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Preservice Elementary Teachers' Understandings of Topics in Number Theory

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PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDINGS OF TOPICS IN NUMBER THEORY

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ABSTRACT

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Research suggests that preservice elementary teachers may lack the mathematics understanding necessary to teach mathematics for understanding. The literature has consistently linked student success in mathematics with teacher pedagogical content knowledge (PCK), and recent study suggested a link between teachers' mathematical content knowledge and student achievement. There are gaps in the literature concerning preservice elementary teachers' understanding of number theory, and little is known about how they develop number theory PCK or the relationship between their content knowledge and their PCK.

The goals of this dissertation were to investigate the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory, the nature of their potential PCK in number theory, and the relationship between the two. To address these goals, I conducted a qualitative, interpretive case study of undergraduate students enrolled in a number theory course designed for preservice elementary teachers, using an emergent constructivist-based theoretical perspective. I gathered observational, interview, and document data and conducted analysis using constant comparative methods.

Many of my findings concerning preservice elementary teachers' understandings of number theory content pertain to their understandings of greatest common factor (GCF) and least common multiple (LCM). In particular, participants were more comfortable creating LCM story problems than creating GCF story problems, but their understandings of GCF story problems were closely related to the two meanings of division. In contrast to their understanding of story problems, participants were more comfortable with procedures for finding the GCF than with procedures for finding the LCM. In response to my other research questions, evidence suggests that preservice elementary teachers do possess potential PCK in number theory, namely potential knowledge of content and students and potential knowledge of content and teaching, and that they are related and influenced by specialized content knowledge, curricular content knowledge, experiences working with students, and epistemological perspectives. My data also suggest that preservice elementary teachers possess a type of PCK that is not explicitly represented by the literature, which I call general mathematical pedagogy.

My findings hold many implications for practice. For example, data suggest a process through which preservice elementary teachers might develop a robust understanding of GCF story problems, which builds on their understandings of division. With this observed development process, instructors can scaffold preservice elementary teachers' understanding of GCF story problems. My results also imply specific ways in which mathematics teacher educators and mathematicians may help preservice elementary teachers develop PCK in number theory. For example, instructors can pose hypothetical student conjectures and ask preservice elementary teachers to reflect on the knowledge necessary to teach the content, determine the validity of the conjecture, identify the concepts the student does and does not understand, suggest how they might respond to the student, and reflect on how they used their content knowledge to do so.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

According to the National Council of Teachers of Mathematics (NCTM, 2000), "students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are all shaped by the teaching they encounter in school" (p. 17). Students build their mathematical foundation during their elementary school years, but research has consistently shown that elementary and middle school students are underperforming in mathematics (Beaten et al., 1996; Hanushek, Peterson, & Woessmann, 2010; Kenney & Silver, 1997; Mullis et al., 1997; NCTM, 2000). In the most recent National Assessment of Educational Progress report (U.S. Department of Education, 2015), only 40% of fourth graders and 33% of eighth graders performed at or above a proficient level in mathematics.

The burden to overcome elementary and middle school students' difficulties in mathematics understanding belongs primarily to their mathematics teachers. Researchers have argued that an effective mathematics teacher is one who not only provides instruction incorporating the shifts in mathematics education set forth by the Common Core Sate Standards Initiative (CCSSI, 2011) and the Every Student Succeeds Act (2015), but who is also knowledgeable of and about the mathematics they will teach (Conference Board on Mathematical Sciences [CBMS], 2012; Darling-Hammond &

Youngs, 2002; Goldhaber & Brewer, 2000; Rowan, Chiang, & Miller, 1997; Simon, 1994). We expect teachers to develop an understanding of the mathematics they will teach prior to entering the field, either during their own school mathematics experience or during their teacher education program. However, researchers suggest that "future elementary school teachers may enter college with only a superficial knowledge of K-12 mathematics, including the mathematics they intend to teach" (CBMS, 2012, p. 3).

Other researchers suggest that the mathematics courses in their teacher education programs may not provide prospective teachers with the knowledge they need to be effective mathematics educators (Ball, 1990; Ma, 1999; NCTM, 2000; Wu, 2011). As a result, CBMS (2012) has since updated their 2001 recommendations, increasing the number of mathematics course credits that preservice elementary and middle school teachers should take to satisfy their teacher education programs. However, this is just a recommendation, and in the meantime elementary and middle school students are still not learning the mathematics they need (e.g., U.S. Department of Education, 2015). We must consider that preservice teachers may be underprepared to teach mathematics for understanding, in spite of teacher education programs.

This study was designed to contribute to the body of research concerning preservice elementary teachers' understandings of mathematics, specifically in the area of number theory, with a focus on how those understandings relate to teaching mathematics. This was the focus of my study, because how teachers use their knowledge to teach affects student achievement. Findings from this study inform teacher educators on the teaching of number theory content to preservice elementary teachers and they reinforce the necessity of number theory in elementary education programs. In the remainder of this chapter, I discuss the nature of the research problem in more detail, followed by the purpose of the research and the research questions. Afterwards, I state definitions of terms used throughout this report and explicate the assumptions and limitations of this study. Finally, I conclude the chapter with a summary of the significance of the research.

Statement of the Problem

Students learn many number theory concepts in elementary and middle school. According to the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2011), fourth graders learn about factors, multiples, prime numbers, and composite numbers. Fourth graders also find common multiples of two numbers in the context of finding common denominators in order to add or subtract fractions. Students are formally introduced to the concepts of least common multiple (LCM) and greatest common factor (GCF) in sixth grade. While the standards do not explicitly refer to Fundamental Theorem of Arithmetic or prime factorization, they may be a necessary component of middle school mathematics curricula. Because the Fundamental Theorem of Arithmetic is the foundation for understanding the multiplicative structure of numbers, it is critical for supporting the Standard for Mathematical Practice that students should "look for and make use of structure" (CCSSI, 2011, p. 10). Understanding of basic number theory topics is also important for developing fluency in other areas in mathematics, such as addition and subtraction of fractions and factoring quadratic equations. Due to its prevalence in school mathematics, preservice elementary teachers need to be knowledgeable of select number theory topics.

Considering that many preservice elementary teachers may only have a superficial understanding of the mathematics they intend to teach (CBMS, 2012), mathematics content courses need to play an important role in teacher education. Hill, Sleep, Lewis, and Ball (2007) assert that aside from being able to "do" mathematics, teachers need an in-depth understanding of the mathematics they are teaching in order to explain why it works, as well as to interpret and evaluate students' work and alternative approaches. Preservice elementary teachers can gain this in-depth understanding from a well-designed and well-taught mathematics content course (CBMS, 2012).

CBMS (2012) asserts that not only should preservice elementary teachers develop a deep understanding of the mathematics they will teach, but they should master mathematics taught in the middle grades as well. By making connections and attending to how mathematical ideas taught in the elementary grades build to those taught in middle school, they will be better able to prepare their students for middle school mathematics. Teachers should be able to not only build on the understandings students possess upon entering their classrooms, but teachers should also be able to scaffold in preparation for the mathematics that students will learn in years to come. For example, because common factors and common multiples are taught in sixth grade (CCSSI, 2011), and many preservice elementary teachers obtain K-6 certification upon the completion of their program, common factors and common multiples are among the topics they should understand in depth. While elementary school students are not introduced to number theory terms like factor, multiple, prime, and composite until fourth grade (CCSSI, 2011), students can informally explore these ideas in third grade when they learn about multiplication and division. To best prepare students for future success in number theory, a third grade teacher would need a deep understanding of concepts such as prime and composite numbers in order to stress their important characteristics. For instance, while they may not have the terminology to accompany it, with the careful guidance of their teacher, third graders could make a distinction between whole numbers that can be divided by exactly two numbers with no remainder and those that can be divided by more than two numbers with no remainder. In other words, students can differentiate between whole numbers that are products of exactly two distinct numbers and whole numbers that are products of different pairs of numbers.

However, research suggests that in spite of the understandings mathematics content courses should impart on preservice elementary teachers, many such future teachers lack the in-depth understandings they need to teach (e.g., CBMS, 2012; Zazkis, 1998a; Zazkis & Liljedahl, 2004). Zazkis and colleagues (Zazkis, 1998a; Zazkis & Campbell, 1996b; Zazkis & Liljedahl, 2004; Zazkis & Sirotic, 2010) have conducted some studies concerning preservice elementary teachers' understandings of topics in number theory, but with Canadian students. She found that preservice elementary teachers had trouble thinking flexibly about topics in number theory. For instance, in her investigation of preservice elementary teachers' understandings of evens and odds, Zazkis (1998a) found that participants did not make the connection between the evenness of a number and its prime factorization. Zazkis and Campbell (1996b) found that their participants struggled to identify other properties of numbers when presented in prime factorized form, such as whether they were perfect squares or divisible by other numbers. When presented with a product of two primes, more than one third of Zazkis and Liljedahl's (2004) participants identified the product as prime as well. In one of the few studies not connected to Zazkis, researchers found that preservice elementary teachers' understandings of LCM is also weak (Brown, Thomas, & Tolias, 2002). Most participants’ understandings were merely procedural and connected to the phrase “least common multiple”. Prior to my study, researchers had not yet investigated preservice elementary teachers’ understandings of greatest common factor, an equally important and related concept.

Recently, research by Campbell et al. (2014) suggested that there exists a significant relationship between upper-elementary and middle level teachers’ mathematical content knowledge and student achievement. In particular, teachers’ scores on a content knowledge assessment aligned with grades four through eight state standards had a positive effect on their students’ mathematics scores on their state assessments. This also implies that teachers with a weak understanding of the mathematics they teach have a negative effect on student achievement. Recall that number theory concepts such as prime numbers and least common multiple are taught in fourth and sixth grade, respectively (CCSSI, 2011). Thus, preservice elementary teachers need to master the concepts they teach in order to have a positive effect on their students’ achievement. However, Brown, Thomas, and Tolias (2002) suggested that preservice elementary teachers may merely have a procedural understanding, at best, of LCM. Zazkis and Liljedahl (2004) also suggested that preservice elementary teachers’ understandings of prime numbers is weak. Due to the connections between teacher knowledge and student learning (e.g., Campbell et al., 2014), the prevalence of number theory in elementary mathematics education (CCSSI, 2011), as well as implications that preservice elementary educators may be underprepared to teach number theory, further investigation of preservice elementary teachers’ knowledge of and about number theory seems warranted.

Prior to Shulman (1986), education research clearly identified only two domains of teacher knowledge: pedagogical knowledge and content knowledge. Shulman proposed a missing link between the two, an additional type of knowledge he called pedagogical content knowledge (PCK). Researchers have struggled to connect teachers' subject matter knowledge to their effectiveness in the classroom (Wong & Lai, 2006) until recently (Campbell et al., 2014), but many researchers have found evidence to suggest that a teacher's PCK in mathematics can impact student knowledge and/or learning (Davis & Simmt, 2006; Hill, Rowan, & Ball, 2005; Nye, Konstantopoulos, & Hedges, 2004; Speer & Wagner, 2009; Wong & Lai, 2006). Few studies have investigated preservice teacher PCK, but those that have conducted such investigations suggested that preservice teachers possess little to no PCK because they do not yet have teaching experience (Van Driel & Berry, 2010). Van Driel and Berry argued that preservice teachers can begin to develop PCK in their education programs, though. To differentiate between developing PCK and the robust PCK of an in-service teacher, I refer to the PCK of a preservice teacher as "potential" PCK. None of the studies that investigated preservice elementary teachers' understandings of number theory have explicitly addressed their potential PCK of number theory.

Some researchers have further conceptualized PCK, building on Shulman's proposal that it includes

> An understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (Shulman, 1986, p. 9).

The most prevalent model for PCK in the mathematics education literature is Ball and colleagues' (e.g., Hill, Ball, & Schilling, 2008) Mathematical Knowledge for Teaching (MKT). This model distinguishes between types of subject matter knowledge and types of PCK. According to Hill, Schilling, and Ball (2004), one of the three constructs of PCK is knowledge of content and students (KCS), which pertains to "knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (p. 17). Another construct, knowledge of content and teaching (KCT), combines knowing about teaching with knowing about mathematics and pertains to instructional decisions as they relate to mathematics. Finally, the model also includes an elaboration of Shulman's knowledge of curriculum, which is a knowledge of programs developed for the teaching of a particular subject, concepts covered at a given level, and instructional materials available. As Ball and colleagues' MKT framework is the most prevalent variation on mathematical PCK in the literature, most studies investigating mathematical PCK in the United States have drawn from this framework.

Purpose of the Study and Research Questions

In an effort to address the concerns about preservice elementary teachers' understandings of number theory, and the gap in the research as it concerns preservice elementary teachers' potential PCK of number theory, I conducted a qualitative interpretive case study (Merriam, 1998). The case and participants for this study constituted the group of students enrolled in a number theory course designed for preservice elementary teachers with a concentration in mathematics. This group of students served as a bounded unit (Merriam, 1998). Over the course of my investigation, I addressed the following research questions:

Q1 What is the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory topics taught at the elementary level?

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

To best address these questions, in accordance with case study methodology, I collected data from multiple sources, such as observational field notes during participants' number theory course, artifact data such as homework assignments and tests, and one-on-one task based interviews. Analysis of those data focused around the three research questions: participants' content knowledge, participants' PCK, and connections between the two.

I made implicit use of definitions of terms and assumptions in the research questions and design. The study I conducted also had certain limitations. I make these definitions, assumptions, and limitations explicit in the following section.

Definitions, Assumptions, and Limitations

Throughout this study, I refer to the ideas 'content knowledge', 'pedagogical content knowledge', and 'number theory.' I also frequently refer to 'preservice elementary teachers.' I define these terms below:

Content knowledge is subject matter knowledge, or the understanding of a subject area that one possesses. For the purposes of this dissertation, the subject matter in question is number theory, as defined below.

Number theory refers to a branch of mathematics devoted to the study of positive integers, their properties, and relationships (Katz, 2004). Elementary topics in number theory range from evenness, prime factorization, and the Euclidean Algorithm to Fermat's Little Theorem, and beyond. Some of these topics are accessible to grade school students, while many are not. For the purposes of this dissertation, *number theory* refers to evens and odds, factors and multiples, primes and composites, prime factorization, greatest common factor and least common multiple, and divisibility, which are typically present in K-6 mathematics curricula.

Pedagogical content knowledge, or PCK, refers to "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p. 9). For the purposes of this research, the content area of PCK is number theory, as previously defined.

Preservice elementary teachers are undergraduate college students enrolled in an elementary education program with an intent to obtain a teaching license. The 'preservice elementary teachers' referred to in the research questions have declared a concentration in mathematics and consequently are required to successfully complete several more mathematics courses than their non-mathematics concentration peers.

During the course of my study, concerns were raised regarding whether or not preservice elementary teachers could demonstrate or even possess PCK, as intended by Shulman (1986). In Chapter III, I support my claim that preservice teachers can both possess and demonstrate mathematical PCK. However, this PCK is not nearly as developed or robust as an in-service teacher. To differentiate between the PCK of an in-service teacher, I refer to the PCK of preservice teachers as “potential PCK”.

Through the design of this study, I made certain assumptions about the participants, how they learn, and the number theory topics taught in their number theory course. I operated under the assumption that the preservice elementary teachers in this study chose a concentration in mathematics because they were strong in mathematics, and perhaps stronger mathematically than their non-mathematics concentration peers, whose number theory understandings may be represented by earlier studies (e.g., Brown, Thomas, & Tolias, 2002; Zazkis, 1998a; Zazkis & Campbell, 1996b). Thus, I assumed that the participants of this study would demonstrate a superior, and thus different, understanding of number theory than the participants of similar studies. The decision I made to focus on preservice elementary teachers with a concentration in mathematics based on this assumption was also a limitation to this study. While I assumed that the nature of participants’ understandings was stronger, by excluding non-mathematics concentration majors I did not have a way to make an explicit comparison.

I also held certain assumptions about the number theory course in which the participants of this study were enrolled. The course was taught using a mixture of lecture and group work, and I assumed that these methods were conducive to participants’ learning needs. I also believe, and therefore assume, that knowledge is constructed. I chose an emergent theoretical perspective based on these assumptions. While participants attended lecture, they constructed their own understandings of what their instructor tried to convey, but during group work knowledge was co-constructed and later internalized differently by each individual. Thus, the theoretical lens for this study incorporated both social and psychological perspectives of constructivism.

Beyond assuming that the number theory course material and design were accessible to preservice elementary teachers, I also assumed that the content of the course was applicable to the interview tasks. In other words, I assumed that topics such as evens and odds, factors and multiples, primes and composites, prime factorization, GCF and LCM, and divisibility would be addressed in the number theory course. At the research site, preservice elementary teachers without a mathematics concentration experienced these topics in a fundamentals of mathematics course that mathematics concentration majors are not required to take. The fundamentals of mathematics course did not address these topics in as much depth as a number theory course would. I assumed that the participants of this study could connect their more abstract number theory instruction to the middle school level tasks I posed in the interview tasks. Again, by excluding students enrolled in the fundamentals of mathematics course I produced a limitation to this study.

In spite of the limitations of this study, it makes a contribution to the existing body of literature concerning preservice elementary teachers' mathematical understandings and potential PCK. The study also further supports the necessity for number theory instruction in a mathematics content course for elementary education majors and informs the instructional goals of this course. I elaborate on the significance of this study in the following section.

Significance of the Research

As I alluded to earlier in the chapter, teachers' understandings have profound effects on their students' learning. According to the CBMS (2012), "prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach" (p. 17). Because number theory is such an integrated area in elementary mathematics, preservice elementary school teachers need to have a solid understanding of certain topics in number theory. As researchers have suggested that preservice elementary teachers' understandings in this area are weak, further investigation of a more holistic nature (i.e., case study) was warranted. I also investigated content understandings that had not yet been explored by the research, such as preservice elementary teachers' understandings of GCF. While Brown, Thomas, and Tolias (2002) investigated preservice elementary teachers' understandings of LCM, further investigation was warranted in order to make connections to preservice elementary teachers' understandings of GCF, a related concept. Evidence suggests that preservice elementary teachers struggle to grasp different aspects of LCM than they do with GCF.

The results of various studies conducted by Zazkis and colleagues were similarly disconnected. For example, Zazkis and Campbell (1996b) investigated preservice elementary teachers' flexibility with identifying divisors of a number from its prime factorization, but they did not connect this to participants' understandings of prime numbers. Data from this study indicate that, in spite of recognizing two as a prime number, half of the preservice elementary teachers appeared to doubt the Fundamental Theorem of Arithmetic when it concerned divisibility by two. In general, by investigating the understanding of a broad array of number theory topics simultaneously, this study provides a much more connected model for preservice elementary teachers' understandings of number theory than previously seen in the research.

Additionally, I investigated preservice elementary teachers' understandings of GCF and LCM story problems. According to Liljedahl (2015), "what is needed is not more abstraction, but more contextualization – and an increased ability to deal with this contextualization" (p. 625). There has been an increased emphasis on teaching mathematics in context, but little is known about preservice teachers' understandings of creating or validating story problems. My study contributes to this area in the literature as well. More specifically, it suggests a process through which a preservice elementary teacher can gain a strong understanding of GCF story problems that builds on his or her understanding of division.

This study also addresses the gaps in the literature pertaining to preservice elementary teachers' potential PCK of number theory. According to a meta-analysis of PCK conducted by Van Driel and Berry (2010), very few examples of topic-specific PCK exist, let alone topic-specific examples of preservice teachers' developing PCK. My findings concerning the nature of preservice elementary teachers' potential PCK of number theory inform this sparse area in the research by providing specific examples of instances of potential PCK in number theory. My study also indicates many influences on participants' potential PCK and its development. Because researchers have consistently shown that PCK is important for teaching (Davis & Simmt, 2006; Hill, Rowan, & Ball, 2005; Nye, Konstantopoulos, & Hedges, 2004; Speer & Wagner, 2009; Wong & Lai, 2006) these findings have implications in teacher education, which I discuss below.

In addition to contributing to a deeper understanding of the mathematical knowledge and potential PCK of preservice elementary teachers, this study also determined the nature of the relationship between the two types of knowledge. While teachers' PCK is clearly linked to student achievement (e.g., Hill, Rowan, & Ball, 2005; Speer & Wagner, 2009), researchers have struggled to establish a relationship between teachers' content knowledge and student achievement (Wong & Lai, 2006) until recently (Campbell et al., 2014). This study further supports this connection by exploring the nature of the relationship between content knowledge and potential PCK. Establishing this connection further necessitates the existence of mathematics content courses in elementary teacher education programs. Additionally, Van Driel and Berry (2010) suggested that subject matter knowledge is a prerequisite for developing PCK, but the researchers also acknowledged that "a strong and well-integrated subject matter knowledge does not guarantee the smooth development of an individual's PCK" (p. 658). The results of this study clarify the relationship between subject matter knowledge and PCK, and inevitably they support the necessity of having a strong understanding of number theory concepts in order to develop strong PCK.

Not only do the results of this study make valuable contributions to the literature, but this study also has important implications in practice and curricula. In particular, identifying the strengths and weaknesses of participants' understandings of number theory concepts and ideas allowed me to make certain recommendations about the teaching of these concepts and ideas in number theory courses designed for teachers. For instance, I found participants struggled to understand the procedure for finding the LCM of two natural numbers, given their prime factorizations. Number theory instructors may be able to preempt similar student struggles by providing additional explanation or investigation. By spending additional class time on the procedure, further emphasizing the reasoning for why the procedure works, or investigating it in a different way, perhaps students will be better able to use and reason through the procedure. Also, the process through which preservice elementary teachers can gain a strong understanding of GCF story problems, which emerged from the data, suggests a way of scaffolding preservice elementary teachers' understandings of GCF story problems specifically, and story problems in general. This would enable preservice elementary teachers to better attend to Liljedahl's (2015) call for more contextualization in mathematics education; by gaining a deep understanding of how to create and validate GCF story problems, preservice elementary teachers might be more equipped to interpret or explain GCF in context for students, which may convince students that learning GCF is relevant to real life.

The interview tasks themselves may also provide number theory instructors opportunities to help their students develop number theory PCK. According to CBMS (2012), "prospective teachers should examine the mathematics they will teach in depth, from a teachers' perspective" (p. 17). To elicit potential PCK from my participants, I posed hypothetical student scenarios that required participants to validate and respond to student reasoning. These validation tasks proved useful in gathering data, and they may also play an important role in getting preservice elementary teachers to think about number theory from a teachers' perspective.

In the following chapter, I synthesize related literature. The literature review focuses on number theory content, as it pertains to its presence at the elementary school level as well as preservice elementary teachers' understandings of the elementary level topics, and the research on PCK, both modeling and identifying it. In Chapter III, I use this information to support my methodology decisions for my dissertation and describe the case study with which I addressed my research questions. In my results chapter, I present the findings from my data analysis and answers to my research questions. Finally, in my discussion and conclusions chapter, I discuss how my results contribute and relate to the existing literature and suggest implications for practice and directions for future research.

CHAPTER II

LITERATURE REVIEW

As I mentioned in Chapter I, I conducted an interpretive case study to investigate the nature of preservice elementary teachers' knowledge related to number theory, with specific interest in their subject matter knowledge and their potential pedagogical content knowledge. I also explored the nature of the relationship between preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level. More precisely, in this research study I addressed the following research questions:

Q1 What is the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory topics taught at the elementary level?

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

To address these research questions, I reviewed literature concerning what is known about (1) number theory education and policy for both PreK-12 students and preservice teachers; (2) teacher and student understandings of number theory; and (3) types of teacher knowledge.

This review of the literature is split up into three sections, the first of which primarily reviews policy concerning the number theory education of both elementary school students and preservice elementary teachers. This review outlines not only what topics are taught at the elementary school level, but I also discuss the recommendations for teaching and learning number theory at the elementary school level.

The second portion of the literature review focuses on the research about elementary school students' and preservice and in-service elementary teachers' understandings of number theory. The bulk of this section focuses on preservice teachers' understandings of topics such as evens and odds, primes and composites, divisibility and multiplicative structure, and least common multiple.

The third portion of the literature review synthesizes relevant research related to pedagogical content knowledge. Shulman (1986) proposed that there is more to a teacher's knowledge than just content and pedagogy. He suggested the existence of a third type of knowledge that connects content and pedagogy called "pedagogical content knowledge", defined as "subject matter knowledge for teaching" (p.9). I outline multiple conceptualizations of this construct, discuss its implications on teaching, and discuss some of the existing literature related to elementary teachers' mathematical pedagogical content knowledge.

Finally, at the end of this literature review, I discuss how the literature review informs aspects of my study, including research questions, participant selection, and data collection and analysis techniques.

Elementary Number Theory Education

Since the conclusion of the "New Math" era in the 1960's, an argument has been made for focusing mathematics education on the "practical utility of mathematics through real-world applications" (Zazkis & Campbell, 2006, p. 2). As a result, number theory has played a diminished role in grade school mathematics. While number theory can be used in context with advanced topics like cryptology and computer science, it does not naturally lend itself well to contexts appropriate for the average primary school student.

This does not mean, however, that number theory *could not* and *should not* play an important role in K-8 mathematics education.

> Topics from number theory, such as factors, divisors, multiples, and congruences provide natural avenues for developing and solidifying mathematical thinking, for developing enriched appreciation of numerical structure, especially with respect to identifying and recognizing patterns, formulating and testing conjectures, understanding principles and proofs, and justifying the truth of theorems in disciplined and reasoned ways (Zazkis & Campbell, 2006, p. 2).

Additionally, Sinclair (2006) suggested that elementary number theory should be taught because not only do students get to work with numbers with which they are most familiar (whole numbers), but they can make surprising and exciting discoveries through them. Thus, the study of number theory can encourage an appreciation for mathematics and its beauty.

National Council of Teachers of Mathematics Recommendations

Number theory is not a focus in K-12 mathematics education per se, but important professional organizations such as the National Council of Teachers of Mathematics (NCTM) recommended that it still plays an important role. In NCTM's 1989 document, "Curriculum and Evaluation Standards for School Mathematics", mention of number theory was limited to the grades five through eight standards. NCTM suggested, “the mathematics curriculum should include the study of number systems and number theory so that students can develop and apply number theory concepts (e.g., primes, factors, and multiples) in real world and mathematical problem situations” (p. 91). NCTM (1989) also acknowledged “number theory offers many rich opportunities for explorations that are interesting, enjoyable, and useful” (p. 91).

In an updated version of the document, “Principles and Standards for School Mathematics”, NCTM (2000) recommended that educators teach certain topics in number theory throughout school mathematics. For example, in grades three through five, teachers should expect students to “describe classes of numbers according to characteristics such as the nature of their factors” (NCTM, 2000, p. 148). For instance, they should know that “even numbers” are integers divisible by two and that when you multiply a number by itself the result is a “square number” (p. 151). By middle school, students should use and understand factors, divisibility, multiples, and so on in many different representations, such as story problems. They should be able to complete tasks like explaining the rule for divisibility by three, and “A number of the form *abcabc* always has several prime-number factors. Which prime numbers are always factors of a number of this form? Why?” (p. 217). And in grades nine through twelve, students should “use number theory arguments to justify relationships involving whole numbers” (p. 290).

NCTM (2000) also recommended that elementary school students become proficient in mathematics topics that make use of number theory. For instance, students in grades three through five should begin to operate with fractions, and they should be able to find equivalent fractions. When students add and subtract fractions, they typically need to find a common denominator, which is just a common multiple of the original two denominators. When they find simpler equivalent fractions, they factor out a common factor from the numerator and denominator. By grades six through eight, students should be able to work flexibly with fractions. Not only should students be able to work with fractions using multiple representations, like strip diagrams and number lines, but they also should be able to use proportional reasoning. Middle grades students should also be proficient at multiplying and dividing fractions, but NCTM acknowledged that this “can be challenging for many students because of problems that are primarily conceptual rather than procedural” (p. 218). If students are more comfortable working with fractions procedurally, they may not recognize the connections between the procedures and the number theoretical ideas behind them.

Every Student Succeeds Act and the Common Core State Standards

Until five years ago, each state maintained their own grade-specific standards for school mathematics, all of which were aligned to the NCTM’s Principles and Standards for School Mathematics (2000) to some degree. However, in recent years there has been a legislative push for the states to share grade-specific standards so as to ensure high expectations and consistency.

Most recently, the Every Student Succeeds Act (ESSA, 2015) replaced the No Child Left Behind Act (NCLB, 2001) and reauthorized the Elementary and Secondary Education Act of 1965. The ESSA mandates that each state set high academic standards to ensure that each child graduates high school ready for college and career. However, the Act stipulated that the Federal Government is prohibited from mandating that states adopt "specific instructional content, academic standards and assessments, curricula, or program of instruction developed and implemented to meet the requirements of this Act" (p. 312). This implies that states are allowed to abstain from adopting the Common Core State Standards developed under the Common Core State Standards Initiative (CCSSI, 2011) a few years earlier. However, of the 50 states, only Alaska, Indiana, Minnesota, Nebraska, Oklahoma, South Carolina, Texas, and Virginia have not adopted the Common Core State Standards. Those states would develop and maintain their own standards, similar to those proposed by CCSSI and NCTM.

The Common Core State Standards for Mathematics were developed by the Common Core State Standards Initiative (2011) in part due to criticisms that mathematics curricula in the United States covered concepts "a mile wide and an inch deep" (p. 3). As a result, the Standards set fewer grade-specific standards, but there is a greater focus on each. The Standards do not dictate the curricula states, school districts, schools, or teachers should use; they merely establish what children need to learn and during what grade level. It is up to the states that have adopted the Standards to choose curricula that align with the Standards.

The Common Core State Standards for Mathematics are meant to prepare children for college and career. To achieve this end, the CCSSI (2011) not only proposed grade-specific content standards, but it also proposed eight Standards for Mathematical Practice:

> (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; and (8) Look for and express regularity in repeated reasoning (p. 10).

These Standards for Mathematical Practice are closely related to some of the Process Standards for School Mathematics presented by the National Council for Teachers of Mathematics (NCTM, 2000).

The CCSSI (2011) proposed content standards that set the foundation for number theory ideas in grades one through three, but the bulk of number theory content is not introduced until grades four through six. In first grade, children learn to count by tens, their first foray into understanding multiples. By second grade, children determine if a group has an even or odd number of objects. Children in third grade learn the basics of multiplication and division, which is the foundation for many number theory ideas. For example, they learn that a total number of objects can be expressed as a product. Third graders also make observations about products and multiples, such as multiples of four are always even.

In fourth grade, students begin to formally explore factors, multiples, prime numbers, and composite numbers. The CCSSI (2011) recommended that fourth graders be able to

> Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite (p. 29).

Fourth graders also explore patterns that follow a given rule. For example, the terms generated by the rule “add three” alternate between even and odd numbers. In fifth grade, students continue to explore patterns by comparing the terms generated by multiple rules, like “add three” starting at zero and “add six” starting at zero. Students also find common multiples of two numbers in the context of finding common denominators in order to add or subtract fractions.

It is not until sixth grade that students are formally introduced to the concepts of least common multiple (LCM) and greatest common factor (GCF). Students should be able to "find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12" (CCSSI, 2011, p. 42). Students should also be able to use the distributive property to factor a sum by the terms' GCF. Sixth graders are also expected to write and evaluate numerical expressions involving whole number exponents.

One major criticism of the Common Core State Standards for Mathematics is that they do not mention the Fundamental Theorem of Arithmetic. This theorem is the foundation for understanding the multiplicative structure of numbers. Introducing the Fundamental Theorem of Arithmetic is critical for supporting the Standard for Mathematical Practice that students should "look for and make use of structure." However, while the CCSSI does not explicitly refer to the Fundamental Theorem of Arithmetic, by sixth grade students will have the prerequisite knowledge to express the prime factorization of natural numbers using exponents.

Implications in Teacher Education

The recent changes to grade school mathematics education policy and state standards have necessitated changes in the education of grade school mathematics teachers as well. The ESSA (2015) requires that

> Prospective teachers… who are enrolled in the academy receive a significant part of their training through clinical preparation that partners the prospective candidate with an effective teacher… while also receiving concurrent instruction from the academy in the content area (or areas) in which the prospective

> teacher… will become certified or licensed that links to the clinical preparation experience (p. 114).

However, ESSA does not dictate aspects of preservice teacher education such as course load requirements. As long as teachers meet all applicable state certification and licensure requirements, the Act deems them to be "highly qualified."

The Conference Board of Mathematical Sciences (CBMS, 2012) recently revised their recommendations on the mathematics education of teachers to accommodate the widespread adoption of the Common Core State Standards. Their policy document is meant to be a resource for preservice and inservice PreK-12 mathematics teachers and mathematics educators. CBMS summarizes the recent changes in mathematics education and makes recommendations concerning the mathematics of teacher education programs.

One of CBMS's recommendations has also been asserted by Hill, Sleep, Lewis, and Ball (2007); aside from being able to "do" mathematics, teachers need an in depth understanding of the mathematics they are teaching in order to explain why it works, as well as to interpret and evaluate students' work and alternative approaches. Prospective elementary teachers can gain this in-depth understanding from a well-designed and well-taught mathematics content courses (CBMS, 2012).

CBMS (2012) acknowledged that the education of elementary school teachers is particularly challenging in that many future elementary school teachers enter college with only a superficial understanding of K-12 mathematics, including the mathematics they will inevitably be asked to teach. The Board claimed that the previously recommended nine credit hours of mathematics coursework were insufficient to alleviate this challenge. Instead, CBMS now recommends that preservice elementary teachers complete 12 credit hours of mathematics coursework specifically designed for teachers. CBMS also drew attention to the fact that many middle level education majors have the same preparation as elementary education majors. Rather, they recommend that middle level majors complete 24 credit hours of mathematics coursework.

CBMS (2012) also had recommendations for the content that future elementary, middle, and high school teachers should learn as part of their teacher education programs. To start, they should study the Common Core State Standards associated with the grades that they would teach. More specifically, "before beginning to teach, an elementary teacher should study in depth, and from a teacher's perspective, the vast majority of Kindergarten through fifth grade mathematics, its connections to prekindergarten mathematics, and its connections to grades six through eight mathematics" (p. 23). Similarly, preservice middle school teachers should gain an in depth knowledge of grades five through eight mathematics and beyond. CBMS warned that this is not sufficient to guarantee high quality teaching; courses in mathematics pedagogy are also needed.

CBMS (2012) made content recommendations that mirrored the grade-specific standards proposed by CCSSI (2011). As the CCSSI had limited grade-specific recommendations regarding number theory, this section's only number-theory related recommendation was that middle level education majors examine concepts of GCF and LCM because these are taught in sixth grade. However, in spite of the CCSS's limited mention of number theory, CBMS recommends that middle level majors complete a three-credit course focusing on algebra and number theory. Factors, multiples, primes and composite numbers, greatest common factor, least common multiple, divisibility tests, and the Fundamental Theorem of Arithmetic are among the number theory concepts that should be included in the course.

CBMS (2012) emphasized the importance of number theory understanding for future middle school teachers over future elementary school teachers. However, because the prospective elementary teachers at the research site can obtain a K-6 certification, they may go on to teach middle school level number theory and should, therefore, understand and be able to explain it. Additionally, the board asserted that to develop a deep understanding of the mathematics they will teach, prospective elementary teachers should master the mathematics taught several grades earlier and beyond what they will teach. This implies that teachers should be able to not only build on the understandings students possess upon entering their classrooms, but teachers should also be able to scaffold in preparation for the mathematics students will learn in years to come. Thus, among the number theory topics that K-6 elementary education majors should see in their mathematics content courses are factors and multiples, primes and composites, and greatest common factor and least common multiple, because these topics should be taught no later than sixth grade (CCSSI, 2011).

Wu (2011) agreed with this last assertion of CBMS and also made further suggestions concerning the mathematics education of teachers. He claimed that

> To help teachers teach effectively, we must provide them with a body of mathematical knowledge that [is]: (A) relevant to teaching, i.e., does not stray far from the material they teach in school; (B) consistent with the fundamental principles of mathematics (p. 373).

Far too often future teachers struggle to apply the mathematics from their content courses to the mathematics that they teach. Thus, an effective content course should bridge this gap by helping students to make these connections. Also, there is a great deal of inconsistency between formal mathematics, where precise definitions, reasoning, and mathematical coherence thrive, and elementary school mathematics and even the mathematics taught to future elementary school teachers. Wu reconciles these inconsistencies by claiming that future teachers “must *know* the content of what they teach their students” (p. 380), and that “*knowing* a concept means knowing its precise definition, its intuitive content, why it is needed, and in what contexts it plays a role” (p. 380), while “*knowing* a technique means knowing its precise statement, when it is appropriate to apply it, how to prove that it is correct, the motivation for its creation, and the ability to use it correctly in diverse situations” (p. 380). Finally, Wu suggests, “teachers learn the mathematics better if it is taught hand in hand with pedagogy” (p. 381).

Organizations like CBMS and CCSSI can make recommendations on what teachers and students, respectively, should learn, but each teacher and student may interpret or learn the content differently. In the following section of the literature review, I summarize the research that has investigated students’ and teachers’ understandings of number theory, with a focus on the number theory understandings of preservice elementary teachers.

Research in Number Theory Understanding

Prior to the mid-1990’s, research on number theory understanding was virtually non-existent. The following decade, however, there was a slight influx due largely in part by Zazkis, Campbell, and colleagues (e.g., Zazkis & Campbell, 1996b, 2006; Zazkis & Liljedahl, 2004). In the past decade, there again has been a lull in research concerning number theory understanding. In the sections that follow, I summarize the existing research related to grade school-level number theory tasks and language as they pertain to learning and interacting with number theory ideas, followed by summaries of elementary students' as well as preservice and in-service elementary teachers' understandings of number theory.

Number Theory Tasks and Language Use

A small handful of articles have addressed grade school-level number theory tasks themselves, with close attention to the language used to pose them. While this body of literature may provide little insight into precisely how students and teachers, both in-service and preservice, understand number theory, it suggests a language with which to speak about number theory tasks and ways in which they might be posed so as to bring about the richest responses and to explore the depths of participants' understandings.

Mason (2006) examined grade school-level number theory tasks in terms of whether or not they are "exemplary". He claimed that worked examples can make a difference in how students learn and understand mathematics; they can discourage students from trying to understand, amaze students into thinking that mathematics is mystical, or they can frame how students approach mathematics and help form their cognitive structures. In his chapter in Zazkis and Campbell's (2006) book, *Number Theory in Mathematics Education: Perspectives and Prospects*, Mason offered four case studies of classes of number theory tasks. Mason's goal was "to illustrate a number of tactics that can be used to promote learner sense-making so that objects become examples of a generality" (p. 43). Mason took examples from within his own experiences and then refined and adjusted them based on his review of the literature and interactions with colleagues who attempted the tasks.

The first of Mason's (2006) four case studies proposed task examples related to the idea of remainder. For example, "write down a number that is one more than a multiple of seven" or "write down a number that leaves a remainder of one on dividing by seven" (p.44). The second case study pertained to tasks about multiplicative closure. An example of this would be, "what aspects or features of the statement, 'the product of any two odd numbers is also odd' could be changed and still the statement would remain true?" (p. 46). The third set of tasks related to common multiples and common divisors. Some examples were straightforward, for example, "find the greatest common divisor or 84 and 90" (p. 49), while others were less standard, for example, "find a number with exactly 13 factors" (p. 50). Another task asked readers to multiply the GCD and LCM of two numbers and compare it to the product of those two numbers. Finally, the fourth case study involved problems about factoring. One of these tasks presented readers with observations like $x^2 + 5x + 6 = (x+3)(x+2)$ but $x^2 + 5x - 6 = (x+6)(x-1)$, and then it asked readers to generate the next few examples and generalize their observations.

In order to best pose examples or tasks that encourage learner sense-making or demonstrate the depth of a learner's understanding, Mason (2006) proposed a list of facets that make examples "exemplary". His first suggestion was that, when asking for an example of something, the teacher or researcher should ask for another, and then another. Research has suggested that by the third generated example, students are more adventurous or thoughtful with their responses (Watson & Mason, 2005). Mason also suggested that you reverse the task or example. For instance, rather than always asking students to find the greatest common factor or least common multiple of two numbers, you could also ask "What pairs of numbers can have 24 as their GCD? What pairs of numbers can have 72 as their LCM?" (Mason, 2006, p. 52). Another suggestion of Mason's was to pose tasks that require students to move from a particular example to more general cases. For example, after asking students to find multiple numbers with an odd number of factors, you could ask them to generalize their findings. He also suggested problems that go in the reverse direction: general to particular. For instance, posing the statement about the product of odd number being odd may encourage learners to generate specific examples.

In his discussion, Mason (2006) suggested that by only posing straightforward tasks or by emphasizing the importance of the correct answer rather than process through which it is found would lessen the pedagogic effect that examples and tasks can have on a learner. He also confirmed what others have suggested about number theory; that it is an excellent domain in which to discover relationships in mathematics and to become familiar with processes of mathematical thinking.

While Mason (2006) focused on the types of tasks one might pose, Zazkis (1998b) discussed issues concerning the wording of such tasks. In particular, many possible number theory tasks make use of the terms “divisor” and “quotient”, which could be problematic due to lexical ambiguity within mathematics contexts. For instance, in division, the divisor is what you divide by. However, in number theory, divisor is interchangeable with factor, which has a somewhat different definition: if a and b are whole numbers and a is a factor (divisor) of b, then there exists a whole number c such that $ac = b$. In the division definition, a, b, and c need not be whole numbers. Similarly, depending on the types of numbers used, there are varying and conflicting definitions of “quotient”.

In her article, Zazkis (1998b) posed a handful of vignettes exemplifying the lexical ambiguity of these words within a fundamental of mathematics course for preservice elementary teachers. In one of her vignettes, Zazkis reported confusion and disagreement among her preservice elementary teachers when she asked for the quotient in the division of 12 by five. Most of her students claimed that the answer was 2.4 or $2\frac{2}{5}$ or $\frac{12}{5}$, citing the definition of division, that the quotient is the answer to a division problem. However, about one-third of the students argued that the answer would be two, because the definition of the Division Algorithm states that the quotient is the whole number portion of the answer.

In another example, Zazkis (1998b) asked a preservice elementary teacher to list the factors of 117. Then Zazkis asked her if there existed a divisor of 117 that was not a factor of 117. The participant replied in the affirmative and declared that one could divide by anything, and that it did not matter if the answer was a decimal. Here the participant's definition of divisor differed from that of Zazkis'. Even after attempting to establish common definitions for terms, Zazkis' participants did not consistently use the number theoretic definitions of divisor and quotient. Finally, Zazkis suggested that number theory students partake in a didactical activity to explore this lexical conflict and then resolve it, so as to reestablish definitions of terms and their appropriate contexts.

In his study concerning in-service teachers' numeracy task design, Liljedahl (2015) described the increasing push for numeracy in mathematics education. The definition of numeracy that emerged from his study was "the willingness and ability to apply and communicate mathematical understanding and procedures in novel and meaningful problem solving situations" (p. 628). Furthermore, Liljedahl stated that "what is needed is not more abstraction, but more contextualization – and an increased ability to deal with this contextualization" (p. 625). While the tasks in Liljedahl's study were not number theory tasks, per se, the sentiment that students be able to contextualize number theory ideas is an important one, and I incorporated it into my own interview tasks.

As the language and design of a task can influence how one responds to a task, I made careful observations of these ideas when discussing the number theory tasks used in the following research. In this next section, I focus on the research concerning elementary school students' understandings of number theory.

Elementary School Students' Understanding

Although, in general, research has shown that students are not learning the mathematics they need (Beaten et al., 1996; Hanushek et al., 2010; Kenney & Silver, 1997; Mullis et al., 1997; NCTM, 2000), there is very little research concerning elementary students' understandings of number theory specifically. Most of the literature in this area outlined lessons for teaching number theory topics (e.g., Johnson, 2001; Kurz & Garcia, 2010). For instance, Johnson suggested a way to connect pitch in music to congruences and proposed a lesson suitable for middle school students. Kurz and Garcia outlined an alternative approach to learning prime decomposition using factor tiles and designed activities appropriate for students in grades three through five. While articles like these provided ways for students to learn about number theory, they say very little about how students understand number theory and even less about what teachers need to know in order to implement these lessons.

In her discussion of pedagogical dilemmas in teaching third grade students, Ball (1993) anecdotally recalled students' ideas regarding evens and odds. The vignette focused on one student's claim that numbers such as six can be both even and odd. As evidence for his claim, Sean partitioned six circles into three groups of two. He concluded that because three was odd, six must be an odd number. However, because you could also partition six in half without making half-circles (the class's working definition of "even" at the time), Sean also concluded that six was an even number. This stems from the idea that while it is true a number is even if it has an even factor, it is not true that a number is odd if it has an odd factor. Some of Sean's classmates seemed skeptical, and the discussion led to the development of a definition for "odd" numbers and to the refinement of Sean's idea: some numbers have an odd number of groups of two. While Ball's experience provided an interesting glimpse of one student's concept of even and odd, it is not at all representative. However, it was interesting that third grade students can stumble across the more sophisticated ideas of divisibility and multiplicity in a discussion of evens and odds.

More thorough research exists related to elementary students' performance with an understanding of fractions, a peripherally related topic. Some studies have focused primarily on individual students' understandings of fractions or operating on fractions (e.g., Saenz-Ludlow, 1994), while others considered the affect of curriculum on students' understandings of fractions (e.g., Moseley, 2005). Neither of these studies made specific reference to number theoretic ideas like common denominators or simplifying fractions by finding common factors. Had they alluded to these ideas, I may have been able to anticipate elementary school students' understandings of common factors or common multiples. However, because none of these studies discuss student understanding of fraction as it relates to number theory understanding, it may be presumptuous to conclude that struggles or successes in fraction understanding imply deficits or accomplishments in number theory understanding, respectively.

While the literature is still incomplete, much more is known about how preservice elementary teachers understand number theory. In the next section, I summarize studies that represent what is known about preservice elementary teachers' understandings of topics in number theory.

Preservice Elementary School Teachers' Understandings

The research on preservice elementary teachers' mathematics content understanding is still limited, but growing. It used to be that researchers gathered data such as grade point average and test scores to determine the degree to which teacher candidates understood the subject matter (Ball, 1990). However, this did not attempt to explain a preservice teacher's depth of understanding, comfort level with the material, or ability to teach for understanding. Traditional elementary education programs require students to pass mathematics content courses, but some researchers' studies have suggested that these courses are not sufficient to guarantee high quality teaching (CBMS, 2012).

Somewhat more is known about future teachers' understandings of number theory than elementary school students' understandings of number theory. An influx of studies were conducted over the course of a decade starting in the mid-1990s. Most of this research has been conducted by Zazkis and colleagues (e.g., Zazkis, 1998a; Zazkis & Campbell, 1996b; Zazkis & Liljedahl, 2004) in Canada. These studies range from preservice elementary teachers' understandings of evens and odds to their understandings of irrational numbers. Many of the studies on number theory understanding suggested that it is somehow related to one's understanding of arithmetic (e.g., multiplication and division) so this is also included in this section.

Elementary number theory. Most of the literature on preservice elementary teachers' understandings of number theory concerns elementary topics in number theory. While there has been an influx of recent research in this area, there are still some fairly large gaps. The available research includes studies that unveil preservice elementary teachers' understandings of prime numbers, prime decomposition, divisibility, even numbers, and least common multiple, but their understandings of topics like greatest common factor and square and triangular numbers is yet to be explored empirically. Even within the mentioned topics, the research is not exhaustive. For instance, even though Zazkis and Liljedahl (2004) acknowledged the importance that preservice elementary teachers recognize that there are infinitely many primes, their study did not address this. Also, none of these studies explored connections between participants' content understandings and their anticipated experience with teaching elementary number theory topics to children, which was a goal of my own study.

Divisibility. Zazkis and Campbell (1996a) investigated preservice elementary teachers' understandings of multiplicative structure and divisibility using the Action-Process-Object (APO) Framework, a theoretical framework based on constructivism. The researchers gathered data by administering a written questionnaire and interviewing participants enrolled in a foundational mathematics course for teachers after their number theory unit. The researchers conducted clinical task-based interviews, for which the problem sets pertained to the divisibility and factors of numbers in whole number and prime factorized forms. Among the tasks were questions such as "Consider the number 12,358 and 12,368. Is there a number between these two numbers that is divisible by 7? By 12?" (p. 543) and "The number 45 has exactly six divisors. Can you list them all? Can you think of several other numbers that have exactly six divisors?" (p. 543). In their data analysis, the researchers coded statements according to their theoretical framework (action, process, or object) so that they could describe participants' construction of mathematical knowledge. They also coded responses according to "their contribution to the purposes for which the interview questions were originally designed" (p. 543). From there, Zazkis and Campbell explored the procedural and conceptual relationships between divisibility and division. The researchers occasionally used time to distinguish between the instances where participants demonstrated conceptual versus procedural understandings. Most tasks were readily resolvable with the appropriate conceptual understanding, which took less time, but participants required significant time and effort to solve the tasks procedurally.

The researchers found that the participants struggled to move away from a procedural understanding of divisibility and onto a conceptual one, perhaps due to the fact that "divisibility is a very complex cognitive structure" (p. 561). Some participants had a conceptual understanding of ideas like evenness or divisibility by five, but struggled to surpass a procedural understanding for ideas such as divisibility by 15. Many of the participants attempted to perform division when it would have been more useful to use the prime factorization, which was occasionally readily available to them. Zazkis and Campbell believed their participants had difficulties with basic arithmetic, which resulted in difficulties understanding divisibility and factorization. They also suggested that the development of a rich understanding of divisibility "must begin by discerning between divisibility as a property and division as a procedure" (p. 555), which many of their participants did not do. The authors further suggested that an understanding of algebra requires a conceptual understanding of both arithmetic and elementary number theory.

Prime decomposition. Using the same data set, theoretical framework, and principles of analysis, Zazkis and Campbell (1996b) investigated preservice elementary teachers' understandings of the Fundamental Theorem of Arithmetic. They asked participants questions such as, "Is 17^3 a square number?" and "Is 11 a divisor of $M = 3^3 \times 5^2 \times 7$?" (p. 207). In most cases, participants were presented with the prime factorization of numbers. Most of the participants could recall the Fundamental Theorem of Arithmetic. They could even decompose large number to find the list of unique prime factors. However, when provided with M, for example, about half insisted on using a calculator to expand M and then divide by the divisor in question to determine divisibility. Of the other half of participants, all could reason that, for example, seven was a divisor of M because they could see it in the prime factorization, but half of those participants had trouble using the same sort of reasoning to explain why 11 was not a divisor. Only about one quarter of participants cited the uniqueness of a number's prime factorization as the reasoning for their responses, while others seemed uncertain of this idea. When questioned about whether $16199 = 97 \times 167$ was divisible by 13, one student in particular mentioned, "because a number is divisible by two primes does not mean that it is not divisible by other primes" (p. 210). This idea that the prime factorization may not be unique persisted upon further investigation by the researchers. Zazkis and Campbell surmised that this difficulty or uncertainty may stem from the participants' experience with decomposing numbers into products of composite factors, which are not unique (e.g., $96 = 16 \times 6 = 8 \times 12$). They also suggested that preservice elementary teachers may have a limited understanding of what a theorem is and are thus unconvinced that theorems are always true. Regardless, about half of the participants did not even recognize the Fundamental Theorem of Arithmetic as a viable solution strategy and another quarter of the participants did not appear to recognize the uniqueness of the prime factorizations of numbers, which indicates that a large portion of preservice elementary teachers have an inadequate understanding of the theorem.

Evens and odds. While the concept of 'even' still falls under 'multiplicative structures', the parity of numbers deserves it own category. Researchers have suggested that students' conceptualization of evenness and oddness does not necessarily occur simultaneously to their conceptualization of divisibility by three, four, and so on (Zazkis & Campbell, 1994). Perhaps it is because there is no other adjective in our language to say that a number is divisible by some p not equal to 2, or perhaps it is because the concept of an even number is the first number theory concept children are exposed to (as evidenced in Bell et al., 2010; Fuson, 2006; Willoughby et al., 2007), and long before other concepts. Either way, even numbers are fundamental. One would think that, of all the number theory topics, preservice elementary teachers would understand even numbers the best of all. However, as Zazkis (1998a) indicates, this may not be the case.

Using the same data set, theoretical framework, and analysis procedures as Zazkis and Campbell (1996a, 1996b), Zazkis (1998a) investigated preservice elementary teachers' understandings of evenness and oddness. In her questionnaire, Zazkis stated a series of numbers using various representations (e.g., whole number and prime factorized form) and asked the students to identify whether they were even or odd. She also asked them to briefly explain their reasoning. Zazkis analyzed students' responses in terms of their strategies. She concluded that students had difficulty perceiving evenness as equivalent to divisibility by two. As a result, they struggled to identify numbers as even or odd for numbers with factored representations, where recognizing divisibility (or non-divisibility) by two would have been useful. For example, rather than recognizing that none of the factors were divisible by two, students used more complicated methods for determining that 3^{99} is an odd number, such as by attempting to multiplying it out.

Primes. Zazkis and Campbell (1996a, 1996b) found that many preservice elementary teachers inadequately grasp the concepts of divisibility and prime decomposition. The researchers surmised that this may be the result of inadequate understandings of more fundamental ideas, like the primality of numbers. It was this suggestion that, in part, spurred Zazkis and Liljedahl's (2004) investigation into preservice elementary teachers' understandings of prime numbers. For their framework, the researchers used a theory on representation that relies on the distinction between transparent and opaque representations. "A transparent representation has no more and no less meaning than the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of ideas or structures but de-emphasizes others" (p. 165). The authors argue that all number representations are opaque, but may have some transparent features. For instance, by representing the number 784 as 28^2, it emphasizes that it is a perfect square, and the representation is transparent in this respect. However, it de-emphasizes the fact that 784 is divisible by 98 and is thus opaque.

Because prime numbers are the veritable building blocks of number theory and because subject matter knowledge is essential in learning to teach for understanding (e.g., Ball, 1996), Zazkis and Liljedahl (2004) asserted that preservice elementary teachers should know a great deal about prime numbers. The researchers claimed that preservice teachers should know: (1) the definition of a prime number; (2) all natural numbers greater than 1 are either prime or composite; (3) if one can represent a number as a product (where none of the factors are 1), then the number is composite; (4) composite numbers have a unique prime factorization; and (5) there are infinitely many prime numbers. To determine their participants' understandings of these ideas, Zazkis and Liljedahl posed three questions to preservice elementary teachers enrolled in a number and operations course for teachers after their number theory unit. The first question pertained to participants' understandings of prime and composite and their perceived relationship between the two ideas. For the second question, the researchers asked participants to determine, with an explanation, if $F = 151 \times 157$ was a prime number. Finally, Zazkis and Liljedahl asked "Consider $m(2k+1)$, where *m* and *k* are whole numbers. Is this number prime? Can it ever be prime?" (p. 170).

In response to the first question (*n* = 18), about half of the participants recalled the formal definition of a prime number from their course (a number with exactly two factors or two distinct factors) while slightly more than half recalled the elementary school definition (a number divisible only by 1 and itself). In most cases, participants supplemented or led their responses with a negative response pertaining to what prime or composite numbers "are not". For instance, a prime number "is not" divisible by anything other than 1 and itself. When asked about the relationship between prime and composite numbers, more than half of the participants alluded to prime decomposition in some way. They claimed that composite numbers can be decomposed into prime factors, while the rest of students seemed to think about the two types of numbers as disjoint and unrelated.

The second question was issued through a written assessment (n = 116) as opposed to an interview like the first and third questions. Nearly two-thirds of the participants correctly identified F to be a composite number, with most participants citing the definition of prime or composite. Many of these participants also made non-essential claims, for instance, that 151 and 157 are prime, which has no bearing on whether or not F is prime. While they correctly claimed that F was composite, some participants felt the need to confirm that F was divisible by 151 and 157 through an algorithm, indicating a weak understanding of the relationship between divisibility and multiplication (Zazkis & Campbell, 1996a). More than one-third of the participants incorrectly claimed that F was prime, and more than one-half of them reasoned that the product of two primes is also prime. Zazkis and Liljedahl (2004) suggest that this may be an indication of "a profound psychological inclination toward closure, that two of a kind produce a third of the same kind" (p. 175). Other participants reasoned that if F were composite it would be divisible by "small" primes or they incorrectly used an algorithm or divisibility rule. For instance, some participants generalized that because the last digit of F was prime, F was therefore prime. Some of Zazkis and Campbell's (1996a) participants made similar incorrect claims in their study.

Due to the general nature of $m(2k+1)$ as a number representation, participants could not rely on algorithms or divisibility rules to determine whether or not it is prime. Thus, in response to the third question, researchers found that most participants either relied on their perceived definitions of prime and composite numbers or they used examples to convince themselves one way or the other. Some participants claimed that $m(2k+1)$ could not be prime because it was written as a product of two numbers, while others recognized the trivial case where $m = 1$. One of these participants also insisted that $m(2k+1)$ was always prime when $m = 1$, possibly confusing her understanding of odd and prime. Only one student recognized the second trivial case where $k = 0$. Of the handful of participants that used examples to convince themselves one way or another, the interviewer posed a follow up question, "A student was here earlier and she claimed that this number cannot be prime because it is divisible by m and by $2k+1$ and it is always divisible by 1 and itself, so it has all these factors so it cannot be prime, what would you tell her?" (Zazkis, & Liljedahl, 2004, p. 179). The participant responded that she would provide the other student with counterexamples, but could not explain why her reasoning would not work.

While all participants could recall some variation on the definitions of prime and composite numbers, there was strong evidence that most participants' understandings were not well connected to their understandings of factors, multiples, and divisibility. For instance, some participants struggled to recognize the equivalence between the statements '*A* is a factor of *B*' and '*B* is divisible by *A*'. Other participants seemed to think that all composite numbers are divisible by small primes or that all prime numbers are small. Another major finding concerns some participants' insistence of what primes "are not". It was perhaps this understanding of prime by exclusion that resulted in so many participants believing that $m(2k+1)$ and even $F = 151 \times 157$ could not be prime. In general, Zazkis and Liljedahl (2006) suggest that the lack of a transparent representation of a prime number may pose a problem in preservice elementary teachers' development of the concept.

Least common multiple. Brown, Thomas, and Tolias (2002) continued with Zazkis and Campbell's (1996a) work with preservice elementary teachers' conceptions of divisibility, posing similar tasks and arriving at similar findings, with one important addition. Brown, Thomas, and Tolias also investigated participants' conceptions of least common multiple (LCM). They posed three tasks related to this idea:

> "[1] Find the smallest counting number that is a multiple of both 72 and 378. Explain; [2] A scientist starts two experiments at the same instant. In the first experiment, a measurement has to be made every 168 seconds, while in the second, a measurement has to be made every 108 seconds. After how many seconds will the scientist have to make two measurements at the same instant? Explain; [3] Find a pair of numbers, each smaller than 200, whose least common multiple is 200. Explain your answer. Find another pair, different from the first pair" (p. 50).

