat?
>
> Christine: Yes, it's not fully a circle.

The first thing they noticed while observing circles with the aid of *GSP* is that the image circle is distorted somewhat and is "not fully a circle" (see Figure 1). This seemed like an important characterization for Christine to make, given that the derivative of a complex-valued function is an approximation of how the function rotates and dilates a given circle. Since the derivative is an approximation, even for small circles, the image of a circle transformed by a non-linear function is also "not fully a circle," but is nevertheless a circle-like closed curve.

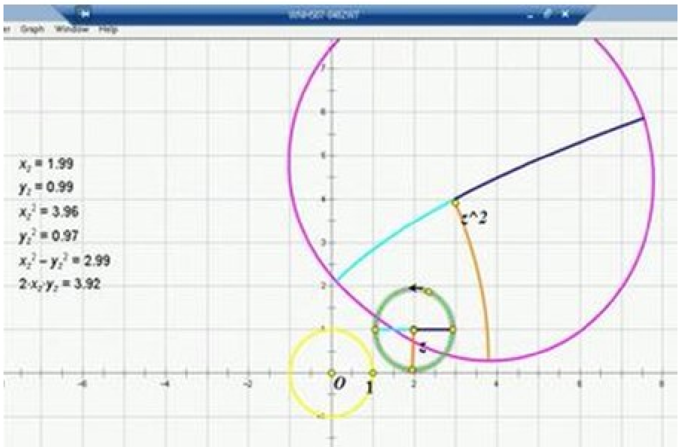


Figure 1. Zane and Christine observe a "flattened" large pink output circle with the aid of *Geometer's Sketchpad*

In another discussion that was coded as geometric, Zane made another similarly important observation while answering the next question, "What do you think the output will look like if the input is a circle where $1 + i$ is in the area enclosed by the circle?" In particular, he observed an instance in which $f(z)$ mapped a circle to a non-circular curve. The following reasoning was coded as geometric due to the fact that they referred to such visual descriptions as "curly" and "loop," as well as the idea that zero is inside a circle.

> Zane: Doesn't that make the, that guy (the image of the input circle) get all curly, in on itself?
>
> Christine: Creates like a loop.
>
> Zane: One point one is within the circle, okay, yeah, makes it non-circular curvy. Let's see.
>
> Christine: It looks like it just doesn't want zero, zero to be in it.

As before, Zane and Christine did not really connect their geometric observations with their geometric reasoning, but these observations are nonetheless both correct and pertinent. Just as Christine noticed before that the image of a circle was "not fully a circle," in the previous exchange Zane pointed out that the image of the input circle became a "non-circular" kind of curve (see Figure 2).

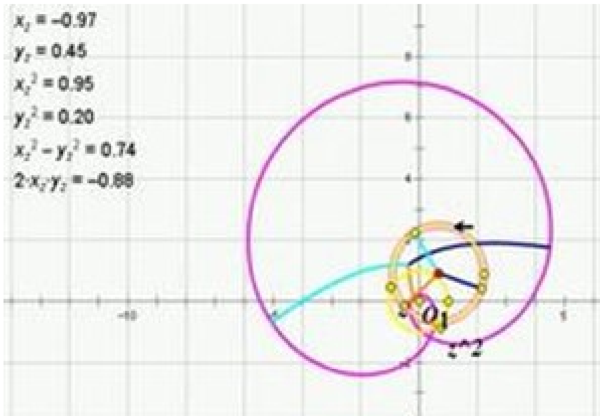


Figure 2. Pink output curve is curvy in a "non-circular" way

Christine followed up this comment by correctly noting that the output curve twists because "it looks like it just doesn't want zero to be in it." This development in reasoning was coded as geometric due to visual ideas such as the output curve twisting and that it did not "want zero to be in it." After this development, Zane and Christine read the following question: "What happens when 0 is in the area enclosed by the circle?" Zane answered this question with geometric reasoning by stating, "Also the input has zero, zero in it, and that's when it starts looping." This reasoning was coded as geometric due to Zane's reference to "looping." Thus, by this point, Zane had correctly identified the origin as the point which causes the output curve to twist on itself, though he still offered no explanation about why this behavior occurred.

The next question required Zane and Christine to change the radius of their input circle and observe what happened to the output. After about one minute of experimentation in *Geometer's Sketchpad* (*GSP*), Zane restated that when the input circle was small, the output was circular, and as the input circle grew, the output circle "runs around the origin.” So, this question did not appear to introduce any new ways of reasoning for Zane and Christine, though it may have clarified their prior developed reasoning and cemented the fact that the origin is what causes the output circle to twist on itself. The following question, which asked them to center their circle on the complex number 2, may have helped them in a similar way. The following reasoning was coded as geometric, due to verbiage such as “oblong” and “loops around.”

Zane: So we start small.

Christine: Just a regular old circle.

Zane: Well, it’s a little oblong I think. Grows until the origin, then loops around.

The additional questions about what happens when particular points are in the area enclosed by the input circle elicited similar responses. One such question was “What do you think the output will look like if $1 + i$ and 2 are both in the area enclosed by the input circle?” Christine only responded with “I feel like it will mostly do what it did when $1 + i$ was in there.” When centering the circle on the origin, they observed some measurement error in the input as a result of what the output looks like.

Zane: Okay, so it’s still doubly looped….That’s not truly at the origin.

Christine: It probably crosses, yeah, crosses at one point.

So, Christine and Zane noticed that the input circle was not perfectly centered because the output was not perfectly double looped. Christine felt that the output probably crosses only at one single point (see Figure 3). Thus, while they did not explicitly state that a circle centered at the origin should have an image that looks like two circles perfectly superimposed, they seemed to have at least some sense of what the image should have looked like to notice the measurement error.

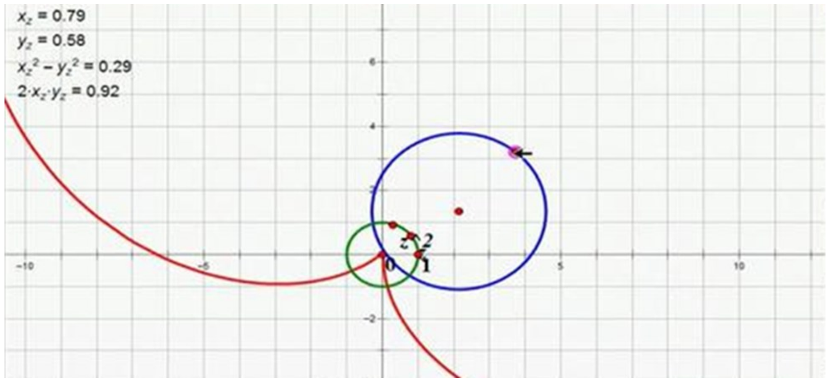


Figure 3. Red output curve under $f(z) = z^2$ touches origin at a single point if input circle intersects the origin.

When I asked Zane and Christine about why the image looked the way it did, some incorrect algebraic reasoning surfaced about what the participants assumed *GSP* was really calculating. The following reasoning was coded predominantly as algebraic due to the participants' repeated references to algebraic formulae.

> Interviewer: Why do you think overall it looks like that?
>
> Christine: I am not sure because the origin is special.
>
> Zane: Because this function is doing the z squared of the z squared, right? Because don't we have the first function to find the circle z to z squared and now we're doing z to z squared again from that other point.
>
> Christine: Like if you brought that radius to a two would that go down to four? Yeah. So it's like squaring this (input circle), then squaring that.
>
> Interviewer: It's putting each point on the circle through the transform z to z squared.
>
> Zane: Well it's this function to this function, so my circle is at the origin. Is that because there's only one, what's it called? (Extends index fingers, moves hands out and in laterally, touching tips of index fingers, holds fists together and extends and retracts index fingers together.) Singularity type thing. Kind of like the whole, when you're doing square roots, you have the whole plus or minus to, er, what's the b squared minus 4 ac?
>
> Interviewer: That's the discriminant.
>
> Zane: That term, yeah, it's got a name. Because then there's only one discriminant as opposed to the two, because you're subtracting zero.

Christine and Zane seemed to reason that *GSP* squared the points on the input circle twice before it plotted the output circle. It is not clear what caused this error in reasoning about *GSP*'s behavior. I responded to this discovery by pointing out that *GSP* calculated the output by squaring each point on the input circle once only. Zane responded by reasoning algebraically that the output curve touches the origin but does not loop around it (see Figure 3) because the discriminant is zero, though he conflated the terms "discriminant" and "root" when he talked about "only one discriminant as opposed to the two". This exploration of changing the radius of the input circle with the aid of *GSP* appeared to help Christine and Zane develop geometric reasoning about the role the origin plays in transforming circles under $f(z) = z^2$. In particular, they discovered that when the input circle contains the origin, the output twists around it, and when the input circle touches the origin, the output curve has a sharp point that also touches the origin (see Figure 3).

After changing the radius at various points, they read from the worksheet I provided them (see Appendix B) to predict what would happen if they moved the input circle along the real and imaginary axes. Christine wondered whether moving the input circle along the real axis would cause the output circle to move, or to become deformed. She decided that this action "probably deforms it a little bit." As Christine referred to geometric properties of a shape, this reasoning was coded as geometric.

While experimenting with the aid of *GSP*, Christine and Zane employed further reasoning coded as geometric for similar reasons. They observed that moving the input circle along the real axis changed the radius of the output circle, and they claimed that moving the input circle along the imaginary axis "sucks the circle in" and that the circle is "looping a little bit." While the first observation that the input circle's distance away from the origin changes the output curve's radius is absolutely true, the second characterization of the output circle "looping a little bit" and getting "sucked in" seems odd. If the input circle is really centered on the imaginary axis, the output circle should be centered somewhere on the real axis, not spiraling toward the unit circle. The output circle should only look like it is getting "sucked in" while the input circle is being moved toward the origin. Perhaps this is exactly what they noticed, though associating this behavior with the input circle being moved anywhere along the axes seems strange because getting "sucked in" has more to do with the direction the circle moves than the locations through which it moves.

During this experimentation with the aid of *Geometer's Sketchpad* (*GSP*), Christine and Zane also started to address the question about how the function transforms the plane by describing how the function maps the quadrants. In particular, Zane observed that one quadrant in the pre-image is mapped to two quadrants in the image and noted, "it's kind of like the whole, when we did the first one around the unit circle it looped around twice." This reasoning was coded as geometric due to Zane's references to the unit circle and looping, but the following reasoning was coded as algebraic due to Zane's references to characteristics of an algebraic inscription. In particular, he attempted to explain this behavior algebraically by stating that one quadrant maps to two because $(x + iy)^2$ has a 2 in the exponent, which suggests doubling something.

Zane may not have felt completely satisfied with this reasoning because he soon after attempted to justify this behavior by reasoning geometrically instead of reasoning algebraically. In this attempt, he stated that the output circle becomes more deformed as it gets closer to the origin. This reasoning was coded as geometric because Zane referred to geometric properties like how “deformed” the circle became. Christine retorted that it actually starts getting deformed *before* it touches the origin. As these were pertinent observations toward developing reasoning about the derivative as a local property, I suggested that Christine and Zane construct and transform spokes to help the students get a sense of how exactly the function $f(z) = z^2$ deforms the transformed circle as the input circle approaches the origin.

Zane and Christine followed directions I provided for building the spokes. These directions resembled those found in the following lab for $f(z) = e^z$ (see Appendix C), which also were intended to highlight the rotation and deformations of output curves. The construction of these spokes with the aid of *GSP* appeared to remind them of dilations and translations.

> Christine: Isn't there like a stretch factor and some other factor?
>
> Zane: Geez, what were all the things? There was stretch, there was.
>
> Christine: Wait, so like a stretch factor would take, if this was our origin, and this was our vertex, like a stretch factor of two would stretch it another one that wasn't just stretching, if it was just moving would move it, um, and there was like negating. So like if we combined all three of those we could get like other things.

As Christine referred to the geometric idea of “stretching” a vector, this exchange was coded as geometric. Christine and Zane added a third transformation of “negating” to their first two transformations “stretching” and “moving,” though she never referred to “negating” again, opting instead to talk about “reflection” in addition to dilations and translations. Rotation is notably absent from this list of transformations. It is possible Christine was cognizant of rotations as a potential linear transformation but had not yet connected her knowledge of such linear transformations to the field of complex numbers.

Thus, at this point she seemed to reason only about dilation, translation, and reflection. It is also possible that Christine was trying to recall the complex analysis discussion of linear fractional transformations, wherein her professor had listed the transformations as dilation, translation, and reciprocation. It is additionally possible that she may have recalled reciprocation first as “negation,” and later as “reflection.”

After this episode, Zane and Christine turned to the question of how the function transforms the plane. Zane tried to answer this question by breaking his investigations into cases—one case for each quadrant. Meanwhile, Christine calculated $(a + bi)^2$ on paper. Both participants’ methods of reasoning during this time were coded as algebraic due to the fact that they appeared to calculate $(a + bi)^2$ via symbolic manipulation. During these investigations, one of Zane’s previous errors returned, and Christine duplicated it with her algebraic calculations by calculating $((a + bi)^2)^2$ while Zane reasoned aloud.

Interviewer: Why are you looking at z squared squared?

Zane: Because that’s what we were supposed to do.

Christine: Is that what the pink thing is?

Zane: That’s what the pink circle is, yeah.

I again corrected this error, which manifested in both my participants’ algebraic reasoning and their geometric reasoning. In particular, Christine algebraically calculated $((a + bi)^2)^2$ as Zane geometrically reasoned that the output curve displayed by *GSP* is the resulting transformation of the input circle under the mapping $a + bi \to ((a + bi)^2)^2$. Thus, Zane’s flaw in geometric reasoning corresponded exactly to Christine’s erroneous algebraic reasoning. This contrasts with previous research (Soto-Johnson & Troup, 2014) which describes examples of a mismatch between algebraic and geometric reasoning when errors occur. Once Christine identified the flaw, she revoiced the correct description of how *GSP* mapped the circles.

> Christine: Okay, so the blue is equal, not equal but blue would be our z, and the pink would be our z squared.
>
> Interviewer: That's right
>
> Christine: So I did that (her $((a + bi)^2)^2$ calculation) for nothing….Can we play with z?
>
> Interviewer: Absolutely.

Thus, once Christine realized that her previous reasoning was fruitless, she turned once again to experimentation with the aid of *GSP*. Her previous reasoning was coded as algebraic due to her symbolic manipulation, and when she shifted to *GSP*, her reasoning was coded as geometric due to her focus on the properties of shapes displayed with the aid of *GSP*. Now that Zane and Christine developed their reasoning about how *GSP* maps functions, I again suggested they construct spokes. After constructing these spokes, Zane and Christine started discussing whether the transformation they observed was a rotation or a reflection (see Figure 4). After this experimentation and observation with the aid of *GSP*, they eventually decided that the transformation was a rotation and not a reflection. The following exchange produced codes for geometric reasoning.

> Christine: It like, flipped (starts with left hand below right hand, rotates hands around each other with left hand in front of right hand, ends with right hand below left hand)
>
> Zane: Rotates it (right hand's fingers start pointing down, rotates hand clockwise until thumb points up)
>
> Christine: Did it rotate, or did it flip?
>
> Zane: well, like, rotates 180 degrees (repeats previous rotation gesture) because orange was down and to the left and now it's up and to the right.
>
> Interviewer: Did you mean is it a rotation or reflection?

Zane: That's a good question. Well see these two are curving, and if it was just a reflection, then (trails off). I had some logic for that a moment ago.

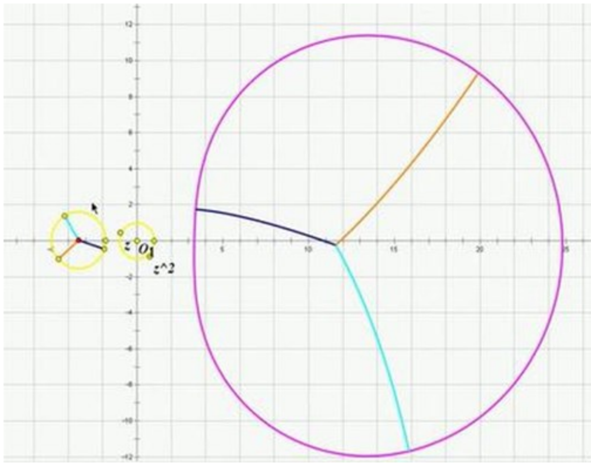


Figure 4. Pink output curve transformed from yellow input curve under $f(z) = z^2$

Initially Christine appeared to reason about the image as a reflection, while Zane stated the transformation was a rotation (see Figure 4). Note also that Christine caught the discrepancy between their reasoning where she suggested a reflection and Zane suggested a rotation. Christine may have noticed this due to the rotation gesture he produced, which was distinctly different from her own produced reflection gesture. However, when I repeated this question about whether the transformation was a reflection or a rotation, Zane tried to justify his reasoning in a way coded as geometric, but admitted he could not reason about now as he believed he had "a moment ago" in the past.

Christine: if it were a rotation it would be, flip (clicks tongue twice) (right arm extends, middle three fingers curl into a fist, thumb and pinky extend left and right respectively) Yeah, does that make sense?

Zane: Yeah, the clicking helps

Christine: Yeah, I think when it goes to a loop that's when it's like coming, cause remember how orientation gets like flipped when it goes into the loop?

For the first time during the task, Christine used reasoning coded as geometric to attempt to explain the difference between a rotation and a reflection. In particular, she incorrectly claimed that the orientation gets “flipped” when the output circle twists on itself. In contrast, she felt the orientation was not flipped when they dragged the circle around the origin instead of through it. Thus, a little later in this same discussion, Christine seemed to reason that whether this transformation was a rotation or reflection depended on the way they moved the input circle. The following reasoning was coded as geometric due to Christine’s references to the various paths through which they dragged their input circle.

> Zane: What were you saying before, reflection and rotation?
>
> Christine: It went around, it was like a rotation, but when it went through it was like a reflection

However, Christine had trouble justifying this reasoning, and when I asked her about it explicitly she seemed to decide that it really was incorrect. The following reasoning was coded as geometric as Christine continued to refer to potential paths through which she could move the input circle.

> Christine: If I start...here, the blue's on the right, and if I go like straight through the origin, and then I curve around, the origin, it’s still on the right. I forgot where I was going with this. Nope, totally lost me. Oh! When I go through…which one? Is it not the same thing? I don’t know. Math is hard.
>
> Interviewer: So you think, where, what the output is path-dependent?
>
> Christine: No, cause that didn't make any sense, and then I tested it and it definitely wasn't. Cause I mean it should just be like point-dependent. So that means if it's a reflection, it's a reflection, if it's a rotation, it's a rotation, but it's not both.
>
> Zane: So you have light blue, dark blue, orange on bottom, you go there, go there…. So if it was just flipped, then the blues would be on the same side.

After Christine realized her reasoning was incorrect, Zane correctly explained via geometric reasoning that the input circle was transformed via a rotation and not a reflection. As Zane referred to geometric properties such as the orientation of the circle to determine whether the transformation was a reflection or a rotation, his reasoning was coded as geometric. Zane's argument was essentially the same as Christine's previous geometric reasoning. Particularly, the orientation of the circle did not change under the transformation. In contrast to Christine's approach, he actually checked that the origin had not changed by checking where each spoke mapped on the output circle.

Despite Zane and Christine both correctly describing this transformation as a rotation, Christine nonetheless concluded this conversation by stating that the output circle is "just getting reflected and stretched." At the end of this first task, she also stated that the output circle "gets really weird near the origin," and Zane voiced some interest in why the circle becomes distorted before it touches the origin. Due to such geometric properties as distortion, reflection, and stretching, Zane's concluding remarks were coded as geometric.

> Zane: Yeah, the origin seems to be the main factor that was distorting it, but there was also the whole, it gets oblong a little bit when it's getting close to it, but (trails off). So yeah, I mean there because before we talked about it, it would just always like, it doesn't contain it in the origin if it wasn't a perfect circle. But if it was a perfect circle then I don't think that would be there. So I don't know why it gets flatter when we're still nearing the origin but not containing the origin (holds palm flat facing computer, raises hand curling fingers, lowers hand uncurling fingers).

**Edward and Melody**

In contrast to Christine and Zane, Melody and Edward seemed much more predisposed to reasoning coded as geometric (see Table 3). For example, Melody remarked that if $z$ was placed on the point $1 + i$, then $z^2$ should be on the circle. This is not correct, but it is notable that she turned first to reasoning coded as geometric, whereas Christine and Zane focused on algebra exclusively to answer the same question. Edward turned to reasoning coded as algebraic to determine where $f(z) = z^2$ maps $1 + i$, much as Zane and Christine did. Edward first determined that the real part of $(1+i)^2$ is $1-1=0$, so $z^2$ should land on the imaginary axis. Thus, while Edward reasoned algebraically to determine the real part, he utilized this algebraic reasoning to inform his geometric reasoning and thereby concluded that $(1+i)^2$ is a point on the imaginary axis. Edward and Melody confirmed this fact with the aid of *Geometer's Sketchpad* (*GSP*).

Table 3

*Task 1 Codes for Edward and Melody*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 174 | E: $x$ squared | x | | Underlines $(x+iy)(x+iy)$ above with right index finger | Writes $(x^2-y^2)$ |
| 175 | E: minus $y$ squared | x | | E Touches the two $iy$'s | |
| 176 | M: and you get two | x | | E Points at first $(x+iy)$ in above line, then places chalk after $(x^2-y^2)$ in next line | |
| 177 | M: $x\ y\ i$ | x | | Has written $z^2=(x+iy)^2=(x+iy)(x+iy)$, touches these last two factors in order with right hand | |
| 373 | E: Oh, when you multiply, oh! | | | | |
| 374 | E: They expand, and they twist, and rotate | | x | Extends left finger and moves hand and finger left | |
| 375 | M: They like rotate | | x | | |
| 376 | E: rotate and dilate | | x | Brings right index finger to his front, turns toward partner, and moves hand and finger up and right | |
| 413 | M: That's on the, on the circle | | x | | Moves $z$ down and left just inside unit circle, then up and right on unit circle at 45° angle from positive real axis. Waves mouse up and left, then down and right, back and forth |
| 414 | E: So, does that make sense? So that is | | x | M Points at screen, then places two fingers of right hand on table next to computer | M mouse moves to origin, then to 1, then rotates ccw along right upper quarter circle to $i$, then moves down and right to bottom of *GSP* window |

*Note.* Alg stands for algebraic, and Geo stands for geometric.

Despite reasoning geometrically early on, both participants began primarily reasoning algebraically in answering the question, "Where do you think you should place $z$ to send $z^2$ to $i$?" Melody created the system $2xy=1$ and $x^2-y^2=0$ from the equation $(1+i)^2=x^2-y^2+2ixy=0+i$ and separating the real and imaginary parts, much as Christine and Zane answered this question. Edward and Melody, however, appeared to have much less success in solving this system. After they appeared to stop progressing toward a solution for this system, I attempted to redirect them to more geometric reasoning.

Note that Edward's reasoning is embodied through both his gesture in the case of dilation and his entire body's motion in the case of rotation. As such, the following exchange is coded as geometric.

> Interviewer: Do you remember what happens to the geometry of two complex numbers when you multiply them?
>
> Edward: When you multiply, Oh! When you multiply, oh! They expand, they twist, and they rotate (points left and moves hand and finger left) (see Figure 5)

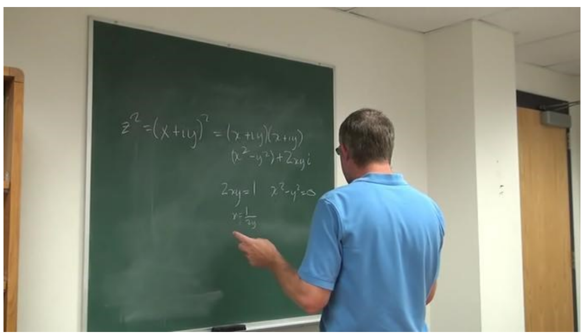

Figure 5. Edward produces a dilation gesture

> Melody: They like rotate
>
> Edward: They rotate (turns toward partner) (see Figure 6) and dilate (flicks right index finger from pointing forward to pointing up while extending arm outward and up) (see Figure 7).

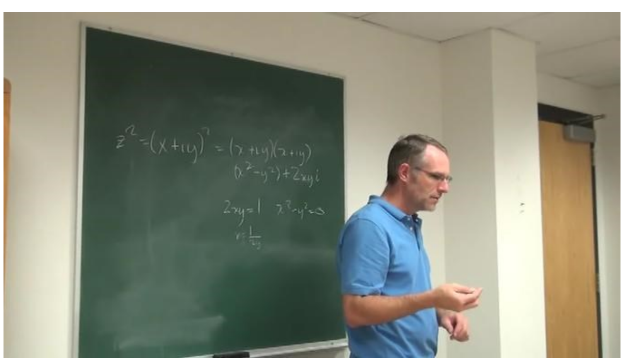

Figure 6. Edward rotates his body while reasoning about rotation

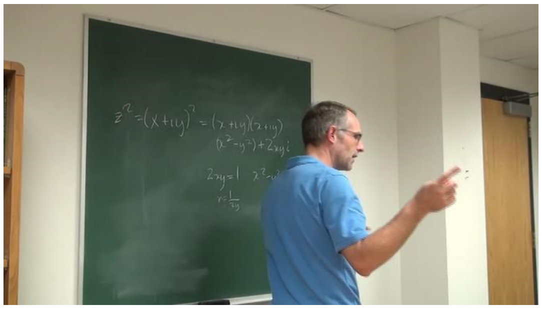

Figure 7. Edward produces a dilation gesture while rotating his body

This redirection appeared to lead Melody and Edward back to experimentation with the aid of *Geometer's Sketchpad* (*GSP*), and they noted that $i$ is mapped to $-1$ under $f(z) = z^2$. They placed $z$ at the correct point on the unit circle to make $z^2$ land on $i$, at which point Edward asked Melody, "Does that make sense that it's on the circle?" Melody responded "I don't know why," and Edward noted that the equation of the unit circle is $x^2 + y^2 = 1$. He claimed that this unit circle equation is "making $x$ and $y$ equal to 1 and they have to be equal, so that's why we're getting the $i$." That is, Edward seemed convinced that the equation of the unit circle was the reason $\left(\frac{1+i}{\sqrt{2}}\right)^2 = i$. He concluded the investigation of this question by stating, "I wouldn't have guessed that without moving that around, okay."

When Melody read the next question, "What do you think will happen to $z^2$ if you move $z$ around the green unit circle once," Edward began trying to reason what would happen as the point $z$ moved to the right following the circle, stating "$x$ will get bigger, $y$ will get smaller," and asked the poignant question, "but will it stay on the circle?" Edward started picking particular values for $x$ and $y$ to determine the real and imaginary parts of the corresponding output. Melody noted that if $x = 0$, "the imaginary part would have to be zero." As such, both Melody's and Edward's reasoning produced algebraic codes. While they did not yet have a complete prediction, Edward suggested, "shall we just try that and just see if that part works," and Melody agreed. This suggestion seems to mirror in some respects Zane's suggestion of "if we just say it'll move around the circle we can do it." Perhaps both Edward and Zane felt that they had done what they could without *GSP* and wanted to transition back to *GSP* to further develop their reasoning, or at least check what predictions they had already made.

When Edward moved the point $z$ around the unit circle, Melody pointed out that the output $z^2$ moved around the unit circle twice. Edward replied that he would not have noticed that behavior without Melody pointing it out, similar to how Christine claimed she would not have noticed the same behavior without Zane's explicit observation. It may be that the fact that $z^2$ travels around the unit circle *twice* when $z$ travels around the unit circle *once* is a subtlety that requires explicit and directed attention to notice. After answering this question, Edward read the next directions about further geometric experimentation with the aid of *GSP* after he and Melody constructed their circles. He then stated, "Okay, move your circle around the graph and observe the output….Okay, now let's mess with this critter!"

Between Edward's reading of the directions and his manipulation of the output "critter," Edward and Melody constructed, transformed, and colored the input and output circles. They colored the input circle blue and the output circle red. During this time, Edward also realized he had an additional output shape (see Figure 8), as he "accidentally did the transform of the transform." That is, he constructed $f(z) = (z^2)^2$. He correctly identified the extraneous output shape and deleted it. It is not clear what caused this error, though it is interesting that this incorrect reasoning was repeated in some respect by both participant groups. Zane repeatedly reasoned about the output shape displayed by *GSP* as $f(z) = (z^2)^2$ despite the fact that the output point was labeled $z^2$, and Edward actually constructed the output corresponding to $f(z) = (z^2)^2$ with the aid of *GSP*. Perhaps this error was motivated in part by the transformation notation I used on the lab worksheets. In particular, the participants had to coordinate the transformation $z \to z^2$ as written on the lab worksheet with the function $f(z) = z^2$ as they wrote the function. Given the potential problems with reasoning about function notation (Tall & Vinner, 1981), it is conceivable that both sets of participants believed that they were to square the input once for the transformation $z \to z^2$ and once again for the function $f(z) = z^2$. However, once the participants realized that *GSP* squares the input only once, they did not repeat this error.

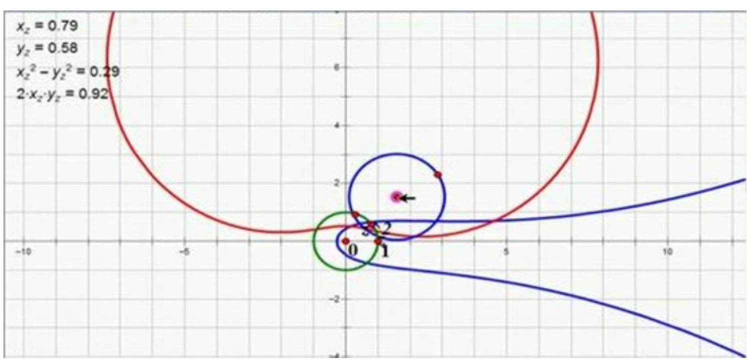


Figure 8. The small blue input circle, the large red image circle of the small blue circle, and the elongated blue image curve of the large red image circle under $f(z) = z^2$

When Edward and Melody started experimenting with the aid of *GSP* again, Melody noticed that the output wraps itself around the origin, and expressed an interest in discovering why the output loops at the origin. With this question in mind, Melody and Edward started investigating the behavior of the output when specific points were in the area enclosed by the input circle.

Edward read the first question of this type, "What does the output look like if the input is a circle with $1 + i$ in the area enclosed by the circle?" Edward asked Melody if she expected that the center would be located at 2. Melody geometrically reasoned that 2 would be in the area enclosed by the output circle, but not necessarily at the center.

> Melody: If you're saying, if this center's (points at screen) at $1 + \mathrm{i}$ (Edward wiggles circle slightly centered at $1 + \mathrm{i}$), then the center will be at 2, but if it's just enclosed in the area, then it's not necessarily at 2.
>
> Edward: Oh, that's right

Melody and Edward experimented and observed with the aid of *GSP* how $f(z) = z^2$ transforms various circles with $1 + i$ inside the circle. When I asked for an explanation, they did not offer any explicit theories about the behavior of the output circle beyond Edward ruminating, "What was our theory? I guess we knew it was going to be on the imaginary."

Melody and Edward began trying to determine how $f(z) = z^2$ maps an input circle where the complex number 2 is in the area enclosed by the input circle. After experimenting with the aid of *GSP*, Edward concluded that they would expect the output to "sort of be in the real realm of things. At least for the center." After this experimentation, they returned to the question that they had previously tried to skip. When Edward asked what the output would look like when the origin was in the area enclosed by the input circle, Melody claimed that "it'll get all messed up," and used *GSP* to demonstrate. Melody's verbiage during this demonstration produced codes for geometric reasoning. It is not clear at what point in her investigations Melody noticed that the origin is a point that causes the output to behave atypically. She characterized this behavior as "wrapping around" while observing the geometry with the aid of *GSP*.

After following the directions on the lab worksheet to observe what happens to the output when the radius of the input circle is changed, Edward noticed that “once we get past the origin it’s going to wrap around.” Edward and Melody appeared to remember this geometric discovery in the next question, “What does the output look like if $1 + i$ and 2 are in the area enclosed by the circle?” In particular, Melody claimed that “including the origin, the output won’t wrap around,” and demonstrated this behavior with the aid of *GSP*. Melody also noted that if the input circle was centered at the origin, the output circle should “loop twice,” a correct geometric prediction (see Figure 9). This episode included reasoning coded as geometric due to references to geometric features of the input and output circles, including when the output circle “wraps around,” “gets past the origin,” or “loops.” Melody and Edward did not yet notice any special relationship between the size of the input and the size of the output when they changed the radius of the circle centered at the origin.

Melody: It’s still looping around twice, it just changes

Edward: the magnitude. If it’s small we go smaller. If it’s big we go bigger.

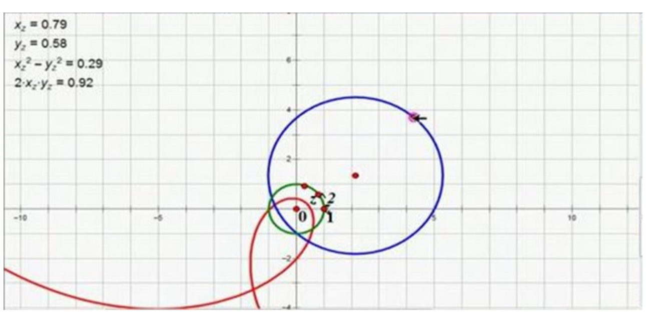


Figure 9. Red image curve under $f(z) = z^2$ twists if blue pre-image is a circle which includes the origin.

When asked about how the output curve changes as the input circle is dragged along the real axis, Edward guessed that “it should sort of unwrap itself, shouldn’t it?” This prediction is correct only under the additional assumption that the input circle is dragged *away* from the origin. This mirrors Zane's and Christine's similarly limited prediction that the output circle would get "sucked in," which is only true if the input circle is dragged *toward* the origin. However, Edward did clarify the cause of the output circle unwrapping itself, and demonstrated that he was correct with the aid of *GSP*. The reasoning in this exchange was coded as geometric.

> Edward: Once we get the blue circle to not include the origin it should just be another circle. (Edward demonstrates in GSP.) Now it's a gigantic circle.
>
> Interviewer: It's just a little bigger than one.
>
> Edward: Oh I forgot we're really zoomed in (pinches index finger and thumb together) on one. That was one thing I was really not getting in class, that when we were talking about small stuff we're talking about really (moves left hand up and right) small stuff. You've got one is huge.

So, not only did Edward reason that the origin is the cause of the output circle wrapping itself, but he also advanced his geometric reasoning to the point where he could say that "small" can mean "really small" (see Figure 10). It is telling that he could not establish this geometric reasoning in class, that even a number as "small" as one can be "huge." It may be that the dynamic nature of *Geometer's Sketchpad* (*GSP*) helped him advance his geometric reasoning in this way, or it may be that this particular aspect of the geometry was not emphasized in his complex analysis class.

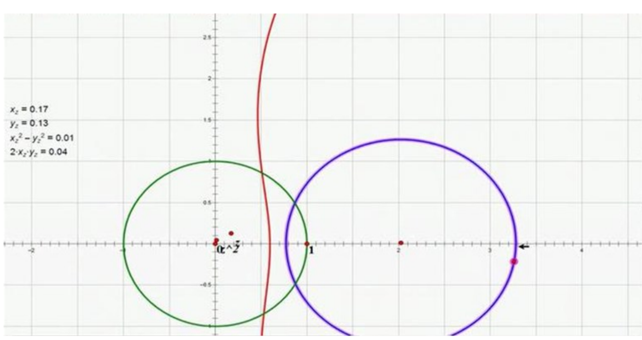


Figure 10. A "gigantic" blue pre-image circle with a radius slightly larger than 1

Edward's gesture is also notable in that he pinched his index finger and thumb together, possibly iconic of the concept of smallness, while noting they were "really zoomed in on one." In contrast, he extended his left arm, possibly iconic of largeness, when referencing "really small stuff," though this gesture could have been intended to match with the subsequent utterance "you've got one is huge." Alternatively, the fact that he used a smallness gesture to reference an object that appeared large and a largeness gesture to refer to objects that previously seemed small may have been an external representation of his internal struggle to reify the precise mathematical meaning of "small" in his geometric reasoning.

When Edward and Melody dragged the input circle along the positive imaginary axis, they observed that the output "turns" or "flips," though they did not argue about whether this "flip" is a rotation or reflection as Christine and Zane did. Melody correctly justified this geometric behavior by noting that algebraically, $i^2 = -1$, so anything along the positive imaginary axis should map to the negative real axis. While dragging the input circle through different quadrants, Edward noticed that the output goes twice around the origin, and associated this behavior with the previous *GSP* experimentation where he moved a single point $z$ around the unit circle.

> Edward: Is it doing that twice thing, like before? So if we start here (moves circle from positive real axis to negative real axis, counterclockwise around outside of unit circle)
>
> Melody: I think it does.

During this experimentation and observation with the aid of *GSP*, Edward and Melody collectively stated a key advancement in reasoning regarding the behavior of $f(z) = z^2$. This reasoning was coded as geometric due to references to circles and such geometric behavior as "wrapping" and "looping."

Edward: So big small (Palms flat facing each other, moves hands toward and away from each other) will map to a big circle, or not necessarily a circle, kind of depending on where you are it'll start wrapping.

Melody: Small where you are.

Edward: Let's test that theory with this thing (Waves mouse up and down through center of input circle. Drags input circle up and right so center is just above the positive real axis). When you get really big (expands input circle to radius 5) there, then if we go through the origin then it loops around (Points at screen and traces one large counterclockwise circle)

Melody: Go through the origin, loop around, and small circle will

Edward: Small circle (Contracts input circle to radius $\frac{1}{2}$) will just, will be a circle, another small circle (Contracts input circle to radius $\frac{1}{4}$)

Melody: Mhm. Unless you include the origin and it wraps around.

Thus, by experimenting and observing with the aid of *GSP*, Edward and Melody both advanced their geometric reasoning to the point where they could verbalize that $f(z) = z^2$ maps a small circle to another small circle provided the pre-image circle does not include the origin and the image curve "wraps around." This ability to notice a small circle mapping to a small circle is crucial for reasoning geometrically about the derivative of a complex-valued function, particularly in the non-linear case. Also note that while Melody only appears to partially revoice Edward's reasoning, the portions of Edward's reasoning she repeated seems to be the same ones he accompanied with gesture.

The last question on the Task 1 (see Appendix B) worksheet asked about how $f(z) = z^2$ transforms the plane. I stated this question by comparing it with the example of how $f(z) = iz$ rotates the plane 90°, and asked for a similar geometric explanation for $f(z) = z^2$. After I stated this question, Edward turned his right hand outward, so his palm faced away from him, and pushed outward along the table. He began manipulating *GSP*, first moving the mouse up and down through the input circle, then moving the circle up into the first quadrant. He collapsed the circle nearly to a point, then expanded it again. He moved the mouse to the center of the circle before dragging the circle back down slightly, still in the first quadrant. In this minute of *GSP* experimentation and observation, Edward and Melody both remained silent, though both appeared to be watching *GSP*. Finally, Melody broke the silence by trying to summarize what was happening to the circle, though at this point she did not yet seem sure of herself.

> Melody: Isn't it like (trails off)? Does it double the magnitude? I don't know.
>
> Edward: Is it always? Oh why don't we put this (input circle) in a place that we know (moves blue input circle so its center rests at 2).

In this observation, Melody had at least given part of the answer to the question I asked, at least with respect to circles, as $f(z) = z^2$ increases the magnitude by a factor of $2z$ and rotates the angle an amount equal to $Arg(2z) = Arg(z)$. Note that $f(z) = z^2$ squares the magnitude of the point $z$, and additionally does not double the size of the input circle unless $|z| = 1$ so that $|f'(z)| = |2z| = 2|z| = 2$. After about another minute of *GSP* experimentation, Edward agreed with Melody's observation despite this inaccuracy.

> Edward: Yeah, so it doubles the magnitude. Or it doubles the magnitude until we get through the origin doesn't it?
>
> Melody: Mhm.

Thus, through geometric reasoning, Edward and Melody decided that $f(z) = z^2$ doubles the magnitude of the input circle, and that the origin is in some way an atypical point for this function. Edward asked me whether I wanted them to talk about the orientation as well as the magnitude, suggesting that he felt to some degree that rotation was also an important geometric aspect of the function. I suggested to Melody and Edward that they should make spokes to highlight the amount the function $f(z) = z^2$

rotated the input circle. Melody initially predicted that the circle should rotate to 90°. However, when Edward placed the center of the input circle at $1 + i$ and pointed the spoke right, Melody observed that the output spoke rested at a $45°$ angle rather than a $90°$ angle. This error might be a result of the relatively common conflation of how $f(z) = z^2$ transforms points and how it transforms circles. The point $z = 1 + i$ would indeed map to a point with argument $\frac{\pi}{2}$, but a short line segment emanating from the point $z = 1 + i$ would map to a nearly straight curve rotated $\frac{\pi}{4}$ from its original direction. At this point, Edward and Melody did not yet seem certain of how much $f(z) = z^2$ rotates a small circle.

> Edward: And when we multiply, am I doing this right? When we multiply (rotates arm about elbow to the left) two complex numbers together it's adding the angles together? Is that right? Because if I move this (moves the center of the input circle closer to the real axis in the first quadrant) to a smaller angle here, you see how
>
> Melody: The line segments are almost like, really close together.
>
> Edward: They're closer….(GSP experimentation here). Oh so this is making a little bit more sense, because we're multiplying by the angle, then that's why it's going around twice. Cause we're doing this angle times two.

Due to Edward's and Melody's experimentation and observation with the aid of *Geometer's Sketchpad*, they synthesized both reasoning coded as algebraic and reasoning coded as geometric to explain that the point $z^2$ or the output circle "goes around twice," because they were multiplying by the angle, which doubles it. That is, they are algebraically multiplying the complex number's argument by 2. After some additional experimentation with the aid of *GSP*, Edward tried to summarize how $f(z) = z^2$ transforms circles, in a way coded as geometric.

> Edward: Is that somewhat of an explanation? That the orientation depends where you are, but the, a small circle goes to a small circle, it's just how it's oriented, isn't it? So should be pretty close to we're on the real line, the orientation, at all

Melody: That's not rotated at all, and then it starts rotating as you go (points at screen with right hand, rotates right arm counterclockwise).

So, by this point, Edward and Melody correctly noted that a small circle maps to a small circle, its magnitude is doubled, and the amount it is rotated increases as the input circle travels counterclockwise through the quadrants. Edward summarized again by saying the function "reorients it (points at input circle) by double this (waves mouse along angle at which blue circle's center is from the origin) angle." Melody asked for clarification of which angle Edward meant. Edward indicated the angle again with the mouse, which Melody described as "from the real axis to the origin of the circle," and Edward agreed with this description. After about 3 additional minutes of *GSP* experimentation, Edward and Melody tried to summarize how the function $f(z) = z^2$ rotates the plane via geometric reasoning. His reasoning was coded as geometric due to the fact that he referred to the geometric idea of how the plane rotates.

Edward: Generally, it's how much the plane gets rotated to the left.

Melody: It's going to rotate

After this observation, Edward additionally mentioned that he thought it was "going to be twice."

Interviewer: What was it doing twice?

Edward: I don't know, for some reason I had it in my mind that was actually going to, that the plane was going to rotate twice.

Melody: That the plane would rotate twice

Edward: Or twice, twice the angle of the origin to the center of the circle, but it is obviously not doing that, so

I asked Edward how he knew that was not what was happening, and Melody retorted, "It is, it's rotating the angle from the origin to the center of the circle." This characterization of the plane rotating twice is a reasonable description of how $f(z) = z^2$ maps the points in the plane, as this $2-$to$-1$ function essentially wraps the plane over on itself. As such, it is not clear why Edward was so convinced this characterization is incorrect, while Melody reasoned that it is. One possibility is that Edward continued to conflate how points are mapped with how circles are mapped by $f(z) = z^2$. This function squares the magnitude of points and doubles their arguments, whereas it also doubles the radii of sufficiently small circles and, as Melody stated, "rotate[s] [the circle by] the angle from the origin to the center of the circle."

## Task 2: Investigating $f(z) = e^z$

### Christine and Zane

At the beginning of the second task, Christine and Zane constructed $f(z) = e^z$ with the aid of *Geometer's Sketchpad* (*GSP*). While investigating this function, they utilized both algebraic and geometric reasoning (see Table 4). Zane and Christine answered the first several questions integrating algebraic reasoning. For example, Zane recalled $e^{\pi i} = -1$ and demonstrated with the aid of *GSP* that he was correct.

Table 4

*Task 2 Codes for Christine and Zane*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 111 | Z: $e$ to the $x$ cosine $y$, so, cosine of 0 is 1, so you just get e to the whatever power. So either it'll approach zero or it'll go out, towards infinity as you grow | x | | index finger extended, moves hand up and right | |
| 112 | C: Yep, and then it just goes around the circle | | x | | |
| 161 | Z: Now make it a little bit off center | | x | | rotates $z$ ccw around unit circle roughly $2\pi$ away from origin, starting and ending at positive imaginary |
| 162 | Z: Okay, see how it spirals? | | x | | |
| 164 | Z: then if you go to the outside of that | | x | points at screen, moves hand left and right | |
| 165 | Z: It just shrinks it, okay | | x | | moves $z$ into 1st quadrant, then down to $2 + 2i$ |

*Note.* Alg stands for algebraic and Geo stands for geometric.

When discussing the behavior that occurs if $z$ is dragged along the axes, Christine and Zane correctly reasoned in a way that produced algebraic codes. Namely, they reasoned that if $z$ moves along the real axis, $e^z$ should approach $\infty$ or $0$ depending on which direction they moved $z$, and that if $z$ moved along the imaginary axis, $e^z$ should go in a circle. Christine did not provide any reasoning for why $e^z$ moves along the unit circle when $z$ moves along the imaginary axis, although she was correct.

> Zane: Okay, well, if x stays positive, it will, so, e to the x, cosy, so cos of 0 is 1, so you just get e to the whatever power. So either it'll approach zero or it'll go out, towards infinity as you grow

> Christine: Yep and then it just goes around the circle.

Once Zane and Christine finished correctly predicting how $f(z) = e^z$ maps the point $z$, they started experimenting with the aid of G*SP* to discover how this function transforms vectors. In particular, they attempted to answer questions on the worksheet for $f(z) = e^z$ (see Appendix C) about how $f(z) = e^z$ maps line segments from the origin stretching into various quadrants. This worksheet provided instructions for how to construct the vector from the origin to the point $z$. After constructing this vector and sending it through the transformation $f(z) = e^z$, Zane noted that the image curve spirals if the vector is "a little bit off center" (see Figure 11).

Given this reference to a geometric idea such as how centered a mathematical object might be, this observation was coded as geometric. While stretching the vector along the imaginary axis, Christine restated this observation, and added that the image curve creates circles if it is kept perfectly straight. Christine and Zane further noted that if the vector is in the first or fourth quadrant, the vector's image spirals outward, and that in the second or third quadrant, the vector's image spirals inward. They additionally noticed that the direction of the spiral changes between the first and the fourth quadrant, or between the second and the third. Given the visually motivated reference to counterclockwise and clockwise spirals, these observations were coded as geometric. After successfully characterizing how the function transforms vectors, they started to construct and transform circles as directed in the Task 2 worksheet.

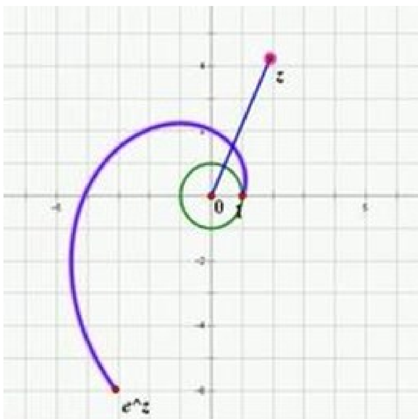


Figure 11. Pre-image vector is transformed to a spiral under $f(z) = e^z$

At first, Zane reasoned that the circle's image would never be "less than the origin," by which he may have meant on the left half of the plane. Under this interpretation, when Zane suggested afterward that the output could not be "doubly negative," he may have meant to ask whether the output could have negative real and imaginary parts simultaneously. If this is indeed how Zane reasoned, it is consistent with algebraic reasoning about the real-valued function $f(x) = e^x$ which never has a negative output. While Zane may have correctly described how $f(z) = e^z$ maps the imaginary axis to a full unit circle around the origin (which intersects the left half of the plane), this instantiation of Danenhower's (2006) *Thinking Real, Doing Complex* may have been too ingrained for Zane to ignore. It is possible that investigating how vectors are transformed felt similar enough to working with real-valued vectors that Zane may have temporarily ignored the fact that he was working with complex numbers, and thus that $e^z = -1$ has a solution. As such, Christine repeatedly surprised him by successfully carrying out actions with the aid of *GSP* Zane had declared impossible.

> Zane: You can't get it anywhere less than the origin though, right?
>
> Christine: Mmm, whoa! Yeah, apparently I can! How did I get over there?
>
> Zane: Just not both doubly negative? Okay
>
> Christine: Basically whatever you say, Zane, I can do.

While Christine was not sure of how she accomplished these supposedly impossible tasks, she seemed able to find counterexamples to Zane's claims nonetheless. After experimenting a little more with the aid of *GSP*, Zane correctly noted that moving the input circle left and right changed the size of the output curve (see Figure 12), while moving the input circle up and down rotated the output curve (see Figure 13). Thus, investigating how the function transforms circles seemed to highlight rotation and dilation for both groups of participants.

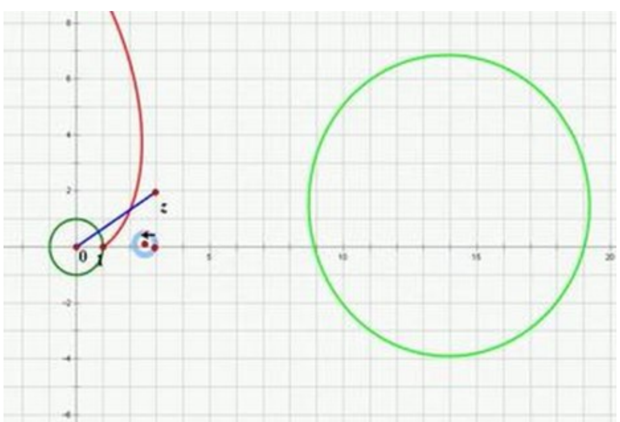

Figure 12. $f(z) = e^z$ transforms small blue pre-image circle centered real axis to a dilated green image circle centered on real axis

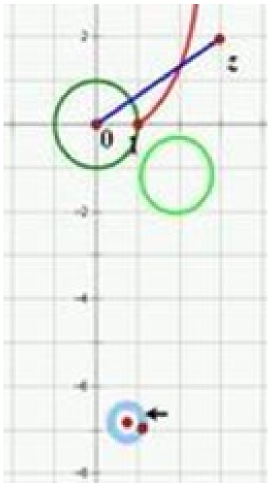

Figure 13. $f(z) = e^z$ transforms small blue pre-image circle along imaginary axis to green image circle rotated counterclockwise from the positive real axis by $Re(z)$ radians.

When Zane and Christine first started investigating what the output of a circle would look like if $1 + i$ was in the area enclosed by the circle, under the function $f(z) = e^z$ they did not offer much description except to say that the output of a circle around $1 + i$ would be another circle. Christine described the output of a circle around 2 geometrically by appealing to how the function maps points and vectors along the real axis. It is not clear whether Zane and Christine considered the point $z$ and the vector from the origin to $z$ as distinct mathematical objects, as I did not question them about this distinction. The following exchange was coded as geometric, due to verbiage about how the circle curves and possibly "gets weird when you go to the origin."

> Christine: I think when we make it real, like it just, it just was a circle on the real that was getting bigger and bigger.
>
> Zane: What if you have a point that's just, er, you just have 2 inside the circle as opposed to on the axis? The new circle's going to be big, yeah? Don't be ridiculous, it's also curved weird. Does it get weird when you go to the origin? It just doesn't go to the origin, that's right. Okay.
>
> Christine: So it doesn't curve around like the other one did.
>
> Zane: Right, it just shrinks.

So, during this time, Christine and Zane discovered that when the circle becomes large, the output shape is "curved weird" (see Figure 14). This appeared to remind Zane of $f(z) = z^2$ looping the output curves on themselves when the origin was in the area enclosed by the pre-image circle, so he asked a related question about this new function. However, he appeared to remember that $f(z) = e^z$ is not 0 for any value of $z$, so he correctly reasoned geometrically that the output curve "just doesn't go to the origin."

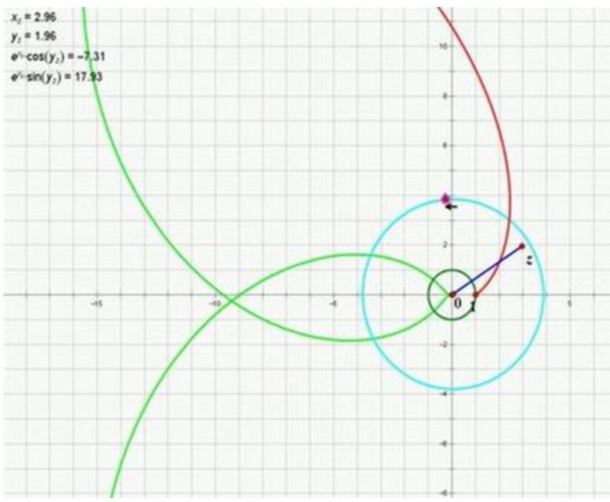


Figure 14. $f(z) = e^z$ maps a larzge blue pre-image circle to a larger green image curve which self-intersects

Despite this discovery, in the subsequent episode Zane felt that if the origin was in the area enclosed by the input circle, then the output shape should become distorted. Christine's observations with the aid of *Geometer's Sketchpad* (*GSP*) disproved this conjecture once again.

> Zane: Okay. What do you think the output will look like if the input is a circle with the origin in the area enclosed by the circle? Yeah, it shrinks and gets misshapen.
>
> Christine: Well now it's the circle. Like, it's not misshapen
>
> Zane: So it depends how far away you are.
>
> Christine: You're making the radius bigger and bigger. The circle ends up, like, it has to go past the zero so it loops around it instead.

This exchange was coded as geometric, due to the references to geometric properties such as whether the output curve is misshapen, whether it shrinks, and whether it loops. While Zane tried to explain the distortion as related to how far away the input circle is from the origin, Christine correctly reasoned geometrically that this distortion is related more closely to the radius than the distance from the origin. Her last comment suggests a reason for why Zane felt the distortion was related to the origin. In particular, they observed that the image of the input circle loops around the origin rather than touching it. They may have seen this behavior as similar to how the image of an input circle enclosing the origin loops around the origin under $f(z) = z^2$. The origin is an important point in both cases, but in the case of $f(z) = z^2$ the origin in the *input* plane is the critical factor, while in the case of $f(z) = e^z$, the image curve avoids the origin in the *output* plane.

As Zane and Christine moved through the Task 2 worksheet and started manipulating the radius of the circle, they observed that as the radius of a circle centered at zero becomes larger, the output curves twist. It is possible Christine or Zane may have still believed the origin plays a role in causing the twists under the function $f(z) = e^z$ partially because of the circle's location at the origin. In truth, the output curve twists if the input circle has a radius of $\pi$ or greater, regardless of location, due to the vertical periodicity of $f(z) = e^z$. Because it was not clear why Christine and Zane believed these twists occured, I asked them about why twists occurred in $f(z) = z^2$. In this case, Christine identified the origin as an important point.

Interviewer: Did you tell me why the twists occurred in z squared?

Zane: In the circles, there were points where the circles twisted in on itself.

> Christine: Probably not directly, but I mean, it had something to do with the origin.

After some further *GSP* manipulation, Zane tried to correspond this reasoning with the behavior they observed for $f(z) = e^z$, by stating, “it’s the same thing here.” Christine seemed skeptical, and asked Zane what would happen if the circle is small enough, or if the circle is large but away from the origin. The following exchange produced codes for geometric reasoning due to imagery such as being far away from the origin (rather than large in magnitude), looking like a circle, or warping due to being too close to zero (rather than due to being small in magnitude).

> Christine: So if you’re away from the origin, you won’t be able to create the twists? And we had a small enough circle, it would just look like a circle, right Zane?....What about a large radius?

> Zane: Well, if the radius is too large it’ll start to warp because it’s getting close to zero.

After experimenting with the location and size of the input circle, they constructed and transformed spokes in the circle and continued similar experimentation. By the end of this session, both Zane and Christine seemed to start experiencing doubt about their original claim that the origin caused the output circle to twist. Zane suggested, “Okay. You said the twists happen when the circle is close enough to the origin.” Despite some concentrated efforts with the aid of *GSP*, neither Christine nor Zane could make a small circle close to the origin map to an output curve with a twist, so they began to wonder why via geometric reasoning. This reasoning was coded as geometric due to references to geometric properties such as twisting or getting close to something.

> Christine: Then why won’t it twist when the small circle gets close? Because the small circle has such a small radius that the output, has such a small radius. But it never interferes with zero.

Zane: And when it finally does interfere with zero it's like so small it becomes a point.

So, while they both realized that a small circle close to zero would not map to a curve with a twist, Christine's and Zane's geometric reasoning was not yet quite correct. Instead, they tried to geometrically reason that by the time the circle interferes with zero, the output is too small to twist. After further experimentation with the aid of *GSP*, they eventually discovered that the radius of the input circle itself is what causes the twists in the output circle. It appears they made this discovery by zooming in and out to find a twist in the small circle and failing.

Christine: Probably, greater than π, maybe, my guess is at π, they touch, and then greater than π, it wraps. (Demonstrates in GSP that this is correct)

Zane: So it looks like it doesn't matter if we're near the origin or not.

Thus, not only did Christine observe that once the radius of the input circle is greater than $\pi$, the output starts to wrap, but Zane finally advanced his geometric reasoning to the point where he could correctly state that the origin has nothing to do with why the output circle wraps. To conclude this task, I asked Christine and Zane to describe how $f(z) = e^z$ mapped the whole plane, and they admitted to preferring algebraic reasoning to geometric reasoning.

Christine: It's really hard for me to picture things.

Zane: Yeah, my mind doesn't really work geometrically either.

Christine: And this one's harder than z squared I think

Zane: I couldn't even picture z squared really.

For the rest of the second day, I had Christine and Zane construct $f(z) = \frac{1}{z}$. As usual, they correctly calculated the real and imaginary parts via algebraic reasoning. Zane correctly noted that closer to the origin, the output is further away, and the output "flips" near the origin. A limitation of *Geometer's Sketchpad* (*GSP*) became apparent here. In particular, a small circle surrounding the origin was mapped to such a large circle that the output curve was represented graphically as a series of connected line segments forming a closed curve. That is, the output curve had sharp corners as a result of how *GSP* calculated the output curve (see Figure 15). This discussion reminded Zane of his numerical analysis class and he correctly explained the measurement error inherent in *GSP*, sketching the sharp corners on the output curve as he did so (see Figure 15). Shortly thereafter, they plotted spokes, and Christine asked me if the spoke showing outside the output curve was also a software bug. I assured her it was not, and she appeared surprised and expressed an interest in discovering why this inversion occurred.

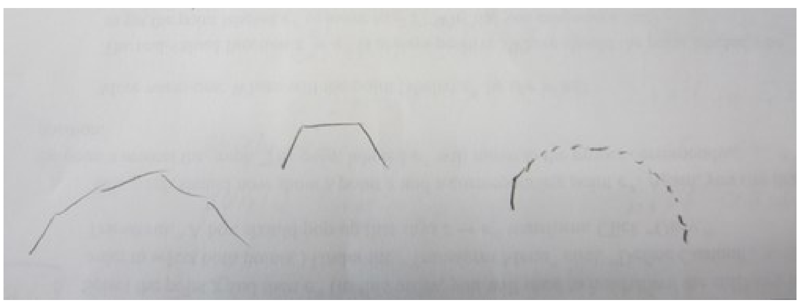

Figure 15. Zane illustrates sharp corners appearing on an output curve that should be smooth

For the rest of this task, Christine and Zane continued plotting and transforming spokes to attempt to discover why the spoke sometimes "flipped" outside the output curve, though they did not offer any explicit reasoning for such behavior at this time. This suggests that Zane and Christine might not yet have developed a geometric interpretation of the division of complex numbers.

**Melody and Edward**

Edward and Melody finished investigating $f(z) = z^2$ earlier than Zane and Christine. That is, Edward and Melody finished this investigation one hour and eleven minutes into the first interview, while Zane and Christine did not finish this investigation until the end of the first interview. They correctly constructed the transformation, after which they moved on to answering the questions listed on the Task 2 worksheet. Examples of codes for this task are included in Table 5.

Table 5

*Task 2 Codes for Melody and Edward*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 1343 | E: When $z$ is real, $e$ to the $z$ is going to be real, so there should be no negative, right, there should be no imaginary part, part of it, so it's just going to be the $x$, so it'll just go along the real axis | x | | Waves flat hand up and down, palm face down. Points at screen, places right index finger on table and drags it right along table off the side. Drags index finger left to right again | |
| 1347 | M: but then when $z$ is, on the imaginary, that's when that | | | | |
| 1354 | M: you just get the cosine y plus i sine y, which is a rotation | x | x | Spins right index finger in two small ccw circles | |
| 1406 | E: What happens if the vector is stretched around the imaginary axis, well rotates along, counterclockwise | | x | | |
| 1407 | M: it just rotates | | x | Spins right index finger in several quick small ccw circles. E Spins right index finger in several large ccw circles | |
| 1408 | M: well, depending | | | | Rotates $z$ ccw to negative imaginary |

*Note*. Alg stands for algebraic and Geo stands for geometric.

To answer the first question, “Where will the point $e^z$ be if the point labeled $z$ is at $\frac{\pi}{2}$,” Melody just positioned the point $z$ at $\frac{\pi}{2}$ with the aid of *GSP* and observed that the image point was “on the circle.”Edward may have initially felt the next question was a trick question, as it read, “$e^x$ is always positive. Where does $z$ need to be for $e^z$ to be at $-1$?” Edward seemed to fixate on, “Always positive,” before furrowing his brows and trailing off. Given this behavior, I emphasized to him that the *real-valued* function is always positive, at which point he recalled that he wanted to use the formula $e^z = e^x(\cos y + i \sin y)$. He correctly identified this as Euler’s formula, and Melody commented that she was “really glad [Edward] remembered that.” With the aid of *GSP*, Melody and Edward observed that if the point $z$ was moved along the positive real axis, the output point remained real and increased rapidly in magnitude. Edward generalized this observation to correctly reason that the output would tend to zero if the input was moved along the negative real axis away from the origin. Melody correctly predicted that if the input point moved along the imaginary axis, the output would "go around the circle." When Melody asked why this behavior occurred, Melody and Edward discussed the particular parts of the algebraic equation that explains this geometric behavior. The following exchange was thus coded as predominantly algebraic.