Similar to many of Zazkis' studies (e.g., Zazkis & Campbell, 1996b), Brown, Thomas, and Tolias (2002) used the constructivist-based action-process-object-schema (APOS) theory, a revision of APO theory, as their theoretical framework. They analyzed their data according to the components of this framework.

Brown, Thomas, and Tolias (2002) found that their participants used one or more of the following three approaches. The researchers referred to the least effective approach for finding the LCM as the *set intersection* approach, which required some brute force; some participants created ordered lists of multiples for each number and then chose the smallest number in common. Another approach was to create one list of multiples for one number and continuously check those multiples for divisibility by the other number. While this approach, referred to as *create a multiple and divide*, still required some brute force, it demonstrated some connections between the ideas of multiples and divisibility. The third and most efficient approach made use of the *prime factorization* of the two numbers, identifying the highest power of each prime factor.

Brown, Thomas, and Tolias (2002) made sure not to use the phrase "least common multiple" until the last question, and it appeared to have an effect on some of the participants' strategies. The researchers observed participants using multiplicative structure to answer the first two questions, through the *set intersection* and *create a multiple and divide* approaches, but then switching strategies to *prime factorization* on the third question. One participant in particular mentioned that he "always" used that method to find the LCM, contrary to the fact that he created a multiple and divided to find the value for the first two problems. The researchers suggested that for preservice elementary teachers like this participant, the "smallest counting number that is a multiple of both" may be a different entity entirely than the "least common multiple". They claimed this may have been the result of a very procedural understanding of both least common multiple and prime factorization.

Other participants recognized that they were solving for the LCM in the first two problems, and many of them attempted to use the prime factorization method when they did. Not all were successful and most of those that were did not adequately explain why this method worked. Only one participant acknowledged the relationship between the prime factorizations of the original two numbers and that of the LCM. She identified each number as a factor of the LCM and accounted for the extra factors by looking at the prime factorization of the other number. Using this reasoning, she was also able to explicate the necessary and sufficient conditions of the algorithm she used to find the LCM. As a result of her clear understandings of LCM and prime factorization, the participant had no trouble answering the third LCM question and explaining her reasoning.

For students to possess a conceptual understanding of LCM, Brown, Thomas, and Tolias (2002) suggested that they first need a flexible understanding of prime factorization and how it relates to factors, multiples, and divisibility. The authors also suggested that students require a connected understanding of LCM across representations. From a pedagogical standpoint, instructors could move from the *set intersection* approach to the *create a multiple and divide* approach with enough ease. But then they should reconstruct the *create a multiple and divide* approach using prime factorization and allow students to develop intuition about the *prime factorization* approach in that way.

The more of A, the more of B. Zazkis (1999) investigated whether or not the common misconception 'the more of *A*, the more of *B*' (e.g., the greater the perimeter of a figure, the greater its area) persisted with number theory concepts. She conducted two phases of interviews with preservice elementary teachers enrolled in a mathematics course for teachers that addressed the elementary number theory topics they might have the opportunities to teach at the elementary school level (e.g., primes, factors, divisibility, etc.). In the first phase of interviews, she asked them questions like, "what are the factors of $117 = 3^2 \times 13$? Can you list all of them?" (p. 198) and "[consider] numbers $A = 3^2 \times 7$ and $B = 3^2 \times 17$. What do you think is the number of factors of *A* and *B*? Is the number of factors of *A* larger than, smaller than, or equal to the number of factors of *B*?" (p. 199). In the second phase, Zazkis asked participants to reflect on phrases like "if a natural number *a* is bigger than a natural number *b*, then the number of factors of *a* is bigger than the number of factors of *b*" (p. 200).

Of the phase one responses, seven of the fifteen participants claimed that *A* and *B* had the same number of factors, but three of these participants based their claims on the idea that each of these numbers had only two (prime) factors. The other four participants attempted to explain their reasoning using the prime decomposition of *A* and *B*, but it was less logical reasoning than intuition. And alas, eight participants felt that *B* would have more factors, because it was a larger number. To the contrary, in the phase two responses, four of 58 participants felt the statement was true. Of the 54 participants who thought the statement was false, 48 justified their claim with a valid counterexample. However, 42 of these counterexamples made use of one or two prime numbers. When asked about the same statement, but with composite numbers rather than natural numbers, 13 participants felt the statement was true. Of the ones who identified the statement to be false, only 35 found a valid counterexample.

While more of her participants responded accurately to the phase two tasks, Zazkis (1999) was skeptical that they had abandoned their intuitive belief in the rule, ‘the more of *A*, the more of *B*’. Upon further analysis, she found that many of the participants who felt the phase two statements were false still had the general belief that larger composite numbers had more factors; they wrote off their counterexamples as exceptions to the rule. Thus, it appears that number theory as a context is no exception to the plight of this common misconception.

Learning number theory through repeated patterns. Zazkis and Liljedahl (2006) investigated preservice elementary teachers learning of elementary number theory through repeated patterns tasks. The task that researchers focused their analysis on was: “Consider a 1,000 car toy train in a 7-color repeating pattern (red, orange, yellow, green, blue, purple, white). What is the color of the 800th car?" (p. 104). Zazkis and Liljedahl observed two major strategies: (1) participants divided 800 by seven and used the remainder to determine the color; (2) participants found a multiple of seven and counted up to determine the color. Zazkis and Liljedahl (2002) observed similar strategies in their study, where participants were asked to determine whether or not 704 was a member of the infinite arithmetic sequence eight, 15, 22, … From other studies (e.g., Zazkis & Campbell, 1996b), we see that the concept "every nth number is divisible by n" and, in general, the partitioning of numbers modulo *n* is not always easily grasped. But, Zazkis and Liljedahl (2006) found that their participants had more overall success with understanding these concepts than has been reported in previous studies. Overall, the researchers claimed "students' engagement with repeating patterns can help them in acquiring and strengthening concepts of elementary number theory and in establishing connections among these concepts" (p. 99).

Advanced topics in number theory. While not taught at the elementary school level, preservice teachers' understanding of more advanced topics in number theory may relate to their understanding of more elementary concepts in number theory. Zazkis and Sirotic (2010) used an interpretation of constructivism, the distinction between transparent and opaque representations, to analyze preservice mathematics and science teachers' understandings of irrationality. They defined transparent representations of numbers as those that emphasize certain characteristics of the number, while opaque representations de-emphasize certain characteristics. During task-based interviews, Zazkis and Sirotic found that the participants approached the representations with one or more dispositions. They asserted that participants had a fractional disposition if fraction representations were preferred and decimal representations went unrecognized. A participant had a decimal disposition if the opposite was true. Participants had a balanced disposition if they were able to use both fractional and decimal representations while addressing the tasks. However, Zazkis and Sirotic also identified a "missing link" in most of the participants' understandings of irrationality. This link stems from an understanding of and a connection between the two definitions of irrational numbers.

Related topics: Operations. Because many concepts in number theory relate to operations (e.g., multiplicity and divisibility), it may be worthwhile to take a closer look at research concerning preservice elementary teachers' understandings of operations. Zazkis and Campbell (1996a) even suggested that their participants' struggle with understanding the multiplicative structure of numbers might be the result of difficulties with arithmetic. Thus, to better understand the effects on number theory understanding, one might consider the research on preservice elementary teachers' understandings of arithmetic ideas, such as division.

Ball (1990) conducted research focusing on the depth in which preservice teachers understood division. "When studying division, students can learn about rational and irrational numbers, place value, the connections among the four basic operations, as well as about the limits and power of relating mathematics to the real world" (Ball, 1990, p. 451). Ball claims that because students typically struggle to understand division, their teachers should understand it well.

The meaning of division is related to forming groups, and there are two ways to think about this (Ball, 1990; Beckmann, 2008). Ball calls the first way the *measurement model*: when forming groups of a certain size, the number of groups is your dividend.

The second is the *partitive model*: when forming a certain number of groups, the number of objects within each equal group represents the dividend. A flexible understanding of division requires one to understand both meanings of division (Beckmann, 2008).

To investigate preservice teachers' understandings of division, Ball (1990) administered a questionnaire to 217 participants when they were about to complete their degree program. Thirty-five of these participants also partook in one-on-one interviews. On the questionnaire, only 30% of prospective elementary teachers and 40% of prospective secondary teachers correctly identified story problems relating to a fraction division statement, and 30% of those students also identified an incorrect story problem. Nearly 10% of all students did not identify a story problem at all. Ball reported that many participants misinterpreted division. Similarly, students in this study confused the same ideas during the interview. Only four of the 35 participants were able to generate an appropriate story problem, all of which were awkwardly worded. Of the individuals who correctly represented the problem, not one was a prospective elementary teacher. This issue is especially disturbing considering the teachers most likely to teach division of fractions are preservice elementary teachers. In general, the participants in Ball's study were particularly weak with division of fractions, and she concluded they did not possess the knowledge and mastery of division necessary to teach for understanding.

Other researchers have conducted similar studies to investigate preservice elementary teachers' understandings of operations (Graeber, Tirosh, & Glover, 1989; Simon, 1993; Tirosh & Graeber, 1989), and all suggested that the participants' conceptual understandings of certain topics in mathematics are weak. The claim that preservice teachers struggle with understanding such vital concepts in elementary mathematics education suggests that perhaps similar difficulties exist in other areas of mathematics, for example, number theory.

In-Service Elementary Teachers' Understandings

Very little research exists that accounts for in-service elementary teachers' understandings of number theory, one of which was conducted by Leiken (2006). Using "learning through teaching" as her theoretical framework, Leiken asked "Nurit", a fourth grade teacher, to teach her students about prime and composite numbers so that Leiken might observe any progress in Nurit's understanding of the material. Nurit chose to introduce the topic to her students through the "Sieve of Eratosthenes", a task that allows students to discover prime numbers through eliminating composite numbers. While Nurit claimed to know what a prime number was, she had never encountered the Sieve task before. To document any change to Nurit's understanding, Leiken conducted a pre-lesson interview so that she could observe Nurit's planning. Then the researcher observed the actual lesson with Nurit's students. Finally, Leiken conducted a post-lesson interview so that Nurit could reflect on differences between the planned and enacted lesson, student learning, and personal learning.

Initially in the pre-interview, Nurit claimed that all she needed to do was recall the content and insisted that she already knew it. However, after completing the task herself, which she found unexpectedly challenging, she and Leiken realized that Nurit's understanding of the primality of numbers was not as rich as Nurit anticipated. Throughout the pre-lesson interview, Nurit was concerned with not knowing more than her students and wanting to be able to anticipate what her students might discover through the activity and struggle with during the activity. She insisted that, "if I don't understand I can’t teach this as I should” (Leiken, 2006, p. 127). Among other things, Leiken noticed that the struggles Nurit anticipated her students having with the task related to her own lack of confidence with it. For instance, Nurit did not believe that her students would be able to discover the definition of a prime number without scaffolding. To relieve these struggles, Nurit reviewed these ideas in greater depth. On one occasion during the pre-lesson interview, Nurit made a false conjecture (a prime raised to a power is also prime), but then convinced herself otherwise. When one of her students made the same conjecture during the lesson, she was able to help him recognize his mistake.

While Nurit made the most progress in her understanding of prime numbers during the pre-lesson interview, she did make an important spontaneous connection while teaching the lesson. While she attempted to scaffold a student’s learning, Nurit suggested using divisibility rules to determine whether or not a number was prime. There were also occasions during the lesson that Nurit neglected to capitalize on student curiosity and limited their learning. For instance, one student asked about the number one, is it prime or not? Rather than engage her students in a meaningful discussion about this, Nurit responded that they should cross off the number one because the directions say so. Later, during the post-lesson interview, Nurit made no advances in her understanding of number theory. In fact, contradictory to the observable evidence, she insisted that she had not learned anything new and that the topic was easy.

We can glean several implications about the number theory education of preservice elementary teachers from Leiken’s (2006) case study of Nurit and the Sieve of Eratosthenes. For one, if Nurit’s understanding of the primality of numbers is any indication, the depth in which preservice elementary teachers are exposed to prime number concepts is insufficient. While typical texts used in preservice elementary teacher mathematics content courses, such as Beckmann (2008) and Sowder, Sowder, and Nickerson (2010), present the Sieve of Eratosthenes and pose multiple questions related to it, preservice elementary students may not see this in their mathematics content courses. More importantly, understanding that the number one is neither prime nor composite is extremely important, and Nurit may have done her students an injustice by not addressing this. Preservice elementary teachers should absolutely see ideas like this in their coursework.

Nurit mirrored a common belief among educators, that to adequately teach a lesson and be able to scaffold student understanding, a teacher needs to know more than her students. Leiken (2006) and others (Shulman, 1986; Hill, Ball, & Schilling, 2008) acknowledge that this "knowledge" is not limited to merely an understanding of content. Shulman suggests that teacher knowledge consists of subject matter knowledge, pedagogical content knowledge, and curricular content knowledge. Having already thoroughly explored number theory as a subject matter topic and preservice elementary teachers' understandings of that topic, the next major section of this literature review is devoted to the second type of teacher knowledge, pedagogical content knowledge.

Pedagogical Content Knowledge

Initial Conception

While content knowledge is still essential for future teachers (CBMS, 2012), Shulman (1986) proposed that teachers need knowledge beyond that of the content they teach. In particular, they require a certain type of knowledge that weaves together content and pedagogy, pedagogical content knowledge or PCK. Shulman proposed that PCK went “beyond subject matter knowledge per se to the dimension of subject matter knowledge for teaching” (p. 9). Included in this is a battery of algorithms, examples, representations, and explanations, both standard and alternative, that may help students understand a certain concept. PCK also includes

> An understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (p. 9).

Additionally, Shulman suggested that teachers need curricular knowledge or knowledge of how topics are arranged within a school year and over a longer period of time.

Since its initial conception, many researches have attempted to further conceptualize PCK (e.g., Hauk, Jackson, & Noblet, 2010; Hill, Ball, & Schilling, 2008) in mathematics by establishing subconstructs and developing frameworks. In the following sections, I elaborate on two such frameworks.

Mathematical Knowledge for Teaching

Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008) expanded on Shulman’s (1986) conceptualization of PCK as well as what comprised subject matter knowledge. They called this model mathematical knowledge for teaching or MKT (see Figure 1), and defined it as “the mathematical knowledge needed to carry out the work of teaching mathematics” (Ball, Thames, & Phelps, 2008, p. 395). Hill et al. proposed that subject matter knowledge is broken into three constructs: common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. While these constructs appear to relate to Shulman’s conceptualization of PCK, Hill et al. suggested that they are separate from PCK because one does not need to possess knowledge of teaching or students in order to possess content knowledge. The types of knowledge necessary for teaching mathematics that relate to teaching and students are referred to as knowledge of content and students, knowledge of teaching and students, and knowledge of curriculum.

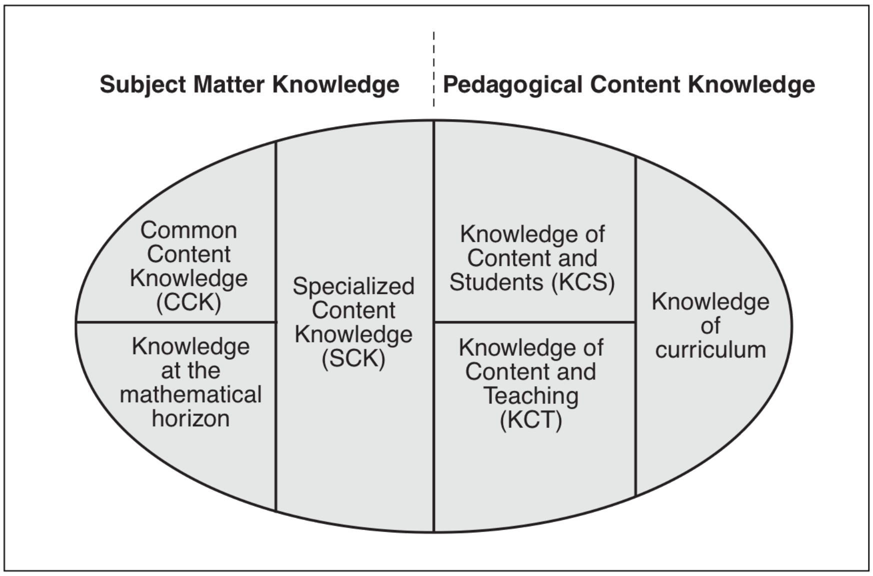


Figure 1. Domain map for Mathematical Knowledge for Teaching (Hill et al., 2008).

Most of the research that cites the subject matter knowledge portion of the MKT framework only refer to common content knowledge (CCK) or specialized content knowledge (SCK) (e.g., Hill, Schilling, & Ball, 2004; Morris, Hiebert, & Spitzer, 2009). Few researchers have conducted studies related to knowledge at the mathematical horizon (e.g., Mamolo & Zazkis, 2011), which is defined as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball, Thames, and Phelps, 2008, p. 403). Mathematical CCK is knowledge of the mathematics that is commonly known and used in various occupations. For example, most individuals know how to factor a whole number. As mathematics teachers need to be able to make accurate computations, this type of knowledge is important for teaching.

In contrast, SCK is not common knowledge and appears to be unique to the ways in which mathematics arises in the classroom. SCK is "the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (p. 377-8). For instance, a teacher demonstrates SCK by creating story problems that accurately represent the ideas of GCF and LCM in the context of real life.

Hill, Ball, and Schilling (2008) drew directly from part of Shulman's definition of PCK when they conceptualized their PCK constructs: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum. In particular, they drew from the idea that PCK is

> An understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching (Shulman, 1986, p. 9).

KCS pertains to "knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004, p. 17). It is important that teachers know the topics with which students would struggle, and the ways in which they would struggle, as well as what topics they will like. This area of MKT essentially makes up teachers' anticipations of students, their actions and understandings, as they relate to mathematics. Ball and colleagues have noted that there tend to be interactions between this type of knowledge and SCK, especially when evaluating student understanding from their statements or work. SCK plays a role in determining the mathematical accuracy of student work, while KCS is necessary for determining student understanding and depth of knowledge. In spite of this interaction, some quantitative studies (e.g., Hill, Ball, & Schilling, 2008) have attempted to isolate these constructs in instrument items, which proved problematic when ascertaining validity and reliability. For instance, a teacher demonstrates SCK by recognizing that a student's claim (e.g., the product of two numbers is equal to their LCM) is false and being able to explain why the claim is false. However, that teacher would demonstrate KCS by acknowledging why a student might think that that particular claim is true (e.g., the claim is true for relatively prime numbers, and it is possible that the student recognized this pattern).

Knowledge of content and teaching (KCT) combines knowing about teaching with knowing about mathematics and pertains to instructional decisions as they relate to mathematics. This includes knowing how to sequence the content for instruction, like which examples to use when introducing a topic versus when attempting to deepen students' understandings. KCT also includes being able to weigh the advantages and disadvantages of the different ways with which to represent mathematical concepts as well as being able to make in-the-moment decisions about whether or not to pause for clarification or pose a new task to further student learning. This requires "coordination between the mathematics at stake and the instructional options and purposes at play" (Ball, Thames, & Phelps, 2008, p. 401). For example, a teacher may demonstrate KCT by presenting GCF story problems to students who are struggling to understand the definition of the concept or by presenting students with a manipulative model for finding GCF if they are struggling to understand the procedure.

As Ball and colleague's MKT framework is proposed as a refinement to Shulman's (1986) categories, the construct of knowledge of curriculum in their framework directly coincides with Shulman's description of curricular knowledge. However, the MKT model accounts for Shulman and colleagues' later proposal that curricular knowledge be part of PCK (Ball, Thames, & Phelps, 2008). Hill, Ball, and Schilling (2008) define knowledge of curriculum as knowledge of programs developed for the teaching of a particular subject, concepts covered at a given level, and instructional materials available. According to Shulman (1986), this includes knowledge of lateral and vertical curriculum, that is, the knowledge of what is taught across content areas within a grade level and what is taught across grade levels within a content area, respectively. In its most robust form, knowledge of curriculum is similar to what Ma (1999) calls the "profound understanding of fundamental mathematics".

Consider the following example as a way of understanding the constructs of Ball and colleagues' MKT model in context: teaching adding fractions with unlike denominators. CCK is necessary for adding the fractions correctly to get the correct answer; SCK is necessary for knowing why finding a common denominator is useful in calculating the answer, or to determine why a student's alternative method will always work; KCS is needed to anticipate common student errors, such as adding across numerators and denominators; KCT is needed to decide how to help students correct these misconceptions, and knowing which representations would be most influential or accessible; Knowledge of curriculum is needed to determine grade-level appropriate ways with which to discuss adding fractions with unlike denominators, or knowing for instance that using number theory topics like prime factorization and least common multiple would not be appropriate at the third grade level.

Limitations. While Hill et al.'s conceptualization of PCK is currently the most prominent in mathematics education research, it is not without its flaws; when the researchers attempted to develop a measure of this knowledge, they encountered challenges directly related to how they defined their constructs. In their 2008 article, Hill, Ball, and Schilling reported on their methodology for creating a measure of KCS. The items for the measure related to elementary mathematics concepts like decimal numbers, fractions, and operations. The researchers also proposed general criteria for developing measures of a particular type of knowledge. They proposed two sets of criteria in creating measurement tools; the first set pertained to conceptualizing the construct, while the second set pertained to analyzing data collected with the measurement tool to determine its validity and reliability in quantifying the construct.

First, researchers should clearly conceptualize what they are trying to measure by defining it, finding evidence of its existence, and relating it to similar constructs. Next, researchers should develop items testing this particular knowledge, careful to ensure content validity. Hill, Ball, and Schilling (2008) created multiple-choice items closely related to knowledge of content and students. As the leading experts in their own conceptualization of this construct, they established content validity by identifying how each item related to their construct. The researchers found that their items fell into four categories of knowledge of content and students: common student errors, students' understanding of content, student developmental sequences, and common student computational strategies. After creating their instrument, the researchers administered it to a large sample of elementary school teachers participating in professional development institutes.

To meet the second set of criteria, Hill, Ball, and Schilling (2008) analyzed the data. Among the methods the researchers used were factor analysis and item response theory (IRT) measure construction, which pertained to scaling. After finding clear proof of multi-dimensionality, the authors admitted that their construct was not clearly conceptualized, keeping the measurement tool from differentiating between it and other forms of teacher knowledge. Hill et al. suggested the lack of research concerning knowledge of content and students may have negatively affected how they conceptualized the construct. Were there additional information available concerning knowledge of content and students, the authors may have been able to differentiate it more clearly. According to the framework the authors used, the items are dependent on the criteria for developing them, which is dependent on the conceptualization of the construct. Because the items themselves were unreliable, the authors admitted that very little of their results were valid. The only result in which the authors felt confident was that KCS exists and is complex in nature.

Other Conceptualizations of Pedagogical Content Knowledge

Implicit in the Hill, Ball and Schilling's (2008) model of MKT is an assumption that separation or sanitization between constructs is possible. Hauk, Jackson, and Noblet (2010) suggested that this may have contributed to Hill et al.'s difficulties in developing a measure that produces valid and reliable results. They proposed an alternative conceptualization of PCK that may alleviate some of the trouble Hill et al. experienced in their instrument development. Among the differences between the two proposed frameworks, Hauk et al. proposed that the constructs within PCK are not mutually exclusive, and thus have slightly different definitions and names, and they proposed an additional construct within PCK. Hauk et al. referred to KCS as anticipatory knowledge, KCT as implementation or action knowledge, and knowledge of curriculum as curricular content knowledge (see Figure 2 for the proposed mapping between the MKT model and Hauk et al.'s constructs).

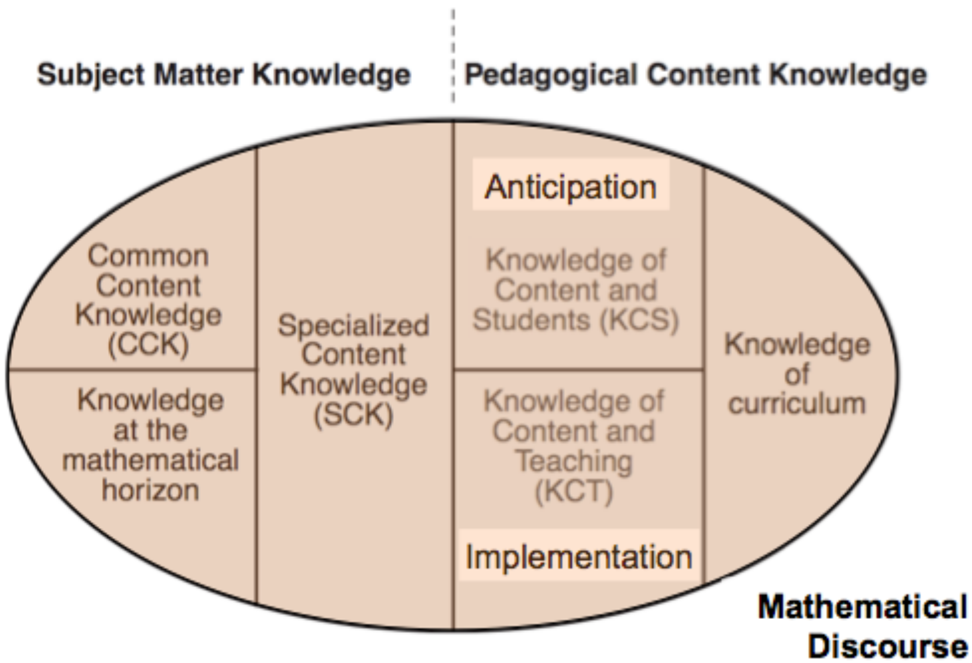


Figure 2. Comparison between Hill, Ball, & Schilling's (2008) Mathematical Knowledge for Teaching and Hauk, Jackson & Noblet's (2010) teacher knowledge constructs.

The primary difference between Hill, Ball, and Schilling's (2008) and Hauk, Jackson, and Noblet's (2010) constructs were whether or not they could be isolated from other types of mathematical knowledge for teaching; Hauk et al. proposed that they could not. This is an especially important distinction to make when creating instrument items that investigate PCK. While an item may pertain to one construct more than the others, each construct is so integrally related to the others that their presence may be implicit.

For instance, while an item may not target a particular construct, a teacher participant may draw from that type of knowledge in an unanticipated way to respond to the item.

Due to the subtle difference in the underlying assumptions of the two theories on PCK, Hauk, Jackson, and Noblet (2010) defined some of their constructs differently than Hill, Ball, and Schilling (2008). *Anticipatory knowledge* is an awareness of, and responsiveness to, the diverse ways in which learners may engage with content, processes, and concepts. This view of anticipatory knowledge is similar to KCS though the “focus is on relational understandings teachers have more than declarative or procedural knowledge about students and content” (Hauk, Jackson, & Noblet, 2010, p. 3). *Action knowledge,* like Ball and Bass’ (2000) notion of “knowledge for practice” and KTC includes knowledge about how to adapt teaching “according to content and socio-cultural context and *enact in the classroom* the decisions informed by content, discourse, and anticipatory understandings” (Hauk, Jackson, & Noblet, 2010, p. 3).

Hauk, Jackson, and Noblet (2010) also proposed an additional construct not accounted for by Hill et al., discourse knowledge, knowledge of the culturally embedded nature of inquiry and forms of discourse in mathematics (both inside and outside the educational setting). This type of knowledge includes knowledge of syntax and symbols, but it also includes knowledge of navigating within the realm of mathematics, how to generate examples using a mathematical definition, for example. The researchers proposed that this construct was embedded throughout Hill et al.’s domain of mathematical knowledge of teaching (see shaded region in Figure 2).

Another alternative to Ball and colleagues’ MKT, Davis and Simmt (2006) offered a theoretical discussion of something they called “teachers’ mathematics-for-teaching", using complexity science as a framework for interpretation. They illustrated four intertwining aspects of teachers' mathematics-for-teaching: "mathematical objects," "curriculum structures," "classroom collectivity," and "subjective understanding." They conjectured (1) that a particular fluency with these four aspects is important for mathematics teaching and (2) that these aspects might serve as appropriate emphases for courses in mathematics intended for teachers. While they appear to be proposing a different but similar-looking theory on PCK, Davis and Simmt did not make connections to some of the other already well-received theories pertaining to mathematical PCK, namely MKT (Hill, Schilling, & Ball, 2004).

Pedagogical Content Knowledge
and Teacher Effectiveness

Many researchers have shown that teachers have an effect on student learning (e.g., Hill, Rowan, & Ball, 2005; Nye, Konstantopoulis, & Hedges, 2004; Speer & Wagner, 2009; Wong & Lai, 2006). However, they have struggled to find a clear connection between teachers' subject matter knowledge and their students' achievement (Wong & Lai, 2006). Hill and Ball (2004) suggest that teacher quality may not be determined by standardized test performance so much as the connectedness of their knowledge, whether it is procedural or conceptual, or the "interplay between teachers' knowledge of students, their learning, and strategies for improving that learning" (p. 332). There does appear to be evidence linking student achievement to teacher PCK (Hill, Rowan, & Ball, 2005; Speer & Wagner, 2009; Wong & Lai, 2006). While researchers who study PCK continue to find that subject matter knowledge influences PCK (e.g., Ball, 1990; Deon, 2009; Ma, 1999; Morris, Hiebert, & Spitzer, 2009), this influence has yet to be substantially investigated enough to also establish the link between teachers' subject matter knowledge and student achievement.

Nye, Konstantopoulis, and Hedges (2004) created and administered a survey to determine teacher effectiveness amongst K-3 classrooms. They found statistically significant ($p < .05$) differences in student gains in mathematics between classrooms and schools, but not within classrooms. Given that the researchers accounted for student factors like socioeconomic status and classroom size, this could have implied that there exist teacher factors that affect student achievement in math. While Nye et al. explored teacher factors such as education and experience on student achievement, they only found statistical significance amongst third grade students. One factor that Nye et al. did not explore was teacher knowledge, both content and PCK.

Hill, Rowan, and Ball (2005) developed a measure focusing on SCK as well as skills used in teaching mathematics (PCK). Using first and third grade student achievement scores, the researchers conducted a quantitative study and found that teachers' mathematical knowledge was significantly related to student achievement gains in both first and third grades. Wong and Lai (2006) used different types of data and had similar findings. They investigated factors affecting mathematics teaching effectiveness among pre-service primary mathematics student-teachers. Data sources included the Mathematics Teaching Supervision Form and post lesson interviews. In a statistical analysis, this study found that PCK, and not subject matter knowledge, is a crucial factor in effective mathematics teaching.

Speer and Wagner (2009) examined the nature of the knowledge required for college mathematics instructors to perform proficient analytic scaffolding during classroom discussion in response to evidence in the literature describing teachers' difficulty mastering analytic scaffolding. For their theoretical perspective, the researchers used a combination of research perspective on teacher knowledge (e.g., Hill, Ball, & Schilling, 2008) and orchestrating classroom discussion (e.g., Williams & Baxter, 1996). Speer and Wagner conducted a case study of one college mathematics instructor, Gage, who had experience teaching differential equations from a traditional differential equations text and was teaching from an inquiry-oriented text for the first time. The inquiry-oriented text influenced the use of small group work and whole-class discussion more so than the traditional text. Data included video-recorded classroom observational data and interviews between one researcher and the instructor. The researchers used grounded theory methods to analyze their data, focusing their coding on important concepts from the research like, PCK, SCK, CCK, and social and analytic scaffolding. Speer and Wagner defined analytic scaffolding as “scaffolding of mathematical ideas for students” (p. 534) and defined social scaffolding as “scaffolding norms for social behavior and expectations regarding discourse” (p. 534). One limitation of Speer and Wagner’s study is that constructs like SCK and KCT are defined for K-12 teachers who have taken pedagogy courses, not college instructors. Using these constructs in their analysis without reconceptualizing them may not have been reasonable.

Speer and Wagner (2009) found that, in spite of Gage’s superior understanding of the mathematics of the course, he struggled to use analytical scaffolding to advance students’ understandings of differential equations during classroom discussion. To better understand the nature of this struggle, the researchers identified four key components involved in analytic scaffolding. The first component involves understanding the ideas that students present during discussion, something that Gage occasionally struggled with. This greatly hampered Gage's success in identifying relevant connections between students' ideas and the discussion goals, as well as any attempt to scaffold students' understandings so as to achieve those goals, components 2 and 3, respectively. The fourth component, the prudent selection of "which contributions to pursue among all those available" (p. 536), was also limited by Gage's struggle to understand students' ideas and reasoning. These component practices of analytic scaffolding appear to be linked to the subcontructs of PCK, such as KCS and KCT. However, Gage's struggles with each of these components suggests that mathematical understanding alone is not enough to scaffold students' understandings of mathematics concepts.

As the research has identified a link between teachers' mathematical PCK and students' mathematical understandings, it is important to summarize studies that investigate teachers' mathematical PCK. In the following section, I chose to summarize two studies pertaining to this area of research.

Research Investigating Teachers' Pedagogical Content Knowledge

Most of the research on PCK pertains to theory on the construct or is primarily quantitative in nature. Qualitative research methods have been used for more exploratory studies, like with Deon's (2009) dissertation where she proposed frameworks with which to investigate PCK. More often, qualitative methods have been used in support of quantitative findings, as with Hill, Ball, and Schilling's (2008) attempt to develop a measure of KCS. Quantitative studies such as theirs usually share the goal of either developing a measure of PCK or using a measure of PCK to determine the effects of some sort of professional development. In the sections that follow, I have synthesized a sampling of the research related to elementary teachers' PCK.

Qualitative studies. Very few studies make extensive use of qualitative research methods to investigate teachers' PCK, let alone qualitative studies that investigated preservice elementary teachers' PCK in mathematics. Among those few studies is a dissertation by Rhoda Deon (2009), who developed frameworks for investigating how preservice and in-service K-8 teachers developed and used their PCK pertaining to assessing student-generated combinatorial representations, and a study by Morris, Hiebert, and Spitzer (2009), who explored preservice elementary teachers' experiences unpacking lesson-level mathematical learning goals. In a third study, Fauskanger and Mosvold (2013) investigated teachers' understandings of the equals sign by opening up MKT test items.

Deon's participants consisted of preservice and in-service elementary teachers. Some of her preservice participants were working towards mathematics concentrations, but most were not. She conducted interviews and administered an open response survey to collect the majority of her data. During the one-on-one interviews with 20 participants, Deon presented participants with a counting problem, "You have 4 different colored buttons: red, black, tan, and purple. How many groups of 3 different colored buttons can you make?" (p. 198). Deon did not require participants to solve the problem themselves, but rather asked them to consider two different student solution strategies to the problem. After comparing and contrasting the solutions, she asked participants to provide an alternative solution. Both of these tasks were designed to gain insight to how the participants perceive the diverse ways in which students can solve combinatorial problems, an aspect of anticipatory knowledge as described by Hauk, Kreps, Judd, Deon, and Novak's (2006) study. Their proposed model for mathematical PCK is an earlier version of Hauk, Jackson, and Noblet's (2010), with the primary difference being in terminology. What Hauk, Jackson, and Noblet refer to as "discourse knowledge", Hauk, Kreps, and Judd et al. referred to as "syntax knowledge".

In the next part of her interviews, Deon (2009) asked her participants to consider what they would say about the two student solutions if they were talking to a junior colleague. For instance, she asked how the participants might help their junior colleagues understand the students as learners and how they might help their junior colleagues to help the students with their understandings. These tasks provided participants an opportunity to demonstrate their action knowledge (Hauk, Kreps, & Judd et al., 2006). Because there are some that claim one cannot demonstrate action knowledge or KCT unless actually in the act of teaching (e.g., Hill, Ball, & Schilling, 2008), it may be more precise to refer to this as pre- knowledge for action.

Deon proposed frameworks relating representation to PCK as her inductive hypothesis and used her data to verify or falsify the frameworks and adjust them as necessary. The final version of her Framework for Representational Activity (FRA) (depicted in Figure 3) proposes the ways in which mental structures of internal and external representations and of concept image and concept definition (Tall & Vinner, 1981) for combinatorial representational activity interact.

Deon differentiated between thinking about mathematics (internal representation) and communicating about mathematics (external representation), insisting that while internal representations cannot be observed like external ones, they can be inferred. Both are an attempt at grasping a concept, which is why Deon's FRA depicts interaction between the representational activity and the concept image. Because the research of representation theory (e.g., Goldin, 1998) proposes that both cognition and affect influence both internal and external representation, and due to undeniable evidence of this in her pilot study, Deon accommodated for these constructs within her framework.

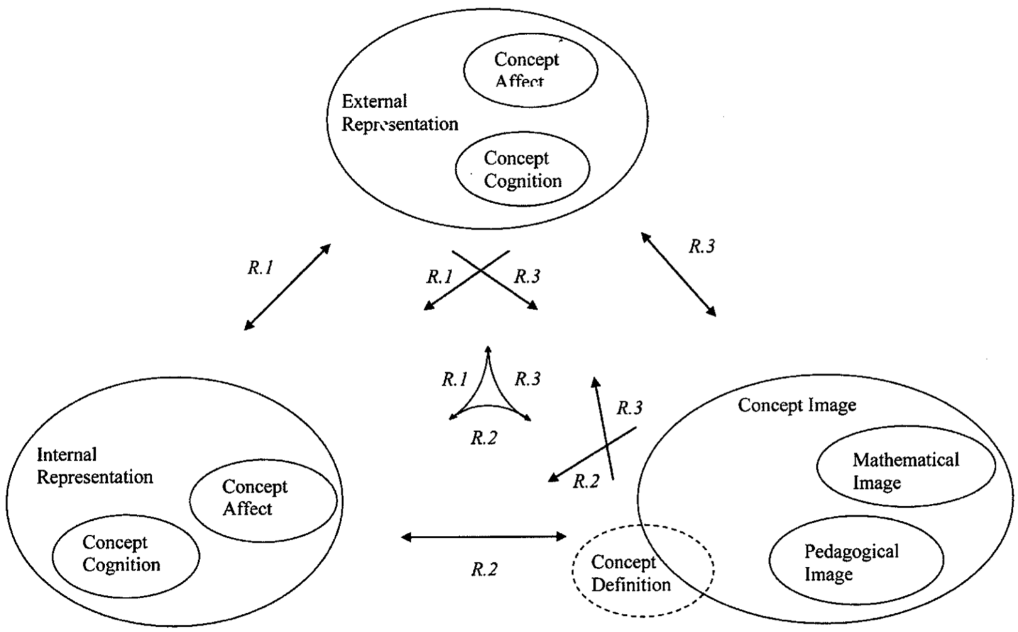


Figure 3. The Framework for Representational Activity (Deon, 2009).

Deon's (2009) second proposed framework, the Intentional System for Teaching Practice (ISTP), attempts to represent the mitigation or interaction between teachers' anticipations about teaching, referred to as preparation and intentions, and their actual actions or implementation of that knowledge (see Figure 4). In addition to theories on teacher PCK (Hauk, Kreps, & Judd et al., 2006), the underpinnings of this framework stem from research regarding self-regulation, or "the ability to create and maintain the intentions and commitments necessary to achieve stated goals" (Deon, 2009, p. 24). This involves mitigating factors such as thoughts, feelings, and actions until the objective is achieved. The goal for classroom instruction is to "unpack a concept to promote student learning and aid students in packing the knowledge for themselves" (p. 27). However, when interacting with student work, the opposite is true, and the goal becomes unpacking student work to evaluate understanding. The "I" interactions depict the different ways in which a teacher's preparation and practice interact to achieve the various goals of teaching. Deon also acknowledged that the framework is situated in cultural milieu, that environmental factors will influence cognition, affect, and ideas about what constitutes effective classroom strategy.

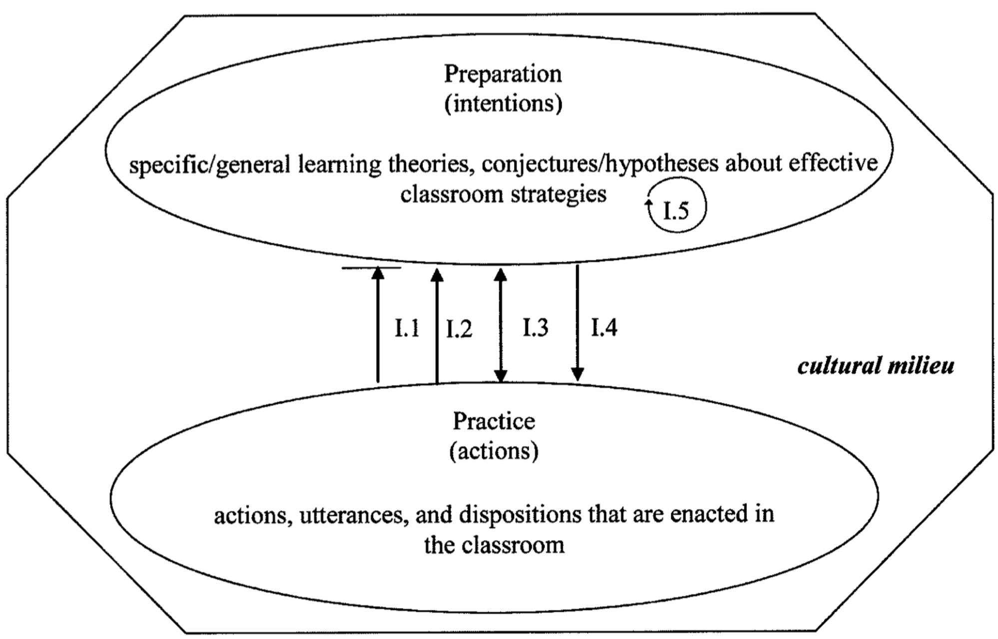


Figure 4. The Intentional System for Teaching Practice (Deon, 2009).

Deon (2009) designed her two frameworks to use as lenses through which she could examine how preservice and in-service elementary teachers built their anticipatory and action knowledge. They were the core result of her dissertation, but she made a few other important observations as well. For instance, her data supported the findings of some other researchers (e.g., Ball & Bass, 2000; Ma, 1999) that mathematics content knowledge influences mathematical PCK. However, the combination of Deon's frameworks provided a unique perspective on how teachers may be using and building their PCK. Deon's frameworks allowed her to differentiate between the participants various attempts to validate student work. Not unsurprisingly, the more packed or compact a student's combinatorial representation was, the more Deon's participants appeared to struggle in unpacking it. However, representations that demonstrated at least some of the student's process, that is, partially unpacked representations, posed less of a cognitive and affective struggle. Deon also found that participants with a mathematics concentration tended to draw from their mathematical image more than their non-mathematics concentration counterparts.

In their study, Morris, Hiebert, and Spitzer (2009) claimed that preparing future teachers to learn from teaching was tantamount to being successful in teaching. Towards this aim, the researchers designed a study that allowed them to evaluate how preservice elementary teachers unpacked learning goals into their constituent parts, or "subconcepts" (p. 493). Morris et al. claimed that recognition of these subconcepts would enable teachers to develop lesson plans and assessment tasks around these ideas as well as to develop their anticipations about student learning with regard to the overall learning goal. In particular, the researchers claimed that their study investigated an area within SCK, but they cited the importance of studying SCK based on its relationship to KCS and KCT (Hill, Ball, & Schilling, 2008).

To investigate preservice elementary teachers' successes and challenges in unpacking lesson goals, during two 2-hour sessions thirty undergraduate participants of Morris et al.'s (2009) study completed four written tasks pertaining to fraction and decimal concepts. One task asked that participants produce ideal student responses to given problems, responses that would convince them that the student understood the learning goal. A second task asked participants to read a lesson transcript and evaluate an incorrect student response by stating what she should understand but does not, which relates to unpacking lesson goals. The third task posed correct student work, and the participants were asked to evaluate what subconcepts the student does and does not necessarily understand. The last task required participants to read a lesson transcript, evaluate the lesson, revise part of it, and justify their revisions. Again, the goal of the task was to determine if participants could accurately deconstruct the lesson goal and identify the subconcepts within the lesson.

While their data collection tool was open response, Morris, Hiebert, and Spitzer (2009) coded and scored responses and used a quantitative analysis to support their qualitative findings. The researchers found that participants did not spontaneously unpack the learning goals in order to address the tasks. For some tasks, this occurred in spite of evidence that the subconcepts were accessible to all participants. For others, it was evident that some participants did not grasp the subconcepts themselves, the concept of one whole when working with fractions, for example. This further supports other's findings that subject matter knowledge influences PCK (e.g., Deon, 2009; Ma, 1999). Morris et al. also found that while most participants referred to one or more subconcepts during each task, they were more likely to reference them if they were present in a more supportive context. For example, if a participant solved a problem themselves, they could more easily identify the subconcepts from their solution. In all, the researchers found that unpacking learning goals was not a skill that came naturally for preservice elementary teachers and they suggested that instructors design lessons with the goal of developing this skill explicitly due to its importance in preservice elementary teachers' future careers.

In a third study, Fauskanger and Mosvold (2013) explicitly drew from Ball and colleagues MKT constructs and multiple-choice items. Feeling that multiple-choice items provide limited insight into teacher knowledge, the researchers investigated the question, "What can be learned about teachers' knowledge of the equal sign by analyzing their responses and written reflections to MKT items?" To answer this question, Fauskanger and Mosvold asked 30 teachers of a various grade levels to solve five open-ended problems. Additionally, the researchers asked participants to reflect, in writing, whether the content of the problems was relevant to the grade they teach and whether the problems reflect knowledge that is important to them as teachers. These tasks were designed to ascertain participants' relational and operational understandings of the equals sign, drawing from Knuth et al.'s (2006) definitions of relational and operational understanding. The reflection questions posed in the tasks were designed to elicit PCK.

Twenty-six of the thirty participants answered all five problems correctly. Three middle school (fifth to seventh grade) teachers answered one problem incorrectly, and one teacher who taught grades eight through ten answered three tasks incorrectly. The three middle school teachers demonstrated operational understandings of the equals sign, but not relational understandings. The item they incorrectly responded to, $14 + 5 = 19 + 5 = 24 + 5 =$ ___, made use of the equals sign operationally like an "equality string." However, the relational meaning of the equals sign is not maintained throughout the mathematical statement, which is problematic. The middle school teachers did not recognize the statement as being problematic. The researchers claimed that the teacher who incorrectly answered three of the five tasks demonstrated a low MKT of the equals sign, especially where its relational meaning is concerned.

Upon first impression, the fact that a majority of the participants answered all items correctly could indicate a sound understanding of the operational and relational meanings of the equals sign. When the researchers analyzed participants' reflections they got a richer picture of the teachers' MKT. Most participants further demonstrated operational and relational understandings of the equals sign in their reflections. Additionally, 12 participants discussed how the content of the items was a precursor to algebra, which the researchers coded as relating to horizon content knowledge (Ball, Thames, & Phelps, 2008). Other teachers demonstrated knowledge of content and curriculum, KCT, and KCS (Ball, Thames, & Phelps, 2008) in their responses. Fauskanger and Mosvold (2013) concluded that more can be learned about teachers MKT by opening up test items. In particular, teachers can draw upon different aspects of MKT when responding to tasks, and sometimes these constructs might differ from those that researchers intended to test.

Quantitative studies. As with Hill, Ball, and Schilling (2008), many quantitative studies related to PCK focus on the development of a measure of PCK. As Ball and colleagues were at the forefront of this movement, other researchers (e.g., Bell, Wilson, Higgins, & McCoach, 2010) borrowed items or frameworks from them to ensure the validity and reliability of their own instruments. Those researchers conducting quantitative studies whose main goal is not to create an instrument typically focus on using their instrument to determine the success of professional development or some other teacher-knowledge related treatment, as it was with Bell, Wilson, Higgins, and McCoach's (2010) study.

While Hill, Ball, and Schilling (2008) found it difficult to ensure reliability and validity when creating an instrument that isolated a particular kind of PCK, there has been a reasonable amount of success in creating more general measures of PCK (Hill, Schilling, & Ball, 2004; Hill, Rowan, & Ball, 2005), even outside of Hill and Ball's research (Bell, Wilson, Higgins, & McCoach, 2010). Bell et al. conducted a quantitative study to measure the effects of a specific professional development program, Developing Mathematical Ideas (DMI). DMI is a widely used professional development program for K-8 teachers in mathematics. During DMI seminars "teachers work[ed] with a trained facilitator to learn about specific mathematics, learn about children's ideas about that mathematics, and analyze how to approach these ideas in the classroom" (p. 482). Drawing from the theoretical and empirical research of Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008), Bell et al. addressed the question of which MKT teacher-participants acquired through their participation in the DMI program.

The researchers created their instrument using multiple-choice items developed and validated by the Learning Mathematics for Teaching Project (LMT) and open-response items adapted from those initially developed by the DMI program experts. Because the DMI program was designed to develop specialized content knowledge, knowledge of content and students, and knowledge of content and teaching, each of the items chosen for the instrument was designed to measure one of these types of knowledge. To choose the multiple-choice items from the LMT item database, the researchers identified items belonging to the content strand related to a particular DMI seminar that were designed to measure specialized content knowledge and knowledge of content and students. From these, the researchers selected the items with the highest reliability in previous studies, eliminating others due to overrepresentation and focusing on “questions that required analysis of student work, instructional tasks, representation tools, and content” (p. 491). The final product was a compilation of 13 items pertaining to specialized content knowledge and seven items pertaining to knowledge of content and students. The researchers relied on the already proven integrity of the items they chose to assert reliability and validity.

Discussion

Prior to this dissertation, no one had yet conducted research with the intention to investigate preservice elementary teachers’ PCK in number theory. One could use the aforementioned research findings concerning preservice elementary teachers’ content knowledge in number theory (e.g., Zazkis & Campbell, 1996a) and their PCK in other areas of mathematics (e.g., Deon, 2009) to propose some theories. For instance, due to participants’ apparent struggle in unpacking various representations of number theoretical ideas (e.g., multiplicative structure, Zazkis & Campbell, 1996a), these same participants may struggle to unpack student work related to these ideas and in turn impacting their KCS (Hill, Ball, & Schilling, 2008) or anticipatory knowledge (Hauk, Jackson, & Noblet, 2010). As Deon (2009) found, both preservice and in-service teachers intentionally use their anticipations about students to inform practice, this struggle to unpack number theory representations may also impact participants’ KCT (Hill, Ball, & Schilling, 2008) or action knowledge (Hauk, Jackson, & Noblet, 2010). However, as no one had focused their investigations on the content knowledge or PCK of preservice elementary teachers emphasizing in mathematics, I could not predict their understandings of number theory nor their anticipations or intentions for teaching it. Deon suggested that these participants may make more use of their mathematical images than their non-mathematics concentration counterparts, but because mathematical images allow for non-standard ideas about mathematics, this is hardly indicative of participants' knowledge for teaching. We can, however, hope that their increased coursework positively impacted their content knowledge and mathematical PCK in general.

In the following chapter, I discuss the methodology for my dissertation and justify my choices with evidence from my pilot study and other research findings.

CHAPTER III

METHODOLOGY

This chapter outlines the study's methods, which are designed to answer the research questions:

Q1 What is the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory topics taught at the elementary level?

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

I refer to the pedagogical content knowledge (PCK) of preservice teachers as "potential PCK" to distinguish it from the more robust or well-developed PCK of an in-service teacher. The existing research concerning preservice elementary teachers' potential PCK of number theory is limited. According to Patton (1990), "in new fields of study where little work has been done, few definitive hypotheses exist and little is known about the nature of the phenomenon, qualitative research is a reasonable beginning point" (p. 193). In cases such as these, Patton calls for *exploratory* qualitative research to investigate the nature of the phenomenon in question. Qualitative inquiry helps researchers understand

and explain the *meaning* of a phenomenon, while interfering as little as possible (Merriam, 1998). This method of inquiry is often holistic in nature, as opposed to quantitative inquiry, which breaks apart the phenomenon into its component parts and represents them as variables (Merriam, 1998). As the nature of one's understanding cannot truly be represented through numbers and variables, qualitative inquiry was better suited to address my research questions.

To begin the chapter, I describe the qualitative approach that allowed me to best answer the research questions, outlining the major design components. Next, I state the theoretical framework for the study and discuss my role and perspective as a researcher. I then discuss the setting, participant selection, data collection, and data analysis. I also address the ethics of my study and the efforts made to ensure the quality and credibility of my research.

Interpretive Case Study

I employed a case study design "to gain an in-depth understanding of the situation and meaning for those involved" (Merriam, 1998, p. 19). I focused my investigation on the number theory understandings of students enrolled in a single section of a number theory course designed for preservice elementary teachers. Students in this course shared experiences with the same instructor and with the same material, although they may have interpreted their experiences differently. Each student's understanding of number theory was not only likely to be influenced by the text and the instructor, but by other student's understandings as well. This shared experience constituted the "bounded system" required of a case study (Creswell, 2007; Merriam, 1998).

My study best fit what Merriam refers to as an interpretive case study, because I explored preservice elementary teachers' understandings. The descriptive data of the case study are used to,

> Develop conceptual categories or to illustrate, support, or challenge theoretical assumptions held prior to the data gathering. If there is a lack of theory… a case study researcher gathers as much information about the problem as possible with the intent of analyzing, interpreting, or theorizing about the phenomenon (Merriam, 1998, p. 38).

As stated previously, little is known about preservice elementary teachers' understandings of number theory, especially as it relates to potential pedagogical content knowledge (PCK). Thus I collected, analyzed, and interpreted task-based interview and observation data, and theorized about my interpretations.

Case study methodology does not dictate data collection or analysis procedures (Merriam, 1998). However, the data collected during a case study should be dense enough to provide thick, rich descriptions of the phenomenon in question. Oftentimes, this means collecting multiple types of data. I conducted interviews and observations as well as gathered samples of student work to accomplish this. As with most case studies, I gathered a tremendous amount of data over a period of time. I needed to continuously manage and organize these data, which allowed me to simultaneously conduct a preliminary analysis of the data. Any synthesis of these data should feel authentic to the reader (Merriam, 1998), thus results of my analysis included dense description, interview quotes, and images from documents to support my findings. I more thoroughly describe these procedures in the sections to come.

Constructivism and Emergent Perspective

I approached my research from a constructivist epistemology, believing that "meaning is not discovered, but constructed" (Crotty, 2003, p. 42). This lends itself well to the social constructivist perspective that acknowledges that historical and cultural norms aide in the participants' formation of meaning (Creswell, 2007; Crotty, 2003). This also blends well with Merriam's (1998) concept of interpretive research. She claimed that "school is a lived experience" and that "multiple realities are constructed socially by individuals" (p. 4). I investigated participants' content knowledge for teaching, and given teachers are highly reliant on their individual understanding of the content that they teach, participants' individual understanding is also an important aspect of this study.

There is both a psychological and a social aspect to how a student constructs his or her understanding of a concept, but psychological perspectives, such as radical constructivism, and social perspectives, such as social constructivism or socioculturalism, have competing value systems. In his comparison of the constructivist and sociocultural theories, Cobb (1994) identified areas in which the theories differ. For example, constructivists claim that learning is a matter of individual cognition, while socioculturalists claim that learning occurs in the "individual-in-social-action" (p. 13). The two groups also differ in the meaning of certain words, such as activity. "Activity," to a constructivist, refers to a student's "sensory motor and conceptual activity," while the same word refers to "participation in culturally organized practices" to socioculturalists (p. 14). Another difference lies in different meanings of the word "understanding". To socioculturalists, understanding is a shared meaning that has been co-constructed by the collective, but to constructivists understanding means to construct knowledge individually, thus an understanding is unique for each person. In spite of such fundamental differences, Cobb suggested that the theories can be used pragmatically to complement one another.

In an effort to demonstrate the complementary aspects of constructivism and socioculturalism, Cobb (1994) referred to Rogoff's (1990) idea of internalization and von Glasersfeld's (1995) ideas of reflective and empirical abstraction. Rogoff drew from the sociocultural perspective to claim that social interaction and internalization are not separate processes, implying that mathematical learning is a process of active construction. Cobb suggests that one can reach a similar conclusion from interpreting von Glasersfeld. His notion of empirical abstraction involves learning "outside" the body through an object, while reflective abstraction is learning that takes place "inside" (Cobb, 1994, p. 16). Cobb claimed that his discussion of the various interpretations of socioculturalism and constructivism "indicated that sociocultural analyses involve implicit cognitive commitments, and vice versa. It is as if one perspective constitutes the background against which the other comes to the fore" (p. 18). Cobb then surmised that if we want our theorizing to be reflexively consistent with the theories we develop, we must take a more pragmatic approach. Like "wielding vocabulary," this may permit us to better understand what we see rather than suggest the nature of the things we are naming.

Later, in a publication with Yackel, Cobb (1996) proposed an emergent perspective that incorporates both the psychological and social perspectives to varying degrees. Through their empirical investigations, Cobb and Yackel identified certain aspects of classroom microculture and their corresponding psychological constructs that all contribute to the construction of student understanding at the classroom level, shown in Table 1. The researchers did not mean for "the classroom level" to be a physical location; instead it refers to the types of activity in which students are engaged. Cobb and Yackel claimed that researchers can use this framework to explain the social influences on the individual's developmental understanding.

Table 1

An interpretive framework for analyzing individual and collective activity at the classroom level (Cobb & Yackel, 1996)

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

The "students" in my study are preservice elementary teachers, and the developmental understandings that I attempted to capture via my investigation related to both content and PCK. As each teacher brings her or his individual understanding into her or his classrooms, the individual perspective is the focus of this study. However, as Cobb (1994) and Cobb and Yackel (1996) have identified, the social aspects of student learning can help to explain an individual's understanding. The preservice elementary teachers in this study were co-participants in the construction of the number theory understandings of the students in the class, especially during group work. While each individual may have internalized this co-constructed knowledge differently, Cobb suggests that the end result was highly affected by the social interaction that took place. For example, if a group of students is given a challenging task, they may be more successful working on it together than they would individually. That joint success might affect each individual's understanding of the task or concept differently than an individual failure. This aligns well with my own views on understanding; I believe that an individual constructs knowledge differently based on their own experiences, but that social interaction can frame and influence those experiences.

The theoretical framework that I have used for my study is a slightly altered version of Cobb and Yackel's emergent perspective, shown in Figure 5. The number theory classroom observations I conducted allowed me to document any collective understandings, such as norms and classroom practices. Collective understanding was not the focus of this study, so I have shown the facets of the social perspective within a dotted circle. Similarly to Cobb (1994) and Rogoff (1990), I operated under the assumption that each participant internalized these ideas in his or her own way. I have represented internalization through the solid line in the figure. This internalization was evident in the mathematical activity within individual assignments and during the one-on-one interviews. Individual understanding is the focus of my study, thus I have shown the facets of the psychological perspective within a solid circle.

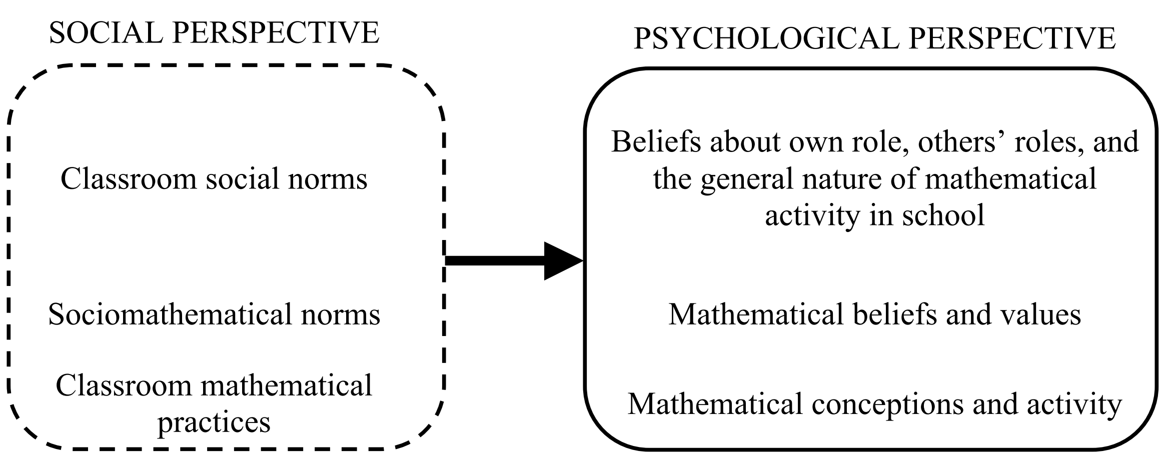


Figure 5. Adjusted emergent perspective.

Role and Perspective of the Researcher

The credibility of qualitative research hinges largely on the perspectives of the person doing the fieldwork. Patton (1990), among others, claim that “objectivity… can limit one’s openness to and understanding of the very nature of what one is studying” (p. 48). Rather, he suggests that researchers take a stance of empathic neutrality, which incorporates understanding through empathy and acknowledging ones biases to attempt neutrality. An important aspect of this is delineating personal experiences that may inform or hinder data collection and analysis, which I address in this section.

Several experiences have influenced my perspectives towards this research. Like the participants in this study, I was once a student in a teacher certification program. However, I completed a secondary certification program in mathematics rather than an elementary certification program with a concentration in mathematics. The coursework varies between the two, but each contains content, methods, and teaching experience components. My own participation in a mathematics education program provides insight to my participants’ mentalities as preservice mathematics educators. For me, my certification program was another set of hoops to jump through before I could seek employment as a teacher. My academic goals were grade-oriented rather than learning-oriented. It was not until I was almost through my student teaching experience that my perspective changed. I regretted not attending to my coursework with the point of view, “how can I use this in my teaching? How will this help me as a teacher?” I have observed through my interactions with preservice elementary teachers that most of them are grade-oriented. Through my own personal experiences, I can both identify and sympathize with this perspective.

My teaching experiences have also molded my perspective. While my high school teaching experience was limited to teaching singular lessons for practicum courses, through my student teaching experience, and substitute teaching, I know what it feels like to be a beginning school teacher. I have also taught number theory concepts to preservice elementary teachers on many occasions. Thus, I have some understanding of how students develop their understandings of these concepts: where they tend to struggle and succeed, what questions I can pose to redirect them, and what activities I can facilitate to help them to further their understandings. These insights and experiences not only contributed to the design of the data collection tools, but they made a valuable contribution to dialogue (which occurred during group work) during the classroom observations and task-based interviews with participants. While I have noticed that my understanding of number theory and related PCK is by no means complete, I used my understanding of each to identify the connections participants may or may not have made between concepts. I used this information to ask participants the questions that revealed a full and rich description of their understandings. I also used my own number theory and PCK understandings to analyze student responses.

Setting and Participants

As with my pilot study (refer to Appendix A), I chose the Mountain State University (pseudonym) as the site for my dissertation because of convenience. My status as a PhD student at the university helped me access professors and their students. This made the data collection process easier than if I were not a student at the school.

Mountain State University elementary education majors enrolled in a number theory course for preservice elementary teachers during the Spring 2012 semester constituted the participants of this study. This shared experience constituted the “bounded system” required of a case study (Creswell, 2007; Merriam, 1998). Recall that the literature has already documented many aspects of the number theory understandings of preservice elementary teachers *without* a mathematics concentration. This study adds to the research by soliciting participants *with* a concentration in mathematics. Not only were these participants required to take more mathematics courses, but they were the only preservice elementary teachers required to take a course explicitly pertaining to number theory. I anticipated their mathematics-intensive experiences to contribute to participants’ rich responses, as it did in my pilot study. Each participant was an adult, none of whom were vulnerable. I obtained all participants on a volunteer basis.

Description of the setting. The number theory course met for 50 minutes three times per week during the Spring semester, and the course was taught by Dr. S (pseudonym) using a number theory textbook by Silverman (2001). The rectangular classroom contained 36 desks, but there were only 24 students.

Silverman’s (2001) text was an introductory number theory text designed to lead students “to think mathematically and to experience the thrill of independent intellectual discovery” (p. v). The text lent itself well to a student-centered classroom where individual and small group activity and investigation is a mainstay. It was originally designed as the text for a mathematics topics class for non-science majors. Aside from high school algebra, the course required very little background in mathematics. Thus, it appropriately started with more elementary number theory topics and got progressively more challenging. According to Silverman, the first eleven chapters were basic and included topics like categories of natural numbers (e.g., even, odd, prime, composite, square, cube, and triangular), Pythagorean triples ($\{x,y,z|x^2 + y^2 = z^2\}$, where x, y, and z are natural numbers), divisibility, greatest common divisor, the Euclidean algorithm (a process that uses linear equations to find the greatest common divisor), factorization, the Fundamental Theorem of Arithmetic (any integer greater than one can be uniquely written, independent of order, as the product of prime numbers), and congruences. Later chapters included a more thorough investigation of prime numbers, perfect numbers (numbers that are equal to the sum of their proper divisors), and the Euler phi function (phi of n is equal to the number of natural numbers less that n that are relatively prime to n), among other topics. Each chapter provided students with definitions, related theorems, examples, and explanations, as well as tasks with which students may investigate the topics further. However, the text did not make any connections to elementary school number theory teaching or learning. The target audience for the number theory course was future teachers, so the instructor supplemented the text with other materials.

Dr. S designed the course to be a problem-set driven, collaborative learning environment. On the first day of class, Dr. S provided students with a list of number theory statements that he asked students to classify as (1) false and can be shown to be false, (2) true and can be shown to be true, or (3) true, and cannot be shown to be true. The list of statements included famous unsolved conjectures such as Goldbach's conjecture and the twin primes conjecture as well as false statements that students could disprove using a counterexample (e.g., "every number of the form $n^2 - n + 41$ is prime") and true statements that can be explained (e.g., "every number greater than six is the sum of two non-primes").

Dr. S explained that much of number theory, like these statements, is understandable, but that their solutions can be challenging. He also said, “if you don’t feel lost a little bit, I’m not trying hard enough.” He instructed students to look at the ideas presented in this course “as elementary as possible” and “in as many ways as possible”. While formal proof structure was discussed when necessary (e.g., proof by induction), Dr. S emphasized reasoning that was easy to understand and “picture proofs.”

The format of the number theory course consisted of a mixture of lecture and group work. Oftentimes, the lecture informed the group work, but occasionally the group work informed the lecture. After presenting students with a problem set, Dr. S either let students attempt the problems in groups or he lectured on related concepts first. He usually instructed students to focus on specific problems first, because they were most accessible. In subsequent lectures, Dr. S provided additional information so that students could attempt new problems. When a majority of students were struggling with an idea, Dr. S paused group work to address the class with a brief lecture or a “hint” to help guide students in the right direction.

On a typical class day, Dr. S presented new information or “hints” regarding specific problems on a problem set. Then students got together in their groups to work diligently on those problems. Students formed these groups on the first day of class, and they remained in these groups for the remainder of the semester. Most students participated in group discussion, but as is the nature of group work, some students were more vocal with their ideas than others. Each student recorded her or his own work and solutions in notebooks.

While students worked through problems, Dr. S walked around the classroom visiting each group and addressing questions. More often than not, rather than directly answer students' questions Dr. S pointed students in the right direction or gave them a question or idea to ponder. This non-directive approach to teaching encouraged students to be dependent on themselves and each other rather than on Dr. S's expertise. At the end of class, Dr. S answered questions and instructed students what to work on and/or finish before the next class.

This problem-solving process continued for days; students had more than a week (but typically less than three weeks) to complete each problem set. Although Dr. S encouraged students to discuss their work with others in their groups, each student submitted her or his own problem set on the due date. After students turned in their problem sets, Dr. S distributed typed solutions to the problem sets. When he returned graded assignments, he also occasionally presented solutions to tasks for which students struggled.

For each of the three in-class exams, Dr. S provided students with copies of old exam problems to use as a study guide. Typically, for one or two class days before an exam, students worked through the old exam problems as if they were problem sets. Dr. S answered questions and presented solutions to problems on which students struggled. The final exam was a take-home assignment that students had over a week to work through.

My role as the researcher also affected the classroom culture. Dr. S announced early in the semester, prior to me announcing my official purpose for attending the class, that I would be observing the course and sitting in on group work. Dr. S and I discussed my role as a non-direct participant-observer. I took notes along with the other students during lecture, but I also passively participated in group work. Each class day, I chose a different group to sit with and observe. If students asked me questions, I gave them hints, pointed them in the right direction, or asked them questions to ponder, but Dr. S asked me not to give them the answers. By attending every class, and by periodically sitting with each group of students, I believe that my presence in the class became routine or "normal". I felt like students accepted my role in the classroom and they never relied on my expertise during group work. Students collaborated with each other to complete their assignments; they rarely asked me for help.

Soliciting participation. I solicited the participation of students enrolled in the number theory course by addressing them in their class. I first obtained the permission of the instructor to observe the number theory course with the Instructor Consent to Participate in Research Form (Appendix B). In the third week of the semester, I introduced myself to the class and informed them of my role and my research. I planned to collect and photocopy artifacts of student work throughout the semester, so I solicited participation from students at this time. Not all of the students were preservice elementary teachers with a mathematics concentration; some students were enrolled in the number theory course as an elective. I specified that I was conducting an investigation on the number theory understandings of preservice elementary teachers so that the students taking the number theory course as an elective would not volunteer for the study. I provided each volunteer with an Informed Consent to Participate in Research Form (Appendix C) whose purpose was to obtain consent to collect artifacts. Thirteen preservice elementary teachers volunteered: Brit, Cara, Dani, Eden, Gwen, Hope, Isla, Kris, Lucy, Nina, Roxy, Shea, and Tess (pseudonyms). I refer to these participants as "document participants". Over the course of the entire semester, I photocopied each of their problem set assignments and exams. Photocopying occurred before Dr. S graded the documents, except for the first two homework assignments. Thus, I did not include participants' problems set and exam scores in my data.

Over the next few weeks, I observed the thirteen document participants to determine which ones might be particularly adept at communicating their understandings of number theory. Students were split up into five groups of four to six students during group work. The social aspect of my theoretical perspective assumes that because participants worked within a group, they shared a collective mathematical understanding with their groupmates. These assumptions, coordinated with those of the psychological aspect of my theoretical perspective, led me to believe that there were similarities (due to the collective understanding) and differences (due to how each individual internalized information) between groupmates' understandings. With the intention of adding another dimension to my analysis, I attempted to choose pairs of participants within the same group with whom to conduct one-on-one interviews. In other words, the six document participants I proposed to select for interviews came from three groups, two participants per group. This would have allowed me to compare individual participant responses with each other, but also between groups and within groups, which assumes that participants' individual understandings were affected by their groups interactions.

To select the interview participants, I first sorted the thirteen document participants according to the five groups that had been formed in the number theory classroom (see Table 2). Four of those groups had two or more document participants in them. At the time I hoped to solicit interview participants, the number theory students had completed two problem set assignments and one in-class exam. To ensure that I had as much documentation from interview participants as possible, I only wished to choose from participants who had submitted both problem set assignments. This eliminated Hope and Roxy, who were each missing an assignment. I also wanted to ensure that any interview participants would successfully complete the course. In other words, I did not wish to select participants at risk for earning lower than a C and possibly withdrawing from the course (which would lead to incomplete data). To ensure this, I only chose from document participants who had passed the two assignments and the exam. Luckily, this did not eliminate any of the remaining eleven document participants. However, after the removal of Hope and Roxy, I was left with three groups with two or more document-participants: Group 1, Group 3, and Group 5.

Table 2

Document participants organized by group

Group	Document Participants
1	Brit, Dani, Eden, Shea, and Tess
2	Cara
3	Gwen and Isla
4	Hope, Kris, and Roxy
5	Lucy and Nina

To allow for a comparison of participant responses between and within groups, I needed interview participation from two document participants (out of five) from Group 1 and all of the document participants from Groups 3 and 5. Groups 3 and 5 each had a participant (Gwen and Lucy) that was mathematically stronger than the other, in other words, their work and explanations were more frequently mathematically accurate, and as a result they scored higher on assignments. To mirror this dichotomy, I wished to also select one stronger participant and one less strong participant from Group 1. I anticipated that choosing one stronger and one less strong participant from each group would ensure that I collect a wide variety of data from participants with similar understandings (due to their participation in group work) yet understandings that clearly differed (as evidenced by their varying performance on the assessments). I chose to interview Brit (stronger) and Shea (less strong), because I felt like I had a better rapport with them than the other participants within Group 1. Each time I worked with this group, Brit and Shea were more likely to communicate their questions, ideas, and understandings in my presence and directly to me.

In the first third of the semester, after the first exam had been graded and returned, I solicited the participation of six document participants (Brit and Shea, Gwen and Isla, Lucy and Nina) for one-on-one task-based interviews. To solicit their participation, I sent an email similar to the one I sent pilot study participants (Appendix D). Unfortunately, my purposeful sampling plan had complications. Shea turned down the offer due to scheduling conflicts. Eden, another member of Group 1, appeared to have the same level of understanding as Shea, so I solicited her participation instead and she accepted. Nina also turned down the offer due to scheduling conflicts. However, because she and Lucy were the only two participants in Group 5, I had to resort to soliciting my sixth participant from another group. I had the best rapport with Cara, from Group 2, and she agreed to participate.

My attempts to add another dimension to my analysis fell short, because I no longer had three pairs of participants with shared group work experiences. Being able to compare participants' responses to the interview tasks between and within groups would have strengthened my analysis from the social perspective of my theoretical framework. However, it was not included in my proposed research design; it was merely an opportunity on which I had hoped to capitalize.

Ethics of participation. I informed students, verbally and in the consent letters, of their right to decline participation, of the procedures taken to ensure confidentiality, and of the benefits of participation. Participants were informed that they could decline participation at any time, without loss of benefit or entitlement. To protect their identities, I assigned each of the participants a pseudonym.

My advisor is storing signed Informed Consent forms in a locked file cabinet, and I am storing the original audio files from interviews on my personal, password-protected computer. After storing audio files on my computer, I immediately deleted them from the recording device. I used participants' pseudonyms to label all memos, artifacts, and interview transcriptions. To label the artifacts, I photocopied documents, blacked out the names on the copied, and placed a sticker label over the blacked-out name. I also used these pseudonyms to refer to participants during the audio-recorded interview. The pseudonyms only indicate the gender of my participants; they do not indicate ethnicity or any other identifiers.

The only document linking the participants' names to their respective pseudonyms is stored on my personal, password-protected computer. This document was destroyed immediately after all data had been recorded and cataloged according to participants' pseudonyms. I was the only one to have access to the original audio files and document linking participants to their pseudonyms. My dissertation committee may be allowed access to memos and interview transcriptions, all of which refer to participants' pseudonyms rather than their actual names. My research advisor has access to the signed consent forms, because she is storing them until the completion of my dissertation, after which she will destroy them. I will destroy the audio files after 5 years. I may retain de-identified data, including interview transcriptions and notes, indefinitely for future use and publication purposes.

I offered interview participants their choice between a $15 gift certificate to Starbucks or a $15 gift certificate to Barnes & Noble. I only selected interview participants for my dissertation from the group of participants contributing to my artifact collection, so those participants all had the opportunity to benefit from a $15 gift certificate. However, those not selected for an interview still received their choice between a $5 gift certificate to Starbucks or a $5 gift certificate to Barnes & Noble. As an additional benefit to all participants, I offered to debrief them on my findings once my study is complete. Participants did not miss instructional time, but interview participants needed to volunteer two sessions of 60-90 minutes of their time outside of class. Interviews occurred on campus, at times convenient to participants, so there were no monetary costs to participants concerning interviews.

Description of Participants. All of the thirteen document participants were elementary education undergraduates with a concentration in mathematics. To fulfill the mathematics concentration, students are required to complete: (1) a course in algebra, statistics, and probability that was designed for preservice elementary teachers; (2) a Euclidean geometry course designed for preservice elementary teachers; (3) first semester Calculus; (4) Discrete Mathematics; (5) Modern (non-Euclidean)

Geometry; (6) a mathematical problem solving course designed for preservice elementary teachers; (7) a two credit mathematics education course that focuses on number and operations; and (8) the number theory course.

Three of my participants, Cara, Gwen, and Lucy, completed the mathematics education course prior to enrolling in the number theory course. The text for that course, Van de Walle, Karp, and Bay-Williams' (2010), focused on teaching and learning mathematics as opposed to conveying mathematical content. The authors claimed that "the fundamental core of effective teaching of mathematics combines an understanding of how children learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning on a daily basis" (p. ix), so a large portion of the text was devoted to these ideas. The rest of the text focused on content: how students understand it, ways for conveying the material to students that stresses the development of conceptual understanding, and ways to connect the material to real-life, technology, and other disciplines. To supplement the text, the instructor provided video files to share with the preservice elementary teachers. These videos showcased examples of standards-based teaching, student-centered activities, and actual student thinking. They provided preservice elementary teachers with an authentic idea of what to expect from themselves and their students in the classroom. The text contained very little mention of number theory. However, it did outline how to lead an activity through which elementary school students could learn about factoring. Van de Walle, Karp, and Bay-Williams suggested selecting a composite number with several factors, then ask students to write as many multiplication expressions for the number as possible, to break up that number of counters into equal groups in multiple ways, and to create multiple rectangular arrays using that number of tiles.

As part of the mathematics education course, students observe a course on number and operations designed for preservice elementary teachers (the first course in a three-course series required of non-mathematics concentration majors) on a weekly basis. When the students of the number and operations course worked in groups, the mathematics concentration majors acted as peer tutors within the groups. When the mathematics education course met, teaching philosophy discussions emphasized collaborative and discovery learning, non-direct instruction, and building on students' prior knowledge. The number and operations course covered some number theory content, but there was more of a focus on operations with whole numbers, fractions, and integers.

The constructivist emphasis of the mathematics education course had an impact on my pilot study participants, three of whom were enrolled in the course at the time of the study. It also appeared to impact Cara, Gwen, and Lucy, the dissertation participants who had completed the course the previous semester. During interview tasks, they each referred to how the course influenced their responses to hypothetical students in student scenario tasks.

While Brit, Eden, and Isla had not taken the mathematics education course at the time of the study, they had completed the number and operations course designed for non-mathematics concentration majors. The text for the course, Beckmann (2008), devoted an entire chapter to number theory, addressing evens and odds, factors and multiples, greatest common factor and least common multiple, prime numbers and prime decomposition, as well as divisibility. The text went beyond presenting definitions and standard mathematical methods in number theory. It also posed hypothetical grade school student work for the prospective teachers to comment on, represented the number theory ideas using multiple representations, and challenged prospective teachers to think about number theory in context by solving and creating story problems. The text also connected common factors and multiples to simplifying, adding, and subtracting fractions. While this course typically explores number theory topics for one or two weeks at the end of each semester, the number theory topics were temporarily removed from the course content during the semester that Brit, Eden, and Isla most likely took the course.

Table 3

Interview participants' completed mathematics coursework and tutoring experiences

Course or Experience	Brit	Cara	Eden	Gwen	Isla	Lucy
Number & Operations Course	X		X		X	
Mathematics Education Course		X		X		X
Modern Geometry Course		X	X	X		X
Problem Solving Course		X	X	X		X
Other Mathematics Courses*	X	X	X	X	X	X
Grade School Tutoring			X	X	X	X
Peer Tutoring	X	X		X		X

* "Other Mathematics Courses" refer to Calculus, Discrete Mathematics, a Euclidean Geometry Course, and an Algebra, Probability, and Statistics course. The last two courses were specifically designed for preservice elementary teachers.

For most of the interview participants, the number theory course was their last mathematics course needed to satisfy the mathematics concentration. In addition to needing to take the mathematics education course, Brit and Isla still needed to complete a course in Modern Geometry and another course in problem solving. For a complete list of the course requirements participants had satisfied, refer to Table 3.

All of the interview participants claimed to have had experiences tutoring mathematics. Eden, Gwen, Isla, and Lucy said that they had experience tutoring grade school students, either in a one-on-one setting or by volunteering in classrooms or at after school programs. Brit, Cara, Gwen, and Lucy also had experience tutoring their peers in mathematics. In particular, they had all worked with students enrolled in the number and operations course. These four participants were among the stronger number theory students, which could possibly relate to their experience tutoring number and operations concepts.

Data Collection Procedures

As mentioned earlier, I began artifact collection a few weeks into the Spring 2012 semester. Artifact collection occurred over the entire semester. I also attended the number theory course for the entire semester, recording field notes as I did. In my field notes, I took lecture notes, but I also made observations about group discussions and classroom norms and expectations. Not only did this aid me in my description of the setting, but it also contributed to data triangulation because it gave me the capability to compare participants' number theory understandings with the content and discussions I had observed in the classroom.

I also conducted task-based, one-on-one interviews with select document participants. The first interview took place throughout the middle third of the Spring 2012 semester. While the dates the interviews took place varied greatly, the inconsistency should not have affected participants' responses. All interview participants had already seen, worked on, and been assessed on the related course material prior to when I solicited their participation.

I set up a time to meet with each participant at her convenience. Interviews took place in a neutral environment on campus. When I first met with each participant, I provided her with the Informed Consent to Participate in Research Form (Appendix E) and reviewed it in detail. Upon the participant's consent and signature, I conducted and audio-recorded the first one-on-one task-based interviews, which lasted 70-90 minutes, depending on the length of time each participant spent on the tasks. Before posing the tasks, I asked each participant a few background questions to ascertain their coursework and teaching experiences. After posing the tasks, I asked a few follow-up questions through which participants could reflect on their responses to the tasks. A complete list of questions and tasks can be found in the First Interview Question Set (Appendix F). I discuss the tasks of the first interview in detail in the following section.

I emailed the interview participants one month before the end of the Spring 2012 semester to set up a time to conduct the second interview. Again, due to scheduling difficulties, the days and times of these interviews were sporadic and took place as late as finals week. The second interview was also audio-recorded, task-based, and one-on-one. This interview took up to 60 minutes. Not every participant was able to complete the tasks, but I allowed enough time to ask each participant reflective follow-up questions. A complete list of questions and tasks can be found in the Second Interview Question Set (Appendix G).

I chose to conduct one-on-one, semi-structured interviews for a number of reasons. Conducting one-on-one interviews allowed me to investigate each student's understanding of number theory individually, which coincides with my constructivist perspective. In addition, the semi-structured nature of the interviews allowed me to use various prompts to enrich the participants' responses, meanwhile staying focused on my research purpose. Many of the interview questions concerned mathematical tasks, so I provided the participants with scratch paper. As a result, I also had artifacts to analyze from interviews. In the data collection of case studies, Patton (1990) calls for multiple forms of data. From the types of interviews I created, I had memos, interview transcriptions, and artifacts to contribute to my findings. These characteristics of the one-on-one, semi-structured interview allowed me to provide thick, rich description of participants' responses and an illuminated understanding of my experience to my readers.

The content of the interview tasks varied. The tasks in the first interview related to greatest common factor (GCF) and least common multiple (LCM). Additional GCF and LCM tasks were added to the second question set to follow up on each participant's understanding of the first round of tasks. However, most of the tasks in the second question set related to prime number concepts. Both sets consisted of elementary number theory problems, to ascertain the participant's content understanding of number theory, and questions pertaining to hypothetical student scenarios and modeling number theory ideas, to ascertain aspects of the participant's PCK. At the end of each interview, I also asked participants to reflect on how they responded to student scenarios to better understand the relationship between the participant's content knowledge and PCK, and to ascertain any impact Dr. S or the course may have had on the participant's PCK. Throughout the interview, I used probing questions to help participants demonstrate the full range of their understandings. However, I neither confirmed nor denied their responses until after the interviews were complete. At the completion of the second interview, I answered any and all questions the participants had concerning the tasks and my research. I completed my dissertation interviews by May 2012.

Interview Tasks, Rationale, and Connections to the Literature

As evidenced by my review of the K-6 Common Core State Standards (CCSI, 2011), number theory is typically restricted to evens and odds, factors and multiples, primes and composites, and greatest common factor and least common multiple, most of which appears in middle school. However, the Conference Board of Mathematical Sciences (CBMS, 2012) recommends that elementary school teachers know and understand the Fundamental Theorem of Arithmetic, prime factorization, and divisibility as well as those number theory topics recommended by CCSSI. Research investigating preservice elementary teachers' understandings of number theory is limited to prime numbers (Zazkis & Liljedahl, 2004), prime decomposition (Zazkis & Campbell, 1996b), divisibility (Zazkis & Campbell, 1996a), even numbers (Zazkis, 1998a), and least common multiple (Brown, Thomas, & Tolias, 2002), but their understandings of topics like greatest common factor had yet to be explored empirically.

Even within the mentioned topics, the research is not exhaustive. Many of these researchers suggested that participants' understandings of other areas of number theory, or even arithmetic, may have been connected to their findings. For instance, Brown, Thomas, and Tolias (2002) suggested that preservice elementary teachers first need a flexible understanding of prime factorization and how it relates to factors, multiples, and divisibility to possess a conceptual understanding of LCM. However, these suggested connections, as with other studies, were underexplored.

Perhaps the largest understudied area in the research is preservice elementary teachers' understandings of number theory as it relates to PCK. Prior to my study, no one had explicitly investigated preservice elementary teachers' number theory PCK. Also, none of the aforementioned studies explored connections between participants' content understandings and their anticipated experience with teaching elementary number theory topics to children. The closest any of the researchers came to putting number theory in the context of an elementary school classroom was Brown, Thomas, and Tolias (2002), when they asked their participants to create a story problem representing LCM.

It was in response to these needs that I, Dr. Soto-Johnson, and Dr. Karakok created the pilot study interview tasks (refer to Appendix H). The tasks covered enough topics to allow me to investigate participants' understandings between number theory concepts, but I also asked enough questions per topic to afford me the thick, rich description necessitated by an interpretive case study (Merriam, 1998). Most of the tasks were posed so that they would not only reveal participants' content understandings, but aspects of their potential PCK as well. To accommodate for all of these goals, the interview tasks were split into two question sets: (1) GCF and LCM and (2) Prime numbers.

As evidenced by my pilot study, most of the interview tasks did in fact elicit rich responses from my pilot study participants. However, my results necessitated me to make a few small changes to the wording of some of my tasks. Later, after the first interview had been conducted with my dissertation participants, Dr. Soto-Johnson, Dr. Karakok, and I opted to add a few new tasks to the second question set with which to follow up on participants responses to the first question set. I discuss each task, my rationale for the task, its connection to the literature, and how it connects to my research questions in depth in the following sections.

While conducting this study, a question arose in multiple discussions with researchers in my field: How do I purport to find evidence of PCK in individuals who have yet to become teachers? Embedded in this question are two more: (1) Is PCK observable outside of the classroom, and (2) Can preservice teachers possess PCK? Ball and colleagues, the researchers who conceptualized Mathematical Knowledge for Teaching (MKT) did so with the intent to measure constructs of MKT outside of the classroom. For example, Hill, Ball, and Schilling (2008) reported on their development of a measure of specialized content knowledge (SCK) and knowledge of content and students (KCS). While some might argue that teachers may only demonstrate true knowledge of content and teaching (KCT) in the classroom, others suggest that demonstrations of KCT in a clinical interview may be a sort of pre-knowledge or a subset of the knowledge they could demonstrate in the classroom (Hauk, Jackson, & Noblet, 2010). Even Hill (2010), a contributor of MKT, developed and implemented PCK test items that proposed to elicit KCT outside of a classroom setting.

Addressing the second prong of the question is more complicated. Many studies in the sciences have purported to investigate (and find evidence of) the PCK of preservice teachers. For example, Schmidt et al. (2014) developed a measure of preservice teachers' technological pedagogical content knowledge (TPACK), which they suggest can be used to assess preservice teachers' development of TPACK over time (even before student teaching). De Jong, Van Driel, and Verloop (2005) investigated preservice teachers' PCK of using particle models in teaching chemistry. The data collection occurred both before and after participants taught the material, suggesting that preservice chemistry teachers' PCK can be investigated and assessed, even prior to teaching. Researchers discovered, however, that most participants' PCK deepened through the teaching experience. From their synthesis of the early research on PCK, Cochran, King, and DeRuiter (1993) found that inexperienced teachers (including preservice teachers) "have incomplete and superficial levels of pedagogical content knowledge" (p. 265). However, they suggested that preservice teachers *can* and *should* be given opportunities to develop PCK throughout their teacher education program.

To that end, Zazkis, Liljedahl, and Sinclair (2009) studied the use of "lesson play" as a tool for professional development and as a way for researchers to investigate pre-service teachers' MKT. A "lesson play" is an imagined and potential script between teacher and students where students' difficulties or faulty conceptions emerge and the teacher resolves them. "Although the lesson plan makes quite clear the content in focus, the lesson play and the dialogue between the teacher and the students draw much more attention to the process through which that content will be communicated in the classroom" (p. 43). Lesson play focuses on two mathematical features: precision in the use of mathematical language and making explicit the various forms of mathematical reasoning that might emerge in the classroom. It also draws attention to the structure of teacher-student communication in the classroom. "Lesson players" (those creating the lesson play) not only need to imagine the choices and moves of the teacher, but they need to think and talk like a student.

While Zazkis, Liljedahl, and Sinclair did not explicitly suggest how lesson play might elicit the specific constructs of MKT (i.e., KCS and KCT), by attending to certain aspects of lesson play players could demonstrate these types of knowledge. For instance, lesson players could demonstrate KCS by revealing common student misconceptions and KCT by having the teacher respond in helpful ways in their lesson plays. The PCK tasks I posed in the interviews were also meant to elicit these types of MKT in these ways, albeit less extensively and using a different format.

A review of the literature suggests that all constructs of PCK are observable outside of the classroom and that preservice teachers can possess and demonstrate it. Their PCK may not be nearly as robust or well-developed as a practiced teacher, so I do not claim that my participants demonstrated PCK per se. Instead, I claim that potential, or developing PCK, is observable given the right conditions. As with lesson plays, my PCK tasks require that participants work within a student scenario. Both lesson plays and my PCK tasks elicit discussion about students' understandings and misunderstandings and require the participant to respond to the hypothetical student. In the following sections, I detail the task of each question set. For each of the tasks designed to elicit potential PCK, I describe how I purported to do so.

I refer to the PCK of preservice teachers as "potential PCK" to distinguish it from the more robust or well-developed PCK of an in-service teacher. In the following sections, when I refer to PCK, I mean "potential PCK". Similarly, when I refer to KCS or KCT, I mean "potential KCS" and "potential KCT", respectively.

First interview question set. The content of the first set of interview tasks was focused on GCF and LCM with the goal of addressing parts of all three research questions. As preservice elementary teachers' understandings of GCF are unexplored in the literature, it was especially important that some of my tasks relate to this concept.

However, as GCF is deeply rooted to the concept of LCM, and in an effort to achieve connectedness among concepts, I also included tasks addressing LCM and the relationship between GCF and LCM. Brown, Thomas, and Tolias (2002) suggested that a preservice elementary teacher requires a connected understanding of LCM across representations, in other words, the various ways of representing and finding LCM, creating story problems and using prime factorization to find LCM. As they are so connected, the same can be said for GCF. I posed most of the GCF/LCM tasks in an effort to determine the connections between participants' understandings of these topics. See Appendix F for a detailed account of the interview tasks from the GCF/LCM question set. In the following sections I discuss each task, its connections to the existing research, and which research questions I hope to address with it. For those tasks I adjusted after my pilot study, I discuss my rationale for doing so.

Problem 1 and 2. Ball (1990) found that preservice elementary teachers struggled to create story problems representing division of fractions. One implicitly uses GCF and LCM while operating on fractions. Thus, because GCF and LCM are as complex as division of fractions, I anticipated preservice elementary teachers would struggle to put these concepts in context as well. In the first two tasks of the first question set, I began by asking participants to create story problems representing LCM and GCF, and then I asked them how to model the ideas with diagrams or manipulatives. If the participant struggled with these tasks, I planned on asking a few prompting questions, such as "what is GCF/LCM?", "How is it used?", and "Can you think of a context where this idea might be useful?"

Finally, at the end of each task, I presented participants with four story problems and asked them to identify which, if any, related to GCF or LCM. Ball (1990) used a similar technique in her study. Some of her participants correctly identified story problems even if they could not create one themselves. I posed this part towards the end of the task so that it was not suggestive when participants wrote their own problems.

For Problem 1, which pertained to LCM, I made changes to the story problems from my pilot study. Story problem (c) related to GCF, and I found it was too suggestive. Pilot study participants Amy and Jen made reference to it while they were working through Problem 2, which pertained to GCF. Instead, I included another LCM problem with a slightly different format. In my pilot study, Amy kept using the phrase "happening at the same time" in reference to LCM. While she may not have meant 'time' literally, I wanted to provide an LCM story problem that did not reference time to see if my dissertation participants would recognize it as an LCM problem. Also, story problem (d) did not represent the LCM of 6 and 8, because their units are different, so I felt I needed at least one correct LCM story problem.

I also made changes to the story problems in Problem 2. Both pilot study participants quickly determined that story problems (b), (c), and (d) did not relate to GCF, so I replaced all three of them with story problems that might elicit more about participants understandings of GCF. During the pilot study, I found that participants held understandings of GCF similar to those of the measurement and partitive models of division. Beckmann (2008), the text with which participants would have been most familiar, referred to these as the *"How many in each group?"* the *"How many groups?"*

interpretations of division. (A detailed description of these interpretations and participants' understandings of them is provided in Chapter IV.)

To accommodate for the two interpretations of division, I changed one of the story problems to account for a *"How many in each group?"* interpretation of GCF. For another story problem, I changed the wording on the *"How many in each group?"* problem. Amy (pilot study participant) brought up the idea that by minimizing the number of groups, we are maximizing the number of objects in each group. Thus, I altered the story problem so that one of the conditions was to minimize the number of groups. I anticipated that this wording might reveal if someone were simply looking for the *greatest* common factor, rather than demonstrate an awareness of the inverse relationship between the number of groups and the number of objects in each group. The third new story problem was an LCM problem. Both pilot study participants acknowledged that they occasionally mixed up these concepts, so I anticipated this problem might reveal any such confusion. Also, to reduce confusion, I reworded story problem (a) so that it was clear that the number of milk chocolates in each goodie bag did not need to equal the number of dark chocolates in each goodie bag, a clarification I needed to make verbally in my pilot study interviews.

The first two tasks were intended to not only help me to establish participants' basic understandings of GCF and LCM, but the majority of these tasks pertained to participants' specialized content knowledge (SCK) (Ball, Thames, & Phelps, 2008). Knowing how to create and identify story problems, as well as representing mathematical ideas with diagrams and manipulatives, are parts of content knowledge fairly unique to teachers. Thus, these tasks were primarily geared toward addressing my first research question pertaining to the nature of participants' content knowledge. Again, any SCK or PCK that participants demonstrated was developmental, as opposed to the more robust, well-developed nature of in-service teachers' PCK. However, there was potential for the data I obtained from these tasks to also contribute to answering my third research question; in order to investigate connections between participants' number theory PCK and their content knowledge, it was important that I pose questions like tasks one and two.

Problem 3. Another way participants can demonstrate their potential SCK is by validating students' solutions and conjectures (Ball, Thames, & Phelps, 2008). In this third task, I posed a hypothetical student's conjecture that one can find the LCM of two numbers by multiplying them. This claim is only true when the two numbers are relatively prime (i.e., their GCF is 1), so this task could have also revealed some of the connections participants made linking the concepts of GCF and LCM, addressing aspects of my first research question.

After participants validated the hypothetical students' claim, I asked them why they felt a student might believe this method is valid, which requires them to access their potential knowledge of content and students (KCS) (Ball, Thames, & Phelps, 2008). Next, I asked whether or not the conjecture is ever valid, addressing SCK again. Finally, to investigate participants' knowledge of content and teaching (KCT), I asked them how they might respond to the student to help him correct his misconceptions. I also asked how they knew to respond in that way to help determine any influences on participants' development of KCT. As this problem addressed aspects of PCK in addition to content knowledge, responses contributed to answering my second research question as well.

This task also had the potential to address the connection between those two types of knowledge, which pertains to research question three.

Problem 4. In the fourth GCF/LCM task, I shared a hypothetical student's geometric strategy for finding GCF using a diagram. I broke the student's work into four stages, presenting participants with an "unpacked" version of the solution strategy. Then, I proceeded to ask participants questions similar to those in the third task. I asked them to validate the student's strategy, to determine if it would always work, and to justify their answers, which required the use of SCK. As this particular strategy for finding GCF always works, I also asked participants why a student might not be convinced that this is a valid method for finding GCF. Anticipating student struggles is an aspect of KCS (Hill, Ball, & Schilling, 2008), so this prompt gave participants the opportunity to demonstrate this type of knowledge when responding to this task.

While the hypothetical student's strategy may look relatively unpacked because I presented it in stages, it is actually quite complex. The diagram the student makes is actually a picture proof of the Euclidean algorithm for finding GCF. The algorithm and its proof rely on the fact that the GCF of two numbers, A and B, is equivalent to the GCF of A and $|A - B|$, as well as the GCF of B and $|A - B|$. Through recursion we can eventually find the GCF, and through transitivity we can show that this is also the GCF of the original two numbers. Explaining this idea requires an in-depth understanding of the Euclidean algorithm and its proof, which is part of SCK. I anticipated that this task would demonstrate the effect participants' SCK has on their PCK, as it would be challenging to convince students of a method that they themselves do not understand. Similarly to Problem 3, I anticipated that this task would help to answer all three research questions.

Problem 5. In a change of pace, the fifth task did not require number theory SCK to respond. I gave students the GCF and LCM of two numbers, told them what one of the numbers was, and asked them to determine the other. This task was related to one that Mason (2006) posed, requesting participants to multiply the LCM and GCF of two numbers and compare it to their product. Mason's research suggested that this was an exemplary task that would reveal a great deal of mathematical understanding. Participants already had the opportunity to acknowledge the relationship between these two products in Problem 3, so I decided to extend this idea with this fifth task. While there are many solution strategies for this task, the most efficient ones rely on knowing the product of two numbers is equal to the product of their GCF and LCM. This task helped me to determine whether or not participants knew and understood this relationship, and therefore helped me to address the first research question.

Problems 6 and 7. GCF and LCM are not only connected to each other, they are connected to other areas of elementary mathematics content as well. I designed the last two tasks to investigate participants' understandings of these connections. I began the sixth task by asking participants if they could think of any other areas in mathematics where GCF or LCM might play a role and how. As with the pilot study, if participants could not think of any, I suggested adding and multiplying fractions and working with ratios in general. In the pilot study, neither participant recognized the nonstandard, but frequently used, method for multiplying fractions by simplifying across fractions before multiplying. This led me to wonder if participants were familiar with other nonstandard strategies for operating with fractions. In particular, I wondered if participants were familiar with the common denominator method of fraction division, where the answer is the quotient of the numerators once you have found equivalent fractions with common denominators. Thus, I added a prompt about the relationship between GCF, LCM, and fraction division. As with the pilot study, I also asked if using GCF or LCM was absolutely necessary for fraction addition, multiplication, division, or working with ratios in general and why or why not. I had hoped that this task would help me to address the first research question, but it also had the potential of helping me to address the third when paired with the response to the last task, Problem 7.

The last task provided participants the opportunity to use number theory relationships to make a seemingly complicated problem much simpler. I posed a student scenario where a student resorted to using a calculator rather than add two fractions by hand. I then asked participants why they thought the student would have such a reaction or aversion to solving the problem by hand, which required participants to draw from their KCS (Hill, Ball, & Schilling, 2008). Next, I asked participants what questions they would ask the student to help guide him through the problem without a calculator. Not only did this question access participants' KCT, but their guiding questions helped me to investigate their own solution strategies. Problem 7 did not require participants to validate a claim, conjecture, or proof, so it did not elicit SCK like Problems 3 and 4. Thus, Problem 7 was most likely to only address the third research question.

In Table 4, I summarized how each GCF and LCM task related to the literature and my research questions. As the PCK-oriented tasks were designed to elicit both participants' SCK and PCK, I hoped to address all three research questions with those tasks. However, I fully anticipated that other content-oriented tasks would help to establish the connection between participants' content knowledge and PCK in number theory. Also note that none of these tasks addressed the second portion of the second research question regarding opportunities participants had to develop PCK in their number theory class. Any information about this that arose in interviews would have been purely anecdotal. Classroom observations were used to triangulate participants' accounts.

Table 4

How the greatest common factor and the least common multiple tasks relate to the literature and research questions

Task	Connection to Literature	Connections to Research Questions
1	Ball (1990); Ball, Thames, & Phelps (2008)	Q1
2	Ball (1990); Ball, Thames, & Phelps (2008)	Q1
3	Ball, Thames & Phelps (2008)	Q1, Q2a, Q3
4	Hill, Ball, & Schilling (2008)	Q1, Q2a, Q3
5	Mason (2006)	Q1
6	N/A	Q1
7	Hill, Ball, & Schilling (2008)	Q3

Second interview question set. The second question set primarily served two purposes. It was an opportunity for me to follow up on participants' responses to the first question set, and it allowed me to investigate each participant's understanding of prime numbers. Per my committee's request, I also posed a task that was accessible at the middle school level but incorporated more advanced number theory ideas that participants would have explored in their number theory class.

Most of the tasks in the second question set pertained to prime numbers, because of the emphasis that Zazkis and Liljedahl (2004) placed on the importance of preservice elementary teachers' understandings of them. The researchers asserted that preservice elementary teachers should know a great deal about prime numbers: (1) the definition of a prime number; (2) all natural numbers greater than 1 are either prime or composite; (3) if one can represent a number as a product (where none of the factors are 1), then the number is composite; (4) composite numbers have a unique prime factorization; and (5) there are infinitely many prime numbers. They investigated preservice elementary teachers' understandings of some of these concepts, but not all, and certainly not in depth. This question set attempts to address all of these ideas in one way or another as well as participants' PCK associated with primality.

See Appendix G for a detailed account of the interview tasks from the second question set. In the following sections I discuss each task. The first five tasks were new additions since my pilot study; the first four were follow-up tasks that resulted from the first question set, and the fifth (a more advanced number theory task) satisfied the aforementioned request from my committee. For each of these tasks, I discuss the necessity and rationale for adding them and which research questions they address. The remaining tasks in this question set were tested in my pilot study. I discuss their connections to the existing research and which research questions they help to address.

Problem 1. A preliminary analysis of the first question set revealed that, in response to Problem 2, participants held understandings of GCF similar to those of the measurement and partitive models of division. Beckmann (2008), the text with which participants would have been most familiar, referred to these as the "How many in each group?" and the "How many groups?" interpretations of division. I observe this with both my pilot study and my dissertation data. To better understand this phenomenon, I needed to pose a task that elicited each participant's understanding of division. This allowed me to determine if participants recognized or understood both the “How many in each group?” and the “How many groups?” models of division models. This task also gave me the opportunity to establish any connections between a participant’s understanding of division and GCF. As a result, this task was intended to contribute to answering my first research question.

Participants’ different models for GCF arose in Problem 2 of the first question set, so I decided to structure this problem similarly. I asked participants to pose a division story problem and then to explain their reasoning for the phrasing. I also asked them to model division using a diagram or manipulatives. At the end of the problem, I also posed a series of story problems and asked participants to identify valid story problems requiring someone to divide 12 by four. All four of the posed problems were valid division problems, but the third problem did not accurately represent 12 divided by 4 because it required unit conversion. Each story problem incorporated different contexts and interpretations of division. The second story problem was the only one to draw from a “How many in each group?” interpretation of division, because it seemed to be the most dominant type of model in participants’ modeling of GCF. I wanted to thoroughly determine participants’ familiarity with the “How many groups?” interpretation of division, in case it related to why this model was less present in their responses to the GCF story problem tasks.

Problems 2 and 3. Problems 2 and 3 were follow-up tasks to Problems 1 and 2, respectively, from the first question set. My preliminary analysis of the first question set revealed participants’ LCM and GCF story problems were rarely carefully worded, and participants also rarely offered specific reasoning as to how they chose to structure their story problems. With these tasks I hoped to elicit participants' thoughts about the structure and care that goes into phrasing LCM and GCF story problems. To do so, I posed pseudo student scenarios.

For problem 2, I stated "You have asked your students to create LCM story problems about light A and light B that blink every four seconds and every six seconds, respectively. Two of your students' story problems are provided below." I asked participants to validate each story problem and to explain their reasoning. Each of the students' story problems had left out necessary conditions of LCM. The first story problem did not specify a starting point (i.e., that the lights blinked as the same time, a condition that participants had also frequently overlooked in the first question set). Also, the question, "when will they blink together" was vague – participants used a similar vagueness in their questions. The story problem also did not ask for the *first* time that the lights would blink together again, which made the question even more ambiguous. Rather than ask for the number of seconds until the lights blink together again (the LCM), the second student's story problem asked about the number of times each light blinked until then (the LCM divided by four and the LCM divided by six). This was something a couple participants stumbled on in creating their own story problems.

For Problem 3, I stated, "You have asked your students to create GCF story problems about making single-colored bunches of balloons with eight red balloons and 12 white balloons. Two of your students' story problems are provided below." As most of the participants created (flawed) GCF story problems with a "How many in each group?" interpretation, I felt like this scenario might be more accessible to them. I did not want to run the risk of them struggling to even recognize that I was giving them GCF story problems. In each of these story problems, I also left out necessary conditions of GCF. This first student's story problem was missing a statement that maximizes the number of balloons per bunch (or, equivalently, minimizing the number of bunches). Without this statement, the story problem merely asks for a common factor, as opposed to the *greatest* common factor. The second story problem maximized the wrong quantity, and it did not have a statement about using all of the balloons, which is necessary for requiring readers to work with factors. Participants made similar mistakes during their first question set interviews.

Problems 1 and 2 from the first question set elicited rich responses from participants, but they left specific questions about participants' understandings of the structure of LCM and GCF story problems unanswered. These two pseudo student scenario tasks provided an opportunity for me to investigate the answers to these specific questions. As these tasks did not require participants to explain the hypothetical students' understandings and they did not require participants to respond to the students, these tasks primarily elicited SCK. As such, they contributed to answering my first research question.

Problem 4. This task resulted from one participants' response to Problem 2 in the first question set. When I asked her to model the GCF of two numbers using manipulatives, Brit laid out two groups of colored counters, each group represented one of the numbers. She then attempted to use their difference to help her break each group of counters into smaller, equal-sized groups of counters. While Brit's strategy worked for her example, I asked her whether her strategy would always work. She said no after she came up with a counterexample. Thus, I asked her if there was a relationship between the

GCF and the difference, to which she also said no, citing 10 and 13 as a "counterexample". As this relationship could actually be very useful in helping students to find the GCF using manipulatives (or even in understanding Eva's diagrammatic method for finding the GCF in Problem 4 from the first interview), I thought it might be useful to explore other participants' conceptions of this relationship.

To elicit their thoughts, I posed another pseudo student scenario. This scenario described one student's strategy of using the difference between two groups of colored counters to regroup the counters and find the GCF. The hypothetical student then posed a conjecture, that the difference of two numbers is also their GCF. I asked participants to validate this conjecture and to determine the relationship (if any) between the GCF and the difference of two numbers. I anticipated participants' responses to the second part of the task to be rich, because the relationship between the difference of two numbers and their common factors was something that participants had explored in their number theory course. Thus, the task offered an opportunity for participants to draw from their coursework. The task was only designed to elicit participants' SCK and not their PCK, so I anticipated that this task would contribute to answering my first research question.

Problem 5. During my proposal defense, one of my committee members requested that I add a task that incorporated some of the more advanced modular arithmetic ideas that participants would have explored in their number theory course. In order to stay consistent with the general "elementary" theme of the interview tasks, another committee member and I designed this task to be accessible to middle school students.

Problem 5 proposed that a factory production line completed the assembly of a part at exactly every *n* hours, where *n* is a whole number, without delay. I then asked that if the factory opened at noon on the opening day, for which values of *n* could a part ever be completed at exactly one o'clock. Given there are infinitely many possible values of *n* that satisfy this condition, I asked participants to conjecture about the solutions in general. I then made the problem more challenging by supposing that this factory relocated to another planet, where clocks were broken into *m* hours. Similarly, I asked participants if the production started at *m* o'clock, for which values of *n* could a part ever be completed at exactly one o'clock. My final request was that participants prove their answer, which would involve mathematics that participants had seen and used in their number theory course.

I scaffolded the question so that participants could explore the scenario in a concrete way, but I also gave them opportunities to generalize and theorize about the mathematics involved. While a middle school student might have been able to explore and conjecture about the initial part of the task, I expected participants' responses to be much more rooted in the number theory they experienced in class. This task purely elicited content knowledge, and thus, only contributed to answering my first research question.

In the Table 5, I summarize my reasons for adding each of the previously discussed tasks to the second question set. I also state the research question to which each task contributes. The remaining questions from the second question set all pertain to participants' understandings of prime numbers.

Table 5

Reasons for adding new tasks to the second question set and their contributing research questions

Task	Reason for Adding	Contributions to Research Questions
1	Follow-Up to Problem 2, First Question Set	Q1
2	Follow-Up to Problem 1, First Question Set	Q1
3	Follow-Up to Problem 2, First Question Set	Q1
4	Follow-Up to Problem 2, First Question Set; Connections to Coursework	Q1
5	Connections to Coursework	Q1

Problem 6. Similar to what Zazkis and Liljedahl (2004) did in their study of preservice elementary teachers' understandings of primes, I asked participants "What is a prime number?" to determine the rigor of their working definition of the concept. I asked about the importance and role of prime numbers to investigate whether participants had made any readily accessible connections between prime numbers and other areas in mathematics. This task not only helped me to establish how each participant's understanding of "prime number" related to the concept definition (Tall & Vinner, 1981), but it also provided some insight to the span, or breadth, of their concept image. This task primarily addressed the first research question.

Problems 7 and 8. An important step in understanding the concept of prime numbers is being able to identify what is and what is not a prime number (Zazkis & Liljedahl, 2004). Composite numbers and the number "1" are among the numbers that are *not* prime. In my years teaching fundamental mathematics for preservice elementary teachers, I found that many of my students believed the number one to be prime. Those that knew one is neither prime nor composite rarely produced a convincing argument explaining why that is. As this evidence is merely anecdotal, the seventh interview task allowed me to investigate further.

The eighth interview task was inspired by a common misconception concerning composite numbers. Zazkis and Liljedahl (2004) found that more than one-sixth of their preservice elementary teacher participants (n = 116) incorrectly identified the product of two prime numbers as also being prime. The researchers suggested that this may be an indication of "a profound psychological inclination toward closure, that two of a kind produce a third of the same kind" (p. 175). This misconception most likely perpetuated from the participants' own school mathematics experience. The eighth interview task allowed me to investigate this idea with my own participants.

Problems 7 and 8 were two-fold; they provided insight to content as well as pedagogical content knowledge. In the context of interpreting student claims and responding to the student, I explored participants' SCK, KCS, and hints of KCT, as with many of the tasks in the first question set. Unpacking students' mathematical reasoning and determining its validity involves SCK, while determining what this infers about a student's understanding makes use of KCS, and knowing how to respond to the student requires KCT (Ball, Thames, & Phelps, 2008). I also asked participants how they knew to respond the way they did. This helped me to determine participants' PCK. As a result, both Problems 7 and 8 addressed all three research questions.

Problem 9. In this task, I posed two student strategies for factoring 540. For my pilot study, both students accurately decomposed the number into its prime factors using different techniques but insisted that their own answer was the correct one. The first prompt asks participants to discuss why the students might have this conflict. This provided participants the opportunity to discuss the surface features of the students' strategies, such as how they organized their work. I also hoped to elicit participants' understandings of the uniqueness of prime factorization. Zazkis and Campbell (1996b) found many of their participants struggled with this concept. For instance, when presented with a large number's prime factorization, some of the preservice elementary teachers were not convinced that the number was not divisible by primes that were not in the prime factorization.

The pilot study design of this problem did not elicit rich responses with respect to the Fundamental Theorem of Arithmetic. I thought that perhaps because both hypothetical students' processes were complete and accurate that this task drew more attention to their methods than their mathematics. I redesigned the task so that one of the hypothetical students' factorizations was incomplete. The intent was that this would draw attention to the fact that this student needs only to complete their factorization to get the correct prime factorization. This modification made room for the follow-up question, "how could you be sure that the students would get the same prime factorization?" This alteration in the task gave me an opening to investigate participants' understandings of the uniqueness of prime factorizations, an incredibly important theorem in number theory.

I also prompted participants to compare, contrast, and determine the validity of the students' methods and answers. When Deon (2009) posed a similar task in her dissertation interviews, she found her participants more readily accepted the solution that was more “unpacked”, in other words, the method that revealed more of the student’s process, even though both students’ solutions were correct. Zazkis (1998a) and Zazkis and Campbell (1996b) found that many participants preferred to work with whole number representations to determine divisibility rather than the prime decomposition of a number. Through this ninth task, I found that my participants preferred one of the students’ solution strategies over the other, and I gained some insight to that preference through participants’ comparison of the two methods.

I also asked participants to explain the hypothetical students’ dilemma, so this task also revealed aspects of participants’ KCS and KCT (Ball, Thames, & Phelps, 2008). Because validating student work also incorporates SCK, this task addressed all three research questions.

Problem 10. Here I posed a number in its prime factorized form, $M = 3^3 \times 5^2 \times 7$. For my pilot study, I asked participants if *M* is divisible by two, seven, nine, 11, 15 and 63. One of my pilot study participants, Zoe, exhibited a curious misconception about divisibility by two as it relates to the prime factorization of a number. Even though “2” was not in *M*’s prime factorization, she multiplied pairs of other prime factors to see if their products were divisible by two. To be able to more clearly identify misconceptions like Zoe’s, I wanted participants to also test *M* for divisibility by 14 and 26. One of 14’s prime factor is in the prime factorization of *M*, and if participants like Zoe feel that the factor of two can come from a “factor pair” within the prime factorization, then they may decide that *M* is divisible by 14. As for 26, neither of 26’s factors are factors of *M*, but someone with Zoe’s misconception could try looking for “factor pairs”, other than two and 13, whose product is 26.

I adapted this task from Zazkis and Campbell's (1996b) study, whose participants were in their first fundamental mathematics course for elementary teachers. Posing this task to a new population of preservice elementary teachers, those that are attempting a concentration in mathematics and are enrolled in a number theory course, was likely to produce some different results. For instance, about half of Zazkis and Campbell's participants insisted on calculating *M* and dividing by the divisors in question. Of those participants who attempted to reason through the task using the prime decomposition of *M*, half struggled to reason why *M* was not divisible by 11. I did not anticipate either response from the participants in this study, but I thought I might encounter a participant with responses like Zoe's.

I made one other major alteration to this task since my pilot study. Due to adding four new tasks regarding GCF and LCM to my second question set, my interview tasks were leaning largely to investigating participants' understandings of GCF and LCM. However, I had yet to ask them about their understandings of GCF and LCM with regards to prime factorizations. Thus, after asking participants about the divisors of *M*, I told them that another number, *N*, had a prime factorization of $2 \times 3^2 \times 5^3 \times 13$. I then asked participants to find the GCF and the LCM of *M* and *N* using the prime factorizations of *M* and *N*. So that this task would reveal more than just a procedural understanding, I also asked participants to explain the rationale for the procedure; why does it work? I felt that this would round out the data I collected concerning participants' understandings of LCM and GCF, having questioned them about GCF and LCM in story problems, visual and concrete models, basic procedures, and now prime factorizations.

Problem 11. Along the same lines as tasks seven and eight, the eleventh task asks participants to discuss different strategies for determining the primality of large ($n = 853$) and small ($n < 50$) numbers. To determine the primality of large numbers, the least sophisticated method is testing *n* for divisibility by whole numbers less than *n*. Someone with a more developed understanding of prime factorization may recognize that by testing for divisibility by prime numbers, we eliminate the need to test for divisibility by multiples of prime numbers. Furthermore, knowing that factors come in pairs eliminates the need to test for divisibility by primes less than $\sqrt{n}$. For small numbers, there are other ways of determining primality through the use of manipulatives, which may reveal participants' SCK (Ball, Thames, & Phelps, 2008). This eleventh task may have revealed any number of connections that participants had made between primality, divisibility, factors, multiples, and manipulatives. I did not provide participants with manipulatives during the pilot study interview, but I did bring them to the dissertation interview to encourage responses about how to use them. This task focused on content, so it addressed aspects of the first research question.

Problem 12. Zazkis and Liljedahl (2004) claimed it is important for preservice elementary teachers to recognize that there are infinitely many primes, yet their study did not address this. I agree with Zazkis and Liljedahl; it is important that preservice elementary teachers understand this idea, but it is also important that they know why there are infinitely many primes and that they can make sense of this idea to a middle school student. This last part of the task may incorporate KCS (Ball, Thames & Phelps, 2008). I created this interview task in response to these goals. The task asked participants to demonstrate content knowledge and PCK in number theory, so it was possible that it addressed all three research questions.

In Table 6, I summarized how each prime number task related to the literature and my proposed research questions. As with the GCF and LCM tasks, the PCK-oriented tasks were designed to elicit both participants' SCK and PCK and had the potential to address all three research questions. However, I fully anticipated that other content-oriented tasks would contribute to establishing the connection between participants' content knowledge and PCK in number theory. Also note that, as before, none of these tasks addressed the second portion of the second research question regarding opportunities participants had to develop PCK in their number theory class. As any information about this that arose in interviews would have been purely anecdotal, and without classroom observation to triangulate participants' accounts the data is not trustworthy.