> Melody: So we're going on the real axis, then y is zero. So you don't get any of the rotation because, right, er.
>
> Edward: Well, when we're on the, when z is real (waves hand up and down in front of screen), e to the z is going to be real, so there should be no negative (drags index finger left to right along table), right, there should be no imaginary part, part of it, so it's just going to be the x, so it'll just go along the real axis (drags index finger left to right again). But then when z is on the imaginary, that's when that
>
> Melody: Yeah, that rotation
>
> Edward: That rotation, that cosine,
>
> Melody: Yeah that rotation….Because x is zero, right….?And you just get the cosine plus i siny, which is a rotation.

Thus, Edward and Melody successfully employed reasoning coded as algebraic in conjunction with gesture to advance their geometric reasoning, to inform their experimentation and observation with the aid of *GSP*, and to offer explanations to the other partner. After this discovery, they constructed a vector and transformed it according to the Task 2 worksheet (see Appendix C). Melody and Edward experimented with this vector and its image with the aid of *GSP* and expressed some surprise about their observations.

> Melody: Really? So the farther away this is

> Edward: So it spirals, but when we're close to the, wait, go along the real. I want to see what it does there.

Melody even summarized what happens if the vector includes both real and imaginary components, by stating, "so when you have both, it's just, it's wrapping but it's also getting large." This seems to be a fairly reasonable description of an outward spiral, which is how $f(z) = e^z$ maps a vector with a positive real component. So, just as for Christine and Zane, *Geometer's Sketchpad* (*GSP*) appeared to encourage Edward and Melody to conduct their own investigations and come up with their own conjectures to test. While they primarily focused on the questions I gave them rather than generating their own, they developed their own methods for finding solutions to these questions. After this particular session of experimentation and observation, they were able to explain clearly how the function maps vectors.

> Edward: What happens if the vector is stretched around the imaginary axis, well, rotates along (see Figure 16)
>
> Melody: It just rotates
>
> Edward: Okay, what happens when you go along the real?
>
> Melody: It just, it gets larger that way.
>
> Edward: Larger if it's positive (see Figure 17) and tends to zero if it's negative (see Figure 18)

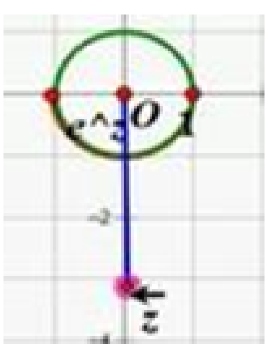


Figure 16. $f(z) = e^z$ transforms blue input vector along negative imaginary axis to green unit circle in clockwise direction

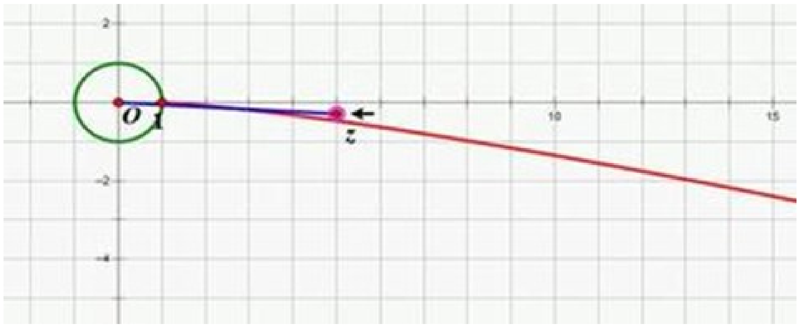


Figure 17. $f(z) = e^z$ transforms blue input vector along positive real axis to stretched red output vector along real axis

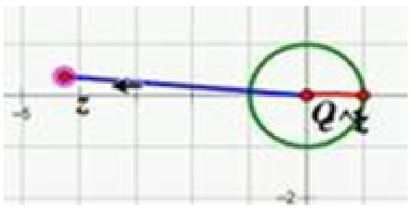


Figure 18. $f(z) = e^z$ transforms blue input vector along negative real axis to red output vector "tending to zero" along positive real axis

Experimentation and observation with the aid of *GSP* as directed by the Task 2 worksheet allowed Melody and Edward to discover that a vector in quadrants I and IV both map to an outward spiral, but the image of a vector in quadrant IV spirals in the opposite direction from the image of a vector in quadrant I (see Figure 19 and Figure 20). They additionally found that a vector in quadrants II or III maps to a spiral inside the unit circle rather than outside it (see Figure 21 and Figure 22).

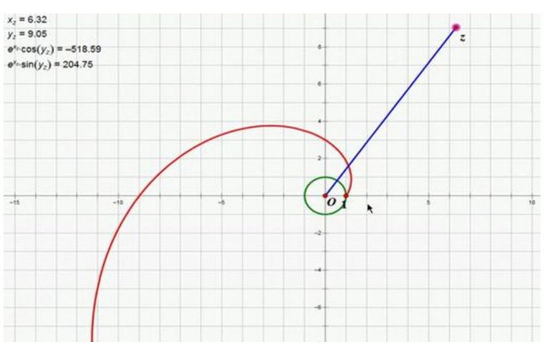


Figure 19. $f(z) = e^z$ transforms blue input vector in quadrant I into red outward counterclockwise spiral

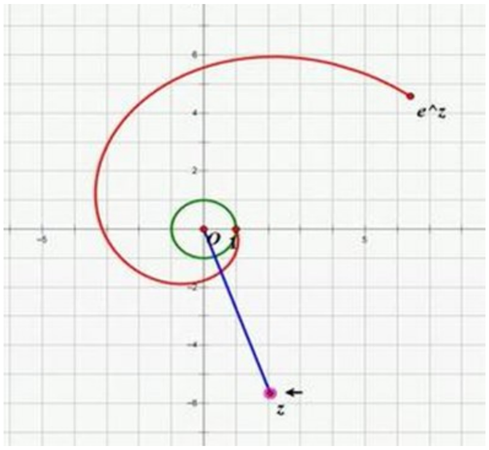


Figure 20. $f(z) = e^z$ transforms blue input vector in quadrant IV into red outward clockwise spiral

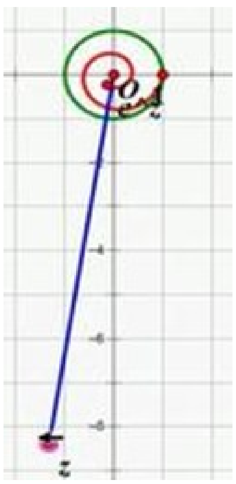


Figure 21. $f(z) = e^z$ transforms blue input vector in quadrant III into red inward clockwise spiral

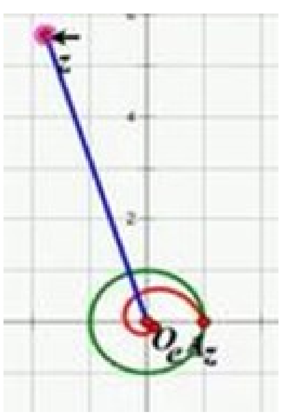


Figure 22. $f(z) = e^z$ transforms blue input vector in quadrant II into red inward counterclockwise spiral

Once Melody and Edward correctly summarized how $f(z) = e^z$ maps vectors, they constructed and transformed a circle as directed in the Task 2 worksheet (see Appendix C). During their experimentation with the aid of *GSP*, Edward observed that the output curve grows as the input circle moves away from the origin, which he justified by referencing the real part, $e^x$. This behavior only occurs if the input circle is moved away in the positive direction. If the input circle moves in the negative direction, the output curve becomes smaller. Given Edward's geometric reasoning, which was coded as geometric, Edward decided he was incorrect when Melody moved the input circle into quadrant II from quadrant I. Perhaps to try to make sense of his previous reasoning, Edward reiterated how $f(z) = e^z$ maps the axes by stating, "On the real, and on the imaginary it just stays the same, so, it goes, goes around, okay." That is, he appeared to remember that the real axis maps to the real axis, and the imaginary axis maps to a circle.

To answer the questions of what the output looks like if particular points are included in the area enclosed by the input circle, Melody and Edward experimented with the aid of *GSP* extensively. During this time, Melody asked a very relevant question about the output looping on itself, possibly even identifying the reason why the looping occurs in her phrasing of the question (see Figure 23). The following exchange produced codes for geometric reasoning due to references to circles "looping" and "wrapping."

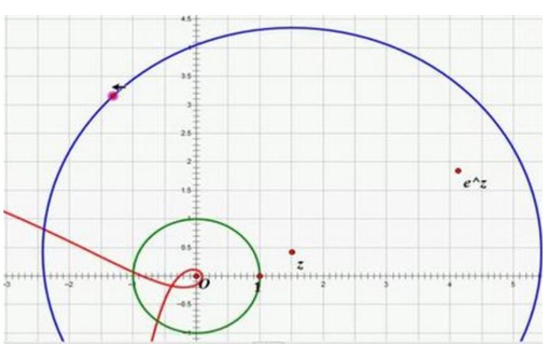


Figure 23. $f(z) = e^z$ maps large blue input circle to "looping" red output curve

Melody: But if we make it larger, will it start going around in loops?

Edward: It's there, because you eventually get the whole

Melody: It loops (demonstrates in GSP)

Edward: Make it big, and it's starting to wrap around, hold it, does z actually go into the unit circle eventually? Oh it does, okay. Oh, there, we have the whole plane. Yay! Almost

Melody: Wrapped around itself.

Thus, Edward and Melody appeared to reason correctly that if the input circle is large enough, then the output plane would loop. It is worth noting that Edward and Melody made no mention of the origin in answering this question, while Zane and Christine were convinced for at least part of the same task that the origin caused the output to loop for $f(z) = e^z$. Edward followed up this investigation by correctly reasoning that the output curve gets "real dinky," as the input circle moves along the negative real axis.

While he answered the next question about what happens when the radius of the input circle changes, Edward and Melody introduced Zane and Christine's error in reasoning when he claimed that the origin caused the output curve to twist.

Edward: Investigate when you change the radius of the circle at these points. Okay, so

Melody: The center until it crosses, oh, and then it, so it starts curving when it crosses the circle and then the origin.

Thus, mirroring Christine and Zane's reasoning for this same question, Melody claimed that the output curve starts to twist when the input circle crosses into the unit circle and ultimately the origin. Edward read the next instructions off the Task 2 worksheet (see Appendix C), which contained directions to center the circle at the origin and manipulate the radius. Melody and Edward noted that as the circle gets larger, the output starts to twist, and further realized that the amount the function $f(z) = e^z$ rotates the circle is dependent on where their input circle is located, though they did not explicitly describe exactly how much the function rotates the circle at a particular location. When I asked them for the angle of rotation, Melody responded it should be the angle from the positive real axis. I noted that if that were true, every circle along the same radial line from the origin should rotate by exactly the same amount. They saw with the aid of *Geometer's Sketchpad* (*GSP*) that this was not the case.

Melody and Edward started experimenting further with the aid of *GSP* and made a series of observations that connected the amount the image circle rotates with respect to the pre-image to the imaginary coordinate more and more closely. Melody observed that, "Oh that probably has to do because, when you have, when we had like the real and imaginary, then you got the rotation from the line segment." Edward decided a little later that, "oh, it's doing the pi thing, isn't it?" Melody correctly noted that when the $y$ −coordinate became $\frac{\pi}{2}$, the output circle rotates $90°$ with respect to the input circle. Finally, Melody made the key observation, "so let's just say like, 2 pi, would go all the way around." With this discovery in mind, they were able to answer why twists in the output occurred, and Melody reasoned that it had nothing to do with the origin.

> Interviewer: What causes the twists in your circle? Before it was your input getting close to zero.
>
> Melody: It's not moving it close to zero anymore. It's when you get really large, isn't it? Because it doesn't do it when we this, like, small circle, doesn't wrap.

Thus, Melody correctly identified that the reason the output circle wraps is related to the size of the input circle. This reasoning was coded as geometric due to references to geometric properties such as getting close to zero or twisting and wrapping. At the end of

Edward's and Melody's first interview, I asked them how large the circle needed to be to make the output wrap. Edward initially suggested that the radius had to be greater than 1, but with a small amount of experimentation with the aid of *GSP* and Melody's explicit counting of the radius size when the output twisted allowed Edward to say, "Oh, exactly the, close to pi." Edward and Melody again utilized algebraic reasoning by referencing the formula $f(z) = e^z = e^x(\cos y + i \sin y)$ in order to explain why a radius of $\pi$ in the input circle causes the output circle to wrap. In particular, they claimed that $e^x$ yielded the radius of the transformed curve and that "$e$ to the $\pi$ is where it wraps around," which is close to correct. This is not exactly precise because $x$ is a point, not a radius, and the wrapping of the image is due to the periodicity of $\cos y + i \sin y$, rather than a dilation factor of $e^\pi$.

Edward and Melody completed Task 2 during the beginning of their second interview. Edward recalled that I had previously asked them to explain why a radius of $\pi$ caused a wrap in the output circle. Melody and Edward experimented with circles in *GSP* some more, dragging their input circle to various quadrants, changing its radius, and observing the results. Edward correctly explained the $\pi$ radius by recalling the geometric fact that the height on the imaginary axis of a circle determined the amount the circle rotated.

> Edward: Yeah, it goes that way because the imaginary axis is the rotation, and so that's why it, it wraps around, because 2 pi would be a full rotation….That's our guess, or, my guess anyway, of why it starts wrapping around.
>
> Melody: I like your theory.

Edward thus offered correct reasoning coded as geometric for Task 2. Given this resolution, I asked them how the orientation of the circle's image was determined.

Edward offered further reasoning coded as geometric due to his references to circles "wrapping" and "expanding" after producing an iconic gesture for rotation.

> Interviewer: Alright, and did you figure out how the orientation went?
>
> Edward: Well, Well I guess it kind of….(trails off) (Touches fingertips together, moves hands in a seesaw-like motion so that the base of palms are touching. Touches both hands together, palms facing each other, moves fingertips apart while turning palm outward again, touches palms together again. Rotates hands so palms face outward, back together sharply, then back outward again.) The plane kind of wraps around itself. It gets expanded more when farther away from the origin you get and then it wraps around.

Via gesture without speech, Edward appeared to reason geometrically that the function $f(z) = e^z$ wraps the plane around itself, and claimed that moving away from the origin, the function expands the plane more and more until it finally coincides with itself. This reasoning is close to correct, though only if moving vertically. The function $f(z) = e^z$ can be described as a one-to-one function by "wrapping" the plane into a horizontally oriented cylinder with circumference $2\pi$, and on this cylinder points are rotated by an amount equal to their imaginary part, so there are at least two different ways $f(z) = e^z$ can accurately be said to wrap the plane around itself. It seems here that Edward was referring to the latter. Edward's gestures also appeared to play a role in allowing him to verbalize how he felt the function transformed the plane. In particular, he tried to start describing how the function transforms the circle, but trailed off, went silent, and started producing rotational gestures. After these rotational gestures, he verbalized that the "plane kind of wraps around itself." It is possible that his silent gesturing allowed him to give voice to his geometric reasoning.

After this description, Melody reiterated that when the input circle's radius is more than $\pi$, the output curve wraps around on itself, so I asked her if she knew which points map to the output's self-intersection point. Edward asked me, "can we experiment to find out," and I let him know that they certainly can. Edward initially seemed to feel that the intersection had something to do with the wrapping of the plane they had just discussed, while Melody appeared to explain this behavior as a result of the real and imaginary "distances" being equal. The following exchange thus produced further codes for geometric reasoning due to references to circles rotating and the production of iconic gestures for rotation.

> Edward: Um, I'm thinking it has to do with the, the rotation, being a full, (moves finger as though tracing spokes of a circle through the air while speaking)
>
> Melody: They're at the same distance on the imaginary and the real, which would make it rotate (Twists right wrist in clockwise rotation, traces circle with index finger) the same (third finger extends and moves index and third fingers like cutting scissors), the same distance (Faces right palm flat toward screen, turns palm to face left, and pushes right hand to the left)

Given their rather different theories, I asked Melody and Edward what would happen to the self-intersection point if the circle was left at the same size and in the same orientation and just moved off the origin. Melody predicted the points that map to the same point will change, and Edward offered no competing theory, so they experimented with the aid of *Geometer's Sketchpad* (*GSP*) to determine the behavior. Despite their differing theories, both Melody and Edward seemed surprised when they observed that the relative locations of the points on their circle that mapped to the same point did not change. Edward correctly reasoned geometrically that this occurred because the relative location of the points on the circle are more relevant in determining which two points map to the same point than their distance away from the origin. Edward further stated that he originally felt that the distance from the origin was the controlling factor, but that *GSP* showed him otherwise.

It is worth noting that both groups wanted to dynamically smoothly rotate the entire input circle, spokes and all. However, neither of the participant groups nor I could determine how to carry out this action with the aid of *GSP*. We attempted to select all the spokes inside the circle and rotate them as a group. However, this caused the spokes to rotate at different rates around the circle rather than in unison as desired. Though Edward and Melody made progress in their reasoning about when two points on a circle map to the same point, they never quite precisely verbalized the proper conditions. Edward concluded the second task by stating that "it has to have that full rotation and then have the same line…same radius from the center," which is not quite accurate. In fact, for $f(z) = e^z$, if $z_1$ and $z_2$ are points on the same circle, $f(z_1) = f(z_2)$ precisely when $z_1$ and $z_2$ have the same real component and are vertically separated by some multiple of $\pi$.

## Task 3: Investigating Linear Complex-Valued Functions and the Derivative of Complex-Valued functions With and Without the Aid of *Geometer's Sketchpad*.

### Christine and Zane

To begin Task 3, I asked the participants to describe how they reasoned geometrically about the derivative of a complex-valued function. Table 6 provides a sample of codes generated by participants' reasoning during this task. Because this question was meant to establish a baseline of what the participants knew about the derivative of a complex-valued function, I did not initially allow them access to *GSP*. Instead, I informed my participants that the opening question was very broad and they should feel free to take it in whatever direction they wished.

Table 6

*Task 3 Codes for Christine and Zane*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 14 | Z: as far as breaking it into real and imaginary | x | | | |
| 15 | Z: You can uh...take partial derivative with respect to | x | | | |
| 16 | Z: what is it? If you break the $z$ into $x$ plus $iy$ and do it with respect to $x$ or $y$ | x | | | |
| 89 | Z: the derivative is 3 plus $2i$, cause you just drop a | x | 89 | | |
| 90 | Z: $z$ essentially | x | 90 | | |
| 234 | C: so we did it in like three different steps. We did a stretch factor, and a stretch factor, and a rotate factor, and, one of the stretch factors, and then we added our two stretch factors together, so it gave us an angle that | | x | points at left middle of paper, then middle, then waves pen flat in a circle around middle of paper | |
| 679 | C: every single circle on, every single point on the bigger circle has been stretched and rotated by the factor that we multiplied by $z$ and then it's been translated | | x | | |

*Note*. Alg stands for algebraic and Geo stands for geometric.

When I asked Christine and Zane what they knew about the derivative of a complex-valued function, Christine responded with, “I’m not even sure what thing we did when we found the derivative.” Zane asked me to repeat the question, then responded with many things they had done in class to determine properties of the derivative, though neither participant felt they had a good sense for the types of problems for which the derivative is actually utilized.

> Zane: Well you can use the Cauchy-Riemann equations to find out if they're holomorphic, where they’re complex differentiable so they’re not necessarily differentiable everywhere if they are differentiable. As far as breaking it into real and imaginary, you can, uh, take partial derivative with respect to, uh what is it?... it did $u_x + iv_x$
>
> Christine: Not really. I just kind of understood, "this is what you use when I ask this question," I barely got by.

Thus, Zane and Christine were both aware that they did not have a good sense of how to reason about the derivative of a complex-valued function beyond some basic procedural algebraic reasoning involving the Cauchy-Riemann equations. Zane even seemed to have difficulty recalling these equations. I asked Christine and Zane if they could reason geometrically about the derivative, and they both told me they "never really worked with graphs," except in the context of discussing singularities. The following exchange produced algebraic and geometric codes (see Table 6). In particular, references to algebraically motivated quantities such as rate of change were coded as algebraic (as opposed to the equivalent geometric idea of the slope of the tangent line), while invoking geometric properties such as determining whether a singularity was inside or outside a circle was coded as geometric.

> Interviewer: Tell me what you know about how, what the derivative tells you about the graph of the function, like what you were playing with yesterday
>
> Zane: So you mean the graph of the original function that you're taking the derivative of, or the graph of the derivative…? Okay, so, I mean derivative's just typically like the rate of change…. Cause we never really did too much interpretation from graphs. We'd pretty much always have a function that was defined and never have to really graph out too much.
>
> Christine: Yeah, the only time we really worked with graphs was when we looked at singularities and we were just trying to decide if the singularity was inside the circle or not.

While Christine and Zane may not have known how to reason about the graph of a function given the derivative, at least Zane had developed enough reasoning to distinguish the graph of the original function and the graph of the derivative. He additionally appeared to draw on his reasoning about the derivative of real-valued functions to say that the derivative usually describes the rate of change. When I asked Christine and Zane to described how they learned about the derivative, Zane listed a lot of typical complex analysis class concepts such as "holomorphicity, analyticity, homework problems, check if it's holomorphic, check if it's differentiable, or I guess where it's differentiable if it's harmonic." They additionally noted that the derivative gives the slope of the tangent line.

While trying to generalize this idea with a function, $\mathrm{f}(z) = (3 + 2i)z$—Zane noted within two minutes of the problem statement that such a generalization was difficult. Namely, he stated that the derivative of $f(z)$ is $3 + 2i$, "because you drop a $z$," and he said he did not know how to reason about a rate of change of $3 + 2i$. This reasoning was thus coded as algebraic. However, he at least started reasoning in the right direction when he further stated that reasoning about this rate is difficult because he had to think of $z$ growing in multiple directions. That is, he started developing the geometric reasoning that leads naturally to investigating the ways in which circles are mapped by complex-valued functions. This progression may have occurred because circles provide a way to represent graphically precisely how $z$ grows in multiple directions. While trying to generalize, Zane indicated that such reasoning was not clear to him during the complex analysis course.

> Zane: I haven't really thought too much about the behavior of that as far as on a graph, or geometrically, or anything like that, because I was just kind of "find the derivative." "Oh, you found the derivative. Cool."
>
> Interviewer: So just general impressions. What does that $3 + 2\mathrm{i}$ tell you?
>
> Christine: Is it like the stretch factor of z….? [Our professor] used to say things about stretch factors from like one graph to another.
>
> Zane: Because that is based off like stretch, rotation.

So, while Christine and Zane may have felt convinced they possessed no geometric reasoning about the derivative, at least they remembered their complex analysis professor saying something about "stretch" and "rotation." On the other hand, they did not appear to have well-developed geometric reasoning about this stretching and rotating. For example, when I asked them what was being stretched and rotated, Christine replied, "$z$, whatever you plug in for $z$." This reasoning that the point itself is stretched and rotated, rather than an $\epsilon$ −neighborhood around the point, seems to be a common error throughout all sets of participants, including those in my pilot study.

Zane and Christine introduced a third transformation as well, when Christine claimed that the derivative is "imbue[d] with a displacement property." She also voiced curiosity about whether the stretch or rotation transformation is applied first. When I asked them exactly how the point is stretched and rotated, Christine drew a vector and proceeded to transform the vector according to the function $f(z) = (3 + 2i)z$ (see Figure 24). While Christine's reasoning immediately prior to the following exchange was coded as algebraic, her reasoning in the following exchange was coded as geometric.

> Christine: So then that, so I think that i is what rotates it, cause it, well it rotated and the 2 stretched it. In this case if it's just 2i, cause it became 1 to 2i, so it got stretched to length 2 and then rotated like 90 degrees and then the 3, the 3 just stretched it….So we did it in like 3 different steps. We did a stretch factor, and a stretch factor, and a rotate factor, and, one of the stretch factors, and then we added our two stretch factors together, so it gave us an angle that, I mean I don't know, that's probably 45, but I can't. I don't know what that is.

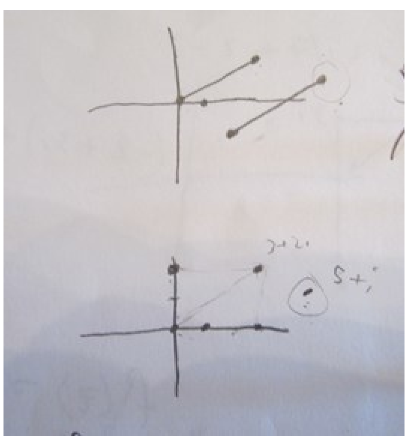

Figure 24. Christine transforms a vector under the transformation $f(z) = (3 + 2i)z$

It appeared that Christine was saying that she rotated her vector by 90° because that is what $i$ does, then stretched the result by 2, stretched the original vector by 3 and finally added the two stretched vectors together. This may be mostly correct geometric reasoning motivated by the algebraic equation $f(z) = (3 + 2i)z = 3z + 2iz$, which is indeed the sum of the original vector stretched by a factor of 2 and rotated by $90°$ and the original vector stretched by a factor of 3. So, in attempting to reason about the derivative, Christine correctly transformed the vector based on the function equation itself. Also note that this way of transforming a vector mirrors an example her complex analysis professor covered in class. That is, her professor noted that for the function $(7 + 2i)z = 7z + 2iz$, the $7z$ "stretches" the vector $z$ by 7, and that the $2iz$ "turns 90°" and "doubles it." This example was discussed in detail in Chapter III under Classroom Setting. Finally, I asked Zane and Christine to generalize their reasoning to $f(z) = z^2$ and $f(z) = e^z$, and Christine noted that her procedure described above did not work for $f(z) = z^2$. Her procedure may have failed because $f'(z) = 2z$ is not quite as algebraically similar to $f(z) = z^2$ as $f(z) = (3 + 2i)z$ is to $f'(z) = 3 + 2i$.

As Zane and Christine did not seem able to make further progress by paper and pencil, I provided access to *Geometer's Sketchpad* (*GSP*). Christine and Zane utilized *GSP* to construct the function $f(z) = (3 + 2i)z$. During her *GSP* investigation, Christine claimed her described procedure only worked if $z$ was real, not complex. Given that her procedure was developed from the equation of a linear function, this statement is not quite true. Her procedure actually works only for linear functions, but for all $z$ within this context. When asked about how to reason geometrically about the derivative of a complex-valued function, Christine just correctly explained again via vector addition how *GSP* maps a vector under the function $f(z) = (3 + 2i)z$, claiming she also had to take "displacement" of the vector into account. After verifying her procedure with the aid of *GSP* (see Figure 25), she stated, "Now I'm even more confident that it works at all points. I just don't understand why it's not working for $z^2$." When I asked Christine and Zane to explain why their procedure did not work for $z^2$, they did not seem sure how to respond, so I suggested they look at how $f(z) = (3 + 2i)z$ transformed circles and vectors with the aid of *GSP*.

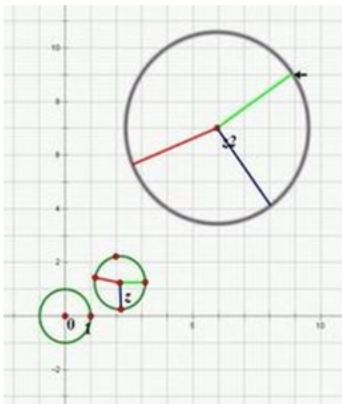

Figure 25. $f(z) = (3 + 2i)z$ transforms green input circle and spokes to blue output circle and spokes of corresponding color

While experimenting with the aid of *GSP*, Zane noticed that the output circle does not rotate as the input circle changes location, and Christine suggested that $z^2$ might involve a reflection in addition to a stretch and rotation. Christine characterized circles as, "really just a bunch of points" during this time as well, suggesting she might still be reasoning about individual points being rotated and dilated by the function rather than $\epsilon$ −neighborhoods. Perhaps to make some additional progress, Christine reasoned geometrically about how the function $f(z) = (3 + 2i)z$ transforms squares on graph paper (see Figure 26), and finally admitted that she too was having trouble generalizing her geometric reasoning about the derivative of a real-valued function as the slope of a tangent line to the complex plane. She stated simply, "I don't know like what slope means in complex world."

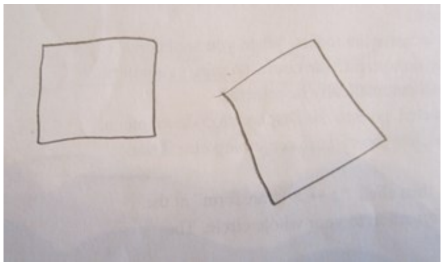

Figure 26. Christine illustrates how $f(z) = (3 + 2i)z$ transforms squares

When I asked them to describe the relationship between the input and the output circle, Christine provided further evidence that she was reasoning about individual points being rotated and dilated, and not how the circle was being rotated and dilated. That is, she noted, “Every single circle on, every single point on the bigger circle has been stretched and rotated by the factor that we multiplied by z and then it's been translated.” Therefore, this reasoning was coded as geometric. Since Christine was fixated on how the function transforms points, I asked them explicitly to construct spokes and tell me how the function rotates the circle itself. I suggested Zane and Christine look back at their linear function in *GSP* and identify properties of the function that do not change, since the derivative is constant.

At the end of the third task, Christine and Zane noted that in the linear function, the stretch and rotation factors do not change, which Christine seemed to take as confirmation of their earlier discoveries. However, they did not verbalize at this time which mathematical entities are stretching and rotating. Thus, it is possible that Christine and Zane were still viewing the individual points as being stretched and rotated by the function. The third task ended with Christine again describing how $f(z) = (3 + 2i)z$ transformed vectors, just as she had reasoned twice before.

**Edward and Melody**

After I removed access to *GSP*, I asked Edward and Christine to describe how they reasoned geometrically about the derivative of a complex-valued function. Edward and Melody first recalled rotation and dilation, though they initially had difficulty determining what object would rotate and dilate. See Table 7 for some coding examples from Edward's and Melody's progress in Task 3.

Table 7
*Task 3 Codes for Edward and Melody*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 372 | E: The amount it twists and the amount that it expands and dilates depends on that point, your function that you're | | x | Places thumb on table next to index finger. Extends index finger forward away from thumb and back. Retracts thumb and touches table at a point again. Touches table several more times, moves index finger left and backward, then back, touches table a few more times | |
| 410 | M: the, would it be the point | | x | | |
| 411 | E: this, this point here, no it, yeah | | x | Taps index finger on table | |
| 412 | M: rotates | | x | E moves right hand left, then points at partner. M Holds hands palms facing each other, fingers curled into C shape, rotates (imaginary ball) to the left while extending her fingers into a curled position | |
| 344 | E: Just the fact that it's the one point is kind of like doing a derivative with a constant. It just does the same thing to all the $z$'s. Which is why when we're moving it around it doesn't change | | x | Holds left palm, fingers curled forward toward screen, holds right hand palm down near left hand, then moves outward and inward with both hands several times. Holds left palm toward screen, doorknob twists once left and right, raises hand slightly and clenches all fingers together | Waves mouse up and down around top right of input circle. Moves mouse left to left side of input circle, back to origin |
| 345 | E: the $f$ doesn't change size | | x | Extends and re-clenches fingers twice more (obscured by screen) | |

*Note.* Alg stands for algebraic and Geo stands for geometric

The following episode produced geometric codes, due to references to geometric actions such as dilation, and twisting.

> Edward: I just remember rotate
>
> Melody: rotate and dilate something?...Does it rotate one eighty always, or, I don't remember
>
> Edward: Well I think that depends, it's kind of like, um, real number, it becomes a complex number that multiplies the point you're looking at that you do the differentiation at… The amount it twists and the amount that it expands and

> dilates depends on that point, your function that you're…So, it would differ, on I think different points that you're at.
>
> Interviewer: What do you think Melody?
>
> Melody: I don't even remember. I just remembered the rotation dilation. I thought I remembered something that rotated…one-eighty. Oh, circles to circles and lines to lines, right?
>
> Edward: That's Mo, the Mubius, remember, Mobius

Melody and Edward correctly recalled that something was rotated and dilated by an amount that depended on the location of the point of interest, though they could not fill in all the details. They further recalled Möbius transformations, which map circles and lines to other circles and lines. When I asked them what exactly rotated and dilated with respect to the derivative, Melody said "it would be the point," and Edward agreed, though a little later, they amended their response to "the whole plane." This response is not technically incorrect, as the derivative function describes how a small $\epsilon$ −neighborhood around a point rotates and dilates. Considering all points at once in some sense yields a description on how every $\epsilon$ −neighborhood across the whole plane rotates and dilates, though this reasoning is different from reasoning about the whole plane rotating and dilating consistently at each point, as it would under a linear transformation.

I asked them to demonstrate this rotation and dilation with the function $f(z) = z^2$ on the blackboard so they could fill in the gaps in their reasoning before applying it to the linear complex-valued function I had prepared in advance. In response, they drew two sets of axes to represent two separate planes and started graphing circles (see Figure 27). On the left set of axes they graphed the input $z$ −plane, and on the right set of axes they graphed the output $w$ −plane as given by $f(z) = 2z$. Using these geometric inscriptions, Melody and Edward started reasoning geometrically about their pictured transformation

$z \to f'(z)$. They first transformed a circle centered on the origin, which they noted dilated, but did not rotate from the input $z$ −plane to the output plane under $f(z) = 2z$, as one might expect. Edward calculated the real and imaginary parts of $z^2$, but suggested afterward that maybe they just picked a circle that happened not to rotate. They transformed a circle centered on the complex number 1 under $f'(z) = 2z$, and then a circle centered around the complex number $2i$. Melody decided that the circles centered around the complex numbers 1 and $2i$ still just dilated, which is correct (see Figure 28). In fact, the function $2z$ itself has a constant derivative of 2, so the function just dilates every circle by a factor of 2.

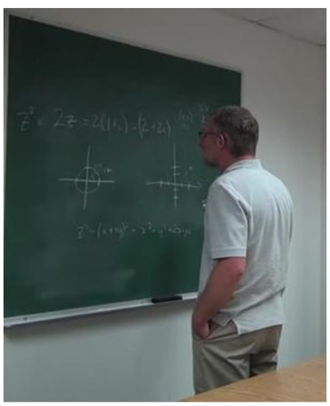

Figure 27. Edward transforms circle on leftmost graph to (undrawn) circle of radius 2 on rightmost graph

Figure 28. Edward transforms a circle under $z \to 2z$

Melody appeared confused about why no rotation occurred, saying "I don't get the rotation part I guess," so I asked why they plotted the function $z \to f'(z)$ rather than $z \to f(z)$. Melody stated simply that I asked them about the derivative, though Edward started wondering whether they should look at a different, but related function. Melody initially defended their choice of function, but eventually decided that looking at the "wrong" planes was the reason no rotation occurred. Thus, Melody directed Edward to plot the output plane under $f(z) = z^2$ on the left set of axes and the output plane under $f(z) = 2z$ on the right, essentially creating the mapping $f(z) \to f'(z)$ (see Figure 29), repurposing the output plane under $f(z) = z^2$ as an input plane for this other function $f(z) \to f'(z)$. In order to accomplish this, Edward first plotted the point $1 + i$ and labeled it $z^2$, then calculated algebraically that $(1 + i)^2 = 1 + 2i - 1 = 2i$. On the plane represented in the rightmost graph in Figure 29, he plotted a point at $(0,2)$ labeled $z$.

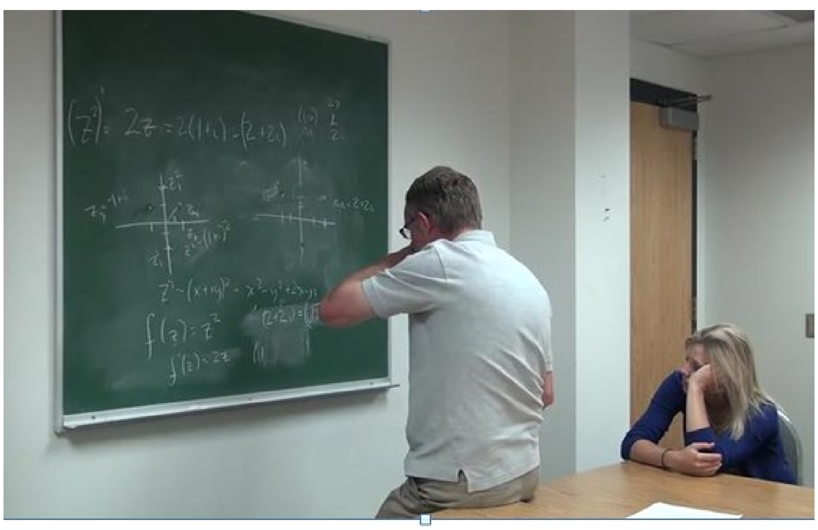

Figure 29. Edward maps $z = 1 + i$ to $z^2 = 2i$ on the left graph and $2z = 2 + 2i$ on the right graph.

Note that despite Melody's instructions, Edward in effect mapped the point under the function $f(z) = z^2$, rather than mapping a point transformed under $z^2$ on the left graph in Figure 29 and a point transformed under $2z$ on the right graph in Figure 29. Melody noticed this breach in her directive, so she stood up to go to the board with Edward. Because she wanted to map the point $1 + i$ under $f(z) = z^2$, she instructed Edward to plot the point $2i$ on the left graph and the point $2 + 2i$ on the right graph (see Figure 30). Thus, she instructed Edward to plot $f(1 + i) = (1 + i)^2 = 2i = (0, 2)$ on the left graph and $f'(1 + i) = 2(1 + i) = 2 + 2i$ on the right graph.

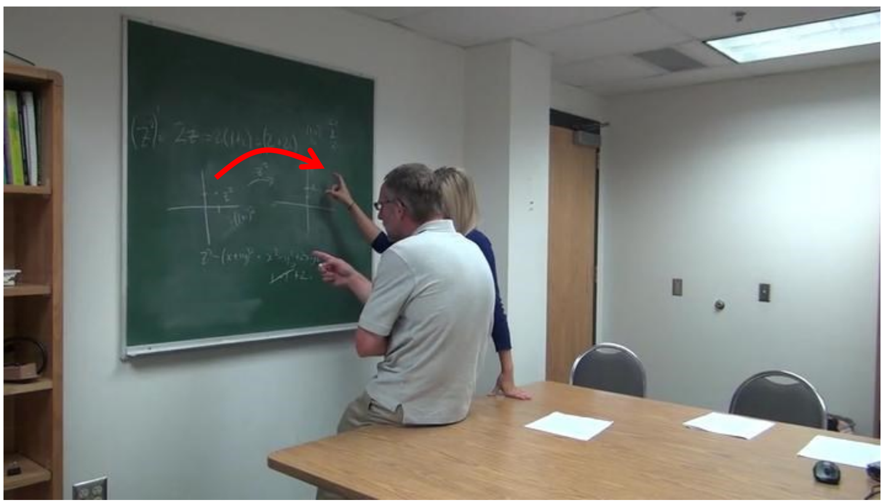

Figure 30. Melody directs Edward to plot a point $z = 1 + i$ transformed under $z^2$ on the left graph and the same point transformed under $2z$ on the right graph

Once Edward and Melody finished constructing this $z^2 \rightarrow 2z$ transformation on the blackboard, Edward said they were still stuck on how to reason about rotation. To address this problem, he converted their points from Cartesian to polar form, which he referred to as Euler's form. Note that polar form does highlight the argument of a point better than Cartesian form, so perhaps Edward signified some ability to reason well about when to switch forms, in contrast to Danenhower's (2006) findings. Edward started trying to explain the rotation in the context of the polar form, though without much success. Reasoning about the polar form was coded as algebraic, while reasoning about how points rotated and dilated was coded as geometric. Thus, this episode produced codes for both algebraic and geometric reasoning.

To determine the proper rotation amount, Edward suggested multiplying $z = 1 + i$ by $2 + 2i$. While doing so, he pointed first at the left-hand side of the equation $2 + 2i = \sqrt{8}e^{\frac{\pi}{4}i}$, then at the point $z = 1 + i$ in the left-hand plane (see Figure 31), and finally at the point $2 + 2i$ in the right-hand plane (see Figure 31). Melody objected to this reasoning, asking, "Why would you multiply?" Edward sighed and responded, "well, maybe I'm still trying to think too much real." It is possible he connected rotation to the polar notation on the right-hand side of the equation $2 + 2i = \sqrt{8}e^{\frac{\pi}{4}i}$, and thus felt multiplication was appropriate. In addition, Edward may therefore be more aware of the dangers of "*Thinking Real, Doing Complex*,*"* (Danenhower, 2006) as he voiced this concern more directly than did the other participants, especially this early on in the interview sequence.

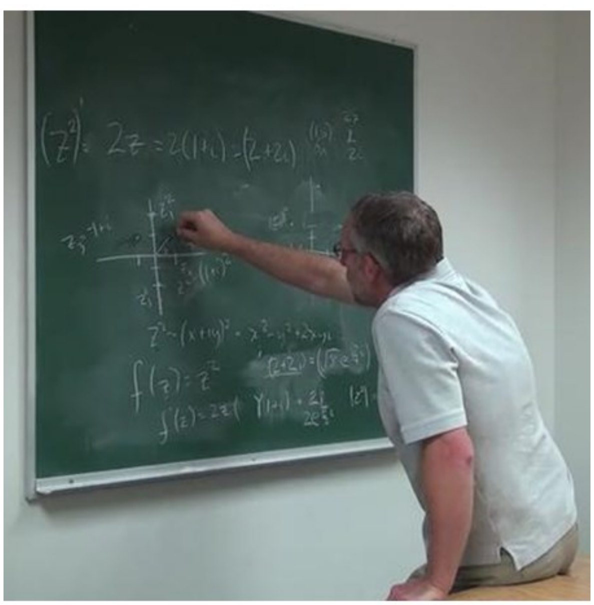

Figure 31. Edward considers multiplying $z = 1 + i$ by $2 + 2i$ to determine the rotation around $2 + 2i$ on the right-hand plane

Given that Edward and Melody mapped $f(z) \to f'(z)$, that is $z^2 \to 2z$, as well as the fact that they mapped points, not circles, the amount their point rotated did not correspond to the argument of the derivative. This also created some difficulty in discussing the function as they are really corresponding the output plane of $f(z) = z^2$ to the output plane of $f(z) = 2z$, though as seen in the following exchange, they appeared to consider the output plane of $f(z) = z^2$ as the input plane for the function $f(z) \to f'(z)$, and the output plane of $f(z) = 2z$ as the output plane. However, Edward still correctly summarized how a point changed from $z^2$ to $2z$. The following exchange produced a code for geometric reasoning, though it was difficult to code much of this exchange as either algebraic or geometric. Rather, it seemed that the algebra informed the geometry.

> Edward: We're mapping from, from the square of z to the derivative over there. In the process of doing that map, mapping it's rotating to the right and then it's, it's doing a dilation, expanding it out to whatever the square root.
>
> Interviewer: Alright, and how do you know how much you're rotating and dilating by?
>
> Edward: Is it doing it by the amount that it is? It's doing it by 45°, $\frac{\pi}{4}$, which is, which is this amount here (points at polar form of $2 + 2\mathrm{i} = \sqrt{8}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\right)$)

Thus, Edward correctly described that from the output plane of $f(z) = z^2$ to the output plane of $2z$ a point $z$ rotates clockwise by the argument of $z$. Because $z^2$ rotates $z$ counterclockwise by the argument of $z$, essentially doubling the angle, and $2z$ has the same argument as $z$, moving from the output plane of $f(z) = z^2$ to the output plane of $f(z) = 2z$ would effectively return the point $z$'s original argument. Following this realization, they further determined the point $z = i$ rotated clockwise by 90 degrees from the $z^2$ plane to the $2z$ plane. Through this experimentation and Melody's subsequent observations, Edward and Melody determined how points rotated from the output plane under $f(z) = z^2$ to the output plane under $f(z) = 2z$. References to geometric transformations like rotation was coded as geometric, though as before it was difficult to tell here whether Edward and Melody were reasoning geometrically or algebraically when referring to the real and imaginary parts of $z$ and $1 + i$. They may have been referring to the $x -$ and $y -$ coordinates on the complex plane, or they may have been referring to the real and imaginary coefficients in the Cartesian representation of a complex number. The first case would have been coded as geometric due to the reliance on a graph, while the second would have been coded as algebraic due to the reliance on an algebraic inscription. Because Melody and Edward did not indicate whether they were utilizing the complex plane or an algebraic inscription in this part of their reasoning, I could not determine whether this reasoning should be coded as algebraic or geometric.

> Edward: I think, so, the rotation is whatever z is
>
> Melody: What the imaginary part of z, not the real part.
>
> Edward: Yeah, because, whatever, whatever this angle is here (indicates angle from origin to the vector 2z)
>
> Melody: Not necessarily the angle, but just the imaginary part of z, like the imaginary part of z was zero, and that wouldn't, didn't rotate. The imaginary part of z was i, that one rotated 90 degrees.
>
> Edward: Yeah, but the imaginary part of $1 + \mathrm{i}$ is
>
> Melody: Was the, oh, that's the 45, okay, yeah, you're right.
>
> Edward: So let's just pick another, and see if it verifies what we're.

Edward correctly described the amount of clockwise rotation as the argument of the vector $z$. Furthermore, despite the fact that Melody was insistent that the amount of rotation was the same as the imaginary part of $z$, Edward eventually convinced her of his geometric reasoning by appealing to the counterexample $1 + i$, which rotated $45°$ clockwise from $z^2$ to $2z$, and not $90°$. Melody's and Edward's example seemed to convince them that their conjecture was indeed correct, and they still seemed very aware of which two planes they were relating to each other. The following exchange was coded as geometric due to the participants' discussion of the angle of rotation while indicating a geometric inscription for this angle via gesture.

> Edward: Whatever z we choose, whatever the angle is from the positive real axis, whatever this angle is (indicates the argument of z), is how much
>
> Melody: The transformation from $\mathrm{z}^2$ to 2z will rotate.

However, after this correct summary, Edward expressed doubt about the direction in which the rotation occurred. To address this, Edward traced through an example with the point $z = -1 + i$. He calculated $(-1 + i)^2 = -2i$ and plotted that point. Then he calculated $2(-1 + i) = -2 + 2i$ and plotted that point on a different plane. Throughout the discussion, Edward held his hand up to the plane and moved his hand counterclockwise in a twisting direction while trying to explain the transformation's behavior. Toward the end of this episode, he twisted his hand clockwise and laughed, "well if you go this way it works," but at this point Melody objected because $-2i$ "is negative," while before the points were above the real axis. Edward noted that he knew the angle was right but he was still uncertain about direction. Melody seemed to be able to alleviate whatever fears he was experiencing, though it is unclear how she did so. That is, it appears she just said that the clockwise rotation direction is indeed correct, and Edward said "okay cool," and smiled, as shown in the following exchange, which was coded as geometric given the participants' references to angles of rotation.

Melody: Oh, because, we were saying that the angle is from, the real, yeah it is right because it was at that this angle, because if it's rotating that way then it's going back up

Edward: It is, it’s rotating that way (traces clockwise from $-2i$ to positive real axis), it goes that way, okay cool.

Thus, I let them know that polar notation gives the amount of counter-clockwise rotation, and asked them why they thought the points rotated clockwise instead. In response, Melody suggested that the angle given in polar notation should be measured toward whichever part of the real axis is closer to the point, while Edward started searching his algebraic inscriptions for something that might have “switched the minus signs.” He reasoned that there must be a missing minus sign to account for a clockwise rotation rather than a counterclockwise one. This reasoning was coded as algebraic due to Edward’s focus on finding an algebraic error in his symbolic calculations. Edward asked Melody if he was missing a sign, and she responded, “I don’t think so, no, because we were measuring this way,” while sweeping her index finger counterclockwise. This is odd because immediately beforehand they had both agreed a moment before that the same rotation occurred in the clockwise direction. Edward appeared to remain unsatisfied, as he continued inspecting his algebra for a missing sign. Given that his reasoning was correct, he could find none. Thus, after a long silence and Edward noting “I’m having a brain freeze”, I assured him that his conversion was correct so he could continue attempting to answer the original question about why the rotation occurred clockwise for a positive exponent in polar notation.

Edward and Melody looked at the point $z = 1 + i$, so $z^2 = -2i = 2e^{\frac{3\pi i}{2}}$ to try to determine why the rotation went in the clockwise direction. After using algebraic calculations to plot the points at the appropriate locations, Edward determined that they were in fact rotating in the negative direction. These calculations were coded as algebraic due to their reliance on symbolic manipulation. Before Edward could offer an explanation for this behavior, Melody contradicted him, saying they were actually rotating counterclockwise by $\frac{3\pi}{4}$, which is incorrect. Moving from $Arg(-i) = \frac{3\pi}{2}$ to $Arg(1 + i) = \frac{3\pi}{4}$ requires a clockwise rotation of $\frac{5\pi}{4}$. However, Melody seemed able to convince Edward to abandon his own correct reasoning and adopt her incorrect reasoning through the use of gesture, apparently just by producing a counterclockwise rotation gesture in the following exchange.

> Edward: So we're actually kind of looking backwards aren't we? Or we were before. We actually showed just, so I think we've determined that it's not rotating positively, it's actually doing the negative by whatever the rotation in z is
>
> Melody: So wait, what, not negative. That's still positive
>
> Edward: But it's doing the same
>
> Melody: It's positive rotating this way, right? (rotates hand ccw)
>
> Edward: That's the, usual way
>
> Melody: Yeah
>
> Edward: Oh, maybe I was misunderstanding your question in the first place. Counterclockwise is the usual
>
> Melody: So it's rotating positive by whatever the angle of z
>
> Edward: is.

At the beginning of this exchange, Edward seemed to realize that they were considering the transformation in reverse, rendering the planes backward in some sense, with $z^2$ on the left and $2z$ on the right, which is much closer to an inverse function of $f(z) = z^2$. However, Melody dissuaded Edward from his reasoning and seemed to convince him by the end of this exchange that the point did not in fact rotate backward. The question I asked that started the whole discussion was in particular about *why* they thought the point rotated the opposite direction, so it is interesting that they seem to have decided the premise of the question was flawed rather than actually trying to answer it. Another possibility is that Edward decided he had misunderstood my question, and adopted Melody's reasoning. When I asked them if there was anything they were still confused about, Melody noted that she expected the whole plane to rotate by the same amount, but she discovered that the amount of rotation was dependent on $z$'s location.

The decision to map $z^2 \to 2z$ caused further problems when I asked them what they knew about a function if they were told it had a constant derivative. Edward initially just said, "well, we wouldn't rotate…real line, the constant would be on the real line" and I clarified that I meant a complex-valued constant derivative, at which point Edward noted that in that case rotations could occur. However, when I gave Melody and Edward the function $f(z) = (3 + 2i)z$ to investigate, they continued their pattern of input and output planes and mapped $(3 + 2i)z \to 3 + 2i$. Melody correctly noted, "so that means everything goes to that one point $3 + 2i$." Edward seemed to forget about his previous statement that rotations could occur and again sided with Melody's reasoning. To this end, he noted that "if the derivative is a constant it has to go to one place over there." This observation may be true in the currently discussed context, but did not particularly seem to illuminate what a constant derivative describes about the original function.

Because the first thing Melody and Edward told me about the derivative was that it described a rotation and dilation, I asked them to reason about how they would determine what they had to rotate by in the function $f(z) = (3 + 2i)z$. Melody dejectedly stated, "I don't think we know what we meant," but Edward correctly reasoned geometrically that any point they picked in the output plane under the function $f(z) = (3 + 2i)z$ had to rotate by the amount required to get to $Arg(3 + 2i)$. This reasoning was coded as geometric due to indications via deictic gestures of points in the plane while discussing geometric ideas such as rotation. Melody noted after this explanation that "the rotation wouldn't be constant for everything either." Because she pointed out the rotation would not be constant but the derivative was, I pointed out that if this was true, the derivative could not possibly describe this rotation. I asked her what it did describe, and at this point she seemed relatively convinced that it did not describe a rotation at all, stating blandly, "it's just a mapping."

A little later, Melody reasoned that to get from $(3 + 2i)z$ to $3 + 2i$, the amount of the rotation would have to be $Arg(z)$, which is correct if the rotation is clockwise. Edward commented that he was "on the cusp of understanding it," though I suspected at the time that significant confusion had arisen due to mapping $f(z) \to f'(z)$ instead of $z \to f(z)$. To help alleviate this potential source of confusion and to redirect them back to a setting more closely related to the concept of an amplitwist, I asked them why, exactly, they were mapping $f(z) \to f'(z)$. Melody responded that this choice was appropriate because they are supposed to look at the derivative, and to obtain $f'(z)$ from $f(z)$ one would have to take a derivative, so it makes sense to plot $f(z)$ alongside $f'(z)$ to visually see how taking a derivative changes the function.

Thus, in effect, Melody and Edward were reasoning fairly correctly about the geometry involved in the change from the $f(z)$ plane to the $f'(z)$ plane, though they both admitted that this investigation of how the derivative $f'(z)$ differs from $f(z)$ told them little about the original function $f(z)$. In fact, I asked Edward and Melody why the derivative is so important that they covered finding derivatives in detail in their complex analysis class, and Edward's response was striking. He said that given the importance placed on it in class that "the derivative's got to do something. It's actually nothing." That is, though class led him to believe that the derivative is an essential mathematics concept, he felt at this stage in the task progression that the derivative really was not quite so important as his complex analysis class implied.

I provided access to *GSP* to Edward and Melody and asked them again to determine how to reason geometrically about the derivative. Melody decided to construct a circle and transform it in *GSP*, while Edward suggested adding spokes to the circle. Once they had this plan established, they constructed the function $f(z) = (3 + 2i)z$ with the aid of *GSP*, the same linear function they investigated previously on the chalkboard. Many of their previous problems simply disappeared because they used *GSP* to map $z \to f(z)$ as I originally intended, so Melody and Edward did not have to worry about whether they should map $f(z) \to f'(z)$ or $z \to f'(z)$ as they had previously. Of course, they could have duplicated these with the aid of *GSP*, but that would have required more intentional construction of these transformations than it did with the blackboard. In particular, I am not certain they were particularly conscious of the fact that they were considering more than one of these transformations during their blackboard investigations. They just told me they were constructing the function $f(z) = z^2$ by plotting a point $z$ and a corresponding point $z^2$ on one plane and another corresponding point $2z$ on a different plane. This suggests that they did not realize they were considering three different possible transformations with this method of graphing.

Perhaps due to this set-up, Edward and Melody were inconsistent in which point they considered the input and which point they considered the output. Duplicating this setup with the aid of *GSP* would have required Melody and Edward to be more purposeful in constructing these transformations. They would at least have had to construct the two distinct transformations $z \to z^2$ and $z \to 2z$, whereas before they thought they only had one transformation to consider, but three points to plot. Thus, just using *Geometer's Sketchpad* (*GSP*) to construct the transformation may have removed the considerable obstacles introduced by trying to investigate the highly related transformations $z \to z^2$, $z \to 2z$, and $z^2 \to 2z$, allowing them to focus exclusively on $z \to z^2$ as originally intended.

While experimenting and observing with the aid of *GSP*, Edward made the critical observation that the input circle is rotated the same amount, regardless of location, in stark contrast to what they had discovered previously about the rotation amount's location dependence under the mapping $(3 + 2i)z \to 3 + 2i$. He made additional important observations that the output circle is never deformed or twisted, and that the dilation is also always the same relative to the circle (see Figure 32). These observations were coded as geometric and allowed him to precisely state what a constant complex-valued derivative told him about how a circle was transformed under the associated linear function.

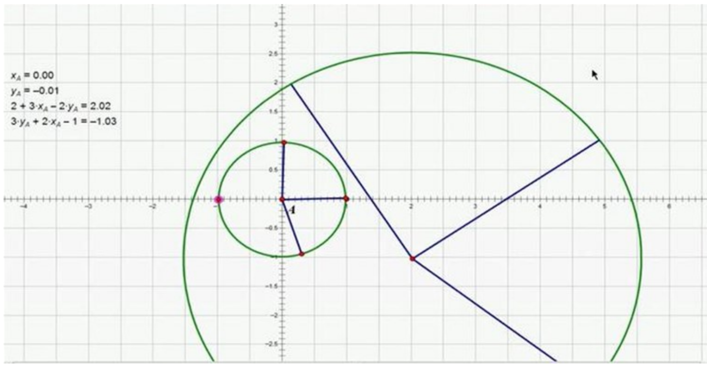


Figure 32. Edward observes that a circle is mapped to another circle under $f(z) = (3 + 2i)z$

In particular, Edward stated, “the derivative is rotating this consistently wherever it is and then it’s expanding it out whatever the length of the derivative is. I guess we can see if that’s true.” This reasoning was coded as geometric due to references to geometric actions such as rotating and expanding. Melody and Edward verified this fact in *GSP* and Edward commented that he did not know if there was anything more they needed to do with this function. On the other hand, Melody felt that they had not yet really considered the derivative. It is possible she wanted to consider the transformation $f(z) \to f'(z)$ as she had on the blackboard previously, rather than just $z \to f(z)$, which does not explicitly involve the derivative.

While Melody seemed to have difficulty letting go of reasoning about $f(z) \to f'(z)$ mapping as she had previously, Edward seemed to realize that the derivative says something about how the transformation $z \to f(z)$ “twists” and “amplifies” the circle. Also note that Edward produced a gesture iconic of an amplitwist in the following exchange. Furthermore, in this exchange, Melody’s reasoning was coded as primarily algebraic, as her main objection was essentially that the transformation $z \to f(z)$ did not include the symbol $f'(z)$, and she additionally referred to the symbolic derivative $f'(z) = 3 + 2i$, for which Melody had not yet offered a geometric interpretation. On the other hand, Edward's reasoning was coded as geometric due to references to geometric properties such as twisting and the fact that the output circle "doesn't change" when dragging around the input circle.

> Melody: Well I guess we only really looked at f of z, to like f of z. We didn't really look at the derivative
>
> Edward: Well I think the derivative is doing, is doing that twisting, twisting the amplification part
>
> Melody: even though it's just going to one point? Cause the derivative of f of z is just the 3 plus 2 i.
>
> Edward: Just the point. I think, just the fact that it's the one point is kind of like doing a derivative with a constant. It just does the same thing to all the z's. Which is why when we're moving it around it doesn't change. (Edward both moves the mouse in GSP here and moves his hands in a circle and then away from each other). The f doesn't change size… and then it's twisting it by whatever that angle, because the derivative is that, that, what 3
>
> Melody: 3+2i, mhm

Notice that while Edward talked about the mathematical objects that the function does not change, he produced a dynamic gesture which may signify rotation and dilation. This gesture suggests that at least some aspect of Edward's reasoning successfully related the derivative of a complex-valued function to a local linearization as described by an amplitwist. It is particularly significant that Edward produced a gesture signifying the relevant types of changes related to the derivative in order to explain why no change occurred in their current situation. In contrast, note that in this same episode Melody was still experiencing difficulty reasoning about the mappings she was considering, as she claimed she and Edward were investigating the identity mapping $f(z) \to f(z)$, which does not make sense in conjunction with their discussion about how the mapping twists and amplifies circles.

However, not only did Edward convince Melody that they had correctly reasoned about the argument and magnitude of the derivative as describing the rotation and dilation of the circle, respectively, but he also related their previous chalkboard investigations to their current discoveries. Namely, he noted that in $f(z) \to f'(z)$, everything mapped to the single point $3 + 2i$, and corresponded that discovery with their observation with the aid of *GSP* that under the mapping $z \to f(z)$, all circles rotate and dilate by the same amounts; amounts that were given by the derivative $3 + 2i$. Melody finally agreed that the transformation $z \to f(z)$ informs us about rotation and dilation better than the transformation $f(z) \to f'(z)$ after some further experimentation with rotating spokes with the aid of *GSP*.

> Melody: So are you saying here, are you saying because like, from z to f of z, since that's like, like a constant in preserving. That's why it goes to the one point?
>
> Edward: Well I think that because the derivative's constant, that whatever it's mapping from the z to the f
>
> Melody: 3 + 2i, mhm.
>
> Edward: It's always doing the same thing to every z in the f plane, so that's why we're getting the same rotation no matter where we are. So when we move around, it doesn't matter, I mean this doesn't get smaller or bigger and it doesn't twist while we're moving around, cause it's constant, I guess
>
> Melody: Okay. That makes sense.

Edward's reasoning in the exchange above was coded as geometric for the same reasons as before: he referred to geometric properties such as rotating, getting "smaller or bigger," and not twisting or dilating when the input circle is moved. I asked them once again to find the rotation and dilation factors, and this time Melody and Edward calculated the length of the vector corresponding to the point $3 + 2i$ as $\sqrt{3^2 + 2^2} = \sqrt{13}$, and experienced difficulty calculating $\theta = Arg(z)$, so I let them know they could use

$\theta = \tan^{-1}\frac{y}{x}$, which seemed to satisfy them. Afterward, Melody and Edward summarized what the derivative told them about $f(z)$. The following reasoning was coded as geometric.

> Melody: So just that the derivative, is, basically, how much it dilates and rotates
>
> Edward: The original
>
> Melody: The original z
>
> Edward: That object thingamajiggy. That's where we finally got the twists from.

Thus, despite the fact that for the entirety of the task up to this point, Melody and Edward investigated how $f(z)$ transformed circles, Melody still felt the rotation and dilation applied to the point $z$. However, Edward's geometric reasoning seemed to advance during his *GSP* investigations to the point where he could say the "object thingamajiggy" is what is being rotated and dilated. His pointing suggests the "thingamjiggy" he referenced is the input circle itself. As he did not just echo Melody's phrasing, it could be that he is was in fact beginning to develop geometric reasoning that the derivative describes how $f(z)$ rotates and dilates a circle.