When I proposed adding Problems 1 through 5 to the second question set, my advisors and I discussed the possibility of running out of time. According to my IRB, the second round of interviews were only to last 60 minutes. While the original question set took approximately 45 minutes with each of my pilot study participants, I acknowledged that adding five new tasks ran the risk of me running out of time. My advisors and I agreed that that would be alright, but we prioritized the prime number tasks so that I could get to the ones that elicited richer responses. Problems 11 and 12 did not elicit rich responses from my pilot study participants, so they were moved to the end of the interview. As anticipated, some of the participants were not able to address these last two questions prior to the end of the 60-minute interview. Data were incomplete for these tasks, so I did not include them in my results.

Table 6

How the prime number tasks relate to the literature and research questions

Task	Connection to Literature	Connections to Research Questions
6	Zazkis & Liljedahl (2004)	Q1
7	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3
8	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3
9	Ball, Thames, & Phelps (2008); Deon (2009); Zazkis & Campbell (1996b)	Q1, Q2a, Q3
10	Zazkis & Campbell (1996b)	Q3
11	Ball, Thames & Phelps (2008)	Q1
12	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3

Follow-up prompts. At the conclusion of both sets of interviews, I asked follow-up questions to enrich the data and inform my perceptions of participants' number theory and PCK understandings. After the first set of interview tasks, I followed up by asking about participant's experiences creating the GCF and LCM story problems. As Ball (1990) found, creating story problems can be a challenge for preservice elementary teachers. To gain more insight to this experience, I asked participants what prior knowledge a student might need to create GCF and LCM story problems. I also asked why a student might struggle with creating these types of story problems and how they, as the teacher, might alleviate this struggle. Not only did this enrich my observations of the participants' experiences creating GCF and LCM story problems, but it informed my understanding of their PCK related to this activity.

For both interviews, I asked participants about any influences on their responses to tasks, such as coursework or experiences. I asked them to be specific about which experience(s) influenced which response(s), how, and why. While the findings from these data were anecdotal, they suggested how participants may have developed their content and pedagogical content understandings.

In the following section, I describe how I conducted my analysis. I also establish connections to my theoretical framework, and I describe how this analysis will address the research questions.

Data Analysis Procedures

Although the case study framework does not claim specific data collection or analysis methods, qualitative case studies are meant to be thick with description and heuristic in nature, illuminating the reader's understanding of the phenomenon (Merriam, 1998). It is customary for case study researchers to gather multiple forms of data to achieve this level of description. According to Merriam (1998), case study analysis should occur concurrently with data collection. Data analysis, either informal or formal, for this study occurred throughout all levels of data collection.

My informal data analysis process began with the first type of data, my field notes from the number theory course. My observations of the number theory students' collective experiences, namely the content and the classroom norms, played a large role in cuing my analytical lens. In this way, I drew from the social lens of my theoretical framework. It contributed to my data analysis, because I kept in mind the collective experiences of the number theory students and the types of expectations to which they were accustomed. Whenever possible, I informed my analysis of participants' individual responses with their collective experiences. This is evident in how I wrote-up my results; I summarized participants' relevant number theory experiences in each section.

After the first round of interviews, I reflected on the participants' responses as part of my initial analysis. It was important that I capitalize on the second round of interviews by asking follow-up questions concerning any emergent themes from the first round of interviews. After identifying emergent themes, I designed follow-up tasks to add to the second question set. My informal analysis continued as I transcribed the audio-recordings from the interviews; I kept a journal on my thoughts and observations.

My more formal analysis process began after all of the interviews were transcribed. In coding the interview responses, I drew heavily from the psychological lens of my theoretical framework. I was constantly asking, "What individual understanding did the participant demonstrate?" I began by coding each task using the codes established in my pilot study, but they were not fine-grained enough; my pilot study codes (see Appendices I and J) did not allow for the variation in participants' responses to the tasks or the differences between the tasks themselves. Each task was designed to elicit something different about a participant's understanding of number theory, so naturally each task revealed new codes. To improve upon my codes, it made sense to recode each task separately using open thematic coding (Corbin & Strauss, 2008). I discuss some of the larger-grained emergent codes and their definitions in the following sections. I discuss finer-grained codes related to specific interview tasks in Chapter IV when I discuss the evidence for such codes.

I began by coding Brit's responses to one task, and then I attempted to code Cara's response to the same task using the same codes. If Cara's response revealed something new, something that was not represented in the codes that I had already developed, I created a new code. I then reviewed Brit's transcription to see if any of her responses could have been better represented by the new code. I then moved on to Eden's response to the same task and repeated the process. I used these constant comparative methods until I achieved saturation (Patton, 1990). After my coding was complete, I went back through my codes to try and collapse some of them. I also identified codes that were only specific to one participant. While these instances made for interesting cases (some of which I detailed in my results), they did not typically make valuable contributions to my coding scheme as a whole.

I initially began recording my coding process as I did for my pilot study (see Appendices I, J, and K), but it revealed to be inefficient. I soon changed tactics and created a spreadsheet with codes separated by interview task. As new codes emerged, I reorganized the codes within each task. Through reorganization, I established a hierarchy and placed finer-grained codes beneath larger-grained codes. To account for which participants' responses were coded using which codes, I had six columns (one for each interview participant) and I placed an "X" in the cell for each participant for whom I used a code. Frequently, I attached comments to these cells with memos concerning the participants' responses and/or quotes that justified or represented the code used. This helped to establish an audit trail (Merriam, 1998) as well as aided in writing my results. The spreadsheet that I used to record my codes can be found in Appendix L. I discuss these codes and their definitions in the sections that follow.

While most of my codes were specific to tasks, I found that some of my tasks had overarching themes. After identifying these themes, I was able to cluster tasks into three umbrella themes: story problems, other number theory content, and pedagogical content knowledge. Within the story problem theme, I was able to merge multiple tasks in order to convey a coherent subtheme. However, tasks pertaining to each of the other two themes were too unique to merge, despite groups of tasks clustering around smaller subthemes.

I used my field notes and the documents that I photocopied from participants to inform and confirm the results from interview responses. I coded the content of my field notes and the documents according to their relevance to specific interview tasks. For example, I coded a homework problem or a set of notes as "Interview 1: Problem 4" if the content was similar or seemed like it contributed to participants' responses. I then summarized the instances where I used each code and included these summaries in Chapter IV. I also coded my field notes according to the social norms and instructor expectations using the code "norms and expectations". I summarized these observations in my description of the setting and referenced these observations occasionally in Chapter IV.

The final step in my analysis occurred while I was writing my results according to the three themes. I was able to further collapse codes and prioritize findings. In the following sections, I outline my codes for each of the three themes or clusters of tasks. The organization of this section mirrors the organization of Chapter IV.

Number Theory Content Theme
and Related Codes

The vast majority of the interview tasks contributed to the number theory content theme. However, the story problem related tasks were more related to participants' understandings of modeling GCF and LCM. The rest of the tasks were not. After removing story problem related tasks and all PCK related prompts and responses from the content tasks, what was left contributed to the development of these codes.

Problems 3 and 4 from Interview 1, and Problems 4, 7, and 9 from Interview 2, all required that participants validate some sort of student claim or conjecture. In coding these tasks, the "validation" code was prevalent, but a few other codes from my pilot study also applied. The rest of the content tasks were not in the context of validating student thinking; for the most part, they were simply number theory tasks that I asked participants to respond to or solve. The pilot study codes that applied to these tasks were more various due to the large variation in content. Table 7 records the pilot study codes I used to initially code the number theory content tasks.

The two validation subcodes, "counterexample" and "verification" needed to be expanded. "Counterexample" was too specific and I needed a code to more generally represent instances when participants identified inaccuracies in student conjectures or claims. In many cases, participants used counterexamples to do this. I also altered how I defined the code "verification". There were a few tasks for which participants were asked to identify the cases for which student conjectures or claims were valid. So, rather than interpreting "verification" as verifying that a student's conjecture was valid, I reinterpreted it as verifying the cases in which a student's conjecture was valid.

Table 7

Initial coding of content related tasks

Task	Content	Codes	Subcodes
Problem 3 Interview 1	LCM LCM/GCF	Validation* Relationship	* Counterexample, Verification
Problem 4 Interview 1	GCF	Validation	Verification
Problem 5 Interview 1	GCF/LCM	Relationship	N/A
Problem 6 Interview 1	GCF, LCM, Other	Fractions Misconceptions	N/A
Problem 7 Interview 1	GCF, LCM, Other	Fractions Misconceptions	N/A
Problem 4 Interview 2	GCF	Validation	Counterexample, Verification
Problem 5 Interview 2	GCF	Relatively Prime	N/A
Problem 6 Interview 2	Prime	Personal Definition	N/A
Problem 7 Interview 2	Prime	Validation Personal Definition	Counterexample
Problem 9 Interview 2	Factoring	Validation	N/A
Problem 10 Interview 2	Divisibility GCF/LCM	Methods Method for Finding	Factorization Prime Factorization

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.

As opposed to the content of the tasks related to the story problem theme, the mathematics of these tasks varied quite a bit. I developed much finer-grained codes to differentiate between participants' responses to the different content. In Section I of Chapter IV, I provide examples of those codes and I present code summaries for each task.

Story Problem Theme and Related Codes

The tasks that contributed to the story problem theme were Problems 1 and 2 from the first question set and Problems 1, 2, and 3 from the second question set. In particular, Problem 1 from the first question set elicited participants' understandings of creating LCM story problems, modeling LCM with pictures and manipulatives, and identifying valid LCM story problems. Problem 2 did the same, but for participants' understandings of GCF. In the second question set, Problem 1 elicited participants' understandings of division modeling and story problems, in my effort to inform my analysis of participants' GCF story problems. Problem 2 elicited participants' understandings of validating LCM story problems, and Problem 3 elicited participants' understandings of validating GCF story problems. On my first round of analysis, I coded these tasks using codes developed from my pilot study. It was not appropriate to code the division story problem task with any of the GCF or LCM codes or subcodes, so I created "division" content code category with similar codes and subcodes to those used for GCF and LCM. Table 8 records the pilot study codes and new division codes used to initially code these five tasks.

Codes "personal definition", "modeling", and "validation" were the most frequent to appear under the umbrellas of the content codes (LCM, GCF, and Division). Similar to my pilot study, I coded responses with the "personal definition" code if the participant referred to what LCM or GCF "is" or "means". I used "modeling" when a participant used or referred to a non-numerical method for representing LCM, GCF, or division. Modeling methods included story problems, pictures, and manipulatives, each of which represented a subcode. Another code, "validation", arose when participants validated or identified story problems. In determining the validity of a story problem, participants would respond about the accuracy and appropriateness of it. All of these codes also pertained to participants' potential specialized content knowledge (SCK), or content knowledge that may be specific to the teaching profession (Hill, Ball, & Schilling, 2008). As a result, responses to all five of these tasks were also coded as "SCK". Recall that I do not claim that participants demonstrated the more well-developed and robust SCK of an in-service teacher, but rather the developing or potential SCK of a preservice teacher.

Table 8

Initial coding of story problem related tasks

Task	**Content**	**Codes**	**Subcodes**
Problem 1 Interview 1	LCM	Personal Definition, Modeling*, Validation	* Story Problems, Pictorial, Manipulatives
Problem 2 Interview 1	GCF	Personal Definition, Modeling*, Validation	* Story Problems, Pictorial, Manipulatives
Problem 1 Interview 2	Division	Modeling*, Validation	* Story Problems, Pictorial, Manipulatives
Problem 2 Interview 2	LCM	Validation	N/A
Problem 3 Interview 2	GCF	Validation	N/A

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.

After coding all of participants' instances of modeling, I realized that I could collapse the "pictorial" and "manipulatives" modeling codes into one, "visual" modeling. I was justified in collapsing these codes because participants tended to draw pictorial models exactly how they modeled them using manipulatives. I further coded the types of division and GCF models participants demonstrated. I coded division models with "how many groups" and "how many in each group" codes. These phrases are specific to Beckmann's (2008) interpretations of division. In my pilot study, I had also used these phrases to code GCF models. While there are similarities between the two types of GCF models and the two types of division models, it no longer seemed appropriate to use the word "group" for GCF models. My own participants struggled to differentiate between the "groups" of objects representing two numbers and the "groups" into which one would subdivide each "group" in order to find the GCF of the two numbers. For clearer phrasing, I chose to reword the "how many groups" code as "how many subgroups". Similarly, I reworded the code "how many in each group" as "how many in each subgroup". This also allowed me to differentiate between my GCF and my division modeling codes. Participants' models of LCM all had a similar structure, so they did not need to be differentiated with additional codes. I discuss additional codes specific to the story problem theme, which specific examples, in Section II of Chapter IV.

Pedagogical Content Knowledge
Theme and Related Codes

My participants were preservice teachers. As such, their PCK was in the developing stages. Officially, I call this "potential PCK" to distinguish between the more robust or well-developed PCK of an in-service teacher and the less-developed PCK of a future teacher. In this section, when I refer to PCK, I mean "potential PCK". Similarly, when I refer to KCS or KCT, I mean "potential KCS" and "potential KCT", respectively.

Most of the pedagogical content related tasks were part of validation tasks. These tasks included prompts concerning "student reasoning" and "student challenges" or misconceptions. These prompts elicited knowledge of content and students (KCS). Other prompts, which required that participants respond to the hypothetical students, elicited participants' knowledge of content and teaching (KCT). During my initial coding of the PCK related portions of the tasks, I found my pilot study codes to be poorly defined and insufficient. Table 9 records my best attempt at using the pilot study codes to initially code the PCK related task.

I coded a statement "KCS" if it pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). This differs from the "SCK" code in that SCK plays a role in determining the mathematical accuracy of student work, while KCS is necessary for determining student understanding and depth of knowledge. The KCS codes that arose from my pilot study were "student solution strategy", "student reasoning", "student challenge/error", and "prerequisite knowledge". I used the codes "student reasoning" and "student challenge" during my initial coding process, but "student solution strategy" and "prerequisite knowledge" proved to be less useful. I coded KCS statements with "student reasoning" if the participant referred to *why* a student might believe a statement, claim, or conjecture about number theory is true or false. I coded KCS statements "student challenge" if the participant acknowledged a specific difficulty or misconception that a student might have related to a certain task or concept. However, neither of these KCS codes could be used when a participant acknowledged the student's valid mathematical conceptions. As such, I coded these cases as "student conceptions".

Table 9

Initial coding of pedagogical content knowledge related tasks

Task	Content/PCK	Codes
Problem 3 Interview 1	KCS KCT	Student Reasoning
Problem 4 Interview 1	KCS KCT	Student Challenge
Problem 7 Interview 1	KCS KCT	Student Challenge
Problem 7 Interview 2	KCS KCT	Student Reasoning
Problem 8 Interview 2	KCS KCT	Student Challenge
Problem 9 Interview 2	KCT	N/A

"KCS" refers to knowledge of content and students and "KCT" refers to knowledge of content and teaching.

The KCT code was insufficient in coding all of participants' instructional responses to hypothetical students. To elicit KCT, I asked participants how they "might respond to the students to help them recognize their misconceptions." This was a deliberately leading request; I wanted participants to address the mathematics of the scenario by attempting to further the hypothetical students' understandings of the mathematics at hand. I coded statements with "KCT" if a participant demonstrated a "knowledge of content and teaching", as described by Ball, Thames, and Phelps, 2008. This includes knowing how to sequence the content for instruction, like which examples to use when introducing a topic versus when attempting to deepen students' understandings. KCT also includes being able to weigh the advantages and disadvantages of the different ways with which to represent mathematical concepts as well being able to make in-the-moment decisions about whether or not to pause for clarification or pose a new task to further student learning. To code a participant's response as an instance of KCT, their response needed to be an instructional response to the hypothetical student in the scenario and it needed to pertain to the specific mathematics at stake.

Some of participants' responses to the students in the scenarios were too general to qualify as KCT – they did not address the specific mathematics of the situation. Of these general responses, I only classified a few as general pedagogical knowledge (GPK), or pedagogy that transcends subject matter. Rather, most of these general responses were only applicable in the realm of mathematics education. I coded these responses as General Mathematical Pedagogy (GMP). I coded most of the participants' responses to the hypothetical students in the scenarios as either KCT or GMP. I elaborate on this proposed construct of PCK in Section III of Chapter IV, as evidence of its existence arose from the data.

Another KCT related code that I created for my dissertation was "insight to KCT". I had asked participants to reflect on their responses to the hypothetical students. What were their motivations for their responses? This added a metacognitive layer to the analysis. Participants' responses to this type of question either related to their epistemology, how they perceived students to learn best, or to the specific mathematics of the task.

Each of the previous sections discussed the larger-grained codes that coincided with each of the three major themes. Each task was unique, so I typically used multiple finer-grained codes to parse apart participants' responses. While these codes were not necessarily used to compile or connect subthemes, they did allow me to more easily discuss my results in Chapter IV.

Quality Criteria

While quantitative researchers attempt to establish validity or "rigor" through their procedures, qualitative researchers' goal for ensuring quality is called "trustworthiness". Lincoln and Guba (1986) suggested "credibility as an analog to internal validity, transferability as an analog to external validity, dependability as an analog to reliability, and confirmability as an analog to objectivity" (p. 76-77), and together these address trustworthiness. I addressed these criteria through various procedures, beginning with establishing my own role in the research. By acknowledging and discussing my own biases I have acknowledged my own subjectivity. As a qualitative researcher, I play the role of the research "tool" and my perspectives, personal and theoretical, affect how I interpret the data. So that these perspectives do not undermine the credibility of my findings, I give preference to my participants' voices.

As is customary with qualitative case studies, I report my findings using thick, rich description (Merriam, 1998). Along with quotes and images of participants' work, this will help to establish the authenticity of my findings. I also ensure the accuracy and credibility of my findings through data triangulation, confirming my findings through multiple sources of data, and investigator triangulation, confirming my findings through other researchers (Patton, 1990). I collected observational, interview, and document data from multiple participants. I also shared my sanitized data with colleagues so that they may confirm my own findings. This also helped me to establish an audit trail (Patton, 2002). Additionally, my memos, coding document, and binders of sanitized data served as documentation with which to determine the dependability of my dissertation.

In Chapter IV, I answer my research questions in three parts. Sections I and II address my first research question. In Section I, I present the results of the number theory tasks that do no relate to story problems. In Section II, I present the results of the story problem tasks and answer the first research question. In Section III, I present the results of the PCK related tasks and answer my second and third research questions.

CHAPTER IV

RESULTS

In this chapter, I present my results and answer my research questions in three sections. Recall that my research questions are:

Q1 What is the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory topics taught at the elementary level?

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

I answer Research Question 1 in Sections I and II of my results. Most of my interview and classroom data pertained to participants' content knowledge. The number theory content varied greatly, so I focus my results around the content explored in the interview tasks (see Table 10). The data concerning participants' understandings of modeling greatest common factor (GCF) and least common multiple (LCM) using pictures, manipulatives, and story problems were rich enough to warrant their own section (Section II). I discuss participants' understandings of all other number theory content in Section I.

Table 10

Topics in number theory addressed in each interview task

Question Set	Problem	Topic	Question Set	Problem	Topic
1	1	LCM	2	3	GCF
1	2	GCF	2	4	GCF
1	3	LCM & GCF	2	5	Modular Arithmetic
1	4	GCF			
1	5	LCM & GCF	2	6	Primes
1	6	LCM & GCF	2	7	Primes
1	7	LCM & GCF	2	8	Primes
2	1	Division*	2	9	Factoring
2	2	LCM	2	10	Divisibility, LCM, & GCF

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.
* This question was designed to inform participants' understandings of modeling GCF.

As previously mentioned, I discuss each interview task as it relates to number theory content in the first two sections of my results. I also discuss participants' related experiences from their number theory course and how they might have influenced participants' responses, incorporating the social lens of my framework. Then, I summarize participants' responses to these interview tasks and my coding of them.

At the end of Section II, I discuss overarching themes that emerged from my data on participants' understandings of number theory content. I also summarize my answer to Research Question 1. I found that participants appeared to be more successful with portions of tasks where they could more clearly connect to their coursework experiences. I also found that there appeared to be a disconnect between the activity of working with a task at a concrete level and the activity of applying theory to explain these explorations. Lastly, I noticed that the representations with which participants best understood GCF were different than the LCM representations participants best understood.

Section III of my results addresses my second and third research questions pertaining to preservice elementary teachers' potential number theory pedagogical content knowledge (PCK) and how it relates to their understandings of number theory content. I discuss participants' responses to three tasks designed to elicit PCK. The data suggest that preservice elementary teachers' number theory PCK is indeed influenced by their number theory content knowledge. I elaborate on the influences on participants' observed PCK. In my analysis of participants' responses to the interview tasks, I observed a type of PCK that was different from Ball and colleagues' well-defined constructs of PCK (e.g., Hill, Ball, & Schilling, 2008). I call this type of knowledge general mathematical pedagogy (GMP), and I discuss my evidence for this different type of PCK in Section III of my findings. At the conclusion of Section III, I summarize my answer to Research Questions II and III. In partial answer to these questions, I propose a model for how preservice elementary teachers' various types of knowledge, as they relate to number theory, contribute or influence their PCK.

Section I: Number Theory Content Understandings

In this section, I answer my first research question concerning the nature of mathematics concentration preservice elementary teachers' number theory content understandings. The majority of the interview tasks elicited number theory content knowledge from participants. In the following subsections, I include the results of these tasks, without the responses to the tasks in which I asked participants to create or validate story problems. To “set the stage” for participants’ responses, and also to incorporate the social lens of my framework, I also include descriptions of related course content, such as lecture material and homework assignments. At the end of this section, I discuss major content related themes that emerged from the data.

Participant Responses

I have organized each of the following subsections by first summarizing the task and stating content-related prompts. I then present participants’ responses to each prompt, presenting evidence and detailed descriptions throughout. When possible, I group participants’ responses and provide a representative quotation. The focus throughout these subsections is on participants’ number theory content understandings. Typically, this appears in the form of specialized content knowledge (SCK), which is “the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Ball, Thames, & Phelps, 2008, p. 377-8). Wherever appropriate, I inform the reader of participants’ relevant number theory course material. In some cases, I elaborate on number theory content to help the reader better understand references to course material.

Interview 1: Problem 3. In this student scenario, Mark suggested that the least common multiple (LCM) of two numbers, say *A* and *B*, is equivalent to their product. Five of the interview participants immediately recognized that Mark’s claim was invalid, and I coded their responses as “invalid”, because the participants recognized that Mark’s conjecture did not always work. Four of the participants cited the counterexample where *A* and *B* are six and eight, a problem they had worked on earlier in the interview. Given that this was an example of an accurate counterexample, I coded these as “accurate counterexample.” Gwen, however, initially thought the claim to be true, citing a supporting example of four and five. In an attempt to generate a second supporting example, Gwen tried two and 10, an “accurate counterexample.” Here, she realized her mistake and decided that Mark’s claim was invalid after all.

All six interview participants easily determined that Mark’s claim does in fact work for a subset of numbers, receiving the validation code “sometimes valid.” When asked to generalize the cases in which Mark’s conjecture is valid, Brit, Cara, and Lucy stated with confidence that the product of relatively prime numbers is equivalent to their least common multiple. I coded their responses as “conjecture works when *A* and *B* are relatively prime”, which is not only a “validation” subcode, but it also spoke to participants’ understandings of the “LCM/GCF relationship.” As this is the largest subset of whole numbers for which Mark’s conjecture is valid, these participants demonstrated specialized content knowledge (SCK) of LCM.

If Brit, Cara, and Lucy had explained why Mark’s conjecture only worked for relatively prime numbers, then I could have documented that they exhibited stronger SCK. Brit thought of it visually, using Cuisenaire rods, explaining that with relatively prime numbers the trains would not match up before they were the length of the product. Cara thought of it discretely, using groups of objects, explaining that with *A* and *B* relatively prime we cannot rearrange an array of groups of *A* into *B* groups unless there are *B* groups of *A*. While both of these explanations provided valuable insight into how the participants thought about the LCM of relatively prime numbers, they did not explain *why* the conjecture works in this case.

Lucy was also unsuccessful in her attempt to explain why Mark's claim only worked for relatively prime numbers (using an inarticulate modular arithmetic argument), but she did appear to have the most articulate understanding of the relationship between two numbers' GCD and their LCM. Lucy described how an *A* by *B* area model relates to the LCM and GCD of *A* and *B*. Lucy said, "It shows that the least common multiple goes into the large area model so many times. It goes into that the GCD amount of times, between the two numbers." Lucy was effectively describing how the product of two numbers is equivalent to the product of their LCM and GCD. This idea can easily be used to adjust Mark's conjecture and make it valid for all whole numbers, but none of the participants got quite this far. Cara did suggest, however, that in order to use Mark's conjecture, one would have to divide each *A* and *B* by their greatest common factor. For example, given the numbers six and eight, using Cara's suggestion one would divide each by two. This would result in the numbers three and four, which are relatively prime. While multiplying these numbers (e.g., three and four) would not always result in the LCM of *A* and *B*, it appeared as if Cara's goal was merely to make *A* and *B* relatively prime and not to find their least common multiple. She did not suggest multiplying the numbers that resulted from dividing each *A* and *B* by their GCF.

When asked to generalize the cases in which Mark's conjecture works, Gwen considered her supporting example of four and five. She recognized that the two numbers were relatively prime, but she did not attribute this to *why* Mark's conjecture worked in this case. Instead, Gwen inevitably decided that Mark's claim was valid for *consecutive* whole numbers and prime numbers, which I coded as "conjecture works when *A* and *B* are consecutive" and "conjecture works when *A* and *B* are prime." However, upon reflecting why two and 10 produced a counterexample, Gwen said: "Maybe because these aren't prime... there's something that goes into them...[but with four and five] two and four won't go into five." Here, Gwen brought out the idea that factors, and common factors in particular, have something to do with the validity of Mark's claim, but she did not quite make the connection. Furthermore, in this instance, she did not seem to recognize that two is prime.

Like Gwen, Eden and Isla incorrectly claimed that Mark's conjecture only worked for prime numbers, which I coded as "conjecture works when *A* and *B* are prime." Isla, however, alluded to common factors playing a role, while Eden did not.

> Isla: OK, so, I'm going to go back to the 10 and 20 example. Since two and five also go into both of those, and then if you times 10 by two, times 20 by two, you can move up in smaller increments than just the 10 times 20. Whereas if you have prime numbers, you can't just move up by twos or by fives, because they're never going to hit, because they don't have any other common factors, I think that's what it is called.

Isla incorrectly claimed that the LCM of 10 and 20 was 40, alluding to the idea that the least common multiple must be larger than the numbers themselves, but she accurately determined that the common factors of two numbers allowed for an LCM smaller than the product.

Table 11 summarizes my coding of participants' responses to Problem 3 from Interview 1. All of the codes fall under the umbrella code "validation" and the content code of "LCM." As presented in Chapter III, I had previously developed two validation subcodes, “counterexample” and “verification”, whose definitions can be found in the code book (see Appendix N). The code “accurate counterexample” is a subcode of “counterexample”, and the codes I used when participants identified the cases when Mark’s conjecture was valid are subcodes of “verification.” The code “conjecture works when *A* and *B* are relatively prime” is also a subcode of the “LCM/GCF relationship” code. All content codes, except for as “conjecture works when *A* and *B* are consecutive” and “conjecture works when *A* and *B* are prime”, can also be dually coded as “SCK.” The codes as “conjecture works when *A* and *B* are consecutive” and “conjecture works when *A* and *B* are prime” represented an incomplete understanding of the content, so I did not code them as SCK.

Table 11

Interview 1: Problem 3 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Invalid	X	X	X	X	X	X
Counterexample						
Accurate Counterexample	X	X	X	X	X	X
Verification						
Sometimes Valid	X	X	X	X	X	X
Conjecture works when *A* and *B* are:						
Relatively Prime	X	X				X
Consecutive				X		
Prime			X	X	X	

In summary, all participants recognized that the product of two natural numbers is not always equal to their LCM. However, only half of the participants appropriately determined the cases in which that conjecture is true.

Interview 1: Problem 4. In this student scenario, Eva demonstrated a geometric way of finding the greatest common factor (GCF) of two numbers. She started by drawing a rectangle, whose dimensions equaled the two numbers given. Then she subdivided the rectangle into squares, starting with the largest square possible, until the rectangle was completely subdivided into squares (refer to Figure 6). At this point, Eva declared that the side length of the smallest square was the GCF of the two numbers.

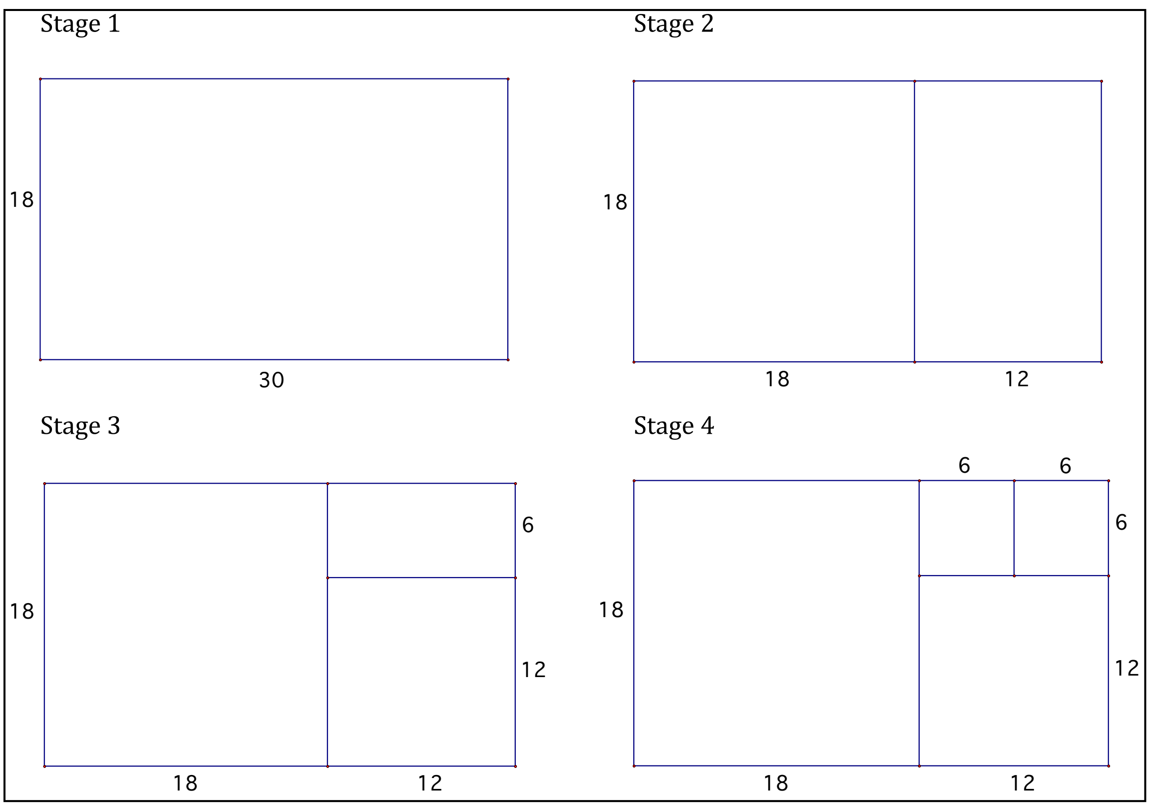


Figure 6. Eva's method for finding the greatest common factor.

I asked the interview participants if they believed whether or not Eva's method for finding the GCF would always work. All interview participants except for Isla were confident that Eden's method for finding the GCF of two numbers always works. I coded these statements with the "validation" code of "valid", under the content code of "GCF." All of the participants acknowledged that they had seen the method in their number theory course. I then asked the participants to explain their reasoning for why Eva's method works. There are many subtleties about this process that make it work, and there were many questions that I was looking for participants to answer here. For instance, why do we use squares? Why is it that we can "square off" sections of the rectangle and only focus on "what's left"? How can we be sure that the last square in this process gives us the GCF? Participants' responses suggested that while they were confident that Eva's method works, they had a tentative grasp on why it works.

Brit, Gwen, and Lucy cited the connection between Eva's method and the Euclidean Algorithm, with Gwen and Lucy explaining how each step in the algorithm corresponded with the stages of Eva's diagrams. I coded these statements under the "validation" subcode of "verification." I also used a finer-grained code of "geometric representation of the Euclidean Algorithm" to specify how participants verified Eva's method. While the connection to the Euclidean Algorithm was an important one, the follow-up question, "if Eva's method works because of the Euclidean Algorithm, then why does the Euclidean Algorithm work?" is needed to further explore participants' understandings of their connection. Brit and Cara were the most articulate in answering my (unasked) question about why we use squares in this process. They both conveyed how the squares ensured that they were doing the same thing to both dimensions of the rectangle. I coded this type of "verification" as "equal sides of square take off common amount."

Brit: The actual squares show what they have in common. So when she split up the first one, she knew that if she split up 30 into 18 and 12, she'd have one square and she could kind of cross that one off, because they, like, have a common... well, they're the same. So you can move on to what you have left.

Both Brit and Eden made the claim that we only need to focus on “what’s left”, that “we don’t have to worry about the perfect squares”, but neither could explain why we could ignore the squares we blocked off or why “what’s left” could help us find the GCF. Brit, Cara, Eden, Gwen, and Lucy did say that the process stops when the entire rectangle is “squared off” and that the smallest square gave the GCF. Some participants, Cara for instance, were careful to say that while this six by six square was the smallest square used in the process, it is not necessarily the smallest square that could evenly divide the 18 by 30 rectangle. Rather, it is the *greatest* square that would evenly divide the 18 by 30 rectangle.

Cara: Once you get it to break down into amounts that are equal to each other, you don't have to go any further, because that's the biggest number. You could obviously break it down to three, if you wanted to break down those sixes. But you want the greatest.

Brit, Cara, Eden, Gwen, and Lucy also showed that six is the GCF of 18 and 30 by tiling the 18 by 30 rectangle with six by six squares and showing that there were no gaps. I coded these verification responses as “tiled rectangle.”

While tiling the rectangle with six by six squares may convince someone that six is a common factor of 18 and 30, it does not necessarily follow that six is the *greatest* common factor. Conclusive reasoning might rely on the fact that any linear combination of 18 and 30 is divisible by their GCF and that the linear combination that divides both 18 and 30 must be their GCF. Brit, Cara, and Lucy did make some progress towards this end. They recognized that the other squares (whose dimensions were linear combinations of 18 and 30) could all be broken into six by six squares (i.e., that the side lengths of the other squares were all divisible by the GCF). Lucy's description of this was even reminiscent of the technique participants used to show why the Euclidean Algorithm worked. Lucy said, "You could work backwards and you could divide the 12 [by 12] rectangle into six by six rectangles as well as the 18 by 18, er, square, by six by six squares." Another important observation that Brit, Eden, and Lucy made was that this process will always stop by the time you reach a one by one square, because any two whole numbers have at least one as their GCF. Lucy, however, made the somewhat common mistake of saying that you could always get down to one by one squares if there "was no GCD", as if relatively prime numbers did not have a GCD.

Most of the participants made very little progress towards explaining *why* Eva's method works, but Isla demonstrated more struggles about the "squaring off" method. Even though this process was introduced and discussed in their number theory class. Isla admitted that she did not understand how the rectangle gets broken up. She knew that it corresponds to the Euclidean Algorithm, but if the smaller number goes into the larger number two or more times, Isla claimed that you would block off a rectangle, not a square. She gave 60 and 18 as an example. Because 18 goes into 60 three times, Isla claimed that rather than blocking off an 18 by 18 square, you would make an 18 by 54 rectangle. Isla never picked up on the idea that you could "square off" using multiple squares of the same size. I coded these statements of Isla's as a validation "misconception." As a result, Isla was not very productive on this task. Given that she believed Eva's method only worked "sometimes", I asked her to elaborate on which cases she felt it would work. Her answer was not entirely consistent with her observations, however. While she did say that cases where the smaller number only “goes into” the larger number once might work, she also said that cases where the two numbers were relatively prime might also work, and she did not explain why.

Eva’s method for finding the GCF was one that participants had seen, used, and explored in class. Almost all of the interview participants openly acknowledged this, which I coded as “course reference.” Their professor demonstrated the “squaring off” method multiple times and had students try their own examples. As part of one of their homework assignments, participants even had to explain why the method works. Their answers varied and merely connecting the method to the Euclidean Algorithm was an acceptable response. However their instructor gave more rigorous notes on why the Euclidean Algorithm works during lecture. While participants may have struggled to make the connections, they had been exposed to, and even proved, theorems that would have been useful in understanding this task. They had seen and used the Euclidean Algorithm and had even worked through a proof of it. But they had also proven the theorem that states “if d divides a and d divides b, then d divides the sum and difference of a and b.” This theorem plays a big role in understanding why Eva’s method works. Both numbers are multiples of their GCF, so any linear combination of the two numbers would be a multiple of their GCF as an extension of this theorem.

Table 12 summarizes my coding of participants’ responses to Problem 4 from Interview 1. Almost all of the codes fall under the umbrella code “validation” and the content code of “GCF.” I also used the validation subcode “verification”, whose definition can be found in the code book (see Appendix N). I used the finer-grained “verification” codes “geometric representation of the Euclidean Algorithm”, “equal sides of square take off common amount" and "tiled rectangle" when participants used one of these ways to verify Eva' method for finding the GCF of two numbers. All content codes can also be dually coded as "SCK", except for Isla's "misconception." The only code that did not directly refer to content was "course reference", which noted that a participant made an explicit connection to the number theory course, such as "we did [this] in our number theory course."

Table 12

Interview 1: Problem 4 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Valid	X	X	X	X		X
Misconception					X	
Verification						
Geometric representation of the Euclidean Algorithm	X			X	X	X
Equal sides of square take off common amount	X	X				
Tiled Rectangle	X	X	X	X		X
Course Reference		X	X	X	X	X

In summary, most of the participants determined Eva's method for finding the GCF to be valid, mostly due to their experience with the method in their number theory course. Participants also verified that the method worked by tiling the rectangle using the smallest square. However, participants were less familiar with *why* Eva's method works.

Interview 1: Problem 5. In this task, I told the participants that $GCF(a,b) = 42$ and $LCM(a,b) = 2352$. I then told them that $a = 336$ and asked them to find the value of

b. Most participants found the value of b eventually, but they used a wide array of methods to arrive at the solution. The most efficient method utilizes derivations of the equation $a \times b = GCF(a,b) \times LCM(a,b)$, which relates the GCF and the LCM. The quickest way to determine the value of b is by dividing the product of the GCF and the LCM by the value of a. However, none of the participants used the most direct method of solving this problem. Given that the content of the task was oriented around this relationship, I used the code "LCM/GCF relationship" to code the content. All participants, except for Cara, eventually arrived at the correct solution to the problem, which I coded as "correct solution."

While very few of the participants used or attempted efficient methods to solve this task, all of them acknowledged (explicitly or indirectly) two relationships that are necessary for making progress: (1) Both a and b must be divisible by their GCF (i.e., b must be a multiple of 42); and (2) the LCM of a and b must be divisible by both a and b (i.e., 2352 must be divisible by b). While these connections are direct results of the definitions of GCF and LCM, they are not necessarily obvious.

Brit was the only participant to use some form of the equation that relates the GCF and the LCM, $a \times b = GCF(a,b) \times LCM(a,b)$, but first she investigated the task from a different perspective. Brit recognized that both a and b are divisible by 42, because 42 is their greatest common factor. She began by finding multiples of 42, knowing that these multiples were possible values for b. She also acknowledged that the product of a and b is a common multiple; it was perhaps not the least common multiple, but certainly divisible by the LCM. She used this idea to make further progress on this task.

> Brit: What I was doing was I was taking these numbers that could possibly be *b*, [multiples of 42], and multiplying them by 336 and seeing if they were divisible by 2352. Because then… I was thinking that 2352 would have to be a multiple… yes. So I was taking *a* times "*b*" and dividing by 2352 and then seeing how many times it went in. But it went into all of them, which makes me question whether that is an accurate way of doing things or not.

Because of the numbers in this particular problem, *a* times any multiple of 42 (the GCF) will always be divisible by the LCM. (This is not always the case.) As a result, this method can not be used to identify *b* - unless you are specifically looking for the quotient $(a \times b) \div LCM(a,b)$ to be equal to the $GCF(a,b)$, which Brit was not. I coded this as an "inconclusive attempt" to solve the problem.

Brit then changed tactics to make further progress on this task. She remembered seeing in class how the combined factorizations of A and B were equal to the combined factorizations of the GCF and the LCM. While this merely represents the factorized forms of the *products* $a \times b$ and $GCF(a,b) \times LCM(a,b)$, she did not seem to recognize that she could multiply them in whole number form. Her understanding of the relationship $a \times b = GCF(a,b) \times LCM(a,b)$ seemed limited to the prime factorized form. Although slightly limited, Brit's understanding of the relationship between the GCF and the LCM enabled her to find $b = 294$ using this less efficient strategy. To check her answer, Brit later took each of the multiples of 42 she had listed earlier, multiplied by *a*, and divided by the LCM to find that the correct multiple of 42 gave the GCF as the quotient. I coded Brit's strategy as "$a \times b = GCF(a,b) \times LCM(a,b)$", because she used this relationship to successfully determine the solution to the task.

Cara's initial approach relied on an invalid conjecture we had explored earlier in the interview (Mark's Conjecture, from Problem 3). I coded this as an "inconclusive attempt.” She began by conjecturing that the $LCM(a,b)$ could be found by multiplying *a* and *b*. This gave Cara $b = 7$, which contradicted her understanding that *b* must be divisible by 42 (the GCF), which in turn reminded Cara that her conjecture only works when *a* and *b* are relatively prime.

Cara took a break from this task, and we revisited it at the end of Interview 1. After her initial exploration, Cara recognized valuable connections between the GCF and LCM of two numbers. She changed tactics, and used similar reasoning to Brit: *b* must be a multiple of 42 (the GCF). For 42 and 84, Cara used listing methods to find the $GCF(a,"b")$ and $LCM(a,"b")$. When the GCF and/or the LCM were not equal to those given in the task, it told her that neither 42 nor 84 could be *b*. Cara shortened her lengthy exploration by realizing that she could check the LCM for divisibility by “*b*” and double-check that the $GCF(a,"b") = 42$ to confirm the value of *b*. This proved to be too time intensive and the interview ended before Cara could determine the value of *b*. While it was one of the less efficient methods of doing so, Cara’s approach would have eventually helped her in solving the task. I coded Cara’s last approach as “check LCM for divisibility by multiples of GCF and double-check GCF.”

Eden arguably exhibited weaker SCK of LCM and GCF than most of the other participants, yet she knew just enough to solve this task with relative efficiency. Although she struggled to articulate it, she recognized that *b* must be a multiple of 42 (because 42 is a factor of *b*) and that 2352 must be divisible by *b* (because 2352 is a multiple of *b*). However, she could not recall how to find the LCM of two numbers, so she ruled out the trial and error method that Cara used. Instead, Eden relied on her stronger understanding of GCF. She knew that 336 was 42 times eight and that *b* must be

42 times *something*. Using reasoning, she decided that *something* must be relatively prime to eight so that the GCF remained 42. She tried $42 \times 5 = 210$ and $42 \times 7 = 294$. She ruled out 210 because it did not divide the LCM, but determined that $b = 294$ because it does divide the LCM. While she did not directly rely on the relationship between the GCF and the LCM, Eden was able to coordinate what she knew about each to solve the problem. I coded Eden's strategy as "pick multiple of GCF for *b* so that $GCF(a,b) = 42$ then check LCM for divisibility by *b*."

It should be noted that simply testing the 2352 for divisibility by "*b*" once you have confirmed that the GCF of *a* and "*b*" is 42 is sufficient in this example, because seven (the quotient of *b* and the GCF, or the LCM and *a*, as both are equal) is prime. There are no other possible values of *b* that would lead to this result – aside from the trivial case, where $"b" = 42$, which was obviously not an option for participants. However, if this quotient were composite, it would also be necessary to show that the LCM of *a* and "*b*" is in fact 2352. For example, if I had constructed the task so that $a = 336$, $GCF(a,b) = 42$, and $LCM(a,b) = 16464$, the quotient would have been 49 ($16464 \div 336$). Thus, $"b" = 294$ would have yielded a GCF of 42 *and* it divides the LCM. However, the correct value of *b* would be 2058. The design of this task is limited, because it would not catch participants making this error. As a result, it is unclear whether or not participants fully realized *why* divisibility of the LCM by "*b*" was sufficient.

Gwen's first instinct was to use the lattice figure from class to relate the different pieces of the problem, which is a fairly direct approach that draws from the relationship between the GCF and the LCM. She drew an *a* by *b* rectangle and drew a diagonal, stating that the diagonal passed through 42 "lattice points" (refer to Figure 7). She also clearly marked the "first lattice point", and cordoned off two rectangles by drawing a vertical line and a horizontal line through the first lattice point. She claimed that the two rectangles' areas were equal, but she could not remember what that area represented or how to use the figure to help her solve the problem. In her sketch, Gwen mistakenly labeled the first lattice point as the LCM, but she admitted that she was unsure of this decision. I coded Gwen's attempt at using the lattice method as an "inconclusive attempt", because Gwen's labeling kept her from obtaining a correct result using this strategy.

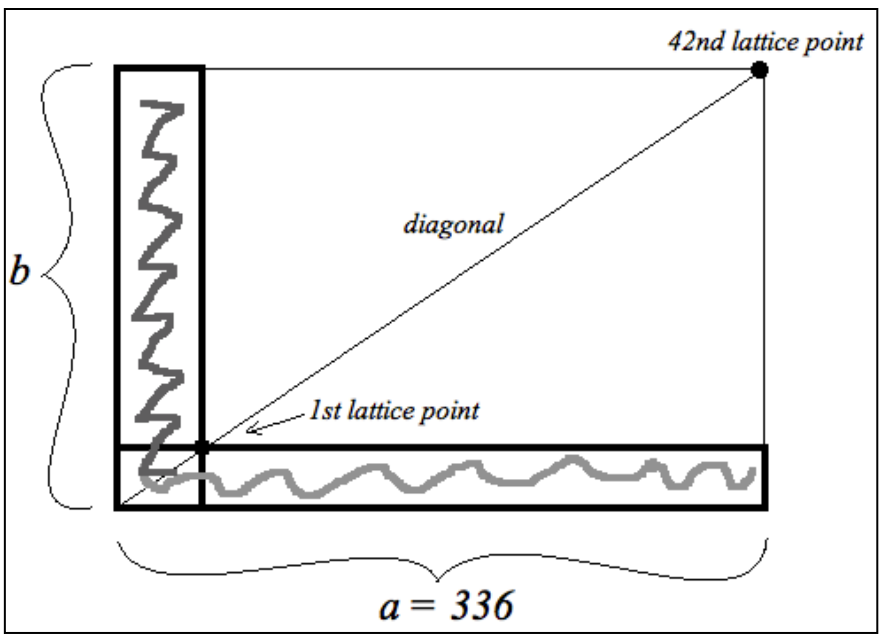


Figure 7. Depiction of Gwen's sketch of a lattice diagram.

Each of the two shaded rectangles in Figure 7 has an area equal to the LCM (2352). Consider the rectangle whose base is 336 units. The height would be seven units. Given that the lattice points are equally placed along the diagonal, the height differential between each would be seven units. This gives us that $b = 42 \times 7 = 294$ units. Lucy was the only other participant to attempt to use the lattice diagram to solve this task, and she remembered it in a slightly different way.

> Lucy: So we have some rectangle, and we don't know the area of it. And we know that one side length is 336 and we don't the other side length. But we do know that the least common multiple is 2352, so my first question was, what multiple of 336 gets you 2352? And that's seven. So I knew that I could shade a rectangle within the larger rectangle of the 336 by seven to get that least common multiple. And then I know that that goes into the whole area the GCD amount of times, because that would mean that you have that least common multiple times the 42... so you'd have the 2352 times the 42 would get you the full area. And so then, doing that 42 times gets you seven times 42 for the other side length, which is 294, so you know that the *b* side length is 294.

Lucy's understanding of the diagram hinged on the idea that the smaller rectangle whose area was equal to the LCM could fit into the *a* by *b* rectangle the GCD number of times. In other words, the area of the large rectangle can be calculated using $a \times b$ or $GCF(a,b) \times LCM(a,b)$. Lucy's sketch (see Figure 8) was a simplification of Gwen's, in which Lucy disregarded the diagonal and lattice relationship for the recollection that the LCM could fit into the area the GCD number of times. I coded Lucy's strategy as "graphical lattice method."

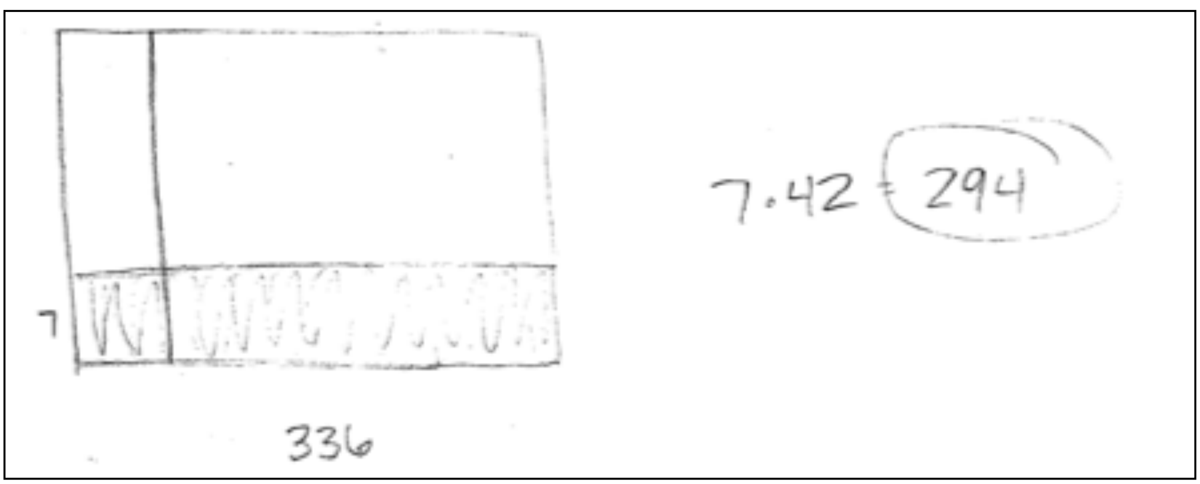


Figure 8. Lucy's sketch of a simplified lattice diagram.

As Gwen was unsure how to accurately interpret her lattice diagram, she moved on to a different strategy. She thought back to the processes for finding the GCF and the LCM using prime factorizations and worked backwards to construct the value of *b*. She knew from the process for finding the GCF that both *a* and *b* must have the prime factorization of 42 in common – and that is it. So *b* must contain $2 \times 3 \times 7$. When she found the prime factorizations of *a*, which is $336 = 2^4 \times 3 \times 7$, and the LCM, which is $2352 = 2^4 \times 3 \times 7^2$, Gwen recalled that we "pull out the biggest factors out of the primes of *a* and *b*." What she probably meant to articulate was that the powers for each prime factor in the prime factorization of the LCM represent the largest power for that factor from either *a* or *b*. Gwen then implied that because *a* does not have a factor of 7^2, *b* must have a factor of 7^2. Combining these two observations, Gwen arrived at the prime factorization of $b = 2 \times 3 \times 7^2 = 294$. While finding the prime factorizations may be a little time consuming, Gwen demonstrated a rather well-connected understanding of GCF and LCM with this strategy, which I coded as "worked backwards from prime factorization of LCM."

Isla began this task by admitting that she was "really probably not going to do this any way [she] learned in number theory" and that she "[did] not really understand the relationship between the greatest common factor and the least common multiple, but [she knew] that there is a big relationship." Then, like Gwen, Isla thought back to the processes for finding the GCF and the LCM using prime factorizations and worked backwards to construct the value of *b*. Because of its similarity to Gwen's method, I coded Isla's strategy as "worked backwards from prime factorization of LCM." Isla seemed less sure of her recollection of these procedures than Gwen did, so Isla's investigation took more time and trial and error, but she was eventually able to arrive at the correct solution.

Table 13 summarizes my coding of participants' responses to Problem 5 from Interview 1. Almost all of the codes fall under the content code "LCM/GCF relationship" and I coded participants' responses according to whether they had a "correct solution" or made an "inconclusive attempt" at solving the problem. I also coded the valid strategies that participants used. All of these codes, except "inconclusive attempt", can also be dually coded as "SCK."

Table 13

Interview 1: Problem 5 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Correct Solution	X		X	X	X	X
Inconclusive attempt	X	X		X		
Appropriate Strategies						
$a \times b = GCF(a,b) \times LCM(a,b)$	X					
Worked backwards from prime factorization of LCM				X	X	
Checked LCM for divisibility by multiples of GCF and double-checked GCF		X				
Pick multiple of GCF for *b* so that $GCF(a,b) = 42$ then check LCM for divisibility by *b*			X			
Graphical lattice method						X

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.

This task seemed, at first glance, fairly straightforward. However, it provided opportunities to observe such a wide variety of approaches. And it certainly supports the perspective that each individual constructs his or her own understanding given the same or similar instruction. In summary, most participants demonstrated valid solution strategies to problem solve in a way that incorporated their conceptions of GCF and LCM. While not all of these solution strategies were especially efficient, participants demonstrated a willingness to problem solve and draw from various experiences in their GCF/LCM repertoires.

Interview 1: Problem 6. In this task, I questioned participants on their knowledge of number theory connections to other areas in mathematics. Specifically, I asked if they knew if and how the notions of greatest common factor (GCF) and least common multiple (LCM) played a role in adding fractions, multiplying fractions, dividing fractions, and working with ratios in general. The content of this task overlaps fractions with GCF and LCM, so each of the participants' responses was coded as "fractions" and "GCF" or "fractions" and "LCM." Overall, participants recognized the use of the LCM in finding a common denominator to add fractions and the use of the GCF in simplifying ratios, but they were unfamiliar with alternative algorithms for multiplying fractions and dividing fractions that made use of the GCF and the LCM, respectively.

All six interview participants readily acknowledged that to add and subtract fractions, we need common denominators. Brit, Cara, Isla, and Lucy claimed that the (least) common denominator is the least common multiple of the two denominators, which I coded as "LCM gives the least common denominator." Eden and Gwen both stated that the common denominator would be the GCD, but it became obvious that they both meant the LCM, which they each eventually corrected. Brit, Cara, and Lucy recognized that while using the LCM is the most efficient common denominator, any common multiple would work. I coded these statements as “any common multiple gives a common denominator.” Eden, Gwen, and Isla also recognized that there were alternatives to finding the least common denominator: finding the product of the denominators. I coded these statements as “the product gives a common denominator.” By not acknowledging that any common multiple would work, this demonstrated slightly weaker SCK of common multiples. Brit exhibited the strongest SCK of common multiples here by voluntarily discussing *why* it is necessary to have common denominators before adding or subtracting fractions. This vastly contrasted the other five participants’ strictly procedural discussion of the role that the LCM plays in adding and subtracting fractions.

When asked about the role that the GCF or the LCM may play in working with ratios, all participants claimed that simplifying ratios and fractions required dividing the numerator and denominator by a common factor. I coded these statements as “dividing numerator and denominator by common factor gives a simplified fraction.” All participants except Eden also acknowledged that the most simplified fraction (i.e., a fraction in “lowest terms”) can be obtained by dividing the numerator and denominator by their GCF. I coded these statements as “dividing numerator and denominator by GCF gives the most simplified fraction.” My question about “working with ratios” was purposely vague, giving participants an opportunity to make connections of their own to other operations on fractions. For example, *comparing* fractions or ratios can be done using common multiples. However, for all participants, “working with ratios” seemed to remind them of simplifying fractions.

Brit and Gwen were the only two participants who recalled a non-standard algorithm for fraction multiplication that made use of common factors. With this algorithm, you can simplify the product before multiplying across by dividing diagonals (the numerator of the first fraction and the denominator of the second fraction, for instance) by common factors. When done appropriately, this eliminates the need to simplify your final answer. I coded these statements as “nonstandard multiplication algorithm uses common factors.” Isla may have alluded to this procedure, but her understanding appeared to be conflated with cross-multiplication (another procedure pertaining to “diagonals”). Isla said, “Because with multiplication [of fractions] you can cross-multiply - or you can… oh I haven't worked with fractions in forever. I think you can just multiply the top times the bottom. And you don't have to change anything.” Cara seemed to think you could "get out the smallest number you could make them", but could not articulate what she meant by that or think of an example where she could use the GCF or the LCM for multiplying fractions. It seems like she may have been referring to using the GCF in some way, but there is not enough data to support this claim. After being unsuccessful in describing an alternative approach to multiplying fractions that may use the GCF or the LCM, Cara and Isla both settled on the claim that the GCF and the LCM did *not* play a role in multiplying fractions, which I coded as “GCF/LCM do not play roles in multiplying fractions.” I also coded these statements as “misconceptions” due to their inaccuracy. Eden and Lucy exhibited slightly stronger SCK by at least acknowledging that it may be necessary to use common factors to simplify the product of two fractions.

Lucy made a similar claim, that common factors may be necessary for simplifying your answer, when I asked her about dividing fractions. Cara, Eden, and Isla claimed that neither the GCF nor the LCM could be used when dividing fractions, which I coded as "GCF/LCM do not play roles in dividing fractions." Again, I coded statements like this as "misconceptions." Brit and Gwen claimed that once fraction division is changed to multiplication, they could use common factors with the alternative multiplication algorithm, which again referenced their understandings of the nonstandard multiplication algorithm for fractions. None of the participants were aware of the nonstandard division algorithm that required common denominators.

While most of the participants were fairly successful at this task, aside from their unfamiliarity with nonstandard algorithms, some participants struggled. Isla admittedly did not remember or understand how to multiply or divide fractions, let alone how to use the LCM and the GCF with those operations. Cara struggled to use appropriate terminology when discussing fractions. She referred to fractions without actually using the terms numerator, denominator, or simplify. Everything was just "number."

Table 14 summarizes my coding of participants' responses to Problem 6 from Interview 1. The content for these codes was typically in the overlap between "fractions" and "LCM" or "fractions" and "GCF", with the exceptions of the two "misconceptions." All of these codes, except for the "misconceptions", were also dually coded as "SCK."

In summary, all participants seemed fairly familiar and proficient with the roles that the LCM and the GCF play in adding fractions and simplifying fractions, respectively. They were less aware of how the GCF can play a role in multiplying fractions, and participants were wholly unaware of how the LCM can be used to divide fractions. The multiplication and division algorithms that make use of these number theory concepts are both nonstandard, so perhaps participants were just unfamiliar with those procedures.

Table 14

Interview 1: Problem 6 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Fractions and LCM						
LCM gives the least common denominator	X	X	X	X	X	X
Any common multiple gives a common denominator	X	X				X
The product gives a common denominator			X	X	X	
Fractions and GCF						
Dividing numerator and denominator by common factor gives a simplified fraction	X	X	X	X	X	X
Dividing numerator and denominator by GCF gives the most simplified fraction	X	X		X	X	X
Nonstandard multiplication algorithm uses common factors	X			X		
Misconceptions						
GCF/LCM do not play roles in multiplying fractions		X			X	
GCF/LM do not play roles in dividing fractions		X	X		X	

“GCF” refers to greatest common factor and “LCM” refers to least common multiple.

Interview 1: Problem 7. In this student scenario, Remi expressed anxiety towards adding eighteen-fifty-firsts and eleven-thirty-fourths by hand and instead reached for a calculator to add the two fractions. While I asked the interview participants several questions about Remi's struggle and how they might help him, I do not discuss those responses here. Those prompts were designed to elicit potential PCK, not SCK. I had not originally designed this task for participants to solve themselves, but all of them at least described their process for adding the two fractions, which I do discuss here because it demonstrated SCK. Operations on fractions were not a focus in the participants' number theory class, but most of them had recently taken a course where number and operations were a focus. And Problem 6 served as a baseline for what they knew about using number theory when operating on fractions.

Four of the participants (Eden, Gwen, Isla, and Lucy) began by suggesting they would try simplifying the fractions first. All four participants then claimed that this was not possible here, which is inaccurate. I coded these statements as "misconceptions" with the subcode "can not simplify 18/51." Eden, Isla, and Lucy cited "51 is prime" as the reason for which the first fraction cannot be simplified, another "misconception." When I asked each of them how they knew that 51 was prime, each cited that it was not divisible by anything that they could "see."

> Lucy: We know that two doesn't go into them… So some of those division rules that we know about numbers, you can kind of run through them in your head and see if they have any common divisors… two, three, four, five… I don't go much past 10.

Isla also suggested that she would test 51 for divisibility by two, three, four, five, six, and so forth, but that she could not "see" any of these dividing 51. Eden tested 51 for divisibility by various numbers, in no particular order, using a calculator. All three appeared to overlook the fact that 51 does satisfy the divisibility test for three; perhaps three seemed too small of a number to be a divisor of this seemingly prime number. All three also neglected to use some of the major tenets of divisibility in their process for determining 51 to be prime. A concept that appeared frequently in the number theory course was that if a number is not divisible by *a*, then it is also not divisible by any multiple of *a*. Incorporating this idea would have made Lucy, Eden, and Isla's processes much more efficient.

Eventually, Eden and Gwen realized that they could simplify $^{18}/_{51}$ by a factor of three. After a while of testing random numbers in her calculator, I reminded Eden that to simplify we need common factors - so what are the factors of the numerator? (This may have been too great of a reminder, because there is evidence that it made an impact on a follow-up question.) Gwen only took another minute of looking at the fraction before she realized that 51 satisfied the conditions of the divisibility test for three. I coded both Eden and Gwen's statements as "simplified $^{18}/_{51}$." After simplifying $^{18}/_{51}$ to $^{6}/_{17}$, Gwen immediately recognized that she could multiply 17 by two to obtain a common denominator and completed the addition. I coded Gwen's statement as "found lowest common denominator/LCM." However, Eden suggested that the easiest way to find a common denominator between 34 and 17 would be to multiply them, which was reminiscent of her response to the previous problem, that "the product gives a common denominator."

Isla and Lucy did not make any progress on simplifying the first fraction. Isla listed multiples of 51 and 34 to find that 102 was the LCM, which I coded "found lowest common denominator/LCM." From there, she was able to find equivalent fractions and then add the fractions. Lucy, however, suggested that the least common multiple of 51 and 34 is their product “because their GCD is one.” If this claim were true, her process would have been more efficient than Isla’s. But as it was, the numbers in the problem became too large, and Lucy merely suggested what she would do to finish out the fraction addition. I coded Lucy’s statement as “the product gives a common denominator”, because while she was incorrect in stating that it would be the LCM, she knew that it was a common multiple.

Without really discussing the fractions themselves, Brit and Cara went straight to “procedure mode”, suggesting the steps for how to add the fractions. They both suggested that while multiplying the two denominators to find a common denominator would be a valid approach (which I coded “the product gives a common denominator”), listing multiples would enable them to find the least common multiple. I noticed that Brit and Cara were the only two participants who did not mention trying to simplify the fractions first, and they were also the only two to *not* make any actual calculations towards finding the sum; they merely discussed what they *would* do. It is unclear whether or not they did not suggest simplifying the fractions because they had already eliminated that option in their heads, rather than verbally as the other participants did. It is also possible that participants did not take this route because it was not what the task explicitly required. It is possible that they only thought of what they would do in order to address the student.

Table 15 summarizes my coding of participants’ responses to Problem 7 from Interview 1. The content for these codes was typically in the overlap between “fractions” and “LCM” or “fractions” and “GCF”, with the exceptions of the two “misconceptions.” All of these codes, except for the “misconceptions”, were also dually coded as “SCK.”

In summary, much of participants' responses confirmed what they demonstrated to know about operating with fractions in Problem 6 of Interview 1. The only new observation revealed from this task was that participants were less proficient operating on fractions with relatively large denominators, especially when one of those denominators was identified as prime.

Table 15

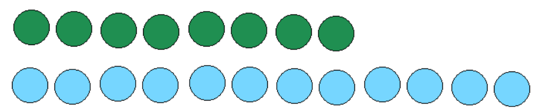
Interview 1: Problem 7 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Fractions and LCM						
Found lowest common denominator/LCM		X		X	X	
The product gives a common denominator	X	X	X			X
Fractions and GCF						
Simplified 18/51			X	X		
Misconceptions						
Can not simplify 18/51			X	X	X	X
51 is prime			X		X	X

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.

Interview 2: Problem 4. This student scenario (see Figure 9) exemplifies a new, uncommon student conjecture for finding the GCF. In this scenario, Maria is using colored chips to determine the GCF of eight and 12, only to discover that the difference of eight and 12 is also their GCF. I asked participants to validate Maria's conjecture that the difference of two numbers is the same as their GCF. If a participant determined Maria's conjecture to be invalid, I planned to then ask her if any relationship between the difference and the GCF of two numbers exists. Most of the codes I used for this problem were "validation" codes under the content code "GCF."

You have given each of your students 8 green chips and 12 blue chips and asked them to use the chips to find the GCF of 8 and 12. When she paired up the green and blue chips, Maria, noticed that there were 4 blue chips left.



As a result of this observation, Maria then tried making groups of 4 green chips and 4 blue chips. When she was successful, Maria conjectured that the difference between any two numbers is also their GCF.

Figure 9. Maria's conjecture that the difference of two numbers equals their greatest common factor.

In response to the first part of my question, all six participants determined that this conjecture did not always work, because they were each able to find a counterexample. For these statements, I used the validation code "invalid", the "verification" code "sometimes valid", and the "counterexample" code "accurate counterexample." Lucy determined that Maria's conjecture would not work for pairs of prime numbers, which always have a GCF of one. I coded Lucy's counterexample as "prime counterexample." Lucy did not acknowledge that the prime numbers two and three represent a confirmatory example. The other five participants produced counterexamples where the pair of numbers are relatively prime: 10 and 13, one and four, seven and 12, eight and 11. I coded these counterexamples as "relatively prime counterexample." Cara, Isla, and Lucy also produced counterexamples where the two numbers were *not* relatively prime or both prime: six and 10, two and 12, three and 12, respectively. I coded these as “counterexample with non-relatively prime and composite numbers.”

While all six participants recognized that Maria’s conjecture was invalid, it was unclear whether they believed their counterexamples exemplified the only cases in which Maria’s conjecture would not work. For instance, because Brit, Eden, and Gwen did not produce counterexamples where the two numbers were not relatively prime (like three and 12), I cannot determine if they knew that such counterexamples exist. Asking a follow-up question pertaining to the cases where Maria’s conjecture does not work would have helped me gain more insight into the counterexamples participants chose. However, I did ask participants to conjecture about the relationship between the difference of two numbers and their GCF. Cara suggested that the difference will always be bigger or equal to the GCF, but stated that she could not determine a more specific relationship between the two numbers. I coded Cara’s response as “the difference is greater than or equal to the GCF.” Brit, Eden, Gwen, and Isla also suggested that no clear relationship existed. I coded these responses as “no clear relationship.” Only Lucy recognized that the difference of two numbers is divisible by their GCF.

> Me: So for example with your three and 12 counterexamples, is there any kind of relationship between the GCF and the difference?
> Lucy: Um, the difference and the GCF… The difference is divisible by the GCF.
> Me: Is that always going to work?
> Lucy: Um, no. It wouldn't... Well, let me think. Yeah, I think it would... I feel like the difference would always be divisible, because even if the GCF was one, the difference would be divisible by one.
> Me: OK, so you kind of think it might work... Any idea how we can know for sure?

Lucy: Um, just looking at the examples and seeing if there's a counterexample. I can't think of a way to know for sure off the top of my head.

I coded Lucy's response as "the difference is divisible by the GCF." Even though Lucy's conjecture that the difference of two numbers is divisible by their GCF is correct, she could not think of a way to validate this conjecture. However, Lucy and the other participants worked extensively with a theorem in their number theory class that could easily support her conjecture: If d divides m and d divides n, then d divides $m - n$ and $m + n$. Given that the GCF of two numbers is a divisor of both numbers, it should therefore divide their difference.

Table 16 summarizes my coding of participants' responses to Problem 4 from Interview 2. Most of the codes fall under the umbrella code "validation" and the content code of "GCF." The code "accurate counterexample", as well as the other codes I used to classify participants' counterexamples, is a subcode of "counterexample." The code "sometimes valid" is a subcode of "verification." The codes I used to categorize participants' responses concerning the relationship between the GCF of two numbers and their difference fell under the content code "GCF/difference relationship." All codes, except for as "no clear relationship" were dually coded as "SCK."

In summary, participants accurately determined that the difference of two natural numbers is not equal to their GCF. They also recognized that this is occasionally true of some pairs of numbers. Even through investigating counterexamples to Maria's claim, participants struggled to identify the relationship between the difference of two numbers and their GCF. However, one participant accurately determined that the difference of two numbers is divisible by their GCF. It is possible that her counterexample with non-relatively prime and composite numbers helped her to recognize this relationship.

Table 16

Interview 2: Problem 4 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Invalid	X	X	X	X	X	X
Verification						
Sometimes Valid	X	X	X	X	X	X
Counterexample						
Accurate Counterexample	X	X	X	X	X	X
Prime Counterexample						X
Relatively Prime Counterexample	X	X	X	X	X	
Counterexample with non-relatively prime and composite numbers		X			X	X
GCF/Difference Relationship						
The difference is divisible by the GCF						X
The difference is greater than or equal to the GCF		X				
No clear relationship	X		X	X	X	

"GCF" refers to greatest common factor.

Interview 2: Problem 5. The goal of this task was to provide participants with an opportunity to demonstrate their understandings of modular arithmetic and congruencies in a context conducive to exploration, and thus suitable for middle school students. I posed the following scenario and question:

> At a factory, the production of a certain part takes *n* hours (where *n* is a whole number). Production starts at 12 o'clock on opening day and continuously puts

> out one part after another. For what values of *n* will a part ever be completed at exactly 1 o'clock?

I clarified that participants should assume that production starts at noon on opening day and never stops. This context was tricky to phrase and required clarification; for each interview, I described how the factory would put out one complete part after *n* hours, then another, *n* hours later, and so on and so forth. I also clarified that the part did not need to be completed at 1am and 1pm everyday, but that the part could be completed at one o'clock on any day, and perhaps not every number *n* would allow for this to happen.

Once participants had an opportunity to explore and arrive at their answer, I asked a series of follow-up questions – if they had made enough progress initially. The first follow-up question required participants to generalize their results. What did all of these values of *n* have in common? Why did these values work and the others did not? If participants were successful here, I altered the scenario and supposed that the factory was on another planet ("alien factory") whose clocks were partitioned into *m* hours. Then I asked them the same question: for which values of *n* is a part ever completed at one o'clock, assuming production starts at "noon" (the hour before 1 o'clock) on opening day and never stops? If a participant answered correctly, I asked them to prove their conjecture. I coded all responses to this task under the content code "modular arithmetic and congruences."

Participants' solution processes fell into two categories when investigating this task: those who used brute force (Cara, Eden, Gwen, and Isla), and those who reasoned through the task more abstractly (Brit and Lucy). Eden and Isla both drew clocks, then selected possible values for *n*, and finally counted out that many hours over and over again until they were convinced that one o'clock would or would not ever occur. Rather than use a visual, Cara listed out the times after adding the value in question over and over (e.g., "five o'clock, 10 o'clock, then three, then eight, then one"). Gwen used a chart to record her investigation, a partial image of which can be found below in Figure 10. These four participants went through nearly every possible value for *n* less than 12 using these strategies, which I coded as "brute force investigation."

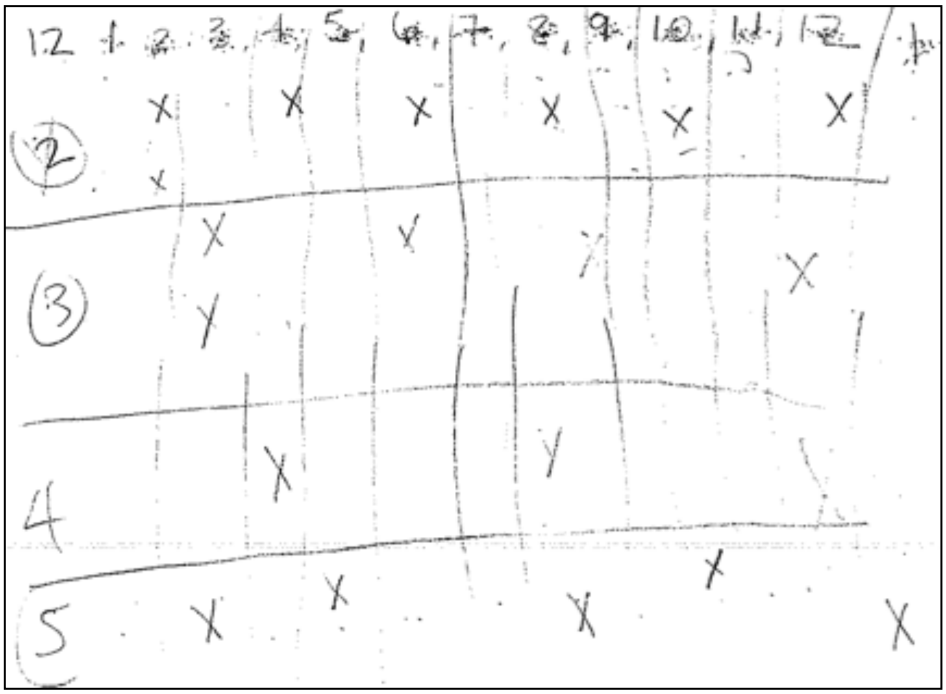


Figure 10. Gwen's chart for investigating the factory problem.

Brit's and Lucy's investigations were much more efficient. By reflecting on their minimal calculations, they were able to eliminate unnecessary investigation. I coded their approaches to this task as "efficient investigations." Brit and Lucy were the only ones to explicitly refer to mods. I coded these statements as "explicitly used mods." Cara, Gwen, and Isla *described* mods in their investigations (e.g., "multiples of 12 plus one" or "one, and then 13, and you could keep adding 12"), but never actually used the terminology or the notation of mods. I coded these statements as "implicitly used mods." Eden never even considered that *n* could be larger than 12, so she did not refer to mods explicitly *or* implicitly.

In Eden's investigation using the picture of a clock, she successfully eliminated two, three, four, six, eight, nine, and 10 as possible values of *n* (less than 12) by recognizing a repeating pattern in the hours she got as she counted around the clock. Her final solution to the problem was that *n* could be one, five, seven, and 11, which was coded as "*n* is 1, 5, 7, or 11." Eden made a couple of observations about the numbers in her solution, but she never tied them back to the number 12, which plays a key role in this scenario. For instance, she recognized that only odd values would work, but could not explain why three and nine did not work.

Eden recognized that the numbers were relatively prime to each other - but she did not realize that (more importantly) they are all relatively prime to 12. Eden also saw a pattern: After one you skip three whole numbers to get five, then you skip one whole number to get seven, and skip three whole numbers to get 11. If this pattern continued (and I do not think that was Eden's intention) Eden would have generated a complete set of solutions. As Eden did not make many of the connections necessary to fully understand the factory problem, she did not make any progress when I asked her about the alien factory. She suggested that she could use her pattern (start at one o'clock, skip three, skip one, skip three, skip one, etc.) to generate the list of possible *n* values, again completely ignoring the role that the partitions of the clock (the number of hours) play in the problem.

Cara made more useful observations about factors in her investigation, which focused on eliminating possible values of *n*. While she was able to successfully eliminate two, three, four, six, and eight, she had trouble articulating the conditions for which a number could be eliminated, especially once she got to the number nine. After investigating a couple even values, Cara said, “I don't think it would work for factors of 12 because you're always going to be an even number and one o'clock isn't even." This was not quite accurate, as three is an *odd* factor of 12. But then Cara went on to say, "With three, it'll be 12, three, six, nine, 12, three, six, nine… it'll keep repeating the same exact numbers and you'll never… get to one o'clock." Cara seemed content with her elimination of two, three, four, and six, but eight and nine proved to be challenging for her. "[Eight] does not work… because it would keep repeating the same numbers again. I don't know how that one figures into my theory, because that one's not a factor of 12." When Cara investigated whether or not nine was a possible value of *n*, this further challenged her theory in a way that she struggled to resolve.

> Cara: So then like nine would be… so… so that'll be nine, six o'clock, and then… so that'll be three o'clock… so I think nine would work. It doesn't have factors of 12 in it though. The only common factor is three. Otherwise… yeah, it doesn't work…
>
> Me: So nine does work, or it doesn't?
>
> Cara: It does work, but it doesn't - the only factor that it shares with 12 is three, and one. So maybe if the numbers below 12 are… only have one or two common factors with it… because three is just one, but then eight had a lot more in it. So it just depends on how many factors they share. Because 12, the GCF was four, for eight and 12. So maybe if the GCF is an even number, they won't work. Because 12 and 9, the GCF is three. So that is odd. So maybe you get a different number... something like that.

It was unclear whether or not Cara felt nine was a possible value of *n*, but she had successfully concluded that “multiples of 12 plus one” as well as five and seven were possible values. Thus, I coded Cara’s final solution as “*n* is 1 mod 12, and 5 and 7.” I did not feel like Cara had made enough progress to ask her the follow-up question about the alien factory.

Gwen investigated possible values of *n* using a chart (refer to Figure 13 above). She successfully eliminated two, three, four, six, eight, nine, and 10, but never explained *why* these numbers did not work. "Well, I got one, five, and seven to work, but I couldn't get three to work. Or nine. Because I was going to go with 'odd numbers'." She also implied that numbers that were one mod 12 would work and that 17 would work. I asked Gwen if she could make any conjectures about the numbers that work, and she said that she could not, because she was stuck on the fact that not all odd numbers were possible values of *n*. I coded Gwen's final solution as "*n* is 1 mod 12, and 5, 7, 11, and 17." I did not ask Gwen about the alien factory.

Isla was the only successful participant to investigate the factory problem using a brute force method. Her success was rooted in making small, but accurate observations and connections with each number that she investigated. "So [for two], it would be two, four, six, eight, 10, and then I go back to 12. So I don't think [*n* could be] even numbers, because an even times an even will always be even." Isla also determined that three would not work because 12 is divisible by three. When Isla found that nine was not a possible value of *n*, she attributed this to the fact that nine and 12 have a common factor of three. She successfully determined that possible values of n were one, five, seven, and 11. To generate all of the possible values of *n*, Isla said that she would "keep adding 12" to each of them. Even though Isla did not refer to mods explicitly, I coded her final solution as "*n* is 1, 5, 7, 11 mod 12."

Isla tentatively suggested that these values of *n* are all relatively prime to 12. She said, “The numbers that don't work aren't relatively prime to 12. And the numbers that do work *are* relatively prime to 12. But I don't know if every relatively prime number would work.” In spite of Isla’s uncertainty, I coded this statement as “*n* is relatively prime to 12.” I considered Isla to be successful with the first part of this task, so I asked her about the alien factory. Isla suggested that *n* had to be relatively prime to *m* (the number of hours the alien clock was partitioned into), otherwise “you’re always going to end up with the same… multiples?” Isla was referencing her experience with numbers like two, whose multiples mod 12 repeated in a pattern (two, four, six, eight, 10, 12, two, four, etc.) and “one” was not amongst it. I coded Isla’s accurate solution to the second part of the task as “*n* is relatively prime to *m*.” While Isla seemed reasonably convinced that *n* should be relatively prime to *m*, she could not think of a way to prove it.

Brit started her investigation by listing numbers that were one mod 12. Brit appeared to recognize the multiplicative nature of the problem; she knew that a multiple of *n* mod 12 had to be equivalent to one mod 12. Brit was the only participant to use this kind of strategy; she was also the only participant to not begin with a “process of elimination” mindset. Brit determined that because 25, 49, 85, and 121 were all equivalent to one mod 12, their factors were possible values of *n*. As a result, she found that five, seven, 17, and 11, respectively, were possible values of *n*. After her initial investigation, Brit conjectured that *n* could be any prime number, but immediately criticized this idea because “one is included but two and three are not.” She soon recognized that two and three did not work, because they were factors of 12. She also recognized that any number that was one mod 12 would also be one mod three (and thus not divisible by three) and one mod four. While she did not explicitly say so, this appeared to draw from Brit's understanding of the Chinese Remainder Theorem.

Brit then adjusted her conjecture: *n* must be a *prime* that is relatively prime to 12. Brit realized that this also did not include one. She was struggling to recognize that one was relatively prime to 12: "They're all relatively prime [to 12]. Except for one. I don't understand one. It's kind of a loner out here, and it doesn't really fit with the rules." After a few minutes, Brit determined that one and "any combination" of primes that were relatively prime to 12 were also possible values of *n*. While this describes a complete set of solutions for *n*, Brit never actually summed up with "*n* is any whole number relatively prime to 12." In spite of her wording variation, I still coded Brit's final solution as "*n* is relatively prime to 12." Given that Brit was successful in her initial investigation, I asked her about the alien factory. She claimed that "any numbers that are relatively prime to *m* or any combination of those said numbers" could be possible values of *n*. Again, her solution was superfluous, but accurate. I coded it as "*n* is relatively prime to *m*." When asked if she could prove it, Brit said that she could not think of anything.

Lucy was the only other participant, in addition to Brit, who explicitly recognized she could use modular arithmetic to think about the factory problem. As a result, Lucy was able to eliminate possible values of *n* with minimal calculations by reflecting on their relationship to 12. For example, "three would not work, because it goes into 12 evenly. So any number that's not… doesn't go into 12 evenly would work, mod 12." Lucy jumped to a false conclusion after only working with two and three, which both divide 12. So then I asked her about nine. Lucy responded,

> Nine is never going to work, either. So it has to be numbers that are relatively prime to 12 to work. Otherwise they… if they're not relatively prime they're always going to be the same couple of hours over and over and over again.

Lucy immediately recognized her error and realized that *n* must be relatively prime to 12, which I coded as "*n* is relatively prime to 12." When I asked her about the alien problem, Lucy said that *n* must be relatively prime to *m*. I coded this solution as "*n* is relatively prime to *m*." Drawing from her investigation, she claimed that if *n* was not relatively prime to *m* she would get repetition like she did with nine. Lucy could not think of any productive way to prove it though.

Some participants made use of their understandings of modular arithmetic from the number theory course in a couple of ways: (1) by recognizing that one o'clock can be interpreted as one mod 12, and (2) by recognizing that some multiple of *n* must be equivalent to one mod 12. However, participants also had experience with finding multiplicative inverses for specific mods and proving that a multiplicative inverse exists. None of the participants seemed to recognize the role of multiplicative inverse in this task. Brit and Lucy made the most connections to their number theory class and seemed to have the best understandings of the factory problem, perhaps because they explicitly recognized the role that "mod 12" played. Had they also recognized that finding numbers whose multiples were equivalent to one mod 12 meant that they were finding numbers with multiplicative inverses mod 12, they may have been able to draw further from their number theory experience.

There were multiple occasions during their number theory class where participants explored the linear combinations of two natural numbers and how they will always be multiples of their GCF. When Dr. S assigned their second homework assignment, he asked students in the number theory class to generate linear combinations of two numbers and conjecture about the outcomes. As a class, they determined that the only possible outcomes were multiples of the GCF. Dr. S. also asked students to use trial and error to find a linear combination that would result in the GCF exactly. In other words, if *m* and *n* were two whole numbers, Dr. S. asked students to find integer solutions for *h* and *k* if $hn + km = GCF(m,n)$. Later, he showed them a more analytical way for finding solutions to these problems. Participants learned that this linear combination can be found by working backwards through the Euclidean Algorithm once you have used it to find the GCF. Students had to then use this procedure to find solutions to this type of equation on their second homework assignment and their first exam.

In their number theory course, participants also explored the special case where the two numbers were relatively prime. More specifically, if *n* and *m* are relatively prime, then there exist integers *h* and *k* such that $hn + km = 1$. They even went as far as using the statement to prove that *n* has a multiplicative inverse mod *m*, which is the crux of the factory problem. By taking the mod *m* of both sides of the equation, we find that *hn* is equivalent to 1 mod *m*, which gives us that *h* is a multiplicative inverse of *n* in mod *m*.

Table 17 summarizes my coding of participants' responses to Problem 5 from Interview 2. All of the codes fell under the content code of "modular arithmetic and congruences." All codes were dually coded as "SCK."

In summary, either through a brute force or more theoretical approach, all participants produced at least some of the solutions to this task. All of the participants that explicitly referred to mods in their initial investigations successfully found all solutions to the first and second part of the task. The one participant that did not make reference to mods at all was the least successful in identifying the solutions to the task. In short, participants' understandings of modular arithmetic and congruences contributed to their success in addressing this task.

Table 17

Interview 2: Problem 5 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Brute force investigation		X	X	X	X	
Efficient investigation	X					X
Explicitly used mods	X					X
Implicitly used mods		X		X	X	
N is 1, 5, 7, and 11			X			
N is 1 mod 12, and 5 and 7		X				
N is 1 mod 12, and 5, 7, 11, and 17				X		
N is 1, 5, 7, 11 mod 12					X	
N is relatively prime to 12	X				X	X
N is relatively prime to *M*	X				X	X

Interview 2: Problems 6 and 7. For Problem 6, I asked participants: what is a prime number, and why are they important? I coded participants' responses to these questions with the content code "prime." This task was meant to establish a baseline understanding of what it means to be prime so that I could better interpret their responses to a hypothetical student with an invalid conception of prime (Interview 2: Problem 7). Participants' textbook defined a prime number as "a number $p \geq 2$ whose only (positive) divisors are 1 and *p*" (Silverman, 2001, p. 44). While Dr. S never wrote the definition on the board, he reviewed different aspects of the definition of prime. On the first day of

class, students investigated conjectures about prime numbers and Dr. S said, "One is not prime. This is important." On participants' third homework assignment, they were asked to "Explain what 'primes' are in at least three different ways. Give some reasons why we do not want to consider 1 a prime." As a partial answer to this problem, students referenced the textbook definition. For example, consider part of Lucy's response in Figure 11. Prime numbers were the building blocks for the majority of the number theory coursework, so there were many different connections participants could have made regarding their importance.

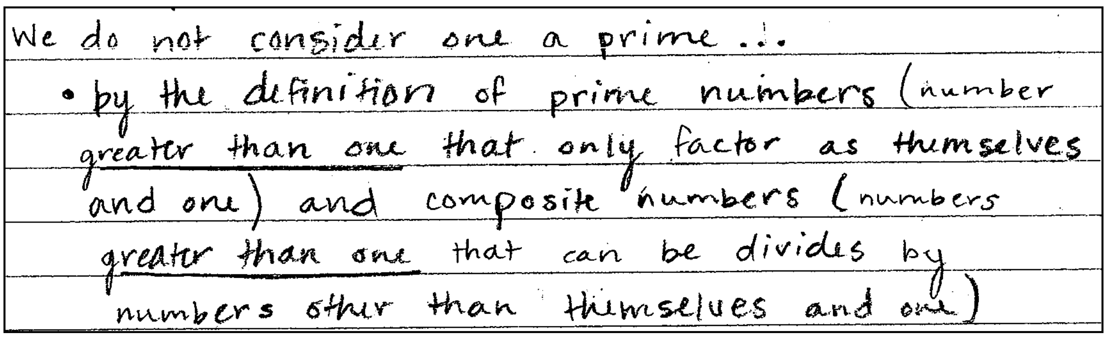

We do not consider one a prime ∴
• by the definition of prime numbers (number greater than one that only factor as themselves and one) and composite numbers (numbers greater than one that can be divides by numbers other than themselves and one)

Figure 11. Lucy's reference to the definition of 'prime' on homework assignment three.

The standard elementary school definition of *prime number* is "a counting number greater than 1 that has exactly two whole-number factors, 1 and itself" (Bell, et al., 2012, p. 266). When asked to define this term, all six participants produced something similar (some referred to divisibility rather than factors) – but *none* of the participants mentioned that prime numbers are greater than 1. For example, Lucy defined prime as "a number that has no other factors besides 1 and itself." I coded each participant's "personal definition" with "its only factors are one and itself." Even though participants produced very similar definitions of "prime", Gwen and Isla understood it differently than the others, which I discuss in my analysis of Problem 7. Brit made the only caveat to her personal definition of prime when she specified that primes are all odd except for two.

I then asked participants why prime numbers are important, and I coded each of their responses with the "prime" subcode of "importance." Almost all of the participants referred to prime factorization in some way, mentioning that they used that quite a bit in their number theory class. I further coded these responses as "prime factorization." In addition to her comment about using primes to factor composite numbers, Brit mentioned how the multiplicative structure of numbers can be used to compare two numbers. Eden was the only one who did not bring up factorization. Instead, she discussed the set of natural numbers. "We have to use them… they're part of everyday… Without them, we'd only have one, four, six, etc. It'd just seem very choppy." I coded this response as "primes are part of natural numbers."

In the student scenario, Problem 7, Shayna conjectures that the number one is a prime number, because its only factors are one and itself. This is a reasonable conclusion when considering the common 'definition' of *prime*: a number is *prime* if it is only divisible by one and itself. Many elementary school curricula add that prime numbers are *greater than* one, but all six participants neglected this caveat when asked to define *prime* in the previous task. Participants explored reasons why one could not be prime and ways for distinguishing it from prime numbers in their number theory class on homework assignment three. Although this misconception was discussed in participants' number theory course, when asked about it during an interview task some participants still wavered in their reasoning. Some admitted to thinking that one was prime prior to the number theory class, recalling that their elementary school teachers *taught them that one* *was prime.* Lucy said, "Shayna is not correct, but that's a tricky one - because that's what my [elementary] teacher told me - that one *is* prime." Building on their previously stated definition of prime numbers, many participants added that one is a "special case" or “an exception to the rule.” All participants did inevitably determine that Shayna’s conjecture was invalid, and these statements were coded using the “validation” subcode of “invalid.”

After all participants stated that one is not a prime number, I asked them to explain their reasoning. While not technically counterexamples, reasoning for why something could not be true most closely resembled my validation code of “counterexample.” Lucy could not explain why one was not prime. "I think we went over it in number theory, but I can't remember. I'm not sure, actually." In spite of this, she did not seem to waver on her belief that one was not prime elsewhere in her interviews. While not ideal, by merely recalling that someone of authority told Lucy that one is not prime”, she considered it to be true. In this way, Lucy’s experience in the number theory course informed her response. Brit, Eden, and Gwen said that one could not be prime because factor trees would never end. For example, the number three could factor into three and one. The number three could further factor into three and one, and one could further factor into one and one, and so forth. I coded these responses as “factor trees would never end.” This was an example participants explored in their number theory class. In spite of her memory of this, Gwen was not convinced.

> Gwen: I've heard it talked about that when you're creating factor trees, if one was prime they would just go on forever, they wouldn't stop… you'd just keep going on because you'd get to prime factors. But I've always kind of thought that it contradicts itself, because you stop when you get to prime factors. And if one is prime, then you'd just stop. But I can reiterate what people tell me...

Gwen's point again provided evidence on how each participant constructed her own meaning of in-class activities. Instead of thinking about never-ending factor trees, Gwen and Isla both preferred to rethink the definition of prime number as only having two *distinct* factors, one and the number itself. Thus, because one violated the definition, it could not be prime. This caveat provided additional insight to Gwen and Isla's "personal definition" of prime, which I coded "prime numbers have two distinct factors." The other four interview participants thought of the number one as an exception to the definition, rather needing to redefine the word prime. I further discuss this difference in participants' personal definitions of prime in Section III of this chapter.

Brit gave a couple other observations in addition to those mentioned thus far: She noticed that every other square number was not prime, so "1" should not be prime either. I coded this reasoning as "square numbers cannot be prime." Brit also noted that if one were prime, "it would mess up all of our other assumptions about primes." Aside from the never-ending factor trees, she could not elaborate. Cara's reasoning was similar to Brit's in that one being prime would violate some previously held understandings. "If we look at one as a prime number, then there are no other prime numbers in math." Cara was convinced that, for example, two could not be prime if one was, because then two could be further factored into primes, making it composite. I coded these two statements as "changes understanding of prime."

None of the participants cited the violation of the Fundamental Theorem of Arithmetic, or the Unique Factorization Theorem, as a reason for why one cannot be prime, in spite of the fact that there were quite a few in-class examples and homework problems related to this idea. For instance, if one was prime, then the prime factorization of 10 could be 2×5, $1 \times 2 \times 5$, $1^2 \times 2 \times 5$, and so forth. The prime factorization of 10 would not be unique. Lucy even referenced this idea on her answer to homework assignment three (see Figure 12). Because participants did not mention the Fundamental Theorem of Arithmetic in this task, Interview 2, Problem 9 was designed (in part) to elicit participants' understandings of this theorem.

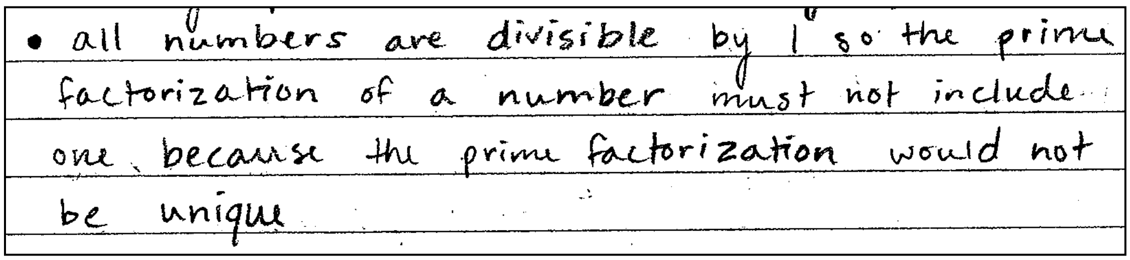


Figure 12. Lucy's reference to unique factorization on homework assignment three.

Table 18 summarizes my coding of participants' responses to Problems 6 and 7 from Interview 2. All responses fell under the content code of "prime." I also used the codes "personal definition", "importance", and "validation." All codes were dually coded as "SCK."

In summary, all participants were relatively confident that one is not prime. They had been reminded of this multiple times in their number theory course, starting with the first day of class. Some participants struggled to remember the reasoning behind this; some even acknowledged that they never understood the reasoning presented in class. In Section III, I discuss how participants' personal definitions of prime affected their responses to the student scenario.

Table 18

Interview 2: Problems 6 and 7 coding summary

Code	B	C	E	G	I	L
Personal Definition						
Its only factors are one and itself	X	X	X	X	X	X
Prime numbers have two distinct factors				X	X	
Importance						
Prime Factorization	X	X		X	X	X
Primes are part of the natural numbers			X			
Validation						
Invalid	X	X	X	X	X	X
Counterexample						
Factor trees would never end	X		X	X		
Changes the understanding of prime	X	X				
Square numbers cannot be prime	X					

Interview 2: Problem 9. In this task, I presented participants with two hypothetical students' work towards factoring the number 540 (see Figure 13). These students, Talisa and Tom, disagreed with each other's work and claimed that their own answers were correct. I discuss participants' validation of Talisa and Tom's work here, and I discuss participants' responses to the PCK-related questions in Section III.

As this was a student scenario problem requiring participants to validate student work, I used "validation" codes under the content code "factoring." All six participants were more familiar with Talisa's method of factoring than they were with Tom's method. Only Lucy had previous experience with Tom's method; she had tutored a student a week prior to the interview, and the student used this method of factoring. All six participants were able to explain Tom's work, though, and recognized it to be valid. I coded participants' validations as "both valid", because all six participants eventually determined both Tom and Talisa's methods to be valid. They also all noticed that while Tom completely factored 540, Talisa still had some work to do. She needed to factor 6 and 9 in her factor tree before she and Tom had the same prime factors. I coded these statements as "Talisa did not factor completely."

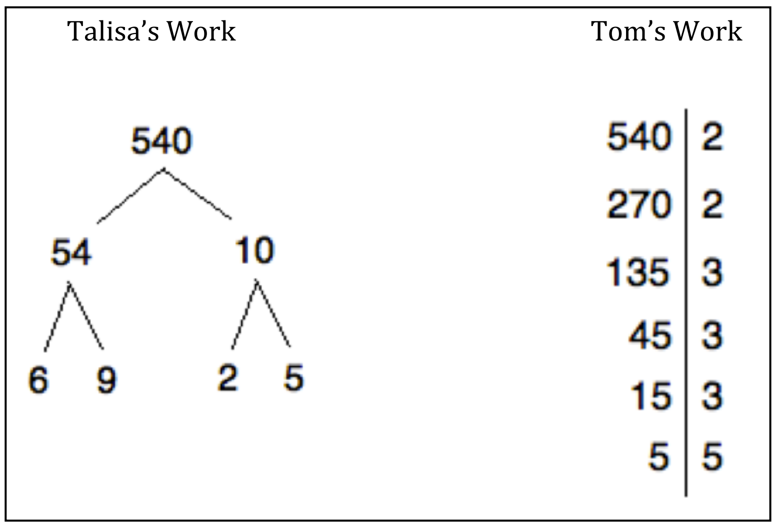


Figure 13. Two students' factorizations of the number 540.

I asked all participants *how* they knew that, once Talisa's factor tree was complete, both factorizations would be the same. Cara's answer was the least specific, stating that as long as Talisa and Tom factored correctly, they would "get the same answer." I rephrased my question a couple of times, but Cara got no closer to describing the Unique Factorization Theorem. The other five participants described the theorem or even called it by name. Brit called it a "unique fingerprint for that number", which was uncommon, but appropriate, language. Eden, Gwen, and Isla all explained that prime factorizations are unique "in this world", as Eden put it, as opposed to "E-world", which was an alternate number system participants explored in their number theory class. In "E-world", odd numbers do not exist, and even numbers can only be factored if they can be written as the product of two even numbers. This allows for multiple "prime" factorizations of the same number. For example, consider Isla's factorizations of 360 from homework assignment three (see Figure 14). Lucy simply stated that every whole number has its own prime factorization. If a participant referenced unique factorization in some way, I coded their statement as "unique factorization."

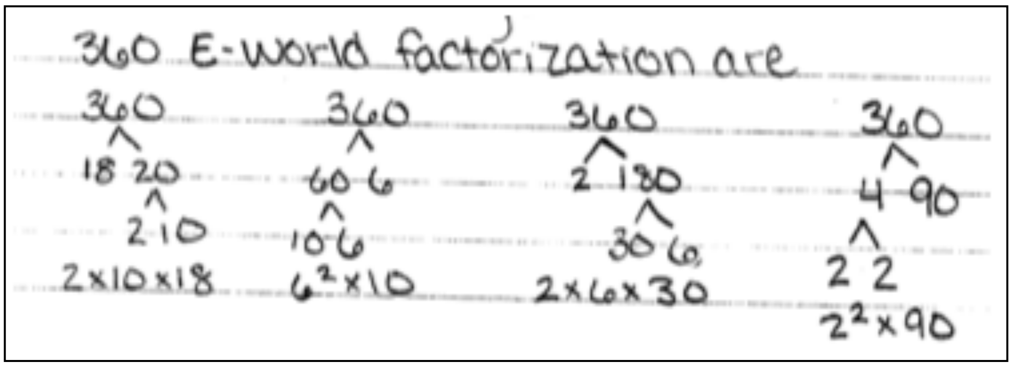


Figure 14. Isla's "E-World" factorization of 360 from homework assignment three.

This was not meant to be a challenging task for participants, which it was not. Rather, it was designed to elicit participants' thoughts on the Fundamental Theorem of Arithmetic, as it is quite often taken for granted by learners. It was also good to see that three of the six participants recalled the juxtaposition that working in E-world presents.

Table 19 summarizes my coding of participants' responses to Problem 9 from Interview 2. All responses fell under the content code of "factoring", as well as the code "validation." All codes were dually coded as "SCK."

Table 19

Interview 2: Problem 9 coding summary

Code	B	C	E	G	I	L
Validation						
Both valid	X	X	X	X	X	X
Talisa did not factor completely	X	X	X	X	X	X
Unique factorization	X		X	X	X	X

Interview 2: Problem 10. For this task, I presented participants with the prime factorization of a number, $M = 3^3 \times 5^2 \times 7$, and I asked that they discuss *M*'s divisibility by two, seven, nine, 11, 14, 15, 26, and 63. The most efficient way to address this task is to try and identify the prime factorization of each potential divisor in the prime factorization of *M*. While most participants did this, there were some instances that suggest some participants were not entirely convinced by the implication of the Fundamental Theorem of Arithmetic, or the primality of two. As a follow-up question, I presented participants with another prime factorization, $N = 2 \times 3^2 \times 5^3 \times 13$, and asked participants to find the greatest common factor (GCF) and the least common multiple (LCM) of *M* and *N*. For the first part of the task, I coded responses using the "factorization" code under the content code "divisibility." For the second part of the task, I coded responses using the "prime factorization" under the content codes "GCF" and "LCM."

All six participants recognized that *M* is not divisible by two, which I coded "divisible by 2", but their reasoning varied as well as their conviction. Cara, Gwen, and Isla immediately recognized that because two is a prime number it needed to be in *M*'s prime factorization if it were to be a divisor of *M*. I coded this as "2 is not in the prime factorization." Brit, Eden and Lucy, however, wavered. They each cited the fact that none of the prime factors of *M* are even, so they could not factor out a two. According to Lucy, "Two is not going to go into it… because we have all odd factors. Because when we're multiplying odd numbers that is going to equal an odd number." While not incorrect, Brit, Eden, and Lucy did not use the most efficient reasoning, and it brings into question their understandings of the primality of two and its role in the prime factorizations of even numbers. I coded these statements as "none of *M*'s factors are even."

This confused reasoning about divisibility by two was perpetuated in participants' responses about *M*'s divisibility by 14 and 26. All participants stated that *M* is not divisible by 14 or 26, which I coded as "not divisible by 14" and "not divisible by 26." The participants who noted that because two was not present in the prime factorization, *M* was not divisible by two (Cara, Gwen, and Isla) also recognized that this resulted in 14 not being a factor of *M*. Brit, Eden, and Lucy maintained their stance that 14 cannot be a factor of *M* because none of *M*'s factors were even, which I again coded as "none of the factors are even." They used similar reasoning in determining that 26 was not a factor of *M*. Cara and Gwen efficiently decided that because two and 13 were not in the prime factorization, their product would not be a factor.

> Cara: I think the prime factorization of 26 is two and 13, and since neither of those numbers are in there [the prime factorization of M], I don't think you can multiply and of those - the three, five, or seven - to get 26, so I don't think it's divisible by 26.

Isla, however, did not use her previously demonstrated efficient reasoning about the divisibility by two to help her address *M*'s divisibility by 26.

> Isla: Um, 26… 5^2 is 25, 3^3 is 27… 21… 35… So I don't think 26 is. And what I was doing was just multiplying 5^2, 3^2, 3^3, 7 and 5, 7 and 3. And I never, out of all those combinations – I never got 26.

When I asked Isla what “combination” gave us 26, she realized that two times 13 was the only way to get 26 and that it would have been more efficient for her to use that in her reasoning.

Participants were all much more successful in determining whether *M* was divisible by seven, nine, and 15. All six participants easily decided that seven is a divisor of *M*, because it is in the prime factorization, or “equation”, as Eden put it. I coded these statements as “divisible by 7.” All of the participants also successfully claimed that nine is a divisor of *M*, because they could “pull out” 3^2 from the prime factorization, which I coded as “divisible by 9.” All were also successful in identifying 15 as a divisor, because they could “pull out” a three and a five, which I coded as “divisible by 15.”

Eden was the only participant to struggle in determining whether 11 or 63 were factors of *M*. Eden waivered a little in determining that 11 was not a factor of *M*, but eventually determined that 11 could not be a factor because she could not “pull out a three, or a five, or a seven.” She felt that 11 would have to be divisible by one of the prime factors of *M* if 11 were to itself be a factor of *M*. While this reasoning is not completely invalid, it proved problematic when she determined that 63 was *not* a factor of *M* "because you wouldn't be able to pull out a five." Eden seemed a bit mixed up, thinking that 63 had to be divisible by *each* of M's factors, rather than the other way around (*M* must be divisible by each of 63’s factors). All of the other participants successfully and efficiently determined that 11 was not a factor because it was not one of the primes in the prime factorization, which I coded as “not divisible by 11,” and 63 was a factor because they could “pull out” 3^2 and 7 from the prime factorization, which I coded as “divisible by 63.”

Participants worked with prime factorizations quite a bit in their number theory class. Aside from the "E-World" factorizations mentioned earlier, participants found prime factorizations on homework assignment two in order to determine the GCF and the LCM of two numbers (see Figure 15). Participants also had to use the properties of exponents to fully factorize numbers such as $a = 10^8 \times 30^5$. Implicit in all of participants' work with prime factorizations was the idea that numbers are divisible by their prime factors and by the product of some of their prime factors. I observed that they had retained a lot of what they had learned about the Fundamental Theorem of Arithmetic and its implications concerning divisibility.

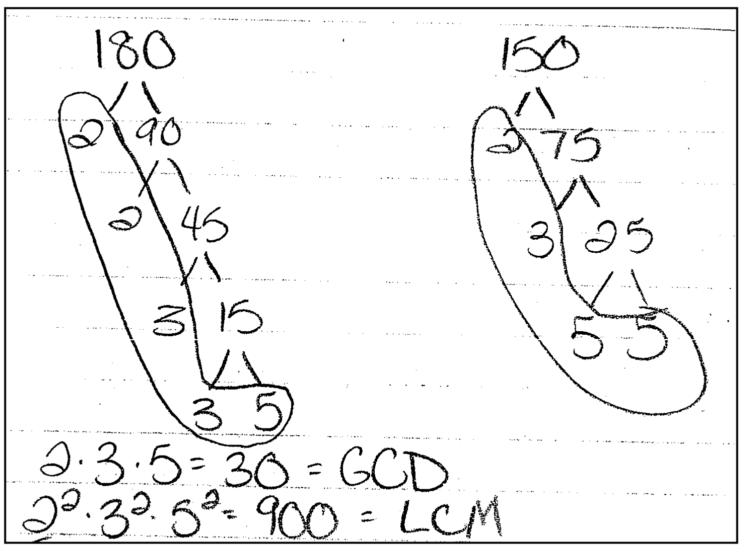


Figure 15. Cara's use of prime factorization to find the greatest common divisor and the least common multiple on homework two.

During the second part of the task, each participant quickly determined the GCF of *M* and *N*, $3^2 \times 5^2$, which I coded as "correctly determined GCF." Participants were also very successful in reasoning through how to find the GCF (identify the common prime factors and the largest prime powers that they have in common). Because participants gave well-reasoned accounts for why the procedure for finding the GCF using prime factorizations works, I coded their explanations as "conceptual reasoning."

Participants were less successful in identifying and reasoning about the LCM of *M* and *N*. Eden and Gwen immediately and accurately found the LCM of *M* and *N*, which was $2 \times 3^3 \times 5^3 \times 7 \times 13$. I coded this as "correctly determined LCM." Eden was not sure why we find the LCM in the way we do; "that's just what we do." Gwen recognized, however, that the LCM must have all of the factors of *M* and *N* (in order for the LCM to be divisible by *M* and *N*). Gwen did not discuss the powers in the factorization.

Brit and Cara initially had incorrect LCMs, but could not reconcile them with what they knew to be true about the LCM. This led them to rethink how they found their LCMs and eventually found the correct value, which I coded as "correctly determined LCM." Brit initially thought the LCM to be $2 \times 3 \times 5 \times 7 \times 13$, because it is "what's left over" once you divide *M* and *N* each by their GCF and multiply. However, Brit was convinced that the product of *M* and *N* was equal to the product of their LCM and GCF (a fact participants explored in their number theory class), which was not true for her LCM. "In this case [N], it [the LCM] has to include at least two 3s, at least three 5s, and a 13. And for here [M], it includes the three 3s, at least two 5s, and a 7." This observation helped Brit find the correct LCM, but she was not confident in her answer. It was not until she multiplied *M* and *N*, then divided by the GCF, and found the same value for the LCM that she trusted her answer. Cara initially could not decide whether the LCM was $2 \times 7 \times 13$ or $2 \times 3 \times 5 \times 7 \times 13$. She eventually decided on the latter, reasoning that all of *M* and *N*'s prime factors need to be represented in the LCM of *M* and *N*. When she tried to explain her reasoning to me, Cara brought up the example of 6 and 8, which caused a conflict because the product of their prime factors, 2×3, was not equal to their LCM, 24. By figuring out how to use the prime factorization of 6 and 8 to find their LCM, Cara was able to correctly find and reason about the LCM of *M* and *N*.

Neither Isla nor Lucy were successful in finding the correct LCM of *M* and *N*. Isla insisted that the LCM was $2 \times 3^2 \times 5^2 \times 7 \times 13$, because it accounts for all of the prime factors of *M* and *N*, and it takes the smallest of the exponents. Isla could not explain why that made sense. Lucy acknowledged that you multiply *M* by something to get the LCM of *M* and *N* and you also multiply *N* by something to get the same number. When I asked her what that meant for the prime factorizations, she suggested that we take the prime factors that *M* and *N* have in common and multiply them, which would give her 15. This seemed completely unrelated to the valid conception she had only moments before. When I reminded Lucy of what she said about having to multiply *M* by something, she immediately realized that 15 could not possibly be the LCM, but she could not think of a way to find it, and we moved on. Because there were so many failed attempts to determine the LCM of *M* and *N*. I went back and coded each time a participant unsuccessfully determined the LCM as “unsuccessful attempt to determine LCM.” Also, while participants reasoning for their procedures varied, only Brit appeared to have a relatively complete understanding of why the procedure for finding LCM works, so her reasoning was the only one I coded as “conceptual reasoning.”

Table 20 summarizes my coding of participants’ responses to Problem 10 from Interview 2. I condensed the divisibility “factorization” subcodes quite a bit, because participants had similar responses. I condensed “not divisible by 2”, “not divisible by 11”, “not divisible by 14”, and “not divisible by 26” into the single code “not divisible by

2, 11, 14, or 26." I also condensed the codes "divisible by 7", "divisible by 9", and "divisible by 15" into the single code "divisible by 7, 9, and 15." I could not include divisibility by 63 in this new code, because Eden did not recognize *M*'s divisibility by 63. For the second part of Problem 10, I used the "prime factorization" code and developed subcodes for successfully determining the GCF and the LCM and also for conceptual reasoning. All codes were dually coded as "SCK", except for "unsuccessful attempt at determining LCM."

Table 20

Interview 2: Problem 10 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Divisibility – Factorization						
Not divisible by 2, 11, 14, or 26	X	X	X	X	X	X
Divisible by 7, 9, and 15	X	X	X	X	X	X
Divisible by 63	X	X		X	X	X
2 is not in the prime factorization		X		X	X	
None of *M*'s factors are even	X		X			X
GCF – Prime Factorization						
Correctly determined GCF	X	X	X	X	X	X
Conceptual reasoning	X	X	X	X	X	X
LCM – Prime Factorization						
Correctly determined LCM	X	X	X	X		
Unsuccessful attempt at determining LCM	X	X			X	X
Conceptual reasoning	X					

"GCF" refers to greatest common factor and "LCM" refers to least common multiple.

In summary, participants were fairly successful in identifying the divisibility of a number, given its prime factorization. For half of participants, however, there seemed to be some confusion about divisibility by two and multiples of two. Participants looked for ways to get an even number from the product of odd numbers, which is concerning. All participants successfully found the GCF of two numbers, given their prime factorizations. They also successfully explained their reasoning for this procedure. Participants were less successful using the procedure to find the LCM from prime factorizations, let alone explaining it, which is concerning given their course experience with this procedure.

In Section II, I present the rest of the interview data pertaining to participants' number theory content understandings. The number theory content in Section II focuses on modeling GCF and LCM and, ultimately, participants' understandings of modeling GCF and LCM story problems. The following section continues to address my first research question. At the conclusion of Section II, I summarize an answer to my first research question.

Section II: Understanding Least Common Multiple and Greatest Common Factor Story Problems

Content knowledge of number theory topics taught at the elementary level can often be classified as specialized content knowledge (SCK), a construct of mathematical knowledge for teaching conceptualized by Ball and colleagues. Recall that SCK is "the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Ball, Thames, & Phelps, 2008, p. 377-8). For instance, a teacher demonstrates SCK by creating story problems that accurately represent the ideas of greatest common factor (GCF) and least common multiple (LCM) in the context of real life.

Story problems contextualize the mathematical structures of the symbolic problems they represent and pose a question or task for the reader to answer. Understanding story problems, creating and critiquing them, lie within the realm of SCK. Creating and critiquing story problems require content knowledge specific to teachers, but as many of my participants acknowledged, story problems are also useful in "helping kids understand" concepts, suggesting that their use in the classroom may demonstrate pedagogical content knowledge or PCK (Shulman, 1986). According to Chapman (2006),

> Word problems can be used as a basis for application and a basis of integrating the real world in mathematics education. They can provide practice with real life problem situations, motivate students to understand the importance of mathematics concepts, and help students to develop their creative, critical and problems solving abilities (p. 212).

During the first two interview tasks, I asked participants to create GCF and LCM story problems, model GCF and LCM using pictures and manipulatives, and identify whether a set of given story problems represents valid GCF or LCM story problems. In this section of my results, I present participants' understandings of these various models of GCF and LCM, with a focus on their understandings of GCF story problems. My data suggested a general process for developing an understanding of GCF that would enable the participants to create and validate GCF story problems.

To incorporate the social lens of my framework, I begin by describing any coursework and classroom experiences that may have contributed to participants' understandings or ability to model GCF and LCM using story problems, manipulatives, or pictures. This informs the individual responses of my participants, which I analyze using the psychological lens of my framework. I then present a summary of participants' responses to the LCM story problem task. Participants modeled LCM with pictures, manipulatives, and story problems in their number theory course. Possibly due to their familiarity with these LCM representations, the interview data were not rich enough to suggest how participants developed their understandings of LCM story problems. However, the data suggested participants' processes for creating GCF story problems. I then describe participants' responses to tasks that required them to critique GCF story problems. Afterwards, I present each interview participant's process for understanding GCF in a way that would enable her to create GCF story problems.

At the end of this section, I summarize overarching themes that emerged from my data on participants' understandings of number theory content. These themes incorporate my results from Sections I and II. Finally, I summarize answers to Research Question 1.

Contributing Coursework and Tutoring Experiences

As evidenced in my field notes, and participants' assignments and tests, participants did not have an opportunity to create, or even answer, GCF story problems in their number theory class. They were, however, given the opportunity to briefly explore visual, concrete, and story problem representations involving LCM. Participants were asked to create and analyze an LCM problem for elementary or middle school students to solve as part of their second homework assignment. Some participants designed tasks that required students to use Cuisenaire rods to find the LCM of two numbers (a procedure introduced in their number theory course) while others wrote story problems involving LCM in other ways. Much later in the course, participants also solved and created story problems related to the Chinese Remainder Theorem.

It is important to note that as part of their mathematics education course and while they were helping out in the number and operations course, Cara, Gwen, and Lucy may have seen various representations of GCF and LCM. After successfully completing the number and operations course herself, Brit tutored other students enrolled in the course. She may have had the opportunity to see various representations of LCM and GCF during her tutoring experiences. As a student grader for the number and operations course, Lucy may have also had additional opportunities to explore GCF and LCM representations.

Overall, participants had more experience with modeling LCM using elementary school level methods than they did with GCF. Perhaps as a result of this, the interview participants had more immediate success in responding to the LCM tasks than the GCF tasks. However, their limited experience with GCF story problems presented me with a unique opportunity to observe their processes for trying to understand this concept in a novel way. In the next section, I briefly synthesize participants' responses to the LCM story problem task.

Modeling Least Common Multiple: Story Problems, Pictures, and Manipulatives

For the first interview task (refer to Appendix G), I asked participants to create LCM story problems that would require someone to compute the LCM of six and eight. I used the content code "LCM" for all participants' responses. I also used the code "modeling" and subcodes "story problem" or "visual" for most responses. Because I asked participants to validate given story problems, I also used the code "validation."

Perhaps because participants experienced this in their number theory course, there were fewer mistakes and struggles with this task than when I asked participants to model

GCF story problems. The most common struggle with modeling LCM story problems was that most participants posed a question asking for *any* common multiple, rather than the *least* common multiple. Participants were otherwise successful in modeling LCM story problems. Only two participants began with pictorial or manipulative models or the meaning of multiplication before suggesting an LCM story problem. Due to the relative ease and familiarity with this task, I did not have enough interview evidence to suggest how participants built their understandings of LCM story problems, as I did with GCF story problems. Participants saw the definition of LCM in class, and they were shown how to model LCM using Cuisenaire rods and pictures. It is likely that these experiences contributed to their ability to produce LCM story problems, but I did not witness the process like I did with participants' experiences with GCF story problems.