## Task 4: Investigating the Derivative of Non-Linear Complex-Valued Functions $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$

### Christine and Zane

At the beginning of Task 4, I asked Christine and Zane to generalize their geometric reasoning about the derivative of linear complex-valued functions back to the non-linear functions $f(z) = z^2$ and $f(z) = e^z$. Table 8 provides a selection of codes from this task for Zane and Christine. Recall that Zane and Christine never explicitly verbalized how to reason geometrically about the derivative of a linear complex-valued function. For $f(z) = z^2$, Christine simply summarized, "Last time we found out that it was like that the stretching and rotating thing gave us twice the point we were looking for." They opened their *Geometer's Sketchpad* (*GSP*) lab for $f(z) = e^z$ as well and noted that $f(z) = e^z$ is not typically a nice value, so they focused on values such as $z = \pi i$ for which they knew the value of $e^z$. After some experimentation with the aid of *GSP* on $f(z) = e^z$, Christine lamented, "I mean before there was a nice pattern. In this one it's not a nice pattern." Given their difficulty with $f(z) = e^z$, Christine and Zane returned their attention to $f(z) = z^2$, using *GSP* once again to investigate. Christine noted that the center point of the input circle does not always map to the "center" of the output curve. This fact is due to the way a non-linear function distorts large circles, though Christine did not seem to know this, as she asked Zane why it occurs. Zane said he wasn't sure why those points were special.

Table 8

*Task 4 Codes for Christine and Zane*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 461 | C: I mean, like that whole unit circle is just like folded in on itself, right? | | x | traces unit circle ccw with index finger, extends third finger alongside index finger and moves both fingers in a ccw upper semicircle | |
| 465 | C: the whole unit circle goes to itself, but just like if we'd flipped it over the $x$ −axis | | x | | moves point over to $-1$ and traces lower semicircle of unit circle |
| 490 | Z: Okay, so the definition of derivative | | | | |
| 491 | Z: $f$ of $z$ plus $h$ | x | | | tries to calculate derivative via difference quotient on paper |
| 521 | I: What are you calculating? | | | | |
| 522 | Z: oh just the derivative, the $f$ plus $z$ $h$ minus $f$ $z$ over, er, quantity over $h$ | x | | | |

*Note.* Alg stands for algebraic, and Geo stands for geometric.

I suggested to Zane and Christine that they explore how $f(z) = \frac{1}{z}$ maps a small circle around two different points with the same derivative with the aid of *GSP*. During this investigation, Zane correctly observed that the function transformed each of these circles in the same way. He continued trying to determine how to predict where the "center" of the output curve would be, attempting to reason geometrically.

> Zane: Yeah, just trying to think of how to determine where the center would be, or where the center would translate to. Just, kind of the opposite, where if you're outside the unit circle the point that's furthest away is going to have the shortest distance inside of the new circle.

Given their relative lack of progress, I tried to have Zane and Christine change tactics. Because they had previously suggested that the derivative is the slope of the tangent line and the rate of change, and these are ideas from real-variable calculus, I attempted to help them generalize these ideas. To do so, I wrote the limit definition of the derivative $f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$, and asked them what this algebraic inscription meant to them in a geometric context with respect to the function $f(z) = \frac{1}{z}$. Zane referred to the limit definition as a ratio that became more and more precise as $h$ became smaller. Afterward, Zane started calculating the derivative of $f(z) = \frac{1}{z}$ algebraically while mumbling something about how *GSP* rotated the circles. Near the end of the fourth task, I let them know a derivative of 1 at a point means that a small circle around that point would not stretch or rotate, in the hopes that they might be able to reason geometrically about the derivative at a different point. They chose to look at a point with derivative $-1$ (see Figure 33) and described the circle as either inverting itself or rotating by $180°$, a distinction with which they previously had difficulty. For example, in the following exchange, Christine seemed to believe the transformation inverts the circle, while Zane was initially unsure. The following exchange was coded as geometric due to Christine's references to geometric transformations such as reflection or folding a circle on top of itself.

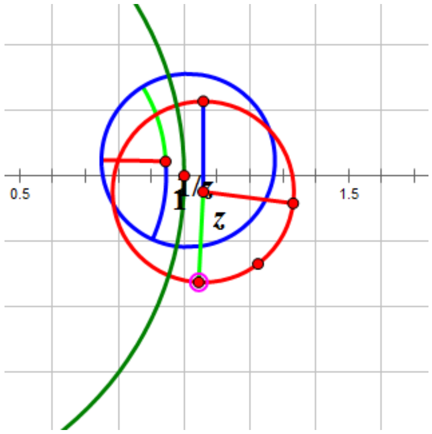


Figure 33. Zane and Christine describe a circle around a point with derivative value $-1$ as either inverting or rotating.

> Christine: The whole unit circle goes to itself, but just like if we'd flipped it over the x $-$axis. See what I'm saying?
>
> Zane: No
>
> Christine: No? Really?
>
> Zane: I guess you could just say it again?( points at right side of unit circle and places left thumb on right side and left index finger on left. Retracts fingers to fist in front of origin and places hand above unit circle, pointing right, palm down)
>
> Christine: So like if we took this arc (moves hand down and places palm flat underneath circles) and we just like took this and folded it down, if you fold it on top of itself, so everywhere is z on this side (traces upper half of circle clockwise starting at right side)
>
> Zane: Okay, yeah, I see what you're saying.

Thus, despite the fact that the transformation is in fact rotating the circle, both Christine and Zane became convinced during this episode that the transformation instead reflects the circle about the $x$ $-$axis. Afterward, I asked them what they thought would happen to a circle that encloses $1 + i$, and told them incorrectly that the derivative at that point was $2i$. This error was not purposeful, and was close to correct, as the derivative at $1 + i$ is in fact $-\frac{1}{2i}$. Furthermore, as I was unaware of the error at the time, I did not correct it. Zane claimed a circle around this point would magnify $2$ and rotate 90° which is correct assuming the slightly flawed derivative value I gave him. Christine, on the other hand, said that in *GSP* "it looked like it halved it and then rotated it by $90$ I think." This is correct given the actual derivative value, and Zane agreed with Christine's assessment. They tested derivative values and geometric behaviors around $z = 1 - i$ where the derivative is $\frac{1}{2i}$ and $z = 2i$ where the derivative is $-\frac{1}{4}$. Around $z = 1 - i$, Zane correctly claimed that a circle around $1 - i$ maps to a circle rotated 90° and dilated to about half the radius, and Christine verified this claim with the aid of *GSP*. Around $z = 2i$, Zane claimed incorrectly that the derivative is $-4$ and that the function does not rotate the circle, though he did correctly note that the function dilates the circle to a quarter of the radius.

At this point, we were close to the end of our two-hour time allotment for the final task, so I concluded the task and let them ask me questions. During this time, I explained that the derivative of a function describes how the function rotates and dilates a small circle around a particular point. I told them that the amount of rotation was given by the argument of the derivative and the amount of dilation was given by the magnitude of the derivative, and Christine commented, "we almost got there." Finally, I let them know that this geometric reasoning about the derivative only applied to very small circles, and they nodded. I concluded the task by showing them a *GSP* lab I had previously constructed of a rational function. They moved circles around in it briefly, commented that the output shapes were "cool," as they were oddly distorted in some cases, and informed me they were glad I did not ask them to work with this last rational function (see Figure 34 and Figure 35). This rational function formed the basis of Task 5 for my second group.

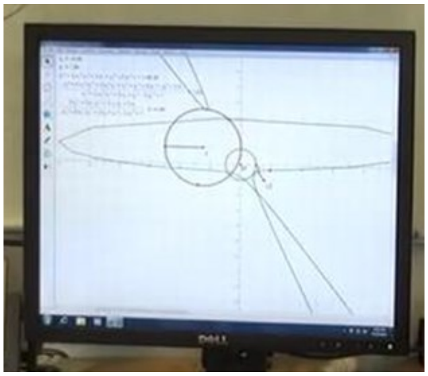

Figure 34. Zane and Christine investigate a rational function’s behavior in transforming circles

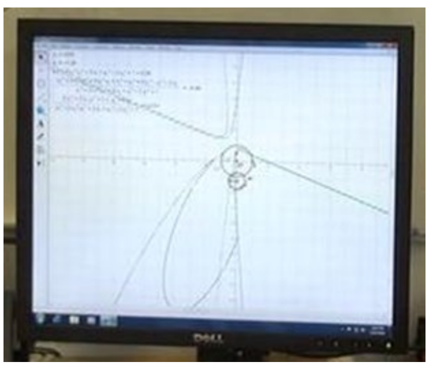

Figure 35. Christine and Zane use a small circle near $(0, -1)$ to locate roots and zeroes of a rational function

**Melody and Edward**

For Task 4, I asked Edward and Melody to generalize their discoveries about the derivative of a linear complex-valued function to the derivative of non-linear complex-valued functions such as $f(z) = z^2$ and $f(z) = e^z$. For sample codes from this section of the interview, refer to Table 9. Similar to Zane and Christine, Melody and Edward opted to begin with the function $f(z) = z^2$. They tried to reason geometrically about the derivative $f'(z) = 2z$ without picking a point $z$ on which to focus. As such, Melody felt that the dilation that occurred would always be by a factor of 2, and only the rotation should depend on the point $z$. Edward corrected her by noting that the dilation also depends on the point $z$.

Table 9

*Task 4 Codes for Melody and Edward*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 494 | M: Yeah. Mhm. Dilating by 2, and rotating | | x | Slides hands along table closer together (reforming circle) and pulling back apart. E Raises hands, left hand palm down fingers point right and forward, right hand palm up fingers point right and forward | |
| 495 | E: And then, I'm trying to remember, (we didn't?) determine what rotated, I'm trying to remember this | | x | Windshield wipes hands twice (from pointing up to pointing right) Drops right hand and leaves left hand in pointing up position, palm right. M Points at screen with right hand | Moves mouse to right end of blue circle, up ccw along circle slightly, back to right end (of circle and spoke) |
| 496 | M: Is it, not (but?), oh it rotated twenty degrees, right? | | x | Points at screen with right hand. Points again at screen with right hand | Rotates blue spoke ccw so it points left |
| 643 | M: So basically here the derivative tells us that it dilates | | x | | E Moves center up (expands) and back down (contracts). Moves blue circle cw into first quadrant |
| 644 | For like, from $z$ to $f$ of $z$ it dilates by $2z$ and it rotates by the angle of wherever $2z$ is at | | | E Points at screen with left index finger. M Taps index finger on table twice (once in front of her and once slightly right), places fists on table and extends second fingers forward and thumbs toward each other | |

*Note.* Alg stands for algebraic and Geo stands for geometric.

During the following exchange, which was coded as geometric due to references to geometric transformations such as rotations and dilations, Edward explicitly admitted that he did not remember exactly which "thing" rotated and dilated.

> Melody: So that would mean that it would, what did that mean? It's the, it's dilating by 2?
>
> Edward: By 2 whatever the length of z is wherever you're at at the time.
>
> Melody: Yeah. Mhm. Dilating by 2, and then rotating
>
> Edward: And then, I'm trying to remember, we didn't determine what rotated, I'm trying to remember this.

Melody algebraically reasoned that the rotation is dependent on $z$ because the derivative $2z$ has a $z$ in it. Afterward, Edward and Melody experimented with the aid of

*GSP* to try to determine precisely what the function rotates and dilates. At first, they conjectured again that the vector $z$ rotates and dilates by the argument and length of $2z$, respectively. Melody then suggested moving the input circle away during their investigations by stating, "what if you move it so it's not wrapping around?" This suggestion may have been motivated by previously established geometric reasoning about how the origin causes strange mappings to occur under the function $f(z) = z^2$. This suggestion was coded as geometric due to references to geometric actions that could be taken with the aid of *GSP* and the geometric idea of "wrapping."

However, after Melody used *Geometer's Sketchpad* (*GSP*) to verify that this prediction did not match what she observed, she made the critical observation that when the circle's center has an argument of $45°$, the circle itself rotates $45°$. Given that this observation involves the geometric property of how a circle rotates, this observation was coded as geometric. Edward followed up this observation by suggesting that if the input circle's center stays on the same radial line from the origin, the amount the function rotates the circle should remain constant. Both Melody and Edward verified this conjecture by experimenting with the aid of *GSP* (see Figure 36).

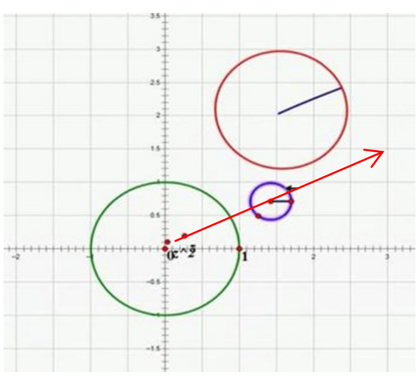

Figure 36. Edward moves blue pre-image circle along ray from origin and observes that rotation of red image circle does not change

After this episode, Melody seemed to reason geometrically in a way that successfully distinguished the rotation and dilation of the circle with the way the function maps the point $z$ itself. However in the following exchange, Edward seemed to conflate the rotation and dilation of the point with that of the circle, though he seemed to feel that something in his geometric reasoning did not quite make sense.

> Melody: So basically here the derivative tells us that it dilates, for like, from z to f(z) it dilates by 2z and it rotates by the angle of wherever 2z is at.
>
> Edward: So z is 1, so it would get amplified to 2. See, that's what I'm, that's what I'm confused about, because I was, if, that this would be at 2
>
> Melody: Like the center (points at (2,0)) would be at 2? So the center of the circle doesn't necessarily depend on the derivative. Like where the output one is located doesn't depend on the derivative (curls fingers slightly into claw, beats toward screen while moving arm counterclockwise in an upper circular arc) The output one is just the size (hand makes claw shape, extends fingers outward and back in) of it and (twists hand first clockwise like a doorknob then back counterclockwise) the rotation not the, like the location would depend on the z squared.

During the entirety of the exchange above, Edward used *GSP* to move circles around on the screen, while Melody tried to explain her reasoning to him via a gesture signifying dilation and a variety of rotational gestures. These exchanges were indications that the participants were utilizing geometric reasoning. For example, Melody repeatedly retracted and extended her fingers to and from a claw-like hand position to signify dilation (see Figure 37). She both twisted her hand like a doorknob and moved her entire hand counterclockwise to signify rotation (see Figure 38). It is further possible that the beat gestures directed toward the screen that occurred while Melody swept her hand in an upper counterclockwise arc were in fact also small extensions and retractions. If so, this particular gesture could have indicated simultaneous dilation and rotation. Thus, Melody's reasoning here was coded as geometric.

Figure 37. Melody produces a “claw-like” gesture for dilation

Figure 38. Melody produces a “doorknob” gesture for rotation

Melody additionally noted that their large circle enclosing $1 + i$ rotates and dilates more oddly than a small circle. Edward expressed confusion about this distortion, but Melody offered a reasonable geometric justification. That is, she observed, “Well wouldn't it like technically dilate differently in different places because the dilation depends on z, so like here, is going to dilate out farther than like here. It only goes like.”

She clarified later that she meant that each part of a large circle would rotate and dilate differently because there was a different $z$ there. This is essentially an accurate justification for why the derivative of a complex-valued function only provides a *local* linear approximation of the function. As this property is a discovery that Christine and Zane seemed to experience difficulty making, I encouraged Melody and Edward to continue advancing their reasoning in this direction by asking them how this behavior compares to the way a small circle rotates and dilates. Melody correctly noted that if the circle is small, each point is very close together so each part of the circle would dilate mostly the same way. She further elucidated that large circles have points farther away from the origin, and instead of offering a further explanation, she asked to use *GSP* to investigate a large circle centered at the origin.

This desire provides further evidence that *Geometer's Sketchpad* (*GSP*) seems to help students develop their geometric reasoning about the derivative of a complex-valued function. Furthermore, while Melody previously distinguished rotation and dilation of a point from rotation and dilation of a circle, during this episode she seemed to interchange these two terms of circles and points. This suggests that differentiating a point from an $\epsilon -$neighborhood is in fact a somewhat subtle distinction given how small an $\epsilon -$ neighborhood is, and that speaking of rotation and dilation of a point as Zane and Christine did during the associated task is not so much incorrect as it is out of context.

Despite the fact that the distinction is subtle, it appears that the ability to reason correctly about the difference is critical to reasoning geometrically about the derivative of a complex-valued function. Christine and Zane never quite made this distinction, and similarly did not quite ever identify what entity the function rotates and dilates. In contrast, Melody appeared to make this leap, and successfully generalized rotation and dilation of a large circle as rotation and dilation of several different small areas of the circle. While she spoke of rotation and dilation of points, she previously voiced that the location where $z$ maps depends on the function $f(z) = z^2$ rather than the derivative. Furthermore, if we take her rotation and dilation of points to mean rotation and dilation of several smaller $\epsilon -$ neighborhoods, her description of why larger circles distort more under $f(z) = z^2$ is entirely accurate—because there are $\epsilon$ −neighborhoods which are far apart and thus rotate and dilate by noticeably different amounts.

When Melody looked at a large circle centered on the origin with the aid of *GSP*, she repeated her reasoning, saying that when the circle is large and not around the origin, different parts of the circle rotate and dilate differently. I asked Edward and Melody if their previous geometric reasoning about the derivative applied to $f(z) = z^2$. Edward suggested, “let’s calculate,” and asked, “which point will be easy?” They experimented and observed how various circles transformed under $f(z) = z^2$ in *GSP*, but had not yet said anything definite, so I asked them what they were wondering. Melody asked why a circle around the origin did not dilate by 2, and ended up reasoning geometrically that the circle wrapping around the origin twice basically *is* a dilation by a factor of 2, as if the radius is twice as large, so is the circumference. This reasoning was coded as geometric due to Melody’s justification. Edward did not seem to have any questions he wished to ask or was able to vocalize. At this point, the third interview concluded, though Task 4 was as yet unfinished.

At the beginning of their fourth and final interview, Melody and Edward started by investigating $f(z) = e^z$ armed with their new geometric reasoning about the derivative of linear complex-valued functions and of $f(z) = z^2$. They noted that $f'(z) = e^z$ as well, and revoiced their previous discoveries about this function, such as the fact that moving in the real direction changes the magnitude of $e^z$, and moving in the imaginary direction changes the argument of $e^z$. These repetitions of previous reasoning again involved references to geometric ideas such as moving circles and points away from the origin, or along the axes, and thus again produced primarily codes for geometric reasoning. Edward and Melody switched to symbolic manipulation to try to determine why this movement along the imaginary axis rotates the point, which appeared to remind Edward of Euler's equation $e^z = e^x(\cos y + i \sin y)$ if $z = x + iy$. As such, this reasoning was coded as algebraic.

On the other hand, Melody recalled that the argument of $z$ was given by the arctangent of some quantity, though she seemed to have trouble remembering the precise equation and she appeared to reason that this calculation would give a different result than Euler's equation. However, Melody correctly summarized how $f(z) = e^z$ transforms $z$ by stating, "I would just say the rotation is $e^{iy}$ and you just plug in the, whatever $z$, like the $y$ part of $z$ is and that's the rotation." This reasoning was coded as both algebraic and geometric. They attempted to check this prediction with the aid of *GSP* with the point $z = 1 + i$, but had difficulty calculating the argument of $e^i$. To obtain an approximate value for $e^i$, they plotted the point $(\cos y, \sin y)$, which was the point on the unit circle with argument $y$. This extra measurement of the proper argument allowed them to verify their prediction with the aid of *GSP*. Melody also eventually correctly reasoned in a way coded as algebraic that the proper amount of rotation of $z$ under $f(z) = e^z$ is $\tan^{-1}(\tan y)$, and I helped her reason that $\tan^{-1}(\tan y) = y$, but I did not explain the intricacies of branch cuts and multiple rotations around the circle. Melody revoiced her discovery while working on this same task:

> Melody: Your y's are going to be close if it's a small circle, and then if you have a large circle, your y's are going to be farther apart and so if you have a line segment here then over here they're going to rotate different ways. They're going to rotate differently.

This reasoning was coded as geometric due to references to geometric objects such as line segments and circles and geometric actions such as rotating, and geometric properties such as being far apart. Even Edward, who appeared to have difficulty advancing his geometric reasoning in this respect, seemed to begin to follow Melody's reasoning. He noted the difference between a point and an $\epsilon -$neighborhood by reasoning aloud, "I was just trying to picture really small. But even if it's really small it's more than zero." So, at the same point in the task progression as Zane and Christine ended, Edward and Melody were able to distinguish the rotation and dilation of points and vectors from the rotation and dilations of $\epsilon -$neighborhoods around those points. The ability to make this distinction may be the primary reason Melody and Edward managed to advance their geometric reasoning about the derivative of complex-valued functions so much further than Zane and Christine. Given that they finished this last task with time to spare in their final interview, while Zane and Christine did not, I administered one final task for the remainder of Melody and Edward's final interview. This task was administered only to Edward and Melody.

**Task 5: Investigating an Unknown Rational Function $\boldsymbol{h(z) = \frac{f(z)}{g(z)}}$**

For Melody's and Edward's final task, I told them I had constructed some function of the form $\frac{f(z)}{g(z)}$ where $f(z)$ and $g(z)$ are polynomials. I asked them to determine where this function was differentiable, and furthermore to determine the value of the derivative at a point of their choosing. Finally, I asked them to construct an algebraic inscription for this rational function. As discussed in Chapter III, this task was primarily motivated by my previous observations of students' repeated difficulties in developing geometric reasoning about the derivative of a complex-valued function, particularly in reasoning about it as a local transformation.

That is, I noticed that all my previous tasks involved a given function and an easily calculated derivative. As part of the tasks, participants described the relationship between the value of the derivative at certain points and how the function transforms small circles around those points. For samples of codes generated during this task, see Table 10. Given that Sfard (1992) suggests that well-developed reasoning necessarily includes the ability to consider mathematical operations in reverse, I decided to include a task that required them to identify both the function and the derivative given only geometric data gleaned with the aid of *GSP*. That is, *GSP* could be used to determine how the function transforms small circles around a given point. In essence, the set-up of this task asks the same question as Tasks 1 and 2 in the reverse direction.

Table 10
*Task 5 Codes for Edward and Melody*

| Line # | Verbiage | Alg | Geo | Gesture | Tech |
|---|---|---|---|---|---|
| 71 | M: That can't be differentiable there | | | | |
| 74 | M: It is weird | | x | | |
| 76 | E: Well that's, that's doing an interesting shape right there | | x | | |
| 92 | E: okay, so to know if this is differentiable, we want | | | | |
| 93 | E: to kind of know when there | | | | |
| 94 | E: goes to, small circle goes to small circle | | x | | |
| 132 | M: So that should still be okay because it's just wrapping around itself I think | | x | | |
| 133 | E: That's differentiable? | | | | |
| 134 | M: I think that is, cause that's like inside, and it's just wrapping | | x | | |

*Note.* Alg stand for algebraic, and Geo stands for geometric.

Melody and Edward first constructed and transformed a circle, and used *Geometer's Sketchpad* (*GSP*) to determine places where the function transforms their circle in an unusual way. For example, Edward once remarked, "why is it doing the flippy thing," while Melody noted a location where the image of the circle "blows up."

Given this visual imagery, these observations were coded as geometric. Throughout this task, Melody and Edward appeared to utilize predominantly geometric reasoning such as this. One possible reason for this is that the task itself provided a geometric inscription, and thus motivated participants to reason geometrically. This suggestion thus complements Soto-Johnson and Troup's (2014) hypothesis that providing an algebraic inscription to participants sways them toward reasoning algebraically.

Before they made any definitive predictions, they constructed and transformed spokes on their circle, after which Melody and Edward started further developing their geometric reasoning through a series of observations with the aid of *GSP*. First, Melody associated strange output behavior with non-differentiability by claiming, "that can't be differentiable there…It's weird." Edward seemed to tacitly agree by continuing to search for locations with odd outputs, and eventually found another while commenting, "Well, that's, that's doing an interesting shape right there" (see Figure 39). These observations were again coded as geometric due to various references to geometric properties such as circles being "weird" or "doing an interesting shape."

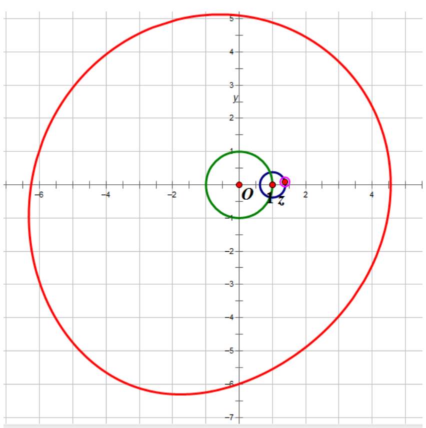

Figure 39. Edward and Melody observe with the aid of *Geometer's Sketchpad* that the red image curve "explodes" as the blue pre-image circle approaches 1

At this point, Edward was able to verbally reason geometrically about what made a point differentiable, by stating, "okay, so to know if this is differentiable, we want to kind of know when, where, goes to, small circle goes to small circle." This utterance is an atypically precise geometric description of the fact that the derivative is a *local* property. Thus, this reasoning was coded as geometric. Through this reasoning, Melody suggested making the circle smaller, possibly to identify points where the circle did not map to a circle more precisely. Through this experimentation and observation with the aid of *GSP*, she identified $(1,0)$ and $(-1,0)$ as non-differentiable points, perhaps due to the slight deformation in the output curve that she observed (see Figure 40).

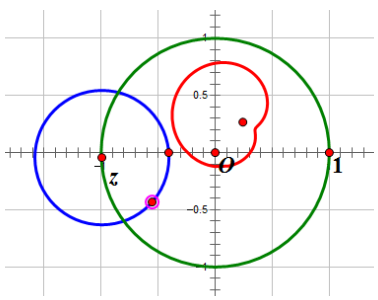


Figure 40. Melody observes a deformation in the red output curve when the blue input curve is centered at $-1$

In contrast, Edward correctly noted that $(-1,0)$ was in fact a differentiable point, possibly by observing that when the input circle is small, the deformation in the output curve diminishes (see Figure 41). Melody later claimed that she had simply misspoken and that she meant $(1,0)$ and $(0,-1)$ were non-differentiable, which was a correct identification. She came to this conclusion by noting that the output curve "explodes" when the input circle is small and centered around these points (see Figure 42 and Figure 43). Melody and Edward also found that near $z = .7 - .7i$, the output was odd in that it wrapped around itself similar to how a circle around the origin under $f(z) = z^2$ mapped to a curve that wrapped around itself (see Figure 44).

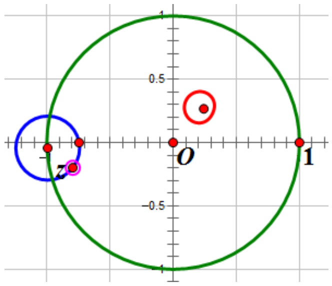


Figure 41. Edward observes a less deformed red output curve when the blue input circle is smaller

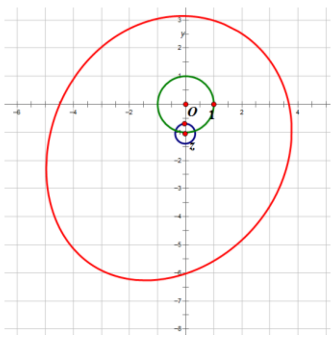

Figure 42. Edward and Melody observe red image curve “explodes” as blue-preimage circle approaches $-i$

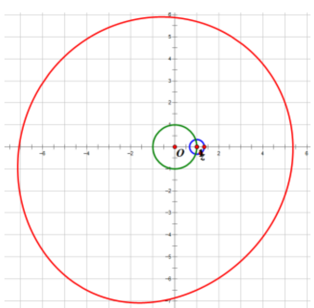

Figure 43. Melody observes red output curve “explode” when small blue input circle is centered at 1

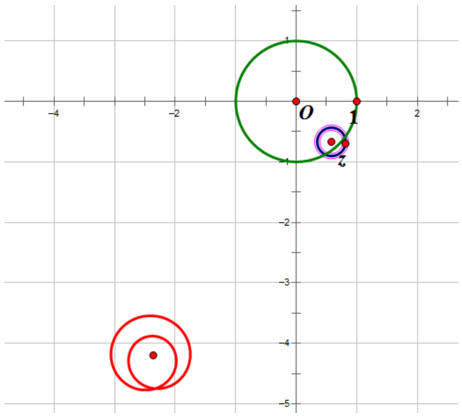


Figure 44. Blue pre-image circle around $z \approx .7 - .7i$ maps to image curve that wraps around twice

This similarity did not appear to be lost on Melody and Edward. The following exchange generated mainly geometric codes due to references to circles “wrapping.”

> Melody: So that should still be okay because it's just wrapping around itself I think
>
> Edward : That's differentiable?
>
> Melody: I think that is, cause that's like inside, and it's just wrapping

Furthermore, Edward incorrectly identified a point near (1,0) as non-differentiable, but Melody correctly countered this claim with her own developing geometric reasoning.

> Melody: I think that’s still differentiable, it’s just huge.
>
> Edward: Did that one blow up yet? Yikes, it still looks (trails off).
>
> Melody: It’s still a circle, it’s just starting to get close to one….It’s still a circle but right when you hit one, I think that’ s when it
>
> Edward: Still makes, still a circle. When it goes crazy, pkow (explosion sound)….So that must mean the g(z) is…z − 1 and z + i

Thus, Edward utilized Melody’s geometric reasoning to develop his own to the point where he could correctly identify $g(z)$ as containing the factors $z - 1$ and $z + i$, because they had found that the rational function was not differentiable at the points

$(1, 0)$ and $(0, -1)$ (see Figure 42 and Figure 43). Both participants' reasoning methods were coded as geometric due to their references to geometric ideas such as "blowing up" or creating a circle. Edward even noted that this task was particularly helpful in developing his reasoning.

> Edward: Now I have to say that this part of the exercise really makes it a little bit easier to understand the whole, you have to keep on shrinking that circle smaller and smaller, to show the differentiable
>
> Melody: The small circle
>
> Edward: I was not getting that in class whatsoever.
>
> Interviewer: So what do you think it was that cemented the necessity of a small circle for you
>
> Edward: Oh, that it has to be a really, well, I, that even when you're…that even though the circle's gigantic here, that we can keep making this one so small that this will eventually be a small circle.

Edward recalled a class conversation that did not make sense to him at the time, but did now. He said he was doing a project where he had a large circle mapped to a "weird thing," and his professor asked him whether a smaller circle might map to a more circular shape. The following exchange was coded as geometric given the references to such geometric imagery as "clover shape" and "shrinking" a circle until it is small enough to map to an image which is also nearly a circle.

> Edward: I didn't quite get it. But, for the particular function, I remember it was kind of like a clover shape, so I was, well, it's a clover shape, because I thought one was small enough, but it isn't small enough because once we start shrinking this down, it gets more and more like a circle."
>
> Interviewer: So what is small enough?
>
> Edward: Small enough to make it a circle, I guess, because it, you know we just get closer and closer and closer to that point. That means just infinitesimal for some things, but not necessarily all things. It just depends on where you are.
>
> Interviewer: Okay, so if you put it on like one

Edward: one, that blows up and you'll just never be able

Melody: Unless like, you're barely off of one, if you're not quite at one

Since Melody and Edward had successfully determined the two points at which the rational function was non-differentiable, and furthermore actually constructed $g(z)$, I asked them to find the value of the derivative at a particular point. In response, Melody and Edward shrunk their input circle down to radius one half and started talking about the rotations and dilations they observed in the output circle. Eventually I realized they were trying to find an actual equation for the derivative, so I asked them once again just to focus on the derivative at a single point.

Edward noted that he was trying to pick an easy point and settled on $z = 0$. He elucidated their strategy to me by saying, "All we have to figure out is for that point, how much it rotates and how much it expands." Melody used *Geometer's Sketchpad* (*GSP*) to measure how much the output circle dilated with respect to the input circle, and claimed that the input circle had radius $.05$, while the output circle had radius $.15$. Edward correctly summarized this by saying the dilation occurred by a factor of 3. Melody also noted that the rotation was "a little more than 3 pi over 2," which is not a bad estimation, as my function $h(z) = \frac{(2z+1)}{(z+i)(1-z)}$ has $h'(z) = \frac{2z^2+2z-(1-3i)}{(1-z)^2(z+i)^2}$, so $h'(0) = -\frac{1-3i}{i^2} = 1 - 3i$. I let them know that the dilation should be $\sqrt{10}$ and that the rotation should be whatever the angle of $1 - 3i$ is.

Edward: One minus 3 i, oh yeah, that's exactly what we have. Yay!

Interviewer: And your dilation is root 10

Edward and Melody together: which is?

Interviewer: Just a little bit bigger than 3

Melody: Okay, yeah

Edward: Okay, wow

Melody: Amazing

Edward: but you have to get that darn circle small

Melody: Yeah

I asked them what questions they would like to ask about this task. Melody asked what the expression for the numerator was, so I asked them to determine a formula. Edward stated they would have to find the zeroes, but algebraically calculated $f(0)$ instead. After some further experimentation with the aid of *GSP*, they determined that $f\left(-\frac{1}{2}\right) = 0$, which is correct. Melody algebraically constructed the polynomial $z + 0.5$, which also has a root at $-\frac{1}{2}$, so I let them know that this was the correct numerator up to a multiplicative constant, as I had used $2z + 1 = 2(z + 0.5)$.

As they had successfully constructed the function, I asked them about some observations they had made previously in *GSP*. In particular, I asked them why they thought the output circle double twisted on itself at about $.7 + .7i$. Initially, Melody reasoned that this twisting occurs when the input circle is large, though they showed with the aid of *GSP* that the double twist forms regardless of the size of the input circle. Thus, Melody and Edward turned their attention to the derivative of that point. Melody suggested calculating the derivative algebraically, while Edward suggested that the derivative should be bigger than 2, because "there's something in the derivative that must be a, um, doubles it up. Doubles up the rate that it rotates." Finally, Melody recalled that the derivative of $f(z) = z^2$ is 0 at the point where the output twists and asked, "so whenever the derivative is zero, it wraps around?" Edward did not provide a response.

At the end of the task, I asked Melody and Edward if their reasoning held for the function $f(z) = \frac{1}{z}$. They correctly noted that $f(z)$ is not differentiable at zero because a small circle around zero should "blow up."

Interviewer: This is one over z, just tell me if anything unexpected happens here

Melody: Well let's see, when z is zero, it wouldn't be differentiable, right?

Edward: Yeah, should blow up

Melody: I guess we should make it small, right? Yeah.

Edward: It's blowing up…So it's not differentiable at z

Melody asked Edward for which points $z$ the derivative of $-\frac{1}{z^2}$ is zero. Edward correctly replied, "nowhere," from which Melody reasoned that "it should never wrap around," which is accurate. Due to such references to geometric actions such as "wrapping around" or "blowing up," this reasoning and the above exchange were both coded as geometric. Finally, Edward and Melody constructed and transformed a circle with spokes and both told me that they would expect the magnitude of the derivative to be the factor by which the circle dilates and the argument of the derivative to be the amount the circle rotates, which is again correct.

However, at the end of the final task, when I asked if there were any questions they would like to ask me, Melody still did not seem sure about the rotation, despite correctly verbalizing correct geometric reasoning just a moment before.

Melody: So, for each function will the derivative, like derivative, the rotation always depend on the argument of…(taps left index finger on table)?

Interviewer: Mhm.

Because she tapped her finger on the table without finishing her question, I assumed she meant to ask whether the rotation is always dependent on the argument of the point a small circle encloses. Therefore, I answered in the affirmative and added that the dilation depends on the magnitude. The full interview sequence concluded with Edward again praising *GSP* for its dynamic nature by saying, "I know you can do some of this stuff in *Mathematica* a little bit but not quite as interactive as this."

## Summary

### Comparison Between Groups

While the first group appeared to favor algebraic reasoning, the second group seemed to prefer reasoning geometrically. This distinction alone may provide a partial explanation for why Melody and Edward appeared to advance their geometric reasoning about the derivative of a complex function further than Christine and Zane. Zane and Christine typically began by reasoning algebraically, then used their algebraic discoveries to reason about the geometric behavior they observed in *GSP*. While Melody and Edward also displayed this progression of reasoning from algebraic to geometric reasoning at times, more often they appeared to begin a new question by reasoning geometrically at first, then transitioning to algebraic reasoning when they wanted to investigate why some specific geometric behavior occurred.

As such, Melody's and Edward's algebraic reasoning often seemed more directed than Zane's and Christine's, though less precise. For example, Melody and Edward referenced the unit circle $x^2 + y^2 = 1$ as an explanation for why $z^2 = i$ when $z = \frac{1+i}{\sqrt{2}}$, though these equations are related only in that $i$ and $\frac{1+i}{\sqrt{2}}$ are both on the unit circle. This more apparently purposeful use of algebra may have helped Edward and Melody connect their geometric reasoning to their algebraic reasoning. In contrast, Zane and Christine suggested at times that algebraic reasoning was preferable to geometric reasoning, almost as though algebra was particularly useful as a way of avoiding geometric reasoning altogether.

Neither group could reason geometrically about the derivative of a complex-valued function at first, though the second group did initially remember the words "dilate" and "rotate," but admitted they did not recall how to reason about the derivative of a complex-valued function via rotations and dilations. Zane and Christine lamented that they knew how to find the derivative, but did not know how or when to use it. By the end of the four-day interview sequence, Melody and Edward characterized the derivative by reasoning geometrically that the argument of the derivative at a point is how much the function rotates a small pre-image circle, and the magnitude of the derivative at a point is how much the function dilates this circle. Christine and Zane, however, never quite verbalized the amplitwist reasoning described in Needham's *Visual Complex Analysis* book, though they did investigate various other stretches and rotations, along with translations and reflections to advance their geometric reasoning.

While Christine and Zane did not advance quite as far as Edward and Melody in either tasks or the development of their reasoning, Melody and Edward may just have possessed more advanced geometric reasoning about the derivative of a complex-valued function than did Zane and Christine at the beginning of their respective tasks. In terms of *Geometer's Sketchpad* (*GSP*) usage, Christine and Zane seemed to utilize *GSP* to answer questions when they were unable to predict, or check conjectures motivated by their algebraic reasoning. In contrast, Melody and Edward appeared to use *GSP* to advance their geometric reasoning about the behavior of the functions with which they experimented, to give them direction about new conjectures to make and new ideas to explore, and to corroborate their geometric conjectures.

Finally, both groups provided additional evidence that students can typically distinguish function behavior from measurement error with the aid of *GSP*, though not always. For example, they both noticed at one point when looking at $f(z) = z^2$ that their pre-image circle was not quite centered on 0 because the image curve was not two perfectly superimposed circles. The image curve was a little off-center, so the students were able to infer that the pre-image circle was also a little off-center. On the other hand, while investigating $f(z) = \frac{1}{z}$, the image curve became so large that the computer started approximating this image curve with a series of contiguous line segments instead of a smooth curve, and both groups expressed interest in discovering the mathematical reason that $f(z) = \frac{1}{z}$ maps a smooth curve to a curve that is not smooth. Furthermore, after being told this was a limitation of *GSP*, both groups asked me whether it was a bug when they observed $f(z) = \frac{1}{z}$ mapped a circle with an internal spoke to a circle with a spoke that appeared to be outside the circle.

**Task 1: Investigating $\boldsymbol{f(z) = z^2}$**

In Task 1, there was a single participant in each of the two groups that noticed with the aid of *GSP* that if a pre-image point $z$ was dragged around the unit circle once, the image point $z^2$ moved around the unit circle twice. In both cases, the other participant commented that they would not have noticed that the image point moved *twice* around the unit circle; they just noticed that it stayed on the circle. As such, it seems that working with the aid of *GSP* in pairs instead of by themselves may have helped them focus on more precise details than they could have on their own. Zane and Christine noticed first that $f(z) = z^2$ maps circles to curves, which are somewhat “oblong,” and “not fully a circle.” So, though they did not progress as far as the other group, they did verbalize at the beginning of the interview sequence that they had noticed an important detail of a circular shape mapping to a shape, which was not quite a circle, but was still in some sense circle-like.

Christine and Zane also recalled that $f(z) = z^2$ maps certain circles to a circle twisted on itself, though they could not at first reason about which circles were mapped in such a way. Through *GSP* investigation, they discovered that circles around the origin are mapped in this way by $f(z) = z^2$. Melody and Edward additionally discovered that the image curve does not loop if the pre-image circle does not contain the origin. Near the beginning of this task, Christine and Zane partly reasoned about the derivative as a local property, apparently one of the hardest advancements in reasoning for my students to make in this interview sequence. Particularly, they noted that $f(z)$ maps a small circle to a “regular old circle,” but the image curve becomes much more distorted when the pre-image circle is large.

Later, when trying to reason geometrically about how $f(z) = z^2$ maps circles, Christine and Zane seemed to have difficulty distinguishing a rotation from reflection, and required experimentation and observation with the aid of *GSP* before they reasoned geometrically about which transformation they observed. Even after this investigation, they appeared to continue to struggle separating rotations from reflections. At the end of the final task, Zane and Christine expressed curiosity about why the output curve “gets really weird near the origin” even before the input curve touches the origin.

Melody and Edward first reasoned geometrically to determine how $f(z) = z^2$ maps various points. Their first noticeable attempt to employ algebraic reasoning was to determine which point $z$ mapped to $i$ under $f(z) = z^2$. However, both their algebraic and geometric reasoning methods were originally incorrect, so they initially had difficulty developing their reasoning on this front. When I suggested they go back to geometry, they recalled that expansion, twisting, and rotation were important geometric attributes to observe, but admitted that they did not know what object they should expand, twist, or rotate.

Both groups of participants experienced some confusion about how to map the function $f(z) = z^2$. Christine and Zane originally calculated algebraically $z \to z^2 \to (z^2)^2$, because in their words, "that's what they were supposed to do." Edward and Melody also constructed exactly this transform at one point accidentally in *GSP*, but seemed to realize that their extra step was unnecessary. However, before using *GSP*, Melody and Edward originally mapped on the chalkboard the transformation $z \to f'(z)$. They noticed that this transformation did not rotate any circles, as $f'(z) = 2z$, so they decided that instead they should map $f(z) \to f'(z)$ to see what the derivative does to the function. Even when using *GSP*, Edward and Melody voiced concern about looking at the function $z \to f(z)$ because this mapping does not seem to involve the derivative function at all.

Melody and Edward verbalized the key observations that a small circle maps to a small circle unless it contains the origin, in which case the image will wrap around the origin. They additionally noted that $f(z)$ maps an input circle to an output curve about double the input circle's size, and suggested finding the amount of rotation next, suggesting they had good judgment about how to advance their geometric reasoning. Edward and Melody even characterized a circle of radius 1 as “gigantic” while they forgot they were zoomed in. Rather than dismiss the point, they took this experience as a learning opportunity to observe that in certain contexts, even a radius of 1 can be “huge”.

Throughout all the tasks, both groups continually conflated rotation and dilation of a single point with the rotation and dilation of a circle or $\epsilon-$neighborhood around the point. As such, both groups predicted that $f(z) = z^2$ maps a circle around $1 + i$ to a circle rotated 90° because $f(z)$ maps $1 + i$ to a point with argument 90°. However, Edward and Melody began to make this distinction near the end of the interview sequence. At the end of the first task, Christine and Zane characterized $f(z) = z^2$ as rotating the entire plane twice.

Finally, both groups sometimes over-generalized their observations made with the aid of *Geometer’s Sketchpad* (*GSP*). In particular, both groups gave geometric explanations about how the image curve moved when the pre-image circle was dragged along either the real or imaginary axis that were only true under certain conditions. In particular, Melody and Edward said that as the input circle moves along an axis, the output curve should “unwrap itself,” which is only true while the circle is moved away from the origin. As such, when they moved the circle toward the origin, they saw that the circle wrapped itself more and decided their geometric reasoning was completely incorrect, rather than simply incomplete. Similarly, Christine and Zane claimed that the output curve should get sucked in as the input circle moved along the axis, which is only true if the input circle is moved toward the origin.

Thus, investigating $f(z) = z^2$ could potentially lead to the following advancements in geometric reasoning related to the derivative of a complex-valued function.

1. $f(z) = z^2$ maps circles around the origin to curves which twist on themselves.
2. $f(z) = z^2$ maps circles which are not around the origin to curves which are not quite circles.
3. $f(z) = z^2$ distorts large circles more than small circles.
4. $f(z) = z^2$ distorts circles close to the origin more than circles away from the origin.
5. $f(z) = z^2$ wraps the plane around itself twice.
6. How $f(z) = z^2$ rotates and dilates circles are relevant characteristics of the mapping to observe.

**Task 2: Investigating $\boldsymbol{f(z) = e^z}$**

While Christine and Zane successfully predicted how $f(z) = e^z$ maps points and used *GSP* to verify their predictions, Edward and Melody just used *GSP* to answer the question directly. On the other hand, Zane and Christine used *GSP* directly to discover how $f(z) = e^z$ maps the axes and various vectors, while Edward and Melody correctly predicted how this function mapped the axes and generalized to a correct description of how the function maps an arbitrary vector. Both groups discovered that $f(z) = e^z = e^x e^{iy}$ maps $z$ by dilating the associated vector by a factor of $e^x$ and rotating it counterclockwise by an angle equal to $y$ if $z = x + iy$. Additionally, both groups discovered that $f(z) = e^z$ maps circles to twisted output curves sometimes, and both assumed that the origin was the cause of the twist in the output, just as the origin was the cause of the twist of some output curves in $f(z) = z^2$. *GSP* experimentation and observation appeared to be necessary for both groups to discover that the origin was not in fact the cause of such twists, and Christine and Zane actually credited their work with the aid of *GSP* with this discovery. Through this same investigation, both groups correctly reasoned that the output curve would have a twist in it when the input circle is large enough.

Even after this discovery, both groups felt the origin might still be the cause of the twists and reasoned geometrically that a small enough circle stayed far enough away from the origin to avoid being mapped in this way. Neither group verbalized that the origin was not the cause of the twist until they discovered that a twist would occur in the output if the input has radius $\pi$ or greater. At this point, both groups noted that the twist was not in fact dependent on the input circle's location as they had originally assumed. Zane and Christine offered no explanation for why a radius of $\pi$ in the input circle causes the output to twist, though Melody and Edward correctly reasoned geometrically that $f(z) = e^z$ rotates circles by an amount equal to the $y-$coordinate of the center of the input circle, and argued that two points on the same circle vertically separated by a distance of $2\pi$ should map to the same point. Only Edward and Melody successfully reasoned geometrically about how the function $f(z) = e^z$ rotates circles. Before reaching the correct conclusion, however, they originally reasoned that $f(z) = e^z$ rotates circles by an amount equal to the argument of the center of the input circle, just as $f(z) = z^2$ rotates circles. With the aid of *GSP*, Edward and Melody determined that this rotation amount did not match their observations, and afterwards no longer claimed that the origin caused twists in the output of $f(z) = e^z$.

Melody and Edward characterized $f(z) = e^z$ as wrapping the plane around itself, which is very similar to its description of how $f(z) = z^2$ transforms the plane. Zane and Christine did not offer an explicit description, saying that they did not feel they had a good description for $f(z) = z^2$ and that the equivalent question for $f(z) = e^z$ felt even harder. Thus, at the end of Zane's and Christine's task, I had them construct the transformation $f(z) = \frac{1}{z}$, and they observed the output "flips" near the origin. Melody and Edward also constructed this function, though at a different time. At one point both groups shrank the input circle origin so small around the origin that the output curve became too large for *GSP* to render it as a smooth curve. As a result, both groups expressed interest in why the output curve had so many sharp corners. I told them there should not be corners and that it was just a limitation of the software. Unfortunately, after this discussion, both groups suspected that $f(z)$ mapping a spoke inside the input circle to a spoke "outside" the output circle was also a software limitation or bug, so I assured them that this particular mapping was correct.

Thus, investigating $f(z) = e^z$ could potentially lead to the following advancements in geometric reasoning related to the derivative of a complex-valued function.

1. $f(z) = e^z$ maps a small circle to a circle-like image curve rotated by $Arg(y)$ with respect to the pre-image circle.
2. $f(z) = e^z$ maps a small circle to a circle-like image curve dilated by $e^x$ with respect to the pre-image circle.
3. $f(z) = e^z$ maps a circle to an image curve with a twist if the pre-image circle has radius $\geq \pi$.

4. $f(z) = e^z$ does not map a circle around the origin to a circle with a twist unless the pre-image circle's radius is large enough.
5. As with $f(z) = z^2$, how $f(z) = e^z$ rotates and dilates circles is an important aspect of the function's behavior

**Task 3: Investigating Linear Complex-Valued Functions and the Derivative of Complex-Valued Functions With and Without the Aid of *Geometer's Sketchpad***

While *Geometer's Sketchpad* (*GSP*) was temporarily unavailable, both groups stated they did not know how to characterize the geometric properties of the derivative of a complex-valued function. Zane and Christine stated that they knew how to find the derivative but did not know what to use it for, while near the end of this task, Melody and Edward claimed that the derivative was "just a mapping" that did not actually tell them anything about the function. Christine and Zane additionally mentioned that the only time they looked at graphs in class was when they were supposed to identify singularities, though they were able to distinguish between the graph of a function and the graph of the function's derivative enough to ask me which of these graphs I wanted them to consider.

During this time, Christine and Zane also mentioned that they thought of the derivative as a rate of change and the slope of the tangent line, and stated they had difficulty reasoning about what a rate of change or slope of $3 + 2i$ means geometrically. Zane and Christine eventually recalled that their professor said something about rotations and dilations, but could not expand on this idea. As with other participants, they initially felt the function's derivative described the rotation and dilation of the point $z$ itself. In contrast, Melody and Edward recalled rotation and dilation at the beginning of this segment of Task 3 before discussing slopes of tangent lines or rates of change. Similar to Zane and Christine, they could not remember what they should rotate and dilate, but they did state that they knew the amount of rotation and dilation that occurred was dependent on the location of the point $z$. Similar to Edward and Melody, Zane and Christine did not consider $\epsilon -$neighborhoods as rotating and dilating originally. Rather, they felt that the rotation and dilation referred either to a single point $z$ or how a function $f(z)$ transforms the entire plane.

While investigating the function behavior of $f(z) = (3 + 2i)z$ without the aid of *GSP*, Christine and Zane added a "displacement" to their stretch and rotation. In particular, they stated that because the derivative $f'(z) = 3 + 2i$, a point $z$ should map to a point constructed by multiplying $z$ by 3, "rotating" a unit vector up to $90°$ because $Arg(i) = 90°$ and doubling its magnitude, and adding these two vectors together to "displace" $z$ by the proper amount. That is, they geometrically described the vector arithmetic involved in calculating $3z + 2iz = f(z)$. Because this describes the mapping of a point and not of an $\epsilon -$neighborhood, this reasoning does not actually involve the derivative $3 + 2i$. As such, Zane and Christine could not successfully generalize this reasoning to $f(z) = z^2$, even after I returned $GSP$ to them. They even stated they were certain of their geometric reasoning about the derivative of $f(z) = (3 + 2i)z$ and were therefore surprised that the same logic did not appear to hold for $f(z) = z^2$. They constructed the linear function $f(z) = (3 + 2i)z$ with the aid of *GSP* and noted that the amount this function rotates and dilates an input circle does not change at all regardless of its location. Similar to Christine and Zane, Melody and Edward also noticed at this point in the task that $f(z) = (3 + 2i)z$ rotates the input circles by the same amount regardless of location and furthermore that the function dilates the circles by a factor of $|f'(z)|$.

During the portion of this task where *GSP* was unavailable, Edward's and Melody's experiences differed significantly from that of Zane's and Christine's. While Zane and Christine asked me which function's graph I wanted them to consider, Melody and Edward became focused on the graph of $z \to f'(z)$, the graph of the derivative function as discussed in the Task 1 summary above, and correctly noted that this function just dilates every circle by a factor of 2. As stated before, they decided this lack of rotation was incorrect and graphed $f(z) \to f'(z)$ instead. Edward graphed $z \to f(z)$ as I intended, but Melody corrected him and changed his transformation to reflect the mapping $f(z) \to f'(z)$ accurately. Melody and Edward even verbalized at one point that they were drawing a diagram of a "transformation from $z^2$ to $2z$."

Edward and Melody seemed to use algebraic reasoning and inscriptions fairly proficiently to determine how to map $f(z) \to f'(z)$ on a chalkboard, and converted to polar form while trying to determine rotations, suggesting they had a better idea of which form to use for certain complex analysis tasks than is typical. Melody and Edward even seemed aware of the difficulty of generalizing from real variable calculus, uttering phrases such as "maybe I'm still thinking too much real." They correctly determined that $z^2 \to 2z$ transforms a point $z^2$ in the $z^2$ plane by rotating it clockwise by $Arg(z)$ and dilating it by 2.

During this discussion, before reaching the correct conclusion, they repeated a common error in this interview sequence by claiming that the rotation is given by the imaginary part of $z$. Melody was adamant that this was correct, and was not deterred until Edward provided a concrete counterexample by pointing out $1 + i$ has real part $i$ but is only rotated $45°$, not $90°$. Furthermore, when I asked Melody and Edward why the rotation was measured clockwise, not counterclockwise as usual, they resolved this conflict incorrectly by stating that actually they were measuring counterclockwise all along, at which point they no longer recognized the same contradiction in their reasoning I had asked them to explain a moment before. Still without *GSP*, Edward and Melody stated that for a constant derivative, the mapping $f(z) \to f'(z)$ would send everything to a single point, which is certainly true. They correctly determined that therefore, this function rotates every point $f(z)$ by $Arg(z)$ clockwise, and could not explain why a function with a constant derivative rotated each point by a non-constant amount.

Returning *Geometer's Sketchpad* (*GSP*) to the participants allowed Melody and Edward to make some advancements in their reasoning as it essentially forced them to concentrate their reasoning on $z \to f(z)$ as I intended, and not on the derivative function $z \to f'(z)$ or the stranger mapping $f(z) \to f'(z)$. Even so, one participant felt that she was looking at the wrong function with the aid of *GSP*, as $z \to f(z)$ does not appear to involve the derivative function $f'(z)$. Nonetheless, the other participant correctly noted that a constant derivative describes how the function $z \to f(z)$ rotates and dilates the input circle, citing the fact that for this function $f(z) = (3 + 2i)z$, every circle is amplified and twisted by the same amount.

At the end of this task, Melody and Edward noted correctly that $f(z) = (3 + 2i)z$ dilates a circle by a factor of $\sqrt{13}$ and rotates it by an angle of $Arg(3 + 2i)$. The participants' utterances did not agree on which object rotates and dilates, as one participant talked about rotating and dilating a point, while the other suggested rotating and dilating "that object thingamajiggy," by which he may have meant the input circle.

Thus, investigating linear functions with and without *GSP* could potentially lead to the following advancements in geometric reasoning related to the derivative of a complex-valued function.

1. The argument and magnitude of the derivative describes how small circles rotate under the function $z \to f(z)$ as opposed to the functions $z \to f'(z)$ and $f(z) \to f'(z)$
2. The argument of the derivative, not the imaginary part, describes how the function rotates and dilates the input circle
3. A linear complex-valued function $f(z)$ rotates and dilates every circle by the same amount regardless of location
4. A linear complex-valued function $f(z)$ rotates every circle by $Arg\big(f'(z)\big)$ regardless of location.
5. A linear complex-valued function $f(z)$ dilates every circle by a factor of $|f'(z)|$ regardless of location
6. The derivative of a linear complex-valued $f(z)$ describes the function rotates and dilates circles, not points.

### Task 4: Investigating the Derivative of Non-Linear Complex-Valued Functions $\boldsymbol{f(z) = z^2}$, $\boldsymbol{f(z) = e^z}$, and $\boldsymbol{f(z) = 1/z}$

Overall, Melody and Edward were much more successful than Christine and Zane in generalizing their geometric reasoning about the derivative of linear complex-valued functions to geometric reasoning about the derivative of non-linear complex-valued functions. During this last task, Christine and Zane noted that the function $f(z) = z^2$ transforms two different circles each centered around different points with the same derivative value in similar ways. However, they became focused on determining which point in the input plane is mapped to the "center" of the output curve, and appeared to make little progress answering this question.

Near the end of Task 4, I informed Christine and Zane that $f(z) = z^2$ did not rotate or dilate a small circle around a point with derivative $f'(z) = 1$. After receiving this new information, they correctly reasoned geometrically about the derivative values $-1$ and $2i$. They noted at a point of derivative $2i$, their circle would rotate by 90° and stretch by a factor of 2, and that at a point of derivative $-1$, their circle would either invert on itself or rotate $180°$. Zane and Christine were not sure which transformation would take effect, just as earlier on they had difficulty distinguishing rotations from reflections. Thus, they could not determine for themselves how the function rotates and dilates small circles, but once I explained a single derivative value to them, they correctly generalized this reasoning to different derivative values.

Melody and Edward originally tried to geometrically reason about the derivative values at all points $z$ in the input plane simultaneously. Because $f'(z) = 2z$, they claimed that the rotation amount should be dependent on $z$ and the dilation factor is 2. Note that this is partially incorrect, as a small circle centered around $z$ should dilate by a factor of $|2z|$, which is only equal to 2 if $z$ is on the unit circle. Initially, Edward and Melody verbalized that they were not sure which thing they should be rotating or dilating. However, they did seem to have some good geometric reasoning about how to use *GSP*. In particular, they suggested that they should move their input circle to a point where the output does not "wrap around." During this experimentation, they observed that the function rotates and dilates the circle itself, when they saw that if the center of the input circle had an argument of $45°$, the function mapped the circle to an output curve that was rotated $45°$ with respect to their input circle. Furthermore, Melody and Edward noted that if this observation is true in general, then the output circle should be rotated by the same amount with respect to the input circle if their input circle is located anywhere on the same radial line the origin. Edward and Melody moved their input circle along a single radial line from the origin and verified that the amount of rotation of the output circle indeed did not change.

During this same experimentation and observation with the aid of *GSP*, Melody again conflated how $z$ maps to $z^2$ with how a circle around $z$ maps to a curve around $z^2$. Edward successfully distinguished between the way the point is mapped from the way the circle around the point is mapped, noting that the point is mapped in accordance with $f(z) = z^2$, while the way the circle rotates and dilates can be extracted from the function $f'(z) = 2z$. Perhaps due to this development of geometric reasoning, Melody and Edward further reasoned that $f(z) = z^2$ distorts large circles more than small circles because large circles contain very different points $z$ within the same large circle. As such, the function $f(z) = z^2$ rotates and dilates each small part of the large circle in its own unique way. Over large distances, this difference in dilation and rotation is noticeable. Thus, in addition to correctly reasoning geometrically about the derivative of linear complex-valued functions, the ability to distinguish a point $z$ from an $\epsilon$ −neighborhood around the same point $z$ may be essential to developing geometric reasoning about the derivative of a complex-valued function. In particular, it seems to be important to have the ability both to reason about how a complex-valued function $f(z)$ maps both a point $z$ and an $\epsilon$ −neighborhood centered on $z$.

Once Melody and Edward felt they could reason about the points $z$ in $f(z) = z^2$ that did not cause a twist in the output, they attempted to extend their reasoning to $z = 0$. They did not check the derivative value at this point. Instead, they reasoned that $f(z) = z^2$ should transform a circle around $z = 0$ by dilating it by a factor of 2, perhaps repeating their previous error of claiming that because $f'(z) = 2z$, the function $f(z)$ should dilate all circles by a factor of 2. They decided this reasoning was correct after observing with the aid of *GSP* that the output circle was twisted twice around the origin, so it had twice the circumference, and that therefore the output was indeed dilated by a factor of 2 with respect to the input.

At the end of this task, Edward and Melody verified that their reasoning developed for $f(z) = z^2$ held for $f(z) = e^z$ in that the derivative value $f'(z) = e^z$ at a point $z$ seemed to predict how the function rotates and dilates a circle around that point $z$, and that the amount of rotation was given by $y$ and the amount of dilation was given by $e^x$. Edward said he made progress in his geometric reasoning about how this function transforms circles by “trying to picture really small,” but “more than zero.” Thus, without directly referring to $\epsilon$ −neighborhoods, he seemed to make good progress toward reasoning geometrically about them, bringing to mind infinitesimals from the historical development of calculus.

Thus, generalizing geometric reasoning about the derivative of a linear complex-valued function to geometric reasoning about the derivative of a non-linear complex-valued function could potentially lead to the following advancements in geometric reasoning related to the derivative of a complex-valued function.

1. A non-linear function $f(z)$ distorts large circles more than small circles because there are many different $\epsilon$ −neighborhoods within a large circle that all rotate and dilate by different amounts.
2. A non-linear function $f(z)$ rotates a small circle counterclockwise by an amount equal to $Arg\big(f'(z)\big)$
3. A non-linear function $f(z)$ dilates a small circle by a factor of $|f'(z)|$
4. The derivative $f'(z)$ does not describe how large circles rotate and dilate. Rather it describes how “small pieces” of this large circle rotate and dilate
5. An $\epsilon$ −neighborhood is in some sense a circle which has a radius that is “really small,” but “more than zero.”

**Task 5: Investigating an Unknown Rational Function $\boldsymbol{h}(\boldsymbol{z}) = \frac{\boldsymbol{f}(\boldsymbol{z})}{\boldsymbol{g}(\boldsymbol{z})}$**

For Task 5, I had previously constructed the transformation $f(z) = \frac{2z+1}{(z+i)(1-z)}$ with the aid of *Geometer’s Sketchpad* (*GSP*) and asked Edward and Melody to determine where $f(z)$ is differentiable, and then to reason geometrically about what the derivative value should be at a point of their choosing. There was not enough time for Zane and Christine to perform this task, but I showed them the function nonetheless. Christine and Zane simply remarked that they were glad I did not ask them to do this task, as they felt they had their hands full with all the functions I had already introduced to them. Melody and Edward, however, vocalized the opinion that this exercise really helped them reason geometrically about why the derivative of a complex-valued function is a local property.

For example, Edward uttered, “Even though the circle’s gigantic here, that we can keep making this one so small that this will eventually be a small circle.” He further recalled a class conversation with his complex analysis professor where he did not realize his circle was too large and that now he could reason that he needed to “make that circle darn small.”

At the beginning of this exercise, Melody and Edward remarked that circles around non-differentiable points should map to outputs that “look weird,” and further characterized “differentiable” to mean that a small circle around that point maps to another small circle. They matched this geometric reasoning with the aid of *GSP* by making their input circle small while looking for strange-looking outputs. Using this method, they correctly identified the two non-differentiable points and additionally noticed a point that mapped to an output that wrapped around itself in the manner of a circle around the origin transformed under $f(z) = z^2$. They did not yet identify this third odd point as a place where the value of the derivative is $0$.