Brit was the only participant whose LCM story problem contained all of the necessary components. I coded her story problem as "valid story problem." When I asked Brit to create a story problem to model the LCM of six and eight, she started by modeling the LCM of six and eight using Cuisenaire rods, which I coded "valid visual model." She determined that the dark green rod was six units long and the brown rod was eight units long. Then she created one-color trains using the rods. Brit presented her solution in terms of what she would have students do, even though the question was not phrased in that context.

> Brit: What I would have kids do… is line up the blocks until they match, which turns out to be three brown ones and four green ones. Then they would do three times eight is 24 and six times eight is 24, would be the least common multiple.

Brit explained that she first saw this representation in the number theory course and that it was a "really good way to visually see it… All you have to do is line up the blocks in a row and connect it back to the math." It seems that Brit used this representation to help her organize her story problem. As a result, I coded Brit's visual model of LCM as "visual model contributed to story problem."

> Brit: So I have a nice society where my only money is $6 coins and $8 coins… I only have those two amounts. So I want to know what amount I can make using the $6 coins, just the $6 coins, can make the same amount as just the $8 coins. And I want the least, because I don't want to carry around that many coins in my pocket at the same time. So I want the least amount that I can make with $6 coins that is the same as what I can make with $8 coins.

While initially wordy, Brit produced a relatively successful LCM story problem. When I asked Brit about her reasoning for phrasing her story problem in this way, she said that she knew to be careful not to use combinations of six and eight; she needed to use *just* $6 coins and *just* $8 coins. Brit explained that she had used an inappropriate monetary context for the LCM story problem on her homework assignment, and that she knew to adjust it for the interview task. This was evidence that Brit's classroom experience contributed to her response. I coded an "appropriate context" as one that necessitated groups of six and eight objects that could not be broken into smaller parts. Although Brit did not explicitly acknowledge it, specifying that the collection of $6 coins be the *least* amount that is equal to a collection of $8 coins was also important for appropriately phrasing her story problem.

Cara explored the meaning of LCM by recalling the meaning of multiplication before attempting a story problem. She thought about how students could find the LCM of six and eight. She said that she would "maybe have them write out the multiples of each number and then see if they can match them up." In her transition to creating a story problem, Cara reminded herself that the LCM of six and eight was 24, which is four times six and three times eight, or four groups of six and three groups of eight. She claimed that her story problem should reference groups of six and groups of eight. In this way, Cara's verbal description of a visual model helped her to construct her story problem, which I coded as "visual model contributed to story problem." Cara chose the context of chocolate chip cookies: some cookies have six chocolate chips and others have eight chocolate chips.

> Cara: So if you have six chocolate chips in this cookie, how many cookies with eight chocolate chips would you need to have the same amount of chocolate chips… er, how many cookies of six do you need, or six chocolate chips, and how many cookies of eight to equal the same amount of chocolate chips?

Like Brit, Cara chose an "appropriate context," and interpreting the problem in terms of the meaning of multiplication (i.e., $a \times b$ means a groups of b objects) appeared to help with her development of the problem. However, her question concerning the number of cookies was not asking for the LCM of six and eight. Rather, it asked the reader to find factors by which we would need to multiply six and eight in order to obtain a common multiple. Cara also did not qualify this multiple. She simply stated that the total number of chocolate chips needs to be the same, not necessarily the *least* number of chocolate chips. Thus, I coded Cara's question as "inappropriate question."

Eden started the interview task by generating a story problem. "The microwave timer will go off every six minutes and the oven timer will go off every eight minutes. When will both timers go off at the same time?" As opposed to groups of quantities like Brit and Cara, Eden used the context of time, which is also an "appropriate context." With time, we have two timelines, one for the microwave and one for the oven. A statement comparing the two timelines is necessary in order to address Eden's question; it is unclear when the microwave and the oven timers last went off at the same time. Because of this, I coded Eden's story problem as "no starting point." Additionally, Eden did not specify that we want the *next* time the two timers will go off, which would imply the *least* common multiple rather than any multiple. Thus, I coded her question as an "inappropriate question." When I asked Eden why she phrased her story problem in this way, she said, "I have no idea. I just thought of, like, when the two would come together. So if this one goes off and this one goes off at the same time, when will they come together?" Here, Eden clarified that she meant for the microwave and the oven to go off at the same time.

Isla also proposed a story problem rather than begin the task by modeling LCM using manipulatives or pictures. And like Eden, Isla's story problem used the context of time. She ran into similar issues with phrasing, but the context was not as successful as Eden's. "Your best friend plays [basketball] in 6-minute intervals, and you play in 8-minute intervals. When are you guys going to play together for the first time?" Given that time intervals overlap, Isla's question did not necessarily ask for a common multiple. Perhaps if Isla would have been more successful had she asked, "if you and your friend start playing right now, when is the next time (in how many minutes) that the two of you will start playing at the same time again?" Because Isla neglected to specify when or if the two basketball players started playing together, I coded her story problem as having "no starting point." I did not code Isla's context as appropriate because it would have taken much more clarification from her to make it appropriate. Also, because her question was unanswerable, I coded it as an "inappropriate question."

When asked about her reason for phrasing her story problem as she did, Isla said that the context made sense in terms of multiples, adding six repeatedly and adding eight repeatedly. She said that it was also similar to something she had done in the number theory course, where they would repeatedly add lengths or Cuisenaire rods until the two sets of lengths were even. While it was clear that Isla's number theory course experiences contributed to how she created her LCM story problem, because she did not begin the task with a visual model, I did not code Isla's response as "visual model contributed to story problem."

Lucy also posed an LCM story problem using the context of time. "Stacy goes to the park every six days, and Edward goes every eight days. So is there ever going to be a day where they are there together?" Lucy's question had similar issues to Isla's because the context refers to intervals of time. Stacy and Edward might never go to the park *together*, per se, but they could go to the park on the same day. Lucy's question also needed to be rephrased so that it was not a yes or no question. The context of time also requires a comparison of the timelines, and the question needs to refer to the *next* day that they are both at the park in order to require the reader to find the LCM. With a little work, the context of Lucy's story problem would have worked, so I coded it as "appropriate context." However, I also coded her story problem as having "no starting point" and an "inappropriate question."

Gwen generated a new type of story problem. She referred to a single quantity and described how to arrange the quantity into even groups of six and also rearrange it into even groups of eight. In other words, this quantity is both divisible by six and divisible by eight. This is different from the multiplicative way that other participants were thinking of LCM. It is also similar to contexts participants had seen in class when working with the Chinese Remainder Theorem. Because she used a verbal description of a visual model to help her create her story problem, I coded this as “visual model contributed to story problem.”

> Gwen: When I arrange a certain number of chairs, they fit into even rows of six going across. And when I arrange them into rows of eight, they fit evenly, without any strays. So how many chairs do I have in all if when I arrange them they fit into six even rows and eight even rows?

Gwen’s context and phrasing were appropriate, but her question asks for any common multiple rather than the *least* common multiple, which I coded as an “inappropriate question.”

Given that most participants, except for Brit, did not explicitly discuss using manipulatives or pictures to model LCM, when they generated their story problems, I made sure to ask them to draw or describe how to model LCM with pictures or manipulatives. Every participant was successful in modeling LCM in both of these ways, and I coded each instance as “valid visual model.”

Typically, participants drew pictures of their manipulative representations. Brit, Cara, Eden, and Lucy all successfully described or demonstrated how to use Cuisenaire rods to model the LCM of six and eight, recalling that they had learned how to use this model in their number theory course. Isla also remembered seeing the Cuisenaire rods used in class, but she could not figure out how to determine which rod represented six and which rod represented eight. She did, however, successfully describe making multiple groups of six objects and multiple groups of eight objects until the two groupings were equal in quantity. Cara and Gwen also described finding the LCM of six and eight colored counters. They recalled using the groups of colored counters to represent multiples in their mathematics education course.

I also asked participants to validate possible LCM story problems (refer to Appendix G) that required one to find the LCM of six and eight. I coded responses using the "validation" code. There were two multiplication story problems and two LCM story problems, but one of them was only invalid in a subtle way and not everyone correctly validated it. It was an LCM story problem, but the units associated with six were different from the units associated with eight. The problem required that the reader convert the units so that they were the same. Once converted, the reader would not be finding the LCM of six and eight. All of the participants were successful in identifying the multiplication story problems and the LCM story problems. I coded participants' appropriate validations of each of the sample story problems as "valid" or "not valid." However, Brit and Isla did not realize the units were different on the second LCM story problem, so they incorrectly determined this story problem to be a valid representation for the LCM of six and eight. I coded their validations of that sample story problem as "valid, incorrect."

Table 21 summarizes my coding of participants' responses to Problem 1 from Interview 1. I used the content code "LCM" for each instance. I also coded valid responses, like a valid model of LCM or an appropriate context, as instances of "SCK."

The data from the LCM tasks were not nearly as revealing as the data from the similar GCF task (the second interview task). Four of the six participants created an LCM story problem when asked, but five participants explored GCF in a different way before attempting to create a GCF story problem when asked. Additionally, participants' processes indicated similarities, which are described in the next sections.

Table 21

Interview 1: Problem 1 coding summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Modeling – Story Problem						
Valid story problem	X					
Appropriate context	X	X	X	X		X
Invalid story problem		X	X	X	X	X
No starting point			X		X	X
Inappropriate question		X	X	X	X	X
Modeling - Visual						
Valid visual model	X	X	X	X	X	X
Visual model contributed to story problem	X	X		X		
Validation						
Story Problem A: Not Valid	X	X	X	X	X	X
Story Problem B: Valid	X	X	X	X	X	X
Story Problem C: Not Valid	X	X	X	X	X	X
Story Problem D: Not Valid		X	X	X		X
Story Problem D: Valid, incorrect	X				X	

Modeling Greatest Common Factor: Story Problems, Pictures, and Manipulatives

The second task I posed to participants during the first interview (Appendix G) was to create a GCF story problem that would require someone to compute the GCF of

28 and 32. All participants (with the exception of Eden) eventually attempted to create a GCF story problem (see Table 22). Each of these story problems drew from one of the two meanings of GCF, which are explained further in the next section. Aside from Brit, who began her response by attempting to create a GCF story problem, participants engaged in different activities to help them respond to this task. Eden and Isla verbally recalled the basic definition of GCF; others used numerical methods to find the GCF of 28 and 32. For instance, Cara and Eden found the GCF by listing all of the factors of 28 and 32 and selecting the largest of the common factors. Isla and Lucy discussed how to use factor trees to find the GCF of two numbers. These strategies did not appear sufficient for helping participants to create story problems, so they switched strategies. While I planned on following-up by asking participants to represent the GCF using pictures or manipulatives, most participants spontaneously created or described visual or concrete representations of the GCF which inspired their story problems.

Because Problem 2 from Interview 1 concerned modeling GCF, most responses were coded using the content code “GCF.” Similar to Problem 1 from Interview 1, I also used codes “modeling” and “validation,” as well as modeling subcodes “story problem” and “visual.” However, in addition, I used the GCF code “personal definition.”

Table 22

Interview participants' greatest common factor story problems

Participant	*GCF Story Problem*	*Meaning of GCF*
Brit	OK, so if I have 28 dinosaur stickers and 32 flower stickers and I want to group the dinosaur stickers and the flower stickers together… and I want to give them to individual students. So I want to know what is the greatest… how many, how many dinosaur stickers and flower stickers am I going to need in each group? I want to use all of them in an equal amount of groups. So I want to know how many stickers are going to be in each group.	"How many in each subgroup?"
Cara	If I want even groups of our class to equal groups of another class, how many groups do we need if I have 32 students, and they have 28 students, can you tell me how many groups each classroom will have?	"How many subgroups?"
Cara	So if we wanted to split our class evenly, our classes evenly, how many people would be in each group if we had one group left over? For the 8, it is one more than 7. So you would have one group of 4 left over… How could we split our groups so that each class has the same number of people in each group?	"How many in each subgroup?"
Gwen	Alex has 28 objects and Kim has 32 objects. If each one decides to group their objects into equal groups, how many objects will be in each group? … So there's… an equal number of things in each group.	"How many in each subgroup?"
Isla	You had 28 things and your friend had 32 things. How could you each group your things where you have the same number in your largest group, but you couldn't have remainders left over?	"How many in each subgroup?"
Lucy	Someone has 32 marbles and then student B has 28 marbles. Can they divide them into the same amount of groups, like the highest number, the same amount of groups?	"How many subgroups?"

"GCF" refers to greatest common factor.

Representing greatest common factor. As with my pilot study, interview participants created two distinct types of GCF representations, similar to the two types of meanings or models of division. Recall that Ball (1990) called the first of these models the *measurement model*: when forming groups of a certain size (the divisor), the number of groups is your dividend. The second is the *partitive model*: when forming a certain number of groups (the divisor), the number of objects within each equal group represents the dividend. All participants would have been familiar with Beckmann (2008) from their experiences with the number and operations course, the textbook routinely used in the number and operations course. Beckmann uses different language than Ball for describing these two types of division representations; her language draws more attention to the value that represents the dividend. She referred to the measurement model as the "How many groups?" representation, where the dividend is the number of groups, and the partitive model was called the "How many in each group?" model, where the dividend is the number of objects in each group.

Initially in my analysis, I referred to the two types of GCF representations using the same language as Beckmann (2008). However, this turned out to be problematic, not only in my attempts at discussing participants' responses, but also in participants' discussions of their own representations. We determine the GCF of two numbers, *A* and *B*, by breaking down *A* objects and *B* objects into equal groups (either an equal number of groups or equal sized groups). However, participants frequently referred to the *A* objects and the *B* objects as "groups" as well. To avoid confusion, I reserve the term "group", in conjunction with modeling GCF, for the groups of *A* or *B* objects. I refer to the smaller groups that amount to *A* or *B* objects as "subgroups." In this section, I discuss the two types of GCF representations in depth and address participants' understandings of them. The two types of GCF representations also served as codes in my analysis.

"How many subgroups?" representations. With "How many subgroups?" representations of the GCF, the groups of *A* and *B* objects are broken up into the same number of subgroups, and this number is maximized. In other words, to find the GCF of *A* and *B*, we find the largest number of subgroups that both *A* and *B* objects can be broken into. Implicit in this language is that all of the objects are equally distributed amongst the subgroups. For a "How many subgroups?" representation of the GCF to be valid, each of these conditions should be addressed. Anytime a participant attempted to model GCF with pictures, manipulatives, or story problems using this structure, I used the modeling subcode "How many subgroups?"

Cara and Lucy modeled or described how to model the GCF to assist them in creating story problems. They both drew from the "How many subgroups?" representation of the GCF. After finding the GCF of 28 and 32, Cara described how to break up 28 and 32 objects to show the GCF is four.

> Cara: So you would end up with four groups of a certain number in it. So for 28, you would have four groups of seven, and with 32 you would have four groups of eight. So the number in your groups would be different, but the amount of groups is the same, showing that that represents the [greatest] common divisor.

Cara's description emphasized that to find the GCF we must equate one factor of 28 and 32, represented by the number of "groups" in this case, but the other factor is immaterial. However, even though Cara described how to model the GCF with manipulatives, she did not describe how one might use manipulatives to *find* the GCF of 28 and 32. Lucy suggested that she would facilitate an activity with elementary school students to help them to find the GCF.

> Lucy: So I'm thinking I would probably show 28 chips, 32 chips, then saying can you put them in groups... you know, starting out in different groups to figure out what their GCF is. Like, how many ways are they divisible with no remainder? Let's say, go through 10. Try splitting them into groups of 1, groups of 2... all the way through 10, and then having the students compare their data. So I would make like tables for that. So... I would say… "How many groups of 1 were there? How many groups of 2 were there?" They could fill in the chart and say whether there were remainders or not. They would do the same for 28, all the way through 10, and then comparing what the GCD would be then, based on their charts if 28 and 32 have that GCD they would be able to see, "oh, they both divided into so many groups." Because you could do this with any number, so they both divided up into so many groups evenly. And that's the highest, so that would be their GCD.

Lucy's activity, which I coded "used to find GCF," could indeed help students recognize the GCF of two numbers by drawing attention to a common number of subgroups. However, her method could draw an equal amount of attention to a common number of objects in subgroups, something that she did not acknowledge. Her choice to find factors "through 10" was also curious, given that checking for factors through the square root of N will generate all factor pairs of N.

"How many in each subgroup?" representations. With "How many in each subgroup?" representations of the GCF, the groups of A and B objects are each broken into subgroups with an equal number of objects in each, and this number is maximized. In other words, to find the GCF of A and B, we find the largest number of objects that can fit into each subgroup of both A and B. Again, implicit in this language is that all of the objects are equally distributed amongst the subgroups. For a "How many in each subgroup?" representation of the GCF to be valid, each of these conditions should be addressed. Anytime a participant attempted to model GCF with pictures, manipulatives,

or story problems using this structure, I used the modeling subcode “How many in each subgroup?”

Cara and Gwen used “How many in each subgroup?” representations of the GCF to assist in creating story problems. Cara compared her model using this representation to the one she described using the “How many subgroups?” meaning of GCF, again emphasizing how one of the factors of 28 and 32 was immaterial.

> Cara: So if we wanted, we could do it the opposite way. Where the groups are even, or the amount in the groups are even, but then the amount of groups might not be even in this case. So you would put 4 in each group.

In Gwen’s description, she began to contemplate how she might use her description to create a GCF story problem, touching on the reasoning necessary to facilitate this transition.

> Gwen: So maybe we have 28 objects and 32 objects… make them into equal groups with the same amount in each group for… So there's going to be 8… this is going to have 7… I'm thinking that this will show that there's 4 in each one. But I would have to word it in a way that would make sense that there are equal groups in each one for the numbers 28 and 32 to have the same amount in each group...

As with Cara’s “How many subgroups?” model of GCF, Cara and Gwen’s discussions of their “How many in each subgroups?” models did not address how they might use the manipulatives to *find* the GCF of 28 and 32. Later on, Gwen revisited her visual model of GCF and explained how to use it to find the GCF.

Rather than use a GCF representation to create a GCF story problem, as Cara and Gwen did, Isla attempted a story problem and then discussed a “How many in each subgroup?” representation of the GCF to explain her reasoning for her story problem. Isla attempted a story problem, albeit confusing and contrived, after she discussed how to find the GCF of two numbers using factor trees, explaining that “you're kind of getting them to make the very bottom of the factor tree by just putting their block into whatever you use in groups." Later, Isla clarified what she meant by referring to how she would group pennies to represent the GCF, but struggled to articulate how she might use this representation to *find* the GCF.

> Isla: If you get somebody with 28 pennies, or 28 little round things, or whatever, and then somebody 32, and you wanted them to group them so then there is an equal number in each group. And… I'm trying to think maybe if you can use... but you would have to know what the GCD is.

Brit also discussed a "How many in each subgroup?" representation of the GCF. However, she was the only participant to *not* use a visual or concrete representation to assist in or explain creating a GCF story problem. She only discussed a concrete model when I later prompted her. As Lucy did in her discussion, Brit carefully explained how she might use manipulatives to find the GCF.

Conflated representations. Eden was initially the least successful in her attempts to create a GCF story problem. After more than five minutes of Eden making little progress and displaying frustration, I suggested that Eden think of a concrete or visual representation, thinking of 28 and 32 as groups of objects. While this seemed to inspire Eden, she struggled to determine the role of the GCF within her model.

> Eden: OK, so you could have 28 objects and 32 objects and you could do… you could have what is the largest number that you could fit into… So if we have 4 circles. So you would have 4 in each one.

When Eden illustrated this idea, she became confused as to why the number of objects did not amount to 28 and 32. She claimed that her goal was for each of the subgroups to have 4 objects, but she felt that the number of subgroups should also amount to 4, the GCF. Eden struggled with the idea that the number of subgroups did not matter.

Later, Eden suggested that she would use Cuisenaire rods to make one-color trains that were 28 units and 32 units long. She associated the largest block that she could use for both trains with the GCF, drawing from a “How many in each subgroup?” meaning of GCF. However, she also broke the trains into four groups of seven and four groups of eight, but she struggled to describe how this related to the GCF. While Eden had hoped that one of her concrete representations of the GCF would help her create a GCF story problem, she felt frustrated by her attempts, possibly due to her conflated understanding of GCF. Eventually, Eden asked to move on. Because of Eden’s inconsistent use of a single representation of the GCF, I coded her attempts at modeling GCF with “conflated representation.”

Table 23 summarizes which types of GCF representations participants created using manipulatives, pictures, or descriptions, as well as the codes I used to identify them. The table also accounts for whether or not participants described how to use their models of GCF to find the GCF of 28 and 32. I also recorded which participants used their model to inform their story problems.

Table 23

Interview 1: Problem 2 visual modeling codes

Code	**B**	**C**	**E**	**G**	**I**	**L**
Modeling – Visual						
Valid: “How many subgroups?”		X				X
Valid: “How many in each subgroup?”	X	X		X	X	
Conflated representation			X			
Used to create story problem		X		X		X
Used to find greatest common factor	X			X	X	X

Understanding of division. After the initial interviews with participants, I wondered if perhaps participants' tendency towards one GCF representation over the other was related to their understandings of the two meanings of division. To investigate this, I began the second set of interview tasks with a division task (refer to Appendix H), where I asked participants to create a division story problem, represent division with pictures and manipulatives, and validate division story problems. As anticipated, some participants demonstrated a predilection to similar representations. For example, a participant with a more refined understanding of "How many in each subgroup?" representations of the GCF had a more refined understanding of "How many in each group?" representations of division. (I describe later what I mean by a "refined understanding.") Unexpectedly, Isla and Lucy demonstrated quite the opposite, which I describe in more detail later in my descriptions of their individual understandings of GCF story problems.

Attempted story problems. Recall that for a representation of the GCF to be valid, (1) *All* objects should be equally distributed amongst subgroups within each group (to establish that we are working with factors); (2) there should be an equal number of subgroups between groups (as with the "How many subgroups?" meaning of GCF) *or* subgroups between groups should have an equal number of objects (as with the "How many in each subgroup?" meaning of GCF); (3) the number of subgroups *or* the number of objects per subgroup should be as large as possible (i.e., maximized). Interview participants struggled more in creating valid GCF story problems than they did in creating valid visual representations of the GCF. This may be due to the various minutiae involved in creating a GCF story problem.

The aforementioned conditions establish the GCF structure of a story problem, but there are other aspects to consider; the narrative of a story problem contextualizes the structure of a mathematical concept, and a story problem also poses a question related to this concept for students to answer. For GCF story problems, this question should be precise enough that the only answer is the GCF. Additionally, to ensure that the story problem is as authentic to real life as possible, the context should necessitate the conditions of the mathematical structure somehow. With GCF story problems, it is not enough to describe breaking up groups of objects into smaller groups; the context should present motivation for doing so. In this section, I discuss participants' story problems (see Table 22) with respect to the necessary conditions for the GCF, contextualization, and the question posed, and their related codes.

Greatest common factor structure. Aside from Eden, all of the interview participants attempted to create GCF story problems, with varying degrees of success. These story problems drew from the meaning of GCF participants used to create their visual or concrete representations of the GCF. Thus, I used the modeling codes "How many subgroups?" and "How many in each subgroup?" to account for the overall GCF structure of the story problems. However, none of the story problems were valid, so I also coded them "invalid." Lucy created a "How many subgroups?" GCF story problem, while Brit, Gwen, and Isla created "How many in each subgroup?" GCF story problems. Cara was the only participant to create both types of story problems. Some participants used ambiguous phrasing, like "even groups" (Cara) and "equal groups" (Cara, Gwen) and "equal amount of groups" (Brit), that required clarification before I could determine the meaning of GCF from which they were drawing.

Occasionally, as with Cara, the word "even" meant that the subgroups were equal-sized within each group, which is necessary to guarantee that the numbers of objects per subgroup are factors of 28 or 32. It is also necessary that all objects are contained within a subgroup, but Brit and Isla were the only participants to explicitly mention this; perhaps Cara, Gwen, and Lucy felt this condition was implied. Lucy frequently used the term "divide" to refer to the process of distributing all objects equally amongst groups.

All of the interview participants established that they were looking for common factors by stating that the number of subgroups or the size of the subgroups between groups should be the same. However, Cara and Gwen neglected to include a statement maximizing the common factor, which I coded as "did not maximize." Recall that both Cara and Gwen deemphasized maximizing the common factor in their descriptions for using manipulatives to model GCF by using the GCF to create their representations (as opposed to using the manipulatives to *find* the GCF). Brit, Isla, and Lucy used the words "greatest", "largest number", and "highest number", respectively, in their story problems to indicate that they were looking for the GCF. However their phrasing was incomplete or unclear. Perhaps, with these participants, the struggle was not in recognizing that they needed to maximize the common factor, but in contextualizing this condition.

Contextualization. While Brit's story problem was somewhat unclear and required clarification, it was perhaps the most contextualized of the story problems. Not only were the numbers 28 and 32 put into a context, but most of her conditions were also phrased consistently with this context. Maximizing the common factors was the only condition that she neglected to phrase in context. Brit also initially hinted at reasoning for grouping the stickers as she did, "I want to give [the groups of stickers] to individual students."

Cara’s story problems were similarly contextualized, but it is unclear why we are grouping students in this way. For instance, if Cara planned on teaming up each of her groups of students with a group from the other class, it would necessitate finding a common factor. Because of how contextualized Brit’s and Cara’s story problems were, I coded them as “contextualized.”

Gwen, Isla, and Lucy posed story problems that were contrived and barely contextualized. Lucy referred to the groups of 28 and 32 objects as marbles, and all three participants assigned one group of objects to one person and the other group of objects to another. However, they did not contextualize any of the conditions for GCF. As a result, I coded their story problems as “not contextualized.” The only discernable difference between Gwen, Isla, and Lucy visual or concrete representations and their story problems was that they posed a question for students to answer.

Questions posed. While the narrative of a story problem sets the stage for the mathematics with which readers are working, the question the problem poses gives readers a clear direction for how to proceed with the mathematics. Without a question, there can be no solution. Without a clear question, there can be many possible solutions. I tasked participants with creating story problems that would require readers to find the GCF of 28 and 32. To do so, it was necessary for participants to word their questions in such a way that the GCF of 28 and 32 is the only solution.

While poorly worded, Brit somewhat conveyed that she wanted readers to find the greatest number of stickers in each group (i.e., the GCF). In spite of her wording issues, Brit’s question was the most specific, and I coded it as an “appropriate question.” Cara’s “How many subgroups?” question was somewhat vague; any common factor would be a sufficient solution for the problem. Gwen's story problem question was similarly vague, because it also lacked a statement maximizing the common factor. I coded Cara's and Gwen's questions as being an "inappropriate question."

Cara had a different problem in phrasing a question for her "How many in each subgroup?" story problem; her final question was even more vague in that it required a description for how to group the objects rather than a numerical solution. Isla's question begged a similar description, and Lucy phrased her question to have a yes or no answer. However, similarly to Brit, Lucy attempted to incorporate conditions for finding the *greatest* common factor into her question, albeit phrased in an unclear way. I coded all of these questions as "inappropriate question."

In Table 24, I summarize the story problem modeling codes that I used when coding participants' GCF story problem attempts. All responses were coded using the content code "GCF", code "modeling", and subcode "story problem." Brit's story problem was the closest to being valid, but it fell short due to unclear wording. Eden did not even attempt to create a GCF story problem.

There are a number of possible reasons for how participants phrased their questions, specifically, and story problems, in general. Unlike the participants in my pilot study who spontaneously wrote and adjusted their story problems, all but Gwen opted to verbalize their GCF story problems. It could be that preservice elementary teachers pay closer attention to the wording of their story problems when writing them. I also wonder if participants phrased their story problems thinking of me as the their audience, assuming I would know what they meant and disregarding precise wording as a result. While most participants mentioned that story problems "help students to understand", perhaps it would have motivated participants to be more precise in their wording if I had emphasized that their audience was elementary school students. Also, as mentioned earlier, some participants deemphasized certain conditions for GCF in both their visual or concrete representations of the GCF and their story problems, indicating potential misconceptions or missed connections. Regardless of the reason, most participants' issues with creating story problems also manifested within their attempts to validate GCF story problems.

Table 24

Interview 1: Problem 2 story problem modeling codes

Code	**B**	**C**	**E**	**G**	**I**	**L**
Modeling – Story Problem						
Invalid: "How many subgroups?"		X				X
Invalid: "How many in each subgroup?"	X	X		X	X	
Did not maximize		X		X		
Contextualized	X	X				
Not contextualized				X	X	X
Appropriate question	X					
Inappropriate question		X		X	X	X

Validating Greatest Common Factor Story Problems

While there are many ways in which to validate story problems, my interview tasks aimed at two types of validation. The first type of validation task was used in the first interview, and it was similar to the LCM validation task. I asked participants to identify GCF story problems from a list of presumably valid story problems. This required a different, and somewhat less critical, lens than would be required of elementary school teachers in the classroom. Not only should a classroom teacher be able to identify a valid story problem, but s/he should be able to critique an invalid story problem. I wanted further exploration, so I created another validation task for the second interview. I asked participants to carefully critique hypothetical student-created GCF story problems. Responses to both tasks further informed my data analysis on participants' understandings of GCF story problems, which I summarize here. I coded responses to both tasks using "validation" codes.

Identifying valid greatest common factor story problems. After I asked participants to create GCF story problems in the first interview, I asked them to identify valid GCF story problems from a list of four story problems, three of which were valid. The first of these three story problems (Story Problem A) was structured using the "How many subgroups?" meaning of GCF, while the other two (Story Problems C and D) used the "How many in each subgroup?" meaning of GCF. A fourth story problem (Story Problem B) required the reader to find the LCM, which all participants successfully identified. I coded participants' validation of this task as "not valid," because they conclusively determined that it was not a valid GCF story problem. Given that the LCM story problem task preceded the GCF story problem task, it is possible that participants' exploration with LCM contributed to their success of this identification.

Cara and Lucy, the only participants to create "How many subgroups?" representations of the GCF, successfully recognize the validity of the "How many subgroups?" story problem after first reading it. Gwen solved the story problem in order to validate it, Isla thought it might be valid but could not explain why, and Eden was not sure. I coded Cara's, Lucy's, Gwen's, and Isla's validation of the task with "valid", because they conclusively believed the story problem was valid. Brit, however, was convinced that it was not a valid story problem because it was "asking the wrong question" (i.e., a "How many subgroups?" question), indicating that Brit did not see the validity in this meaning of GCF. I coded Brit's response as "not valid, incorrect."

Similarly, Cara and Lucy determined that the "How many in each subgroup?" story problems were asking the wrong questions. I coded these validations as "not valid, incorrect" as well. Cara's validation of these story problems was not consistent with her success in modeling GCF representations with both meanings as well as her relative success and versatility in creating GCF story problems. This further suggests that the understanding that is necessary to create GCF story problems is not sufficient for validating GCF story problems. The other participants were more likely to correctly identify the "How many in each subgroup?" story problems, as they created similar representations themselves.

While Story Problem D did demonstrate a "How many in each subgroup?" representation of the GCF, I phrased it to minimize the number of groups rather than maximize the number of subgroups. This confused Brit, who incorrectly thought the story problem was invalid without a "maximization" phrase. The phrasing also confused Lucy into thinking that it was asking a "How many subgroups?" question. So for the wrong reasons, Lucy claimed that this story problem was valid. I coded both participants' responses as "not valid, incorrect", in spite of the fact that Lucy said the story problem was "valid." Had she realized what the question was actually asking, it seems clear from her responses that she would have determined the story problem to be invalid. In Table 25, I summarize the validation codes I used for this part of Problem 2 from Interview 1.

Table 25

Interview 1: Problem 2 story problem validation codes

Code	B	C	E	G	I	L
Validation						
Story Problem A: Valid		X		X	X	X
Story Problem A: Not valid, incorrect	X					
Story Problem B: Not valid	X	X	X	X	X	X
Story Problem C: Valid	X		X	X	X	
Story Problem C: Not valid, incorrect		X				X
Story Problem D: Valid			X	X	X	
Story Problem D: Not valid, incorrect	X	X				X

Critiquing student greatest common factor story problems. While the task described above helped me to identify participants' predilection to a certain meaning of GCF, it did not provide participants with a sufficient opportunity to discuss the various minutiae involved with GCF story problem design. Thus, in Problem 3 of the second interview, I asked participants to critique hypothetical student story problems with various issues (elaborated on in the next paragraphs), inspired by the problems created by participants in the first interview.

As earlier tasks revealed, creating GCF story problems can be a difficult task. Thus, to make the hypothetical students' task more realistic (i.e., one that could reasonably be posed to middle school students) I suggested conditions and a context: "You have asked your students to create GCF story problems about making single-colored bunches of balloons with eight red balloons and 12 white balloons." The condition that bunches of balloons be single-colored best lends itself to a "How many in each subgroup?" GCF story problem structure. Each bunch has the same number of balloons. However, students could also manipulate the scenario to necessitate an equal number of red bunches as white bunches, creating a "How many subgroups?" story problem.

Most participants appeared to demonstrate a better understanding of the "How many in each subgroup?" meaning of GCF. I used this meaning to design the hypothetical student story problems to further investigate participants' understandings with this meaning of GCF. Thus, both of the story problems that I provided participants claimed that bunches of balloons should all have the same number of balloons. The first of the story problems (Story Problem A) did not maximize the common factor (the number of balloons per bunch), allowing the reader to work with any common factor. The second story problem (Story Problem B) maximized the wrong factor (the number of bunches). Additionally, the second hypothetical student left out the condition that all of the balloons be used, a condition necessary for factoring, and posed a question pertaining to the number of bunches rather than the number of balloons per bunch.

In spite of participants' previous success with "How many in each subgroup?" representations of the GCF, they incorrectly determined that these story problems were, for the most part, valid, which I coded as "valid, incorrect." Brit and Lucy were the only two participants to determine that the first story problem was invalid, but for very different reasons. Brit provided a complete and succinct validation, which I coded as "not valid, valid reasoning."

> Brit: Um, [Story Problem] A deals with finding common factors, but not necessarily the greatest common factor, because under these circumstances you could group all of the balloons in groups of two, so like

2 reds and 2 whites. While that *is* a common factor between 8 and 12, it's not the greatest common factor.

Lucy also correctly identified the first story problem to be invalid, but her critique of it drew from her incomplete understanding of the meaning of GCF.

Lucy: The question is how many balloons should be in each bunch. That's not exactly what we're looking for. We're looking for how many bunches that there will be. So this student is a little bit off.

Lucy fixated on the question that the hypothetical student posed in her problem, perhaps because it did not align with Lucy's "How many subgroups?" understanding of GCF. In Lucy's attempt to better understand the student's story problem, she revealed another possible misconception with her own understanding of GCF.

Lucy: She got it right that we want to make bunches of balloons so that each bunch is all one color, and each bunch has the same number of balloons. So we already know that they have the same number of balloons. So then, we don't need to say … how many balloons will be in each bunch, because we already know we have the same number of balloons in each bunch, and we want to know how many bunches we're making.

Although bunches will have the same number of balloons within each color, they will not necessarily have the same number of balloons between colors. And while the number of balloons per bunch may be the *same*, this does not mean that we know that value. This condition distinguishes the meaning of GCF which structures a story problem. As Lucy had displayed a fairly refined understanding of the "How many subgroups?" meaning of GCF in her previous interview, it is unclear whether this comment reveals a possible misconception or confusion with reading the story problem. I coded Lucy's validation of this story problem as "not valid, invalid reasoning" to distinguish from Brit's accurate and complete validation.

Lucy incorrectly determined the second story problem to be valid because it posed a "How many subgroups?" question, indicative of the "How many subgroups?" structure she was oriented towards. I coded this as "valid, incorrect." All other participants correctly determined that the second story problem was invalid. However, none of them provided a complete critique. Brit, Cara, Eden, and Isla all acknowledged that the student in the second story problem "asked the wrong question", but neglected to acknowledge that the student maximized the wrong factor and did not specify to use all of the balloons. Gwen felt that the wording of the story problem was more indicative of LCM, and thus determined it to be invalid. I coded these responses as "not valid, incomplete reasoning." In Table 26, I summarize the validation codes I used to analyze Problem 3 from Interview 2. All codes fell under the content code GCF. Only the "not valid, valid reasoning" code can also be coded as "SCK."

Table 26

Interview 2: Problem 3 code summary

Code	**B**	**C**	**E**	**G**	**I**	**L**
Validation						
Story Problem A: Not valid, valid reasoning	X					
Story Problem A: Not valid, invalid reasoning						X
Story Problem A: Valid, incorrect		X	X	X	X	
Story Problem B: Not valid, incomplete reasoning	X	X	X	X	X	
Story Problem B: Valid, incorrect						X

In general, the results of the validation tasks suggested that overall success (or lack thereof) in creating GCF story problems was not necessarily indicative of success in validating them or vice versa. Consider the cases of Eden and Gwen. Eden felt unable to even create a GCF story problem but demonstrated some, albeit limited, success in validating them. Gwen created a mostly valid GCF story problem but, of the participants with an orientation towards the "How many in each subgroup?" meaning of GCF, she was the least successful in critiquing the hypothetical student story problems.

There was, however, a possible connection between participants' understandings of GCF and their success in validating GCF story problems. Lucy was a prime example of this, as her understanding of GCF was so oriented towards one meaning that she could not acknowledge an alternative GCF structure. It also appeared that participants were less likely to identify faulty or missing conditions in student story problems if they themselves neglected those conditions in their own story problems. For instance, consider that both Cara and Gwen neglected to maximize the common factor in their story problems, and they did not acknowledge that the student in the first story problem made the same mistake.

While participants' processes for understanding GCF story problems were relatively similar, each was also unique. As I have already portrayed, there are many facets to how one may understand GCF story problems. I further analyzed each individual participant's understanding by looking at their responses and the codes I used across the various tasks pertaining to GCF story problems. Contributing factors include one's understanding of the meanings of division, the basic definition of GCF, visual and concrete representations of GCF, the meanings of GCF, numbers in context, and GCF story problem creation and validation. In the sections that follow, I provide a model and explanation for how each participant understands GCF story problems. For each of the facets, I describe how I determined each participant's understanding, as well as describe how it might have contributed to their overall understandings of GCF story problems. While there may be many more contributing aspects to this understanding, my findings are confined by the limitations of my study design and implementation, which I discuss in Chapter V.

Brit's Understanding

Brit demonstrated the most robust understanding of GCF story problems with a "How many in each subgroup?" structure. By making connections between the codes I used to analyze her process for understanding GCF story problems, I was able to model her process (see Figure 16). The figure also serves as an account of the codes I used.

Brit's understanding of GCF drew from a more refined understanding of the "How many in each group?" meaning of division. For the purposes of this study, I coded a participant's understanding of a meaning of division as "refined" if the participant successfully created, described, or critiqued an overt representation (concrete, visual, or story problem) of the meaning of division under consideration. The extent to which participants could demonstrate this understanding was limited by the tasks that I posed. Within the confines of this study, a refined understanding of the meaning of division typically entailed recognizing all of the necessary conditions for division, including that all objects be used and that groups of equal size be created, recognizing the validity of story problems with this division structure, and articulately discussing reasoning for decisions in creating and validating representations of the meaning of division (models) under consideration.

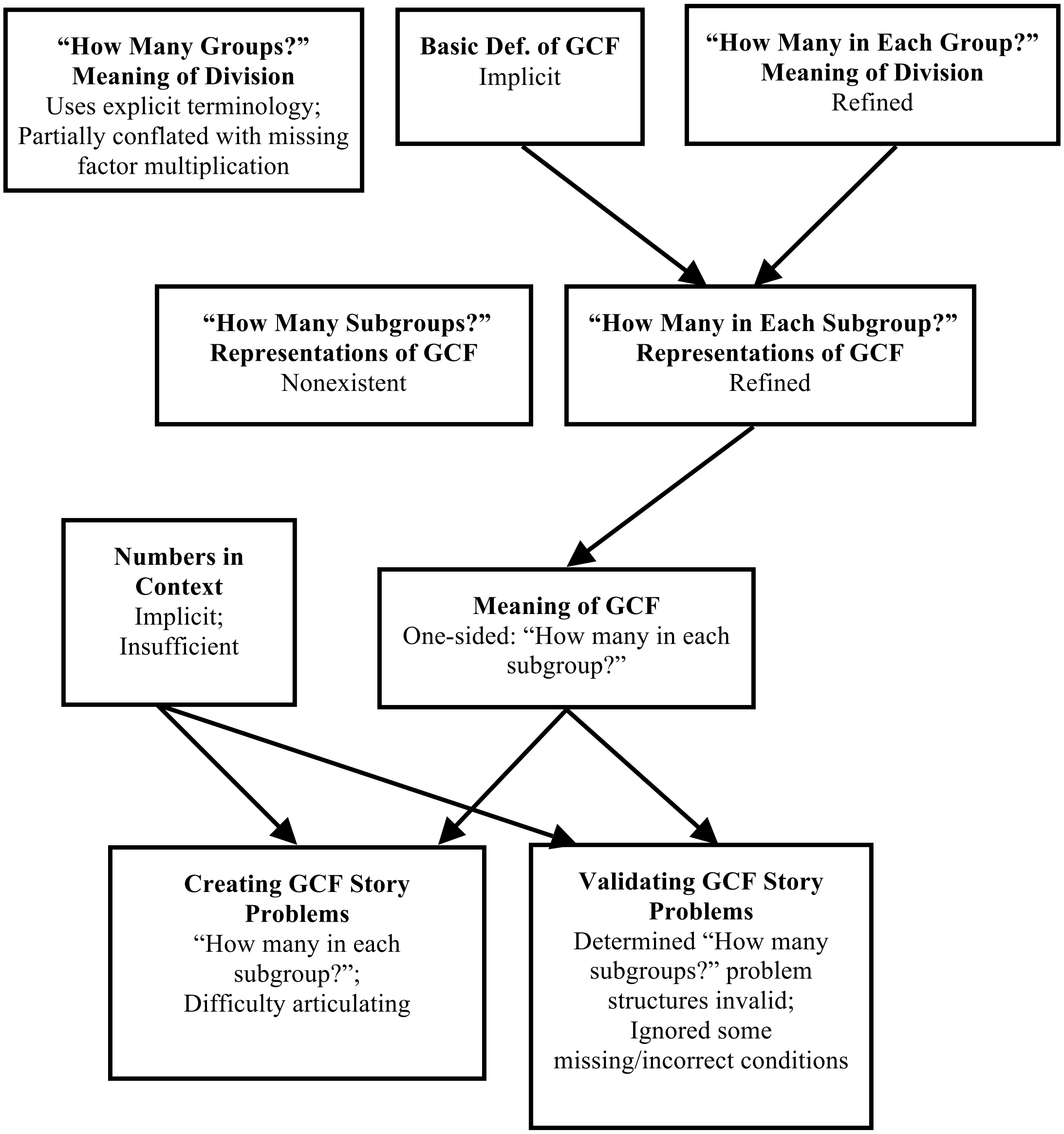


Figure 16. Brit's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

The data suggest that Brit's understanding of the "How many in each group?" meaning of division was refined, because she accurately and completely produced and explained multiple representations of this meaning, including a description of how students might divide 12 by 4 using cookies. The only discrepancy between this description and her division story problem was that she specified, "using up all of the cookies” in her description, but neglected to include a similar phrase in her story problem. Brit also recognized and identified the “How many in each group?” structure of division story problems that I posed to her in the second interview.

As Brit opted to create or describe “How many in each group” representations of division, she only displayed her understanding of the “How many groups?” meaning of division once when validating “How many groups?” division story problems. She succinctly reasoned through the differences in structure when she read the first “How many groups?” story problem.

> Brit: Ok, so the first [story problem] is [a division problem], because you would have the 12 apples divided by four per day, and then in three days you would run out of apples. You would use all 12 of those apples. You're dealing with… instead of 12 divided by groups gives you how many in each group, you have 12 divided by how many in each group gives you how many groups, or days.

However, when she read the second “How many groups?” story problem she identified it as more of a multiplication problem.

> Brit: Most kids would automatically, I think, go from the four to the 12… So four times three is 12, rather than 12 divided by four equals three. But it is an applicable problem for 12 divided by four. I'm just not sure that that's the most straight-forward way to get there.

While missing-factor multiplication is one way of thinking about division, Brit seemed unsure of the connection between the two ideas. Also, Brit did not appear to recognize the similarities between this story problem and the first one, which she eloquently critiqued. Perhaps the difference in Brit’s perception was due to the fact that the “4” preceded the “12” in the second story problem, or perhaps her understanding of “How many groups?” division story problems is dependent on the context. Regardless, Brit’s understanding of this meaning of division appeared less refined than her understanding of the "How many in each group?" meaning.

As mentioned earlier, Brit was the only participant to begin with attempting a GCF story problem when asked, rather than first modeling GCF visually. However, to clarify her slightly confusing story problem, she demonstrated and discussed a detailed "How many in each subgroup?" concrete representation of the GCF. To make her work more manageable using the colored counters, Brit found the GCF of 14 and 16 rather than 28 and 32.

> Brit: OK, so now I have 2 different colors … all lined up, generally, with their little partners here [see Figure 17]… It doesn't matter how many groups I have, I just want the same number in each group so that each of these gets used… I have two extras on the end, so I'm going to try making groups of 2 and see how that works. So all of the 16s are going to be put into groups of 2, and all of the 14s are going to be put in groups of 2. OK, so that's one of my divisors, because they can all go into groups of 2. So now, I want to check, say I don't believe that's the right answer - magically, I don't believe that - I'm going to try for groups of 4. I'm going to connect two of the groups together to make groups of 4. So it works up until the last group, so I now I know it can't go any higher than 2. So I'm going to split them into groups of 2, because I know that's my greatest common divisor because I know that that is what both sets can be equally divided into with no chips left over, and they have the same number in each group - it doesn't matter how many groups I have.

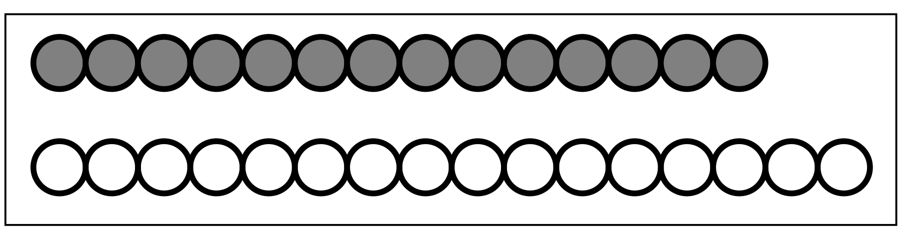


Figure 17. Brit's configuration of the colored counters for finding the greatest common factor of 14 and 16.

Due to the completeness and accuracy of Brit's description, I coded her understanding of "How many in each subgroup?" representations of the GCF as "refined." While she did not initially realize that the GCF of two numbers cannot exceed their difference, I considered this a misconception concerning more abstract concepts related to GCF rather than a misconception concerning the concrete structure of GCF. (Recall that in Section I of my results, I explored participants' understandings of the relationship between the GCF of two numbers and their difference.)

Brit's relative ease in describing this method for finding and representing GCF, compared to her struggle to articulate a valid story problem, suggested that perhaps she drew from this understanding to create her story problem. Her wording used to describe her GCF representation suggested an implicit connection to her understanding of the "How many in each group?" meaning of division as well as her understanding of the basic meaning of GCF.

Brit did not spontaneously demonstrate or discuss the "How many subgroups?" meaning of GCF in any way, and when asked to validate a GCF story problem with that structure, she insisted that it was invalid. This led me to code Brit's understanding of the meaning of GCF as "one-sided." This contributed to her predilection for "How many in each subgroups?" GCF story problems. However, as stated earlier, Brit struggled to articulate her own story problems and fully validate others', especially ones that used a "How many subgroups?" structure.

This struggle may have been due to a difficulty translating her concrete understanding of GCF representations into context. While creating an LCM story problem in a previous task, Brit discussed how some numbers and operations worked in some contexts but not in others. She did not explicitly recall this discussion when she created or validated GCF story problems, but it is likely she used similar reasoning when she chose a context for her GCF story problem. Although her context, in general, was appropriate, the specifics of that context, or in other words, how each facet of the GCF structure translated within that context, seemed to pose a problem for Brit.

Cara's Understanding

Cara's understanding of division appeared very similar to Brit's (see Figure 18); she chose to create concrete and story problem representations of division using the "How many in each group?" meaning and had similar success with them. Cara also accurately determined the "How many groups?" story problems to be valid, implying some understanding of the "How many groups?" meaning of division. As a result, I coded Cara's understanding of the meanings of division the same as I did with Brit, which was "How many groups?."

Unlike Brit, however, Cara had a much more developed understanding of the meaning of GCF, primarily due to her flexibility modeling GCF representations. As mentioned earlier, Cara spontaneously provided both types of GCF representations in her efforts to create story problems. Aside from her reliance on the value of GCF in creating these representations, her understanding appeared refined. The language Cara used in her descriptions indicated that she drew from her understandings of the two meanings of division as well as the basic definition of GCF. Not only did she articulate each of the facets of GCF for each meaning, but she also compared and contrasted them, which led to a fairly differentiated understanding of the meaning of GCF.

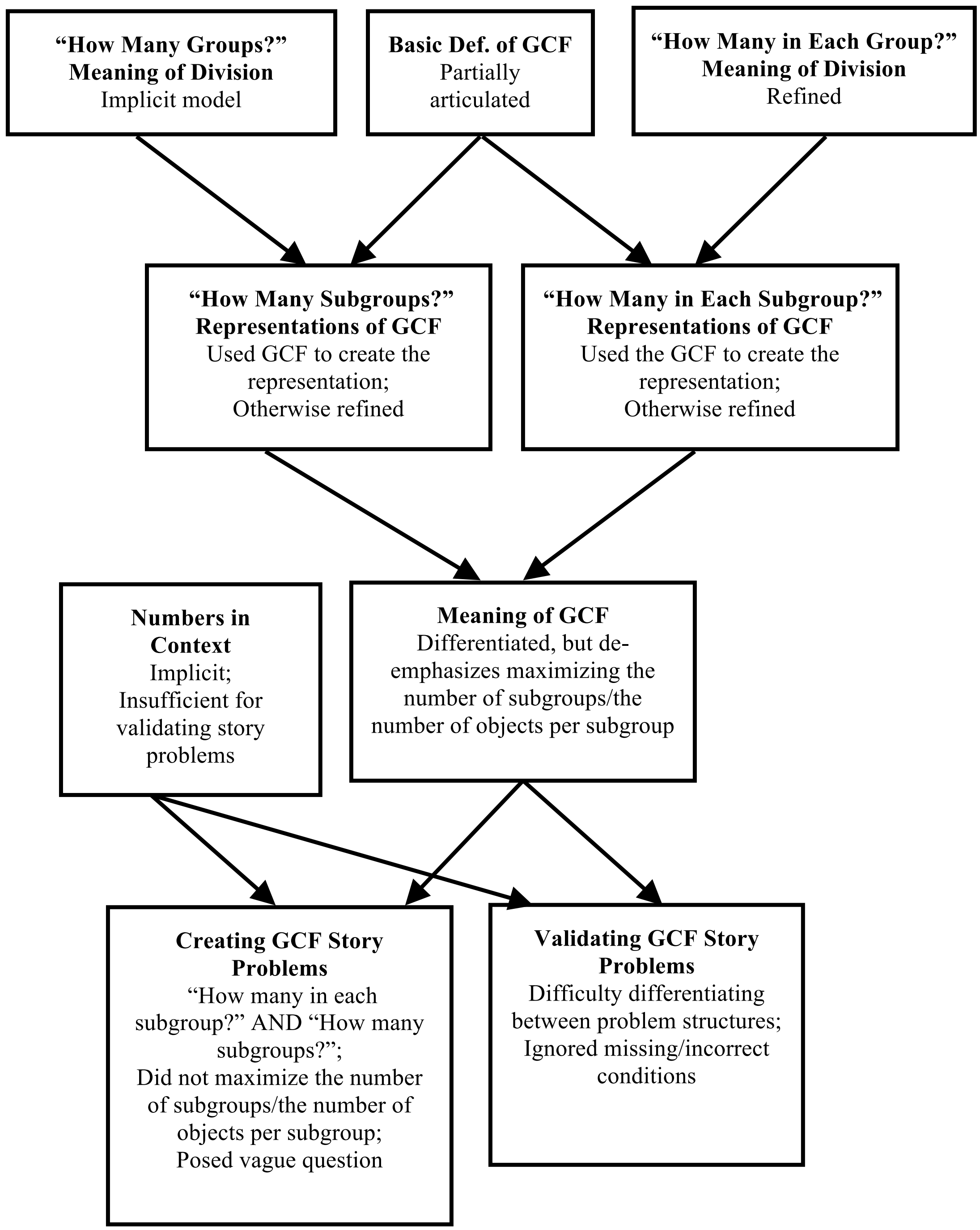


Figure 18. Cara's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

Cara made fairly successful attempts at creating both types of GCF story problems, but her lack of attention to maximizing the common factor in her concrete

GCF representations led to a similar oversight in her story problems. However, she chose appropriate contexts for her story problems and successfully translated the remaining facets of the GCF structure within those contexts. Validating GCF story problems, or more specifically, identifying those facets within other contexts seemed to pose more of a problem for Cara.

When I asked Cara to identify valid GCF story problems from a list during the first interview, Cara made assumptions about their structures rather than using the narrative to identify which type of GCF story problems they were. For instance, Story Problem A was a "How many subgroups?" problem, but Cara attempted to solve the problem assuming that each subgroup had the same number of objects between groups.

> Cara: So for the first [story problem], if she's dividing it evenly, she can put 2 dark chocolates and 2 milk chocolates in each bag, and then she'll have 2 milk chocolates left… She wants to use all the chocolates so she would have 2 left over, so I don't think… So she wants… Oh, ok. I was reading it as the dark chocolates and the milk chocolates had to be the same, and that's not the case. So all she cares about is that there are 6 dark chocolates and 7 milk chocolates in each bag… and then she could only get 2 bags…

Cara knew that she had to use all of the objects in order to find the GCF, and when her assumption conflicted with this condition, she read the story problem more carefully. This led to the identification of her assumption and to her correctly determining the structure of the problem.

Having just worked through Story Problem A, Cara incorrectly assumed that Story Problems C and D were "How many subgroups?" problems. After she forced this structure onto the story problems, I directed her attention to the condition that subgroups have an equal number of objects between groups. Rather than switching the meaning of GCF like she did with Story Problem A, this conflict led Cara to believe that neither story problem was valid. She even suggested that they were “asking the wrong question”, because it asked for the number of objects per subgroup rather than the number of subgroups.

During the second interview, Cara struggled to identify the missing or problematic conditions in the narratives of the hypothetical students’ GCF story problems, which aligns with the fact that she ignored most of these conditions when they were present and accurate. Cara did claim, however, that the second story problem “asked the wrong question,” an accurate validation this time. Curiously, the question asked for the number of subgroups, but Cara thought it should ask about the number of objects per subgroup – opposite from her validation of Story Problems C and D. Overall, Cara displayed difficulty in differentiating between the two GCF problem structures when validating GCF story problems. Given that Cara’s understanding of the GCF meanings themselves was the most advanced of any of the participants, this was most likely a problem with reading comprehension or, more accurately, mathematics-in-context comprehension.

Eden’s Understanding

Eden’s understanding of division was the same as Brit’s and Cara’s, and I used my codes to generate the model of her understanding in Figure 19. As mentioned earlier, while none of the participants explicitly drew from their understandings of division when they described representations of the GCF, their phrasing was similar to that of their descriptions of division representations. However, it appeared necessary to adjust their understandings of division using what they knew about the basic definition of GCF in order to achieve a GCF structure rather than a division one. This was most obvious with Eden, as she began her process by recalling the basic definition of GCF.

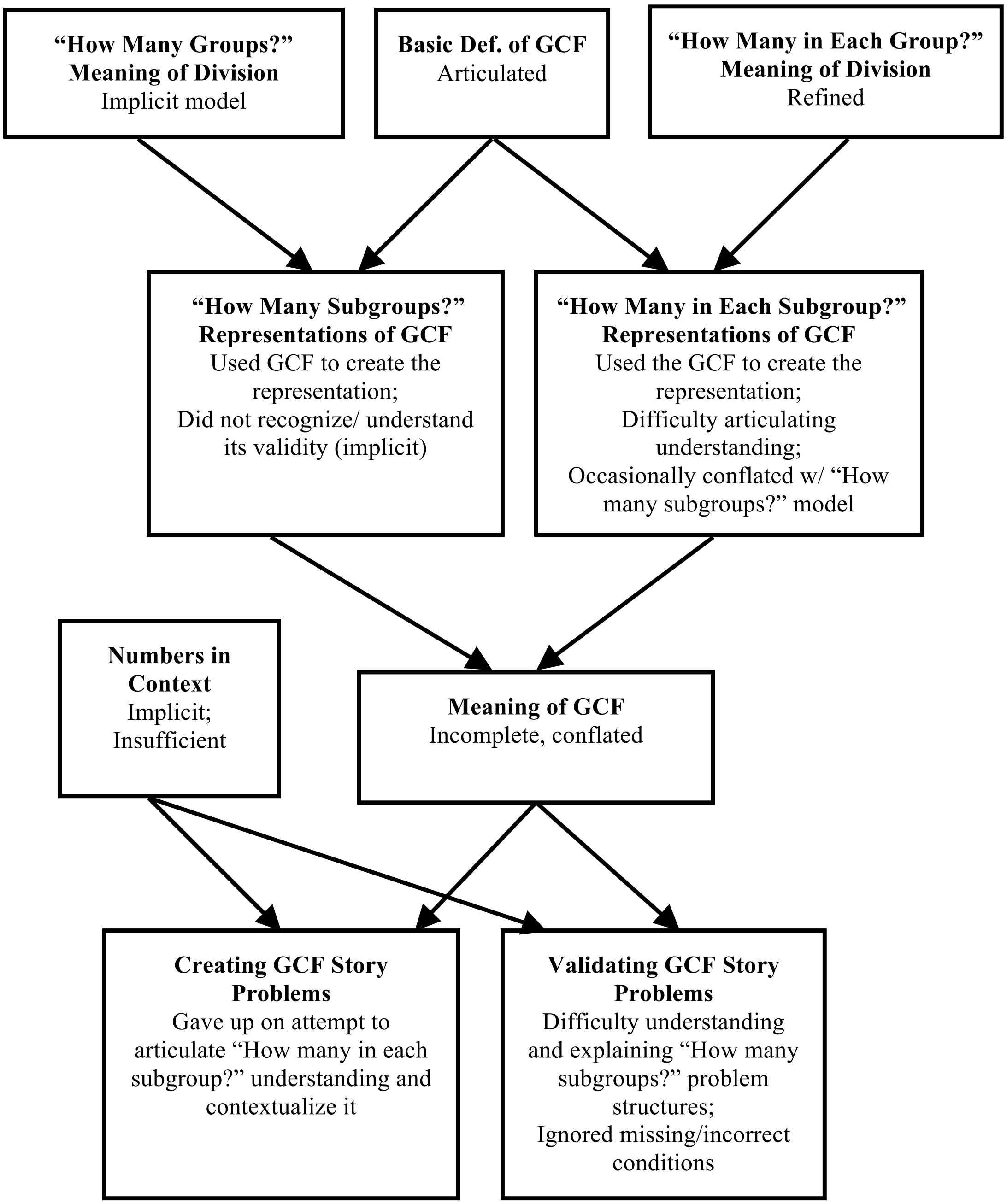


Figure 19. Eden's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

While she eventually described how to represent the GCF using manipulatives, Eden was the only participant to fall short of attempting a story problem, most likely due to her confused and somewhat conflated understanding of the meaning of GCF (described earlier). Even though Eden's understanding of GCF story problems seemed limited due to her inability to create a story problem, she made some progress in validating them. Eden struggled with the structure of Story Problem A from the first interview, but she identified Story Problems C and D as valid GCF problems with confidence. She also provided the same, albeit lacking, validation of the hypothetical students' GCF story problems that Cara did, in spite of having a much weaker grasp on the meaning of GCF.