When I instructed Edward and Melody to find an actual derivative value, they again made their circle small and started discussing rotation and dilation amounts at various points, another good geometric strategy. As they seemed to be trying to determine the derivative *function*, I informed them they could focus on a single value. They correctly reasoned geometrically once again and uttered that all they had to do was figure out for that single point, how much a small circle around the point rotates and how much it expands. Using this method, they correctly estimated the magnitude and argument of the derivative value.

Melody and Edward had more difficulty determining the zeroes of the function than the derivative value, though with some leading questions I helped them determine that they needed to find the points $z$ where $f(z) = 0$. At this point they related the twists in this function with the twists in $f(z) = z^2$, and decided that a derivative value of zero means the output twists. While this is not always true, it was true in the few examples with which they experimented with the aid of *GSP*, so the generalization seemed reasonable.

However, this generalization does suggest that if *GSP* is to be used as a teaching tool, the teacher must guard against the dangers of overgeneralization from a small number of examples. Requiring explanations for the geometric reasoning, rather than just allowing my participants to look for patterns in observations, seemed to protect my participants somewhat from overgeneralizing too much, so this may also be a reasonable strategy in the classroom. Thus, the ability to reason geometrically about why certain behaviors occur may be essential in preventing students from overgeneralizing patterns they merely observed occuring. Regardless, by looking for places where the output twists, Melody and Edward correctly identified where $f(z) = 0$, and thereby completed their construction of the function $f(z) = \frac{(2z+1)}{(1-z)(z+i)}$.

Finally, Edward and Melody verified their geometric reasoning about the derivative of a complex-valued function up to this point for $f(z) = \frac{1}{z}$. They noted that a circle around $z = 0$ "blows up," and thus $f(z)$ is not differentiable at $z = 0$. They also correctly pointed out that because $f'(z) = -\frac{1}{z^2}$ is never zero, the output curve should never twist.

Thus, constructing a rational function from experimentation and observation with the aid of *GSP* could potentially lead to the following advancements in geometric reasoning related to the derivative of a complex-valued function:

1. $f(z)$ maps circles around non-differentiable points in a strange way.
2. $f(z)$ maps small circles around differentiable points to other small circles.
3. If $f'(z) = 0$ at some point $z$, a circle around that point may map to an output that self-intersects.
4. The magnitude and argument of $f'(z)$ at a point $z$ does not give the dilation and rotation amounts, respectively, of a large circle around a point $z$.
5. The derivative of a complex-valued function $f(z)$ at a point $z$ can be used to construct a local linearization of the function $f(z)$ near that same point $z$.

## Conclusion

In this chapter, I discussed results of a four-day interview sequence with two sets of participants. Each interview was approximately 2 hours long, and both sets of participants progressed through the same set of five tasks in roughly the same order. However, the second group spent a great deal more time on the final task, while the first group casually experimented with it and observed results with the aid of *GSP* briefly at the end of their final interview.

In Chapter V, I explore implications of this research for teaching and contributions to existing research. Finally, I discuss limitations of this study and suggest possible directions for future research.

# CHAPTER V

# DISCUSSION

The purpose of this study was to contribute to the literature on algebraic and geometric reasoning about complex analysis, specifically as both kinds of reasoning relate to inscriptions and gestures. In particular, this study explored students' reasoning with inscriptions created with a dynamic geometric environment (DGE) in the field of complex numbers. More precisely, it aimed to address the research questions;

Q1 What is the nature of students' reasoning about the derivative of complex-valued functions?

Q2 What is the nature of the development of students' reasoning about the derivative of complex-valued functions while utilizing *Geometer's Sketchpad* (*GSP*)?

In chapter IV, I reviewed findings from a four-day interview sequence for two groups of students. To summarize briefly, I found that Melody and Edward generally seemed to develop a more complete reasoning about Needham's (1997) amplitwist characterization of the derivative than Zane and Christine. This may have been due to Edward's and Melody's strong geometric focus in their reasoning, as this focus appeared to create more opportunities to develop reasoning about the derivative than Christine's and Zane's predominantly algebraic reasoning afforded.

In this chapter, I interpret my findings from Chapter IV in light of the research questions and highlight the ways in which my chosen theoretical perspective of embodied cognition guided these interpretations. After interpreting these findings, I discuss the implications of these findings for teaching and how this study contributes to research in mathematical education. Finally, I conclude with limitations of the study and directions for future research.

Overall, I answer the two research questions listed above in the following ways. To answer the first research question, I argue that my participants reasoned about the derivative of a complex-valued function via embodied cognition in three distinct ways. In particular, they grounded their algebraic and geometric inscriptions via gesture and speech (see leftmost cycle in Figure 45), integrated their algebraic and geometric reasoning methods via these inscriptions (see center cycle in Figure 45), and further grounded these reasoning methods in both the real and virtual environments (see rightmost cycle in Figure 45). To answer the second research question, I detail three developments in reasoning which arose during their work with *GSP*, and seemed critical to my participants' reasoning about the derivative of a complex-valued function.

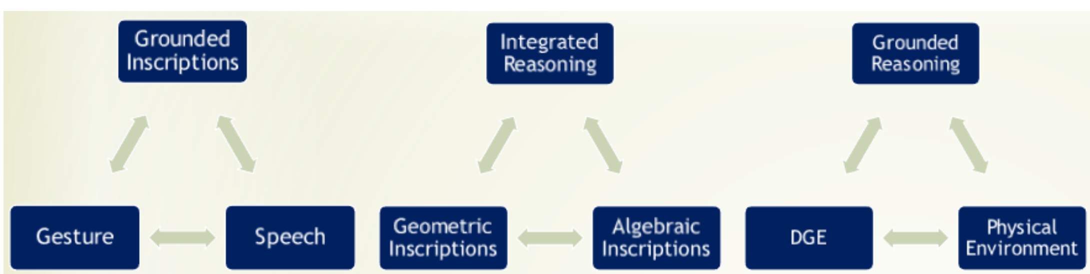


Figure 45. Integration through Embodied Cognition

**Students' Development of Reasoning via Embodied Cognition**

Throughout this project and the progression of the tasks, the theoretical perspective of embodied cognition was leveraged by participants on three fronts (see the three cycles in Figure 45), and capitalized on by the usage of *GSP*. These three types of embodied cognition address the first research question. Thus, in this section, I first elaborate on the three ways in which embodied cognition occurred.

Furthermore, three particular developments in geometric reasoning appeared essential for my participants' developing reasoning about the derivative of a complex-valued function as an amplitwist. These three developments in reasoning address the second research question. Thus, in this section, I additionally elaborate on these three critical developments in my participants' geometric reasoning. I continue by offering support for the conclusion that these three developments in reasoning are indeed an essential part of my participants' reasoning about the derivative of a complex-valued function as an amplitwist.

Finally, my participants displayed the three types of embodied cognition (see Figure 45) throughout these three developments in reasoning. Therefore I additionally offer support throughout this section for the occurrence of at least one of these three kinds of embodied cognition in each of the three critical advancements in reasoning. I conclude this section by discussing each participant group's progress toward reasoning geometrically about the derivative of a complex-valued function as an amplitwist.

## Three Types of Embodied Cognition and Three Essential Advancements in Reasoning

As a result of my dissertation study, I found that my participants' reasoning about the derivative of complex-valued functions leveraged embodied cognition in three distinct ways (see Figure 45). These three ways address the first research question. Furthermore, each of these distinct ways supported three critical developments towards reasoning geometrically about the derivative as an amplitwist. These three developments in geometric reasoning also help in addressing the second research question. Overall, these three types of embodied reasoning (see Figure 45) provide an explanation for participants' usage of *GSP* that seemed not only to help the development of the participants' reasoning but also may have strengthened the participants' abilities to generalize across distinct contexts.

The first type of embodied cognition displayed by participants was their grounding of algebraic and geometric inscriptions via gesture and speech (see leftmost cycle in Figure 45). The second type of embodied cognition occurred when participants integrated their algebraic and geometric reasoning through a combination of algebraic and geometric inscriptions (center cycle in Figure 45). Finally, students demonstrated the third type of embodied cognition by grounding their algebraic and geometric reasoning methods through a combination of the physical and virtual environments (rightmost cycle in Figure 45). These three types of embodied cognition describe the nature of students' reasoning about the derivative of complex-valued functions, and thus address the first research question. Furthermore, they occurred throughout the interview tasks. Therefore, I offer support for these types of embodied cognition while discussing my students' developments in geometric reasoning. That is, I provide support for both research questions simultaneously throughout the following section.

My participants' development of each type of embodied cognition appeared to occur as follows. First, algebraic and geometric inscriptions were grounded via the integrated system of gesture and speech (Goldin-Meadow, 2003). Second, the well-documented gap between algebraic and geometric reasoning (Danenhower, 2006; Dubinsky & Harel, 1992; Panaoura et al., 2006; Sfard, 1991, 1992, 1995; Sfard & Linchevsky, 1994) was bridged via these integrated algebraic and geometric inscriptions (Soto-Johnson & Troup, 2014). Finally, this algebraic and geometric reasoning was grounded in both the physical and virtual environments as both kinds of reasoning take place over both kinds of environment. This cross-environment reasoning may also partially explain why my participants used their experience with *Geometer's Sketchpad* (*GSP*) to make sense of related past class discussions. That is, grounding reasoning across a physical and virtual environment simultaneously may have encouraged them to integrate reasoning from the third environment of their complex analysis classroom as well. Thus, usage of *GSP* or some similar dynamic geometric environment (DGE) may strengthen students' abilities to generalize across distinct contexts, which helps address another long-standing problem in educational mathematics research (Danenhower, 2006; Dubinsky & Harel, 1992; Panaoura et al., 2006; Sfard, 1991, 1992, 1995; Sfard & Linchevsky, 1994). That is, utilizing a DGE seemed to help my participants ground their reasoning in the physical environment via gesture and speech, the virtual environment via algebraic and geometric inscriptions, and their past mathematical experiences (e.g., classroom discussions) via their experiences with the physical and virtual environments in tandem.

The second research question is addressed by the three critical developments in geometric reasoning my participants seemed to require in order to reason about the derivative of a complex-valued function as an amplitwist. These developments address the second research question due to the fact they were all supported by *GSP* as well as at least one type of embodied cognition. These three critical developments are as follows. First, my participants recognized a need for a geometric characterization of linear complex-valued functions, which was supported by grounding algebraic and geometric inscriptions via gesture and speech. Many of these inscriptions were displayed by *GSP*. This development allowed participants to begin extending their reasoning about real-valued functions to the complex-valued case. Before investigating linear complex-valued functions, both groups had difficulty moving beyond reasoning geometrically about how points rotate and dilate.

Second, my participants reasoned geometrically that a linear complex-valued function rotates and dilates every circle by the same constant amounts $Arg\big(f'(z)\big)$ and $|f'(z)|$, respectively. They accomplished this while integrating their algebraic and geometric reasoning through a combination of algebraic and geometric inscriptions. Again, a large portion of these inscriptions wasfor created with the aid of *GSP*. Melody and Edward developed this reasoning with the aid of *GSP*, while Zane and Christine continued to focus on how points rotate and dilate even after correctly observing how circles rotate and dilate. Melody's and Edward's eventual focus on how circles rather than points rotate and dilate may be a main reason why Edward and Melody seemed to develop relatively complete reasoning about Needham's (1997) amplitwist, while Christine and Zane did not develop their geometric reasoning quite as far.

Finally, the participants observed that for an appropriate definition of "small," small circles map to small circle-like objects under any complex-valued function. This observation seemed to be largely supported by my participants' developing ability to ground their reasoning in both the physical and virtual environments. Edward and Melody developed this reasoning most completely, while Zane and Christine had no discussion about what "small" means or even that a "small enough" circle was necessary, though they did at one point observe that circles seem to map to circles. With this final advancement in geometric reasoning, Melody and Edward appeared able to reason geometrically about the derivative as a local property.

By the end of these three critical developments in reasoning, Melody and Edward were able to verbalize the derivative of a complex-valued function as an amplitwist fairly completely, as described in Chapter IV. As Christine and Zane did not develop reasoning about the geometry of constant derivatives completely, or reasoning about smallness at all, they simply verbalized the derivative as a rotation and a dilation of circles, and did not reason about the derivative as a local property.

In the following subsections, I provide examples from both groups' developments in reasoning relative to the three critical developments listed above, as well as offer support for each of the three types of embodied cognition displayed in Figure 45. I additionally discuss my participants' developments in reasoning during the first two tasks of my interview sequence for the sake of completeness. I included these two tasks in my interview sequence to help my students familiarize themselves with *Geometer's Sketchpad* (*GSP*) and with the geometric behavior of complex-valued functions in general. As such, my participants' work with these first two tasks may have contributed to my participants' realization that they first needed to develop geometric reasoning about linear complex-valued functions. Furthermore, within the context of embodied cognition, these initial tasks afforded my participants the opportunity to discuss how to create functions given an algebraic formula and access to a computer program that aided them in investigating the geometry. That is, my participants had the tools required to communicate with each other via gesture and speech and create both algebraic and geometric inscriptions. Thus, in the creation of these first two functions $f(z) = z^2$ and $f(z) = e^z$, my participants may have begun integrating their algebraic and geometric inscriptions with the aid of *GSP*.

Given that my participants' development of their reasoning in these first two tasks may have led directly to their realization that they needed to develop geometric reasoning about a linear complex-valued function, I first discuss students' developments in reasoning before they investigated linear complex-valued functions. I continue by discussing my students' progress toward the three critical developments in reasoning discussed above, and conclude this section by summarizing my participants' progress toward developing reasoning about the derivative of a complex-valued function as an amplitwist.

**Developing Reasoning Prior to Considering Linear Complex-Valued Functions.** While both groups appeared to make beneficial developments in reasoning prior to investigating linear complex-valued functions, neither group could reason about the way circles rotate and dilate under $\boldsymbol{f(z) = z^2}$ or $\boldsymbol{f(z) = e^z}$ before investigating the derivative of a linear complex-valued function. Additionally, although both groups recalled that the derivative was related to "rotation" and "dilation," neither group seemed able to reason about the amount a circle rotates and dilates given the derivative value until after investigating the derivative of a linear complex-valued function. After investigating the derivative of a linear complex-valued function, these difficulties appeared to lessen in both groups.

Within the context of embodied cognition, this may have occurred because participants had possibly not yet sufficiently integrated their algebraic and geometric reasoning to connect the algebraic definition of the derivative with which they were familiar to the associated geometric behavior. That is, while they seemed aware that the derivative $f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, they could not yet reason that the derivative was related to the geometry of small circles under the function $f$. If their algebraic and geometric reasoning had truly been integrated completely, they potentially could have reasoned that in taking the limit in the derivative definition, they would necessarily have to restrict their attention to a small circle around $z_0$. Thus, it is likely that participants either did not reason algebraically about the limit definition of the derivative at all at this early stage, or they had not yet integrated their algebraic and geometric reasoning completely enough to connect the algebraic definition with the expected geometric outcome. As discussed later, this integration appeared to develop while my participants investigated a linear complex-valued function with the aid of *GSP*.

During these initial stages of the interview, my participants reasoned about the functions $f(z) = z^2$ and $f(z) = e^z$. During this time, Zane and Christine repeatedly demonstrated an apparent lack of integration between their algebraic and geometric reasoning. For example, they told me directly they preferred algebra to geometry. In accordance with this self-observation, they consistently relied upon algebra to determine how a complex-valued function maps points, vectors, circles, and line segments. They never converted the Cartesian form to the polar form of complex numbers when determining rotation. Rather, they plotted the Cartesian point and estimated the angle from the real axis based on its location in the plane.

Therefore, Zane and Christine exhibited behavior reflective of Danenhower's (2006) observation that students did not exhibit good judgment when deciding which representation of complex numbers to utilize. This algebraic focus may have thus limited Zane's and Christine's opportunities to develop their geometric reasoning about the derivative of a complex-valued function, particularly at this early stage. They attributed this algebraic focus to their complex analysis class, wherein they discussed how to find the derivative algebraically, how to test analyticity, and how to find non-differentiable points.

Zane and Christine further stated that the only time they looked at graphs in class was to identify singularities. This claim is not true, as documented in the classroom observations detailed in Chapter III. I attended a class where their instructor used a graph to reason geometrically about how a complex-valued function rotates and dilates a small circle with respect to the derivative of that function. However, it seems Zane and Christine did not recall this discussion explicitly and thus could not reproduce this geometric reasoning from memory, though they did state they vaguely recollected a class discussion about rotation and dilation. This memory was so vague that they could not even initially recall these two transformations explicitly, first considering stretches, reflections, and translations instead.

Despite this apparent lack of integration, Zane and Christine did seem proficient in the usage of algebra to avoid geometric reasoning. For example, Christine and Zane utilized algebra to explain to me and to each other various geometric aspects they discovered with the aid of *Geometer's Sketchpad* (*GSP*), such as using *GSP* to explain why $f(z) = e^z$ maps the imaginary axis to the unit circle. However, while they seemed generally able to use algebra to explain and predict basic function behavior such as how points and vectors are mapped, Christine and Zane seemed less skilled in utilizing algebra to explain or predict geometric behavior related to the derivative of a complex-valued function.

In contrast, from the beginning of the interview sequence, Melody and Edward demonstrated proficiency with the algebraic forms in a way Christine and Zane never did. In particular, Edward and Melody converted a complex number from Cartesian form to polar form specifically to consider rotation. Given that polar form highlights how a function rotates and dilates a vector from the origin, this conversion offers evidence that Edward and Melody possessed good judgment about when to use alternative representations of complex numbers. This ability likely helped Melody and Edward integrate their algebraic and geometric reasoning more readily than Zane and Christine. This particular episode additionally provides a direct contrast to Danenhower (2006) research, which suggested that students could not purposefully decide when to convert between such forms as an element of good strategic mathematical reasoning. Thus, my study produced mixed results regarding Danenhower's finding that students exhibited poor judgment about which representation of a complex number to utilize for a given task. The main difference between the groups in this regard appeared to be that Zane and Christine preferred algebraic reasoning, while Melody and Edward favored geometric reasoning.

Melody's and Edward's predilection toward geometric reasoning may have informed their choices about which form of a complex number to utilize. For example, when Melody and Edward attempted to determine how much a circle rotated from $z \to z^2$, Edward noted specifically that the polar notation $z = Re^{i\theta}$ includes information about the angle a vector from the origin would have to be rotated from the real axis to point in the same direction as the vector associated with $z$. This is not to say that Melody and Edward always made such strategic choices. For example, Edward and Melody did not exhibit good judgment about which transformation to investigate while developing geometric reasoning about the derivative by attempting to construct the transformation $z \to (z^2)^2$ rather than the transformation $z \to z^2$. As discussed below, their reasoning for constructing this transformation appeared to fit under Danehower's (2006) classification of *Thinking Real, Doing Complex*.

Thus, while Melody and Edward may have integrated their algebraic and geometric reasoning more than Zane and Christine and Zane at this stage, they still exhibited some incompleteness in this integration. Note also that graphing $z \to (z^2)^2$ for the transformation $f(z) = z^2$ also suggests an incomplete integration between their algebraic and geometric inscriptions. Therefore, in these initial two tasks, both sets of participants appeared to lack some connection between algebraic and geometric inscriptions. Furthermore, through discussion via gesture and speech during these first two tasks, they began to integrate the provided algebraic inscriptions with the geometric inscriptions they constructed with the aid of *GSP*. Furthermore, as discussed below, both groups appeared to generalize their geometric reasoning about complex numbers incorrectly from their geometric reasoning about real numbers.

Given how much teaching in complex analysis courses harkens back to the behavior of real numbers, it should not be surprising that both groups experienced significant difficulties related to real number behavior over the course of their *GSP*-driven investigations, especially before investigating a linear complex-valued function. In particular, they demonstrated Danenhower's (2006) theme of *Thinking Real, Doing Complex* on multiple occasions and contexts. For example, upon seeing instructions to construct the transformation $z \to z^2$ with the aid of *GSP*, both groups constructed the function $f(z) = (z^2)^2$, albeit in different ways. Zane and Christine calculated $(x + iy)^4$ algebraically before realizing that they only had to calculate $(x + iy)^2$. Melody and Edward transformed a circle twice with the constructed transformation $z \to z^2$.

It is possible that this error is simply a case of misunderstood notation in the algebraic expression $z \to z^2$, and the fact that both groups committed the same error may have been caused by the presentation of this expression. It is alternatively possible that this error resulted from each group trying to graph the complex-valued transformation $z \to z^2$ in the same way that they would graph a similar real transformation $x \to x^2$, and something was lost in translation from the field of real numbers to the field of complex numbers.

In the real case, only one graph would be required, with two axes: the input $x$ along the horizontal axis and the output $x^2$ along the vertical axis. Both groups knew that a complex-valued transformation $z \to z^2$ requires two planes to graph, as $z$ is a two-dimensional value itself. However, I believe they may have tried to duplicate the real-valued way of reasoning and reasoned about plotting $z$ as input in one of the planes, and $z^2$ as output on the same plane. This reasoning would have left the second plane empty, and so they may have considered the function $f(z) = z^2$ and reasoned that they should transform the output $z^2$ on the first plane to another output $f(z^2) = (z^2)^2$ to graph on the second plane.

Thus, this strange error could even have resulted from a combination of Danenhower's (2006) *Thinking Real, Doing Complex* and a failure to realize that $f(z) = z^2$ and $z \to z^2$ are equivalent expressions. Regardless of the true reason this error occurred, Melody and Edward exhibited similar reasoning later on in another circumstance. In particular, they argued about what to include on their two planes, further suggesting that they are not entirely sure how to graph a complex-valued function. It is therefore likely that they did in fact draw on their experience reasoning about real-valued functions to answer this unresolved question.

As one final possibility, perhaps because of Danenhower's (2006) *Thinking Real, Doing Complex*, Edward and Melody plotted both the input $z$ and the corresponding output $z^2$ on the same plane, and after deciding $f(z) = (z^2)^2$ was incorrect, they plotted the output of the derivative function $2z$ on the second plane. This setup is even more like the case of relating a real-valued function to its real-valued derivative, where a calculus class might consider the graph of a function on a single plane, and graph next to it the derivative function on a second plane. Thus, in the context of embodied cognition, this method of graphing may have been motivated by Edward's and Melody's previously established embodied reasoning about real numbers. That is, a typical way of introducing the geometry of the derivative of a real-valued function is to graph $f(z)$ on one Cartesian plane and $f'(z)$ on an adjacent Cartesian plane for the purposes of comparison.

Given this education, it is not so odd that Melody and Edward created a nearly equivalent construction for a complex-valued function by graphing the output of $f'(z)$ alongside the output of $f(z)$. As their reasoning was likely strongly grounded in past classes about real-valued functions, Edward and Melody spent considerable time discussing whether this was in fact the appropriate way to graph a function. After tentatively deciding that it was, Edward felt convinced that their investigations in this respect that the derivative "doesn't actually tell you anything." That is, he felt that graphing $z^2 \to 2z$ did not greatly help him in using the derivative to inform him about the function or vice versa. Thus, without graphing the derivative of a linear complex-valued function, Edward and Melody concluded that there was no meaningful way to reason geometrically about the derivative, despite their complex analysis class' heavy focus on calculating the derivative of a complex-valued function in a variety of contexts. This way of graphing essentially vanished once they returned to investigating with the aid of *GSP*, as discussed further below.

While Melody and Edward seemed to experience considerable confusion as a result of graphing $z^2 \to 2z$, Christine and Zane experienced no similar confusion, as they asked me directly whether I wished them to graph the derivative $f'(z) = 2z$ of the function $f(z) = z^2$, or to graph the function itself. Given that I interviewed Zane and Christine before Edward and Melody, I answered this question straightforwardly and they wasted no more time on graphing the derivative function. However, while Christine and Zane appeared to be more directed about how to graph a complex-valued function, Edward and Melody still seemed more aware of the potential dangers of engaging in a style of reasoning reminiscent of Danenhower's (2006) *Thinking Real, Doing Complex*. For example, Edward muttered "Maybe I'm trying to think too much real," when Melody challenged his attempt to connect the rotation of a point about the origin under the function $f(z) = z^2$ to a conversion between Cartesian and polar forms. He had converted to polar notation to determine the amount of rotation as discussed previously, but also suggested finding the rotation by multiplying the input point by the value of the derivative at that point in Cartesian form.

This particular reasoning could have been adapted from the expression of the derivative as a local linearization at a point $(x_0, y_0)$: $f(x) = f'(x_0)(x - x_0) + y_0$. Note that the appropriate way to construct this function is to multiply the value of the derivative at $x_0$ by the distance between $x$ and $x_0$. Thus, it is possible that Edward was simply trying to extend this kind of real-valued formula to the complex setting, but did not recall much of the detail beyond multiplying a point value by a derivative value. Alternatively, his suggestion of multiplying the point by the derivative value at that point may have been motivated by the polar form itself, and the recollection that multiplying two complex numbers adds the angles and multiplies the magnitudes. Whatever the case, Edward in particular portrayed an unusual awareness in the dangers of associating too closely with the real-valued ways of reasoning about similar concepts.

It is strange, however, that Melody and Edward did not repeat their error of referring to the mapping $z^2 \rightarrow 2z$ as the function $f(z) = z^2$ while working with *Geometer's Sketchpad* (*GSP*), as they certainly could have just as easily done this with the aid of *GSP* as they did previously on the chalkboard. While working on the chalkboard, neither Melody nor Edward seemed particularly aware of the fact that the derivative informs us more readily about the behavior of the transformation $z \rightarrow z^2$ than about the behavior of the transformation $z \rightarrow 2z$ or the transformation $z^2 \rightarrow 2z$. Rather, they were insistent that they should graph the output of $z \rightarrow 2z$ somewhere precisely because I asked them to reason about the derivative, and $z \rightarrow 2z$ is the derivative function associated with $z \rightarrow z^2$. This error was thus likely driven both by Danenhower's observed theme of *Thinking Real, Doing Complex* and the nature of the question itself of reasoning geometrically about the derivative of a complex-valued function. This awareness provides further evidence that Edward and Melody were attempting to ground their reasoning about the derivative of a complex-valued function in both their past class discussions about real-valued functions and their current experiences with *GSP* and the physical environment, but had difficulty accomplishing this goal while their algebraic and geometric reasoning were still disparate.

While Danenhower's (2006) *Thinking Real, Doing Complex* summarized a significant barricade to my participants' development of their geometric reasoning about the derivative of a complex-valued function, the usage of *GSP* may have provided ample support to overcome this difficulty. That is, *GSP* may have helped my participants ground their reasoning about the derivative of a complex-valued function in the virtual environment, the physical environment, and their past experiences with real-valued functions. In particular, it may have helped them accomplish this goal by first providing them with an opportunity to integrate their algebraic and geometric reasoning via the tandem usage of algebraic and geometric inscriptions that *GSP* requires.

For example, in one episode, Zane made a string of claims based on the function formula about the behavior of the complex-valued function $f(z) = e^z$ that Christine immediately disproved with the aid of *GSP*. In this episode, when Zane suggested that $f(z) = e^z$ could not take on a negative real value because $e^x$ "couldn't be negative," Christine positioned an input point $z$ so that the corresponding output point $e^z$ had a negative real value. Despite the fact that Zane's comment did not particularly make sense, as complex numbers cannot appropriately be considered positive or negative, Christine still felt she had produced a counterexample to Zane's statement. Christine even laughed and quipped, "basically, whatever you say, Zane, I can do."

Both this particular finding and my participants' interactions with *GSP* as a whole provide supporting evidence for Salomon's (1990) claim that dynamic geometric environments (DGEs) help students by providing interactivity, intelligent guidance, dynamic feedback, and multiple representations of mathematical objects. For example, in the above episode, Christine disproved many of Zane's claims as a direct result of the dynamic geometric feedback provided via *GSP*. Without this instantaneous feedback, Zane and Christine may not have disproved these claims quite so quickly. Salomon's claim is further supported by Edward's observation at the completion of the tasks: "I know you can do some of this stuff in *Mathematica* a little bit but not quite as interactive as this." Furthermore, the barricade posed by participants' tendency to engage in Danenhower's *Thinking Real, Doing Complex* appeared to be significantly reduced by their investigations of a linear complex-valued function with the aid of *GSP*. As such, *GSP* appeared to help them integrate their algebraic and geometric reasoning by helping them coordinate their algebraic and geometric inscriptions.

Thus, my students' reasoning about the derivative of a complex-valued function may have been supported by *Geometer's Sketchpad's* (*GSP*) high level of interactivity. In particular, it seemed to provide them with an opportunity to integrate their reasoning about the geometry of a linear complex-valued function with their previously embodied algebraic reasoning. Without this dynamic feedback, my students may not have quite as easily noticed discrepancies between their algebraic reasoning and their associated geometric reasoning. Thus, dynamic feedback in particular may be an essential characteristic of a DGE in furthering students' geometric reasoning about the derivative of a complex-valued function. This observation is supported by Salomon's (1990) suggestion that dynamic feedback is one particular aspect of DGEs that seems to support students' mathematical reasoning.

Just as algebraic and geometric inscriptions provided by *GSP* may have helped my participants connect their algebraic and geometric forms of reasoning, gesture may have helped my participants ground their embodied experiences in both the physical and virtual environments. In particular, gesture appeared to help my participants reason in an embodied way about the geometric behavior observed with the aid of *GSP*. For example, Zane and Christine both employed iconic gestures to reason geometrically about whether $f(z) = z^2$ rotated or reflected their input circle. Christine believed the transformation was a reflection. While stating this belief she produced a reflection gesture by starting with her left hand below her right hand with palms facing inward, then switching the position of her hands by bringing her left hand up in front of her right hand and above. In contrast, Zane suggested the transformation was a rotation. While explaining his reasoning, he produced a rotation gesture by pointing his right hand's fingers downward, then rotating his hand clockwise until his thumb pointed upward.

In addition to helping Christine and Zane embody their geometric reasoning, this distinction in gesture may have helped Zane and Christine recognize the discrepancy in their reasoning and begin to geometrically reason more purposefully about whether the transformation was in fact a rotation or reflection. Nonetheless, Christine and Zane did not seem to notice that the amount the circle rotated and dilated was directly connected to the derivative value at the points in the area enclosed by the circle. Without investigating linear complex-valued functions, they seemed limited to reasoning simply that the circle rotated rather than flipped, and not reasoning about how much the circle rotated in the context of the derivative value at the appropriate point.

Just as Zane and Christine used gesture to embody their reasoning, Edward similarly embodied his geometric reasoning about how to multiply two complex numbers. In particular, he demonstrated his reasoning about dilation through hand gesture by holding his hands together and then widening them apart (see Figure 5 in Chapter IV). In the same sentence, he embodied geometric reasoning with his entire body when he spoke of how multiplication of two complex numbers rotates the numbers while he turned toward his partner (see Figure 6 in Chapter IV). He again gestured for dilation during this process by pointing forward and extending his arm while stating that multiplying two complex numbers dilates the numbers (see Figure 7 in Chapter IV).

My participants' reasoning about the derivative of a complex-valued function was further supported by gesture's ability to help my participants communicate with each other and themselves (Goldin-Meadow, 2003) when they lacked the appropriate words. In many instances when my participants trailed off a sentence or did not speak at all, they moved their hands around during the silence, often with a contemplative or thoughtful expression showing on their faces. These gestures may not have always been iconic of the concept they were trying to recall. For instance, while Zane attempted to explain why a circle intersecting the origin maps to a curve with a sharp point, he moved his hands laterally apart and together, touching his index fingers' tips together. Still silent, he held his fists together and extended and retracted his index fingers. Finally, he stated the behavior was because there was only one "singularity type thing." He further clarified that he meant the mathematical entity represented by the algebraic inscription $b^2 - 4ac$, whereupon I informed him he was speaking of the discriminant. Thus, the usage of gesture helped Zane communicate his reasoning that the output curve possesses a sharp point because the discriminant is zero due to the fact that their input circle intersects the origin. This episode provides further support for the claim that gesture and speech helped my participants integrate their algebraic and geometric inscriptions.

In summary, during the first two tasks, my participants mainly seemed engaged in integrating their algebraic and geometric inscriptions via gesture and speech with the aid of *GSP*. On the other hand, they struggled in integrating their algebraic and geometric reasoning in these early stages, even with the aid of *GSP*. For example, while my participants made many intriguing algebraic and geometric observations, they also verbalized that they could not adequately reason about a complex-valued function before considering functions such as $f(z) = (3 + 2i)z$ in detail. They further mentioned that they felt this difficulty arose because they did not understand what a "line" is in the field of complex numbers. They seemed to arrive at this impasse by first considering that the derivative of a real-valued function describes the slope of a tangent line. This recollection led them to consider reasoning about the tangent line of a complex-valued function, which in turn led them to realize that they could not verbalize the meaning of "tangent" or "line" in this new context.

As discussed below, participants began to reason about the amount circles rotate and dilate under a given function after investigating complex-valued linear functions with the aid of *Geometer's Sketchpad* (*GSP*). In particular, they characterized complex-valued linear functions as functions, which always rotate and dilate a given circle by the same values, regardless of the size or location of the circle. Thus, participants could not adequately reason geometrically about the derivative of a complex-valued function geometrically prior to investigating complex-valued linear functions with the aid of *GSP*.

Therefore, investigation of linear functions with *GSP* was a critical development of reasoning about the derivative of a complex-valued function. In the context of embodied cognition, it seemed that the investigation of linear complex-valued functions helped my participants integrate their algebraic and geometric reasoning via inscriptions provided by *GSP*.

**Reasoning Geometrically about the Derivative of a Linear Complex-Valued Function.** Christine and Zane did not appear able to completely integrate their algebraic and geometric reasoning even while investigating a linear complex-valued function $\boldsymbol{f(z) = (3 + 2i)z}$. Rather, Christine reasoned geometrically about the derivative value $\boldsymbol{3 + 2i}$ by stating that this derivative value tells us that $\boldsymbol{f(z) = (3 + 2i)z = 3z + 2iz}$ rotates an input vector $\boldsymbol{90°}$, stretches it by a factor of $\boldsymbol{2}$, and adds on the vector obtained by stretching the input vector by a factor of $\boldsymbol{3}$. This is a correct explanation of the vector addition of $\boldsymbol{3z}$ and $\boldsymbol{2iz}$, and thus works for the case of linear complex-valued function, but does not appear to generalize easily to the non-linear case without restricting the investigation to vectors within small circles. Indeed, after investigating $\boldsymbol{f(z) = (3 + 2i)z}$ with the aid of *GSP*, Christine stated she was confident her process would work in general, but could not understand why it did not seem to hold for $\boldsymbol{f(z) = z^2}$. Of course, her process would work approximately if she restricted $\boldsymbol{f(z) = z^2}$ to some small enough disk, as then $\boldsymbol{f(z) = z^2}$ would appear approximately linear within the area of that disk.

Zane and Christine did eventually notice with prompting that a linear function always rotated and dilated their input circle by the same amount regardless of location, but Christine seemed to focus on how the function transforms individual points on the circle, rather than the circle itself. Thus, Zane's and Christine's apparent failure to gain insight on how derivative values relate to the rotation and dilation of circles in the case of linear complex-valued functions may have contributed significantly to their continued difficulty in developing their geometric reasoning further about the derivative of a general complex-valued function.

In contrast, once Edward and Melody appeared to make further progress integrating their algebraic and geometric reasoning while investigating a linear complex-valued function. In fact, once they actually mapped the appropriate function $f(z) = (3 + 2i)z$ with the aid of *GSP*, they discovered that the input circle rotates and dilates the same amount, regardless of location. Unlike Zane and Christine, once Melody and Edward focused on the appropriate transformation, they integrated their geometric reasoning and algebraic reasoning about the derivative $3 + 2i$ of this linear complex-valued function. In particular, Edward stated that the function rotates every circle by $Arg(3 + 2i)$ and dilates every circle by a factor of $|3 + 2i| = \sqrt{13}$. Melody initially felt that $z \to (3 + 2i)z$ was the inappropriate function to consider, and that they should instead consider the transformation $(3 + 2i)z \to z$. Edward appeared to convince Melody that his reasoning was correct through a combination of gesture, speech, information from *GSP*, and a prior discussion during this interview sequence. Thus, informed by his own grounded reasoning, Edward helped Melody embody her reasoning by providing her the means to ground it in his gesture and speech.

In particular, Edward moved the mouse cursor around a linear complex-valued function constructed via *Geometer's Sketchpad* (*GSP*), then moved his hands in a circle and then apart from each other, likely signifying rotation and dilation. Simultaneously, he claimed that having the same derivative at every point results in the function doing the same thing to every circle. He also related this observation to their previous investigations of the constant derivative function $(3+2i)z \to (3+2i)$ that sent everything to the single point $3+2i$ by noting that because the derivative is constant, the function $z \to (3+2i)z$ always rotates and dilates every circle the same way. He additionally noted that as a result of a constant derivative, the function does not cause twists such as those seen in $f(z) = z^2$ or $f(z) = e^z$.

Therefore, while investigating a linear complex-valued function, Edward and Melody utilized their algebraic and geometric inscriptions provided via *GSP* to integrate their algebraic and geometric reasoning. After this breakthrough, Edward and Melody precisely related the derivative value $3+2i$ to the rotation and dilation of an input circle by noting that an input circle would rotate by $Arg(3+2i)$ and dilate by a factor of $\sqrt{13}$ under $f(z) = (3+2i)z$. Possibly as a result of their ability to make this connection, Edward and Melody were able to develop their geometric reasoning about the derivative as a local property further than Christine and Zane.

**Developing Reasoning that Small Circles Map to Small Circles.** Another critical advancement in reasoning geometrically about the derivative involved the realization that a function always maps a small enough circle to a circle-like output. Algebraically, if $\boldsymbol{f'(z) = \lim_{z_0 \to 0} \frac{f(z)-f(z_0)}{z-z_0}}$, then rearranging this formula gives us $\boldsymbol{f(z) \approx f'(z)(z-z_0) + f(z_0) + \epsilon(z)}$, where $\boldsymbol{\lim_{z \to z_0} \epsilon(z) = 0}$. That is, $\boldsymbol{f(z) = f'(z)(z-z_0) + f(z_0)}$ as $\boldsymbol{z \to z_0}$. Geometrically, this means that $\boldsymbol{f(z)}$ is approximately linear in the neighborhood of $\boldsymbol{z_0}$. Therefore, this development in geometric reasoning is needed to notice that the derivative of a complex-valued non-linear function evaluated at a particular point only describes how the associated function rotates and dilates small circles. Thus, without this realization, participants could not reason about the derivative as a local property. This realization is also meaningful in that it helped Edward integrate his *GSP* investigations with a prior class discussion between him and his instructor about small circles, as discussed below.

Melody and Edward seemed to first make steps toward the realization that small circles map to small circles while investigating $f(z) = \frac{2z+1}{(z+i)(1-z)}$. During this investigation, Edward recalled a lab in class when he investigated a function that mapped his input circle to a clover shape. He noted that when he asked his instructor about this behavior in class, his instructor asked him what might change if his input circle was smaller. Edward claimed he did not really understand at the time, but said using *Geometer's Sketchpad* (*GSP*) to quantify geometric behavior for the purpose of reconstructing the algebraic form of an unknown rational function helped him reason about why the input circle needs to be small.

It is also significant that Melody and Edward made this critical observation while Christine and Zane did not. This distinction could be related to the fact that Edward and Melody appeared to more fully encapsulate Needham's characterization of an amplitwist than Zane and Christine. In the context of embodied cognition, it therefore appears that Edward and Melody really started grounding their developing geometric reasoning about the derivative of a complex-valued function in both the virtual and physical environments, as well as related past classroom discussions.

My participants' ability to ground both their previously established reasoning and their developing geometric reasoning appeared to be further supported by *GSP*'s ability to display algebraic notation alongside associated geometric figures (Heid & Blume, 2008; Marrades & Gutiérrez, 2000; Pea, 1987; Salomon, 1990; Zazkis, Dubinsky, & Dautermann, 1996). For example, while Melody and Edward attempted to determine the locations of non-differentiable points for the function $f(z) = \frac{2z+1}{(z+i)(1-z)}$, the numerical output of *GSP* helped my participants determine reasonably precise estimates for these points. The desire to obtain accurate decimal approximations for these non-differentiable points seemed to provide them additional motivation to keep their input circle small. Thus, the combination of geometric and numerical output provided by *GSP* seemed to help Edward and Melody develop reasoning about the derivative of a complex-valued function specifically in the context of small circles. Edward additionally noted that this exercise helped him reason about how the circle needs to be small enough to avoid all the non-differentiable points.

As noted above, while working with *GSP*, my participants were reminded of past classroom discussions related to geometric reasoning about the derivative of a complex-valued function. For example, Edward experienced a "flash" of insight when he reasoned that a radius of one can in fact be considered rather large. He may have recalled a previous class discussion about considering rotation and dilation amounts of small circles, and *GSP* could have helped him ground his reasoning in that past class discussion. This possibility seems especially likely in light of the fact that on another task Edward also commented on a prior class discussion about how he did not understand his instructor when he was directed to consider a smaller circle as part of one of his classroom lab activities. In this instance, he actually uttered that he "didn't get it" at the time, but *GSP* really helped him reason about why that circle has to be "darn small," and further helped him establish context regarding what "small" really meant.

In a previous episode, Edward used embodied gesture to reason geometrically about the apparently difficult concept of smallness in the context of the derivative of a complex-valued function. In this episode, Edward pinched his fingers and thumbs together while talking about being “zoomed in” on the complex plane, which made the constructed geometric objects appear larger. Similarly, Edward extended his arm while talking about the fact that the circles a complex-valued function transforms must be smaller than he expected to relate sensibly to the derivative of the function. Note that Edward utilized a gesture that seems iconic of “smallness” when talking about geometric objects appearing larger, and a gesture that seems iconic of “largeness” when talking about the necessity of considering a small circle. Through this embodied gesture and reflection on his investigations with the aid of *GSP*, Edward formed the correct conclusion that a radius of one may in fact be too large for the purposes of relating a complex-valued function’s transformation of a circle to the function’s derivative.

In another previous case, Edward observed that a large circle maps to another large circle. While making this claim, he moved his hands toward and away from each other and revised his statement by clarifying that a large circle will not even necessarily map to another circle. In the same episode, Edward used both his index finger and the mouse cursor to point at a circle constructed with the aid of *Geometer’s Sketchpad* (*GSP*). In both cases, Melody revoiced part of Edward’s geometric reasoning. Furthermore, she appeared to repeat the speech Edward accompanied with gesture in particular. Thus, gesture helped Edward and Melody reason together about the derivative of a complex-valued function. As such, they may have helped ground each other’s geometric reasoning about the derivative of a complex-valued function rather than progressing independently.

Melody and Edward were not the only participants to ground their reasoning in previous mathematical discussions. Zane, for example, recalled computational concepts from his numerical analysis class when *GSP* displayed a jagged circle when it should have shown a smooth curve. Zane noted that the *GSP* display reminded him of how he was taught to build a circle in his numerical analysis class out of a large number of line segments. He drew pictures to explain how using too few line segments to construct the curve results in the same jagged appearance that *GSP* displayed, and that as the line segments became short enough and numerous enough, the sequence of short line segments began to resemble a smooth curve. While I thought he might have connected this approximation of a circle with small enough line segments to a linear approximation of a derivative within a small enough circle, he did not provide any such elaboration on his explanation.

Thus, while it is possible that working with *GSP* helped students ground their prior knowledge and their current discoveries in these related past conversations, my participants were also reminded of classroom discussions they did not successfully connect to reasoning geometrically about the derivative of a complex-valued function, even in cases where such a connection was possible. For example, near the beginning of the interview, both participant groups recalled that the derivative value described a rotation and dilation, but had difficulty describing specifically that it described a rotation and dilation of a small circle. They certainly did not seem able initially to explain how small a “small” circle needs to be in this context.

**Developing the Concept of Amplitwist.** By the end of the tasks, Edward and Melody reasoned that complex-valued linear functions rotate and dilate a circle by the argument and magnitude of the appropriate derivative value, that a small circle always maps to a small circle, and that the input circle needs to be small enough to stay away from the "bad" points. As a result, they successfully reasoned that the magnitude of the derivative value of a function at a point is the factor by which the function dilates a small input circle, and the argument of the derivative value of that function at that point is the angle by which the function rotates a small input circle counterclockwise. Thus, Melody and Edward essentially developed Needham's (1997) amplitwist characterization of the derivative of a complex-valued function over the course of the interview sequence.

In contrast, while Christine and Zane reasoned that a linear complex-valued function rotates and dilates a circle, they did not quite connect these amounts to the appropriate derivative value's argument and magnitude. As a result, they also did not reason about the necessity of having a "small" input circle or why this is the case. Thus, Zane and Christine only verbalized that the derivative has something to do with how the function rotates and dilates a circle, and did not clarify that the circle must be small. It is therefore possible that successfully reasoning precisely about the relationship between the magnitude and argument of the derivative to rotation and dilation of a small circle in a complex-valued function aided Melody and Edward in characterizing the derivative as a local property. Indeed, for larger circles the amount of rotation and dilation is far less predictable.

One related advancement in reasoning that may have helped Melody and Edward develop their reasoning about the derivative of a complex-valued function more completely than Christine and Zane is their specificity about which mathematical entities rotate and dilate. In particular, Zane and Christine talked about how much the function rotates and dilates, while Edward and Melody talked more precisely about how the function rotates and dilates a point, or how the function rotates and dilates a circle. This distinction between how points are transformed and how circles are transformed seemed critical to developing geometric reasoning about the derivative of a complex-valued function.

More precisely, Zane and Christine never identified exactly which particular mathematical entity the function rotated and dilated. When I asked them to specify, they seemed unsure, guessing that the function rotated and dilated "the entire plane" by the derivative value at a particular point, or perhaps just rotated and dilated that one point. Of course, their guess that the entire plane rotates and dilates by the derivative value is accurate in the case of linear functions, but as one might expect, they experienced difficulty generalizing this geometric reasoning to the case of non-linear complex-valued functions.

In dealing with non-linear functions, it seemed Christine and Zane focused on how the function rotated and dilated particular points, and thus did not quite develop the geometric reasoning necessary to consider an amplitwist until I gave them the significant hint that a derivative value of one at a point means the function does not rotate or dilate a small circle around that point. However, they still did not gain any specificity in their geometric reasoning about rotating and dilating circles in particular. Thus, while Zane and Christine successfully characterized the amount of rotation and dilation given a derivative value, they did not identify the entity that rotates as necessarily a small $\epsilon$ −neighborhood, or even as a circle.

Similar to Christine and Zane, Melody and Edward initially reasoned about points rotating and dilating rather than circles while working through geometric transformations on the blackboard. Edward and Melody additionally recalled that the derivative value has something to do with rotation and dilation, but could not verbalize the nature of this connection at the beginning of the tasks. As such, they too experienced difficulty relating the derivative value at a point to how the function rotates and dilates a vector associated with that point. This difficulty may have motivated their thorough investigations of the related transformations $f(z) = z^2$, $f'(z) = 2z$, and $f(z) \to f'(z)$.

However, once they began focusing on $f(z) = z^2$ and $f(z) = (3 + 2i)z$ with the aid of *Geometer's Sketchpad* (*GSP*), they began reifying their investigations with recalled discussions from their complex analysis course. They additionally began successfully relating the derivative value at a point with the manner in which a complex-valued function rotates and dilates a circle at a point. Then, while Melody and Edward attempted to determine an algebraic formula with the aid of *GSP* for an unknown rational function I had previously constructed, they noted that the input circle must be small to adequately relate its rotation and dilation to the derivative value at a point the input circle surrounds. This particular breakthrough would have been difficult to achieve without the prior realization that the derivative value at a point describes something about how the function rotates and dilates a circle. Thus, Melody and Edward's specificity in what geometric object the function rotates and dilates was likely crucial in their development of their geometric reasoning about the derivative of a complex-valued function as an amplitwist as described by Needham (1997).

Over the course of the tasks, Christine and Zane did start reasoning about rotations and dilations, although they did not explicitly verbalize the amplitwist concept in the context of small circles in particular. Rather, they developed their geometric reasoning about the derivative of a complex-valued function enough so that they could state that a particular derivative value at a point for some particular function means that the function dilates a circle around that point by the magnitude of the derivative value and rotates the circle by the argument of that point. In other words, they were able to reason that a derivative value describes exactly how a small circle turns around and stretches out. Notions of local linearization were never explicitly verbalized. Thus, while the classroom lectures were not enough to support my participants' reasoning about rotation and dilation, Zane's and Christine's active participation in a *GSP*-aided investigation of the way functions rotate and dilate circles allowed them to develop their reasoning further. These results led me to suggest one of my teaching implications. Briefly, I suggest that mere information acquired from lecture is insufficient for students to develop geometric reasoning; some active engagement on the part of the learner is also required. This teaching suggestion will be revisited later in the chapter.

In the context of my study, my participants needed to ground their reasoning, and working within virtual and physical environments simultaneously seemed to help them ground their reasoning in both these and previous classroom discussions. Before they were able to do this, it appeared that they needed to integrate their algebraic and geometric reasoning, which they may have accomplished by working with algebraic and geometric inscriptions simultaneously. This integration appeared to be further aided by *GSP*, a dynamic geometric environment (DGE) that can give dynamic feedback about the correspondence between different kinds of inscriptions. Of course, in order to accomplish this integration, my participants must have already grounded their algebraic and geometric inscriptions together, which they appear to be able to accomplish via a combination of gesture and speech.

Unlike Christine and Zane, Melody and Edward explicitly reasoned about the derivative of a complex-valued function as a local linearization expressed as an amplitwist. In particular, Edward and Melody noted that the complex value of the derivative of a complex-valued function at a given point describes how the function rotates and dilates a circle around that point. That is, the function dilates the circle by the magnitude of the value of the derivative at the point and rotates the circle by the argument of the value of the derivative at that point. Christine and Zane articulated this much about the rotation and dilation of a circle, but Edward and Melody added the crucial additional observation that this rotation and dilation amount applies only to small circles.

Furthermore, while both groups of participants were reminded of prior class discussions, only Melody and Edward seemed to completely ground their recollections of these past discussions together with their current *GSP* investigations. Thus, developing good judgment about when to reason with various forms of complex numbers may have helped my participants develop their geometric reasoning about the derivative of a complex-valued function as both a local linearization and as an amplitwist. We must therefore be careful not only to help students how to convert forms, but also impress upon them the reasons for doing so in order to develop in them the ability to reason both algebraically and geometrically about complex numbers. This implication will be further discussed in the Teaching Implications section later in this chapter.

In conclusion, it appears that my participants needed to make three main developments in their reasoning about the derivative of a complex-valued function as an amplitwist, and that these three advancements correspond to each of the three forms of embodied cognition discussed previously. Furthermore, each of these developments correspond to some important relationship between the limit definition of the derivative $f'(z) = \frac{f(z)-f(z_0)}{z-z_0}$ and the geometric behavior of $f(z)$.

The first necessary development was that participants had to realize that they needed to reason geometrically about a linear complex-valued function. Appropriately, this is related to one of the first statements made about the algebraic definition: that it represents a local linearization. In particular, given the limit definition, we can then construct a linear function around $z_0$, namely $f(z) = f'(z)(z - z_0) + f(z_0) + \epsilon(z)$. The arrival at this stage seemed to be associated with participants successfully utilizing gesture and speech to integrate their algebraic and geometric inscriptions. In particular, they discussed information from *GSP* with each other to help them learn how to associate algebraic and geometric forms of complex numbers. Once they did so, they arrived at the realization that they were unsure of how to characterize a complex-valued "line," and that thus they needed to investigate a linear complex-valued function to obtain this information.

The second development in reasoning seemed to be that linear function always rotates and dilates every circle by the argument and magnitude of the derivative respectively, regardless of location. This stage is associated with the algebraic definition of the derivative in that our function $f(z) = f'(z)(z - z_0) + f(z_0) + \epsilon(z)$ is exactly linear when $\epsilon(z) = 0$ for all $z \in \mathbb{C}$. Thus, an investigation of a linear function with a complex-valued derivative helped my participants reason geometrically about the derivative $f'(z)$ in some linear formula of the form $f(z) = f'(z)(z - z_0) + f(z_0)$. This stage also necessitated the successful coordination of $Arg\big(f'(z)\big)$ and $|f'(z)|$ with the rotation and dilation of circles, respectively. As such, this stage seemed associated with my participants' ability to integrate their algebraic and geometric reasoning together coherently with the aid of algebraic and geometric inscriptions provided by *GSP*. That is, at this stage participants began to become more proficient in relating their algebraic and geometric reasoning with each other.

The third and final advancement in reasoning appeared to be that small circles always map to small circles, for an appropriate definition of the word "small." In context of the algebraic definition, this advancement is equivalent to the realization that $f(z) = f'(z)(z - z_0) + f(z_0) + \epsilon(z)$ becomes approximately the linear function $f(z) = f'(z)(z - z_0) + f(z_0)$ provided that $\epsilon(z) \approx 0$, which occurs if we restrict $f(z)$ to a "small enough" radius around $z$. That is, recognizing that $\lim_{z \to z_0} \epsilon(z) = 0$, for any $\epsilon > 0$ we decide on as an acceptable error, we can always find a disk around $z_0$ so that $|\epsilon(z) - \epsilon(z_0)| < \epsilon$. By "small enough" we mean any disk that satisfies this requirement. This advancement seemed particularly aided by my participants' investigation of an unknown rational function $f(z) = \frac{2z+1}{(z+i)(1-z)}$. I constructed the function first and asked them to reconstruct the function formula with the aid of geometric information obtained via *Geometer's Sketchpad* (*GSP*). My participants utilized *GSP* to realize that they needed to make their input circle small enough to stay away from non-differentiable points and obtain an output point that was approximately a circle. They utilized the chalkboard, speech, and gesture to reconstruct the appropriate algebraic function and gain some insight in how the function's geometric behavior is related to its derivative. Finally, they recalled past classroom discussions about the necessity of using a "small" circle. Thus, this seemed by far the most difficult advancement in reasoning for students to achieve, and to do so it seemed they had to rely on the virtual environment, the physical environment, and their past classroom discussion.

Therefore, while embodied cognition was leveraged on all three fronts throughout the interview sequence, each of these fronts appeared to aid at least one of the three critical developments in reasoning toward the derivative of a complex-valued function. First, participants integrated their inscriptions via gesture and speech, and realized the need for an investigation of a linear complex-valued function. Second, they integrated their algebraic and geometric reasoning via algebraic and geometric inscriptions, and reasoned that linear functions always rotate and dilate every circle by the same amount. Finally, they grounded their embodied reasoning in both the virtual and the physical environment with the aid of past classroom discussions, and reasoned that small circles always have to map to small circles, where "small" means that the function looks approximately linear at that point. With the successful completion of these three steps, participants realized that the derivative can be characterized as an amplitwist. Formally, within $B(z_0, \epsilon)$, an $\epsilon -$neighborhood of $z_0$ for some appropriately small $\epsilon$, $f(z) \approx f'(z)(z - z_0) + f(z_0)$, where $f(z)$ rotates the image $f\big(B(z_0, \epsilon)\big)$ counterclockwise $Arg\big(f'(z)\big)$ and dilates it by a factor of $|f'(z)|$ with respect to the pre-image $B(z_0, \epsilon)$.

## Connections to Literature

My dissertation study builds on existing research in three ways. First, it extends research on reasoning about the derivative of real-valued functions to reasoning about the derivative of complex-valued functions. Second, it adds to research on the field on the teaching and learning of complex numbers in general, which is still relatively sparse. Finally, and perhaps most significantly, it contributes to the growing theoretical perspective of embodied cognition.

The participants of my dissertation study regularly utilized reasoning about the derivative of a real-valued function to inform their reasoning about the complex-valued derivative. This was seen when both groups of participants claimed the complex-valued derivative could be characterized as the slope of the tangent line, and when Melody and Edward compared the output graphs of $f(z)$ and $f'(z)$ side by side for the sake of comparison. As such, research on the derivative of real-valued functions will likely inform potential student difficulties and successes while developing reasoning about the derivative of a complex-valued function.

For example, after my participants reconstructed the function $f(z) = \frac{2z+1}{(z+i)(1-z)}$ utilizing only information obtained from *GSP*, students reasoned successfully about the necessity of considering a small enough circle when reasoning geometrically about the derivative of a complex-valued function. This setup reversed the process established by all prior tasks in the interview, where students were given an algebraic formula and asked to determine geometric information relative to the derivative of the given function. This result corroborates Sfard’s (1992) suggestion that reification does in fact occur when processes can be successfully reversed.

On the other hand, participants also occasionally seemed to overgeneralize a geometric finding or ignore the context of the derivative. Zane and Christine seemed to downplay the role of the derivative while explaining the geometry associated with the derivative $3 + 2i$ of the complex-valued function $f(z) = (3 + 2i)z = 3z + 2iz$ exclusively in terms of vector addition of $3z$ and $2iz$. They additionally overgeneralized this finding by reasoning that this same kind of vector addition should hold in the case of a non-linear function, and expressed surprise when they saw that their reasoning did not hold.

Similarly, Melody and Edward seemed to overgeneralize their reasoning about the real-valued derivative wherein they compared the graphs of $f(x)$ and $f'(x)$ to develop reasoning about how the geometry of $f(x)$ and the geometry of $f'(x)$ were associated. When reasoning about the derivative of a complex-valued function, they tried to create a similar set-up, but only compared the output planes of $f(z)$ and $f'(z)$, apparently not realizing that they lost half their geometric information by excluding the input plane of each function from their investigation. This apparent tendency to either overgeneralize or ignore the appropriate context supports the research of both David, Tomaz, and Ferreira (2014) and Kuo, Gupta, and Elby (2013).

**Connections to Studies on Complex Numbers**

While the research on real-valued functions is diverse and far-reaching, the teaching and learning of complex numbers still represents a small but growing field of research. Thus, my research helps make this field of research a little less sparse in a few ways, and additionally corroborates the previously existing research on complex numbers.

At first glance, it may appear that my dissertation study conflicts with Harel's (2013) assertion that students had considerable difficulty reasoning geometrically about the addition and multiplication of complex numbers. This apparent conflict is due to the fact that my participants demonstrated no such difficulty reasoning about addition of complex numbers as vector addition and multiplication of complex numbers as a "rotation dilation"—add the angles and multiply the magnitudes, respectively. However, I believe this conflict is primarily due to a difference in populations. In particular, Harel reported findings about how in-service and pre-service math education teachers reasoned about complex numbers. These findings are further supported by Karakok, Soto-Johnson, and Anderson's (2014) research, which also utilized in-service high school teachers.

On the other hand, I reported findings about how undergraduate math and physics majors previously enrolled in a complex analysis course reasoned about complex numbers. It is unsurprising that students who had taken a course focusing on complex numbers reasoned geometrically more proficiently about complex numbers than teachers who had likely never taken such a course. This conclusion is further supported by Soto-Johnson and Troup (2014), who also interviewed students previously enrolled in a complex analysis course. These students were also generally able to reason geometrically about the multiplication of two complex numbers.

My study also produced mixed results in the light of Danenhower's (2006) dissertation study. In particular, Zane and Christine demonstrated poorer judgment in which form of a complex number to utilize. For example, they considered vector addition of a Cartesian representation at a point when it may have been more useful for them to consider angles of rotation and magnitudes of dilation, which should have made the polar form $z = Re^{i\theta}$ look particularly appealing. In direct contrast, Melody and Edward consistently recognized that the polar notation $z = Re^{i\theta}$ was especially useful when trying to determine angles of rotation, and even verbalized this strategic geometric reasoning. Thus, sometimes my participants developed good judgment about which representation of a complex derivative to utilize while investigating the derivative of a complex-valued function, while at other times they did not.

While my study produced mixed results about students' patterns in shifting representations, my study significantly corroborated Danenhower's (2006) theme of *Thinking Real, Doing Complex*. To begin with, and perhaps somewhat unsurprisingly, my participants' first attempt to reason geometrically about the derivative of a complex-valued function was as the slope of the tangent line. While this definition is essentially accurate, participants were unsure of how to generalize correctly from the case of the real-valued function to the case of the complex-valued function. For example, they could not provide a geometric characterization of a tangent line with slope $3 + 2i$.

In what is likely another case of *Thinking Real, Doing Complex*, Melody and Edward investigated the output planes of $f(z)$ and $f'(z)$ side by side, mirroring the real-valued case of comparing the function graphs of $f(x)$ and $f'(x)$ side by side to develop geometric reasoning about the derivative function with respect to the original function. Again, my participants could not develop further geometric reasoning about the derivative of a complex-valued function, possibly because the generalization was imprecise.

Finally, my dissertation study contributes to research on complex numbers by supporting Nemirovsky et al.'s (2012) assertion that students reasoned geometrically about complex numbers successfully when supported in an embodied way. In Nemirovsky et al.'s research, students reasoned about the multiplication of complex numbers via a physical representation of a complex plane in which students could move around and place points or vectors with stick-on dots and strings. In particular, Nemirovsky et al. noted that students noticed when their algebraic and geometric reasoning conflicted while utilizing this "embodied" complex plane.

In my research, my participants seemed to recognize conflicts between their algebraic and geometric reasoning while they utilized *Geometer's Sketchpad* (*GSP*). In one case, Zane reasoned algebraically about the geometric behavior he expected from the function $f(z) = e^z$, and Christine provided counterexamples to all Zane's claims (such as the idea that $f(z)$ cannot take on a negative real value). When confronted with contradictory output from *GSP*, Zane accepted Christine's demonstrations as true counterexamples, rather than reason that she had simply found an exception to the rule. This reasoning contradicts Harel & Sowder's (2005) finding that students tend to dismiss counterexamples as exceptions to a mathematical rule rather than a disproof of a conjecture. Perhaps this occurred due to my participants' usage of embodied reasoning with the aid of *GSP*. As argued in Chapter III, *GSP* and other similar dynamic geometric environments (DGEs) can reasonably be considered a tool that supports my participants' embodied cognition. This finding additionally supports Soto-Johnson and Troup's (2014) assertion that working with algebraic and geometric inscriptions in tandem helped their participants recognize a conflict in associated algebraic and geometric reasoning.

**Connections to Embodied Cognition**

My research thus helps contribute to research about the field of teaching and learning of complex numbers and provides an additional bridge between the prolific research about real-valued functions to the comparatively sparse research about complex-valued functions. Furthermore, it contributes significantly to the theoretical perspective of embodied cognition itself. In particular, each of my three fronts of embodied cognition discussed at the beginning of this chapter contributes to a different aspect of research related to or motivated by embodied cognition.

First, my research corroborates Goldin-Meadow's (2003) findings that gesture is a single integrated system. In particular, my participants grounded their algebraic and geometric inscriptions via embodied reasoning by producing gestures and communicating with each other regarding these inscriptions. For example, throughout the interview sequence, both sets of participants produced many gestures iconic of dilation (e.g., moving hands apart from each other, extending arm outward, holding a sphere and moving hands as though this sphere was growing) and rotation (e.g., tracing circles in the air with the index finger, holding a sphere and rotating it) while discussing the multiplication of two complex numbers, or the way a function turns and stretches a circle. In the process of grounding these inscriptions, they were able to realize the necessity of first reasoning about the geometry of a linear complex-valued function in order to develop reasoning about the derivative of a complex-valued function. Thus, reasoning both geometrically and algebraically with inscriptions via gesture and speech helped my participants embody their reasoning, much as Goldin-Meadow (2002) describes in her research on gesture.

Second, my research corroborates findings from both Zazkis et al. (1996) and Soto-Johnson and Troup (2014). Both research papers report that translating regularly between algebraic and geometric reasoning helps students integrate both kinds of reasoning into a single integrated system, much as gesture and speech already naturally function as such an integrated system. Similarly, at the beginning of my interview sequence, my participants seemed to view their algebraic and geometric reasoning as relatively disparate. In fact, Zane and Christine told me directly that they preferred to utilize algebra whenever possible, and to avoid geometry completely. Melody and Edward did not appear quite so averse to geometry, but still experienced difficulty relating their geometric reasoning to their algebraic reasoning. For example, they could not explain why a complex number with positive argument appeared to rotate clockwise on their function graph of $f(z) = z^2$. Rather than realize that this behavior resulted from the fact that their graph represented the graph $f(z) \rightarrow f'(z)$, they decided that their algebra and geometry together suggested that they originally reasoned incorrectly and that the clockwise direction was in fact associated with positive angles.

However, as the interview sequence progressed, Melody and Edward became more proficient in integrating their algebraic and geometric inscriptions coherently. In the process of integrating these inscriptions, my participants developed geometric reasoning that a linear complex-valued function rotates every circle by the argument of the function's derivative, and dilates every circle by the magnitude of the function's derivative. Thus, my findings corroborate Zazkis et al.'s (1996) assertion that cycling through algebraic and geometric reasoning allowed their participants to integrate both kinds of reasoning into an integrated system. Similarly, they support Soto-Johnson and

Troup's (2014) findings that cycling through reasoning with algebraic and geometric inscriptions helped their students develop an integrated system of reasoning that utilizes both types of inscriptions simultaneously.

Thus, Goldin-Meadow's (2003) research suggests that gesture and speech can help ground algebraic and geometric reasoning together as one (see leftmost cycle in Figure 45), and Zazkis et al.'s (1996) and Soto-Johnson and Troup's (2014) research suggests cycling between algebraic and geometric inscriptions can further integrate this reasoning (see center cycle in Figure 45). My dissertation study adds the similarly structured claim that oscillating between reasoning with the aid of the virtual environment and reasoning with the aid of the physical environment can help ground both kinds of reasoning together as a single embodied system (see rightmost cycle in Figure 45). That is, I argue reasoning with the virtual environment and physical environment simultaneously is similar to utilizing gesture and speech simultaneously, or viewing associated algebraic and geometric inscriptions as two aspects of the same mathematical reasoning.

Under this view, at the beginning of my interview sequence, students likely viewed information garnered from *Geometer's Sketchpad* (*GSP*) as disparate from their existing grounded mathematical reasoning, as this new information had yet to be grounded in anything. For example, Edward and Melody did not initially seem to notice that they constructed $f(z) = z^2$ differently on the chalkboard than they did with the aid of *GSP*. In particular, Edward and Melody graphed $f(z) \to f'(z)$ on the chalkboard and represented $z \to f(z)$ with the aid of *GSP*, but claimed both graphs represented the function $f(z) = z^2$.

However, as they reasoned via gesture, speech, and inscriptions in the physical environment while simultaneously reasoning via mouse movements and inscriptions produced with the aid of the virtual environment, students began to integrate the two environments and thereby ground their reasoning developed via *GSP* together with the mathematical reasoning they had previously grounded in the physical environment. For example, as the interview sequence continued, Edward eventually reasoned that the derivative informs the geometry of the function $z \to f(z)$ more meaningfully than the geometry of the function $f(z) \to f'(z)$. In particular, he stated that because the derivative is constant, $z \to f(z)$ always rotates and dilates every circle by the same amounts, regardless of location. In contrast, Melody insisted that they should investigate $f(z) \to f'(z)$ because this expression actually involves the derivative, and that the fact that the derivative is constant means that every point in the complex plane should map to the same location $f'(z)$. Through a combination of reasoning with the aid of *GSP* and allusions to their previous chalkboard investigations, Edward was able to persuade Melody toward his line of thinking.

Thus, oscillating between the physical and virtual environments allowed my participants to ground their reasoning in both environments simultaneously. One apparent result of this integrated grounded reasoning is that students recalled related class discussions. This result was particularly apparent in Edward and Melody's investigation of a rational function unknown to the participants that I had previously constructed, when Edward connected his newly developed geometric reasoning with a previous classroom discussion. This function was $f(z) = \frac{2z+1}{(z+i)(1-z)}$.

While attempting to determine the location of non-differentiable points, Edward noted that their input circle should be small to allow them to accurately pinpoint the locations on the complex plane that seemed to cause odd output behavior (i.e., something other than a rotation and dilation of the input circle). This reasoning was further enforced when they attempted to determine the derivative value for a specific point of their choosing given only the geometric output of *GSP*. They again observed that reasoning about a rotation and a dilation was only quantifiable when their output shape was close to circular, which only occurred when their input circle was small. This observation led them to correctly estimate the derivative value at the point they chose, and further influenced Edward to recall a prior classroom discussion.

In particular, he recalled an episode where he saw that a function transformed his input circle into a clover shape rather than a circle. When he asked his instructor about this, he was told to consider a smaller circle as an input. He reflected that at the time this admonition confused him, but that he now understood as a result of his investigations with $f(z) = \frac{2z+1}{(z+i)(1-z)}$. Thus, not only did Edward ground his reasoning garnered from a virtual environment in the physical one, but he also grounded reasoning that had previously confused him together with his new discovery. Therefore, the combination of reasoning in the virtual and physical environments may help ground reasoning from both together in much the same way as the combination of gesture and speech can help ground algebraic and geometric reasoning together, and the combination of algebraic and geometric inscriptions may help integrate both kinds of reasoning together.

**Additional Connections to Literature**

A few other ways my dissertation study contributes to the research is by supporting Kaschak, Jones, Carranza, and Fox's (2014) claim that language comprehension is by nature embodied. The claim that language comprehension is embodied is again supported by Edward's and Melody's investigation of $f(z) = \frac{(2z+1)}{(z+i)(1-z)}$. In particular, Edward noted that they had to "stay away from bad points," and that keeping the circle small accomplished this goal. Thus, Edward's reasoning about the necessity of a small circle was likely grounded in his reasoning about the physical world. Namely, to keep our body from touching something we wish to avoid, say, in a shop full of fragile equipment, we reduce the relative size of our body in order to move around more freely without damage. Edward further observed that "you've got one is huge." That is, he noted that the reason this advancement in reasoning was so difficult for him was that he did not realize that the number one could actually be quite large in the context of the derivative of a complex-valued function, and he was used to reasoning that one was fairly small.

My study additionally supports the *Focusing Framework* (Lobato, Rhodehamel, & Hohensee, 2012; Lobato, Hohensee, & Rhodehamel, 2013), which Lobato, Rhodehamel, and Hohensee developed to catalogue seventh grade students' reasoning about slope in terms of what they noticed from a class on linear functions. Lobato, Hohensee, and Rhodehamel refined this framework and posited that transfer could occur across contexts as a result of the aspects on which students focused during class. That is, students seemed more likely to transfer aspects of slope they had previously developed in class to the new context she presented to them. Danenhower's (2006) *Thinking Real,* *Doing Complex* in particular could be deepened via this framework. That is, my students' reasoning about the derivative of a complex-valued function in my interview sequence may have been representative of the reasoning they had previously developed in their classes on real-valued functions.

Given that it is likely that teachers of these classes emphasized the derivative as descriptive of the slope of a tangent line, it seems likely that students would have particularly focused on this geometric aspect of the derivative. Lobato, Rhodehamel and Hohensee's (2012), and Lobato, Hohensee, and Rhodehamel's (2013) *Focusing Framework* thus suggests that my participants were likely to reason about the derivative of a complex-valued function as the slope of a tangent line, and indeed they did so. They also likely noticed the way the graphs of $f(x)$ and $f'(x)$ were compared for real-valued functions, and attempted to recreate this comparison for the graphs of $f(z)$ and $f'(z)$. Thus, Danenhower's *Thinking Real, Doing Complex* can be recast in the light of Lobato, Rhodehamel, and Hohensee's, and Lobato, Hohensee, and Rhodehamel's *Focusing Framework*.

This framework further explains students' ability to find a derivative, but not reason about it. In particular, my participants claimed that their complex analysis class taught them many ways in which to find a derivative, but not why the derivative was useful. My participants also recalled the words "rotate" and "dilate" from their class, but could not originally explain precisely what they meant by these terms. Indeed, in several of the classes I observed, their professor called attention to "rotations" and "dilations" while discussing the multiplication of two complex numbers and the way a function maps a small circle. Thus, it seems that my participants focused on the words "rotation" and "dilation" themselves without precisely focusing on details such as when points rotated and dilated and when entire circles rotated and dilated. They were also not originally aware of how much the circle rotates and dilates under a given function. Thus, the reasoning with which my students entered into my interview sequence was likely a direct result of the aspects of reasoning on which they focused in their previous math classes.

One final connection to Lobato, Hohensee, and Rhodehamel's (2012) and Lobato, Rhodehamel, and Hohensee's (2013) *Focusing Framework* can be found in the nature of my participants' investigations undertaken with the aid of *Geometer's Sketchpad* (*GSP*). *GSP* is capable of producing a large quantity of geometric information, and my interview sequence was largely student-driven. Therefore, my students' reasoning was likely highly influenced by the aspects of geometry produced by *GSP* on which they chose to focus. For example, Christine and Zane focused on how points rotated and dilated under the given functions throughout the interview. Even when explicitly told to focus on how the function mapped circles, they reasoned about the circles as an infinite collection of points. This characterization of circles is essentially correct, but their focus on points seemed to make it difficult for them to reason about the quantities by which the circles themselves rotated and dilated. In contrast, Edward and Melody focused on a circle as a single mathematical object in its own right, and thus more successfully transitioned from reasoning about the way points rotate and dilate to the way these circles rotate and dilate under the given functions.

Therefore, my dissertation study extends research on real-valued functions to the field of complex numbers, builds on the still relatively young body of research on complex numbers, and adds to the theoretical perspective of embodied cognition.

Namely, it supports prior claims that gesture and speech help ground algebraic and geometric inscriptions together and that algebraic and geometric inscriptions help integrate both kinds of reasoning, and that reasoning in virtual and physical environments simultaneously help further ground both kinds of reasoning.

## Implications for Teaching

The findings described above suggest a few implications for teaching the derivative of a complex-valued function. First, it provides necessary components of a potential learning trajectory for students seeking to develop their geometric reasoning about the derivative of a complex-valued function. Second, it suggests that students work beneficially with DGEs when placed in pairs and allowed some opportunity for free exploration of related algebraic and geometric reasoning. Finally, my research suggests that students' focus must be directed to key points over the course of their mathematical investigations, even when aided by a dynamic geometric environment (DGE).

### Learning Trajectory

Curriculum designers should particularly take note of the potential learning trajectory my dissertation study demonstrated toward reasoning about the derivative of a complex-valued function as an amplitwist. This learning trajectory progresses as follows: students should develop reasoning about the geometry of lines in $\mathbb{C}$, reasoning geometrically about a constant derivative in terms of rotation and dilation, and reasoning about the need to reason specifically about small circles.

First, note that the ability to reason geometrically about multiplication of two complex numbers as a rotation and dilation is a critical starting point to this investigation. Without significant development of this initial geometric reasoning, reasoning about rotations and dilations of small $\epsilon$ −neighborhoods proved difficult. Second, investigation of a complex-valued linear function allowed my participants to extend their reasoning about the derivative of real-valued functions as the slope of a tangent line to the complex-valued case. In other words, students had difficulty generalizing their pre-existing geometric reasoning about the derivative as the slope of a tangent line until they investigated the complex-valued analog of a line. Finally, extending this geometric reasoning once again from complex-valued linear functions to general complex-valued functions proved to be an equally essential advancement.

This generalization occurred in one of my participant pairs while they worked on a task, which reversed the typical problem template from my dissertation interviews. In particular, Melody and Edward first gathered geometric information about a pre-constructed function with the aid of *GSP*. They then utilized this information to reconstruct an algebraic formula for the given function, identify non-differentiable points, and estimate the function derivative's value at a chosen point. This task proved essential in allowing my students to reason geometrically about the derivative of a complex-valued function as a local property. Before this task, they were able to verbalize that the magnitude of the derivative value at a point is the factor by which the function dilates a circle around that point, and the argument of the derivative value is the angle by which the function rotates a circle counterclockwise around that point. After this last task, my participants additionally verbalized that this property holds only for very small circles.

Previously in my proposal study, I discovered that reasoning geometrically about complex-valued linear functions as an essential step toward reasoning geometrically about the derivative of a complex-valued function as a more general amplitwist. This finding was further supported by the findings of my dissertation study, as both Edward and Melody and Christine and Zane verbalized their difficulty in reasoning geometrically about the derivative before investigating a complex-valued linear function with a complex-valued derivative. They further stated that they believed this difficulty stemmed from the fact that they were not sure how to characterize a "line" geometrically in the field of complex numbers. Thus, they could not specify what they meant when they stated that the derivative value is "the slope of the tangent line."

Investigating a complex-valued linear function appeared to shift their focus from reasoning about slope to reasoning about rotations and dilations of circles. The critical observation about linear functions that my participants seemed to require was that a linear complex-valued function transforms a circle by rotating it by an amount equal to the argument of the function's derivative, and dilating it by a factor equal to the magnitude of the function's derivative. This observation appeared to lessen my participants' dependence on characterizing the derivative of a complex-valued function as the slope of a tangent line. Thus, instructors should take note to emphasize this particular geometric behavior in their complex analysis courses.

In particular, students first need to develop geometric reasoning about the fact that a "line" looks different in the complex plane than it does in the real plane due to differences in dimensionality and how functions are graphed in each setting. Note that this advancement requires that students first develop geometric reasoning about multiplication of two complex numbers $z_1 = R_1 e^{i\theta}$ and $z_2 = R_2 e^{i\phi}$ as the composition of a rotation and dilation. In particular, the resultant vector $z_1 z_2 = R_1 R_2 e^{i(\theta+\phi)}$ is obtained by multiplying the magnitudes and adding the arguments of the input vectors $z_1$

and $z_2$. Participants seemed likely to notice that they were not familiar with the geometry of a "line" in the field of complex numbers during *GSP* investigations of various complex-valued functions such as $f(z) = z^2$ or $f(z) = e^z$. While these functions are themselves not lines, the participants noticed that they could not geometrically reason about the tangent line at a given point in either of these functions. In particular, note that "the slope of the tangent line" is not always obvious for complex-valued functions. (For example, what does a line with a slope of $1 + i$ look like?) This realization tended to arise naturally when asking participants to describe how they reasoned geometrically about the derivative of a complex-valued function.

Given students' apparent tendency to rely heavily on slope while reasoning geometrically about the complex-valued derivative, at least at first, instructors may be able to capitalize on this tendency. In particular, they may wish to consider first demonstrating a real function graphed from $\mathbb{R}$ to $\mathbb{R}$, rather than in the Cartesian plane $\mathbb{R}^2$. They could then discuss the geometry of the real derivative in this context. This could potentially leverage the power of Danenhower's (2006) *Teaching Real, Doing Complex* in a beneficial way by giving students the opportunity to generalize correctly from geometric reasoning about the derivative of real-valued functions to geometric reasoning about the derivative of complex-valued functions. Once students have developed reasoning about this difficulty in generalizing from the real plane, they should develop geometric reasoning about complex-valued linear functions (i.e., "lines" in the complex plane). The goal here is for students to reason that complex-valued linear functions have a constant derivative, and that such a function will always rotate and dilate a circle by the same amount.

With this geometric reasoning about linear complex-valued functions in place, students need to notice that a complex-valued linear function will always rotate an input circle counterclockwise by an angle equal to the argument of the derivative and dilate the input circle by a factor equal to the magnitude of the derivative. This and the previous goal can be met via investigations with the aid of *Geometer's Sketchpad* (*GSP*) of complex-valued linear functions such as $f(z) = (3 + 2i)z$. Linear functions with complex-valued derivatives seemed better for this purpose than linear functions with either purely real or purely imaginary derivatives. This is due to the fact that exposing participants to a function with a purely real derivative seemed to lead them to reason that the real part of the derivative value at a point is the amount the function stretches a circle around that point. Similarly, participants' explorations of a function with a purely imaginary derivative seemed to encourage them to reason that the imaginary part of the derivative value at a point is the amount the function rotates a circle around that point. In contrast, utilizing a function with a complex-valued derivative seemed to motivate students to more appropriately relate the amounts of rotation and dilation to the derivative value's argument and magnitude, respectively, rather than to the real and imaginary parts.

In this study, I uncovered what appears to be a similar critical step on this learning trajectory: reasoning about "smallness." This objective was accomplished by asking my participants to reason about derivative values given only geometric behavior. In particular, asking students to utilize geometric information to reconstruct algebraic inscriptions appeared to help them develop their reasoning further in this regard. As stated in Chapter III, most tasks called for students to construct a given complex-valued function and investigate geometric behavior given the function formula. That is, I typically asked students to determine geometric information given an algebraic inscription. Recall further that one new task turned this format around by supplying only geometric behavior via *GSP* and requiring students to determine a specific derivative value at a point of their choosing. While only one group had enough time left at the end of the interview to explore this task in detail, one participant from this group observed that this task in particular helped him realize that the derivative value only describes how small circles around the point are affected. This observation is significant because in my previous iteration, I identified participants' ability to reason about the derivative as a local property as one of the main recurring obstacles to their geometric reasoning.

In contrast, in this study, while working on Task 5, Edward commented that he realized that a circle of radius one could be considered quite large in certain contexts. He further observed that he needed to use a small enough circle so that he both "stayed away" from bad points and obtained a roughly circular output. At the conclusion of the task, he related a classroom discussion he had experienced with his complex analysis professor in which he remembered the professor telling him directly that the circle needed to be small. He reflected that he had not understood at the time, but that as a result of this task, he felt he finally had a grasp on what his professor meant by utilizing a smaller circle. Edward additionally credited his ability to make this discovery to the dynamic nature of *GSP*, commenting that *Mathematica* is "not as interactive as this." With this discovery, Edward and Melody were able to focus on smaller circles in other functions as well and thus more precisely characterize the derivative of a general complex-valued function as an amplitwist.

Thus, this task in particular may help students reason geometrically about the derivative as a local property, much as investigating complex-valued linear functions helped students reason geometrically about how the derivative relates to circles in general. That is, this task may help students move past reasoning geometrically about how functions map all circles just as investigating complex-valued linear functions may help students move past reasoning geometrically about the derivative of a complex-valued function as the slope of a tangent line. *GSP* is noteworthy in that its usage made this task possible. Without such a program to give immediate feedback to my participants' manipulations, providing such complete geometric information without any algebraic information would likely have proved impossible. This fact in itself is suggestive of Olive's (2000) discussion of how the existence and usage of computer programs amplify and reorganize mathematical investigations by making different types of tasks possible.

Thus, contrary to Kieran's (2007) and Lagrange's (n. d.) concerns, my participants' usage of *GSP* did not appear to lessen their focus or proficiency on symbolic forms. In fact, in task 5, my students' reasoning about the correct algebraic function formula was largely motivated by their experiences with *GSP*. In previous tasks, Zane and Christine tried to leverage their algebraic reasoning to predict the geometric behavior of a complex-valued function, then used *GSP* to check these hypotheses. For example, Zane and Christine algebraically reasoned correctly about why the function $f(z) = e^z$ maps the imaginary line to the unit circle by reasoning about the real and imaginary parts $e^x(\cos y)$ and $e^x \sin y$, respectively. Conversely, Melody and Edward used the geometric behavior from *GSP* to inform their algebraic inscriptions, most notably in Task

5 when they used *GSP* to correctly reconstruct a formula for an unknown rational function $f(z) = \frac{2z+1}{(z+i)(1-z)}$. As previously noted, Edward credited his ability to do this to the important fact that *GSP* is "more interactive than *Mathematica*."

Thus, Melody and Edward may have been able to accomplish Task 5 specifically due to the feedback possible from *Geometer's Sketchpad's* (*GSP*) allowance of relatively unconstrained movement of mathematical objects. That is, they were able to create a circle, make it smaller, and move it to various locations via clicking and dragging in smooth motions. Thus, far from harming Melody and Edward's abilities to reason algebraically, direct usage of *GSP* appeared to accomplish the direct opposite by strengthening their ability to reason both geometrically and algebraically about the derivative of a complex-valued function. Instructors may thus wish to consider allowing their students the opportunity to freely explore mathematical concepts with the aid of a DGE in the hopes of further integrating their algebraic and geometric reasoning about these topics.

In particular, students need to develop their geometric reasoning to the point that they can verbalize that their reasoning about linear complex-valued functions only holds for small circles in the general case. In particular, students need to have the ability to reason about what "small" means in this context. This observation may help them connect to the algebraic definition of the derivative $f'(z) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$. Namely, it may help them realize that the limit requires considering all $z$ "close" to $z_0$, which is strongly related to the idea of "smallness". Like the previous goal, this is another critical development for students to make in reasoning about the derivative of a complex-valued function as an amplitwist, and it appears to take some time to help students develop this reasoning. The best way I found in my study to meet this goal was to create a rational function with the aid of *GSP*, and ask my participants to identify a derivative value at a point of their choice. I additionally asked them to identify non-differentiable points.

Through the course of this investigation, they reasoned that utilizing small circles had practical benefit in both staying away from "bad" non-differentiable points and giving them more precise information about the one point on which they really wanted to focus. Finally, students need to develop geometric reasoning that a point is differentiable and non-zero exactly when the function maps a small circle to another nearly circular output shape. This development in geometric reasoning seemed to occur naturally in my dissertation study, particularly while participants used geometric information to construct an algebraic inscription for a function with an unknown formula.

**Pairs of Students**

Placing my participants in pairs seemed to help them advance their reasoning about the derivative of a complex-valued function in a few ways. First, this pairing allowed students to take on distinct roles within the group, where one participant manipulated *GSP* directly, while the other participant observed and reflected on the results and strategized about future direction. These roles were not rigid, as participants in both group switched roles with each other several times over the course of the tasks. While I initially directed them to switch, over the course of the tasks they began to switch roles with each other without such direction.

Second, my participants informed me that they were glad that they were not required to tackle such an advanced topic alone, and that they were likely to feel unmanageably frustrated were they to undertake these tasks without the help of a fellow student. Thus, pairing the students helped mitigate this inherent frustration. Usage of *GSP* itself may have helped mitigate frustration and advance both geometric and algebraic reasoning in the light of my theoretical perspective of embodied cognition.

Finally, it should be noted that I typically did not manipulate *GSP* for the participants, and I allowed them to wander "off-course" so as to allow them to fully engage in an exploration of mathematics. Thus, I suspect that it is essential to allow students in a classroom to carry out their own investigations with the aid of a dynamic geometric environmentrather than simply watch the teacher manipulate a program at the front of the classroom for them, or follow overly detailed instructions. By allowing my students to engage in explorations such as the behavior of $f(z) \to f'(z)$, they learned properties of transformations that were likely never addressed in their complex analysis classroom, or at least did not fit nicely under the typical characterization of the derivative of a complex-valued function as an amplitwist.

Thus, instructors may wish to consider pairing students together when allowing students to independently engage in the mathematics, to prevent their students from feeling too overwhelmed with the information they acquire. More generally, this implication suggests that students are unlikely to develop their geometric reasoning through lectures alone. Rather, students appear more likely to develop such reasoning if some sort of active engagement is required of them. Usage of a DGE such as *GSP* could greatly encourage this sort of active engagement.

Therefore, given the implications for teaching and research, curriculum designers should keep in mind Pea's (1985) discussion about how the usage and existence of mathematical technology changes the nature of mathematical investigation itself. In particular, my participants' usage of *GSP* naturally focused them on specific aspects of mathematics, some of which are not covered in the traditional complex analysis classroom, such as the behavior of the transformation $f(z) \to f'(z)$. The existence of technology enables unique and beneficial mathematical tasks, such as a task that requires students to reconstruct a derivative value at a point and an algebraic inscription for the function given only dynamic geometric information garnered from *GSP* investigations.

**Directed Focus**

Given that a large amount of such information can be gathered from *Geometer's Sketchpad* (*GSP*) and other similar DGEs, instructors must take care to help students focus on relevant aspects of the mathematical concepts they are investigating. In the context of research, instructors could leverage the power of Lobato, Rhodehamel, and Hohensee's (2012) and Lobato, Hohensee, and Rhodehamel's (2013) *Focusing Framework* by calling students' attention to the most relevant details of a mathematical topic. For example, to encourage development of geometric reasoning about the derivative of complex-valued functions, instructors of such classes might consider first constructing a graph of a real-valued function $f: \mathbb{R} \to \mathbb{R}$ as mapping from an input real line to an output real line, mirroring the typical way of graphing complex-valued functions. In fact, Needham describes exactly this construction in *Visual Complex Analysis* (1997). If students successfully reason about this way of mapping real functions, they may therefore more readily reason about the graphs of complex-valued functions.

Graphing a function $f: \mathbb{R} \to \mathbb{R}$ instead of in the Cartesian plane $\mathbb{R}^2$ also allows for discussions on the geometric properties of the derivative in relation to this way of graphing $f$. In particular, students might notice that a larger derivative value at a point suggests that a small interval around that point maps to a larger interval around the image of the point. That is, for a real-valued function, the magnitude of the derivative value is the factor by which $f$ dilates a small interval around that point. This aspect of the derivative thus generalizes precisely to the complex-valued case. If students are directed to focus on this new way of graphing a real-valued functions, the only new aspect of reasoning about the derivative of complex-valued functions involves how much the function rotates a small circle around a point. Establishing reasoning about a function as graphed from the real line to the real line would allow students the opportunity to develop geometric reasoning about the derivative as a local dilation of an appropriately small $\epsilon -$ neighborhood.

This way of graphing a real-valued function could thus represent a reasonable transitionary step between reasoning geometrically about the derivative of a real-valued function and reasoning geometrically about the derivative of a complex-valued function. While actually investigating the geometry of the derivative of a complex-valued function, instructors should call attention to the fact that linear functions always rotate and dilate every circle by the same amount, and that these amounts are given precisely by the argument and magnitude of the linear function's derivative. They should additionally help students focus on the fact that investigating small input circles with the aid of *GSP* both increases the precision of their measurements with the software and helps them avoid non-differentiable points. With this guidance, students' investigations with *GSP* or a similar DGE may significantly help students develop their geometric reasoning about the derivative of a complex-valued function.

If students are guided in such a way, *Thinking Real, Doing Complex* might actually function as a beneficial tendency if students are guided toward proper generalization. Students may more easily generalize from a real-valued graph as represented from $\mathbb{R}$ to $\mathbb{R}$ than a graph in $\mathbb{R}^2$, as the former set-up is more closely related to the typical representation of the graph of a complex-valued function. However, my participants either tended to generalize incorrectly or were aware they did not know how to generalize correctly. Thus, as in Danenhower's (2006) research, students' tendencies toward *Thinking Real, Doing Complex* seemed more of a hindrance than a help.

Furthermore, my participants' ability to reason effectively with various forms of complex numbers may have helped them develop their geometric reasoning about the derivative of a complex-valued function not only as an amplitwist, but also as a local linearization. Therefore, in the classroom, it may be beneficial for lecturers to help students notice the variety of strategic reasons for converting between various forms of complex number. For example, they may consider explicitly directing attention to the fact that while the Cartesian form simplifies vector addition, the polar form highlights rotation and dilation amounts. Through this careful direction of students' attention in the classroom, lecturers may be able to help students further develop their ability to reason about complex numbers via both algebraic and geometric reasoning methods.

Therefore, curriculum designers may wish to consider including a "lab" section for complex analysis courses, or allowing all sections of the course access to university computers to enable students within the course to carry on their own mathematical investigations as they see fit within the context of the course. My research thus suggests that there is the potential for a considerable improvement in students' geometric reasoning about the derivative of a complex-valued function as an amplitwist when they are allowed to use *GSP* in pairs in a semi-directed setting.

**Limitations and Future Research**

As this is a case study, we cannot expect the results to generalize beyond the specific circumstances surrounding my participants, research setting, and the tasks themselves. Some constraints of this study involve how student pairs were selected, the differing time frames of the two groups' interviews, and occasional technological difficulties.

In my dissertation study, only five students volunteered, and I could not obtain a sixth with which to form a third pair. Therefore, my participants were essentially self-selected. This constraint could be addressed by sampling from a larger complex analysis class or a set of similar complex analysis classes. Students' typical reason for not volunteering was that the eight-hour interview sequence represented too large a time commitment too close to finals week.

Another constraint involved the timing of the interviews. To improve my proposal, I stated that I would interview my pairs of students relatively close to the conclusion of the complex analysis class rather than the end of the subsequent semester. As Edward and Melody were also concerned about the time commitment and their impending final exams, I could not schedule their interview sequence at a mutually agreeable time for both of these participants at the end of the same semester. Thus, I interviewed this group at the beginning of the following semester. Zane and Christine, in contrast, were interviewed at the beginning of the summer, immediately after their complex analysis class had concluded.

One additional significant difficulty involved the difficulties imposed by technology such as *Geometer's Sketchpad* (*GSP*) and the cameras used to record video data. In my proposal, I asked other graduate students to help me ensure appropriate camera angles. However, I could not find any graduate students to help me with the camera again, so I attempted to change the camera angle where appropriate myself. This mostly involved rotating the camera to point at the chalkboard when participants worked at the chalkboard, and at the computer when they worked with *GSP*, and I was able to do this for the most part. On a few occasions, though, I neglected to rotate the camera, so I may have lost some data involving participants' gestures. I believe I successfully captured all of their chalkboard formulas with the camera.

There was a related problem with the screen-capture software, in that it failed to record my participants' *GSP* work for a small number of intervals in the interview sequence due to the fact that the computer crashed immediately prior to these intervals. During this time, the camera was placed correctly and recording, so at all points in the interview I was missing at most one of my two data sources. In the majority of the interview, all data-gathering devices were functioning correctly. Despite all these difficulties, sufficient data were collected to help address the research questions. However, many of these conditions could be improved in a future research study.

One possible direction for future research is to increase the breadth of these results by implementing dynamic geometric environments (DGEs) on a large scale in real classrooms and collecting quantitative data on student performance on tasks related to reasoning about the derivative of a complex-valued function as an amplitwist. Such research would theoretically allow these or related results to achieve some level of generality. Another possible direction is to increase depth of the results in this case study by iterating on the last task specifically. My participants reported beneficial effect from constructing algebraic information from exclusively geometric information from *GSP*, so further research on the effects of similar tasks for other students could prove highly interesting.

Further study is needed on this last task in particular because only Melody and Edward devoted a substantial amount of time to this task. My other pair of participants, Christine and Zane, looked at the rational function briefly at the conclusion of the interviews and merely stated they were glad they did not get to that task given how intimidating it appeared. Thus, iteration on this last task, even in another case study, could strengthen the results of this study by providing more information about how other students react both to the presentation and the execution of the task. Christine and Zane may have felt differently about the task had they actually carried out the necessary investigations, and may even have further developed their geometric reasoning about the derivative of a complex-valued function.

While my dissertation study was focused on Needham's (1997) characterization of the derivative of a complex-valued function as an amplitwist, he describes other possible geometric characterizations of the derivative. Thus, future iterations could potentially build tasks around these other possible characterizations. For example, future participants could develop reasoning about the derivative with respect to a vector field described by the Jacobian matrix, or discover a way to reason geometrically about the Cauchy-Riemann equations directly. Another possibility is to direct students toward investigating conformality more completely in the context of the derivative of a complex-valued function. That is, participants could utilize *GSP* to investigate how functions transform shapes other than circles with and without spokes.

For example, future participants might be asked to investigate how the function $f(z) = z^2$ transforms a square grid that covers the Argand plane. Participants might also investigate the deformation of visual pictures to help them meet this goal. While transforming a circle with spokes with the aid of *Geometer's Sketchpad* (*GSP*) seems to have been enough to allow my participant to reason geometrically about rotation as well as dilation, transforming a grid or solid picture may highlight even more geometric facets of a given transformation. For example, transforming a more solid picture may encourage students to develop focused geometric reasoning about the localized distortions they see in larger copies of the picture, or they may more quickly notice that smaller copies of the picture are relatively undistorted by the transformation. Thus, transforming a grid or solid picture may particularly help students develop geometric reasoning that allows them to verbalize that this aspect of the derivative of a complex-valued function is a local property.

One additional concept Needham (1997) discusses is how to reason geometrically about differentiating a power series. Therefore, future participants could potentially be asked to build up geometric reasoning about the derivative of a complex-valued function by first reasoning about the derivative of complex-valued polynomials and then building up to the general derivative via Taylor series. One final potential improvement for future iterations is to conduct an exit interview after giving students the opportunity to ask whatever questions they wished. This interview could provide more in-depth affective information about how students reacted to the tasks and what they personally thought they learned.

With these improvements in place, future iterations may give rise to additional discoveries about how students develop geometric reasoning about the derivative of a complex-valued function. Different variations on the tasks or transformed geometric objects may result in a different rate of development of geometric reasoning. As a naturally biased human, my role as interview represents another significant constraint on the study. As such, more consistent and intentional probing may help students develop more complete geometric reasoning about the derivative of a complex-valued function as an amplitwist. Students may discover a variety of new ways to utilize *GSP* to beneficial effect. In general, future iterations are likely to discover different variations on the learning trajectory described in this chapter. They may additionally confirm previously discovered relationships and discover new relationships between students' usage of *GSP*, gesture, speech, inscriptions, and reasoning. These relationships may then be leveraged to improve teaching practices, inform future research, and guide the development of geometric reasoning involving the derivative of a complex-valued function.

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APPENDIX A

IRB FORMS

**Title: Students' Development of a Dynamic Conception of the Derivative of a Complex Function**

## A. Purpose

1. The purpose of my research is to discover students' naturally occurring thought patterns and strategies while they are attempting to develop a more dynamic idea of a complex derivative. Sfard (1998) argues through her own research that having a metaphor of mathematical concepts involving motion is essential to viewing all aspects of a particular topic as a single, coherent whole. She adds that experts tend to possess such a dynamic metaphor, while Danenhower (2006) suggests that students may not be able to see two aspects of a concept as part of the same idea, and may fixate on a more static aspect of the concept-a view emphasized by formal mathematical definitions. The means by which students travel to and arrive at such a view of any mathematical concept is therefore a topic of great interest. Núñez (2004) suggests that fictive motion-the idea that we regularly refer to things that do not move as possessing motion (e.g., the fence runs along the road)-may be a reasonable way to connect static formal textbook definitions with useful intuitive dynamic metaphors of a given mathematical concept. Furthermore, while there is an established body of literature related to the derivative of real functions, much of which has proven quite useful, the literature on the derivative of complex functions seems to be relatively sparse, so extending the research on real functions to the analogous case on the complex plane seems natural. Finally, I intend to incorporate the gesture students produce during the task-based interviews into my analysis to help further understand each student's developing patterns of thought and strategies (Goldin-Meadow, 2002). This aspect of my research will help extend the existing research on gesture, which has previously been conducted primarily on elementary students and teachers. This project will help extend those findings to the realm of undergraduate students and teachers.

2. My project falls within the expedited category, since I am collecting video and audio data of the participants, which are data sources which could potentially identify the participants of the study and breach confidentiality. However, I am neither researching a vulnerable population nor increasing risk to participants beyond what is typical of enrolling in an undergraduate complex analysis course, and talking about the associated experiences. We will exclude students under 18 from participation in the study.

**B. Methods – Be specific when addressing the following items.**

1. Participants
   The professor who taught the most recent undergraduate complex analysis course will be asked for their recommendations of students to participate in the guided discovery of a dynamic conception of the derivative of a complex derivative. These recommendations will be utilized to select about 4-6 students to take part in the study. Participation in the survey is voluntary and students selected will have the opportunity to decline or discontinue participation at any point in the study without additional risk or loss of benefit.

2. Data Collection Procedures
   The researcher will observe and video-tape an undergraduate complex analysis course to obtain data regarding what concepts the participants learned prior to the start of this project. Permission to collect video data will be obtained from the students in this course via consent forms for these non-interview participants (Document 1).These data will additionally be utilized to inform the selection of students for participation in this study. The researcher will begin data collection from the selected students by distributing consent forms for participants (Document 2) to the students recommended by the previous undergraduate complex analysis instructor, which will inform the participants of their right to decline or withdraw from the study, as well as provide information regarding the purpose of the study. No professor other than my advisor will know the correspondence between pseudonym and student, so it is unlikely that any findings reported will affect the students' grades in future college courses. Students will be interviewed in pairs, to allow for the possibility of support and collaboration and because students are more likely to share their thinking processes with another student. The researcher will present each pair of students with mathematical tasks on which to work related to complex analysis, some of which will give the students opportunity to explore mathematical ideas via the Geometer's Sketchpad computer program. The students will be asked to explain their thinking regarding each task and may be asked additional follow-up questions designed to clarify the students' ideas to the researcher. (See Document 3 for outline of tasks and potential categories of follow-up questions.) Each student pair will be interviewed over the course of several days, based on their availability. At the moment, it is estimated that no more than four 2-hour long interviews will be necessary. Data will be video and audio recorded, and the researcher will not refer to any participant by name in any written findings. Another graduate student will help me record the interviews by controlling 2 cameras, one in the front of the room, and one in the back. Finally, a computer will be utilized to record the students' work on Geometer's Sketchpad.

3. Data Analysis Procedures
   The professor's recommendations and the researcher's observations and video data of students will be utilized as a criterion for purposeful selection of participants for the guided discovery of additional mathematical ideas related to the complex derivative. Since the purpose of the study is largely related to how exactly students learn new material regarding the derivative of complex functions, the researcher wishes to obtain a sample with students that will successfully make progress through some of the presented mathematical tasks. Such a sample also serves to minimize the risk and discomforts associated with mathematically induced anxiety. Preliminary data from each interview will be considered in conducting the next, providing potential insight regarding students' current thoughts and suggesting further possible questions to ask the students or ways to help them progress. Therefore, it will be necessary to record all the students' work regarding the questions asked, including the work they do on the computer in Geogreba or Geometer's Sketchpad. The researcher will individually code video and audio data, attempting to produce mutually exclusive and exhaustive categories in which to sort data. The computer program ELAN will be used to transcribe all participants' hand gestures and spoken words. ELAN will also be used for analysis, as it contains the capability of associating the transcriptions with the time at which it occurs in the video files. In addition to synchronizing gesture, speech and verbiage, ELAN also allows for interpretations to be written within the program and associated with the relevant spoken phrases or gestures. After sufficient refinement of the data, each participant and each pair of participants will be described in terms of these produced categories. In particular, the patterns and progressions of thinking about the presented tasks utilized by each student and by each student will be described as completely and accurately as the data makes possible. The overall goal of the data analysis is to discover one or more possible ways in which students could reasonably develop an intuitive dynamic sense of the derivative of a complex function. If possible, the researcher will ask the participant for verification of their analysis. Data analysis will be overseen by the researcher's advisor.

4. Data Handling Procedures
   Each pair of students will be video and audio recorded, and these data will be stored on a secure folder on a password-protected computer which is located in a locked office. The video and audio data of the class will also be stored on a secure folder on the same password-protected computer within the same locked office. The researchers will assign pseudonyms to each participant when presenting data to further increase confidentiality. Only the researcher and his advisor will have access to these data, and only the researcher and his advisor will know which

pseudonym corresponds to which student. The researcher's observation notes will utilize only pseudonyms, and thus will not refer to any student by name. These notes will be kept in a locked office; only the researcher will have access. Any list corresponding pseudonyms to participants will be kept on a secure folder on a password-protected computer located in a locked office, and at most one such list will be generated. The list of pseudonyms will be destroyed upon completion of the study, and the video and audio data will be destroyed 5 years after the completion of the study. The researcher's advisor will keep consent forms for 5 years after the completion of the study.

**C. Risks, Discomforts and Benefits**

There are no foreseeable risks beyond the risks normally associated with enrollment in a college-level class on complex analysis. Pairs of students will be presented with a variety of mathematical tasks related to the complex derivative in an effort to help the students naturally develop a more dynamic view of that concept. Students will be asked questions related to their work on the given problems, but remain free to decline answering any question posed or withdraw from the study completely at any time. There are no direct benefits to students for participation in the study. However, the qualitative nature of our study may lead to knowledge regarding the nature and effectiveness of naturally occurring student thought patterns and progressions regarding the derivative of complex functions. The knowledge developed may benefit both students and teachers of future complex analysis classes.

**D. Costs and Compensations**

Each student will be compensated for participation in this study with a $25 Starbucks gift card. Participation is voluntary. The only additional foreseen costs are the time costs associated with the implementation and execution of the teaching experiment, and the time cost resulting from the subsequent video data analysis.

**E. Grant Information (if applicable)**

We have not and will not apply for any grants or any funding regarding the execution of this study.

**F. References**

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**Documentation:**

Document 1: Consent form for Non-Interview Participants

Document 2: Consent form for Interview Participants

Document 3: Interview topics and types of tasks

**Document 1: Consent Form for Non-Interview Participants**

UNIVERSITY of NORTHERN COLORADO

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH

UNIVERSITY OF NORTHERN COLORADO

Project Title: Students' Development of a Dynamic Conception of the Derivative of a Complex Function

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Purpose and Description: The primary purpose of this study is to determine potential effective ways to teach the concept of the derivative of a complex derivative to undergraduate students enrolled in complex analysis. We are particularly interested in how students develop a geometric understanding of the derivative of a complex-valued function. In order to investigate this phenomenon, we request that you allow us to video-tape you while you are in the complex variables class (Math 460).

This video data will allow us to see various ways in which you communicate your understanding of complex variables; this recording will allow us to watch these offered explanations multiple times, and will be utilized to corroborate my classroom observations. I will be present every day during class, but will not be an active participant during class. I will only video-tape you and take observation notes. You do not need to worry about saying anything incorrect, as we are solely interested in how you reason through and communicate your geometric ideas about complex variables and the derivative of a complex-valued function.

Given the purpose of our research, we would like to share portions of your video-clips during presentations and it is possible that we may want to incorporate photos that illustrate your gestures and/or diagrams in a publication. Thus, we are requesting permission to do so, but if you would prefer that we protect your identity, then we will honor your request. In such a case, we will only use your responses and assign you a pseudonym – care will be taken to protect your identity.

Please note that you are not under any obligation to participate in this research and your decision to not participate in this research will not impact your course grade. You also have the option to participate in different aspects of the research. You may choose to:

a. participate in the video-taping where we are allowed to use episodes showing your face and where we are allowed to use your student work,
b. participate in the video-taping where we are NOT allowed to use episodes showing your face but where we are allowed to use your remarks and your student work,
c. not participate in the video-taping but allow us to use your student work, or
d. not participate in the research at all.

All data will be stored on my computer, which is password protected. Thus, no one will have access to these data other than me or Dr. Soto.

We foresee no risks to participants beyond those that are normally encountered in a classroom setting and possibly some discomfort if you do not feel comfortable being video-taped or are embarrassed by your work. It is possible that we may accidentally video-tape you, especially if you are working closely with someone who has agreed to be video-taped. In such circumstances, we will attempt to edit the video accordingly. Please feel free to contact us if you have any questions or concerns about this research at jonathan.troup@unco.edu. We appreciate your willingness to help us with our research.

So that you may benefit from this study, participants may ask for a copy of the report after the completion of the study.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

**If willing to participate in classroom video-taping and willing to disclose your identity i.e., agreeing to have your video shared with others at conference presentations, classes, publications, etc. please complete the following.**

______________________________________________________________

Name (please print) Signature Date

Jonathan
Troup______________________________________________________

Researcher's Name Research's Signature Date

**If willing to participate classroom video-taping but prefer to have identity protected, please complete the following.**

______________________________________________________________

Name (please print) Signature Date

Jonathan
Troup_______________________________________________________________

Researcher's Name Research's Signature Date

**If not willing to participate in the research, please complete the following.**

_______________________________________________________________

Name (please print) Signature Date

Jonathan
Troup_______________________________________________________________

Researcher's Name Research's Signature Date

**Document 2: Consent Form for Interview Participants**

UNIVERSITY of NORTHERN COLORADO

CONSENT FORM FOR HUMAN PARTICIPANTS IN RESEARCH

UNIVERSITY OF NORTHERN COLORADO

Project Title: Students' Development of a Dynamic Conception of the Derivative of a Complex Function

Researcher: Jonathan Troup, M.S., School of Mathematical Sciences

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Research Supervisor: Dr. Hortensia Soto-Johnson, School of Mathematical Sciences

Phone: 970-351-2425 E-mail: hortensia.soto@unco.edu

Purpose and Description: The primary purpose of this study is to determine potential effective ways to teach the concept of the derivative of a complex derivative to undergraduate students enrolled in complex analysis. No more than four separate 2 hour interviews will be conducted in which you and another student previously enrolled in complex analysis will be asked to collaborate on various mathematical tasks related to the derivative of a complex function. Since I am particularly interested in how you and other students develop purposeful strategies and ways of thinking regarding complex analysis, these tasks are designed to be somewhat challenging. To help offset the increased difficulty, you will be interviewed in conjunction with another student who has also taken complex analysis, and with whom you may collaborate on any task or question posed. You will be asked to explain your methods in approaching the problems presented, and may be asked additional clarifying follow-up questions based on your responses. You may decline to answer any question posed, and if you become too uncomfortable with the tasks presented or the interview questions asked, you may choose to stop the interview and/or withdraw from the study at any time without additional risk or loss of benefits. We will document each interview with video and audio recording equipment, to which only the researcher and his advisor will have access.

If you participate in any aspects of the research, then we will compensate you with a $25 Starbucks card at the conclusion of the interviews. We will assign each participant a pseudonym to help protect confidentiality. Pseudonyms will also be used to report data; no one will be referred to directly by name. Only the researcher and his advisor will know which pseudonyms correspond to which student. Data collected and analyzed for this study will be kept in a secure folder on a password-protected computer located in a locked office, which is only accessible by faculty and graduate students.

We foresee no risks to participants beyond those that are normally encountered in a classroom setting. Please feel free to contact us if you have any questions or concerns about this research at jonathan.troup@unco.edu. We appreciate your willingness to help us with our research. So that you may benefit from this study, participants may ask for a copy of the report after the completion of the study.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

______________________________________________

Subject's Signature Date

______________________________________________

Researcher's Signature Date

**Document 3: Interview Topics and Types of Task**
**Guide to Tasks and Interview Questions**

Tasks:

1. I will begin by asking the participants how they think about the derivative of a complex valued-function.
2. I will ask them questions related to the geometry of repeated multiplication by a complex number. (I.e., What happens to the complex plane when it is multiplied by $i$? How much is the complex plane rotated? How much is the complex plane dilated? What happens if I multiply by the same complex number several times. This concept of iterated multiplication is related to spiral Nautilus pictures) In this step and the next, students will utilize a program such as Geometer's Sketchpad or Geogebra as an aid to their inquiry regarding the above questions and the derivatives of various complex functions.
3. Students will continue to experiment with derivatives of various complex functions in a program like Geometer's Sketchpad or Geogebra to help them develop an intuitive and dynamic sense of how the derivative behaves in the complex plane. This step should help the students think more readily about the derivative of complex functions in later tasks. This computer program will enable students to see visually and dynamically on the computer screen how the derivative of complex functions affects the complex plane. Ideally, this program will allow the students to relate the concept of the derivative to the function's behavior more directly and intuitively.
4. The next natural step is to guide the students through an exploration of quadratic complex functions and their linearizations (closely related to derivatives) at various local points. (I.e., what is the linearization of the function $f(z) = z^2$ at the point $z = i$? At the point $z = 0$? What about the other points of the complex plane? In addition, they will explore questions similar to the above with other functions such as $f(z) = z^3$
5. If time permits, I will explore the exponential function, by asking the students (and nudging them at appropriate times) to design a Taylor polynomial that approximates it. (E.g., $W = f(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3}$

Possible Interview Question Types

Note: This is certainly not an exhaustive list, as probing questions must naturally arise from the current context of each interview, and the kind and amount of progress made by each student. However, this list should at least give a good idea of the kinds of questions I intend to ask.

1. Can you relate your algebraic work to a geometric picture?
2. How are you thinking about this problem?
3. What are you thinking right now?
4. How do you know that's true?

5. Can you repeat what the other student just said?
6. Do you agree (with the other student)?
7. Why do you agree/disagree (with the other student)?
8. Do you have anything to add (to what the other student said)?
9. How else could you think of this problem?
10. How else could you think of this concept?

# APPENDIX B

# LAB WORKSHEET FOR TASK 1

Lab 1:

Instructions:

We will begin by constructing a graph and unit circle.

1. First click the Graph drop-down menu and select "Show Grid"
2. Click the A toolbar ($4^{th}$ from the bottom) and double-click on the red point at the origin. Type "O" in the Label field in the pop-up window
3. Double-click on the red point at (0,1) and label this point 1.
4. Now click the "Construct circles" icon on the left toolbar ($3^{rd}$ icon from the top on the left) and click on the origin.
5. Now drag the mouse away from this point to increase the radius to 1. Click the circle again when the radius is at the proper size.

Note: You can always zoom in or out by selecting the point 1 and moving it closer to or farther away from the origin. Be careful not to move the point too close to the origin (i.e., zoom too far away), or it may be difficult to reselect this point when you need to.

Next we need to construct the transformation $z \rightarrow z^2$.

1. Select the Point tool ($2^{nd}$ icon from the top on the left) and click once somewhere on the grid to place the point there
2. Select the A toolbar and double-click on this new point. Label it $z$.
3. Select this point (if it isn't already) and go into the "Measure" dropdown menu. Select "Abscissa(x)." This will output the $x$ −coordinate of $z$.
4. Make sure that only the point is still selected (you may have to unselect the value you just measured) and go into the "Measure" dropdown menu. Select "Ordinate(y)" to output the $y$ −coordinate of $z$.
5. Go to the "Number" dropdown menu and select "Calculate." You can click on the coordinates you just measured to input them into the calculator. Use this calculator to calculate the real part of $z^2$ with an appropriate expression. Click "Okay" when you're done. Now calculate the imaginary part of $z^2$.
6. Go to the "Graph" dropdown menu and select "Plot points." Click the real part of $z$, then the imaginary part of $z$, and click "Okay." Your new point should now be on the graph. Click "Done."
7. Label this new point $z^2$.
8. Select the point $z$ and then $z^2$ (in this order; you will need to hold down the shift key in order to select both points.) Under the "Transform Menu" click "Define Custom Transform." A box should pop up that says $z \rightarrow z^2$ transform. Click "Okay."

This graph should now show a point $z$, and the corresponding point $z^2$. Try dragging $z$ around to various points on the graph. You can select a point with the Transformation Arrow tool at the top of the left toolbar.

> Warm-up Questions: What do you expect the point labeled $z^2$ to go if you put $z$ on $1 + i$? Why? Did it go where you expect?
>
> Where do you think you should place $z$ to send $z^2$ to $i$? Test your theory.
>
> What do you think will happen to $z^2$ if you move $z$ around the green unit circle once? Test your theory.

Now we will construct a circle and apply the transform $z \to z^2$ to the whole circle.

1. Click the "Construct circles" icon on the left toolbar (3rd from the top) and click somewhere on the graph to place the center of your circle there (Don't worry too much about location; you will be able to move it later.)
2. Now drag the mouse away from this point to increase the radius. When you are happy with the size of your circle, click the mouse again to create the circle. (Again, you will be able to change the radius later.) Your circle will automatically be selected.
3. Without unselecting the circle you just constructed, go into the Display drop-down menu, and select a "Color" for your circle. (I used red, but you can use something else if you like.)
4. Now, go into the Transform drop-down menu, then click "$z \to z^2$ transform" at the bottom of the menu. This will apply this transformation to your whole circle. The "output" shape will automatically be selected.
5. Go into the Display drop-down menu again and choose a different color for the "transformed circle." (I used blue, but again, you can pick a different color.) This is intended to help you keep track of your input and output shapes more easily.
6. Remember to click on the Transformation Arrow tool again before you start trying to drag your circles around! (Otherwise you'll just end up making more circles)
7. Move your circle around the graph and observe how the output shape changes as a result. Try to predict the behavior of the output in advance.

Some pointers:

- If you select the center point and move it, the other point you created (the one actually on the circle) will remain fixed, but the radius will change.
- If you select the point on the circle, the center point will remain fixed and the radius again will change.

- You can also select the circle itself. This will preserve the radius of the circle. (i.e., make sure to select the circle itself, and not the points, if what you want to do is drag the circle around the graph without changing anything else about it.

Questions: What do you think the output will look like if the input is a circle where $1 + i$ is within the area enclosed by the circle? Test your theory.

What do you think the output will look like if the input is a circle where 2 is within the area enclosed by the circle? Test your theory,

What do you think the output will look like if the input is a circle where the origin is within the area enclosed by the circle? Test your theory.

Now we will investigate what happens when we change the radius of circles at these points.

Center your input circle around $1 + i$ (so that $1 + i$ will be within the area enclosed by a circle centered at $1 + i$ of *any* radius.) Try changing the radius of your circle (Move the point on the circle so the center stays fixed). What happens to the output?

Center your circle around 2. Try changing the radius of your circle. What happens to the output?

What do you think the output will look like if the input is a circle where $1 + i$ *and* 2 are both in the area enclosed by the circle? Test your theory.

Center your circle around the origin. What happens to the output?

What happens to the output when your circle is inside the unit circle? What about when your circle is outside the unit circle.

Try dragging your circle along the real axis. What happens? What about when you drag your circle along the imaginary axis?

Try dragging your circle to different quadrants. What happens?

Now, try to summarize what you think is happening. What do you think a large circle around a point $x + iy$ in the complex plane will map to? What about a small circle around the same point?

# APPENDIX C

# LAB WORKSHEET FOR TASK 2

Lab 2:

Instructions:

Select Show Grid under the "Graph" dropdown menu, label the origin and 1, and create a unit circle centered around the origin as you did in the previous lab.

Now we want to construct the mapping $z \to e^z$.

1. Create a point and label it $z$.
2. Measure the $x-$ and $y-$ values as you did in the previous lab. (Use Abscissa(x) and Ordinate(y) in the "Measure" dropdown menu.)
3. Before we actually start calculating $e^z$, we will need to tell GSP to interpret angle measurements as radians instead of degrees. You can do this by selecting "Preferences" in the "Edit" dropdown menu, make sure the Unit tab is selected, and change the field marked "Angle:" from degrees to radians. Click "OK" once you've done this.
4. Now we need to calculate the real and imaginary parts of $e^z$. (Recall that if $z = x + iy$ then $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$.) Select "Calculate" in the number dropdown menu to input the appropriate formulas. (You can find $e$ in the "Values" dropdown menu on the calculator and the functions sin and cos in the "Functions" dropdown menu on the calculator.)
5. Plot the point $e^z$ as you did in the previous lab by selecting "Plot points" in the graph dropdown menu and inputting the real and imaginary parts in the $x-$ and $y-$ coordinate boxes, respectively. Click "Plot" then "Done". Label your point $e^z$.
6. Select the point $z$ and then $e^z$ (in this order; you will need to hold down the shift key in order to select both points.) Under the "Transform Menu" click "Define Custom Transform." A box should pop up that says $z \to e^z$ transform. Click "Okay."

This graph should now show a point $z$ and a corresponding point $e^z$. Again, you can drag the point $z$ around the graph. The point labeled $e^z$ will move to the proper corresponding position.

More warm-ups: Where will the point labeled $e^z$ be if $z = \pi i$?

The real-valued function $x \to e^x$ is always positive. Where should the point labeled $z$ be to get the point labeled $e^z$ to move to $-1$? Why did you conjecture that?

What do you think will happen if you drag $z$ along the real axis? What about the imaginary axis? Why does this happen?

This time (before we start mapping circles) we will send the vector defined by $z$ through the transformation $z \to e^z$.

1. Click the "segment straightedge" tool on the left toolbar ($4^{th}$ icon from the top)
2. Click the origin
3. Click the point labeled $z$. Your vector should now be created
4. In the Display dropdown menu, select your "input" color to make your newly created vector that color.
5. Now in the Transform dropdown menu, select "$z \to e^z$ transform" at the bottom .This will send your vector through this mapping.
6. Select your "output" color to change the color of the newly created curve.
7. Re-select the transformation arrow tool. Now you can click and drag the point $z$ to various points and watch how the output changes!

   Questions: What happens if the vector is stretched along the imaginary axis?

   What happens if the vector is stretched along the real axis?

   What happens if the vector is stretched in the first or fourth quadrant?

   What happens if the vector is stretched in the second or third quadrant?

Now we will investigate how circles are mapped at various points under this transform. You will follow essentially the same steps as you did in the last lab.

1. Click the "Construct circles" icon on the left toolbar ($3^{rd}$ from the top) and click somewhere on the graph to place the center of your circle there (Don't worry too much about location; you will be able to move it later.)
2. Now drag the mouse away from this point to increase the radius. When you are happy with the size of your circle, click the mouse again to create the circle. (Again, you will be able to change the radius later.) Your circle will automatically be selected.
3. Without unselecting the circle you just constructed, go into the Display drop-down menu, and select a "Color" for your circle. (I used red, but you can use something else if you like.)
4. Now, go into the Transform drop-down menu, then click "$z \to e^z$ transform" at the bottom of the menu. This will apply this transformation to your whole circle. The "output" shape will automatically be selected.

5. Go into the Display drop-down menu again and choose a different color for the "transformed circle." (I used blue, but again, you can pick a different color.) This is intended to help you keep track of your input and output shapes more easily.
6. Remember to click on the Transformation Arrow tool again before you start trying to drag your circles around! (Otherwise you'll just end up making more circles)
7. Move your circle around the graph and observe how the output shape changes as a result. Try to predict the behavior of the output in advance.

Tip Reminders:

- If you select the center point and move it, the other point you created (the one actually on the circle) will remain fixed, but the radius will change.
- If you select the point on the circle, the center point will remain fixed and the radius again will change.
- You can also select the circle itself. This will preserve the radius of the circle. (i.e., make sure to select the circle itself, and not the points, if what you want to do is drag the circle around the graph without changing anything else about it.

Questions: What do you think the output will look like if the input is a circle where $1 + i$ is in the area enclosed by the circle? Test your theory.

What do you think the output will look like if the input is a circle where 2 is in the area enclosed by the circle? Test your theory.

What do you think the output will look like if the input is a circle where $1 + i$ *and* 2 are both in the area enclosed by the circle? Test your theory.

What do you think the output will look like if the input is a circle where the origin is in the area enclosed by the circle? Test your theory.

Try putting the point on the circle itself along the positive real axis. What happens to the output if you drag the center along the negative real axis?

Now we will investigate what happens when we change the radius of circles at these points.

Center your input circle around $1 + \frac{\pi i}{2}$ .Try changing the radius of your circle. What happens to the output?

Center your circle around $-1 + \pi i$. Try changing the radius of your circle. What happens to the output?

Center your circle around the origin. What happens to the output?

Now, try to summarize what you think is happening. What do you think a large circle around a point $x + iy$ in the complex plane will map to? What about a small circle around the same point?

# APPENDIX D

# FINDINGS FROM PROPOSAL STUDY

**Between-Group Comparisons**

As my groups for the pilot study were different sizes, I begin this section detailing the differences and similarities I noticed between groups. After summarizing the differences and similarities between groups, I provide a discussion of each group's development of geometric reasoning about the derivative of a complex-valued function, utilizing the essential ideas I isolated through my analysis methods detailed above. These findings are grouped first by task, then by participant. Thus, summaries of each interview task are provided in sequence. Within each task, Karen's progress is discussed first, and followed by Joshua and David's progress within the same task. After I have discussed all tasks, I provide common themes that occurred for each group and for both groups in the subsequent section.

By the end of the interview sequence, both participant groups appeared to develop the ability to geometrically reason about the derivative as a local linearization, or at least vocalized all the requisite intuitive ideas. In particular, both groups verbalized the following:

- Circles are mapped to other nearly circular shapes.
- The magnitude of the derivative predicts the factor by which the output shape is dilated with respect to the input circle.
- The argument of the derivative predicts the angle by which the output shape is rotated with respect to the input circle.

Furthermore, placing two participants in a group appeared to ease some of the frustration that Karen seemed to experience with my repeated requests to determine how to reason about the derivative of a complex-valued function in a geometric way. Having a partner additionally seemed to encourage them to explore the presented topics more deeply. Joshua and David often appeared to consider a particular question for a longer amount of time and attempted to approach the question from several different possible solution paths. In contrast, Karen sometimes verbalized that she did not know how to think about a particular question or seemed to stop reasoning through the question because she was unsure of how to proceed. Additionally, David appeared to synthesize algebraic and geometric reasoning in some of his explanations, particularly his explanation of how $f(z) = z^2$ transforms the complex plane. Joshua and Karen may both have combined algebraic and geometric reasoning at some points in the interview, but both seem to do so to a lesser extent than David.