Gwen's Understanding

As with Brit, Cara, and Eden, Gwen also chose to create "How many in each group?" story problem, picture, and concrete representations of division. She was less likely to articulate the conditions for division in her descriptions, however. While the picture she created illustrated using all of the objects to create equal sized groups, Gwen never mentioned using all of the objects and only mentioned making "equal groups" once between the three descriptions. Thus, while she was successful in performing the tasks, the data did not suggest that Gwen's understanding of "How many in each group?" division was as refined as the others' (see Figure 20).

Also similar to Brit, Cara, and Eden, Gwen's demonstrated for the first time what she knew and understood about the "How many groups?" interpretation of division was in validating the division story problems. Gwen compared the first story problem, a "How many groups?" story problem, to one that she created in which 12 candies were split equally amongst 4 people.

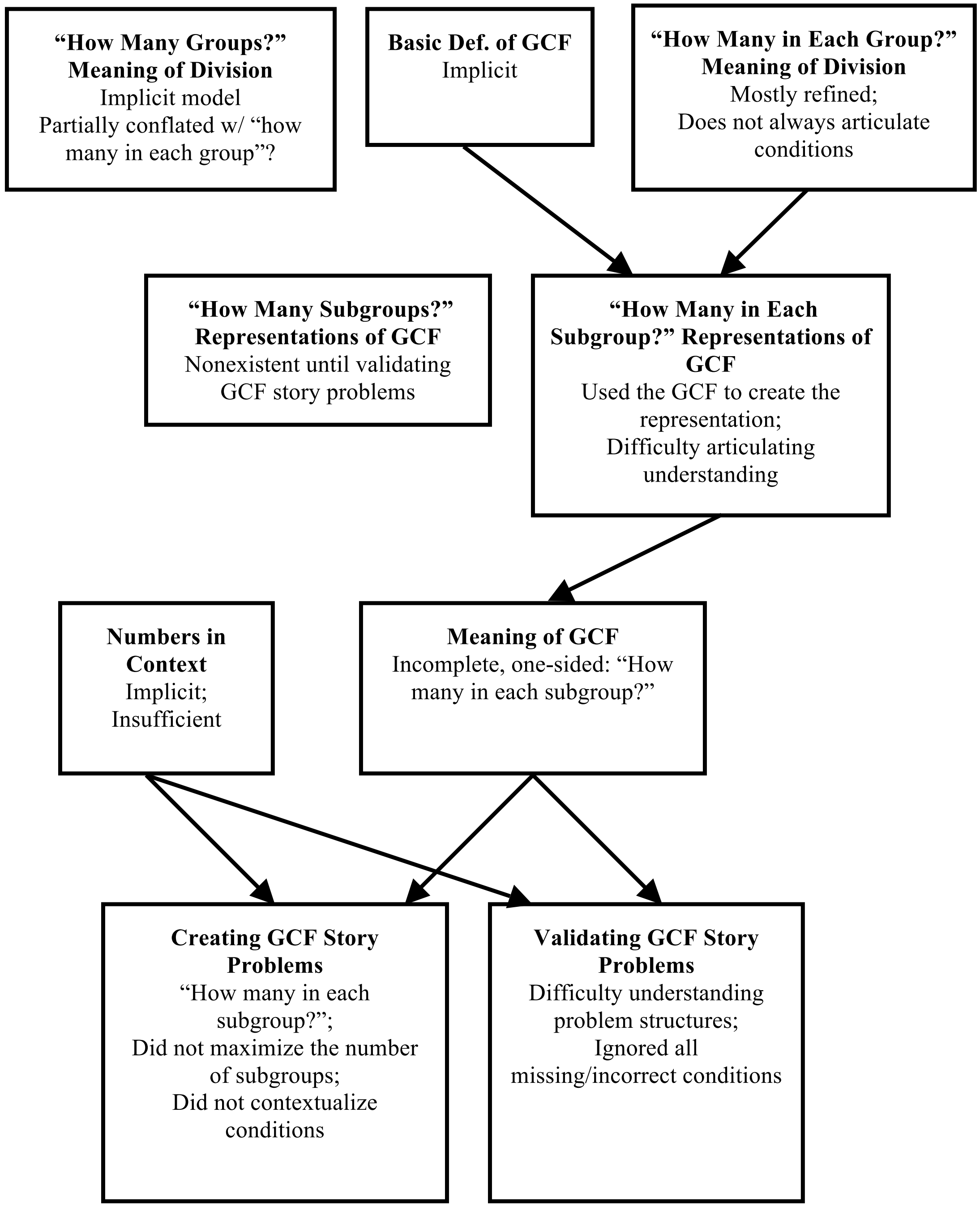


Figure 20. Gwen's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

> Gwen: I think yes for [Story Problem] A because she's… you're trying to find out how many [days] it takes before she runs out of apples when she's feeding her horse 4 a day. She will eventually run out. I feel like it's like my [story problem]… You're splitting it up into different groups, but in this one you're splitting it up into different days…

While the act of splitting up 12 objects is the same, Gwen appeared to conflate the meanings of division in this statement. In her story problem, the quotient was the number of objects in each group rather than the number of groups. Perhaps, Gwen struggled to see the distinction.

Gwen appeared to draw from the "How many in each group?" meaning of division to create a "How many in each subgroup?" representation of the GCF. As with Cara and Eden, Gwen used the GCF value to create the representations. She, however, later described how she might find the GCF using the manipulatives: "Maybe start by grouping them into 2s and then… I guess I'd go up to 4." While she struggled to articulate her understanding, Gwen eventually described all of the conditions for a "How many in each subgroup?" representation of the GCF. Overall, this amounted to an incomplete, one-sided understanding of the meaning of GCF.

When it came to creating her own GCF story problem, Gwen neglected to articulate or contextualize many of the conditions necessary to answering the story problem, as she did with her division story problem. When asked to identify valid GCF story problems during the first interview, Gwen struggled to interpret the narratives. Similar to Cara, Gwen imposed a "How many in each subgroup?" structure on Story Problem A in an attempt to solve it, but struggled with the idea that there would be objects left over. Gwen required more prompting and clarification than Cara before she recognized that the number of subgroups per group represented the GCF. She required similar prompting before recognizing the Story Problems C and D were also valid. Gwen also incorrectly deemed both student GCF story problems valid in the second interview. This may have been due to a combination of things: a weak understanding of the meaning of GCF, or the conditions for GCF in particular, and a struggle with understanding mathematics in context, which would hinder identifying those conditions.

Isla's Understanding

Isla was the only participant with a predilection towards the "How many groups?" meaning of division. She created a near-valid story problem: "If we have… 12 students and we want 4 students in each group, how many groups can we make?" When I asked Isla to describe or create pictorial or concrete representations of division, though, she described rectangular arrays that could be easily interpreted as 4 groups of 3 or 3 groups of 4. She even referenced both ways in her description as if they were interchangeable. When I asked Isla to identify valid division problems, she did not draw attention to the different structures or even compare them to her own story problem. Due to Isla's lack of distinction between the two meanings of division, I coded her understanding as being "partially conflated" (see Figure 21).

While Isla's understanding of division leaned towards the "How many groups?" interpretation, her understanding of GCF was exclusively oriented towards the "How many in each subgroup?" meaning. While this, at first, appeared to be quite a contradiction, the similarities between Isla's concrete representations of division and GCF were remarkable. When she modeled division, she used the divisor to determine the size of the groups, because the divisor was known to her. Likewise, when Isla modeled GCF, she determined the GCF ahead of time and used it to determine the size of the subgroups. While this allowed her to create a model, she did not know how to use the model to find the GCF. She struggled to articulate her understanding, thus making her understanding of the meaning of GCF incomplete and one-sided.

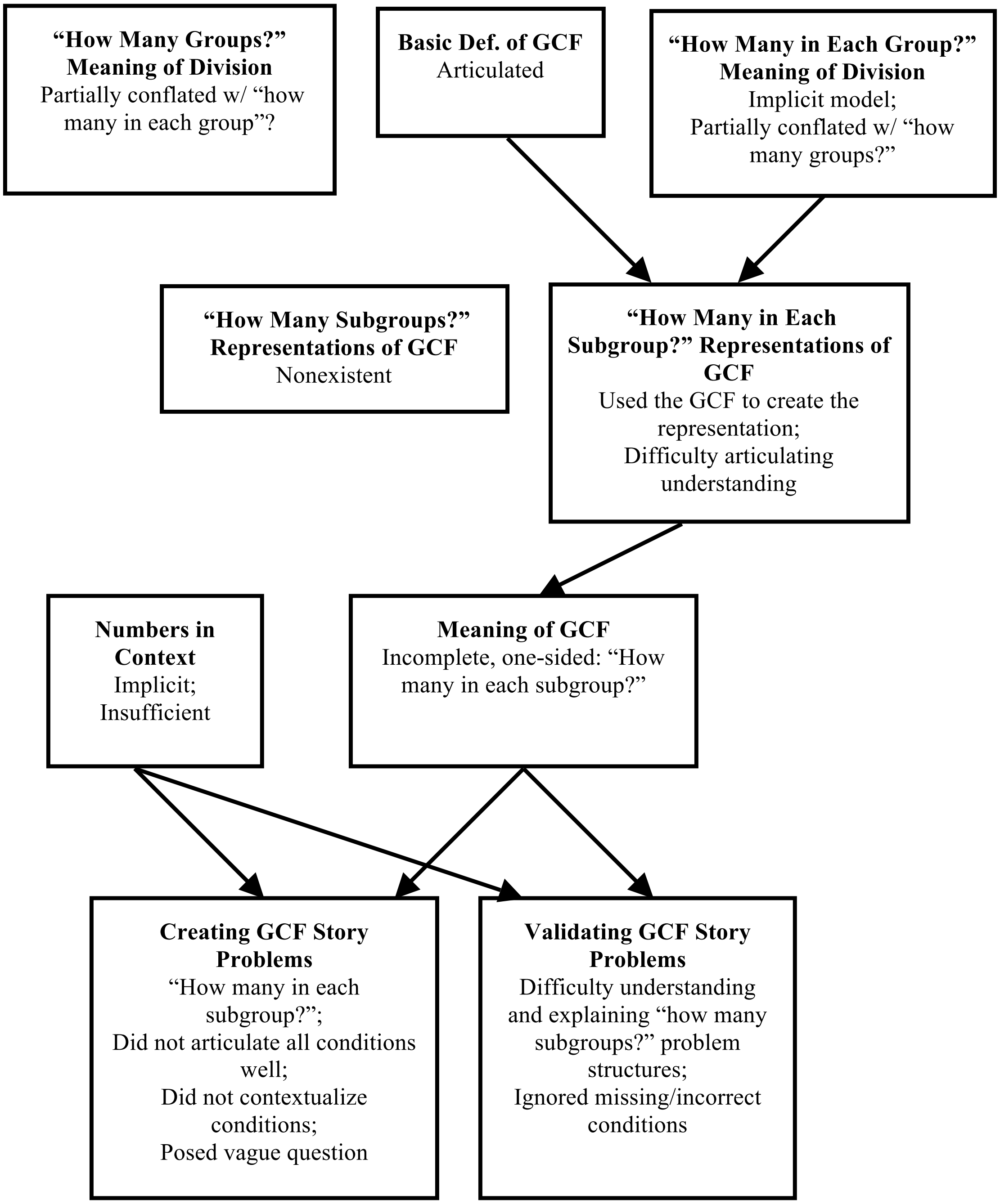


Figure 21. Isla's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

Similarly, Isla struggled to put her GCF story problem into words, stating incoherent conditions and neglecting to put them in context. She also posed a vague question that seemed more oriented towards grouping the objects than determining the GCF. This makes sense when we consider her focus on making groups with the manipulatives over determining the GCF through the experience. As with most other participants, Isla struggled to understand the “How many subgroups?” GCF story problem, uncertain whether it was valid. She also provided the same incomplete validation of the hypothetical students’ GCF story problems that Cara and Eden did in the second interview.

Lucy’s Understanding

Like many other participants, Lucy demonstrated an understanding of the “How many in each group?” meaning of division, but she did not always articulate the conditions. As a result I coded her understanding of the “How many in each group?” meaning of division to be “mostly refined” (see Figure 22). When I asked Lucy to validate the list of division story problems, she was surprised that some of them were so different from her own. In spite of recognizing this distinction between the two meanings of division, Lucy obviously preferred the “How many in each group?” meaning.

However, Lucy also consistently demonstrated that this understanding was not connected to her understanding of GCF. Instead, Lucy drew from a “How many groups?” understanding of division when modeling GCF. Her understanding of the “How many subgroups?” meaning of GCF was quite clear in her description of how to model GCF using counters, as described earlier. As with most participants, this led to a fairly one-sided understanding of GCF. Lucy stood out as the only participant with a predilection towards the "How many subgroups?" meaning of GCF.

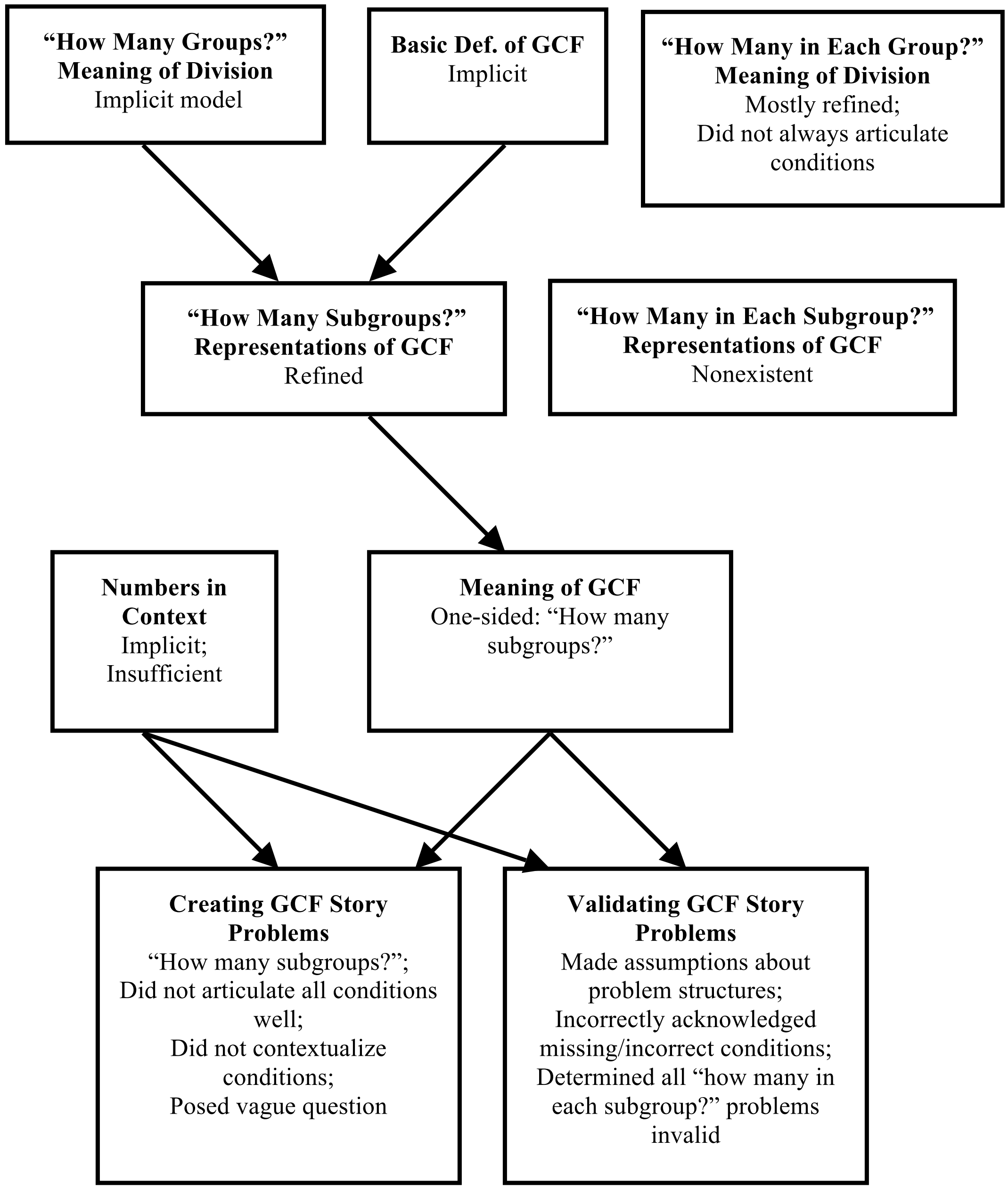


Figure 22. Lucy's understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

Lucy struggled to contextualize her understanding of GCF through a story problem. She also posed a vague question. This may have resulted from an insufficient understanding of the meanings of GCF or a weak understanding of the contexts in which GCF might be deemed useful. Lucy's one-sided understanding of the meaning of GCF also perpetuated her story problem validation. She quickly and accurately determined that story problem A was a valid problem, but she attempted to conform the other story problems to her "How many subgroups?" understanding or disregarded them as being invalid because they did not ask the "right" question.

Lucy seemed unwilling to allow for the other meaning of GCF, something she perpetuated in her response to the follow-up questions in the second interview. The second hypothetical student's story problem posed the wrong question, but because it asked about the number of subgroups, Lucy determined that the story problem was valid. Lucy also neglected to recognize all of the missing conditions of GCF in both story problems. Overall, Lucy demonstrated a partially accurate and partially complete understanding of GCF story problems.

After analyzing the content data from each interview tasks and incorporating the relevant number theory course data, I analyzed my data from a more holistic perspective; I looked for overarching themes. In the following section, I discuss these themes and reference evidence from Sections I and II. After discussing these themes, I present a more succinct answer to Research Question 1.

Emergent Content Themes

Several overarching themes emerged from my analysis of the content tasks from the interviews. I made observations concerning how the content of the tasks related to the content of participants' number theory course, whether participants explicitly or implicitly made these connections, and related it to participant success on the related tasks. I found that participants appeared to be more successful with portions of tasks that they could more clearly connect to their coursework experiences. I also found that with tasks that allowed participants to explore using concrete methods, but then required them to apply abstract theory to explain their results, participants were much more successful at the concrete portion. There appeared to be a disconnect between the activity of working with a task at a concrete level and the activity of generalizing their observations. Lastly, I noticed that the representations with which participants best understood GCF were different than the LCM representations participants best understood. I describe and support these observations here, but I provide further discussion in Chapter V.

Connecting Content to Coursework. As I supplemented my analysis of the interview tasks with participants' number theory coursework experiences, I noticed that participants did not always recognize or recall the connections between the content they successfully demonstrated knowing in the number theory course and the content of the interview tasks. When the content of an interview task was similar to a task that participants explored in their number theory class, they were more likely to identify it as something they had seen before, and they were also more likely to demonstrate success with the task. However, participants could only demonstrate success with these tasks at surface level. In other words, they could not provide or recall the more rigorous reasoning that was provided to them in class.

For example, participants all demonstrated some success at the LCM story problem task (Interview 1, Problem 1). Brit was the only participant to create a fully valid, albeit unpolished, story problem, but the other participants made reasonably strong attempts. All participants were successful in describing or modeling LCM with manipulatives or pictures. This was expected, given that participants worked with manipulative and story problem models of LCM in their number theory course, as I discussed earlier in Section II. Creating an LCM story problem was a surface level task for participants, especially given their previous experiences with this process.

The task in which Eva used a geometric "squaring off" procedure to find the GCF of two numbers (Interview 1, Problem 4, discussed in Section I) was also very familiar to participants. They recognized Eva's procedure as one they had done in class, and (almost all of them) were therefore confident that Eva's method for finding the GCF of two numbers always works. The proof of the Euclidean Algorithm is also connected to this task, but it is a more abstract connection. Participants were informed of this connection in their number theory course, and they were asked to explain it with a specific example on a homework assignment. Four of the participants recalled, without prompting, that Eva's method is a geometric version of the Euclidean Algorithm, but when asked why her method works they did not reference the proof they discussed in class. Participants demonstrated success with the surface level aspects of this task that were clearly connected to their coursework, but they struggled to apply the more abstract reasoning that was provided in the number theory course.

Another task that was similar to a problem participants saw in their number theory coursework was Shayna's claim that the number one is prime (Interview 2, Problem 7, discussed in Section I). This was a conjecture that participants explicitly discussed in their number theory class; they explored and demonstrated multiple reasons why Shayna's claim must be false. The connection between Shayna's claim and the participants' number theory experience was an obvious one to them, and they each demonstrated a higher degree of success with it than with many of the other tasks. All participants recognized Shayna's claim to be false, and many of them recalled they had discussed reasons for this in their number theory course. However, the reasoning participants recalled did not demonstrate deep understandings. "Factor trees would never end" and "it would mess up all our other assumptions about primes," without being able to explain, do not constitute rigorous reasoning.

The task where Talisa and Tom each factored 540 in different ways (Interview 2, Problem 9, discussed in Section I) was another task in which participants demonstrated great success with the surface level content they connected to their coursework. Participants had many successful experiences with factor trees and prime factorizations in their number theory course. The concept of factorization is the obvious connection between the number theory course and this task. All participants were successful at determining the validity of the different factorizations. A more implicit connection could have also been made to the Fundamental Theorem of Arithmetic. It took more rephrasing of my question than with participants' validations of the factoring methods, but eventually five of the participants made reference to the Fundamental Theorem of Arithmetic.

Interview 2, Problem 10 (discussed in Section I) was a task that was virtually identical to tasks participants had been asked to complete for their number theory course. It required participants to determine the divisors of M, given its prime factorization. Then participants were asked to find the GCF and the LCM of M and N given their prime factorizations. Participants demonstrated incredible success with this task, perhaps because of their equally successful experiences in their number theory course. Curiously, the only part of the task that seemed to be a struggle for participants was finding the LCM of *M* and *N* using their prime factorizations, which was a procedure participants successfully demonstrated on their assignments.

There were also a few tasks that differed a bit from participants' coursework, but not in a significantly challenging way. Mark's conjecture that the LCM of two numbers is equal to their product (Interview 1, Problem 3, discussed in Section I) is one such task. Participants investigated the relationship between the GCF and the LCM in many different ways and for multiple assessments in class, but this task did not explicitly ask participants to validate Mark's conjecture using this relationship. Perhaps because of this, participants did not leap to use this relationship to determine the cases for which Mark's conjecture works. Instead participants attempted to find a pattern in the examples and counterexamples they generated. Using this indirect method, three participants successfully determined that the GCF of the two numbers must be one in order for Mark's conjecture to work and successfully explained why that makes sense.

Another task whose most efficient solution utilized the relationship between the LCM and the GCF of two numbers was Interview 1, Problem 5 (discussed in Section I). Given the $GCF(a,b)$, the $LCM(a,b)$, and the value of a, participants were tasked with finding the value of b. Brit recalled that $a \times b = GCF(a,b) \times LCM(a,b)$ from the course material and successfully used it to find the value of b. The relationship between GCF and LCM also contributed to Lucy's solution, but she used the graphical lattice method participants had seen in class. Although the remaining four participants did not use methods they had seen in class to efficiently address this task, the strength of their understandings of GCF and LCM allowed them each to identify a productive strategy, and three participants even correctly solved the task. It is likely that participants' extensive experiences with the various representations of GCF and LCM in their number theory course contributed to their more flexible understandings of GCF and LCM in this task. Where participants' understandings of LCM and prime factorizations lacked, their understandings of GCF and prime factorizations compensated.

In Interview 2 (discussed in Section I), there were a couple tasks whose connections to the course content were not at all obvious. Maria's conjecture (Interview 2, Problem 4), that the difference of two numbers is equivalent to their GCF, was such a task. The connection between Maria's conjecture and participants' coursework was subtle. The greatest common factor of two numbers is a common divisor of each. Participants proved and used the theorem that if d divides m and d divides n, then d divides $m - n$ (and $m + n$). One participant successfully determined that the GCF of two numbers divides their difference, but she did so by discovering a pattern in her counterexamples, not by using this theorem.

The factory problem (Interview 2, Problem 5) was another task whose connection to the coursework was subtle. Additionally, some of the content connected to this task was more sophisticated. One of the connections that participants could have made was that the solutions to the task were all relatively prime to 12. Participants had seen many situations in which numbers being relatively prime was a powerful condition or result. This was the more obvious connection to their coursework, and three participants did make it. However, this was a surface level connection. As Brit stated during her interview, "when in doubt, the answer is always 'prime' or 'relatively prime.'" Participants also worked with modular arithmetic on a number of occasions, and the connection between a clock and mod 12 was another more obvious connection that participants could have made. Brit and Lucy explicitly referred to mods and all but Eden demonstrated modular thinking explicitly or implicitly. However, none of the participants recalled the more rigorous reasoning related to the task.

The most challenging of the connections that could have been made to participants' coursework was to the concept of multiplicative inverses mod 12. All of the solutions to the task were relatively prime to 12 *because* they were the only solutions that had a multiplicative inverse mod 12. By the time of the interview, participants had explored multiplicative inverses mod n in class and on their homework assignments, and they had been tested on it, but their course work did not look similar to the factory problem. All participants incorporated multiplication, at least implicitly, by using repeated addition. However, none of the participants exhibited an understanding of the fact that when one of these products resulted in 1 mod 12, it meant that there existed a multiplicative inverse mod 12. Making this connection may have also helped them to recall the proof they had used to prove that if n is relatively prime to m, then there exists a multiplicative inverse of n in mod m.

Due to the accessibility of the task, many participants demonstrated a degree of success with it. The task was designed in such a way that a solution could be obtained using brute force, but an explanation of the solution required an elevated understanding of number theory concepts. It is possible that this type of abstraction interfered with participants' success.

Connecting Concrete Representations to Abstract Content. Participants demonstrated success in performing tasks when concrete model- or example-exploration were available, but they could not generalize to an abstract or rigorous level, even though underlying conceptualizations were provided in class. In Interview 2, Problems 4 and 5 (discussed in Section I), participants were presented with tasks they could explore at a concrete level. For Problem 4, Maria's conjecture, participants were provided colored counters with which to explore the claim that the GCF of two numbers is given by their difference. It seems that the concrete nature of this task made it accessible. Participants were all successful in determining that the claim was false, because the task was easy enough to explore using manipulatives or testing numbers. They all struggled when they were asked to abstract from their concrete observations and determine the actual relationship between the difference of two numbers and their GCF. All of the participants referred to their examples and counterexamples to determine the relationship, rather than referring to their number theory repertoire to help them better understand the situation. For one participant, Lucy, searching for a pattern to determine the relationship proved to be fruitful. She correctly determined that the GCF of two numbers divides their difference. However, without referencing the theorem that if d divides m and d divides n, then d divides $m - n$, Lucy could not "think of a way to know for sure."

Problem 5, the factory problem, was also very accessible to participants in that they could explore it in a concrete way by repeatedly adding the same number of hours on a clock. This brute force method was tedious but successful. Similarly to Problem 4, however, this concrete exploration of the task could only get participants so far. Half of the participants successfully recognized from their observations, again, by finding a pattern from concrete examples, that the solutions were all relatively prime. When asked to prove that all relatively prime numbers will result in a solution, however, participants were at a loss. They admittedly could not think of a way to transition from their pattern-finding activities to a theory-based proof.

For both of these tasks, it was curious how participants opted to work with concrete examples and look for patterns when the tasks required them to exhibit an abstract understanding of the concepts. Looking for patterns is practical and especially useful at the elementary school level. But understanding *why* something is valid, like the fact that the GCF of two numbers divides their difference, is also important, and it is dependent on an ability to apply theory to concrete observations.

Understanding greatest common factor versus understanding least common multiple. The concepts of GCF and LCM appeared in most of the interview tasks, and they took many different forms. I asked participants to model these concepts using pictures, manipulatives, story problems, and with numerical examples. They used both GCF and LCM to solve problems, like Problem 5 from Interview 1. I also asked them to find the GCF and the LCM of two large numbers given their prime factorizations. Participants also validated students' claims about GCF and LCM. I made additional observations about participants' treatment of GCF and LCM, such as how they occasionally used "factor" when they meant "multiple", or vice versa, but there appeared to also be some differences in how participants understood GCF and LCM across the various representations.

Concerning modeling with pictures, manipulatives, and story problems, participants' demonstrated more success with LCM than with GCF. They demonstrated or described how to use manipulatives and pictures to represent LCM with relative ease perhaps because they had practiced this in their number theory course. All participants easily described how to find the LCM with pictures or manipulatives, but their descriptions for how to model GCF took more time and thought. And only half of the participants actually described how to use those models to *find* the GCF. Participants were fairly successful in creating and validating LCM story problems (discussed earlier in Section II). They struggled a great deal more to create and validate GCF story problems. Perhaps all of these observations can be attributed to the fact that participants had seen a variety of LCM representations in their number theory class, but they had not seen any concrete or story problem representations of the GCF. Another possibility is that the complexity of modeling GCF in a concrete way makes GCF models and story problems more challenging to understand and demonstrate than those of LCM.

However, when participants were required to work with the concepts of GCF and LCM numerically, participants demonstrated greater success with GCF. Participants successfully demonstrated finding the GCF by listing factors, using factor trees, and by using prime factorizations. Participants were less successful with finding the LCM using factor trees and prime factorizations. This was most obvious in Problem 10 from Interview 2 (discussed in Section I). Given the prime factorizations of *M* and *N*, all six participants easily determined the GCF, and each could correctly explain why the procedure for finding the GCF worked. In contrast, only Eden and Gwen immediately and accurately determined the LCM of *M* and *N*, and only Gwen accurately explained why the procedure for finding the LCM works. Eden said, “that’s just what we do.” Brit and Cara struggled but eventually found the correct value of the LCM by using something they knew to be true and adjusting their original LCM. For instance, Brit knew that the product of the LCM and the GCF would be equal to the product of *M* and *N*, so she was able to use *M*, *N*, and the GCF(*M*, *N*) to adjust the value that she found using an incorrect procedure. However, neither Isla nor Lucy could recall the appropriate procedure to find the LCM or use their other understandings about GCF and LCM to find the LCM in a different way.

Overall my data suggested that participants were more successful finding the GCF using numerical methods than they were representing the GCF using manipulatives, pictures, or story problems. Moreover, it seemed that participants *understood* GCF numerically more so than they did through elementary modeling. The opposite appeared to be true regarding participants' comfort and success with LCM. Discussion of these observations, and others, will occur in Chapter V. In the following section, I summarize my answers to Research Question 1.

Answers to Research Question 1

My first research question pertains to the nature of preservice elementary teachers' number theory content understandings. Given the exploratory nature of my study, this is not a question that can easily be answered. Each of the content tasks revealed different aspects of my participants' understandings of number theory, and I do not have evidence to suggest that all of these understandings are related. However, I summarize some of these understandings here.

Participants were fairly successful in validating number theory related claims and conjectures. For instance, all participants recognized that the product of two natural numbers is not always equal to their LCM, the difference of two numbers is not always equal to their GCF, and one is never prime. However, some participants struggled to explain why some claims were incorrect (why one cannot be prime) or identify cases when conjectures were valid (numbers whose LCM is their product) or identify valid relationships between concepts (the relationship between the difference of two numbers and their GCF). Participants also demonstrated success in validating various number theory procedures, like Eva's method for finding the GCF and Tom's method for finding the prime factorization of a number. Participants sufficiently explained why Tom's method worked, but they struggled to explain why Eva's method was valid.

In straightforward number theory tasks, like Problem 5 from Interview 1, most participants demonstrated valid solution strategies where they problem-solved in a way that incorporated their conceptions of GCF and LCM. While not all of these solution strategies were especially efficient, participants demonstrated a willingness to problem solve and draw from various experiences in their GCF/LCM repertoires. On Problem 10 from Interview 2, participants were fairly successful in identifying and justifying the divisibility of a number, given its prime factorization. For half of participants, however, there seemed to be some confusion about divisibility by two and multiples of two. All participants successfully found the GCF of two numbers, given their prime factorizations. They also successfully explained their reasoning for this procedure. Participants were less successful using the procedure to find the LCM from prime factorizations, let alone explaining it, which is concerning given their course experience with this procedure.

On Problem 5 from Interview 2, which incorporated more sophisticated number theory ideas, all participants produced at least some of the solutions to this task, either through a brute force or more theoretical approach. All of the participants that explicitly referred to mods in their initial investigations successfully found all solutions to the first and second part of the task. Participants' understandings of modular arithmetic and congruences contributed to their success in addressing this task.

Participants also demonstrated varying success in validating and creating LCM and GCF story problems. By all accounts, participants had never validated or created GCF story problems prior to their participation in this study. I presented models suggesting the processes each participant underwent to validate and generate these story problems earlier in Section II.

In general, I found that participants appeared to be more successful with portions of tasks that they could more clearly connect to their coursework experiences. I also found that participants were much more successful at the concrete portion of tasks that allowed participants to explore number theory ideas using concrete methods, but then required them to use theory to explain their results. There appeared to be a disconnect between the activity of working with a task at a concrete level and the activity of applying theory to explain these explorations. Lastly, I noticed that the representations with which participants best understood GCF were different than the LCM representations participants best understood.

Section III: Number Theory Pedagogical Content Knowledge

In this section, I answer the second and third of my research questions:

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential

> pedagogical content knowledge of number theory topics taught at the elementary level?

I designed several interview tasks to elicit participants' potential number theory PCK. As my participants were pre-service teachers, their PCK was not nearly as developed or robust as in-service teachers. Anytime I discuss their PCK, I am referring to their potential or developing PCK. Typically, participants demonstrated this potential PCK during the student scenario tasks. After posing the student scenario and asking participants to validate the student's claim, conjecture, or thinking (which elicits potential SCK, as discussed Sections I and II), I asked questions designed to elicit knowledge of content and students (potential KCS) and knowledge of content and teaching (potential KCT), constructs of PCK as outlined by Ball and colleagues' model for Mathematical Knowledge for Teaching (Hill, Ball, and Schilling, 2008). However, Ball and colleagues' model was designed with in-service teachers in mind, another reason I distinguish between the type of PCK demonstrated by my participants and the mathematical PCK of in-service teachers. In Chapter III, I supported my claim that preservice teachers can demonstrate potential KCS and potential KCT in an interview setting by discussing the existing literature on preservice elementary teachers' PCK. At the very least, the KCS and KCT that participants demonstrated was a subset of their potential KCS and KCT.

In this section of the results, I begin by reviewing how I elicited KCS and KCT and describing my efforts in coding instances of potential PCK as either potential KCS or potential KCT. Then I provide evidence for participants' potential PCK by elaborating on their responses to three student scenario tasks. Next, I suggest themes related to participants' potential number theory PCK. Finally, I summarize my findings to address my second and third research question. In the following sections, the reader should interpret PCK to mean "potential PCK", KCS to mean "potential KCS", and KCT to mean "potential KCT."

Eliciting and Coding Knowledge of Content and Students

Knowledge of Content and Students (KCS) is the "knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004, p. 17). To elicit KCS, during the student scenario tasks I asked questions like "why might the student think this?" or "what does the student understand about the concept?" or "what misconceptions might the student have about the concept?" These types of questions were designed to specifically elicit KCS, because they provided participants an opportunity to demonstrate their understandings of student reasoning and understanding. This differs from the type of understanding participants demonstrate when validating student claims, because validation merely determines the accuracy of the mathematics involved – it does not attempt to explain *why* a student may have made the claim.

As described in Chapter III, I coded participants' statements as "KCS" if they pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). If a participant provided a reasonable explanation for why a hypothetical student made a number theory claim or conjecture, I coded the statement as "student reasoning." However, if a participant provided a reasonable explanation for why a hypothetical student might demonstrate a specific difficulty in understanding number theory or a number theory-related misconception, I further coded the statement as "student challenge." The code "KCS" and its subcodes, "student reasoning" and "student challenge" emerged from my pilot study analysis (see Appendix N). I present additional examples of, and justifications for, these codes later in this section.

Eliciting and Coding Knowledge of Content and Teaching

> Knowledge of Content and Teaching (KCT)
>
> Combines knowing about teaching and knowing about mathematics, … [such as] sequenc[ing] particular content for instruction… evaluat[ing] the instructional advantages and disadvantages of representations used to teach a specific idea and identify[ing] what different methods and procedures afford instructionally. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning… Each of these decisions requires coordination between the mathematics at stake and the instructional purposes at play (Ball, Thames, & Phelps, 2008, p. 401).

To elicit KCT, I asked participants how they "might respond to the students to help them recognize their misconceptions." This was a deliberately leading request; I wanted participants to address the mathematics of the scenario by attempting to further the hypothetical students' understandings of the mathematics at hand. Had I asked participants a more vague question like "how might you respond to the student?", the responses may have been less likely to reveal potential KCT. To classify a participant's response as an instance of KCT, the response needed to be an instructional response to the hypothetical student in the scenario and it needed to pertain to the specific mathematics at stake. For instance, in Problem 3 of Interview 1, participants said they would give the hypothetical student a specific counterexample to help him recognize his misconception. I coded these statements as "KCT" because not only were they hypothetical instructional responses, but by strategically picking a specific counterexample the statements pertained to the specific mathematics related to the misconception.

Some of participants' instructional responses to the students in the scenarios were too general to qualify as KCT – they did not address the specific mathematics of the situation. For example, if a participant told the hypothetical student to try and find a counterexample, the mathematics of the statement would have been too general to target the specific mathematics of the students' misconception. Statements such as these also did not demonstrate participants' understandings of the mathematics at stake. Either way, I could not code such statements as "KCT", because they did not fit the definition of the code. Of these general responses, very few could be qualified as general pedagogical knowledge (GPK), or pedagogy that transcends subject matter. Rather, most of these general responses were only applicable in the realm of mathematics education. I coded these responses as General Mathematical Pedagogy (GMP). Most of the participants' responses to the hypothetical students in the scenarios were coded as either KCT or GMP. I elaborate on this proposed construct of PCK in later sections, as evidence of its existence arises from the data.

As a final follow up question to each student scenario task, I asked participants to reflect on their reasoning for responding to the hypothetical students in the ways that they did. Occasionally, this metacognitive line of questioning revealed the various influences on participants' KCT. More often than not, it provided insight into participants' GMP.

Participant Responses

Three of the student scenario tasks richly revealed participants' PCK and demonstrated the wide variety of responses to the KCS- and KCT-eliciting follow-up questions. Here, I provide detailed evidence of participants' number theory PCK in response to these three tasks.

Interview 1: Problem 3. In this student scenario, Mark suggested that the least common multiple (LCM) of two numbers is equivalent to their product. Recall that all participants determined that Mark's conjecture was incorrect, and found appropriate counterexamples, but only Brit, Cara, and Lucy correctly determined that Mark's conjecture works for pairs of relatively prime numbers.

When I asked participants why they thought Mark might believe his conjecture to be true, they responded with a variety of insights, which I coded as "KCS" if the statement pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). I also coded the KCS statements as "student reasoning" if the participant referred to *why* a student might believe a statement, claim, or conjecture about number theory is true or false.

All six participants had acknowledged at some point during the interview that Mark's conjecture works for some pairs of numbers. Cara, Eden, Gwen, and Isla explicitly cited this as a reason for why Mark may have formed his conjecture. I coded this as "KCS", and "student reasoning", more specifically, because it was a reasonable explanation for why Mark might have believed his conjecture to be true. Here, the participants' KCS drew from their mathematical understandings of the scenario (i.e., their SCK), which was discussed in depth in Section I. Two participants, Cara and Gwen, recognized that younger students typically work with small numbers, many of which provide confirmatory examples for Mark’s conjecture. They surmised that Mark could have been drawing from his experience with these examples to form his conjecture.

> Cara: I think we start off kids with the notion that we can just multiply for the most part, and we're going to get [the LCM]. Then we move to greater numbers, but we haven't necessarily told them, or they may not necessarily know, or they may not understand that you [need] a GCD of one.

I coded these statements as “KCS” and “student reasoning”, because again they were reasonable explanations for why Mark believed his conjecture to be true. Cara and Gwen’s curricular content knowledge, as well their SCK, contributed to these instances of KCS. By discussing how students progress with a concept by using small numbers at first and then moving on to larger numbers, Cara and Gwen were referencing scaffolding and curriculum. Thus, I also coded these statements as “curricular content knowledge”, another code that emerged from my pilot study analysis. While I did not design my interview tasks specifically to elicit this type of knowledge, it did occasionally appear to contribute to participants’ PCK.

Three different participants (Brit, Isla, and Lucy) also drew from their curricular content knowledge and SCK in their responses, but they focused on a different aspect of the content: the concept of multiple. When elementary students first learn the concept of multiplication, they learn that the product of two numbers is a multiple of each. Brit, Isla, and Lucy, in one way or another, felt that this was the reasoning behind Mark’s claim. Again, this was a reasonable explanation and could thus be coded “KCS” and “student reasoning.” Brit claimed that Mark probably thought that "if you multiply them once that would be the least common one, and that if I multiplied them together multiple times that those would be bigger multiples." While this is a possible reason for Mark's conjecture, Isla and Lucy's insights may have been more probable causes for Mark's reasoning.

> Isla: [The numbers] *a* and *b* are both going to go into *a* times *b*. I think it's an easy way to introduce students to [LCM], and so since they're first introduced to it, they just automatically think that it's always going to work, just because it's the first thing they learned.

> Lucy: I think area models play a lot into this, because I think they see that 6 times 8 is 48 and… that would easily be divided up into 6 groups of 8 and 8 groups of 6… the least common multiple goes into the large area model... the GCD amount of times. So I think that a lot of times students are familiar with the area models, but maybe not looking at them in terms of the least common multiple and that there may be something [a common multiple] below just 6 times 8.

All three of these participants drew from the idea that the product of two numbers gives a common multiple, but they each had a unique perception of how Mark might be thinking about it. Given that each of the participants also referenced number theory curriculum, I also coded their statements as "curricular content knowledge."

After I asked participants why Mark might believe his conjecture to be true, I asked them how they would respond to Mark to help him correct his misconceptions, hoping to elicit KCT. Cara, Eden, and Gwen said they would begin by pointing out that Mark's conjecture works for some pairs of numbers. This seemed to be more of an effort to boost the hypothetical student's confidence, rather than a deliberate instructional decision related to the mathematics at stake. As such, I did not feel justified in classifying these statements as "KCT."

Eden was the only one to suggest that she would explicitly tell Mark which types of numbers worked. However, Eden's limited understanding of Mark's conjecture led to an inaccurate response. "You could go in and say, 'Yes, this method does work but only for certain types of numbers. And these certain types of numbers would be the prime numbers.'" Here, Eden's SCK weakened her KCT.

Gwen suggested that she would discuss a confirmatory example (four and five) with Mark so that he would better understand why it worked. "[Five] is a prime and nothing will go… one, or two won't go into this one, or four, into the five. Maybe that's where I would go to start with. See if he can kind of see that." Even though it may not have pointed out why Mark's conjecture does not always work, Gwen's instructional decision did involve the mathematics at stake, so I coded it as "KCT." Gwen's response suggested that she had a vague understanding that factors had something to do with whether or not Mark's conjecture works for a pair of numbers. Gwen did not suggest an explanation as to why the factors were relevant, and in a hypothetical conversation with a student it is more than likely Mark would have asked why she was discussing the factors of four. However, Gwen struggled to address this question when we discussed the content of the task; her overall conclusion was merely that Mark's conjecture works for prime numbers and consecutive whole numbers. The data suggested that Gwen's understanding of the content may have limited any explanation with regards to the role factors play in finding the LCM of two numbers.

In their responses to Mark, Brit and Cara also suggested they would point out that while the product of two whole numbers is a multiple, it is not always the *least* common multiple. This instructional decision did draw attention to the inaccuracy in Mark's conjecture, so I coded it as "KCT." Brit had claimed earlier that Mark's misconception may have been due to his understanding of what it means to be a least common multiple. So her decision to respond to Mark in this way was purposeful; she seemed to be drawing from her KCS. Cara, on the other hand, did not explicitly intuit that Mark's issue lay with his understanding of LCM; instead, she claimed that he was merely drawing from a pattern of confirmatory examples. So Cara's comment that the product was a multiple, but not always the least common multiple, was not as targeted as Brit's. Perhaps it was just an offhand comment meant to elaborate on the concept, in which case Cara's comment drew more from SCK than KCS.

At some point in their responses to Mark, all participants claimed they would present him with a counterexample to explore. I coded these statements as "KCT" because not only were they hypothetical instructional responses, but by strategically picking a specific counterexample the statements pertained to the specific mathematics related to the misconception. Isla suggested the counterexample of two and six, and she went so far as to explain to Mark *why* it was a counterexample.

> Isla: If we had the numbers two and six, and if you multiply them together, you get 12. But in the sense of the least common multiple of two and six, it can be six, because six times one is six and two times three is six.

If Mark's misconception was due to a misunderstanding of what it means to be a *least* common multiple, as Isla had suggested earlier, then this response would address that. Isla's decision to respond in this way illustrated yet another example of KCT drawing from KCS (and SCK).

Cara initially claimed that she would also provide Mark with a counterexample. However, her tack was very different than Isla's. Isla said that she would fully explain the counterexample to Mark, while Cara insisted that Mark explore the counterexample on his own. Then, Cara backpedaled and said she preferred an even less direct approach.

> Cara: Or… Have him tell me why [his conjecture] works every time, and then just ask him if he can find an example where it doesn't work… Just really have him show me why he thinks it's that way, or why something might work when he thinks it doesn't… So having them respond and not just tell them what they did wrong, but instead see if they can figure it out on their own, kind of thing.

Here Cara suggested that she would have Mark explain his conjecture and explore examples, hoping that he would stumble across a counterexample on his own. This appeared to speak to Cara's general epistemology on how children best learn mathematics, but it did not address the specific mathematics of the task, which is a large component of KCT. Given that having the student try more examples to see if he can find a counterexample is such a general approach to student conjectures, I classified Cara's response to Mark as an instance of general mathematical pedagogy (GMP). This was the only instance of GMP to arise from participants' responses to this particular student scenario, but it did occur quite a bit in response to a task I discuss later.

The other four participants, Brit, Eden, Gwen, and Lucy, suggested that they would give Mark a counterexample to explore using Cuisenaire rods. They all claimed that this would help Mark see, in a tactile and visual way, that his conjecture was not always true. This decision seemed to draw from participants' SCK and experience with Cuisenaire rods from their number theory course. Brit also used this opportunity to draw attention to common factors. "With six and eight, they have that two in common, so they have that stuff to match up before they actually multiply together." Brit went on to say that "we have to look at what they have in common and whether we can match up [the trains] before [the product]." Not only did Brit create an opportunity for Mark to realize that his conjecture was invalid, but she demonstrated KCT by also creating an opportunity for Mark to understand *why* his conjecture does not always work.

Because Brit, Eden, Gwen, and Lucy suggested they would have Mark use a specific manipulative to perform a specific procedure to find the LCM, their responses to Mark demonstrated KCT that drew on their SCK. However, if any of the participants had merely suggested that Mark investigate LCM using any manipulative, this would have demonstrated GMP. This suggestion would not have demonstrated a specific knowledge of the mathematical concept at hand, but rather it would have demonstrated a general knowledge that manipulatives can improve mathematical understanding. Often, when a student is struggling with a procedure, visual or tangible aids can be helpful in improving their understandings. "Try using a manipulative to solve the problem" is a general response that does not exhibit the teacher's specific mathematical understanding of the task at hand.

Participants offered a multitude of reasons for why they responded to Mark in the way that they did. I coded all of these responses as "insight to KCT." Brit, Eden, and Isla all cited their tutoring experiences with elementary and middle school students. Brit said that her response to Mark was "just a natural thing" for her because of her years of experience tutoring students. Eden's tutoring experiences led her to believe that students can be quite adamant that their answers are correct and that it can take a bit of work to convince them of an invalid answer or procedure.

> Eden: I tutor some kids in math, and they always think that their method is right, but you kinda show them that, 'if you do it this way I get this answer and it's not the same as yours. How come?' And you kinda slowly take what they're saying and slowly show them why it's wrong. And hopefully they'll connect to it saying, oh yeah, that is wrong.

Eden was one of the three students (including Brit and Isla) that had not taken the mathematics education course focusing on number and operations the previous semester.

In response to a follow-up question later in the interview, she claimed that her tutoring experiences were the only experiences that contributed to her responses to the hypothetical students in the student scenarios. She also suggested that her strategy when working with students mostly consisted of trial and error. She said she would “see what works and what doesn’t work.” Isla claimed that her tutoring experiences helped her to recognize the conflicts that arise when teachers tell their students that a “rule” always works when, in fact, it may not. She frequently tutored her younger cousin, a 5th grade student, and she witnessed her cousin attempt to make generalizations about her mathematical understandings from earlier grade levels in order to better understand the current material. Isla claimed that this was problematic. She said, “You’re told this rule applies for all, but it really doesn’t.”

Participants provided many other reasons for their responses to Mark, besides their tutoring experiences. Brit, Eden, Gwen, and Lucy – the participants who suggested they would help Mark explore his conjecture using Cuisenaire rods – all claimed that they responded to Mark in this way because they believed that visual and tactile methods help students understand. This seemed to relate to participants’ mathematical epistemologies.

Brit, Cara, and Gwen also said that it was important to them to try and build on or connect to Mark’s understanding of the concepts. Brit said her response was “so [Mark] can use what [he] kind of know[s] and apply that to what [he] *can* know… [Students] will remember it better if it’s a piece of what they know.” These demonstrated examples of constructivist perspective on learning. Brit added that building on what students already know can inspire confidence because they are partially correct.

Most of the participants explicitly mentioned that it was important not to tell Mark, “No, you’re wrong; do it this way.” The only participant that seemed to reference a deficit model was Eden. In contrast to emphasizing that Mark was partially correct, Eden responded to Mark the way she did because she felt it was important for him to recognize that he was wrong. As mentioned earlier, Eden’s tutoring experiences had led her to believe that convincing students of their mistakes was paramount when students have a misunderstanding. Cara, Gwen, and Lucy also felt that it was important that Mark recognize his mistake, but in a non-direct way. According to Lucy,

> If you just say 'no, that's wrong' they're obviously going to question why. And you want to be able to show them. But I think it's often times best for them to see it themselves and to figure out themselves.

Cara, Gwen, and Lucy believed that presenting Mark with an opportunity for cognitive conflict would be the most effective way to help him understand his mistake. Again, this demonstrated a constructivist approach.

Cara, Gwen, and Lucy were the participants who had taken the mathematics education course emphasizing number and operations the previous semester. At one point or another, each of these participants acknowledged the role this course played in their responses to the hypothetical students. The data suggested that these participants’ ideas on constructivism and non-direct instruction resulted from their experiences in that course.

> Cara: That [course] taught us to look at how the kids are learning and what they're thinking. Not change what they're thinking, but help them move along to get the answer, versus coming up with a totally different way for them to learn - being ready to teach different ways.

> Gwen: [That course] really emphasized having the students develop the concept instead of just telling them… having them see if they can

> find their mistake. That way, I think it sticks with people longer if they can find those mistakes and see where you made it and can correct it. Then, I think it sticks with you longer than just being told how to go about doing something.

It is also worth pointing out that the participants who had not taken this mathematics education course (Brit, Eden, and Isla) were the only participants to cite their tutoring experiences as their primary reasoning for their responses to Mark.

Some participants also mentioned that the number theory course influenced their responses to Mark. Brit claimed that using the Cuisenaire rods to model LCM in the number theory course was helpful, and that that experience influenced her suggestion that Mark uses them to explore his conjecture. Cara described her experience with the number theory course as a learner.

> Cara: [In the number theory course], we do a lot of things that might work one way but then if you get a larger number, that doesn't really work, and we're not going to be told "this will always work", even if we think it will, a lot of times we come to the conclusion ourselves that it won't always work or we try to.

Cara claimed that this non-direct method of instruction influenced her responses to the hypothetical students in the scenario.

Most of the participants' reasons for responding to Mark in the ways that they did seemed to stem from epistemological beliefs and influences. This was surprising given the mathematical sophistication of Mark's conjecture; I had anticipated that participants might also suggest that their understandings of the content influenced their responses to Mark. While this appeared obvious to me from their responses, participants were more aware of how their epistemologies influenced their responses, rather than aware of how their mathematical understandings influenced their responses. This was fairly typical of their reasoning for responding to all of the hypothetical students in the scenarios.

Interview 2: Problem 7. In this student scenario, Shayna, a fourth grade student, incorrectly claimed that one is a prime number because its only factors are one and itself. As part of their validation of Shayna's response, all six interview participants correctly determined that Shayna's claim was incorrect. From Problem 6 in the same session, all participants defined a prime number as a (whole) number whose only factors are one and itself. Participants drew from this definition heavily in their responses to the questions of this task. I discussed participants' SCK related to this task in Section I.

To elicit KCS, I asked participants to identify the concepts that Shayna *does* understand. Participants' responses revealed a lot more about their personal definitions of "prime" than their explicit response to my question in Problem 6: what is a prime number? Even though Gwen and Isla defined prime very similarly to the other four participants, it was evident from their responses to Shayna that they considered the factors of a prime number, one and "itself", to be distinct numbers, and *this is part of the definition*. As a result of this understanding, Gwen and Isla claimed that Shayna understood part of the definition, but that she did not recognize the part that makes the factors distinct. This is a reasonable explanation for why Shayna believed her claim to be true, so I coded Gwen and Isla's statements at "KCS" and "student reasoning."

When I asked Gwen and Isla how they would respond to Shayna to help her recognize her misconceptions, they both responded in a way that highlighted their personal definition of prime. Gwen responded, "I'd just try to tell her that prime [numbers] have two different [factors]. They don't just have the one." Isla responded similarly, also emphasizing that the factors must be distinct.

Isla: I think the easiest way is just to tell her it has to have one and itself, but "itself" has to be a different number. So it has to have

> two different [factors]. And I think if she starts thinking about it like that, she'll never confuse one to be a prime number again.

These responses were instructional decisions that addressed what Gwen and Isla believed to be Shayna's specific mathematical misconception, so I coded them as "KCT." They drew not only from Gwen and Isla's SCK, but their KCS. Their responses directly addressed the misconception that they identified using their own understandings of the definition of prime.

When I asked Gwen and Isla why they responded to Shayna in this way, they both indicated that rethinking their definition of prime was part of what convinced them that one could not be prime.

> Gwen: Because I think it would make more sense to me if I was told that in 4th grade. Because I always had trouble with the whole prime number thing. I think that if I was told that prime factors… er, prime numbers only have one and themselves as a factor, and they're always two different numbers, then that would have helped me conceptualize that a little bit better. It wouldn't really help me understand all of the background stuff, but it would help me say 'OK, it doesn't fit that criteria.'

In her response to my question, which I coded as "insight to KCT," Gwen suggested that having a more useable definition of "prime", a definition whose criteria the number one does not fit, would have been preferable. What she did not realize is that the definition of prime *does* eliminate one as a prime number. Gwen also stated that while her response would help a student classify whole numbers as prime or not, it would not "help to understand all the background stuff." I believe that Gwen was referring to an understanding of *why* one is not prime. Isla also attributed some of her experiences in the number theory course as to why she responded to Shayna as she did.

> Isla: I used to be the person that would fight tooth and nail that one was a prime number, until this class. And [the professor] proved me

> wrong. It's all based off of number theory and talking about it in groups and how we all came up with our own ideas of how one couldn't be prime.

Isla said that there was an "equation" that they used in the number theory course that was invalid if one was prime, and that equation was what really convinced her that one could not be prime. While she could not remember what that equation was, the experience gave her the confidence to respond to Shayna with certainty that one was not prime.

Gwen's and Isla's evaluations of Shayna's understanding of the content (KCS) and their responses to her (KCT) hinged on their understandings that a prime number was a whole number with two distinct factors (SCK). To them, this easily eliminated one as a possible prime number. The other four participants did not consider the distinctness of the factors to be part of their definition of prime, and as a result they classified one as "an exception to the rule." This had a significant effect on how Brit, Cara, Eden, and Lucy assessed Shayna's understanding of the content.

> Brit: [Shayna's reasoning] makes sense because of how we define primes. It can only be multiplied by one and itself. So, technically, based on that definition, [one] should be [prime], because it can only be multiplied by one and itself, and 'itself' just happens to be one. So it's not invalid at all. She's just going straight off of that definition.

Cara, Eden, and Lucy all made similar statements claiming that Shayna correctly used the definition of prime. These statements drew from the same weakness in the four participants' SCK. As with Shayna, Brit, Cara, Eden, and Lucy were working from an incomplete definition of prime. Every definition of prime is phrased in such a way that there should be no ambiguity when classifying a whole number as prime or not. Phrases like "greater than one" or "greater than or equal to two" can be found as qualifiers in every textbook definition of prime - including those written in elementary school textbooks and participants' number theory text. Thinking of one as an extra exception discounts the fact that the definition already eliminates one as a possible prime number.

Brit went on to say, "We just have to understand that, just like the 'all primes are odd' rule doesn't apply to two, that certain rule doesn't really apply to one." Again, Brit's SCK weakened her KCS. "All primes are odd" is not a "rule", as she called it; this is a common invalid student conjecture about prime numbers. The other "rule" she referred to is actually the *definition* of prime, which is a statement of fact. And a distinction should be made that one is an exception *because* of the definition rather than *in spite* of the definition. The latter diminishes the role that mathematical definitions play in reasoning and proof, as evidenced by how Brit combined the definition with incorrect student conjectures. Cara, Eden, and Lucy also claimed that the number one is merely "an exception to the rule."