Perhaps simply because she was interviewed alone rather than in a pair, the nature of Karen's inquiry was also different than Joshua's and David's. In particular, Karen seemed content with knowing *why* the output circle sometimes twisted on itself, while Joshua and David seemed insistent on discovering additional information about *how* the circle twisted. That is, Joshua and David spent almost 40 minutes on day 1 trying to discover which two points on the input circle mapped to the same location on the output circle, thereby causing the twist they saw; Karen appeared content once she verbalized that she thought a twist occurred whenever her input circle for $f(z) = z^2$ surrounded the point $z = 0$. Thus, as one might expect, these tendencies seemed to motivate Joshua and David to provide somewhat more information than Karen in their mathematical characterizations. However, Joshua and David sometimes appeared to forget their original goal, perhaps due to their occasional tendency to begin reasoning exclusively algebraically.

On the other hand, Joshua and David's apparent consistent concern with local properties may have helped guide them toward conveying of the derivative as a local property, while it is possible that Karen's seemingly more global way of interpreting the derivative may have partially obscured this fact from her. For example, in the context of the real derivative, interpreting the derivative primarily in terms of slope, as Karen seemed to do, may make reasoning through details involving specific limits more difficult.

**Task Progression Comparison**

Having just detailed the original differences I noticed between a group of one participant and a group of two, I now provide more task-oriented details for each group of participants. Since Karen progressed at a different speed through the interview sequence than Joshua and David, this format provides a more direct comparison of the development of each group on each task than sorting the details by day. In the sections following some select exchanges between participants are presented. For these exchanges, accompanying gestures are described in parentheses, while speech incident with these gestures is bolded. Descriptions of gesture that occurs in the absence of speech are themselves bolded. I additionally document the participants' stage of diagrammatic reasoning in parentheses. The construction stage involves creating an inscription such as a computer-simulated diagram or algebraic equation. The experimentation stage occurs when students manipulate aspects of the inscription to see what happens. Finally, in the observation stage, students reflect on structural properties of the diagram to inform their mathematical reasoning. I refer to the relevant stages in parentheses throughout my task descriptions.

### Task 1: The Function $f(z) = z^2$

Within this task, I asked students via a task worksheet (found in Appendix E) to explore the function behavior of $f(z) = z^2$. I guided students in constructing the function $f(z) = z^2$ (construction), after which they investigated the ways in which the function mapped points and circles of various sizes at varying locations in the domain, including circles surrounding the point $z = 0$ (experimentation and observation). At the end of the task I asked them to characterize how $f(z) = z^2$ transformed the entire complex plane (observation).

Throughout this task, Karen tended to reason about rotation angles correctly and dilation amounts incorrectly or vice versa when multiplying two complex numbers. In the initial construction of this function, she initially attempted to reason algebraically to find the real and imaginary parts. She appeared to struggle with these algebraic calculations before noting that she could just reason geometrically by rotating and dilating. This realization suggests she had already integrated the algebraic and geometric reasoning about complex number multiplication to some degree. However, once she plotted and transformed a point (construction), she expressed surprise at the location of the output (observation), claiming that the point had the correct dilation, but not the proper rotation. That is, she noted that her geometric reasoning disagreed with the inscription provided by *Geometer's Sketchpad* (*GSP*). She started to explain where she believed the resulting point should be, but stopped herself partway through her explanation and suggested that perhaps *GSP* was showing the correct transformation after all.

After this occurrence, she turned her attention to answering the first exploratory question on Task 1: What is $f(1 + i)$ (phrased as "where should $z^2$ go if $z = 1 + i$?" on the worksheet)? Karen answered $(0,1)$, which is a point that has the correct argument, but not the correct dilation. She tested her answer on *GSP* (experimentation), and matter-of-factly noted that she "didn't add," and "thought that $i + i$ [3] was 1 in some strange fashion. It's actually 2." (The reader will note that $1 + i \neq 2$ and $i + i \neq 2$.) This explanation suggests she may have been reasoning algebraically, albeit incorrectly. As in the previous occurrence, Karen again demonstrated a willingness to be corrected by *GSP* and modify her reasoning to match the inscription she saw with the computer program (observation), rather than assume the inscription itself may have been flawed. She also noted the nature of her error via geometric reasoning: "I got the angle right….I didn't account for the dilation." Karen also initially failed to account for dilation in the next question: Where do you need to put $z$ to make $z^2 = i$? That is, for what value of $z$ is $f(z) = i$?

When asked what happens to the output when the input is moved along the unit circle, Karen began by reasoning geometrically. She traced a few circles in the air with her index finger before announcing that the output point $f(z)$ corresponding to the input point $z$ would move in an ellipse, reasoning that $f(1) = 1$ and $f(i) = -2$. Neither the nature of her calculations nor the way she arrived at $f(i) = i^2 = -2$ is clear. Perhaps she reasoned geometrically and doubled both the argument and magnitude of $i$, rather than squaring the magnitude. This geometric reasoning would reflect her previous error of attending correctly to rotation, but not dilation. Perhaps she reasoned algebraically and misremembered the value of $i^2$.

[3] While Karen said "$i + i$", this was likely a slip of the tongue—she probably meant to say $1 + i$.

She briefly suggested that $f(z)$ would follow the unit circle in the range as she moved $z$ along the unit circle in the domain (experimentation) with *GSP*, and appeared to reject this explanation by stating, "except at $i$ it's not going to do that." This utterance suggests Karen still believed that $f(i) = -2$, and thus conflicted with her geometric reasoning about $f(z)$ following the unit circle. It is thus possible that Karen noticed a contradiction between this correct geometric reasoning and her incorrect (possibly algebraic) reasoning about the value of $f(i)$, and attempted to modify her reasoning to address this contradiction. This finding is inconsistent with the same participant's behavior in a prior research study (Soto-Johnson & Troup, 2014) where she noticed a contradiction between her algebraic and geometric reasoning, but attributed the contradiction to the different reasoning styles and thus did not attempt to reconcile the two.

At this point I intended to ask Karen why she believed that $f(i) = -2$, but misspoke and instead asked why she believed $f(-1) = -2$. She answered by stating that "oh, nope, it's not going to $-2$. That was a lie. At $-1$, it'll be $1$, I think….I'm doing math strangely in my head. So, it should just, I think, follow the path of the circle." Thus it would appear that an incorrectly phrased question on my part helped Karen resolve the apparent contradiction and realize via geometric reasoning that the image point of $z$ traces an ellipse as $z$ traces the unit circle.

However, she moved back to her ellipse idea once she reconsidered the point about which I had intended to ask. "Except, at $i$ it's not going to do that. It's going to go to $2$. And at $-i$ it's going to go to positive $2$. So, something like, maybe create an ellipse." Upon stating that $f(i) = 2$ and $f(-i) = -2$, she again notes that these values would mean that the image could not trace the path of the unit circle as she moved the pre-image around the unit circle. As she moved the point $z$ with *GSP* (experimentation) along the unit circle (see Figure 46), she paused briefly when her point $z$ was resting on the point $i$ in the domain (observation) and exclaimed, “Oh it does follow the circle. So again I’m doing math kind of funky.” That is, due to her observation of the inscription provided by *GSP*, she may have recognized that her claim that $f(i) = -2$ was incorrect, as it was this belief that seemed to dissuade Karen from her original suggestion that $f(z)$ would move along the unit circle in the range as $z$ was moved along the unit circle in the domain.

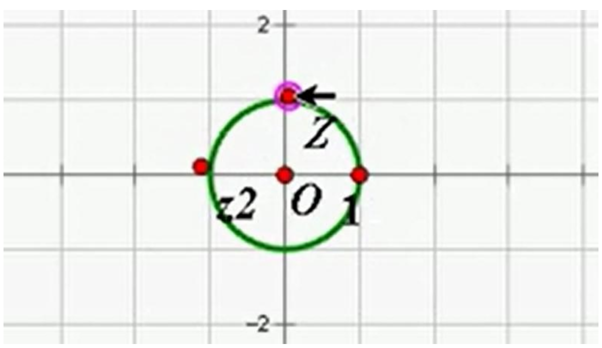


Figure 46. Karen traces unit circle counter-clockwise with a point $z$

In this same exploration, she made an observation via motion-based gesture that she employed later to describe why double circles formed when her circle in the domain contained the origin. I asked her why and she expressed that “because you’re **dilating and rotating each time** with $z^2$ (clasps hands with index fingers extended and moves fingers left and right as seen in Figure 47), it’s **going to go around twice as fast** (traces counterclockwise circles as seen in Figure 48), but your magnitude isn’t changing at all.” This explanation suggests Karen was reasoning geometrically to describe how $z^2$ transforms the unit circle, and was attending to the speed of the vector for which she was enacting a rotation. Since $z^2$ squares the magnitude and doubles the angle, on a circle where all points have magnitude 1, the magnitude is not changed and doubling every angle makes the point "move twice as fast."

Figure 47: Karen moves index finger left and right

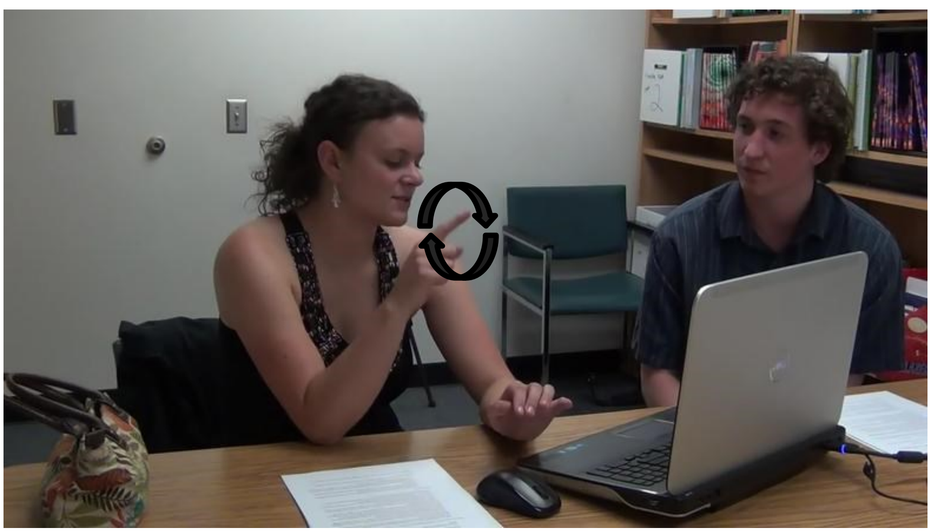

Figure 48. Karen traces counterclockwise circles

When Karen started exploring how circles are transformed by $f(z) = z^2$, she claimed that the image should be "some sort of circle depending on what I do with the rest of the circle." It appeared that Karen, even before investigating this question in

*Geometer's Sketchpad* (*GSP*), had some notion inherent in her geometric reasoning that circles should, in some sense, be mapped to another circle within this particular context. Her explorations with *GSP* appeared to allow Karen to develop the idea that smaller circles are less distorted by the transformation than larger circles; after moving circles of various sizes to various locations on the input plane according to the task's instructions, Karen remarked that "if it's bigger it's going to get all wonky-shaped." When I asked her why this is the case, she replied that $z^2$ changes the magnitude, but did not extend her idea much beyond this concept. However, she did add that double loops occur when the input contains the origin shortly thereafter, thereby identifying (observation) the only place that $z^2$ does not send (small) circles to other circle-like shapes.

After considering the ways in which the output shape might be "distorted" (i.e., ways in which the output looks less like a circle), Karen discovered with *GSP* (experimentation) that even with small circles, "weird" things such as double loops still occur near zero (see Figure 49). Furthermore, Karen noted that the output is "basically a circle," until it gets "close to zero", where the output "flattens out, on the side that's closest to zero (see Figure 50)." She added that farther away from zero "you're just going to get a bigger circle that's dilated and rotated according to the usual fashion." She did not, however, elaborate on what she meant by the "usual fashion". I asked her whether she thought the output was a perfect circle farther away from zero. She replied "No!" and clarified that the distortion is just more obvious closer to zero since the output has flatter sides and more clearly "doesn't want to hit zero."

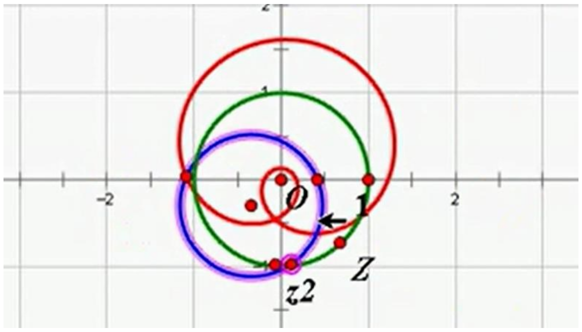


Figure 49. “Weird things” happening with small circles (input is blue, output is red)

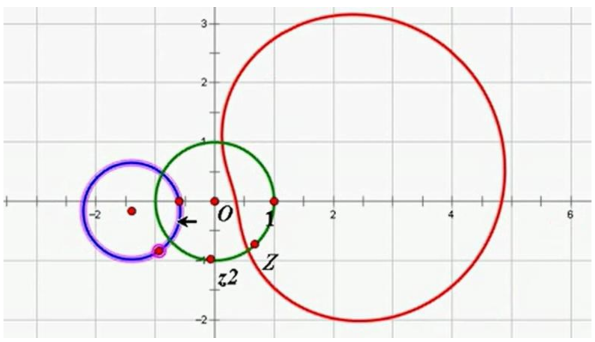


Figure 50. Image (red) of the blue circle “flattens out” near zero

Even though Karen had been attending correctly to how the inputs were dilated throughout the interview thus far, I still had her construct spokes on her circle and transform them (construction) so she could see how the circle would be rotated. I thought investigating the transformations of small circles more thoroughly here might strengthen her understanding of the rotation aspect of the derivative as a local linearization. She began her response by noting that if her spokes had been perfectly spaced, they would be perfectly aligned in the output as well (observation). This observation was significant in that it was not directly reflected by *GSP*’s provided inscription (see Figure 51). She could have interpreted this figure as indicating that spokes directly opposite to each other will not map to precisely the same place in the output plane, though they may be near to each other. Rather, she correctly noted the particular arrangement of spokes in the input circle that would transform to coinciding line segments in the output, though neither of these facets were immediate from *GSP*'s inscription.

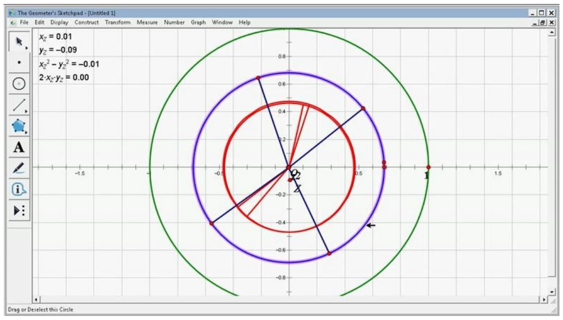

Figure 51. Red spokes are slightly misaligned because blue spokes are not evenly spaced

Thus, Karen appeared able to respond appropriately to *GSP*'s output both when she was correct and when she was not. Karen was willing to change her geometric reasoning when she saw $f(z)$ move in a circle and not an ellipse in response to the movement of $z$ around the unit circle. There were also moments like the previous where Karen explained what might have been taken as an inconsistency between her geometric reasoning and *GSP*'s output *without* changing the way she thought.

That is, Karen appeared to consistently identify in which cases she was wrong and in which cases *GSP*'s output was potentially misleading. In this case, she recognized that since her input was slightly off from where she intended—the spokes were not quite evenly spaced—her output would be slightly mismatched from the intended input as well. This realization suggests that Karen could extend her reasoning beyond the context of *GSP*'s dynamic environment, and that she is not overly dependent on *GSP* to drive her reasoning methods (Salomon, 1990).

While Karen completed task 1 in a little less than an hour on the first day, Joshua and David spent the entire two hours of the first interview exploring various aspects of the given function $f(z) = z^2$. This time discrepancy occurred in large part because David and Joshua attempted to answer many more questions than Karen. Karen answered only the questions listed on the worksheet or questions I specifically asked, whereas David and Joshua generated their own conjectures and explored many aspects of the function to which I never explicitly referred. Thus, while Karen exhibited a more stereotypical progression through mathematical tasks as a "game of mental gymnastics" (Olive, 2000, p. 11), Joshua and David seemed to view their work as a "laboratory science" (Olive, 2000, p.11), in that they "observ[ed], record[ed], manipulat[ed], predict[e]d, conjectur[ed] and test[ed], and develop[ed] theory as explanations for the [interesting] phenomena" (Olive, 2000, p.11). For example, during the first two hour interview at various points, Joshua and David utilized *Geometer's Sketchpad* (*GSP*) to investigate the possibility that the maximum imaginary value on an input circle maps to the maximum imaginary value on the corresponding output shape, or that if an input circle surrounds the point $1 + i$ then the output intersects the imaginary axis exactly twice (experimentation). Furthermore, they discovered the conditions for which there exists two values $z_1$ and $z_2$ located on the circle in the domain such that $f(z_1) = f(z_2)$, and how to determine these two values.

For the first question of task 1, "where does $z^2$ go if $z = 1 + i$," Joshua explicitly suggested calculating the answer algebraically, while David suggested reasoning geometrically by squaring the magnitude and doubling the angle. Using this method, David noted that $f(1 + i) = 2i$, though when Joshua tested this assumption with *GSP* (experimentation), he claimed the output point showed up at $-1$ instead of $2i$ as David expected (observation). Despite *GSP*'s unexpected output, instead of assuming he was wrong, David searched for an explanation for why *GSP* seemed to be supplying an incorrect answer to their question. He found that Joshua had placed the input point at $z = i$ and not at $z = 1 + i$ (observation). Thus, *GSP* had calculated $f(i)$ rather than $f(1 + i)$ as David had originally assumed. Once Joshua corrected the input location, he conceded that David was right. This event is significant in that David could detect and correct discrepancies between his reasoning and the output *GSP* provided, just as Karen demonstrated.

David reasoned geometrically by squaring the magnitude and doubling the angle; he elaborated on this geometric reasoning by reversing this process to answer the following question. When he read "Where does $z$ have to be to get $z^2 = i$", he halved the angle and noted that the input should be on the unit circle since the output was on the circle. Thus, David seemed able to reason geometrically regarding multiplication of complex numbers, just as Karen could at this point in the interview. These two participants may have developed and retained this ability due to their involvement in prior research involving the development of connections between algebraic and geometric reasoning methods via diagrammatic reasoning.

While Joshua appeared to rely on David's geometric reasoning at many points in the interview, he did not always accept it without question. While investigating circles that contained the point $1 + i$ (experimentation), both Joshua and David initially felt that the top of the input circle, would correspond to the top of the output circle. That is, they conjectured that $f(z) = z^2$ would preserve the maximum imaginary value of their input. Joshua found a counterexample (see Figure 52) to this conjecture with *GSP*, namely a circle where $1 + i$ was the top of the input circle, but $2i$ was not the top of the output circle. The existence of this counterexample was enough for Joshua to dismiss their conjecture, but David did not seem willing to accept it as false, even with the evidence provided by *GSP*. Joshua was able to talk him out of his flawed reasoning by referring to *GSP*, saying "because this point (places cursor over the point $1 + i$ at the top of the circle in the domain) is only going to be imaginary things, right (drags mouse along imaginary axis in *GSP*)?" That is, Joshua correctly reasoned that the maximum value of the circle in the domain will have a pure imaginary image, and thus cannot be a maximum of the image of the whole circle (see Figure 52). It is not clear which forms of reasoning Joshua employed to determine that $f(1 + i)$ is a purely imaginary number.

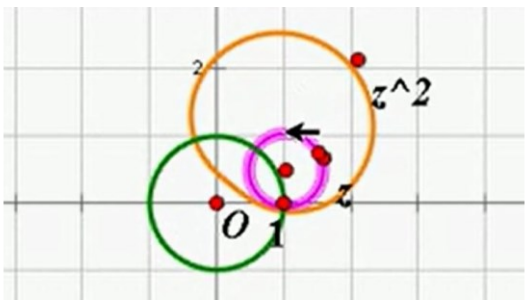


Figure 52. Joshua's counterexample—the point indicated by the arrow does not map to the top of the image (orange circle) of the pink circle.

As Joshua and David had not yet varied the size of their input circle, I asked them what they thought would happen for a larger circle. They responded to this question first by embarking on what appeared to be an in-depth technological exploration (experimentation), dragging the input circle to various quadrants as they had before with the smaller circle, and varying the size of the circle in both directions at each location. During this free-form exploration, Joshua and David discovered that double loops sometimes form (observation). As a result, David and Joshua expressed a desire to learn why these twists occurred. When I asked them why they thought the twists occurred, David referenced the idea that if he traversed a circle once in the domain, he would traverse a circle twice in the co-domain $z^2$, though this explanation did not yet acknowledge the origin as an important point regarding this behavior.

While David attempted to explain what caused double loops, Joshua fixated on trying to discover exactly which two points mapped to the overlap that the twist caused—the point of the output curve's self-intersection. To help himself answer this question, he strategically positioned (experimentation guided by geometric reasoning) the input circle so that the twist in the output was aligned with the positive real axis (see Figure 53). According to Joshua, he believed this positioning would make algebraic calculations simpler. However, Joshua appeared to become a little confused while reasoning through this algebra. For example, he initially stated he needed to find points where the imaginary part $2xy = 0$ and thus where $x = 0$, though *GSP* showed this assumption to be incorrect. Later he said he was looking for places where $y = 0$, but he changed his mind entirely and claimed he should have been looking for places where the real part $x^2 - y^2 = 0$. This last suggestion seemed strange in that Joshua was supposedly looking for input points where the output would be pure real, not pure imaginary. It is possible that Joshua became so involved in his algebraic reasoning that he forgot what exactly he was looking for in the first place.

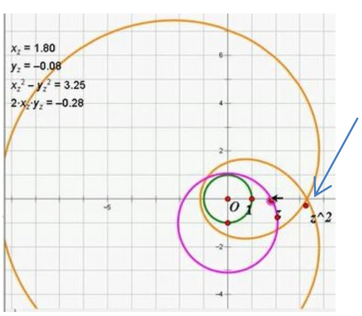


Figure 53. The image's (orange) twist (indicated by blue arrow) located on the positive real axis

Perhaps to place himself back on track, Joshua asked David to summarize their findings thus far, and David responded with geometric reasoning. He reminded Joshua that a small input circle maps to something that looks like a circle, though he still did not acknowledge the origin's role in potentially disrupting this near preservation of circles. Joshua added that small circles were not distorted a great deal because they were able to stay away from "trouble points," though Joshua admitted that he did not yet know where these "trouble points" were exactly. When David investigated the transformation of circles surrounding the point $1 + i$ he noted that "twists" occurred when the input circle passed through the unit circle. Furthermore, while David utilized *Geometer's Sketchpad* (*GSP*) to investigate various circles in the domain surrounding the point $2$ Joshua narrowed in on the origin itself as the point that caused the twists, stating that "It looks like $2$ is not special. The only deformations happen when we get closer to the origin" (see Figure 54). Thus experimentation with *GSP* and the subsequent observation allowed David and Joshua collectively to refine their geometric reasoning to include the fact that the origin somehow causes twists in the output and the point $(2,0)$ does not.

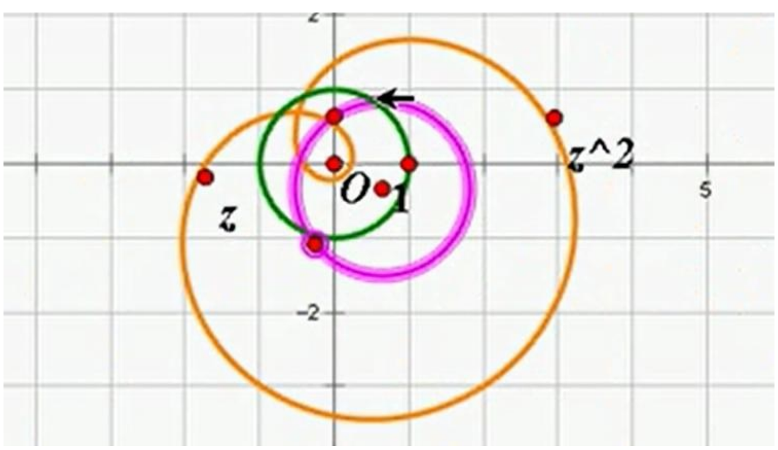


Figure 54. Orange output "twists" when purple circle surrounds the origin

Due to this observation, the next question I asked seemed natural: "What happens when the origin is included in or surrounded by the circle in the input plane?" David commented that "everything near the origin is really small, [so] the magnitude squared makes it even smaller so it's all getting pulled in." This explanation suggests a synthesis of algebraic and geometric reasoning: in this explanation, David speaks both of algebraic operations (squaring the magnitude) and geometric behavior (proximity to origin and being "pulled in").

Now that Joshua knew what the "trouble point" was, he offered geometric reasoning to explain why smaller circles are less distorted: "Once we get below a certain radius the twists go away, because it's when the circle does not contain the origin anymore." He continued this reasoning by saying that when the input circle is centered at the origin, the twists overlap exactly and become the same circle. When I asked Joshua why this occurs, David observed in *GSP* that if the input center stays on the $x-$axis, then the intersection stays on the real axis. Joshua expressed discontent at this offered explanation, arguing "that's not *why*," marking another time when Joshua was not willing to accept David's geometric reasoning. Perhaps due to his extensive technological explorations with *GSP*, Joshua eventually articulated that two input points map to a single output point, thereby causing a "twist," exactly when the two points have the same magnitude and a difference of $\pi$ in their arguments. Joshua and David additionally discovered that $z = 0$ was the only "trouble point."

Joshua suggested that the geometric reason for the twists going away would also explain the dent's dissipation as the input circle moved away from the origin: "We'll get to a place outside the unit circle where no point on our circle will have the same radial distance from the origin." Instead of pointing out to Joshua that his statement was flawed, David utilized this idea to essentially prove via geometric reasoning that 0 is the only non-conformal point of $z^2$. He conveyed that "we should have a lot of points where two different points have the same radial distance, but if it's outside the unit circle then we can't have that $\pi$ separation." This argument successfully showed that the input circle surrounding the origin is a necessary condition to cause twists, and that therefore there are no other similarly problematic points.

Task 1 ended with Joshua and David attempting to summarize via geometric reasoning how the function $f(z) = z^2$ transforms the entire plane, and at this point David recalled their complex analysis instructor "might have used pizza dough to explain this one." David essentially felt that the function $f(z) = z^2$ stretched out the plane and folded the quadrants around on themselves, while Joshua primarily said that each single quadrant in the pre-image was mapped to two quadrants in the image.

**Task 2: The Function $\boldsymbol{f(z) = e^z}$**

As in Task 1, my participants followed instructions contained in a task worksheet (found in Appendix E) to explore the function $f(z) = e^z$. This worksheet was similar in form to the worksheet paired with Task 1. After my participants constructed $f(z) = e^z$, I asked them first to determine how the function transformed various points and lines. They then investigated ways in which this function transformed various circles (experimentation). Finally, they attempted to geometrically reason through how $f(z) = e^z$ transformed the entire complex plane.

Karen finished the second task in the remaining hour of the first day. The first set of questions from the second task required Karen to determine the nature of the output given some input or vice versa. While exploring the function $f(z) = z^2$, Karen appeared to reason predominantly geometrically; she seemed to favor algebraic reasoning to investigate $f(z) = e^z$. For example, when I asked her to determine the value of $z$ for which $f(z) = -1$, she algebraically reasoned that "because you're rotating**…that's** (Points cursor at $e^x$ in the equation $e^x \cos y$) just going to be 1 and $\cos \pi$ is $-1$. So **it's going to – 1** (Waves mouse over real component $e^x \cos y$) **and then this part** (Waves mouse over imaginary component $e^x \sin z$) is going to be 0 if you calculate it out." (See Figure 55)

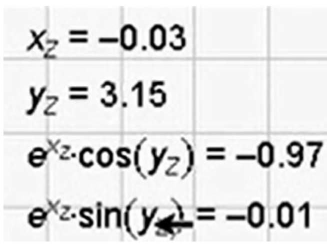


Figure 55. Real and imaginary parts of $f(z) = e^z$ (upper left corner)

It is possible she was utilizing Euler's equation $e^z = e^x(\cos y + i \sin y)$, or perhaps just evaluated each part of the relevant algebraic inscriptions (see Figure 55) provided by *Geometer's Sketchpad* (*GSP*). She appeared to reason algebraically in a similar way when attempting to explain what she thought would happen if her input point was moved along the real axis. She stated, "the cosine and sine wouldn't change, just the $e^x$." Immediately after forming this hypothesis, she confirmed it (experimentation and observation) with *GSP*. Seemingly unsurprised, she remarked, "Yeah. Just kind of shoots off into the distance." When asked what happens if the input point was dragged along the imaginary axis, she once again reasoned algebraically to determine the behavior of the image point $f(z)$: "the $e^z$ is always going to be 1, because you don't have a real value….So it's just going to, oh it moves along the circle there" (see Figure 56). In contrast to Karen's test on the positive real axis, Karen seemed somewhat more surprised at the results of moving the point $z$ along the positive imaginary axis (observation).

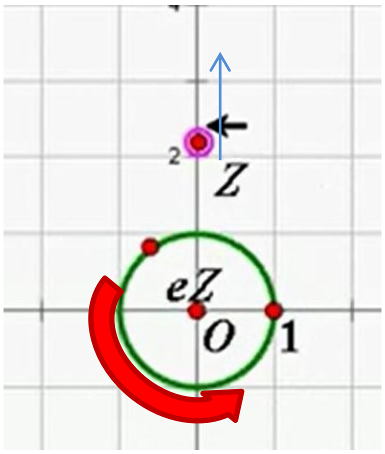


Figure 56. Karen moves her point $z$ along the positive imaginary axis (indicated by blue arrow). The image moves counterclockwise along the unit circle (indicated by red arrow)

After some further experimentation in *GSP*, Karen noted that she had not yet seen any double circles in this function, so I asked her if she thought she could make one. She was not sure, but agreed to try, thus entering a more purposefully directed experimentation phase. She started by dragging a small circle to various locations surrounding the origin in *GSP*, then repeating this action several times, making the circle progressively larger between each repetition. Despite the fact that none of these circles created a twist in the output, she still commented, "I'm thinking that we're going to get a double circle." After about a minute, she found a way to create a twist while zooming out and thereby dramatically increasing the size of her circle in the domain (see Figure 57). Karen clarified that in some way, she felt the twist was different from the double circles that had occurred in $z^2$, perhaps due to her initial difficulty in finding a circle that mapped to an image with a twist. No small circles around the origin created a double circle, as they had in $f(z) = z^2$. She called the twist of $e^z$ a curlicue rather than a double circle. She also realized that the input circle had to be sufficiently large to create these curlicues, and discovered via experimentation and observation with *GSP* that the radius needed to be $\pi$ or greater for a twist to occur. However, even after discovering these "curlicues," she did not appear certain about why they occurred.

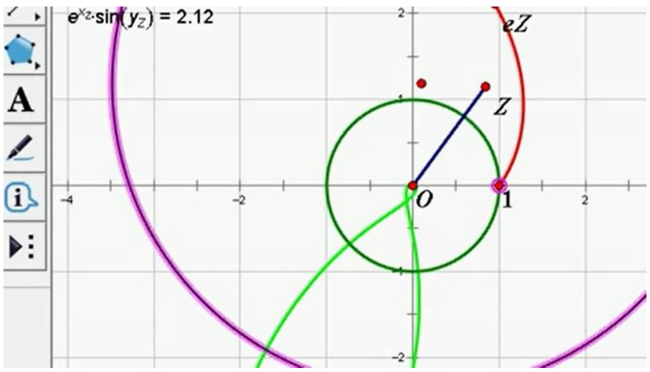


Figure 57. A twist in the image (green curve)

When I asked her if she could obtain a twist in the output with a small circle, she expressed doubt, but suggested that perhaps there are microscopic twists. She found none with *GSP* (experimentation), and after this technological experimentation she stated that no curlicues could occur in the image of a small circle. However, she still did not know *why* the curlicues occurred, though she may have implicitly drawn a parallel to $z^2$ with her terminology. Karen claimed that $f(z) = e^z$ should behave more strangely as a function as the input circle moves closer to zero, and added "turns in zero, we start getting our double loop." While she observed with *GSP* that if the circle in the domain is not zero, its image is "your classic circle," and that "the closer it gets to zero the weirder it becomes," the meaning of her utterance about "turns in zero" remains unclear. Perhaps she thought of the way $f(z) = z^2$ turns the complex plane around on itself at $z = 0$ via its mapping, as she referenced a "double loop," a shape she had previously determined to be distinct from the "curlicues" she was considering at this time.

Motivated by this possible implicit reference to $z^2$, I asked Karen whether a small circle could yield strange output behavior, and she said yes, "if it includes zero." While Karen had previously stated that she did not believe the image of a small circle could have a twist, her geometric reasoning still allowed for the possibility of some other kind of unusual behavior. She even began demonstrating this point in *GSP*, placing the input circle closer to zero, and expressed surprise when her small circle around zero mapped to another relatively normal small circle (see Figure 58). Her surprise may have stemmed from the possibility that she was using the real-valued function $f(x) = e^x$ as a reference point to reason through the analogous complex-valued function $f(z) = e^z$, as near 0 in the real-valued function the graph does not bend a great deal. Alternatively, perhaps her experience with the function $f(z) = z^2$ led her to believe that the point $z = 0$ causes atypical behavior in all functions. Regardless, she spent several minutes in silence after I asked her why no strange behavior occurred in the case of $f(z) = e^z$. She moved the input circle along the negative axis briefly (see Figure 59) and additionally stretched the line attached to the origin to various locations during this time (see Figure 60), but it appeared that her confusion remained.

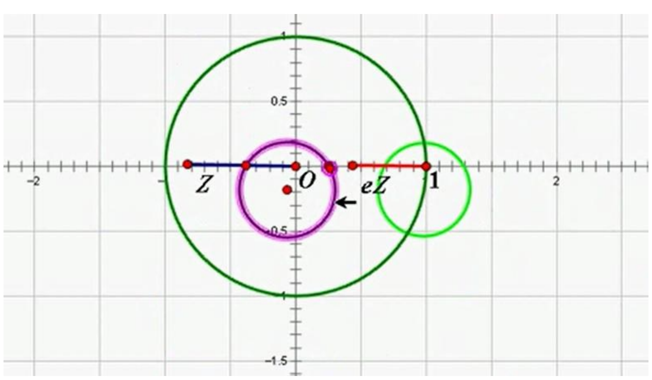


Figure 58. Small purple circle around 0 maps to small green circle around 1

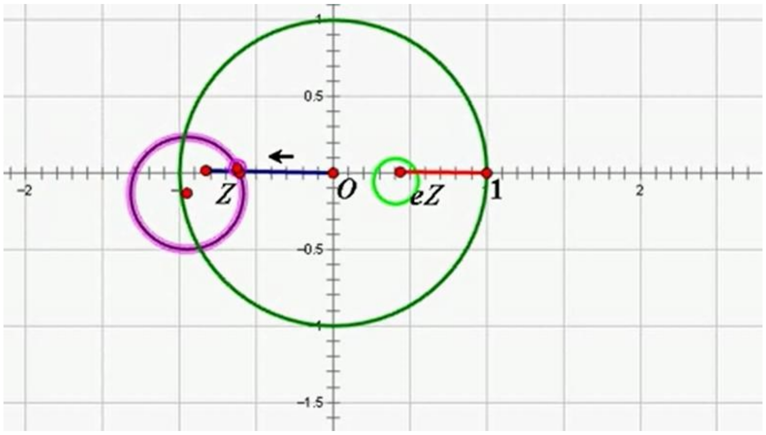


Figure 59. Karen moves purple circle along negative real axis

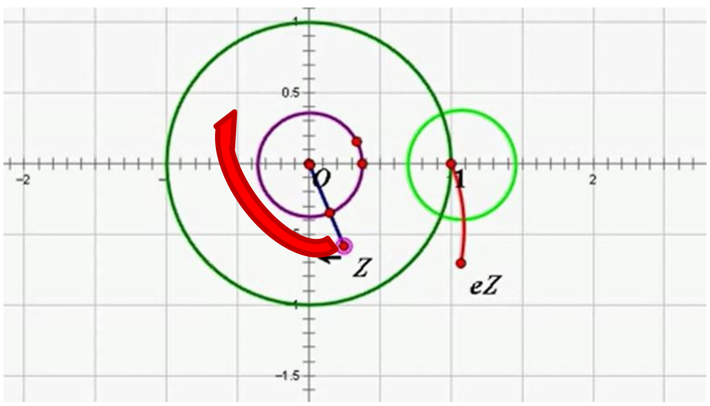


Figure 60. Karen rotates and stretches the vector attached to the origin (following the path of the red arrow))

While moving this vector around (experimentation), she remarked, "the outside [of the unit circle]'s where [the image of the vector] starts to bend in your exponential fashion (observation)." This remark could connect to the behavior of the real function $f(x) = e^x$, which has a graph that also begins to bend more noticeably at points with domain values at greater distances from 0. Perhaps her experimentation with *GSP*'s geometric inscription reminded her of this pictorial resemblance. Immediately after this observation she claimed that "[inside the unit circle] you're going to have a fractional power," suggesting a transition from geometric to algebraic reasoning. She additionally stated that she no longer believed that "weird behavior" could occur in the image of a small circle under the transformation $z \to e^z$. Thus, experimentation with *GSP* once again helped her refine her geometric reasoning for the behavior of a complex-valued function. I ended both the first day and task 2 by asking Karen to calculate the respective derivatives of $f(z) = z^2$ and $f(z) = e^z$, which she correctly supplied immediately.

Like Karen, Joshua and David both correctly predicted how various points and lines were transformed with the function $f(z) = e^z$. David noted that the positive real axis would stretch out and Joshua predicted that the positive imaginary axis would map to a circle. Joshua and David both seemed to struggle to articulate how the function would transform the negative imaginary axis, so both seemed to shift their form of reasoning from geometric to algebraic.

> *Joshua:* ***But when it's negative, it's cosine minus i sine.*** (David traces several circles counterclockwise in the air as shown in Figure 61)
>
> *David:* ***So it'll just*** **(**traces a few more clockwise circles as shown in Figure 61)*, spiral in,* ***or go in the circle the other way*** (traces several larger clockwise circles.)*, or it will be the same. Because at that point we just basically have* $e^{iy}$.

Figure 61. David traces clockwise circles

In particular, Joshua appeared to reference the algebraic formulas for the real and imaginary parts of the output $f(z) = e^z$, as when the imaginary part of the input $z$ is negative, the real part $e^x \cos y$ is positive and the imaginary part $e^x \sin y$ is negative. Joshua does not gesture or make any technological actions at this time, though David appears to trace circles in response to Joshua's statements. Neither participant writes anything down. Thus, Joshua appears to reference an algebraic inscription while David continues to reason geometrically, though it appears David's geometric reasoning may have been influenced by Joshua's more algebraic statements.

The exchange just outlined may have prepared Joshua and David to answer my next question: what would happen if they stretched a vector along the imaginary axis? While Joshua had already correctly predicted, apparently via geometric reasoning aided by gesture, that the function transforms the positive imaginary axis into circles (while tracing counterclockwise circles as shown in Figure 62). David elaborated on this geometric reasoning, adding that with each additional increment of $2\pi$ the vector was stretched would form a new circle—another correct prediction. David utilized no gesture throughout the majority of this explanation except near the end when he referenced the new circle the image would form as the vector was stretched (see Figure 63).

Figure 62. Joshua traces counterclockwise circles

Figure 63. David traces counterclockwise circles

After this exploration I asked Joshua and David what would happen if a vector was stretched into one of the quadrants instead of along the axes. Joshua continued the trend of correctly predicting how $f(z) = e^z$ transformed various points on the plane via geometric reasoning by stating that the imaginary component of the input controlled the amount of "twisting" or "spiraling" that occurred. Continuing his geometric reasoning, he further clarified that the real component controlled the size of the spiral.

As he referenced the $x-$component and size of the spiral, he held his hands apart and facing each other, as though they were signifying length or size (See Figure 64). So, Joshua geometrically reasoned both that $f(z) = e^z$ rotated objects based on the imaginary component of $z$, and that a vector stretched into a quadrant would result in a spiral shape. Joshua added that the spiral would twist in the opposite direction if he moved the vector from the first quadrant into the fourth, and David justified this assertion by stating that "following along our vector, basically we can trace how the imaginary components are either increasing or decreasing." Thus David's integration of algebraic and geometric reasoning appeared to allow him to note that the function $f(z) = e^z$ maps a point to an image whose argument is equal to the imaginary component of its pre-image.

Figure 64: Joshua explains the size of the spiral

While observing the spiral in *Geometer's Sketchpad* (*GSP*) (see Figure 65), Joshua correctly noted that it moved toward 0 fairly quickly (observation) "because it's exponential." This statement is the first reference made to the exponential nature of the function, so the nature of Joshua's reasoning is not entirely clear. It is possible that Joshua reasoned algebraically via the real-valued counterpart $f(x) = e^x$ and drew on the knowledge that the exponential function can have values $z_1$ and $z_2$ such that $|z_1 - z_2|$ is small but $|f(z_1) - f(z_2)|$ is still large. Perhaps Joshua's geometric reasoning suggested to him that the output values of an exponential function can vary drastically for small changes in the input values (as in $\mathbb{R}^2$ where a small horizontal change in the graph of $f(x) = e^x$ can correspond to a large vertical change). Joshua and David also had a brief disagreement regarding exactly *how* the function $f(z) = e^z$ moved in toward zero. Namely, after noting that their output point in *GSP* could move quite close to zero (observation), they discussed whether or not there existed $z_0$ such that $f(z_0) = 0$. This question is similar to another question that has commonly caused problems in the past for undergraduate mathematics students considering the nature of limits. Namely, "Does

$f(z) = e^z$ ever actually attain its limit point 0?" (e.g., Cottrill et al., 1996; Tall & Vinner, 1981).

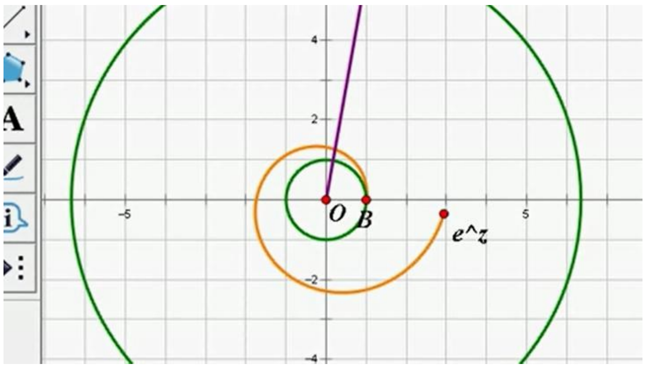


Figure 65. Spiral in *Geometer's Sketchpad*

Previous literature suggests that most students believe that a limit point can *never* be attained by the function. Therefore, it is noteworthy that in the following conversation Joshua committed the opposite error by suggesting that $f(z) = e^z$ *does* attain its limit point 0 when in fact it does not. However, it is important to note that this wording was never employed by either David or Joshua, who only discussed whether the function ever actually reaches 0, as seen in the following conversation.

> Joshua*: Not quite to zero, however, but definitely zero by here* (Left side of the unit circle as seen in Figure 66).
>
> David*: Close to zero*
>
> Joshua*: Close enough….yeah, that we can't visually tell the difference.*

David did not expound upon what he meant by "close to zero," and it is possible that Joshua believed $f(z) = e^z$ actually reaches zero at some point, though this is not clear. His speech seems similar to the zooming metaphor that some of Oehrtman's (2009) calculus students employed, though they never explicitly referenced limits.

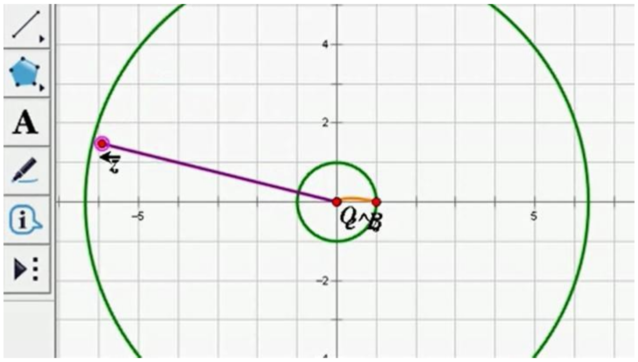

Figure 66. Image (orange) “reaches” 0

Eventually I pulled Joshua and David away from transforming vectors and had them predict how $f(z) = e^z$ transformed circles. David reasoned geometrically that this function should send a closed loop to another closed loop, but said he wasn’t sure why he thought that should be the case. Joshua offered a more specific suggestion that the function should send a circle around $1 + i$ to “a nice oval.” This time, David asked Joshua why he thought that, and Joshua replied, “the points further on the axis are more stretched than the points closer.” Thus, Joshua could have geometrically reasoned that the points further away from zero would spread out further on the output curve and the points closer to zero would end up somewhat closer together, thereby creating a major axis and a minor axis, rather than a uniform radius as a circle would have.

David did not appear satisfied by Joshua’s geometric reasoning, as he began talking about how individual points from the unit circle are mapped, apparently in order to form an idea of what the output should look like. After some discussion and work at a chalkboard (experimentation), they determined that the top part of their input maps to the point on their output curve that has the maximum argument (See Figure 67). Perhaps motivated by this discussion, David asked what would happen to an input circle with radius larger than $2\pi$. David began answering his own question via geometric reasoning by claiming that things would get weird after “the $y$ goes beyond $2\pi$, so it includes more than rotation around, ” and further that “everything makes sense if we have a small circle that stays within $2\pi$ and $-2\pi$. David seemed to want to know exactly how things would get weird if they expanded their input circle beyond a radius of $2\pi$, possibly reasoning geometrically that at this point the maximum angle would have to be more than one rotation around the circle.

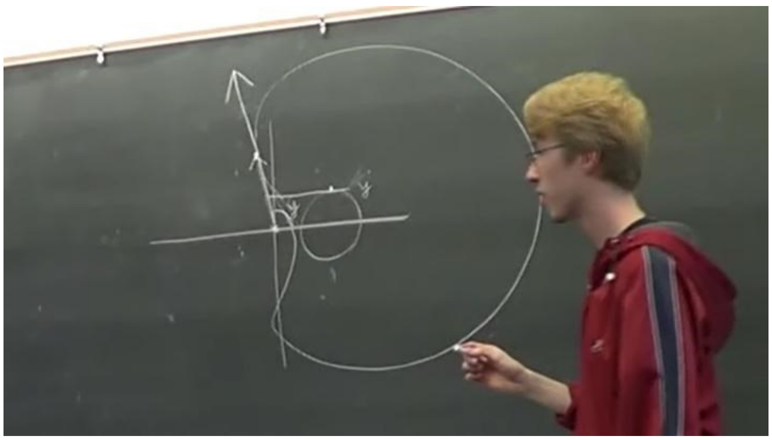

Figure 67. David’s Diagram

David and Joshua discovered via experimentation in *Geometer’s Sketchpad* (*GSP*) that their output curve would “twist” on itself; that is, it would no longer be one-to-one. They eventually characterized exactly which values $z_1$ and $z_2$ on their input circle had the property that $f(z_1) = f(z_2)$. Joshua estimated that points which map to the same place on the twist have a vertical distance of $2\pi$ between them, and suggested that this geometric reasoning also provided an explanation for why circles with a radius smaller than $2\pi$ map to an output with no twists.

I asked David and Joshua why the two points on the other side of the circle do not also map to a single point. This question was misleading because only one intersection point was visible on the screen, and according to David and Joshua’s geometric reasoning, there should have been two. Both David and Joshua understandably seemed troubled by this apparent contradiction to their logic, and David initially attempted utilizing geometric reasoning to explain this discrepancy by suggesting that twists do not occur unless two points are rotated at least $\pi$ in opposite directions: "the image hasn't completed a $\pi$ rotation, so they can't intersect." However, Joshua quickly noticed that this geometric reasoning suggests that the first two points should not have mapped to an intersection point either. Eventually, David commented that a lack of a second intersection point did not make sense, despite what he was seeing in *GSP*. So, he scrolled a little further to the left and actually found the second intersection, which had simply been off-screen. Joshua and David both expressed relief that their initial geometric reasoning really was correct and that there were in fact two twists (observation); one for each pair of points that are vertically separated by a distance of $\pi$ on the input circle.

Both the second day and the second task ended with Joshua and David attempting to geometrically reason through what $e^z$ did to the plane, though both Joshua and David claimed that their summary of this new function was less insightful than their overarching description for $z^2$. Joshua restated that points with a vertical distance of $2\pi$ away from each other are mapped to the same location, while everything else gets turned depending on imaginary values and stretched depending on real values, suggesting another synthesis of algebraic and geometric reasoning. David remembered that lines through the origin are mapped to spirals, and Joshua further added that the transformation $f(z) = e^z$ squeezes in negative values toward zero, before finally reiterating twice more that this function maps points $2\pi$ apart to the same place in the output plane.

### Task 3: Exploring Linear Functions and the Derivative With and Without *Geometer's Sketchpad*

After each participant group completed the first two tasks, I asked them to characterize the meaning of the derivative of a complex-valued function. This task allowed me to establish a baseline regarding facts they already knew about the derivative of a complex-valued function before the beginning of the interview sequence or facts they acquired via one of the first two tasks. At the start of this task, I did not provide any access to *GSP*, thereby encouraging the participants to describe what they already knew, and implicitly discouraging exploration of concepts of which they were not yet sure. Disallowing *GSP* usage aided me in further determining how participants reasoned through the derivative of a complex-valued function. In the context of *GSP*, participants could utilize the technology to help them answer the questions of which they were unsure, whereas without it they may have been more likely to tell me only what they knew already. Additionally, this tactic helped me address some concern documented in previous literature that the knowledge gained with technology may not be retained without the usage of this technology (Salomon, 1990). While my participants did not have access to *GSP*, I asked them to reason about the meaning of the value of the derivative at particular points for particular functions such as a $f(z) = z^2$ or various complex-valued linear function.

After I felt the participants finished describing the derivative of complex-valued functions geometrically as far as they were able, I returned their access to *Geometer's Sketchpad* (*GSP*) so they could continue investigating aspects of this topic they still felt were unfamiliar. Though I did not explicitly plan for it, both groups investigated the nature of linear functions at some point during this task. Karen utilized *GSP* to consider this topic, while Joshua and David first considered them before I returned their access to *GSP*. Due to the different rates of progress between the two groups through the first and second tasks, Karen worked through this task on the second day of her interview sequence, while Joshua and David completed the task on their third day.

At the beginning of Karen's third task, I removed her access to *GSP* and asked her to describe the nature of her reasoning about the derivative of a complex-valued function. She recalled the Cauchy-Riemann equations, but could only remember vague details about a matrix with two entries that had matching signs and two entries with opposite signs. As she could not remember any detail, she began describing the derivative as the slope of the tangent line. While this answer was understandable given her background in calculus, she did not feel that her explanation of the derivative as a slope generalized well to the complex plane for her. She also later stated that a "linearization" felt different than a "line." This way of reasoning geometrically about the derivative as the slope of a tangent line seemed salient to Karen, as it recurred frequently throughout her interview.

After she struggled to geometrically reason about a complex-valued derivative as the slope a tangent line, I suggested she re-summarize how $f(z) = z^2$ transforms the complex plane. She remembered that double circles are created when the input surrounds the origin, and further recalled the indent in the output that becomes more apparent as it moves closer to zero. Karen also told me that small circles "got rotated and dilated to their effect but they still had that weird bump thing happening around the point that was inside the unit circle." So, while she alluded to the idea that small circles, in particular, are both rotated and dilated in a specific way, she did not say anything about what that specific way might be, or how it might relate to the derivative. I re-introduced *GSP* to Karen after she finished summarizing the behavior of the function $f(z) = z^2$

As Karen seemed to be investigating more potentially useful mathematical concepts related to the derivative, I encouraged her to investigate how the function $f(z) = z^2$ transformed circles with *GSP*. During this experimentation stage she conveyed that she was "having trouble grasping what a line means in complex." About 8 minutes later, while investigating the function behavior $f(z) = e^z$ with *GSP*, she finally seemed to realize that even the idea of a line is different in the context of the complex numbers. She elaborated on what she felt was the source of her confusion: "I guess what's really screwing me up is the difference between, because you have a function, you have an input plane and an output plane, you have a function, and then you have a transformation." In short, Karen felt there were too many similar mathematical objects to keep track of at once. Karen may have had difficulty separating the input plane and the output plane because *GSP* displayed mathematical objects in both the domain and the co-domain within one single Cartesian grid. As in the previous two tasks, Karen used different colors to graphically separate the objects in the domain from the objects in the co-domain, but their physical proximity may still have confused Karen somewhat.

Karen further seemed to distinguish between functions and transformations, despite their mathematical equivalence. Perhaps this separation is due to a common difficulty experienced by students of mathematics—that of viewing a function as both an object and a process simultaneously (Sfard, 1992)—and the different words she used may reflect each of these facets separately. Karen utilized the function $f(z) = z^2$ to calculate the output value and derivative value at $z = 1 + i$ via algebraic reasoning, and constructed a transformation according to task instructions with *GSP*, so it is possible Karen viewed a function as a process and a transformation as an object, not realizing that they are both one single mathematical concept.

Since Karen had attempted to reason about complex-valued functions geometrically similar to how she reasoned geometrically about functions in $\mathbb{R}^2$, I reminded her of the limit definition of the derivative for a real-valued function. I asked Karen to consider the point $x = 4$ for the function $f(x) = x^2$ and tell me what $4 + h$ meant. She replied simply that it was $4$ when $h$ goes to zero, perhaps utilizing algebraic reasoning to answer this question, so I asked her what $h$ approaching zero meant. She appeared to transition back to geometric reasoning, claiming that a circle showed what was happening around the points of interest, and asked "Is that why we're doing circles?" Thus, for the first time, Karen appeared to generalize an aspect of geometric reasoning about the derivative of real-valued functions to the setting of the complex plane.

While she did not appear certain of the limit's geometric relationship to circles, this reasoning appeared to develop for Karen as she described to me what "narrowing in" on $\pi i$ in the complex plane looked like geometrically. She initially talked about approaching $\pi i$ from "both" sides, as though there were only two, so I asked Karen about a point on her circle in *GSP* that was located neither left nor right of $\pi i$ but rather somewhere above it. This question seemed to make something click. "Sure! We can have a little circle around $\pi i$. Oh! Circles! Gotcha!"

Following this exclamation, she realized that she "can get as close to $\pi$ as [she] want[s] by changing the diameter of her circle," stating that a small circle will "have a small $\epsilon$ or…$h$," and a large circle would have "a big $h$."The next time I asked her to characterize what information the value of the derivative at a point $z$ gives her about the output $f(z)$, she first reasoned algebraically through the limit definition $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ I had supplied, and eventually transitioned to geometric reasoning. In particular, she reasoned that "the closer we are to [the point z] the more accurate for what our derivative actually is at a single point." That is, she once again successfully enhanced her geometric reasoning about the derivative of a real-valued function to apply to the derivative of a complex-valued function. Through her experimentation with *GSP* of circles of varying size around a point $z$, she realized that she could utilize the same limit definition that defines the derivative of real-valued functions to reason about the derivative of complex-valued functions. Her observation that one can approach points in the complex plane in many directions, rather than just from the left or right as in the case of points in $\mathbb{R}$, seemed crucial to this discovery. After that realization, she suggested that $h$ was controlled entirely by the size of the circle surrounding the complex-valued point of interest.

However, in manipulating a circle centered at $1 + i$, she collapsed the radius to $0$, making the input circle in essence a single point. Despite having just rediscovered a relationship between derivatives, limits, and approximations, she once again returned to geometrically reasoning about the derivative as the slope of the tangent line. This time I told her directly that we should investigate properties of "lines" in the complex plane, recalling her previous exclamation that she did not know what a line really looked like. I therefore asked Karen to construct the function $f(z) = 2z + 1$. While initially convinced that the output circle would change in size as the input circle moved away from the origin, she discovered via experimentation with *Geometer's Sketchpad* (*GSP*) that if the input circle did not change in size or rotate, neither did the output circle. Moving the input circle only changed the output circle's location, not its shape (see Figure 68 and Figure 69). I reminded Karen to use this function to learn what "slope" meant, and pointed out that this function's derivative evaluated to 2 everywhere. Karen responded by suggesting that this value of 2 meant that the output circle is always twice the size of the input circle in this function, successfully reasoning geometrically about an algebraic result.

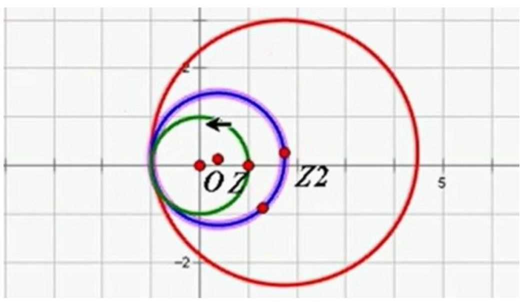

Figure 68. Circle at origin

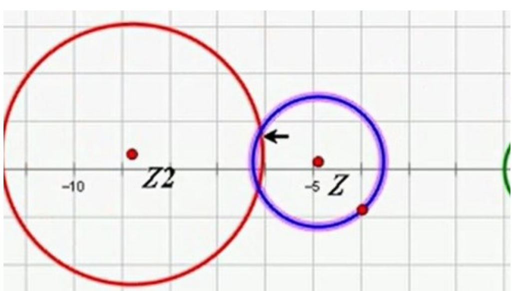

Figure 69. Circle away from origin

While Karen developed some sense of how to reason about the derivative geometrically in the context of a linear function, she felt that this reasoning would not generalize to $f(z) = z^2$, as circles no longer mapped to circles. She added that her rule for linear functions probably would not work here because the output is not a circle and thus "doesn't have a radius really," an objection that David and Joshua also raised briefly. Thus, the geometric reasoning of both groups initially suggested to them that the process of “rotation” and “dilation” applied only to circles—an odd objection, as in previous research both Karen and David spent a great deal of time rotating and dilating vectors, and they additionally knew that multiplication by a complex number corresponds to a rotation and a dilation. Perhaps some facet of their experimentations and observations with *GSP* contributed to my participant’s sense that geometric reasoning about the operations of “rotation” and “dilation” are somehow more difficult if the object undergoing these transformations is not a circle. It is not clear whether they ever truly resolved this issue, as all participants essentially managed to sidestep this problem by noting that the image of a small circle under $f(z) = z^2$ and other functions at least resembled a circle (observation), then reasoning about these images as though they were circles.

Just as Karen developed a sense of how the derivative might affect the dilation factor of an input circle with the function $f(z) = 2z + 1$, the function $f(z) = iz + 2$ seemed to help her develop a sense of how the derivative might affect the degree of rotation of an input circle. However, Karen was incorrect regarding how the derivative could affect the rotation and dilation of the input circle. She claimed that the real part of the derivative impacted the dilation of the image of the input circle, and the imaginary part was the degree by which the function rotated the image of the input with respect to the original circle. This claim is incorrect, as the derivative value’s magnitude, not its real part, impacts the dilation of the image at a point and the derivative value’s argument, not its imaginary part, impacts the rotation. Karen may have made this error because of the particular linear functions I asked her to construct: the first had a pure real derivative to emphasize the dilation aspect of the derivative and the second had a pure imaginary derivative to highlight the rotation aspect.

While Karen originally claimed that the derivative was related to the Cauchy-Riemann equations and described the slope of a tangent line, David and Joshua had no such pretense. When I asked Joshua and David how they thought of the derivative of a complex-valued function, Joshua bluntly stated, “I don’t” and David confessed, “I don’t think I ever really got a good grasp on it, so,” and did not elaborate. After I pressed them further, David tried to recall a limit definition, but neither Joshua nor David seemed able to remember details. Having unsuccessfully attempted to recall an algebraic definition, David tried transitioning to a geometric reasoning approach, and started talking about the slope of a tangent line, just as Karen did. However, unlike Karen, David appeared to believe that this geometric reasoning would not generalize to the complex plane: “We were talking about somehow, the normal real derivative is just slope change in a function. For the complex derivative, I think somehow it had to do with a change but it was a change along a vector, or something like that. Maybe.”

Since neither participant seemed sure how to proceed, I directed their attention to the behavior of the function around the point $z = 1 + i$ in the function $f(z) = z^2$. David simply stated “that was a circle that did weird things,” apparently recalling their investigation of the transformation of circles surrounding the point $z = 1 + i$. Joshua restated the conditions necessary for two points to map to the same place under this transformation, thereby causing a twist in the output. When I asked them to relate their observations to the derivative, David started considering the magnitude and angle of the value of the derivative $f'(1+i) = 2 + 2i$ evaluated at $1 + i$, and suggested the 45° angle of $2 + 2i$ "represents what the point $1 + i$ would get mapped to in $z^2$."

He continued by calculating the magnitude and angle of $1 + i$, possibly to attempt to discover some relationship between these quantities and the magnitude and angle of the value of the derivative of $f(z) = z^2$ at $1 + i$. It is also possible that David still wanted to consider the value of the derivative at $1 + i$ but chose to work with the incorrect value of $1 + i$ instead of $2 + 2i$. David stated that he was attempting to find some relationship involving magnitude and angle through these calculations. I redirected him and Joshua back to the value $2 + 2i$ and asked Joshua and David how the transformation rotated the input circle. Joshua suggested upper bounds for the amount the circle could be rotated, and David appeared to object to talking about rotations at all in this context. Like Karen, David initially felt that the term "rotation" did not make sense for a shape that was not a circle. Shortly after Joshua re-voiced this concern, David seemed to overcome his own objection by geometrically reasoning about the output shape as though it was a circle, stating "So do you mean, rotated as in a circle is taken and you twist it like this," as his hands positioned themselves as though they were holding a sphere, then rotated as though rolling this sphere to the left (see Figure 70).

Figure 70. David rotates a circle counter-clockwise

This discussion appeared to remind David of talking about $\epsilon$ −neighborhoods as a class in his complex analysis course. Armed with this idea, he suggested looking at how a small region around a point is transformed by the function $f(z) = z^2$, rather than how the point itself is transformed. This realization was a major step forward in geometrically reasoning about the derivative as a local linearization, as the inclusion of $\epsilon$ −neighborhoods in their reasoning made it possible to include the idea of approximation of a function in their reasoning as well. Joshua and David appeared to solidify this geometric reasoning about rotating something that is not a circle after they drew several line segments and discussed how they thought each are transformed. In particular, they wanted to know whether their line segments would map to another line segment. David initially posed this question, and Joshua appeared to believe this was not the case, stating, "I guess it does not need to be a straight line to have some idea of rotation." David however, felt that a line would indeed map to a line rather than a circle due to a belief that rotation and magnitude occur separately, apparently forgetting that a different point on the line might be rotated in a different way.

> Joshua*: I mean, but well sure, but we're also rotating each point separately*
>
> David*: Yeah, but you can think of rotating and then,*
>
> Joshua*: I guess, okay no, you're right, the rotation is separate from the magnitude.*

Despite Joshua's seeming reluctance to believe that lines were mapped to lines, observing an inscription that rendered the image of lines as lines (see Figure 71) appeared to impact Joshua's geometric reasoning. In particular, he seemed to believe that the transformation $f(z) = z^2$ preserved certain angles such as those indicated in Figure 72. David and Joshua followed this investigation by trying to develop their geometric reasoning to predict how the circle gets rotated by considering specific angles. Joshua asked, “So we saw that this 45 degree angle was rotated 45 degrees. So would this 60 degree angle be rotated 60 degrees….or is everything rotated 45 degrees no matter what?” After some discussion of how various angles were transformed by $f(z) = z^2$, David and Joshua eventually agreed that the output shape, which was nearly a circle, seemed to rotate by 45° with respect to its pre-image. Just as Karen had originally, David related this occurrence to the previously investigated function behavior rather than to the derivative at that point: “Well in this instance it would be because of circles around 1+i and that already had, that's a 45 degree angle so when you double that we get another 45 degrees. So if we had an angle at, or a preimage a circle at an angle of 30 degrees, the image is going to be rotated 30 degrees.”

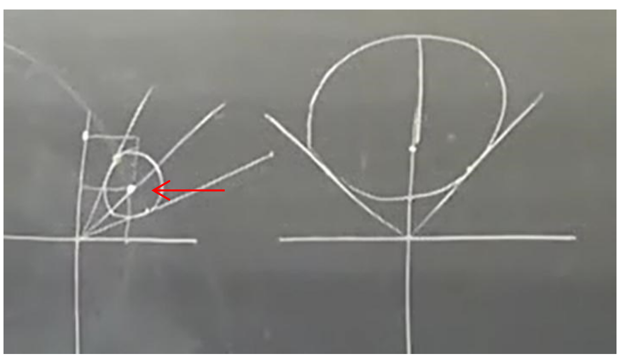

Figure 71. Domain (left) and co-domain (right) of $f(z) = z^2$

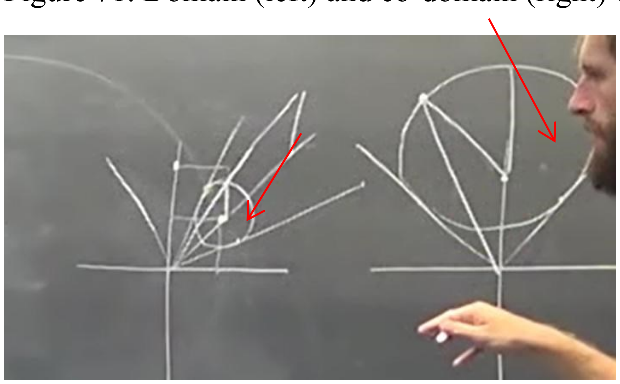

Figure 72. Angles in the domain (left) and co-domain (right)

To pull them away from only reconsidering function behavior, I asked them to consider functions with a constant derivative. This question parallels my attempt to utilize linear functions to discourage Karen from continuing to reason geometrically about the derivative as the slope of a tangent line. At this point, Joshua also reintroduced the idea of slope, and defined it as the change in imaginary over the change in real values—a definition that, though flawed, is still symbolically related to the algebraic definition of a real-valued slope: $\frac{y_2-y_1}{x_2-x_1}$ (see Figure 73). It appeared that Joshua had conflated the idea of input and output values (represented by $x$ and $y$ respectively) with the idea of real and imaginary parts (represented by $x$ and $y$ respectively) due to the fact that the same pair of variables are used to represent both concepts; his algebraic reasoning about these two facets could have interfered with each other due to the similarities of these algebraic inscriptions. While Joshua seemed unconcerned by the idea of slope, David seemed troubled, stuttering "Well no, the sss, are you talking about the ssss, what do you mean by slope?" Joshua tried to defend this way of reasoning geometrically about slope by building the domain out of several tangent lines through the origin (construction) but immediately abandoned this reasoning while looking at the geometric inscription he had just produced (seen in Figure 73). When I asked him why he said he "[did]n't like the idea anymore," he claimed that he "[could]n't connect it to the function $z$."

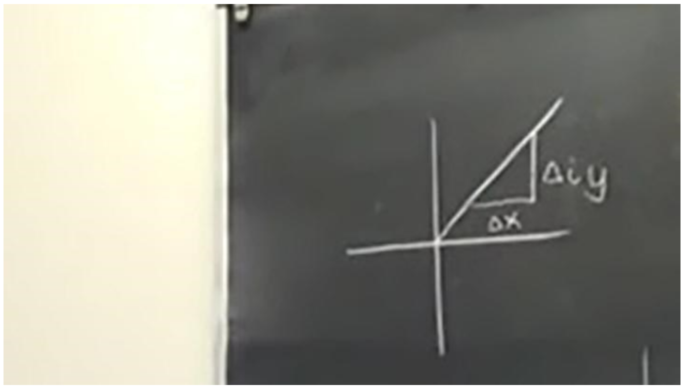


Figure 73. Joshua tries to explain the slope of the tangent line of a complex-valued function

While it was difficult to discourage Karen from reasoning geometrically about the derivative as the slope of the tangent line, David and Joshua seemed so proficient at reasoning through function behavior directly that it felt difficult to encourage them to think of the derivative at all. That is, I created the interview tasks to encourage participants to think about how a complex-valued function maps a small circle around a point $z$, and how the value of the derivative of the function at the point $z$ describes this mapping. Particularly, the magnitude of the derivative value at a point $z$ impacts the amount the image of a small circle around $z$ dilates with respect to its pre-image, and the argument of the derivative value impacts the amount the image rotates. However, David and Joshua seemed able to predict the way a function maps a small circle simply by referencing the equation of the function itself, without ever having to utilize any derivative values. This proficiency at predicting function behavior without the use of the derivative may have caused some difficulty in motivating Joshua and David to see a need to geometrically describe the derivative at all. In this way, such knowledge of function behavior may have conflated with the development of their geometric reasoning about the derivative as a local linearization.

When I asked Joshua and David to consider the function $f(z) = iz$, which had an imaginary derivative, Joshua noted that geometrically the axes switch and so the plane rotates 90° under this transformation, and David justified this answer by algebraically reasoning that "$x$ goes to $ixy$, $iy$ goes to $-y$, so all the real parts get mapped to the imaginary, and all the imaginary get mapped to the negative real parts." Just as I had with Karen, I started asking Joshua and David more pointed questions to highlight the derivative's role. Their response suggests development of geometric reasoning that included the possibility that the derivative is related in some way to rotation and dilation. Joshua thus became interested in whether this geometric reasoning would generalize to non-linear functions such as $f(z) = z^2$

> Interviewer: *What does the derivative tell you about the function?*
>
> David: *So that would tell us that everything gets rotated* ***by 90° because i is*** (rotates open-palmed hand from facing up to facing left) *at* 90°.
>
> Joshua: *So the derivative tells you something about how you stretch and how you rotate.*
>
> David: *Yeah, that makes sense for like* $2z$, $3z$, $4z$,*all that stuff because they're,* ***all the derivatives are on the real line****,* (pinches fingers together with right index finger pointing forward, then pulls hands horizontally apart) *so that gets rotated*
>
> Joshua: *We're not rotating anything, we're just stretching it so, yeah. Does that make sense for* $z^2$ *though?*

Neither Joshua nor David explained which mathematical entities they claimed should be rotated and dilated throughout this exchange. David noted that "everything" rotates 90°, and later claims "that" rotates, without indicating to which objects either "everything" or "that" refers. Joshua introduces a similar ambiguity in commenting that "it" stretches. Throughout this exchange, both Joshua and David appear to focus on the geometric operations that occur, but do not identify any entity on which these operations are acting.

In a following conversation, Joshua duplicated one of Karen's errors by claiming "the derivative if it's a constant tells you about, either a rotation if it's imaginary, or a dilation if it's real," thereby associating the real part of the derivative to a dilation and the imaginary part to a rotation. However, when I asked them what the complex-valued constant derivative $3 + i$ meant, they answered correctly, although they were still vague about which rotation and dilation the derivative describes.

> Interviewer: *Okay, and what if it's complex? So like,* $(3 + i)z$.
>
> Joshua: *Probably some combination.*
>
> David: *Well it'd be whatever angle* $3 + i$ *is at, that's our rotation*
>
> Joshua: *Well shouldn't it's also whatever magnitude* $3 + i$ *is, so it's exactly a combination of magnitude and rotation. So you dilate by root 10 and then rotate by whatever angle* $3 + i$ *is like you said.*

David initially expressed discontent with this rule they developed, stating "actually I'm still not sure on the magnitude part." His discontent could have stemmed from his previous statement that he wanted to think of rotation and magnitude occurring sequentially rather than simultaneously, but here he seemed to relent and accept Joshua's geometric reasoning, admitting, "I don't know, it kind of makes sense that it does it at the same time."

Despite having just correctly geometrically reasoned about the derivative within the context of a linear function, David repeated a previous error when attempting to generalize to the function $f(z) = z^2$. When considering the point $z = 1 + i$ where $f'(z) = 2 + 2i$, he applied their rule to the input point $1 + i$ rather than an

$\epsilon$ −neighborhood around it, citing the fact that $2 + 2i$ has magnitude $2\sqrt{2}$, but the point $1 + i$ maps to $2i$, a point with magnitude 2, not $2\sqrt{2}$. Joshua tried to salvage some of their heuristic by noting that $1 + i$ has a magnitude of $\sqrt{2}$ and "$\sqrt{2} * 2 = 2\sqrt{2}$." In particular, $|1 + i| * |f(1 + i)| = |f'(1 + i)|$. David paraphrased Joshua's claim by saying, "So the magnitude of the point of the pre-image times the magnitude of the image of the $z^2$ is the same as the magnitude of the derivative, is that what you're saying. Joshua confirmed: "I mean, apparently." When I asked Joshua if this rule would hold everywhere, he noted it would not, since $f(1) = 1^2 = 1$, which is not dilated at all even though the function $f(z) = z^2$ at the point 1 has a derivative of 2 under this mapping.

After this conversation, I re-introduced *Geometer's Sketchpad* (*GSP*) to allow David and Joshua to test all the conjectures they had just generated between themselves and with some help from inscriptions on the blackboard (experimentation). This re-introduction of *GSP* and its related inscriptions began with several quick progressions through construction, experimentation, and observation, as Joshua and David built the necessary objects (if needed/not already present) to test their conjectures (experimentation), placed them in their proper locations, and observed the results (observation). To begin with, Joshua noted that lines do not map to lines, though the spokes of the input circle looked close to straight in the output when the circle was small. David similarly observed that "small input circles…look like very circular outputs." They additionally overturned their previous conjecture that an angle located at the origin in the input would be preserved by the function $f(z) = z^2$ in the output, observing that the angle is doubled instead.

After I asked Joshua and David to describe how circles are transformed by $f(z) = z^2$, Joshua verbalized a connection between the magnitude of the derivative and the way in which circles are transformed by the function.

> Joshua: *Oh. I said* ***whatever magnification that is*** (places cursor over input point $z$), *and I said the number, so* ***looks like the input circle is a radius of a quarter*** (waves mouse along radius of input circle)*,, which is what,* $.707$*? So not quite tripled. So I mean, the output is 2 root 2 bigger than the input, which is the magnitude of the derivative.*

He additionally realized shortly thereafter that this information only applied when the input circle was small.

> Joshua: *Maybe the derivative tells you how much bigger the radius is, but then once you start getting deformation it doesn't hold. I guess in an epsilon neighborhood around* $1 + i$*, that you don't need to consider deformations so they will hold, I guess. So the derivative tells you how much the radius grows around an epsilon neighborhood of* $1 + i$.
>
> David: *I guess I can buy that a little bit.*

After this exchange, David spent about 8 minutes back at the blackboard trying to prove via algebraic reasoning and algebraic inscriptions that an $\epsilon$ −neighborhood around the point $1 + i$ should be dilated by a factor of $2\sqrt{2}$ by the function $f(z) = z^2$. After this interlude at the board, Joshua asked a question about the one aspect for which his explanation did not account: "But do we know how the derivative tells us that we're rotating?" David turned to *GSP* to answer this question, choosing a different point as the center of their input circle and attempting to predict how the function would map one of its radii (see Figure 74). Joshua and David appeared to be able to predict that a radius at 0° with respect the center of the input circle maps to a radius at 30° with respect to the center of the image of the circle (see Figure 75). However, it is not clear how they were reasoning about it, nor did they actually verbalize how the derivative relates to the rotation. Additionally, they seemed to have the ability to accurately predict how an input angle would be transformed, though it is not clear whether this ability stemmed from their understanding of the derivative, or simply their experience with *GSP* or even just the function $f(z) = z^2$ itself.

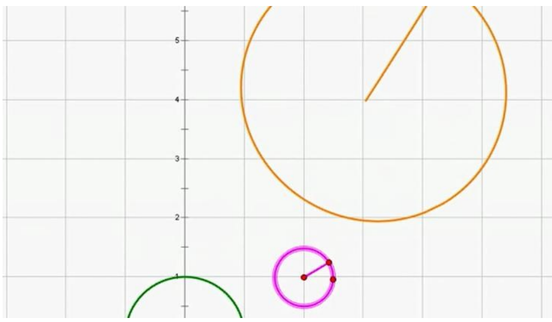

Figure 74. Joshua and David consider a circle (purple) centered around $2 + i$

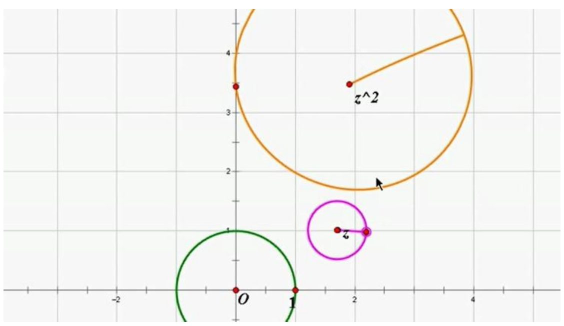


Figure 75. Joshua and David test their rotation prediction

**Task 4: Generalizing and Testing Meaning of Derivative via $f(z) = z^2$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$**

After exploring the transformation of circles under linear functions, both groups of participants seemed able to reason geometrically about the derivative. In particular, within the context of linear functions, both groups noted that the derivative of a linear function allowed them to predict how a circle was rotated and dilated under the function. However, neither group seemed certain how this result generalized to non-linear functions, or if it generalized at all. In this task, I asked each set of participants to apply their geometric reasoning to the derivative of a complex-valued function within the context of the functions presented by the two tasks: $f(z) = z^2$ and $f(z) = e^z$. In particular, I asked them if their current geometric reasoning generalized to these new transformations.

Furthermore, I introduced a function they had not yet investigated: $f(z) = \frac{1}{z}$. I selected this function for two reasons. First, while $f(z) = z^2$ maps circles around the origin to double circles and $f(z) = e^z$ maps larger circles to an image that twists around on itself, the function $f(z) = \frac{1}{z}$ always maps a circle to a circle or a line. Thus, in these ways, $f(z) = \frac{1}{z}$ maps an arbitrary circle somewhat less oddly than either of the other two functions, though $f(z) = e^z$ does always map a small circle to another circle as well. Second, unlike either of the first two functions, $f(z) = \frac{1}{z}$ has a point for which its derivative is not an ordinary complex number; namely $z = 0$. At $z = 0$, the derivative can either be taken as non-existent, or as $\frac{1}{0} = \infty$.

In this task, I discuss the ways in which my participants attempted to generalize their current geometric reasoning about the derivative of a complex-valued linear function to the context of these three functions. The participants' explorations regarding the meaning of a non-existent derivative is discussed further in Task 5. Karen began this task on the third day of her interview sequence, and completed it near the end of her fourth and final day. Joshua and David worked through this task entirely on their fourth day.

Karen began the fourth task by re-summarizing how she reasoned about the derivative of a complex-valued function, both algebraically and geometrically. She recalled what the derivative meant for a linear function, again claiming that the magnitude of the real part determines the dilation and the argument of the imaginary part determines the rotation. She also remembered the limit definition of a derivative and shrinking the input circle all the way down to a single point: “So at a point, like, you're taking the circle and as $h$ goes to $0$ it the output also kind of converged onto the point that we would expect it to. ”

As Karen had just offered a way of reasoning geometrically about the derivative of a complex-valued linear function, I asked her to describe her geometric reasoning about the derivative of $\frac{1}{z}$ at $1+i$. Via algebraic reasoning, she calculated the value to be $\frac{1}{2}-\frac{1}{2}i$ . Note that this is the incorrect value, as the derivative at this point evaluates to $\frac{1}{2}i$. Regardless, she told me that $\frac{1}{2}-\frac{1}{2}i$ is the slope of the tangent line at $1+i$. As before, throughout this task Karen geometrically reasoned about the derivative as the slope of the tangent line many times over, although I was often able to encourage her to find alternate methods of reasoning by asking her to explain what a line with imaginary slope looked like. For example, when I asked her to tell me what a slope of $\frac{1}{2}i$ meant, she replied, “It means that, see this is where I get lost because, can linear functions slope like that? Because input and output are not the same, geometrically this is difficult. So it’s dilating, it’s shrinking, and….” Furthermore, after geometrically reasoning through what it means to multiply a complex number by $\frac{1}{2}-\frac{1}{2}i$ via rotations and dilations, it seems she began to feel once again that her tangent line idea may not easily generalize to the case of complex functions.

> Karen: *That's why I'm getting confused mostly is because I don't know how that would transfer over. If it transfers over like, directly or if*

> *there's something else that happens because your input and output in the Cartesian plane is on the same plane, so you just find your point on that plane and then you draw the tangent line. But since it's on two different planes in a complex, I don't know what that implies as far as tangent lines go.*

She seemed to conclude that the tangent line should exist entirely within the output plane of a complex function: "So it would be on an output plane….because, well then maybe it's not. No it has to be because, if you go back to the algebraic definition of limit, your output is what defines your function, the derivative. Because it's $f(x+h)$ and that's the output, minus $f(x)$ over $h$."

This time, to encourage an alternate method of reasoning and discourage reasoning involving tangent lines which seemed to be unhelpful to Karen, I suggested Karen direct her attention to how circles are transformed by the function $f(z)=\frac{1}{z}$. Via experimentation and observation with *Geometer's Sketchpad* (*GSP*), Karen discovered that $z=0$ is a "weird" point in that a small circle around 0 was transformed into a very large circle. Though apparently initially interested by the discovery that a small circle around 0 was transformed into a very large circle (observation), she decided that this result was unsurprising since "1 over 0 is infinity." She additionally described how $f(z)=\frac{1}{z}$ transformed circles in general while demonstrating this in *GSP*. Particularly, she noted that this function maps circles "inside the unit circle to something bigger because it's less than 1" (see Figure 76) and the function maps circles "inside the unit circle are going to be larger than what they were" (see Figure 77). She additionally noted that if she moved the unit circle farther away from the origin, the output gets closer to zero and it "basically all but disappears into just a single point." (see Figure 78)

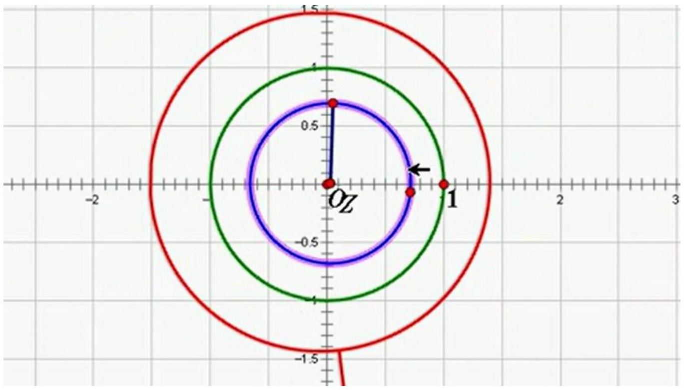


Figure 76. Input circle (blue) inside unit circle (green)

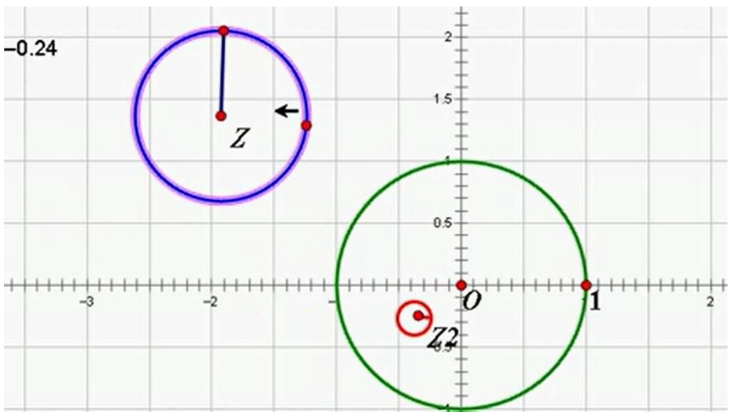


Figure 77. Input circle outside unit circle

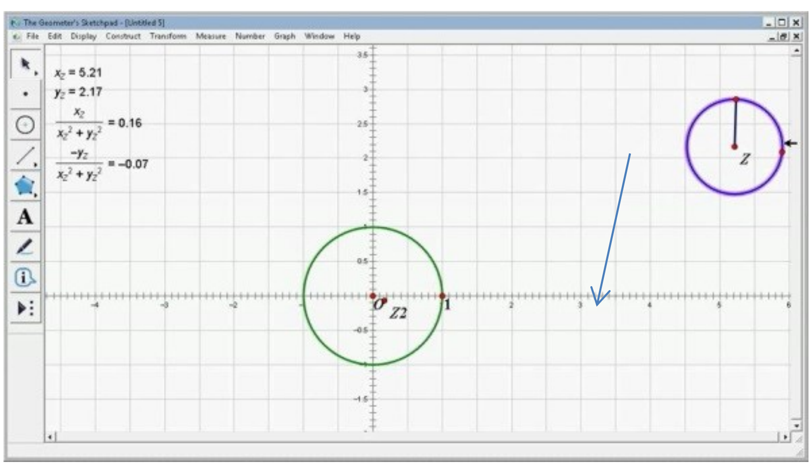


Figure 78. Image collapses “into just a single point”

When I asked Karen to reason geometrically about the derivative of $f(z) = \frac{1}{z}$ at $z = 1$, she calculated the derivative to be 1 and again claimed that this was the slope of the tangent line. I discouraged this explanation once again by asking her to extend this reasoning to the point $z = 1 + i$, which has a complex-valued derivative.

She attempted to answer by drawing a line segment (construction) in the output plane through the point $\frac{1}{z}$ at $\frac{1}{2} - \frac{1}{2}i$ with slope $-\frac{1}{2}$ (see Figure 79). I challenged this reasoning, and again she seemed to decide that she probably did not want to reason via the slope of tangent lines.

> *Interviewer: Why is that what it looks like?*
>
> *Karen: That has to go through that point and it has a slope of.* ***Oh! Dang it*** (strikes table forcefully)*, I'm thinking in terms of Cartesian again. It doesn't look like that. Shoot.*
>
> *Interviewer: Tell me why you don't want to think in terms of Cartesian coordinates.*
>
> *Karen: Well, because I keep thinking that,* ***like negative one half would be*** (palms face each other, fingers tessellate with palms facing inward, then) ***which it is, but it doesn't necessarily*** (rotates palms upward with right palm facing left). *Does it mean rise over run* (right hand rotates on top of left hand, palms coming together) *in complex like it does in Cartesian?*

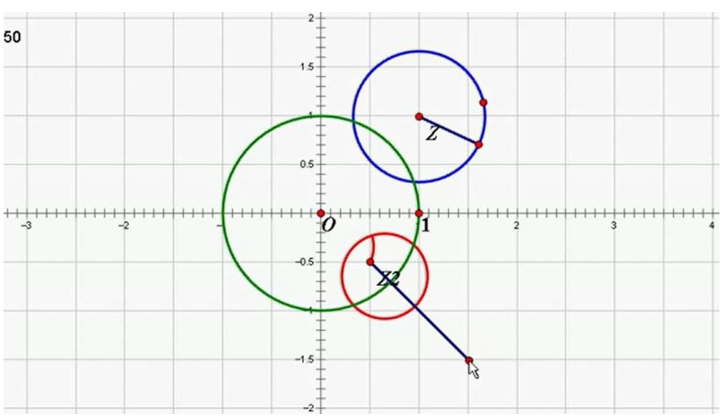


Figure 79. Karen's "tangent line" for the point

Karen continued to spend some time trying to generalize her tangent line reasoning properly to the complex plane, and I tried to help by directing her attention to various components of reasoning about the real-valued derivative. Eventually she returned to the idea of linear transformations.

*Interviewer: What do $x$ and $y$ mean?*

*Karen: The output values over the input values.*

*Interviewer: So where do the tangent lines live?*

*Karen: Like in?*

*Interviewer: Is the tangent line part of the output values or the input values or neither or something else entirely?*

*Karen: On the Cartesian plane it lies tangent to your output values at that specific input value. Does that make sense?*

*Interviewer: So what does a line look like in the complex plane?*

*Karen: Like a linear transformation you mean?*

It is possible that here Karen was committing an error of which Joshua was also guilty. In particular, she may have been conflating the input and output values (represented by $x$ and $y$ respectively in the Cartesian plane) with the real and imaginary parts (represented by $x$ and $y$ respectively in the complex plane). Furthermore, even within the context of the Cartesian plane, Karen did not appear to recognize that the tangent line of a function is tangent to a point that consists of both an $x$ −value and a $y$ −value, rather than just the $y$ −value. This oversight may have contributed to Karen's difficulty in generalizing geometric reasoning about tangent lines to a complex-valued function, as she may not have realized that such a tangent line touched a four-dimensional point $\left(z, f(z)\right)$ consisting of both a two-dimensional input point $z$ and its corresponding two-dimensional output $f(z)$. After this summary, I asked Karen to explain what the derivative at the point $i$ in the function $f(z) = \frac{1}{z}$ told her about the function. She correctly noted that the derivative at $i$ was 1 and claimed this meant that "[the input circle]'s going to stay the same, because it would just dilate by a factor of 1, but that doesn't do anything." When she tested her theory in *GSP* (experimentation), she initially seemed disappointed (observation) by the result (see Figure 80).

> *Karen: So that didn't work* (traces down and right, outlining a line with slope negative one half). *Yeah,* $dz$ *is at* 1. *Why did it do that* (referring to black circle in Figure 80) *though?*
>
> *Interviewer: It's where you want it?*
>
> *Karen :* $dz$ *is where I want it. The circle is not ideal.*

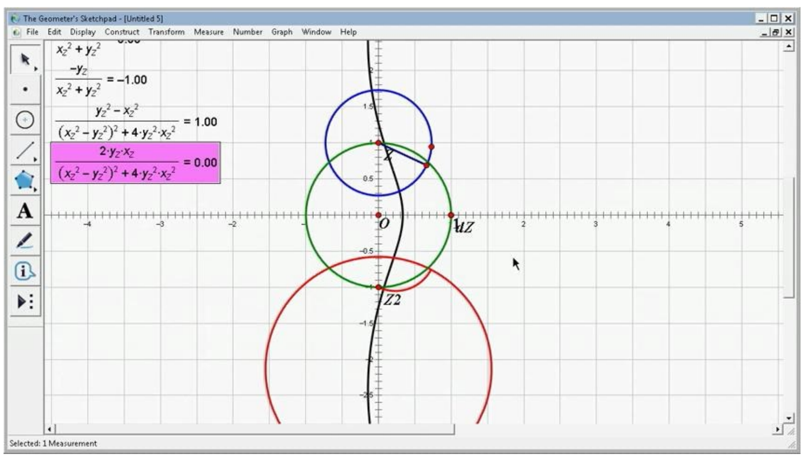


Figure 80. The derivative of $z$ (red dZ point) and the transformation (black) of the blue circle under the derivative function

However, Karen soon decided that a slight distortion should be present in the output shape, because for different "points on the circle, the derivative is going to have a different effect on them because they have a different value." Thus, at this point in the interview, Karen may have had a sense not only that the derivative describes an essentially local property, but also could have developed some geometric reasoning explaining *why* the property is necessarily local in nature. However, Karen still did not seem certain that her geometric reasoning about the derivative at a point in the function

$f(z) = \frac{1}{z}$ was correct, apparently due to the fact that the output of a circle in *GSP* was slightly distorted, stating that "before, when we did linear transformations, it didn't affect the size of your circle that you were looking at." When I asked her what was different about $f(z) = \frac{1}{z}$, she seemed to change her reasoning to allow for the possibility of generalization from linear functions, claiming "…it almost kind of looks like and maybe this is just a coincidence, but if you took this circle (waves mouse over black circle, which represented the derivative; see Figure 81) and you kind of rotated and dilated it to a certain effect, that's kind of what you got with the derivative."

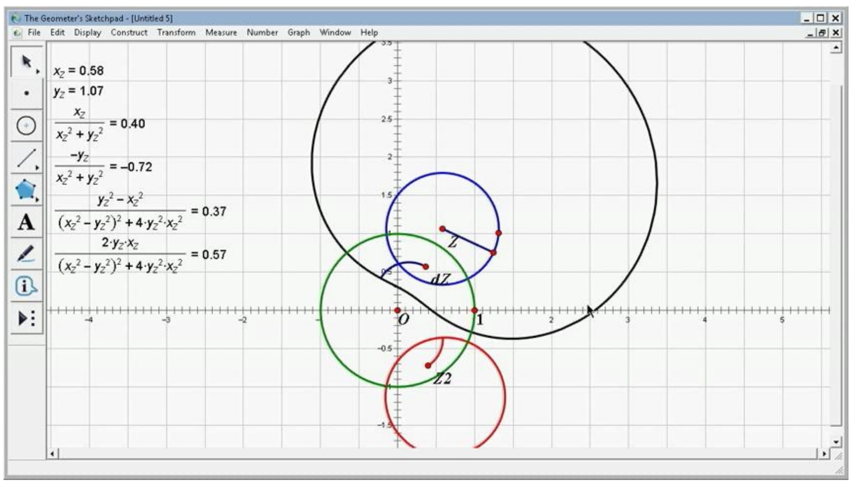


Figure 81. Input (blue), function output (red), and derivative output (black)

However, after apparently developing geometric reasoning about the meaning of the derivative of a linear function and offering geometric reasoning for the meaning of the derivative of $f(z) = \frac{1}{z}$, Karen returned again to geometric reasoning about the derivative as the slope of a tangent line. She tried to refine this reasoning by attempting to explain what the derivative of a real-valued function would tell her about the shape of the function. Additionally, she claimed that the derivative of a complex-valued function really told her nothing, despite having already explained several aspects of her geometric reasoning about the derivative of a complex-valued function.

> *Karen: I don't know what else it would tell you, but I don't necessarily know that it's the same either. So the derivative function really tells me nothing. Well it does, but it doesn't give you any graphical meaning. That's not right either. It tells you the slope of all the tangent lines but doesn't look like any of them. Does that make sense?*

Karen eventually managed to transition back to her previous geometric reasoning about the derivative of complex-valued linear transformations, and even repeated her error that the real part of the derivative controlled the magnitude of the dilation, and the imaginary part controlled the angle of rotation. Karen seemed to take another step toward developing geometric reasoning about the derivative of complex-valued functions when I asked her what it would mean if the derivative at an arbitrary point $z$ under an arbitrary complex-valued function $f$ evaluated to $3 + 2i$. In particular, it seemed she began to realize that the magnitude of the derivative, not the real part, impacted the dilation of the transformed circle, and the argument of the derivative, not the imaginary part, impacted the rotation.

> *Interviewer: What does that tell you about this function?*
>
> *Karen: Nothing. It tells me about that point, but as far as a function goes,* ***just that it's*** *increasing* (palms face inward, fingertips point at each other, fingers fan out) *at that point.*
>
> *Interviewer: What's increasing?*
>
> *Karen: The function itself.*
>
> *Interviewer:* $3 + 2i$ *means increasing? Why is that?*
>
> *Karen: Well, I guess not in complex terms. (***Both hands start palm down. Left hand raises as right hand raises straight and drops in an "S" shaped path***) It means that it's dilating (*Hands move out to shoulder

> width, palms facing each other, fingers slightly curled*) in a positive fashion and rotating in a positive direction. Is that better?*

It seemed clear by the end of the third day that Karen did indeed realize the magnitude of the derivative evaluated at a point described a dilation and the argument described a rotation. However, instead of relating this dilation and rotation to the image of a circle centered around a point where the derivative evaluates to $3 + 2i$, Karen claimed that a derivative value of $3 + 2i$ meant that "the function itself" was "increasing." She elaborated that by "increasing" she meant that "[the function]'s dilating in a positive fashion, and rotating in a positive direction." As Karen had previously associated the real part of the derivative evaluated at a point with this dilation and the imaginary part with a rotation, I asked her to apply this geometric reasoning to a different value to see if she would repeat this error. She did not, instead interpreting "$\tan\frac{b}{a}$ or something like that" as the "angle [she]'s dilating by," and "the magnitude of $a + bi$" as her "dilation factor," where $a + bi$ is the value of the derivative evaluated at some point.

To continue this task on the fourth day, I asked Karen to re-summarize how a complex-valued linear transformation behaved. She recalled that a linear function transforms a circle according to the derivative: dilating the circle by the magnitude of the derivative and rotating it by the argument. My questions grew more direct as well; I started asking Karen to consider how $f(z) = \frac{1}{z}$ would transform a circle at the point $z = i$ if this transformation were linear. Although she had just evaluated the derivative at $z = i$ as 1, stating "because the derivative is $-\frac{1}{z^2}$, and $i^2$ is $-1$," she responded by associating the function $f(z) = -iz$ with this point. Despite this error, her reasoning about the derivative as a linear approximation appeared to develop.

*Karen: "It almost looks like it's a linear map, but it's not. You can tell because the spoke is being dampened and because the spoke isn't straight, and I don't know how I would characterize that.*

*Interviewer: When you say it looks like a linear map, what do you mean?*

*Karen: Well the linear maps we looked at, a lot of them were just a representation of that circle either farther away or rotated or both, and that's kind of what this looks like, but it's a little bit different.*

*Interviewer: Can you tell me which linear transformation this looks like?*

*Karen: No….Actually, if it looked like any transformation, it would look most like* $\cos 90°$ *. I was going to say* $i$*, but that's not right, because otherwise* ***it'd be over here*** (points at the point $-1$ on the unit circle. See Figure 82 for reference).

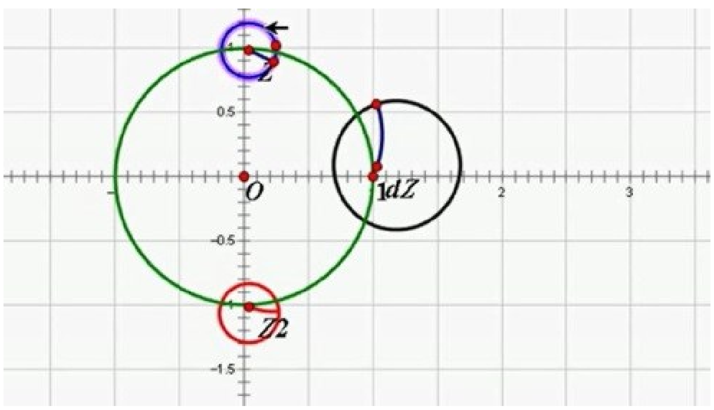


Figure 82. Circle (blue) transformed under $f(z) = \frac{1}{z}$ (red) and $f'(z)$ (black)

Karen even stated at one point during this interview portion that she believed that $f(z) = \frac{1}{z}$ actually was linear, since "the circle isn't changing in any way shape or form, just moving (observation)." She even asserted that she was "pretty sure it's linear. Should still be. It's just different than expected." She continued by correctly applying geometric reasoning to the meaning of a derivative value of $f'\left(\frac{1+i}{\sqrt{2}}\right) = i$: the function does not dilate the image of a circle around the point $\frac{1+i}{\sqrt{2}}$ at all with respect to its pre-image, but does rotate it $90°$ (observation; see Figure 83). However, Karen still seemed to feel that her geometric reasoning was flawed for $f(z) = \frac{1}{z}$ at $z = 4 + 2i$, possibly because *GSP* showed her something slightly different than expected (see Figure 84).

> *Karen:* $4 + 2i$ *I would expect the output to be dilated by the magnitude of that, so it'd be quite big. And it would be rotated by the angle which I think is* $\frac{1}{2}$.
>
> *Interviewer: What does your output look like?*
>
> *Karen: Not that* (see Figure 84).

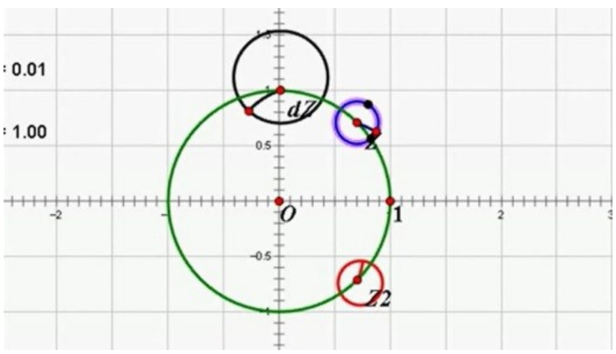


Figure 83. A circle around $\frac{1+i}{\sqrt{2}}$ (blue) and its image (red) under $\frac{1}{z}$

It is not clear what Karen expected to see, as the output did indeed appear to be dilated and rotated approximately as she described (see Figure 84). Perhaps she felt that the output was too small, or perhaps she confused the slope of the line through the origin and the value of the derivative at $2 + i$ with the argument of this value. Karen later told me she was unsure whether the output matched her prediction or not. On one hand she felt that that her circle was not quite rotated by the correct angle, but on the other hand she felt that the dilation was correct, and was not entirely sure whether her angle was correct or not. Karen seemed to develop a sense that the derivative described some local property, but appeared unsure of whether the derivative was in fact a linearization.

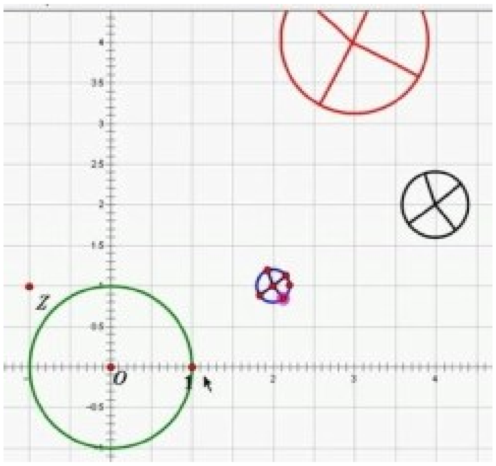


Figure 84. Mapping an input (blue) via $f(z) = 2z + 1$ (red) and $f'(z)$ (black)

While it was not entirely clear whether Karen continued to reason through the derivative as a local linearization by the end of the four-day interview sequence, Joshua and David seemed to develop this knowledge by the beginning of Task 4.

> *D:* ***It would be the*** (Joshua pinches right index finger and thumb together) ***derivative at a single point*** (David pinches right index finger and thumb together) *tells us how* ***the antiderivative functions*** (rotates right hand from facing palm down to facing palm up, then returns his hand to palm down and raises his hand up then lowers it again)*, the function we took the derivative of,* ***tells us how original function*** (pinches right index finger and thumb together, then points left with index finger) *transforms* ***by the angle*** (flips right hand right to face palm up, then left to face palm down) ***and magnitude of the derivative****.* (right hand forming a "C") ***So the angle and magnitude of the derivative*** (rotates right and back left) *at a point* ***tells us how epsilon balls*** (touches curled fingers together as though holding a ball) ***are little circles around*** (hands raise slightly, then left hand drops) ***points will transform*** (right hand fingers point left) ***how they'll rotate*** (right hand fingers point up and rotates left and right about wrist)*,* ***and how they'll expand*** (palms face each other, hands move horizontally away from each other) *or contract* (hands move horizontally back toward each other).

This explanation appears to lack only the specifics regarding *how* the derivative describes the rotation and dilation of their $\epsilon -$balls. In particular, David and Joshua did not explain that the magnitude of the derivative evaluated at a point $z$ is the factor by which the transformation dilates the image of an $\epsilon -$ball around $z$ with respect to their pre-image and the transformation rotates the $\epsilon -$ball at an angle equal to the argument of the derivative evaluated at $z$.

As Joshua and David explained their reasoning about the derivative at a point for $f(z) = z^2$ at a previous point in the interview, I asked them if they thought the geometric rule they developed would hold for $f(z) = e^z$ at $1 + \frac{\pi i}{2}$. Their rule stated that if for a particular point $z$, $f'(z) = w$ for some complex number $w$, then the image of an $\epsilon -$ball around $z$ is $|w|$ times larger than its pre-image, and rotated $Arg(w)$ degrees with respect to its pre-image. After they calculated the derivative to be $ie$ at this point, Joshua correctly stated that this derivative value suggested that "maybe it's a rotation by 90°. That's what it would tell us if the rule stays." Still, Joshua expressed skepticism about the rule holding for this new function, while David felt that their rule would still apply in the context of this new function, at least for small circles.

Joshua seemed to feel that their rule regarding rotation and dilation was rooted in the fact that $f(z) = z^2$ multiplies the input by itself, and multiplication is strongly related to the rotation and dilation of the input point. Joshua claimed that "for $e^z$ I can't really think of it in that way, and I know that multiplication by a complex number is the same as a rotation and dilation." David appeared to have no corresponding reason for why he felt their rule would still hold, simply stating that he believed it would "just because maybe that should make sense." He additionally cited the fact that the output of an exponential function changes drastically even for small changes in the input, and finished his explanation by expressing increasing doubt: "The exponential's where I guess, or maybe not. No. Maybe. I don't know what I'm saying. I retract what I said."

David and Joshua went on to investigate whether the rule for the derivative they had developed in the task for $z^2$ still held for $e^z$, beginning by looking at a circle of radius $.5$ centered around the point $1 + \frac{i\pi}{2}$ (experimentation; see Figure 85). When I asked them if there were any points for which their rule did not hold for $f(z) = e^z$, they expressed brief concern over the twists formed in the output of large circles (observation; see Figure 86). However, this concern appeared to dissipate quickly once David reiterated that they were looking primarily at small circles.

> David*: So, yeah, I don't think that'll come into* ***play since we're taking very small circles around all of the points*** (forms the "OK" gesture with right hand).
> Joshua*: Locally. So if we're taking small enough circles* ***we'll never get the repeated*** (waves mouse over twist in output). *So I guess the answer to our question is nowhere. Or, the rule will hold everywhere, I guess.*

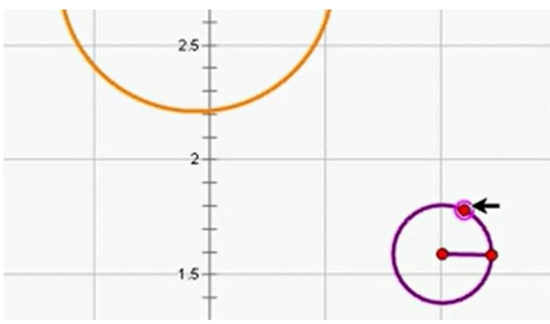


Figure 85. Input circle (purple) centered around $1 + \frac{i\pi}{2}$ in $f(z) = e^z$

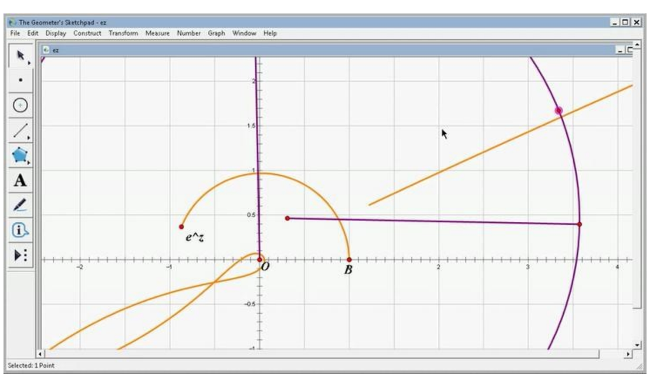


Figure 86. A twist in the image (yellow) of a large circle (purple) in $f(z) = e^z$

David followed this event with a mostly correct description of the derivative's geometric meaning, but reverted to applying the derivative rule to the *image point* rather than the epsilon neighborhood around it—an error that Karen committed as well.

> David*: The image point is found, using the derivative we know that if we take a point, it's rotated* ***by whatever angle that the derivative gives*** (moves right index finger to the left). *That point is radially magnified by whatever the magnitude of the derivative is. Okay, that makes sense.*

Near the end of this task, I asked Joshua and David what the derivative meant geometrically in the context of the derivative of the function $f(z) = \frac{1}{z}$. They explained that the magnitude of the value of the derivative evaluated at a point $z$ still describes the dilation of the image of their $\epsilon -$ balls around $z$ and the argument of the value of the derivative evaluated at $z$ still describes their rotation. When they attempted to validate this heuristic with *Geometer's Sketchpad* (*GSP*) (experimentation and observation), there was some brief concern that the $\epsilon -$ball at which they were looking seemed to be rotated by the right angle but in the wrong direction (observation). However, David was able to discover through some algebraic reasoning that they had committed a minor sign error when originally calculating the derivative. After this correction, David and Joshua both seemed to feel that the rule they had developed for both $f(z) = z^2$ and $f(z) = e^z$ still held for $f(z) = \frac{1}{z}$.

### Task 5: Exploring the Meaning of Non-existent Derivatives Via $\boldsymbol{f(z) = \frac{1}{z}}$ and $\boldsymbol{f(z) = |z|}$

In what little time remained for each group after the conclusion of task 4, I asked participants to try to explain non-differentiability geometrically. In particular, I asked each participant group to explain geometrically what it means when there is a point $z$ for which $f'(z)$ does not exist for a given function $f$. To help them answer this question, I asked them to consider the point $z = 0$ in the function $f(z) = \frac{1}{z}$, and later suggested that they construct the function $f(z) = |z|$ in *GSP*. As this task came at the end of the four-day sequence, neither participant group spent as much time on this task. Karen only spent about 20 minutes considering the meaning of the existence of a non-differentiable point, while Joshua and David spent about 45 minutes. While I expected both participant groups to believe that the derivative did not exist at $z = 0$ for $f(z) = \frac{1}{z}$, both groups instead claimed that the derivative at $z = 0$ for this function evaluated to $\infty$. Furthermore, they seemed able to reason geometrically about this value as describing a dilation and rotation of an $\epsilon -$neighborhood around 0 as they had for points with finite derivative values.

I began this task with Karen by asking her what she thought it meant if the derivative does not exist at some point. To help Karen answer this question, I asked her to construct the function $f(z) = |z|$. She correctly stated that this function is nowhere differentiable, and I further noted that the derivative does not exist at $z = 0$ for the function $f(z) = \frac{1}{z}$. She objected to this statement, however claiming instead that the derivative had a value of $\infty$. She continued with geometric reasoning by saying that this derivative value meant that an input circle around $z = 0$ should be dilated by a factor of $\infty$ and rotated by a negative angle. This reasoning seemed to correspond with her previous geometric reasoning about the derivative as describing how a function $f$ transforms small circles around a point $z$. That is, the image of these circles are dilated by the magnitude of the value of the derivative evaluated at $z$ and rotated by the argument of this same value.

Just as in the other tasks, David and Joshua spent more time than Karen trying to arrive at a satisfactory solution. Additionally, while I had explicitly asked Karen to consider the meaning of a non-existent derivative, David noted the derivative at $z = 0$ for $f(z) = \frac{1}{z}$ does not exist before I directed his attention to that fact. In the previous task, after they determined that their geometric reasoning for the derivative of $f(z) = z^2$ also worked for the derivative of the function $f(z) = e^z$, Joshua suggested testing more points to further verify this reasoning (experimentation). Joshua suggested looking at their "trouble point" $z = 0$ in $f(z) = \frac{1}{z}$, and David added that they "don't have a derivative there." At this point, Joshua agreed, "oh yeah, that's true." However, less than a minute later when I asked Joshua and David what it meant when the derivative does not exist at a point, Joshua offered exactly the same geometric reasoning for $z = 0$ in $f(z) = \frac{1}{z}$ as Karen. That is, he stated that the derivative evaluated at this point yielded ∞, saying "the derivative at zero is infinity right, so we'd expect a magnification by infinity, and that, I guess the angle isn't even relevant at that point. I mean it's just the negative angle right?"

For the function $f(z) = |z|$, instead of verifying their developed heuristic for their geometric reasoning about the derivative as they had in all the previously presented functions, Joshua and David utilized this heuristic to determine what the derivative of this new function at each point of the plane should be. In the following exchange, it appears that they committed an error that Karen had in previous tasks. Namely, they applied the rotation and dilation to a specific point rather than to a neighborhood around the point. Throughout this exchange, neither participant performed any actions in *GSP*. (Their display at the time of the following conversation is shown in Figure 87.)

*David: So for everything on the real axis, not including zero, the derivative would be 1. Right, so I guess by the rule we were using there's*

*Joshua: No magnification*

*David: No magnification or rotation.*

*Joshua: So we know there's no rotation so we don't really have to worry about that.*

*David: So I mean, the derivative where it exists is one. So I mean,* ***all points would get mapped to themselves*** (right index finger touches thumb) ***like we saw*** (right index finger points left and bounces up and down)before*.*

*Interviewer: Okay, so now I'll ask you with that in mind do you think your rule still holds?*

*Joshua: From the looks of things, yeah. Because I mean, if we march out* ***one*** (Points up and outward as seen in Figure 88) *on the positive $x$,* ***we obviously*** (flattens palm)*, our image is obviously* 1. ***Negative 1 goes to 1*** (points right with both index fingers). *The magnitudes are the same I mean.*

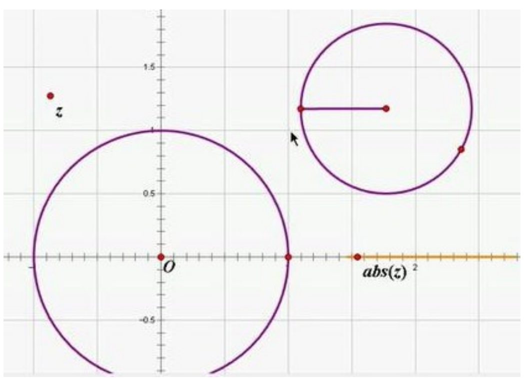


Figure 87. The function $f(z) = |z|$ in *Geometer's Sketchpad*

Figure 88. Joshua gestures "march[ing] out on the positive $x$"

Joshua also repeated the error of believing the imaginary part of the value of the derivative evaluated at a point $z$, rather than the argument, described the rotation of the image of an $\epsilon$ −ball around $z$ with respect to its pre-image. David corrected him, stating that the argument of the derivative at a point describes the rotation. David tried to calculate the value of the derivative along various lines through the origin, suggesting that given one of these lines, the derivative should be the same for every point on that line. David geometrically reasoned that this should be the case since along a line through the origin, each point on the line is "pushed down" by the same amount to the real axis, meaning that each of these points were rotated by the same amount. Finally, David concluded by saying that he believed the derivative existed everywhere. As an example, he offered that the derivative of $f(z) = |z|$ at $z = 0$ should be $0$, since $0$ is mapped to itself and is thus not changed at all. Thus, by the end of this task it appeared that David had forgotten his previous statements that if $f'(z) = 1$, then a neighborhood around $z$ in the domain mapped to an image that was not rotated or dilated with respect to the neighborhood.

## Summary (Themes)

While Joshua and David seemed to investigate more aspects of the behavior of the function $f(z) = z^2$ than Karen in the first task, they still seemed to develop many of the same advancements in reasoning, albeit in a slightly different order. Both participant groups successfully resolved a time when *Geometer's Sketchpad* (*GSP*) appeared to provide an inscription that contradicted previously stated reasoning. Both groups noticed that moving an input point around the unit circle once would result in the corresponding output point moving around the unit circle twice. Furthermore, they noted that a small input circle was transformed by $z^2$ into an output shape that was nearly a circle except when the input surrounded the point $z = 0$. Finally, both groups of participants appeared to consider whether it was possible to reach or approach an output value of zero (i.e., is there a value of $z$ for which $f(z) = 0$ or is at least "close" to zero. Thus, it is possible that the usage of inscriptions provided by *GSP* in task 1 encourages these particular facets of reasoning related to the derivative. Namely:

1. $f(z) = z^2$ transforms the input by doubling its argument and squaring its magnitude
2. One revolution around the unit circle in the domain causes two revolutions around the unit circle in the co-domain. Therefore circles that surround the origin map to "double circles"
3. Circles map to shapes which are nearly circles if they do not surround the origin, meaning that the point $z = 0$ is the only non-conformal point of the complex-valued function $f(z) = z^2$
4. Small circles are distorted less than large circles under the mapping $f(z) = z^2$.
5. Circles further away from $z = 0$ are distorted less than circles closer to $z = 0$ under the mapping $f(z) = z^2$

However, I believe all participants, particularly David and Karen knew how to reason geometrically about multiplication of complex numbers before starting the task, and was thus a pre-existing condition rather than a real result of the task itself. This belief is due to my knowledge that both David and Karen had previously participated in research involving undergraduate students' abilities to connect algebraic reasoning to geometric reasoning via diagrammatic reasoning. Many of these tasks required them to multiply a vector by a complex number; they eventually determined how to carry out this operation on their diagram directly, rather than performing any algebraic calculations first (Soto-Johnson & Troup, 2014). Joshua seemed to have well-developed algebraic reasoning abilities. While he may not have synthesized the two modes of reasoning as completely as David, he still seemed able to reason geometrically about a complex-valued function given the function equation.

In the second task, the trend of Joshua and David exploring more questions than Karen continued, though there were several similarities in their experimentations as well. Both groups discovered a twist could occur when the radius of the input circle was $\pi$ or greater, and both groups felt at one point that there might *always* be twists in the output of a circle under the mapping. Both groups eventually discovered that twists *only* occur when the input circle has a radius of $\pi$ or greater. In summary, it is possible that task 2 encourages these particular advancements in geometric reasoning related to the derivative. Namely:

1. $f(z) = e^z$ dilates the input based on its real part and rotates it based on it imaginary part.
2. A circle's image has distinct points $z_1$ and $z_2$ such that $f(z_1) = f(z_2)$ exactly when the radius of the input circle is $\pi$ or greater.
3. Small circles are distorted less than large circles in $f(z) = e^z$ at every point, including $z = 0$

4. Small circles always map to shapes which are nearly circles, regardless of which point they surround (i.e., "weird behavior" like twists or double circles never occur with small circles.
5. $f(z) = e^z$ never actually reaches zero.

For the third task, I removed my participants' access to *GSP* (and thus the associated inscriptions that a dynamic geometric environment (DGE) provides) and I asked them to first describe how they thought of the derivative of a complex-valued function, and then to try to reason through its meaning geometrically. While Karen only spent about 30 minutes without *GSP* developing this geometric reasoning, David and Joshua spent an entire 1 hour and 15 minutes before they wished to return to *GSP*. During the time without *GSP*, both groups of participants referred to the idea that the derivative is the slope of a tangent line. For Karen, this idea was highly recurrent and difficult to discourage, whereas Joshua and David dismissed the idea themselves almost immediately and without prompting. Karen additionally suggested reasoning algebraically using the Cauchy-Riemann equations as a potential starting point. Both groups recalled their experiences with $f(z) = z^2$ and made some comment about how circles were transformed, but seemed to reason through the nature of these transformations by utilizing the function behavior itself rather than the derivative value at any of the points.

Geometric reasoning about function behavior and about the derivative of a complex-valued function may even have interfered with one another in the development of the participants' geometric reasoning about the derivative of a complex-valued function. That is, both groups tended to relate the behavior of a circle around a given point to the nature of the given function, and not to the value of the derivative of the function at that point. Therefore, motivating any need for geometric reasoning about the derivative may have felt difficult due to both groups' proficiency in determining geometric behavior by considering the function formula only. Finally, both Karen and Joshua each verbalized an incorrect algebraic definition of slope at least once during this portion of the interview. Karen claimed that the slope was the difference between output and input values, while Joshua believed that the slope was the change in imaginary components divided by the change in real components.

I re-introduced *GSP* for different reasons for the two groups: Karen simply appeared to run out of ideas and seemed to be growing frustrated, while Joshua and David had a large number of conjectures they had generated between them and wished to test. After I re-introduced *GSP*, both groups appeared to further develop the idea that the derivative somehow controlled the dilation of an input circle, and again that this input circle needed to be small for the derivative to be accurate.

Thus, as I suggested previously in this chapter, removing *Geometer's Sketchpad* (*GSP*) did indeed seem to help me in determining how the participants reasoned about the derivative of a complex-valued function and in creating a laboratory setting as Olive (2000) described. Before this point in the tasks, I asked the participants fairly specific questions (seen in the task worksheets, found in Appendix E) about specific functions. By asking an open-ended question for task 3, I gave my participants the opportunity to reason about the derivative of a complex-valued function as they wished, and not as I directed. Furthermore, as participants utilized *GSP* to answer previous tasks, restricting their access may have helped encourage this less prescribed reasoning by creating a new learning context further removed from the more pre-determined structure of the first two tasks. Some ideas were common to both groups but differed in when they were first discovered in relation to the presence or absence of *GSP* (see Appendix F for a table of these concepts).

That is, David and Joshua seemed to develop certain concepts without *GSP* that Karen only discovered after its re-introduction. In particular, Joshua and David started talking about $\epsilon -$neighborhoods, and thus the idea that the derivative is essentially a limit or approximation without *GSP*, while Karen did not seem to understand this idea until actually zooming in on a point with a circle. Joshua and David dismissed their reasoning about the derivative as a tangent line near the beginning of Task 3, while Karen returned to this geometric reasoning many times over. She appeared to become dissatisfied with this reasoning only when attempting to explain what a complex-valued derivative evaluation would mean in this context or demonstrating a "complex-valued slope" with a geometric inscription (either with a blackboard or with *GSP*.) Despite these differing circumstances, both groups then committed the same error of attempting to apply their meaning of the derivative to a particular point rather than a small circle/$\epsilon -$neighborhood surrounding that point.

Furthermore, I asked Joshua and David to consider functions with a constant derivative without the use of *GSP*, while Karen did not consider linear functions until I prompted her to construct one with *GSP*. However, neither group seemed to connect the derivative to the rotation or dilation of an input circle before considering linear functions in task 4, instead opting to reason through the given transformation simply by considering function behavior. Thus it appeared that this consideration of linear functions was a necessary prerequisite for the attempted generalization of the participants' geometric reasoning about the derivative of a complex-valued function. For both groups, looking at a linear transformation with a real derivative seemed to highlight the way in which the derivative described the dilation of the input circle, while a linear transformation with an imaginary derivative appeared to help them develop a sense of how the derivative affects the rotation. However, Karen ended this portion of the interview apparently believing that the real part of the derivative describes the dilation and the imaginary part describes the rotation. In contrast, Joshua and David appeared to be more correct in the sense that they connected the magnitude of the derivative to the dilation and the angle of the derivative to the rotation in the context of a linear function, though they too initially committed Karen's error. Neither group was sure of whether this rule generalized to the function $f(z) = z^2$, as this function does not have a constant derivative, thus the output of a circle is no longer always a circle. Because of this distortion, the idea of rotating this output seemed strange to them. That is, the image of a circle is no longer a circle, and is therefore not just a rotation and dilation of the pre-image. Experimentation with *GSP* appeared to allow both groups to develop their geometric reasoning to the point where they could view this image as an approximation of a rotation and dilation of the pre-image.

Following this task, Karen's experience began diverging more from Joshua and David's group after investigating the nature of linear functions. This difference in experience appeared to occur in part because of her continuing tendencies to reason about the derivative as the slope of a tangent line, and in part because she was interviewed alone rather than as part of a pair. Her strong tendency to reason about derivative values as the slope of a tangent line could be an instantiation of a theme Danenhower (2000) titled “Thinking Real, Doing Complex” (p. 184). His participants also seemed to exhibit some tendencies to borrow strategies or facts from one-dimensional real-valued calculus. For example, some of his participants inspected graphs to determine differentiability and assumed all polynomials are entire. Like Danenhower’s participants, Karen appeared to experience difficulty arising from her tendency to reason about the derivative of a complex-valued function in a way similar to the way she might reason about the derivative of a real-valued function. She eventually extended this geometric reasoning to the complex-valued case by developing geometric reasoning about complex-valued linear functions.

While Karen began reasoning about the derivative of a complex-valued function as a tangent line, David and Joshua made only a passing reference to this idea before considering the complex-valued case separately from the real-valued case. However, there were still similarities in the two participant groups’ patterns of development. In particular, while both groups had previously developed a geometric idea of the meaning of the derivative of a linear complex-valued function, both seemed uncertain of whether this meaning would generalize (task 4) to the presented non-linear functions $f(z) = e^z$ or $f(z) = \frac{1}{z}$. While Joshua and David came to the conclusion that their rule did in fact hold for the non-linear functions due to some tests run in *Geometer’s Sketchpad* (*GSP*), Karen’s experience with *GSP* originally seemed to fuel her uncertainty by showing her that circles did not map to perfect circles. She vacillated in her geometric reasoning about the derivative of linear complex-valued functions, initially believing that this reasoning did not generalize to the given non-linear functions, but appeared to end the interview believing that her reasoning was valid even in the non-linear case by viewing the output as approximately correct. Both groups seemed able to correctly predict the proper amount of dilation of the input circle based on a derivative value alone by the end of their collective investigations of $f(z) = \frac{1}{z}$, $f(z) = e^z$, and $f(z) = \frac{1}{z}$.

During these investigations, Karen also made some discoveries that Joshua and David had already observed in prior tasks. For example, I asked her to consider the meaning of a derivative value of $3 + 2i$ for an arbitrary function at some point. In response, Karen began relating the magnitude and argument of the derivative to the dilation and rotation, respectively, of the $\epsilon -$neighborhood, rather than utilizing the real and imaginary parts as she had previously. Furthermore, while Joshua and David seemed able to predict the proper rotation of a particular example based on the derivative value by the end of their third day (where *GSP* was initially restricted and then re-introduced), Karen still seemed uncertain of how to predict the angle by which her input circle should rotate.

I developed the final task of investigating the transformations $f(z) = \frac{1}{z}$ and $f(z) = |z|$ to help Karen, Joshua and David develop geometric reasoning about a non-existent derivative. However, all participants adopted the convention that $\frac{1}{0} = \infty$, and thus stated that the derivative of $f(z) = \frac{1}{z}$ at 0 was $\infty$. Furthermore, they may even have reasoned geometrically about the derivative as a local linearization within this context, treating $\infty$ simply as another number. In particular, Karen and Joshua both claimed that this derivative meant the function should dilate a circle centered at 0 by $\infty$ and rotate it by some negative angle. Joshua added that at $\infty$ the angle may not even be relevant. For the function $f(z) = |z|$, Joshua and David appeared to try to utilize their heuristic to evaluate the derivative at each particular point, and somehow ended up concluding the derivative should exist everywhere.

Thus, it may be that Joshua and David's geometric reasoning about the derivative of a complex-valued function did not yet include the viewpoint that the output of a small circle (an $\epsilon$ −neighborhood) is a roughly circular shape. Neither of my participant groups spent much time on this final task in relation to the time they spent on the previous four tasks. However, given more time, this task may have helped my participants realize that the output of a small circle should again be nearly a circle, especially if they eventually discovered that the derivative did not in fact exist anywhere in $f(z) = |z|$. The other function, $f(z) = \frac{1}{z}$, may not be as useful in investigating the meaning of a non-existent derivative in a complex-valued function, as my participants tended to suggest that for this function, $f'(0) = \infty$, rather than not existing.

Overall, participants appeared to develop geometric reasoning about the derivative of a complex-valued function that included the three points toward which I guided them. Namely,

1. The associated image of the $\epsilon$ −neighborhood is approximately a circle.
2. The function rotates the image of this neighborhood by the argument of the derivative of the function at $z$ with respect to its pre-image.
3. The function dilates the image by the magnitude of the derivative of the function at $z$.

However, Joshua and David appeared to forget about point 1 once they began investigating the function $f(z) = |z|$ while trying to determine the meaning of a non-existent derivative. Furthermore, they appeared to modify their reasoning regarding points 2 and 3 to match what they observed in *GSP* regarding this function's behavior, rather than realizing that their previously developed geometric reasoning did not match the inscriptions provided by *GSP* in this task.

Finally, at some point in the interview sequence, all participants considered the following points in their reasoning through the presented tasks:

4. The behavior of a given function (e.g., how points, lines, or circles are transformed)
5. Local vs. global properties
6. The meaning of "linearization" or "linear" in the complex plane (particularly for Karen)

## Preliminary Implications

### Teaching

In giving students opportunities to manipulate software like *GSP*, I recommend that instructors consider assigning students in pairs. In my research, my pair of students appeared to assume fairly consistent roles, according to who directly manipulated the computer and who watched. It appeared that the observer usually assumed the responsibilities of suggesting questions to investigate and producing explanations when something unexpected occurred within *GSP*. The observer would additionally monitor the actions the other participant took within *GSP*, occasionally checking for accuracy of input (i.e., that the input point or vector was where the participants had said it should be). The observing participant would frequently produce iconic gestures while providing explanations or deictic gestures (particularly pointing at the computer screen) while correcting an error or guiding construction actions taken by the other participant with

*GSP*. The participant manipulating *GSP* built necessary constructs in *GSP* such as points, circles, and lines, and sometimes offered strategic suggestions for solving a particular problem. For example, Joshua placed a “twist” on the real axis in an attempt to simplify the associated algebra, and thus the associated reasoning. The participant in direct contact with the computer tended not to produce gestures, instead producing virtual motion with the mouse cursor, both to signify other objects such as circles or lines and to point at these same objects. Neither participant remained in one role exclusively; both participants in my paired group experienced both roles, though Joshua more often directly interacted with *GSP* while David tended to watch. This difference may explain why Joshua and David seemed to tackle questions of their own invention, while Karen tended to answer only questions on the worksheets or that I asked in particular.

Part of my solo student’s frustration may have been due to the fact that she had to fill both these roles simultaneously at all times throughout the interview. She did not have anyone else to watch; she always manipulated *Geometer’s Sketchpad* (*GSP*) directly, and may have found the dual roles of experimentation with *GSP*’s inscriptions and strategizing about future explorations. This frustration may have been exacerbated during the period of the interview when *GSP* was re-introduced after it was removed. In previous tasks, I asked specific questions for Karen to answer; thus the role of the observer may not have been quite as large a factor. However, in this portion of the interview, I offered little direction, thus implicitly requiring Karen to decide for herself how to proceed. The lack of a second student to help her may explain some of her feelings of frustration. Similarly, some of this difficulty may have been alleviated once I returned to asking her specific questions, as she no longer had to determine her own direction to such a great degree.

Therefore, I would recommend that teachers assign students to pairs, as this seemed to facilitate my students' willingness to explore the mathematics on their own terms, rather than simply perform the tasks I set out before them. A solo student may have difficulty navigating their own course through an unstructured task; a pair of students may experience more success.

Teachers should be aware of common problems that could arise, even while using *GSP* or related dynamic geometric environments (DGEs). Particularly, the idea of "slope" appears to be a salient one for students and seems easy to confuse within the realm of complex numbers. Karen was aware that she did not know what a slope with imaginary value means geometrically, though she returned to this idea with great frequency. Furthermore, both participants at some point stated that the slope was the change in imaginary values divided by the change in real values. It is possible that this error arose from conflated symbolic reasoning. In particular, they may have conflated the meanings of the $x$ and $y$ inscriptions in the formula for slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ and the equivalence between the symbolic and Cartesian form of a complex number $z = x + iy$. They may not have realized that slope is the difference between outputs divided by the difference between inputs due to these symbolic similarities. It does not seem likely that *GSP* alone would help reduce the occurrence of this error: both input and output were superimposed on the same plane, and the fact that $(z, f(z))$ is a 4-dimensional point renders any literal ideas of slope invisible.

Investigation of linear complex-valued functions—those with constant derivatives—seemed essential for my participants in developing geometric reasoning about the derivative of a complex-valued function. In particular, based on their experience with *GSP* and my implicit suggestion that they investigate the behavior of circles, they seemed able to realize that the derivative has something to do with how the function rotates and dilates circles. This investigation additionally seemed to help Karen significantly in generalizing her geometric reasoning about the derivative such that it became more than just the slope of a tangent line.

It is possible that Karen's difficulty in this respect is an instantiation of Danenhower's (2006) "Thinking Real, Doing Complex" (p. 184) theme. Perhaps Karen also had difficulty advancing her geometric and algebraic reasoning in tandem. Past research shows students generally have difficulty integrating differing forms (Danenhower, 2006) and representations (Panaoura et al., 2006) of complex numbers, though a learner may be able to shift between two forms, representations, or styles of reasoning with increasing ease if given the opportunity to practice making these connections (Soto-Johnson & Troup, 2014; Zazkis, Dubinsky, and Dautermann, 1996). In my project, both sets of participants appeared to make substantial progress toward reasoning geometrically about the derivative of a complex-valued function. Therefore, teachers may wish to consider the potential benefits of including direct student experiences with DGEs. Rather manipulating these programs for their students, or creating inscriptions while explaining meaning, instructors may find that students could in theory derive substantial value from creating these inscriptions and experimenting with these programs for themselves, allowing them to opportunity to discover meaning and thus perhaps imbuing their reasoning with a more personal flavor. This may help students avoid errors that occur due to a lack of integration of algebraic and geometric reasoning approaches.

One additional error my pair of students made was to overextend their geometric reasoning about the derivative of a complex-valued function in the context of the function $f(z) = |z|$. At the beginning of this task, Joshua and David correctly stated that this function has no derivative anywhere. Despite this statement, they nonetheless considered various rotation and dilation factors between circular inputs and line segment outputs to reason through what they thought the derivative value should be at various points. Thus, instructors may need to emphasize that for points that do have derivatives, small circles are mapped to shapes which are nearly small circles, and that line segments should not be considered reasonable approximations to circles in this context.

Finally, students may need directed instruction to develop their geometric reasoning to the point where they realize the derivative is an approximation. My pair of students, in particular, admitted reluctance in speaking of the amount of rotation or dilation for a non-circular shape. They eventually appeared to overcome this by assuming these shapes were approximately circular, but this advancement appeared to require considerable discussion and experimentation with *GSP*. Perhaps this particular issue could be addressed in a teaching context by asking students first to predict how an $\epsilon -$neighborhood around some particular point $z$ will be transformed under the transformation $f(z) = |z|$. In particular, the teacher could note that this output will never be a circle, whereas in previously investigated functions, the output was at least nearly a circle, and not a line segment.

More generally, emphasizing the output of a sufficiently small $\epsilon$ −neighborhood around a point as "approximately" a circle may be essential to helping students develop geometric reasoning about the derivative of a complex-valued function. This idea seems even more important to point out in teaching contexts where *Geometer's Sketchpad* (*GSP*) is used, since in some situations, students might falsely attribute the distortion of the output circle to a minor misalignment in the placement of their unit circle, rather than to the behavior of the function itself.

**Research**

As my participants assumed the roles of observer and *GSP* manipulator—roles which are purposefully adopted by professional computer programmers while writing code (Cockburn & Williams, 2000; Williams, Kessler, & Cunningham, 2000)—future researchers may wish to investigate the usage of these roles more closely. Within computer programming, this practice appears to increase accuracy, efficiency, and quality of the final product. It is possible that similar benefits may apply to undergraduate students' investigations of complex numbers. One significant difference between my participants' behavior and the professional practice is that computer programmer pairs switch these roles frequently, whereas my participants did not. Rather, Joshua more often manipulated *GSP*, while David observed. Future researchers may wish to consider explicitly asking the participants to switch roles at pre-determined points in time.

Alternatively, they may wish to investigate how these roles arise without explicit instruction, and how they are negotiated throughout the development of the team's reasoning about the derivative of a complex-valued function. Joshua and David adopted these roles on their own terms despite lack of instruction, and experienced benefits similar to those of computer programmers who design code in teams of two: one person manipulating the computer, and the other observing, strategizing, and troubleshooting.

## Limitations and Improvements for Future Study

As is appropriate for a design experiment, I identified several limitations and weaknesses of the pilot study upon which to improve for my dissertation study. This section is a discussion of those weaknesses and the improvements I intend to carry out for this second iteration. The section parallels the methodology section in its organization: I begin by discussing improvements to my participant sampling, ways to enrich the participants' descriptions, and the formation of participant teams better suited to my study than those found in the pilot. I continue by noting improvements made to the interview tasks, and similarly to the actual structure of the interview. I outline improved data collection and analysis methods, and conclude by suggesting potential ways I could strengthen my results and implications.

### Participants and Setting

In the pilot study, two of the participants took part in a prior research study, which could have primed them to view rotation and dilation as strongly salient aspects of a transformation, and thus may have noticed these properties of the derivative more quickly than a typical student might. To avoid the former problem, I will sample students that to my knowledge have not participated in strongly related prior research. This will increase the possibility that their exposure to geometric properties of the derivative of a complex number is minimal.

Furthermore, the fact that I interviewed one student alone and two students in a pair is a severe conflating factor for the majority of my cross-case findings. For this reason and because the nature of the social interaction between my participants is a relevant data source for a design study (Cobb et al., 2003), in my dissertation study I will not have any participant groups consisting of just one student. Rather, all participants will be placed in teams of two. This grouping will help strengthen my cross-case analysis by eliminating this potential conflating factor. In the pilot study the majority of these differences may merely be because one student worked alone, while the other two did not. However, if these differences re-occur in my dissertation study with no participants working alone, these results would be strengthened.

Furthermore, placing students in groups may increase the amount of data I collect, particularly data regarding technological action and gesture. In my pilot study, Karen, the solo student, tended either to gesture with her hands, or take some action within *Geometer's Sketchpad* (*GSP*), but rarely both at the same time. This observation is in keeping with the suggestion implicit in my theoretical framework that technological actions in *GSP* could conflate with gestures. Placing participants in pairs seems to help address this issue, as typically Joshua performed technological actions within *GSP* on behalf of the group, while David more frequently utilized gestures in his explanations. Even within this setting, neither participant operated in both contexts simultaneously, but at least with two participants it seems to be more likely that I will collect both data involving gestures and data involving *GSP* rather than just one or the other at any given time. Therefore my dissertation study will involve only participant teams of two, and none of just one. I hope to obtain three pairs of students for my dissertation study, thus I will need to sample six students.

I will again sample students based on instructor recommendation, additionally using data collected from my observations of their class to guide this decision. As the class is based on lecture, students were silent and attentive most of the time. However, when the instructor asked questions of the students, some simply answered verbally, while others supplemented their response with gesture. Due to my reliance on gesture as a data source, I will favor those students who demonstrated this stronger tendency to gesture while reasoning through or responding to a question.  However, I will be sampling six or eight students, and there are only 11 students in the complex class total, so it is quite possible I may have to sample all the students that volunteered.

Additionally, I observed all potential participants in an undergraduate complex analysis course during several class days while the derivative of a complex-valued function was discussed. This allows me a better sense of which aspects of my interview tasks are novel for the participants, and which are not novel. To further help with this aspect of my research, I will ask participants when they attended their last geometry class and what they did in that class as an introductory question in the interviews.

**Task Development**

As a result of the pilot study, I now have a more definite itinerary regarding progression through the detailed tasks than I did for iteration 0 of the design experiment. Whereas I developed later tasks of my pilot study during the interview sequence in response to the rate at which my participants completed the tasks, I now have access to these refined and tested tasks for use in my future study. I will more deliberately plan some of the tasks that I had previously spontaneously generated; particularly those involving the investigation of linear functions. These tasks suggested themselves organically during the interview of both sets of participants. The solo student insisted on viewing the derivative as the slope of a line, but became uncertain of how a line behaved within the complex plane, even while working with *GSP*. I thus suggested she look at various linear functions. The pair of students arrived at linear functions more independently than the solo student; they started their *GSP*-free discussion of the geometric properties of the derivative of a complex-valued function by first considering what a constant derivative function would mean. Thus, the linear functions that the two sets investigated were not consistent across groups.

Another problem was that since the linear functions were spontaneously generated, and thus may not have been ideal for students still new to geometric reasoning about the derivative of a complex-valued function. In particular, I tended to suggest linear functions which had a derivative which was either real or purely imaginary rather than functions with derivatives that had both real and imaginary non-zero components. My choice of functions may have been primarily responsible in forming my participants' initial conclusions that the real part of the derivative determines how the associated transformation would dilate an $\epsilon -$neighborhood, and the imaginary part determines the $\epsilon -$neighborhood's rotation.

I am at this point reasonably convinced that investigation of a linear function is essential to building up geometric reasoning about the derivative of a complex-valued function, thus for my second iteration I will sacrifice the organic nature of this task in order to ensure consistency across groups and more general linear functions, such as $f(z) = \left(1 + i\sqrt{3}\right)z$. I will additionally develop a worksheet similar to those for Tasks 1 and 2 to pair with this task.

In addition to including a worksheet for linear functions, I will improve some of the wording contained in my task worksheets. For example, the question "What do you think the output will look like if the input is a circle that contains the origin?" caused some confusion for one of my participant groups, as they told me they were not sure whether I meant that the origin was supposed to be one of the points of the circle, or "surrounded" by the circle. Using their terminology, I could rephrase this question as "What do you think the output will look like if the input is a circle that *surrounds* the origin."

As the participants seemed willing to investigate $f(z) = \frac{1}{z}$ without explicit instruction, I will refrain from developing a worksheet to pair with this function to ensure that my dissertation study participants have at least one function to investigate in a less structured way. The lack of explicit direction allows them to explore the function as they wish; the manner in which they choose to do this, the propositions they suggest, and the questions they seek to answer could all help elucidate the nature of their geometric reasoning regarding the derivative of complex-valued functions. Furthermore, this function's behavior is ostensibly simpler than either $f(z) = z^2$ from Task 1 or $f(z) = e^z$ from Task 2, in that it sends circles and lines to images which are exactly circles and lines, and not simply approximations of circles or lines. However, as my pair of participants from the pilot study noted, for the derivative to accurately predict the rotation and dilation of the pre-image under $f(z) = \frac{1}{z}$, the pre-image circle must still be small. Thus, not all notions of the approximate nature of a local linearization are lost within this function. The fact that it is a truly conformal transformation with simple behavior and a non-constant derivative may even make it ideal for a free-form exploration within the context of my dissertation study.

Finally, I can continue to refine the questions I ask or the wording of my worksheets by considering the data I collect between interviews. This will help me improve future methods by analyzing current data, thus strengthening the reflective aspect of my design experiment.

**Interview Structure**

I will administer similar tasks to participant pairs in my future study. Additionally, I have a greater sense for how quickly my future participants are likely to progress through these tasks. The pair of students in my pilot study progressed through the tasks at the expected rate; thus no additional tasks will be added on to the current interview protocol. I will ask pairs of students to investigate the functions $f(z) = z^2$ on the first day and $f(z) = e^z$ on the second, and then ask them to reason about the derivative of complex-valued functions in general, first without the use of *Geometer's Sketchpad* (*GSP*). This phase of the interview is intended to be free-form, though based on pilot study results I feel it would be beneficial to direct each pair's attention to an investigation of functions with a constant derivative. Furthermore, while I attempted to re-direct Karen's attention away from reasoning about the derivative as the slope of a tangent line, I will not discourage future participants so strongly. Karen was eventually able to generalize this idea to the derivative of a complex-valued function after investigating the properties of linear complex-valued functions. After this stage, I will re-introduce participants to *GSP* based on similar criteria as laid out for the pilot study. I will ask them to verify their conjectures, to explain again how to reason about the derivative of complex-valued functions geometrically, and to demonstrate this reasoning within the context of the previously introduced functions and a newly presented function $f(z) = \frac{1}{z}$. I will end the interviews by asking how to reason about the derivative of $f(z) = \frac{1}{z}$ at $z = 0$ and $f(z) = |z|$ at any point. Investigation of this last function produced interesting findings in my pilot study—they used a rotation and dilation heuristic to geometrically reason through what the derivative needed to be to cause the transformations they observed with *GSP*—so more time could be allocated to this task on the final day of the interview. I will again conclude the interview by allowing participants to ask me about any unresolved questions that may have formed for them as a result of the interview. Finally, since I now have a more well-defined framework, I can adjust the phrasing of my interview questions to match. For instance, while for my pilot study I sometimes asked participants to describe their "geometric interpretation" of the derivative of complex-valued functions, for my dissertation study I will request that they describe the derivative of complex-valued functions geometrically.

**Data Collection**

I will again video-record the interviews to capture gesture and speech and use screen-capture software to document actions taken within *GSP*. Interviews for the participants of the pilot study took place a full semester after the conclusion of their complex analysis course. For my dissertation study, participants will be interviewed near the end of the semester during which their complex analysis course took place. Thus, I expect that participants of my dissertation study may more frequently recall geometric reasoning methods from their complex analysis course than did participants of my pilot study.

**Data Analysis**

In selecting my time segments, while I included some repeated discussions for the sake of helping corroborate my interpretation of the original discussion (see Appendix G), I did not analyze all such repetitions. More typically, I included repeated discussions if some previously unmentioned aspect of reasoning was introduced. However, for my dissertation study, I will additionally analyze these discussions even when no new concept is mentioned. This analysis will provide further supporting or disconfirming evidence of my interpretations of the first instance of each discussion. Furthermore, a record of the times a particular discussion or aspect of reasoning is repeated may help me infer the relative strength of each of these aspects as they exist in my participants' geometric reasoning about the derivative of a complex-valued function.

For my dissertation study, I will include an additional column noting the stages of diagrammatic reasoning as occurring with *GSP* or the usage of a blackboard.

## Conclusion

This concludes my discussion of methods for the first iteration of this project. I described my participants, the development of tasks to use with *Geometer's Sketchpad* (*GSP*), the structure of my interviews, and my data collection and analysis methods. After these sections, I provided my results both in detail and in a summarized form. Finally, I suggested limitations to this first iteration of the study and possible ways to improve the next iteration.

# APPENDIX E

# LAB WORKSHEETS FOR TASKS 1 AND 2

Lab 1:

Instructions:

We will begin by constructing a graph and unit circle.

6. First click the Graph drop-down menu and select "Show Grid"
7. Click the A toolbar (4th from the bottom) and double-click on the red point at the origin. Type "O" in the Label field in the pop-up window
8. Double-click on the red point at (0,1) and label this point 1.
9. Now click the "Construct circles" icon on the left toolbar (3rd icon from the top on the left) and click on the origin.
10. Now drag the mouse away from this point to increase the radius to 1. Click the circle again when the radius is at the proper size.

Note: You can always zoom in or out by selecting the point $1$ and moving it closer to or farther away from the origin. Be careful not to move the point too close to the origin (i.e., zoom too far away), or it may be difficult to reselect this point when you need to.

Next we need to construct the transformation $z \to z^2$.

9. Select the Point tool (2nd icon from the top on the left) and click once somewhere on the grid to place the point there
10. Select the A toolbar and double-click on this new point. Label it $z$.
11. Select this point (if it isn't already) and go into the "Measure" dropdown menu. Select "Abscissa(x)." This will output the $x$ −coordinate of $z$.
12. Make sure that only the point is still selected (you may have to unselect the value you just measured) and go into the "Measure" dropdown menu. Select "Ordinate(y)" to output the $y$ −coordinate of $z$.
13. Go to the "Number" dropdown menu and select "Calculate." You can click on the coordinates you just measured to input them into the calculator. Use this calculator to calculate the real part of $z^2$ with an appropriate expression. Click "Okay" when you're done. Now calculate the imaginary part of $z^2$.
14. Go to the "Graph" dropdown menu and select "Plot points." Click the real part of $z$, then the imaginary part of $z$, and click "Okay." Your new point should now be on the graph. Click "Done."
15. Label this new point $z^2$.
16. Select the point $z$ and then $z^2$ (in this order; you will need to hold down the shift key in order to select both points.) Under the "Transform Menu" click "Define Custom Transform." A box should pop up that says $z \to z^2$ transform. Click "Okay."

This graph should now show a point $z$, and the corresponding point $z^2$. Try dragging $z$ around to various points on the graph. You can select a point with the Transformation Arrow tool at the top of the left toolbar.

> Warm-up Questions: Where do you expect $z^2$ to go if you put $z$ on $1 + i$? Did it go where you expect?
>
> Where do you think you should place $z$ to send $z^2$ to $i$? Test your theory.
>
> What do you think will happen to $z^2$ if you move $z$ around the green unit circle once? Test your theory.

Now we will construct a circle and apply the transform $z \to z^2$ to the whole circle.

8. Click the "Construct circles" icon on the left toolbar (3rd from the top) and click somewhere on the graph to place the center of your circle there (Don't worry too much about location; you will be able to move it later.)
9. Now drag the mouse away from this point to increase the radius. When you are happy with the size of your circle, click the mouse again to create the circle. (Again, you will be able to change the radius later.) Your circle will automatically be selected.
10. Without unselecting the circle you just constructed, go into the Display drop-down menu, and select a "Color" for your circle. (I used red, but you can use something else if you like.)
11. Now, go into the Transform drop-down menu, then click "$z \to z^2$ transform" at the bottom of the menu. This will apply this transformation to your whole circle. The "output" shape will automatically be selected.
12. Go into the Display drop-down menu again and choose a different color for the "transformed circle." (I used blue, but again, you can pick a different color.) This is intended to help you keep track of your input and output shapes more easily.
13. Remember to click on the Transformation Arrow tool again before you start trying to drag your circles around! (Otherwise you'll just end up making more circles)
14. Move your circle around the graph and observe how the output shape changes as a result. Try to predict the behavior of the output in advance.

Some pointers:

- If you select the center point and move it, the other point you created (the one actually on the circle) will remain fixed, but the radius will change.
- If you select the point on the circle, the center point will remain fixed and the radius again will change.

- You can also select the circle itself. This will preserve the radius of the circle. (i.e., make sure to select the circle itself, and not the points, if what you want to do is drag the circle around the graph without changing anything else about it.

Questions: What do you think the output will look like if the input is a circle that contains $1 + i$? Test your theory.

What do you think the output will look like if the input is a circle that contains $2$? Test your theory,

What do you think the output will look like if the input is a circle that contains the origin? Test your theory.

Now we will investigate what happens when we change the radius of circles at these points.

Center your input circle around $1 + i$ (so that a circle at this point of *any* radius will contain $1 + i$. Every circle contains its center.) Try changing the radius of your circle (Move the point on the circle so the center stays fixed). What happens to the output?

Center your circle around 2. Try changing the radius of your circle. What happens to the output?

Center your circle around the origin. What happens to the output?

What happens to the output when your circle is inside the unit circle? What about when your circle is outside the unit circle.

Try dragging your circle along the real axis. What happens? What about when you drag your circle along the imaginary axis?

Try dragging your circle to different quadrants. What happens?

Now, try to summarize what you think is happening. What do you think a large circle around a point $x + iy$ in the complex plane will map to? What about a small circle around the same point?

Lab 2:

Instructions:

Select Show Grid under the "Graph" dropdown menu, label the origin and 1, and create a unit circle centered around the origin as you did in the previous lab.

Now we want to construct the mapping $z \to e^z$.

7. Create a point and label it $z$.
8. Measure the $x -$ and $y -$ values as you did in the previous lab. (Use Abscissa(x) and Ordinate(y) in the "Measure" dropdown menu.)
9. Before we actually start calculating $e^z$, we will need to tell GSP to interpret angle measurements as radians instead of degrees. You can do this by selecting "Preferences" in the "Edit" dropdown menu, make sure the Unit tab is selected, and change the field marked "Angle:" from degrees to radians. Click "OK" once you've done this.
10. Now we need to calculate the real and imaginary parts of $e^z$. (Recall that if $z = x + iy$ then $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$.) Select "Calculate" in the number dropdown menu to input the appropriate formulas. (You can find $e$ in the "Values" dropdown menu on the calculator and the functions sin and cos in the "Functions" dropdown menu on the calculator.)
11. Plot the point $e^z$ as you did in the previous lab by selecting "Plot points" in the graph dropdown menu and inputting the real and imaginary parts in the $x -$ and $y -$ coordinate boxes, respectively. Click "Plot" then "Done". Label your point $e^z$.
12. Select the point $z$ and then $e^z$ (in this order; you will need to hold down the shift key in order to select both points.) Under the "Transform Menu" click "Define Custom Transform." A box should pop up that says $z \to e^z$ transform. Click "Okay."

This graph should now show a point $z$ and a corresponding point $e^z$. Again, you can drag the point $z$ around the graph. The point labeled $e^z$ will move to the proper corresponding position.

More warm-ups: Where will $e^z$ be if $z = \pi i$?

The real-valued function $x \to e^x$ is always positive. Where should $z$ be to get $e^z = -1$? Why did you conjecture that?

What do you think will happen if you drag $z$ along the real axis? What about the imaginary axis? Why does this happen?

This time (before we start mapping circles) we will send the vector defined by $z$ through the transformation $z \to e^z$.

8. Click the "segment straightedge" tool on the left toolbar (4th icon from the top)
9. Click the origin
10. Click the point labeled $z$. Your vector should now be created
11. In the Display dropdown menu, select your "input" color to make your newly created vector that color.
12. Now in the Transform dropdown menu, select "$z \to e^z$ transform" at the bottom .This will send your vector through this mapping.
13. Select your "output" color to change the color of the newly created curve.
14. Re-select the transformation arrow tool. Now you can click and drag the point $z$ to various points and watch how the output changes!

    Questions: What happens if the vector is stretched along the imaginary axis?

    What happens if the vector is stretched along the real axis?

    What happens if the vector is stretched in the first or fourth quadrant?

    What happens if the vector is stretched in the second or third quadrant?

Now we will investigate how circles are mapped at various points under this transform. You will follow essentially the same steps as you did in the last lab.

8. Click the "Construct circles" icon on the left toolbar (3rd from the top) and click somewhere on the graph to place the center of your circle there (Don't worry too much about location; you will be able to move it later.)
9. Now drag the mouse away from this point to increase the radius. When you are happy with the size of your circle, click the mouse again to create the circle. (Again, you will be able to change the radius later.) Your circle will automatically be selected.
10. Without unselecting the circle you just constructed, go into the Display drop-down menu, and select a "Color" for your circle. (I used red, but you can use something else if you like.)
11. Now, go into the Transform drop-down menu, then click "$z \to e^z$ transform" at the bottom of the menu. This will apply this transformation to your whole circle. The "output" shape will automatically be selected.
12. Go into the Display drop-down menu again and choose a different color for the "transformed circle." (I used blue, but again, you can pick a different color.) This is intended to help you keep track of your input and output shapes more easily.

13. Remember to click on the Transformation Arrow tool again before you start trying to drag your circles around! (Otherwise you'll just end up making more circles)
14. Move your circle around the graph and observe how the output shape changes as a result. Try to predict the behavior of the output in advance.

Tip Reminders:

- If you select the center point and move it, the other point you created (the one actually on the circle) will remain fixed, but the radius will change.
- If you select the point on the circle, the center point will remain fixed and the radius again will change.
- You can also select the circle itself. This will preserve the radius of the circle. (i.e., make sure to select the circle itself, and not the points, if what you want to do is drag the circle around the graph without changing anything else about it.

Questions: What do you think the output will look like if the input is a circle that contains $1 + i$? Test your theory.

What do you think the output will look like if the input is a circle that contains 2? Test your theory.

What do you think the output will look like if the input is a circle that contains $1 + i$ *and* 2? Test your theory.

What do you think the output will look like if the input is a circle that contains the origin? Test your theory.

Try putting the point on the circle itself along the positive real axis. What happens to the output if you drag the center along the negative real axis?

Now we will investigate what happens when we change the radius of circles at these points.

Center your input circle around $1 + \frac{\pi i}{2}$ .Try changing the radius of your circle. What happens to the output?

Center your circle around $-1 + \pi i$. Try changing the radius of your circle. What happens to the output?

Center your circle around the origin. What happens to the output?

Now, try to summarize what you think is happening. What do you think a large circle around a point $x + iy$ in the complex plane will map to? What about a small circle around the same point?

APPENDIX F

MATHEMATICAL DISCOVERIES GROUPED BY TASK IN WHICH THEY OCCUR

MATHEMATICAL DISCOVERIES GROUPED BY TASK IN WHICH THEY OCCUR

Table 11
*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| Task 1 | | | |
| | Multiply by a complex number via algebraic calculation | Task 1 | Task 1 |
| | Multiply via rotation and dilation | Task 1 | Task 1 |
| | Discovered algebraic error through GSP | Task 1 | Task 1 |
| | Student Conjecture: moving z along the unit circle might make its image move in an ellipse | Task 1 (in $f(z) = z^2$). Overturned via *GSP* in Task 1 | Task 2 (in $f(z) = e^z$). Overturned via *GSP* in Task 2 |
| | Circles map to circles | Task 1 | Task 1 |
| | Output is distorted from a circular shape when pre-image is close to origin | Task 1 | Task 1 |
| | Small circles are distorted less than large circles | Not stated | Task 1 and Task 2 |
| | Recognized when GSP's output differed from the problem considered | Task 1 | Task 1 |
| | Student Conjecture: The maximum imaginary value of an input circle might be preserved in the image | Not considered | Task 1 |
| | Maximum imaginary value is not actually preserved | Not considered | Task 1 |
| | Twists occur when the circle includes 0 | Task 1 | Task 1 |
| | Two points map to the same spot under $f(z) = e^z$ when two points have the same magnitude and a difference of $\pi$ in their arguments | Not considered | Task 1 |

Table 11, continued

*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| | Including the origin is the only way to cause a twist | Not stated | Task 1 |
| Task 2 | | | |
| | Correctly predicted how points and the real axis map in $f(z) = e^z$ | Task 2 | Task 2 |
| | Had difficulty predicting how the imaginary axis maps | Task 2 | Task 2, difficulty with negative imaginary axis only |
| | Circles in $f(z) = e^z$ twist if the input circle has a radius of $\pi$ or larger | Task 2 | Task 2 |
| | Small circles cannot twist | Task 2 | Task 2 |
| | Student conjecture: small circles will twist if they surround zero | Task 2 | No occurrence |
| | Discovered via *GSP* that small circles never twist | Task 2 | Additionally discovered exactly when twists did occur |
| | Twists occur when there are points on the input circle with the same real part and an imaginary part that differed by a multiple of $2\pi$ | Not considered | Task 2 |
| | The real part of $f(z) = e^z$ determines magnitude and the imaginary part determines argument of the output point | Not stated | Task 2 |

Table 11, continued

*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| | The maximum imaginary value of the input circle maps to the point in the circle's image with the maximum argument | Not stated | Task 2 |
| Task 3 | | | |
| | Derivative is the slope of the tangent line | Task 3, frequently recurred throughout remainder of interview sequence | Task 3, rejected immediately. Recurred once |
| | Used C-R equations to calculate the derivative | Task 3, without *GSP* | No occurrence |
| | Connected difference quotient definition of limit to circles narrowing in on a point in *GSP* | Task 3, with *GSP* | No occurrence |
| | The derivative describes the mapping of a small circle more accurately than a large one | Task 3, with *GSP* | Task 3, with *GSP* |
| | Linear functions always map circles to another circle of the same size and degree of rotation. The kind of circle is determined by the derivative | Task 3, with *GSP* (believes real part determines dilation and imaginary part determines rotation.) Corrects herself in Task 4 | Task 3, without *GSP* (believes magnitude determines dilation and argument determines rotation) |

Table 11, continued
*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| | Relationship between derivative and the mapping of circles in linear functions may not generalize to $f(z) = z^2$ or $f(z) = e^z$ (because the output is not a circle) | Task 3, with *GSP* | Task 3, without *GSP*. |
| | Relationship might generalize because the output is nearly a circle | Not stated | Task 3, without *GSP* |
| | Epsilon neighborhoods of a point | No occurrence | Task 3, without *GSP* |
| | Student conjecture: Real part of derivative describes the dilation of the circle, and the imaginary part describes the rotation | Task 3, with *GSP* | Task 3, without *GSP* |
| | Student conjecture: Magnitude of the derivative describes the dilation of a point, angle describes the rotation | Task 4 | Task 3, without *GSP* |
| | Student Conjecture: $f(z) = e^z$ maps lines to lines and preserves angles | No occurrence | Task 3, without GSP. Overturned with GSP |
| | Derivative at a point describes how e-neighborhood at a point is mapped: dilation factor is magnitude and rotation is argument | Not stated | Task 3, with *GSP* |

Table 11, continued

*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| Task 4 | | | |
| | Small circle around zero maps to a very large circle. This is unsurprising to them because they believe $\frac{1}{0} = \infty$ | Task 4 | Task 4 |
| | Describes how $f(z) = \frac{1}{z}$ maps circles successfully | Task 4 | Task 4 |
| | Derivative is a local property- the output will be slightly distorted, unlike in linear functions | Task 4 | Task 4 |
| | Small circles are almost mapped in a linear way | Task 4 | Task 4 |
| | Successfully generalized their interpretation of the derivative to non-linear functions | Task 4 | Task 4 |
| Task 5 | | | |
| | $f(z) = \frac{1}{z}$ has no derivative at 0 | Argued this assertion | Task 5 |
| | $f(z) = \frac{1}{z}$ has a derivative of infinity at 0 | Task 5 | Task 5 |
| | $f(z) = \lvert z\rvert$ is nowhere differentiable | Task 5 | Task 5 conjecture, rejected |
| | $f(z) = \lvert z\rvert$ is differentiable everywhere. Calculate derivative via interpretation of the derivative as describing | No occurrence | Task 5 |

Table 11, continued
*Tasks During Which Mathematical Concepts Were First Verbalized*

| Task | Ideas Referenced | Task where idea first occurred to Karen | Task where idea occurred to Joshua and David |
|---|---|---|---|
| | A rotation and a dilation | | |

# APPENDIX G

# SAMPLE *ELAN* DOCUMENTATION

| Karen's Task 1 (Day 1) Time | Description |
|---|---|
| 00:07:36-00:08:10 | First reference of "rotation dilation" to calculate the real and imaginary part of $z^2$. Asks if she is right. When interviewer refuses to answer, she appears to trace through algebra to verify |
| 00:09:41-00:10:05 | Plot point appears in a place Karen does not expect. Interviewer asks her to explain why. Something convinces her that actually the point is right. |
| 00:12:38-00:13:30 | Predicts where z will go given the point $1 + i$. Calculates correct angle, forgets about dilation. Knows why she was wrong after trying it |
| 00:14:30-00:16:30 | Tries to predict how $z^2$ will move when $z$ moves around the circle. Says almost correct answer once (move in the path of the circle except at $i$), but sticks with the original idea of tracing an ellipse anyway |
| 00:16:30-00:17:40 | Experiments with GSP-notes that it moves twice in a circle, tries to explain to herself why. |
| 00:23:45-00:24:25 | Karen predicts what will happen to the output if the input is a circle that contains $1 + i$. Says she will get some kind of circle shape |
| 00:28:20-00:28:53 | Karen correctly notes that double circles result when the input surrounds the origin. Calls them bread rolls first. Later called double loops |
| 00:29:22-00:30:12 | Karen describes why she thinks double circles happen |
| 00:31:59-00:32:38 | In response to interviewer prompt, Karen describes what happens with small circle inputs, and again notes double circles occur when the origin is surrounded |
| 00:35:50-00:36:19 | Notes circles outside the unit circle seem to go to circles, and get flattened a little as input approaches unit circle |
| 00:36:30-00:37:50 | Karen starts to explain why output is not a perfect circle but peters out. (Also seems to be thinking somewhat algebraically for a brief time) |
| 00:43:45-00:44:47 | Karen comments that if input is centered at origin then the circles should be exactly on top of each other. Interviewer asks her why. (Circle should be rotated exactly 90 degrees) |
| 00:48:30-00:50:12 | Karen distinguishes small circles and large circles and comments on how distorted the circles will be based on the input |
| 00:51:12-00:51:28 | Karen asks me why the circle is distorted when the output nears zero |
| 00:52:01-00:52:46 | Karen explains why double loops happen |
| 00:53:45-00:54:26 | Karen suggests 0 is weird because it is a fixed point. Interviewer overturns this idea by asking Karen to consider 1 |

| Karen's Task 2 (Day 1) | Description |
|---|---|
| 01:04:04-01:04:42 | Karen explains algebraically why $i\pi$ is mapped to $-1$ |
| 01:04:43-01:05:46 | Karen explains algebraically what happens when input is dragged along real axis. Experiment confirms her answer |
| 01:06:08-01:07:32 | Karen explains algebraically what happens when input is dragged along imaginary axis. Experiment surprises her, but she still provides an explanation for what she observes |
| 01:17:05-01:18:03 | Karen discovers a way to create a loop with $e^z$. Distinguishes from double loops, is unable to explain why they occur |
| 01:19:00-01:20:20 | Karen notices that loops do not occur unless input is large. Even notes that input has radius pi when first loop occurs. Turns to algebra/trig to explain why. |
| 01:21:06-01:21:50 | Responding to interviewer prompt, Karen determines through experimentation that loops do not occur with large circles. |
| 01:31:14 - 01:31:30 | Karen again investigates the possibility of creating double circles and succeeds in making more than one loop. Does not explain why and interviewer does not prompt her |
| 01:37:50-01:39:23 | Karen describes due to prompt what happens to a circle around an arbitrary point, quadrant by quadrant. Ends by demonstrating with GSP |
| 01:39:30-01:40:03 | Karen investigates what happens when input circle is small based on interviewer prompt. Karen is surprised to discover that weird behavior does not occur. Karen at this point is unable to articulate why weird behavior does not occur. |
| 01:42:00-01:42:36 | Karen notes bending occurs in a vector when it is outside the unit circle. Relates this to fractional exponents |
| 01:44:00-01:44:45 | Karen adds what will happen to a small circle at various points, says weird behavior will not occur, and does not offer any explanation about why not. |
| 01:45:14-01:46:18 | Interviewer asks if a circle in $z^2$ is made smaller if the double circle will stop happening. Karen says no, double circles will always happen around zero. |
| **Karen's Day 2 (No *GSP* Portion)** | **Description** |
| 00:00:33-00:01:28 | Karen describes Cauchy Riemann equations based on interviewer prompt asking about definition of complex derivative |
| 00:01:30-00:02:00 | Karen describes derivative as slope when interviewer asks for graphical interpretation |
| 00:02:03--00:02:19 | Karen clarifies what she means by slope |

| Karen's Task 2 (Day 1) | Description |
|---|---|
| 00:03:10-00:04:38 | Karen draws tangent line of $z^2$at $1 + 0i$ in response to interviewer prompt |
| 00:04:38-00:05:47 | Interviewer attempts to disrupt tangent line idea. Karen decides her calculus and algebra are contradictory |
| 00:07:36-00:09:45 | Karen uses interviewer-supplied Cauchy Riemann equations to verify derivative. Is still bothered by geometry |
| 00:11:14-00:14:30 | Interviewer again attempts to disrupt line idea. Karen tries hard to reconcile geometry with her algebra. Karen states for the first time at the end of a long silence that it might not just be a straight line |
| 00:14:30-00:15:02 | Karen describes the answer she "instinctually" wants to give as a tangent line of slope 4 in the output plane |
| 00:16:24-00:18:17 | Karen draws a tangent line of "slope $2 + 2i$" through the point $2i$ in the output plane |
| 00:21:10-00:22:10 | Karen describes what $z^2$ does to input circles in response to interviewer prompt |
| 00:22:10-00:23:40 | Karen describes what $z^2$ does to the complex plane near 1+I in response to interviewer prompt |
| 00:23:51-00:24:16 | Karen describes what $z^2$ does to small circles around 1+I in response to interviewer prompt |
| **Karen's Day 2 (*GSP* Portion)** | **Description** |
| 00:00:12-00:05:11 | Interviewer asks Karen to determine with GSP what it means for the derivative to be $2 + 2i$ at $1 + i$ |
| 00:06:20-00:07:00 | In response to interviewer prompt, Karen describes what happens to output circle when input surrounds $1 + i$ |
| 00:07:42-00:07:54 | Karen makes the salient observation that she is "not sure what a line means in complex." |
| 00:09:50-00:11:41 | Karen attempts to determine how much bigger the output circle is than the input in response to interviewer prompt |
| 00:12:39-00:12:52 | Karen asks to pull up $e^z$as well and experiments in the context of a different function |
| 00:14:31-00:15:31 | Karen starts experimenting with circles in $e^z$. Interviewer suggests looking at a point with a more intuitive derivative and calculates derivative for that point |

| **Karen's Task 2 (Day 1)** | **Description** |
|---|---|
| 00:15:50-00:18:12 | Karen tries to respond to interviewer prompt asking about what the derivative means geometrically here. Identifies pieces that are confusing her, including a distinction between "function" and "transformation" |
| 00:19:27-00:21:45 | Karen describes what algebraic definition of derivative means. interviewer explains way too much about the geometry involved in response |
| 00:21:45-00:23:15 | Interviewer prompts Karen to extend limit/derivative ideas to the complex plane via circles |
| 00:23:20-00:24:03 | Interviewer prompt leads Karen to connect the idea of "close" to the size of the surrounding circle |
| 00:24:28-00:27:10 | Interviewer asks again what the derivative tells about the output values. Karen explores ideas like accuracy and attempts to translate algebraic definition to the geometric setting |
| **Back to $z^2$** | |
| 00:28:01-00:28:18 | Karen confirms that output becomes more accurate as circle shrinks here as well |
| 00:28:18-00:29:57 | Karen attempts to answer interviewer prompt about what the derivative $2 + 2i$ means. She again invokes the idea of slope |
| **First Linear Transformation** | |
| 00:30:00-00:30:59 | Interviewer asks Karen to create the transform $2z + 1$ in GSP for the purpose of investigating "what a line looks like" in the complex plane |
| 00:35:21-00:36:15 | Karen answers interviewer question by saying size will change based on location, but orientation will stay constant |
| 00:36:31-00:37:12 | Karen clarifies how size will change based on input |
| 00:37:15-00:37:25 | Karen is asked to demonstrate the expanding circle |
| 00:38:05-00:38:52 | Karen notes the circle is not expanding or shrinking as expected. Tries to explain why |
| 00:39:24-00:39:44 | Interviewer prompts Karen to think about the derivative (constant) in the context of this function |

| **Karen's Task 2 (Day 1)** | **Description** |
|---|---|
| 00:40:30-00:41:51 | Karen is asked to test radius expansion theory in the context of $z^2$. Talks about rotation/dilation but has difficulty talking about particular numbers |
| 00:44:51-00:45:35 | Karen observes that in a linear function output is always a perfect circle, and so the output expanding by a certain radius in a non-linear function does not make sense |
| 00:46:24-00:46:40 | Karen says derivative has something to do with how the circle expands or contracts. No prompt here |
| 00:49:20-00:52:45 | Karen is asked to consider derivative at a point other than $0$. She looks at $-1$, $1$ and $i$. interviewer asks Karen to contrast $1$ and $-1$ outputs in particular |
| 00:52:45-00:58:18 | Karen investigates where the spokes are sent and suggests after prompt that derivative affects where the spokes go |
| **New Linear Function:** $\boldsymbol{iz+2}$ | |
| 01:01:52-01:03:13 | Interviewer asks what the derivative is of this new function and what it means. Karen responds multiplication by $i$, but continues by simply describing function behavior |
| 01:04:35-01:05:10 | Karen notes that again the circle does not change, just moves |
| 01:05:10-01:05:53 | interviewer asks Karen to put spoke on the circle |
| 01:06:03-01:06:19 | Karen says derivative should dilate and rotate (apparently saying rotate by imaginary part and dilate by real part) |

| **Karen's Day 3** | **Description** |
|---|---|
| **First 2 minutes (separate file)** | |
| 00:18-2:26 | Karen describes her current perception of derivative |
| **(switch to second file)** | |
| 6:28-9:15 | Karen calculates the derivative at $1+i$ and describes meaning as slope |
| 9:15-10:48 | Karen tries to decide what slope of $\frac{1}{2}-\frac{1}{2}i$ and catalogues several the problems with this interpretation |
| 12:24-14:02 | Karen describes the input and output planes and why derivative is problematic as a result |
| 15:01-16:02 | Karen explores circles mapped under $\frac{1}{z}$ and notes it too is funky around zero |
| 16:02-17:08 | Circle exploration continues while Karen explains what $\frac{1}{z}$ does to the complex plane |
| 20:36-21:20 | Karen responds to interviewer question about what does it mean when the derivative does not exist |
| 21:20-23:46 | Interviewer asks Karen about where the tangent line is |
| 23:46-24:30 | Karen steps through meaning of derivative for $\frac{1}{x}$ in response to Interviewer prompt |
| 24:30-25:17 | Karen tries to extend this process for $\frac{1}{z}$ |
| 27:07-28:08 | Karen again references tangent line for derivative at $1+i$, but stalls out when asked to identify tangent line |
| 28:28-29:08 | Karen draws potential tangent line: when asked to explain why that is it, she catches herself "thinking in terms of Cartesian again" |
| 29:22-29:53 | Interviewer asks Karen about why she does not want to think in terms of Cartesian |
| 29:55-30:30 | Karen explains why she thinks Cartesian thinking is not right |
| 30:30-31:00 | Karen tries to discover how to graph a slope in the complex plane |
| 31:30-31:54 | Karen justifies a line of slope $-\frac{1}{2}$ is actually the one she wants despite previous objections |

| Karen's Day 3 | Description |
|---|---|
| 31:54-33:05 | Interviewer revives K's objections to the drawn tangent line |
| 33:52-34:51 | Interviewer tries to help Karen connect meaning of real derivative with meaning of complex derivative |
| 37:20-39:13 | Interviewer asks Karen what a linear transformation looks like in the complex plane and if it is different from a line |
| 39:25-39:53 | Interviewer asks Karen to describe the equation of a complex linear transformation |
| 41:00-41:36 | Interviewer asks Karen to draw a tangent line in the technological environment and she explains why she cannot |
| 42:34-42:57 | Karen says derivative is just another transformation |
| 43:00-44:25 | Karen tries to predict what will happen based on the derivative at i |
| 58:25-59:41 | Karen describes why the derivative transformation looks the way it does |
| 59:41-1:00:55 | Karen again suggests derivative should relate to output more than input values |
| 1:00:55-1:03:00 | Karen describes "2-step process" of interpreting the derivative, then demonstrates in GSP |
| 1:03:34-1:05:25 | Karen predicts GSP will not behave as predicted and tries to clarify her meaning |
| 1:05:25-1:06:57 | Interviewer asks Karen about difference between linear transformation and line |
| 1:07:05-1:07:39 | Karen refers to the derivative as a linear transformation |
| 1:07:43-1:09:04 | Karen describes the difference between here and an actual linear transformation. Comes close to actually describing what the derivative means |
| 1:09:13-1:09:51 | Throws out the idea of derivatives corresponding to linear transformations because $\frac{1}{z}$ is not linear |
| 1:10:15-1:11:44 | Karen describes derivative describing slope of a line on one hand but that the derivative is not always a linear transformation on the other. |
| 1:11:44-1:12:36 | Interviewer points out that a derivative can be non-linear and still describe the slope of a line |
| 1:12:38-1:12:44 | Karen instantly makes the connection to complex: "So the circle is telling me about the slope of a line?" |
| 1:18:45-1:20:07 | Karen expresses uncertainty whether the tangent line in the complex plane is a line |

| **Karen's Day 3** | **Description** |
|---|---|
| **New Linear Transformation: (3+2i)z+4 -i** | |
| 1:21:05-1:23:10 | Karen successfully predicts what a constant derivative means |
| 1:23:10-1:23:49 | Karen claims the circle will be rotated by the real part and dilated by the imaginary |
| 1:27:54-1:28:23 | Karen believes her prediction was flawed since the output is bigger than the input |
| 1:31:12-1:31:59 | Interviewer asks what the derivative of a linear transformation means and Karen notes which aspects of the input/output circles are constant |
| 1:39:05-1:40:05 | Interviewer asks Karen what a derivative what tell her about an unknown function at a single point |
| 1:40:05-1:41:08 | Karen describes derivative close to a point |
| 1:42:30-1:44:27 | Karen describes a set of lines that fit the derivative Interviewer gave her at a single point as well as a parabola |
| 1:45:07-1:46:03 | Interviewer asks Karen what derivative means for a linear function locally |
| 1:46:03-1:46:53 | Karen describes what negative signs and fractions do in the derivative |
| 1:46:53-1:48:11 | Karen clarifies what happens in an instance like 3+1/2i |
| 1:48:11-1:49:52 | Karen applies this knowledge to our particular example |
| **Karen's Day 4** | **Description** |
| 0:00-1:02 | Karen describes what a linear function tells her |
| 1:17-3:37 | Interviewer asks how $\frac{1}{z}$ would behave at $i$ if was linear |
| 3:37-4:11 | Karen compares this to how $\frac{1}{z}$ actually looks at $i$ |
| 4:11-4:53 | Interviewer asks what happens to the output as you shrink the input around $i$ |
| 8:02-9:44 | Karen says it looks like a linear function |

| **Karen's Day 3** | **Description** |
|---|---|
| 9:44-10:50 | Interviewer asks what derivative of linear function tells you |
| 11:44-14:10 | Interviewer asks what the derivative point has to do with the circles |
| 14:10-15:22 | Interviewer asks what the derivative of $\frac{1}{z}$ at I tells you |
| 15:22-16:55 | Interviewer asks what the difference between $\frac{1}{z}$ and a linear function is locally |
| 17:49-22:20 | Karen says linear function output circle is rotated differently at different points |
| 24:45-25:52 | Interviewer asks if $\frac{1}{z}$ looks like a linear transformation near i |
| 26:00-30:03 | Interviewer asks about derivative and behavior of $\frac{1}{z}$ at $1+\frac{1}{\sqrt{2}}$ |
| **$z^2$ at 31:42** | |
| 35:30-39:15 | Interviewer asks about derivative and behavior of $z^2$ at $2+i$ |
| 40:22-43:02 | Interviewer asks if the output at $1+\frac{i}{\sqrt{2}}$ is rotated by the wrong angle and why not |
| 43:50-49:06 | Interviewer asks why $2+2i$ was not rotated correctly but other points were |
| 49:20-50:27 | Interviewer asks how much bigger output circle is than input |
| 50:28-52:20 | Interviewer asks what it means for a linear transformation to have a derivative of $4+2i$ |
| 52:20-54:18 | Interviewer asks if there is similar behavior at other points |
| 54:18-55:39 | Interviewer asks Karen to characterize derivative's meaning for $z^2$ |
| 55:40-56:30 | Interviewer asks how $e^z$ might be different |
| **Switch video and function to $e^z$** | |
| 58:00-1:00:40 | Interviewer asks about derivative in context of $e^z$ |

| **Karen's Day 3** | **Description** |
|---|---|
| 1:08:22-1:08:41 | Interviewer asks what it means if the derivative does not exist |
| **New function \|z\|** | |
| 1:19:53-1:28:20 | After Karen determines derivative of $\|z\|$ does not exist, Interviewer asks what a non-existent derivative means |
| 1:28:47-end | Interviewer ends inquiry and lets Karen ask questions to be answered |

| Joshua and David's Day 1 | Description |
|---|---|
| 10:56-11:07 | Joshua suggests $z$ and $z^2$ are on the same line and instantly corrects himself due to technological exploration |
| 11:16-12:09 | Where do we expect $z^2$ to go if $z = 1 + i$. Joshua suggests calculating while David leans on a geometric interpretation |
| 12:30-12:50 | David describes his geometric reasoning |
| 13:03-14:58 | David again utilizes geometric reasoning to find input given output. Joshua uses algebra to flesh out details |
| 15:06-16:10 | Joshua uses geometric reasoning for the first time to describe why one trip around 0 for input means two trips around for output |
| 23:25-26:12 | David and Joshua try to decide whether $1 + i$ on a circle being transformed will be a max or min in the output (wrt $y$-value). Joshua both introduces and corrects this misconception |
| 27:05-28:15 | Joshua suggests a circle with $1 + i$ on it will intersect the imaginary axis twice. David notes a case where only one intersection occurs. |
| 28:30-28:46 | David notes another case where only one intersection occurs |
| 29:05-33:38 | Interviewer asks about what happens when the input is a large circle. They try to explain why twists or deformations occur |
| 34:00-37:00 | Joshua notes that when twists occur, there are two points getting mapped to the same spot at the overlap. They look at a particular example where the overlap is real-valued |
| 37:04-38:50 | Joshua switches to a new example where the overlap is pure imaginary |
| 38:51-40:00 | Joshua asks about better answering first question. David notes that small circular inputs map to circular outputs |
| 41:07-45:21 | They further explore what it means to have zero inside the input circle |
| 45:21-46:10 | Joshua observes similar twists when circle contains 2 |
| 48:15-50:00 | Joshua looks at circle containing origin, Interviewer asks about circles containing 2 |
| 50:00-53:30 | David talks about what happens near zero regarding twists and spikes |
| 54:15-57:10 | They try to explain where the twisting occurs |
| 58:04-1:00:37 | Further explanation of twisting location |
| 1:01:00-1:03:13 | What happens when the radius changes |

| Joshua and David's Day 1 | Description |
|---|---|
| 1:03:39-1:07:45 | What happens when the circle is centered at 0 |
| 1:11:08-1:12:00 | Joshua and David make interesting observations about where the intersection occurs |
| 1:13:07-1:13:48 | Joshua and David again clarify the ambiguity in using angles |
| 1:14:08-1:15:47 | Joshua suggests algebraic reason for twist |
| 1:16:15-1:19:30 | Joshua feels algebra is backed by software exploration. Davidchallenges interpretation first by clarification then counterexample |
| 1:19:38-1:21:21 | Joshua feels satisfied with twist explanation now but David is not. (Switched) |
| 1:21:30-1:24:26 | Joshua and David try to sort out more angle language |
| 1:25:40-1:27:16 | David explains how to predict where the twist will occur. Joshua explains why this means every point is 2-to-1 when circle is centered at origin |
| 1:28:13-1:31:18 | Joshua and David predict what happens when input circle is inside/outside unit circle |
| 1:38:30-1:42:57 | Interviewer asks what $z^2$ does to the plane |
| 1:43:40-1:44:58 | Joshua and David start summarizing $z^2$ again in terms of quadrants |
| 1:47:30-1:49:15 | Joshua and David try summarizing $z^2$ nicely |
| 1:50:07-1:51:10 | Joshua gives something like a final answer |
| **Joshua and David's Day 2** | **Description** |
| 9:55-12:07 | What happens if you move $z$ along the real axis? |
| 12:07-13:33 | What about the imaginary axis? |
| 18:29-19:30 | What happens if you stretch along the imaginary axis? How many circles? |
| 19:54-24:12 | What happens if you stretch vector in a different direction? They answer in terms of quadrants |
| 24:42-25:26 | What happens when vector is rotated but not dilated? |

| Joshua and David's Day 1 | Description |
|---|---|
| 26:12-28:24 | What happens in 2nd and 3rd quadrants? |
| 37:20-39:57 | What happens when input circle contains $1 + i$ |
| 43:25-46:03 | Joshua struggles to discuss indent in output while David discusses orientation of output |
| 46:23-48:36 | What happens when circle contains 2? |
| 49:13-54:18 | What halves of input and output correspond and what does that mean? |
| 54:52-57:25 | David clarifies his hypothesis |
| 1:05:00-1:07:02 | What happens when input circle includes origin |
| 1:07:12-1:11:24 | Interviewer probes about $y$-values of $2\pi$ |
| 1:12:18-1:15:59 | What happens when input circle has radius$2\pi$? |
| 1:16:00-1:18:32 | Investigating/explaining when output has a twist |
| 1:20:20-1:23:40 | Interviewer asks how far apart the input points are that go to the same place |
| 1:23:53-1:26:15 | Why is it that the other two points on the circle $2\pi$ away do not intersect on output? |
| 1:28:53-1:31:50 | David finds another intersection to resolve the question |
| 1:39:34-1:47:19 | What happens when you change the radius centered at $1 + \frac{\pi i}{2}$ |
| 1:48:25 for question, 1:49:10-1:50:44 | What happens to the whole plane? (David and Joshua decide to investigate circles first) |
| 1:52:38-1:53:59 | What happens to the whole plane? |
| 1:57:00-1:57:50 | Another attempt to summarize transformation |
| 1:58:15-2:00:10 | Summary of transformation. (David nudges edge of cylinder explanation) |

| **Joshua and David's Day 3** | Description |
|---|---|
| 00:19-1:52 | How do you think about derivatives? |
| 2:00-4:21 | Interviewer attempts to prod about derivatives without much success |
| 4:30-7:54 | Connection between derivative and magnitude |
| 16:27-19:30 | Connection between derivative and rotation (long silences) |
| 23:00-31:30 | David talks about epsilon circles and how lines get mapped |
| 36:01-36:44 | Why does a line get mapped to a line? |
| 47:00-50:53 | Does the mapping preserve the angle? |
| 54:12-57:50 | Derivative is like a rotation |
| 59:12-1:02:50 | What does a constant derivative (like 1) mean geometrically? |
| 1:03:12-1:04:34 | Interviewer disrupts tangent line idea |
| 1:04:35-1:07:08 | Interviewer asks about linear function $iz$ |
| 1:07:08-1:08:45 | Joshua says derivative describes how function is stretched and rotated and asks if it applies to $z^2$ |
| 1:08:46-1:11:40 | Interviewer asks if pure imaginary like $2iz$ will dilate the input |
| 1:11:55-1:16:00 | Joshua and David try to calculate by how much the input will stretch |
| **New Video** | |
| 1:21-2:00 | Joshua notes output of a line does not look straight |
| 9:00-12:02 | David notes circles map to circular outputs. Interviewer asks about the line segment |
| 12:47-16:06 | Joshua makes a connection between small circle's radius/epsilon neighborhood and magnitude of dilation |

| | |
|---|---|
| 17:00-17:35 | David reiterates Joshua as his own idea |
| 17:40-22:30 | David attempts to verify algebraically. Something goes wrong |
| 22:30-25:00 | David and Joshua try to debug their algebra |
| 25:10-29:45 | David and Joshua argue about whether derivative says how the input gets rotated |
| **Joshua and David's Day 4** | **Description** |
| 00:00-03:38 | Describes derivative in context of $e^z$ |
| 5:37-8:45 | Uses technology to investigate what happens at $1+\frac{\pi i}{2}$. Constructs circle and spoke and transforms them at this stage |
| 8:45-14:30 | Actually starts exploring. Determines rules of magnification and dilation hold |
| 14:30-16:00 | Do these rules hold everywhere? |
| 16:00-16:42 | What happens at 0? |
| 17:15-19:15 | Given the option to explore what they want, they drag circles along axes. (How do I make dilation go to 0) |
| 27:30-29:00 | What does $\frac{1}{z}$ do to the complex plane? |
| 29:00-32:21 | How do circles get mapped? (Joshua interprets pieces of algebra at end) |
| 33:00-34:37 | Joshua discovers some circles map to lines (i.e. the circle "breaks" at some point) Connects quickly to 0 |
| 34:37-36:11 | Notices spoke flips to outside of circle |
| 36:11-40:19 | What does the derivative mean for $\frac{1}{z}$? |
| 41:20-45:25 | David utilizes polar coordinates to aid his algebra. David and Joshua attempt to explain negative sign discrepancy |
| 46:29-47:56 | What does it mean when the derivative does not exist? |
| 49:00-50:38 | Where is the outside of the circle? |
| 51:00-55:40 | What does $\frac{1}{z}$ do to the complex plane? When do you get broken circles? |
| 1:16:30-1:18:52 | What is the derivative of $\|z\|$ at a point where you think it exists? |

| | |
|---|---|
| 1:21:20-<br>1:23:16 | Joshua and David convince themselves that derivatives on real axis exists and follows their rule |
| 1:30:00-<br>1:31:45 | Joshua now thinks the rest of the plane does have a derivative |
| 1:31:45-<br>1:32:15 | What does it mean when the derivative does not exist? |
| silence | |
| 1:32:50-<br>1:33:27 | Answer to previous question |