When I asked participants how they might respond to Shayna, Brit, Cara, Eden, and Lucy responded very differently than Gwen and Isla. Gwen's and Isla's responses to Shayna were strong in the way that they helped her better understand the definition of prime, but they were weak with respect to helping her understand *why* one could not be prime, which Gwen acknowledged. Brit, Cara, Eden, and Lucy's responses to Shayna were the opposite: weak with respect to helping Shayna better understand the definition of prime and strong with respect to helping her understand why one cannot be prime. Either way, their instructional decisions addressed the mathematics at stake, so I coded these responses as "KCT."

> Brit: I think the factor tree could be really helpful, because if we use the example of 25.We get five times five. Usually we would stop and say, OK, do we know that five is prime. But what if we said that five is divisible by five and one, well then one is prime, and just

keep going and see how crazy it would get if we said that one was prime.

Here, Brit described a method that the participants experienced in their number theory course. If we assumed that one was prime, then factor trees would never end, because we continue to factor for as long as we get prime factors – and we could always find another factor of one. Eden described a similar response to Shayna.

Cara also attempted to describe the never-ending factor tree, but then she took a different tack.

Cara: I think multiplying by itself, like one times one, isn't prime because you're multiplying by itself. And I could have her do two times two. You're multiplying by two things, so that gets you not a prime number. Maybe that's one way to get her to that thinking.

Cara's wording was a little imprecise; I believe she meant that the two numbers being multiplied were the same, not merely that two numbers were being multiplied - because all whole numbers (even primes) can be written as the product of two numbers. But Cara's idea that square numbers cannot be prime had merit. However, given that Cara described one as an exception to the *definition* of prime, Shayna may have required additional explanation as to why one was not an exception to this rule too.

While Lucy struggled to phrase her response to Shayna, it appeared to allude to the Fundamental Theorem of Arithmetic.

Lucy: I think I would just show her its prime factorization because its prime factorization is one and itself. But really it's just one to the first power. You could do that over and over again. You could have one to the first, times one to the first, times one to the first... Whereas prime factorizations are just normally broken down into like two to the 13th power and you know... I think you could show her that it's always repeating I guess. So that's not exactly prime. It's not breaking down as even as other numbers do in their prime factorizations.

When Lucy referred to prime factorizations as being "even" - a term that students tend to inappropriately attribute to many processes - I believe that she was referring to the lack of variability. Prime factorizations are unique, except for the order of the prime factors, by the Fundamental Theorem of Arithmetic. Lucy seemed to hint at the variability of the “prime” factorization(s) of one if one were considered prime. This would contradict the Fundamental Theorem of Arithmetic, proving that one could not be prime. If Shayna was familiar with this theorem, this may have been a valuable way to respond to her. Lucy's confusion with regards to the content, however, weakened her response.

Brit, Cara, Eden, and Lucy’s KCT, demonstrated in their responses to Shayna, all drew from their SCK, their understandings of why one cannot be prime. Their KCT also drew from their KCS. They determined that Shayna did not understand that one must be an exception to the rule; one cannot be prime. Their responses were all specifically designed to help Shayna understand this, perhaps with varying levels of success. Brit, Cara, and Lucy also explicitly drew from their curricular content knowledge. They each discussed how factor trees are part of the fourth grade curriculum. Lucy’s discussion of prime factorizations may have been slightly out of reach for a typical fourth grader, however, because students are only first introduced to prime and composite numbers in fourth grade (Common Core State Standards Initiative [CCSSI], 2011). Also, students might not see exponents until sixth grade (CCSSI, 2011), which is also prerequisite knowledge for determining a number’s prime factorization. I coded these statements as “curricular content knowledge.”

When I asked Brit, Eden, and Lucy why they responded to Shayna the way they did (coded as “insight to KCT”), they each cited the number theory course and how their experiences in that course convinced them that one was not prime.

> Brit: I originally thought, like, I always questioned why one doesn't work, and obviously I still question it a little bit, because it took me a while to get here. But for me it makes sense to see it this way. This is a strong example of why it can't work, so I know it can't ever work.

Brit’s number theory course experience convinced her that one was not prime. This experience also gave her a tool with which to help Shayna understand that concept. Eden and Lucy made similar statements.

When I asked her to explain her response to Shayna, Cara responded more generally. Instead of discussing the mathematics involved in her response to Shayna, Cara discussed her response from a constructivist perspective. She said, “Instead of just telling [Shayna] that she's wrong, showing her ways that she might realize that she is wrong. That way you're not pointing it out and she can still discover it on her own.” Given that Cara framed her response to Shayna in a "you're not quite correct and here's why" way, she did not accurately represent her response as a discovery learning opportunity. However, her response did present an opportunity for cognitive conflict, which aligns with the constructivist perspective.

Interview 2: Problem 9. In this student scenario, two students, Talisa and Tom have factored 540 using two different techniques. All six participants correctly determined that Talisa’s factor tree was incomplete. Isla only tentatively believed Tom’s method of factoring to be valid, while the other participants were confident in their validation of Tom’s method. Talisa and Tom disagreed with each other’s answers, and I asked participants how they would resolve the situation. In contrast to the previous student scenarios, most of the participants suggested that they would respond to Talisa and Tom in very general ways; the majority of their previous responses to the hypothetical students specifically addressed the mathematics of the task, whereas these responses did not. Again, I coded these responses as general mathematical pedagogy (GMP).

Brit was the only participant whose primary response to Talisa and Tom exhibited KCT. She claimed that she would explain each student's method to help Talisa and Tom recognize the validity in each other's methods of factoring.

> Brit: I would just help them see that both of their ways are valid, because they're both kind of doing the same process of finding those small numbers that make [540] up. So while Tom just starts with the small ones and keeps dividing by the small ones, and Talisa starts with the big ones and divides them into small ones. We can still… It doesn't matter which way you do it. It's just whatever is best for you. But we do end up getting those same numbers.

Brit acknowledged that Talisa would have to continue factoring in order for the two students to "get the same numbers", something she would have pointed out to Talisa and Tom. Brit's explanatory response to Talisa and Tom seemed to draw from SCK. When I asked her why she responded to Talisa and Tom in this way, Brit said that it was important that the hypothetical students recognize that they would get the same prime factorizations regardless of what method they used. This scenario presented participants with an opportunity to acknowledge and emphasize the importance of the Fundamental Theorem of Arithmetic, and Brit capitalized on that opportunity. To be sure that Talisa and Tom understood each other's factoring methods, Brit also suggested that she would have them try to perform each other's procedures with another example.

> Brit: I would maybe give them another example, and maybe have them switch. So have Talisa do it Tom's way, and Tom do it Talisa's way, and kind of show that other way and how they got there, so they can both see again the other ways - how the other person's way works and is valid, and how they both get the same answer either way.

Brit's secondary response, while potentially helpful in solidifying Talisa and Tom's understandings of other factoring methods, demonstrated a more general knowledge of how to respond to students who are learning a new process: practice the procedure using a new problem. This strategy for responding to Talisa and Tom is not specific to methods of factoring. Because this response did not specifically pertain to the mathematics at hand, I classified it as an instance of GMP.

Like Brit, Cara also offered two responses to Talisa and Tom. However Cara's primary response exhibited GMP, while her secondary response exhibited KCT. Initially, Cara suggested that she would have Talisa and Tom explain their methods to each other.

> Cara: Well you could have Tom explain to her how and why he did his way, and have Talisa explain to Tom why she did it her way and then they can compare. And maybe at this point she would realize [that she's not done]... But having them explain their methods to each other would be helpful so that they both realize that there's not just one way to do that. You can do it in multiple ways.

This strategy is a general response that can be utilized whenever two students are using different procedures to solve the same problem. Having students reconcile their conflict without any teacher intervention does not require the teacher to demonstrate any mathematical understanding, let alone the specific mathematical understanding of the task at hand. Thus, Cara's primary response was an instance of GMP. After Cara responded in this way, she backtracked, realizing that it would not necessarily ensure that Talisa would recognize her mistake.

> Cara: Have [Talisa] walk you through it before explaining it to Tom, just to make sure she realizes that she does need to go a little further, and then they would get the same answer… But you would have to fix hers before you let her explain it, because then she wouldn't get it right.

Helping Talisa navigate her mistake would require a discussion about the specific mathematics of her factorization, which would exhibit KCT that draws from SCK. When I asked Cara why she responded to Talisa and Tom in this way, she did not address the mathematics of the problem like Brit. Instead, her main emphases were collaborative learning and understanding.

> Cara: I think it's really important to let students work their own way and to discover, so by having them explain to each other, they'll have a better understanding of what they're doing, and they also get to see another method… So just walking yourself through it and talking can really help you understand why you did it. This way even if you did it, you may not know exactly why.

Cara’s epistemology drove her response to this task and her primary response to Talisa and Tom, a clear contrast to Brit’s emphasis on the mathematics of the task.

Like Cara, the remaining four participants all suggested that they would have Talisa and Tom explain their factorizations to one another. Unlike Cara, however, they did not offer any KCT to ensure that Talisa recognized her mistake. Isla’s response is evidence that this strategy, having Talisa and Tom explain their methods, does not even require a strong understanding of the mathematics involved in order to employ it.

> Isla: The first thing would be [ask]ing how Tom did this. And if it was just by chance and he worked it out, then I have no idea how he did it. But if he could explain how he did it and why he did what he did, maybe showing them to each other or putting them in a group together. And letting them realize that both their ways work. And maybe Tom's is easier than Talisa's or takes less time. So maybe having them teach each other their different ways.

Having never seen Tom’s method of factoring before, Isla was admittedly confused by it. In the case where students cannot identify or fix their issues on their own, a teacher's SCK would be necessary to more actively guide the discussion. Otherwise, asking the students to explain their methods to one another may have the same mathematical outcomes, regardless of the teacher's mathematical understanding. While Isla suggested employing this teacher-strategy and hoping for the best, Eden had a clearer vision of how she wanted the discussion to go.

> Eden: I would maybe have them come together and see, maybe, look at each other's work and see how they got their numbers and then maybe one would notice, maybe Tom would notice, six can still be broken down, and nine can still be broken down. And so just maybe having them work together to help them figure out why is there a discrepancy here. So maybe they'll both be able to critique the other one's work and figure out where something went wrong.

Here we see that instances of GMP do not necessarily mean that a teacher does not have a specific mathematical outcome in mind. In other words, SCK *can* inform GMP, but in a much less observable way than with KCT. Had I observed Eden in the classroom, her decision to have the students discuss their methods would not have demonstrated any of her SCK pertaining to factoring. However, any teacher intervention thereafter may have.

When I asked Eden, Gwen, Isla, and Lucy why they responded to Talisa and Tom in this way, they each provided a reason that seemed to relate to their epistemology. Eden, Isla, and Lucy each suggested that collaborative learning would be more meaningful to Talisa and Tom than a teacher-provided explanation. Additionally, Gwen suggested that explaining your mathematical understanding could actually strengthen it; she claimed that it worked well for her. All of the participants at one point during the interview cited their experiences as learners in one of their many mathematics courses that emphasized collaborative learning, including the number theory course. Cara, Gwen, and Lucy also mentioned their discussions on teaching philosophy in their mathematics education course as having an impact on their responses to the hypothetical students in the scenarios. Lucy even explicitly cited the mathematics education course as emphasizing student discussion about mathematical tasks, or "math talks." She said that this strategy could be used to broaden student understanding, help them self-check, or just create a more meaningful learning environment.

> Lucy: I think a lot of problems can be discussed as a class so students can see other students' work and how they did it and then they can also correct their own work if they're like "oh, I did it kind of right, but I didn't finish it, or I didn't get to the right answer." I think they can see it that way... just kind of making everyone's work go together and students realizing that they can learn from their peers.

My analysis of the PCK-related tasks indicated that the nature of participants' number theory PCK was quite varied and complex, but that their understandings of the content – specifically, their SCK – most certainly played a role. A multitude of personal experiences, beliefs, and mathematical understandings had varying effects on the participants' KCS and KCT. In the following section, to further answer my second and third research questions, I summarize my observations about the influences on participants' KCS and KCT. Finally, I answer Research Questions 2 and 3. In partial answer to these questions, I propose a model to better illustrate the influences on and connections between the various constructs of PCK.

Influences on Participants' Pedagogical Content Knowledge

The interview tasks revealed quite a bit about the nature of participants' KCS and KCT. They also revealed a type of knowledge that is certainly a type of PCK, but does not appear to fit with the definitions of the constructs as defined by Ball and colleagues; a construct I have named GMP. While curricular content knowledge, or CCK, is also a type of PCK, the interview tasks were not designed to elicit it, nor did it arise often during the interviews. The data suggested that participants' SCK and epistemological perspectives frequently contributed to their PCK, and that some interaction between the types of PCK existed.

Influences on knowledge of content and students. In most instances of KCS participants drew on their SCK pertaining to the given student scenarios. Depending on the nature of participants' SCK (which was the focus of the first two sections of this chapter), it could strengthen or weaken their KCS. For example, we saw this in participants' understandings and treatment of the mathematical definition of prime in Interview 2, Problem 7. Participants who felt a mathematical definition was merely a rule that could be broken, determined that Shayna's understanding of the definition of prime was correct, and that the number one, while meeting the definition of prime, was an exception to the "rule." Participants who had a stronger sense of the mathematical definition of prime recognized that there exist caveats within the definition that preclude one from being prime, and they correctly determined that Shayna was unaware of these caveats.

Some participants also drew from their curricular content knowledge, or CCK, while exhibiting instances of KCS. The most prominent examples were when Brit, Isla, and Lucy explained why Mark might have believed his conjecture to be true, in Interview 1, Problem 3. All three participants identified a different aspect of the curricular materials typically used to teach the concept of "multiple" and explained how this could have led to

Mark's confusion. The last influence on participants' number theory KCS that arose from their responses to the interview tasks was participants' experiences working with students. Occasionally, participants recognized a "common" mistake in the student scenario task, "common" to them because they had experienced it before as learners. The most prevalent example was Shayna's misconception about the number one being prime. Some participants explicitly acknowledged that this was a common misconception, because they had experienced this misconception themselves and with others.

Influences on knowledge of content and teaching. The interview tasks revealed the influences on KCT to be much more variable. Similarly to its influence on KCS, participants' SCK usually contributed to participants' exhibited KCT, and depending on the nature of their SCK, it could strengthen or weaken KCT. For example, in Interview 1, Problem 3, when Eden responded to Mark, she told him that his conjecture only worked for pairs of prime numbers, because that was her mathematical understanding of Mark's conjecture. Eden's KCS weakened her KCT. However, when Brit responded to Mark, she attempted to help him recognize that the common factors of the two numbers will somehow affect their LCM, which was Brit's understanding of the mathematics in the scenario. While it often contributed, participants did not always draw from *all* of their relevant SCK in responding to a student. In some cases, participants explicitly decided that some of their own ways of understanding the context was too complex. In other situations, it was unclear why participants did not use their SCK to further strengthen the hypothetical student's understanding. For example, even though Cara, like Brit, knew that Mark's conjecture only worked for relatively prime numbers, she never suggested that she would help Mark understand that.

KCS and CCK were also influences on participants' KCT. On occasion, a participant reflected on the curricular materials for a given grade level to determine an appropriate response to the hypothetical students in the scenario. For example, in Interview 2, Problem 7, I told participants that Shayna was a 4^{th} grade student. Lucy explicitly reflected on whether or not Shayna would have worked with factor trees to determine whether or not she could us factor trees to help Shayna understand that the number one could not be prime. In many cases, participants' KCS also played a role in targeting their responses to the hypothetical student. If a participant acknowledged that the student in the scenario had a specific misconception, for instance, then the participant typically responded in a way to correct that misconception or help the student correct that misconception. Participants did not always draw on their KCS when responding to a student, though. For example, in Interview 1, Problem 3, when Lucy explained why Mark could believe his conjecture to be true, she claimed that area models play a role in helping students understand that the product of two numbers is a common multiple of both numbers, but that students like Mark may not always recognize that area models can also be used to find the least common multiple. For some reason, Lucy chose not to build on Mark's supposed understanding of area models to help him recognize his misconception about LCM.

Outside of their mathematical knowledge for teaching, there appeared to be two other influences on participants' responses to the hypothetical students in the scenarios: their past experiences with students (usually from tutoring) and their epistemological perspectives. Many of the participants cited their past tutoring experiences as framing how they responded to the students in the scenarios. While their knowledge typically contributed to the content of their response, their past experiences with students informed participants on how to deliver that content. Another more prevalent influence on how participants delivered their responses to the students was their epistemological perspectives, or how they believed that students learn mathematics. For example, some participants took more direct routes to helping the hypothetical students in the scenarios recognize their misconceptions, while other participants created opportunities for the student to recognize their misconceptions on their own. These decisions, according to the participants, were a direct result of their beliefs about student learning. In discussing these beliefs, it became clear that there were many influences on the participants' epistemological perspectives, including their tutoring experiences, their experiences as a learner (reflecting on what best helped them understand), and the education theory of their mathematics education course (for those that had taken it).

Influences on general mathematical pedagogy. The last type of PCK that the interview tasks elicited was GMP. One of the reasons that this type of knowledge is decidedly different from KCT is that the influences on GMP are very different from those on KCT. The "canned" responses that were typical of instances of GMP did not rely on a participant possessing SCK of the given scenario. Instead, it tended to rely on a more general understanding of the learning of mathematics. For instance, if a student struggles using a procedure to solve a problem, suggest that they try using manipulatives. If a student presents a conjecture, have them check that conjecture by trying to find a counterexample. If a student is struggling with the mechanics of a procedure, suggest that they try more examples. All of these responses to students do not require an in depth understanding of the content at hand. They do however require a vague understanding of mathematics, for example, knowing that a conjecture could be determined invalid if a counterexample existed.

Another strong influence on GMP appeared to be participants' epistemological perspectives. In their responses to Interview 2, Problem 9, almost all of the participants suggested that Talisa and Tom explain their methods to each other in order for them to negotiate the correct factorization. When asked why they responded in this way, participants revealed their beliefs about peer learning. As GMP is an underexplored realm of PCK, there may be more influences on it, and further investigation is warranted, but that will be discussed in Chapter V.

Answers to Research Questions 2 and 3

As my study was largely exploratory, it is not possible to answer my research questions in their entirety. However, in this section I summarize my findings thus far pertaining to Research Questions 2 and 3. Recall that these questions are stated as follows:

> Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?
>
> Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

Research Question 2 is two-fold: (a) What is the nature of my participants' potential number theory PCK?, and (b) What opportunities did their number theory course offer in developing their PCK? The data confirmed that preservice elementary teachers can and do possess PCK, albeit in the developing stages. My syntheses of Problem 3 from Interview 1 and Problems 7 and 9 from Interview 2 presented specific examples of participants' PCK. However, in general, I suggest that the model proposed in Figure 23 best represents the nature of participants' potential PCK.

Figure 23 is a proposed model of the various types of knowledge, beliefs, and experiences that appeared to influence participants' number theory PCK as they demonstrated it during the interviews. (I elaborate on this model in Chapter V.) The solid arrows indicate which constructs explicitly affected participants' PCK, as discussed in earlier sections. The dashed arrow connecting SCK to GMP is dotted to suggest the subtle and very general understanding of mathematics necessary to exhibit GMP. The arrows coming from "Experiences with Students" are also dotted because, while this certainly contributed to participants' responses, participants' experiences with students were limited at the time of the study; participants had not yet had any student teaching experiences. The arrows connecting CCK to KCS and KCT are also dotted because not every participant demonstrated CCK; this influence was not as prevalent as others.

My study did not directly address Research Question 2b. There was no guarantee, or expectation, that participants' number theory content course would provide them with opportunities to develop number theory PCK. I proposed Research Question 2b in the case that the number theory course *did* present participants' with an opportunity to develop their PCK. The only answer, albeit insufficient, that my data suggested was that by influencing participants' SCK, the number theory course indirectly influenced the development of participants' number theory PCK. This implies a direction for future research, which I discuss in Chapter V.

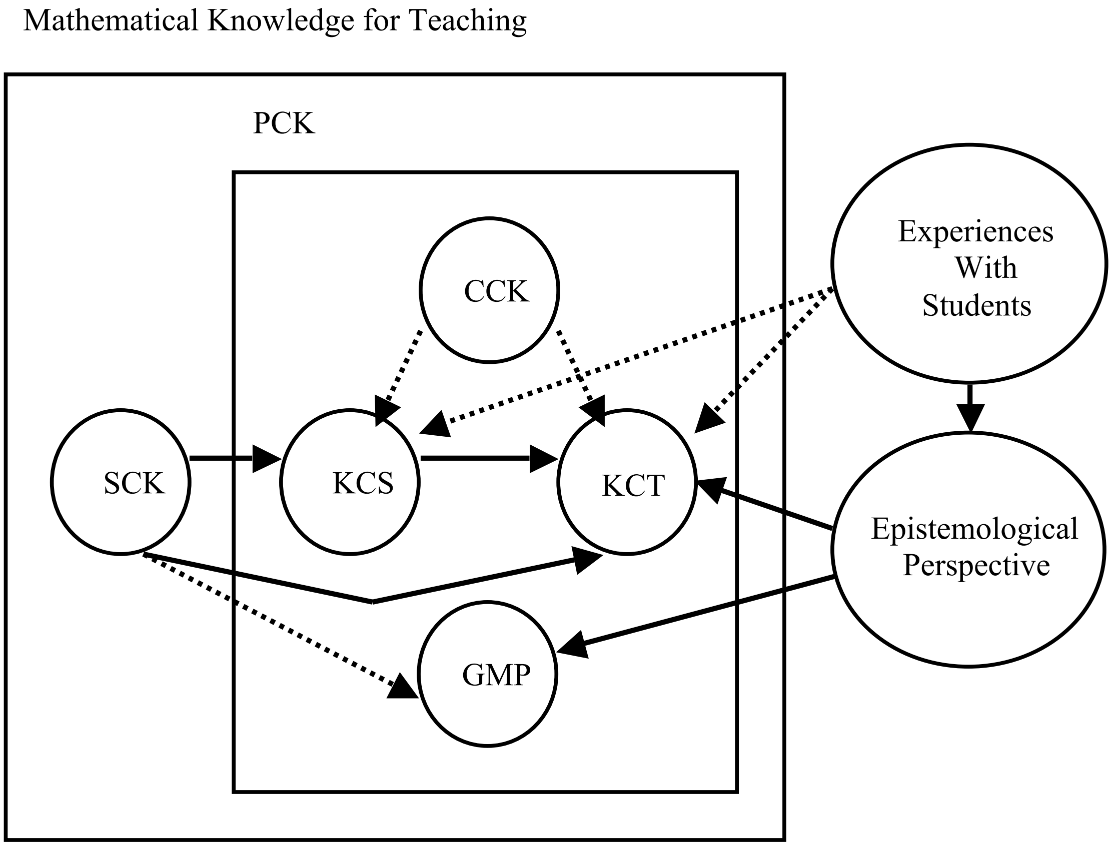


Figure 23. Influences on preservice elementary teachers' number theory pedagogical content knowledge. "CCK" refers to curriculum content knowledge, "GMP" refers to general mathematical pedagogy, "KCS" refers to knowledge of content and students, "KCT" refers to knowledge of content and teaching, "PCK" refers to pedagogical content knowledge, and "SCK" refers to specialized content knowledge.

Research Question 3 essentially asks about the nature of the relationship between participants' SCK and their potential PCK in number theory. As exemplified by my detailed accounts of Problem 3 from Interview 1 and Problems 7 and 9 from Interview 2 in this section, my data suggested that there is a one directional relationship between participants' SCK and their potential PCK. I found that participants' potential KCS drew from their SCK. I also found evidence that participants' potential KCT drew from their SCK in two ways. There were instances where participants' potential KCT drew directly from their SCK, without explicitly addressing hypothetical student reasoning. There were other instances where participants' SCK clearly influenced their potential KCS, which in turn influenced their potential KCT.

The influence that participants' SCK had on their potential PCK, in particular their potential KCT was both positive and negative. I found evidence where strong or robust SCK resulted in relatively sophisticated potential KCT, especially for a preservice teacher. However, there were quite a few instances where incomplete or under-developed SCK weakened participants' potential KCT.

In Chapter V, I begin by summarizing the answers to all three research questions. I then discuss my findings as they relate to the existing literature and my theoretical framework. Next, I discuss the implications of my study on the practice of teaching preservice elementary teachers. Finally, I present limitations and directions for future research.

CHAPTER V

DISCUSSION AND CONCLUSIONS

The purpose of this exploratory case study was to contribute to and broaden the existing research concerning preservice elementary teachers' number theory content knowledge and pedagogical content knowledge (PCK). Recently, research by Campbell et al. (2014) suggests that there exists a significant relationship between upper-elementary and middle level teachers' mathematical content knowledge and student achievement. In particular, teachers' scores on a content knowledge assessment aligned with grades four through eight state standards had a positive effect on their students' mathematics scores on their state assessments. Other studies have shown that teachers' PCK is also important for teaching (e.g., Ball, Thames, & Phelps, 2008; Campbell et al., 2014; Shulman, 1986). Zazkis and colleagues (e.g., Zazkis, 1998a; Zazkis & Campbell, 1996b; Zazkis & Liljedahl, 2004) conducted a majority of the studies pertaining to preservice elementary teachers' understandings of number theory, but they did not explicitly investigate preservice elementary teachers' number theory PCK or the relationship between number theory content knowledge and PCK. Previous studies also do not address preservice elementary teachers' understandings of some topics in number theory, such as greatest common factor.

While little is known about elementary and middle school students' understandings of number theory specifically, research has consistently shown that elementary and middle school students underperform in mathematics (Beaten et al., 1996; Hanushek et al., 2010; Kenney & Silver, 1997; Mullis et al., 1997; National Council of Teachers of Mathematics [NCTM], 2000). In the most recent National Assessment of Educational Progress report (U.S. Department of Education, 2015), only 40% of fourth graders and 33% of eighth graders performed at or above a proficient level in mathematics. Given the link between teacher knowledge and student achievement, the mathematical underachievement of US elementary and middle school students warrants further investigation into the mathematical preparation of future elementary and middle school teachers. In my attempt to do this, I investigated the following research questions:

Q1 What is the nature of mathematics concentration preservice elementary teachers' content knowledge of number theory topics taught at the elementary level?

Q2 What is the nature of mathematics concentration preservice elementary teachers' potential pedagogical content knowledge of number theory topics taught at the elementary level? Also, what opportunities are provided in a number theory course designed for preservice elementary teachers to develop their pedagogical content knowledge?

Q3 What is the nature of the relationship between mathematics concentration preservice elementary teachers' content knowledge and potential pedagogical content knowledge of number theory topics taught at the elementary level?

My study was exploratory in nature, and my research questions were broad enough to capture preservice elementary teachers' understandings of number theory through the emergent perspective (Cobb & Yackel, 1996), which incorporates both psychological and social perspectives. While I cannot answer these questions in their entirety due to the limitations of my study, my findings make substantial contributions to answering the questions and contributing to the existing literature.

In this chapter, I discuss the answers to each of my research questions. Many of my findings concerning Research Question 1 pertain to participants' understandings of greatest common factor (GCF) and least common multiple (LCM). In particular, participants were more comfortable creating LCM story problems than creating GCF story problems, but their understandings of GCF story problems were closely related to the two meanings of division. In contrast to their understandings of story problems, participants were more comfortable with procedures for finding the GCF than with procedures for finding the LCM. In response to Research Questions 2 and 3, evidence suggests that preservice elementary teachers do possess potential number theory PCK, namely potential knowledge of content and students (KCS) and potential knowledge of content and teaching (KCT), and that they are related and influenced by specialized content knowledge (SCK), curricular content knowledge (CCK), experiences working with students, and epistemological perspectives. My data also suggest that preservice elementary teachers possess a type of PCK that is not explicitly represented by Ball and colleagues conceptualization of mathematical PCK (e.g., Ball, Thames, & Phelps, 2008), which I call general mathematical pedagogy (GMP). I briefly discuss how the psychological and social perspectives of my theoretical framework influenced my results as I answer my research questions, but I discuss the role that my theoretical framework played in a separate section of this chapter. Next, I discuss the implications of my study on the mathematics education of preservice elementary teachers. Finally, I present limitations and directions for future research.

Answers to Research Question 1

In this section, I discuss some of my major findings from Sections I and II of Chapter IV, which address Research Question 1. In general, participants demonstrated an understanding of various number theory concepts as they related to validating and creating story problems, solving non-standard problems, and validating student claims. All of this falls in the realm of SCK, defined as "the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Ball, Thames, & Phelps, 2008, p. 377-8). I further discuss how my answers to Research Question 1 corroborate, contradict, and add to the existing literature.

Story Problems

Participants' understandings of validating and creating LCM and GCF story problems varied quite a bit, with participants demonstrating more success with LCM story problems. Participants' collective success can be attributed in part to their number theory course experiences where they modeled LCM in groups and as a class using Cuisenaire rods, pictures, and story problems. My classroom observations did not allow me to isolate individual processes for understanding LCM story problems, but it was clear from the ease of their interview responses that these experiences affected their individual understandings in some way.

In contrast, by all accounts participants had never modeled GCF or validated or created GCF story problems in class prior to their participation in this study. As a result, I was able to observe their processes for understanding GCF story problems in their entirety during the interviews. As evidenced from participants' responses to the GCF story problem tasks, they interpreted GCF in a way that allowed them to produce or describe a visual model of GCF in order to attempt a GCF story problem. Every single participant's model of GCF was similar to how one might model division. I capitalized on this observation in a follow-up task that investigated participants' understandings of division, which I found to be related to participants' understandings of how to model GCF.

Beckmann (2008) describes two "meanings" of division, which can be used to model division with manipulatives, pictures, and story problems: (1) the "How many groups?" meaning of division, similar to the "measurement model" (Ball, 1990); and (2) the "How many in each group?" meaning of division, similar to the "partitive model" (Ball, 1990). I found that either one of these meanings can be adjusted for GCF: (1) the "How many subgroups?" meaning of GCF; and (2) the "How many in each subgroup?" meaning of GCF. While more complex, the grouping structure of a "How many subgroups?" representation of the GCF of two numbers is similar to that of a "How many groups?" representation of division. The same can be said for "How many in each subgroup?" representations of the GCF and "How many in each group?" representations of division.

Three participants demonstrated predominant understandings of one meaning of GCF over the other, and they also demonstrated predominant understandings of the related meaning of division. Cara demonstrated a connected understanding of both meanings of GCF and both meanings of division. However, curiously, two participants demonstrated predominant understandings of GCF that differed from their predominant understandings of division. This proved especially problematic for these two participants, Isla and Lucy, when they were validating story problems.

The research pertaining to how preservice elementary teachers understand story problems and how to create them is sparse. However, researchers have found that preservice elementary teachers' understandings of story problems is related to their understandings of the mathematical concepts represented in the story problems (e.g., Ball, 1990; Crespo, 2003; Goodson-Espy, 2009; Silver, Mamona-Downs, Leung, & Kenney, 1996). They also found that preservice elementary teachers tend to struggle to represent mathematics contextually through story problems. Given story problems can be an important pedagogical tool, especially in elementary and middle schools, investigating how preservice elementary school teachers create and validate story problems can provide valuable insight into mathematics teacher education.

In his study concerning in-service teachers' numeracy task design, Liljedahl (2015) described the increasing push for numeracy in mathematics education. The definition of numeracy that emerged from his study was "the willingness and ability to apply and communicate mathematical understanding and procedures in novel and meaningful problem solving situations" (p. 628). Furthermore, Liljedahl stated that "what is needed is not more abstraction, but more contextualization – and an increased ability to deal with this contextualization" (p. 625).

In general, my data concerning preservice elementary teachers' understandings of GCF and LCM story problems suggests that preservice elementary teachers struggle to create story problems. These struggles included: (1) interpreting the mathematical concepts; (2) contextualizing the mathematical concepts; and (3) posing appropriate questions. These struggles with creating story problems are similar to those that elementary and middle school students have when creating fraction story problems. Barlow and Drake (2008) asked middle school students to write story problems that could be represented by six divided by $\frac{1}{2}$. Some of their participants represented an incorrect expression (such as $6 \div 2$ or $6 \cdot \frac{1}{2}$) due to an insufficient understanding of the measurement or partition models of division. Other issues included not posing a question, posing an inappropriate question, or using an unrealistic scenario. Alexander and Ambrose (2010) made similar insights into students' understandings and misunderstandings about fractions when they tasked upper elementary and middle school students with creating original story problems involving fraction addition.

My participants' processes for developing their understandings of GCF story problems suggest a more general process for how preservice elementary teachers might develop an understanding of GCF story problems which partially addresses these struggles. By understanding the various ways for interpreting GCF (i.e., the "meanings" of GCF), one can better understand how to represent the concept, either through visual or concrete means. Contextualizing this concept then requires some translation, which requires further development and investigation. The language I have used to outline the meanings of GCF may help in posing appropriate questions. For example, if one were to create a "How many in each subgroup?" type of GCF story problem, they would need to represent two quantities, the groups, and describe a scenario that necessitated breaking each of the groups into equal sized subgroups. Then the appropriate question would be phrased "how many in each subgroup?", but in context. I propose and further describe a process by which preservice elementary teachers might develop their understandings of GCF story problems in the implications section of this chapter.

Understanding Greatest Common Factor Versus Understanding Least Common Multiple

As mentioned in the previous section, participants demonstrated significantly less difficulty when modeling LCM visually and with story problems than they did when modeling GCF. I found that the opposite was true when participants attempted to find the LCM and the GCF of two numbers procedurally using prime factorizations. During Interview 2, Problem 10, I presented participants with the prime factorizations of two numbers and I instructed them to find the LCM and the GCF of the two numbers without using a calculator. I also asked participants to explain their reasoning. All six participants found and explained the GCF with ease. Only two participants immediately and accurately found the LCM, but neither completely explained the reasoning for the algorithm. The other four participants had failed attempts at finding the LCM, with two eventually succeeding. Only one participant successfully explained the reasoning behind the procedure.

This result contributes to the body of literature concerning preservice elementary teachers' understandings of number theory, because their understandings of these procedures have yet to be investigated. Prior to this study, no one had investigated preservice elementary teachers' understandings of GCF, but Brown, Thomas, and Tolias (2002) investigated preservice elementary teachers' understandings of various procedures for finding the LCM of two numbers. In contrast to my study, they presented participants with whole numbers rather than prime factorizations and not all participants used prime factorizations to determine the LCM of the two given numbers. Of those participants that elected to use prime factorizations to find the LCM, not all were successful and most of those that were did not adequately explain why the procedure works. My results corroborate those of Brown, Thomas, and Tolias, because my participants also struggled with the procedure and how to reason about it.

It is curious that my participants demonstrated significantly more success when modeling the LCM of two numbers visually and with story problems than finding the LCM using prime factorizations. It is also interesting that the opposite was true of GCF. As mentioned earlier, it is possible that participants' number theory course experiences enabled them to be more successful in modeling the LCM than modeling the GCF of two numbers, because they had done so in class and on assignments. However, participants also found the LCM and the GCF of two numbers using prime factorizations in class multiple times. Yet participants demonstrated significantly more success when finding the GCF than they did when finding the LCM. The reasoning behind these procedures was also discussed in class, and participants only consistently recalled the reasoning for the procedure for finding the GCF.

This suggests that while participants' number theory course experiences may have marginally influenced their responses to these tasks, there may be fundamental differences between these procedures that make finding the LCM more challenging than finding the GCF. From a set theory perspective, the GCF of two numbers is obtained by finding the intersection of their sets of prime factors, whereas the LCM of two numbers is obtained by finding the union of their sets of prime factors. Finding the union of two sets is a more complex process than finding the intersection of two sets. Also, allowing for some duplicate prime factors but not others, especially in the case of finding the LCM of two numbers, further complicates the procedure.

Prime Numbers

Similarly to a study conducted by Zazkis and Liljedahl (2004), I investigated my participants' understandings of prime numbers by asking them to define "prime." Most of my participants defined "prime" using an imprecise version of the elementary school definition, "a number divisible only by one and itself." Two of my participants also defined a prime number as having "two distinct factors." These findings corroborate those of Zazkis and Liljedahl, whose participants primarily defined "prime" in one of these two ways. Unlike Zazkis and Liljedahl, I further investigated participants' understandings of the definition of prime by posing a student scenario that questioned whether one is prime.

I found that participants who defined a prime number as having two distinct factors drew from their definition of prime to discount one as a prime number. Participants who defined a prime number as a number divisible only by one or itself did not use their definition to discount one as prime. Rather, they acknowledged that their definition did not discount one as a prime number. Instead, they considered it "an exception to the rule" and resorted to other reasoning as to why one cannot be prime. A mathematical definition, by nature, should allow one to classify examples and nonexamples. The imprecise nature of some participants' definition of prime proved to be problematic in this respect. Zazkis and Liljedahl claimed that it is important for preservice elementary teachers to know the definition of prime number, but they did not address the importance of the precision of that definition. My results suggest that it is important for preservice elementary teachers to use precision in their definition of prime in order to be able to discount one as prime by definition. Otherwise, reasoning about one as "an exception to the rule" diminishes the role of a mathematical definition. Consider Gwen's statement.

> Gwen: I always had trouble with the whole prime number thing. I think that if I was told that prime factors… er, prime numbers only have one and themselves as a factor, and they're always two different numbers, then that would have helped me conceptualize that a little bit better. It wouldn't really help me understand all of the background stuff, but it would help me say 'OK, it doesn't fit that criteria.'

Further investigation on reasons why one cannot be prime can enhance preservice elementary teachers' understandings of prime numbers. However, without being able to discount one as prime by definition, some of my participants waivered on why one could not be prime. For instance, when I asked Lucy why one is not prime, she said, "I think we went over it in number theory, but I can't remember. I'm not sure, actually." Other participants referenced their number theory course experiences to explain why one cannot be prime. The primary reasoning used in class and on assessments was that if one were prime, the Fundamental Theorem of Arithmetic would be violated. Some participants' individual understandings of this reasoning were that "factor trees would never end." It appeared that participants did not quite grasp the formalism of this concept.

Divisibility by Two

I altered the first part of Interview 2, Problem 10 from existing literature. Like Zazkis and Campbell (1996b), I investigated preservice elementary teachers' understandings of divisibility by presenting participants with a prime factorization and asking if various numbers were divisors. However, my participants and tasks were different from those of Zazkis and Campbell's in possibly important ways. Zazkis and

Campbell's participants were not mathematics concentration students, and they were not enrolled in a 300-level number theory course. Also, while both my task and Zazkis and Campbell's task had prime factorizations that did not contain a "2", Zazkis and Campbell only asked their participants to determine if certain prime numbers, excluding two, were divisors. In contrast, I also asked my participants if two and other composite numbers were divisors.

Zazkis and Campbell's (1996b) participants could typically determine prime divisors if they were part of the prime factorization, but they struggled to discount prime numbers that were not in the prime factorization. Almost all of my participants demonstrated a strong understanding of prime factorization by correctly identifying all divisors of the given prime factorization. This is possibly a result of the fact that my participants were mathematics concentration majors or due to the fact that my participants had seen similar tasks in their number theory course. However, half of my participants seemed confused about using the prime factorization to determine divisibility by two and multiples of two.

While Zazkis and Campbell did not ask their participants to determine if "2" was a divisor of their prime factorization, Zazkis (1998a) conducted a parallel study focusing purely on preservice elementary teachers' understandings of evenness. She concluded that students had difficulty perceiving evenness as equivalent to divisibility by two. As a result, they struggled to identify numbers as even or odd, even in their prime factorization form. For example, rather than recognizing that none of the factors were divisible by two, students used more complicated methods for determining that 3^{99} is an odd number. For example, some participants attempted to multiply 3^{99} out. While none of my participants needed to multiply out the odd factors of the prime factorization to realize that the number was odd, half of them needed to use this kind of reasoning to discount two as a possible divisor. I also observed this kind of reasoning with one of my pilot study participants. It was as if my participants did not trust the Fundamental Theorem of Arithmetic with regards to divisibility by two. This suggests that preservice elementary teachers that have an otherwise robust understanding of how divisibility relates to prime factorizations may understand divisibility by two differently.

Conjectures About Least Common Multiple

I designed some of my interview tasks to specifically build on the existing literature concerning preservice elementary teachers' number theory understandings. I designed additional number theory tasks to investigate areas in which there had been no study, to my knowledge. My student scenario tasks, in particular, are quite different from other interview tasks posed by researchers investigating preservice teachers' understandings of number theory. These tasks required participants to validate student reasoning, which demonstrates SCK in the context of a teaching scenario. In this section, I discuss answers to Research Question 1 that resulted from two such student scenario tasks.

One hypothetical student, Mark, suggested that the product of two natural numbers is equal to their LCM in Problem 3 from Interview 1. All participants recognized that this claim was false, which they each easily confirmed by finding a counterexample. I followed-up by asking participants to describe the cases in which Mark's conjecture is true. Only half of the participants appropriately determined that Mark's conjecture is true for relatively prime numbers. This conjecture was not one that participants had specifically explored in their number theory course, but they had worked extensively with the relationship between the GCF and the LCM, which dictates that the LCM of two numbers is the product of the two numbers divided by the GCF. Participants had also investigated many number theory ideas for which having relatively prime numbers was a necessary condition. Participants' frequent work with relatively prime numbers may have influenced their answers to my follow-up prompt. For instance, when I asked Brit in which cases Mark's conjecture worked, she immediately said that it worked for relatively prime numbers and that "when in doubt, it's always 'prime' or 'relatively prime.'"

In another student scenario, Maria suggested that the difference of two natural numbers is equal to its GCF. As with Mark's conjecture, all participants easily recognized this conjecture to be false, and they were able to confirm this by finding a counterexample. For this task, I followed up by asking participants to determine the relationship between the GCF of two numbers and their difference. In this case, the default "relatively prime" answer would not suffice. Only one participant, Lucy, correctly determined that the difference of two numbers is divisible by their GCF, and only after investigating multiple examples and counterexamples. This relationship is a direct result of a theorem that participants discussed in their number theory course, but none of the participants seemed to recognize the connection.

Specifically, preservice elementary teachers seem to recognize that the product of two natural numbers is not equal to their LCM, but they struggle to identify the cases for which this is true. Preservice elementary teachers also recognize that the difference of two natural numbers is not equal to their GCF, but they struggle to identify the true relationship between the difference and the GCF of two numbers. More generally, it appears that preservice elementary teachers are typically successful investigating number theory conjectures when example-exploration is available to them, but they struggle to generalize their observations, even when the underlying conceptualizations were discussed in class.

Answers to Research Question 2

In this section, I discuss my answers to Research Question 2 and how they relate to the literature. Research Question 2 is two-fold: (a) What is the nature of my participants' potential number theory PCK?, and (b) What opportunities did their number theory course offer in developing their PCK? The data suggested that participants possessed some degree of number theory KCS, defined as the "knowledge of students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004, p. 17). The data also suggested that participants possessed KCT, which entails making decisions that "coordinat[e] between the mathematics at stake and the instructional purposes at play" (Ball, Thames, & Phelps, 2008, p. 401). However, these constructs specifically pertain to in-service teachers' PCK, which is more developed than preservice teachers' PCK. My study contributes to the existing literature by exploring preservice teachers' PCK, which I refer to as "potential PCK." When I refer to preservice elementary teachers' KCS and KCT, I mean "potential KCS" and "potential KCT", respectively.

Influences on Number Theory Pedagogical Content Knowledge

The data confirmed that preservice elementary teachers can and do possess PCK, albeit in the developing stages, which is why I refer to this type of knowledge as “potential PCK.” My data also suggested that participants’ potential PCK, namely potential KCS and potential KCT, drew from many other types of knowledge and experiences, modeled in Figure 24. The interview tasks also revealed a type of knowledge that is certainly a type of mathematical PCK, but does not appear to fit with the definitions of the constructs as defined by Ball and colleagues; a construct I have named GMP. The data suggested that participants’ SCK and epistemological perspectives frequently contributed to their PCK, and that some interaction between the types of PCK existed. Influences and interactions between the different types of knowledge and experiences that were strongly supported by my data are represented in the figure with solid arrows. Other less prevalent influences are represented with dotted arrows. I discuss all influences on participants’ PCK supported by my data in this section.

In most instances of KCS, participants drew on their SCK pertaining to the given student scenarios. Depending on the nature of participants’ SCK, it could strengthen or weaken their KCS. The relationship between KCS and SCK is further discussed in my answers to Research Questions 3. While strong and consistent evidence suggested that participants’ SCK was influenced by their KCS, some of my data implied that participants’ KCS had other influences. One such influence is CCK, another type of PCK. The interview tasks were not designed to elicit it, nor did it arise often during the interviews, but occasionally participants cited specific curricula in their evaluation of student understanding. The last influence observed on participants' number theory KCS that arose from their responses to the interview tasks was participants' experiences working with students. All participants had some degree of experience tutoring or mentoring students, but their experiences varied greatly, and reference to these experiences was not consistent. Occasionally, participants recognized a "common" mistake in the student scenario task, "common" to them because they had experienced it before as learners.

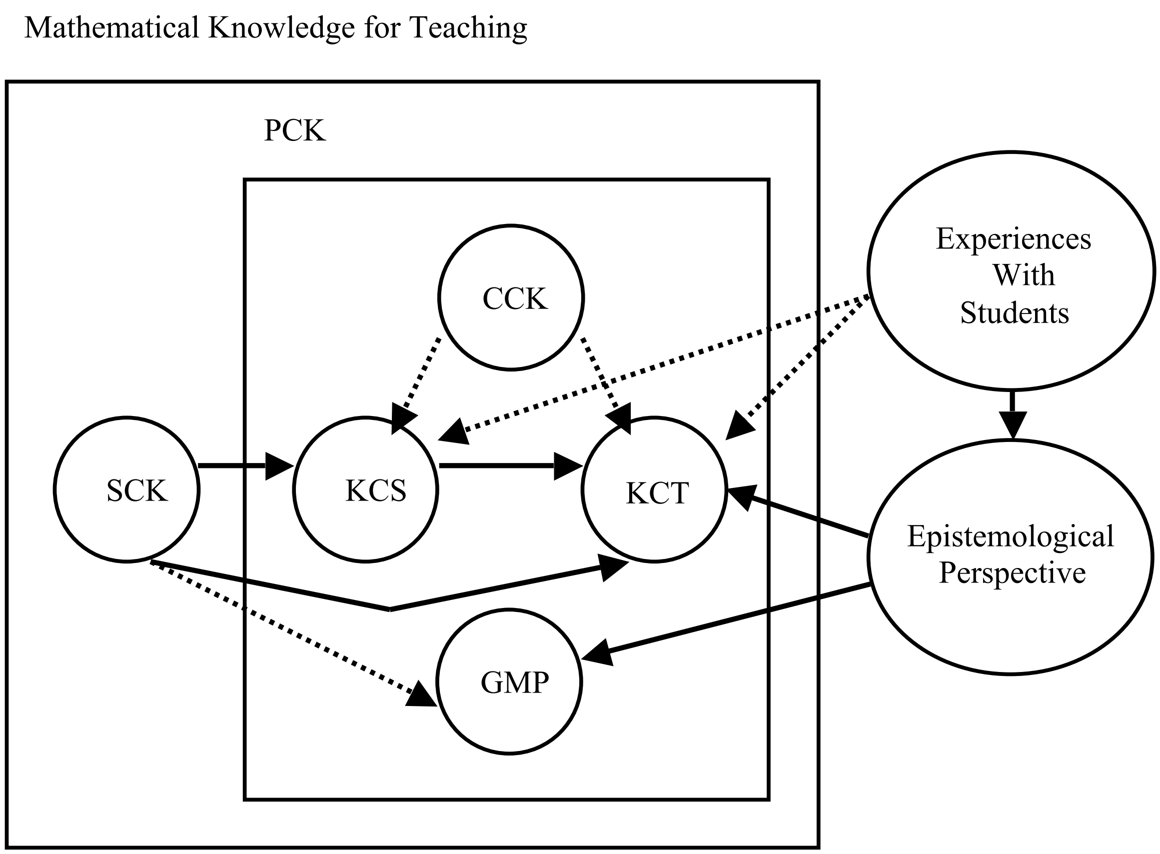


Figure 24. Influences on preservice elementary teachers' number theory pedagogical content knowledge. "CCK" refers to curriculum content knowledge, "GMP" refers to general mathematical pedagogy, "KCS" refers to knowledge of content and students, "KCT" refers to knowledge of content and teaching, "PCK" refers to pedagogical content knowledge, and "SCK" refers to specialized content knowledge.

The interview tasks revealed the influences on KCT to be much more variable. Similarly to its influence on KCS, participants' SCK usually contributed to participants' exhibited KCT. As with the influence of participants' SCK on their KCS, depending on the nature of their SCK, it could strengthen or weaken KCT. I further discuss this relationship in my answers to Research Question 3. KCS and CCK were also appeared to influence participants' KCT. On occasion, a participant reflected on the curricular materials for a given grade level to determine an appropriate response to the hypothetical students in the scenarios. In many cases, participants' KCS also played a role in targeting their responses to the hypothetical student. If a participant acknowledged that the student in the scenario had a specific misconception, for instance, then the participant typically responded in a way to correct that misconception or help the student correct that misconception.

Outside of their mathematical knowledge for teaching, there appeared to be two other influences on participants' responses to the hypothetical students in the scenarios: their past experiences with students (usually from tutoring) and their epistemological perspectives. Many of the participants cited their past tutoring experiences as framing how they responded to the students in the scenarios. While their knowledge typically contributed to the content of their response, their past experiences with students informed participants on how to deliver that content. In their meta-analysis of the literature pertaining to preservice teachers' PCK, Van Driel and Berry (2010) suggested that teaching experiences are paramount for developing PCK. Perhaps each of my participants demonstrated KCS and KCT, in part, due to their experiences working with students. While occasionally working with students in small groups or individually is different than consistently teaching your own class, tutoring is a more feasible authentic teaching experience for preservice elementary teachers.

Another more prevalent influence on how participants responded to the hypothetical students was their epistemological perspectives, or how they believed that students learn mathematics. For example, some participants took more direct routes to helping the hypothetical students in the scenarios recognize their misconceptions, while other participants created opportunities for the student to recognize their misconceptions on their own. These decisions, according to the participants, were direct results of their beliefs about student learning. In discussing these beliefs, it became clear that there were many influences on the participants' epistemological perspectives, including their tutoring experiences, their experiences as a learner (reflecting on what best helped them understand), and the learning theories discussed in their mathematics education course (for those who had taken it).

For most student scenario tasks, if a participant suggested that she would respond to the student in a particular way, I asked her to explain why she responded in that way. This allowed me to gain further insight into participants' KCT. Curiously, participants tended to attribute their responses to their epistemological perspectives more often than their SCK, even when it was evident that their responses to the hypothetical student drew from their specific understandings of the mathematics involved. According to Van Driel and Berry (2010), when an experienced teacher teaches an unfamiliar topic (s)he relies on general pedagogical knowledge, "which can constitute a supporting framework for the development of their PCK" (p. 658). This implies that a novice teacher, with far more limited PCK, might also lean on general pedagogy or learning theories when responding to student conjectures and claims concerning unfamiliar topics. Perhaps there is an underlying reason for why some participants referenced the mathematics involved while others did not, even given similar responses to the hypothetical student. In addition to teaching experience, Van Driel and Berry suggested that self-confidence is another precursor for PCK. Perhaps when preservice elementary teachers are less familiar and less confident teaching a topic, they rely more heavily on general pedagogy or learning theories.

The last type of PCK that my data revealed was a different and unexpected type of PCK, which I called general mathematical pedagogy (GMP). Participants exhibited instances of GMP when they responded to student scenarios in general and somewhat "canned" ways. Each of these responses drew from a general understanding of *mathematics*. Thus, I could not classify them as instances of general pedagogical knowledge, which transcends subject matter. Ball and colleagues' domain mapping of mathematical PCK includes three constructs of PCK: KCS, KCT, and knowledge of curriculum (Hill, Ball, & Schilling, 2008). GMP is most similar to KCT, however KCT requires taking into account the specific mathematics at stake, which instances of GMP did not.

Another reason that GMP is decidedly different from KCT is that the influences on GMP are very different from those on KCT. The "canned" responses that were typical of instances of GMP did not rely on a participant possessing SCK of the given scenario. Instead, it tended to rely on a more general understanding of the learning of mathematics. For instance, if a student struggles using a procedure to solve a problem, suggest that (s)he try using manipulatives. If a student presents a conjecture, have her or him check that conjecture by trying to find a counterexample. If a student struggles with the mechanics of a procedure, suggest that (s)he try more examples. All of these responses to students do not require an in depth understanding of the content at hand. They do however require a vague understanding of mathematics, for example, knowing that a conjecture could be determined invalid if a counterexample existed. One influence on GMP appeared to be participants' epistemological perspectives.

I have discussed the concept of GMP with multiple mathematics education researchers (members of the audience during my RUME 2015 presentation), and they had varying ideas concerning the role GMP plays in preservice teachers' mathematical PCK. Some suggested that GMP lies outside the construct of KCT, while others suggested it is a subconstruct of KCT. If this is the case, then perhaps Ball and colleagues' conceptualization of KCT (e.g., Hill, Ball, and Schilling, 2008) should be more clearly expanded to include such general responses to students. Toney (2015) suggested that perhaps GMP is evidence of "students telling us what they think we want to hear." Drawing from her experiences teaching mathematics education courses, Toney acknowledged that such courses aim to cultivate preservice teachers' epistemological perspectives by discussing constructivist learning theories and the roles they play in the mathematics classroom. Perhaps the "canned" responses that are typical of GMP are manifestations of mathematics education course norms and behaviors. However, this does not explain why participants that had not yet taken the mathematics education course demonstrated GMP.

While my conceptualization of GMP resembles Shulman's (1986) more general definition of PCK, it is unclear how GMP fits into Ball and colleagues conceptualization of PCK (e.g., Ball, Thames, and Phelps, 2008). Evidence of something like GMP suggests that Ball and colleagues' constructs, as they are defined, may insufficiently represent the realm of mathematical PCK. Hauk, Jackson, and Noblet (2010) have also suggested that the MKT model is insufficient. They suggested that underlying all of the constructs of content knowledge and mathematical PCK is another type of knowledge they called discourse knowledge. The researchers defined this as knowledge of the culturally embedded nature of inquiry and forms of discourse in mathematics, both inside and outside the educational setting. This type of knowledge includes knowledge of syntax and symbols, but it also includes knowledge of navigating within the realm of mathematics, for example, how to generate examples using a mathematical definition. Perhaps GMP is related to Hauk, Jackson, and Noblet's construct of discourse knowledge. Regardless, as GMP is an underexplored realm of PCK, further investigation is warranted.

Course Opportunities for Developing Number Theory Pedagogical Content Knowledge

My study did not directly address Research Question 2b. There was no guarantee, or expectation, that participants' number theory content course would provide them with opportunities to develop number theory PCK. I proposed Research Question 2b in the case that the number theory course *did* present participants' with an opportunity to develop their PCK.

Some researchers have suggested that inexperienced teachers, including preservice teachers, "have incomplete and superficial levels of pedagogical content knowledge" (Cochran, King, & DeRuiter, 1993, p. 265). While my participants may not have demonstrated robust PCK, their instances of PCK were not insignificant. Some participants, like Brit and Lucy, demonstrated rather sophisticated PCK considering their inexperience. Overall, it was encouraging to see that preservice elementary teachers can possess pedagogical content knowledge in number theory prior to their student teaching experiences. This suggests that number theory PCK can be cultivated prior to entering the field of teaching. According to Van Driel and Berry (2010),

> Preservice teachers' lack of teaching experience explains why they usually express little to no PCK… However, while teaching experience may promote the development of PCK, the provision of structured opportunities for reflection on the relationship between subject-matter knowledge and classroom practice is also important for facilitating the development of preservice teachers' PCK (p. 658).

Perhaps the problem sets that participants worked on every day in their number theory course constituted structured opportunities for them to reflect on the number theory content. While Dr. S did not discuss elementary school classroom practice, implicit in the learning environment he cultivated was the idea that "you are going to be teachers, so you need to be able to explain your reasoning."

As mentioned in Chapter IV, participants often mentioned their number theory coursework when responding to the mathematics of a task. There is sufficient evidence to suggest that the number theory course helped to develop participants' number theory SCK. As mentioned in the previous section, my data also suggested that by influencing participants' SCK, the number theory course indirectly influenced the development of participants' number theory PCK. Participants explicitly referred to their number theory course experiences in responding to the content of the interview tasks, and my data suggested participants' KCS and KCT drew from their understandings of the content. For instance, consider Isla's statement.

> Isla: I used to be the person that would fight tooth and nail that one was a prime number, until this class. And [the professor] proved me wrong. It's all based off of number theory and talking about it in groups and how we all came up with our own ideas of how one couldn't be prime.

Isla's course experiences, in particular discussing the content in groups, contributed to her individual understanding that one is not a prime number, and she referred to this understanding when responding to the student scenario task.

Answers to Research Question 3

Research Question 3 essentially asks about the nature of the relationship between participants' SCK and their potential PCK in number theory. My data suggested that there is a one directional relationship between participants' SCK and their potential PCK. I found that participants' potential KCS drew from their SCK. I also found evidence that participants' potential KCT drew from their SCK in two ways. There were instances where participants' potential KCT drew directly from their SCK, without explicitly addressing hypothetical student reasoning. There were other instances where participants' SCK clearly influenced their potential KCS, which in turn influenced their potential KCT. Participants' potential PCK, in particular their potential KCT, was either strengthened or weakened by their SCK. I found evidence where strong or robust SCK resulted in relatively sophisticated potential KCT, especially for a preservice teacher. However, there were quite a few instances where incomplete or under-developed SCK weakened participants' potential KCT.

My findings suggest that a strong understanding of number theory is necessary, but not necessarily sufficient, for strong number theory PCK. This is not surprising, but in light of so much inconclusive research attempting to link teachers' content knowledge and teaching, it was encouraging to see that number theory content significantly influences aspects of number theory PCK. This contributes to recent findings by Campbell et al. (2014), who suggested that upper-elementary and middle level teachers' mathematical content knowledge has a positive effect on student achievement. In conjunction with the literature that suggests that PCK is important for teaching (e.g., Ball, Thames, & Phelps, 2008; Campbell et al., 2014; Shulman, 1986), a conclusion can be made that SCK influences PCK, which in turn influences student learning.

Understanding and the Emergent Perspective

My theoretical framework was adjusted from Cobb and Yackel's (1996) emergent perspective (see Figure 25). The researchers' proposed emergent perspective incorporated both the psychological and social perspectives to varying degrees. The individual perspective was the focus of this study, but the social aspects of student learning helped to explain my participants' individual understandings. In other words, while the individual conceptions elicited by my task-based interviews constituted the focus of my data analysis, I also used the classroom mathematical practices elicited by my field-note and document analysis to inform my findings.

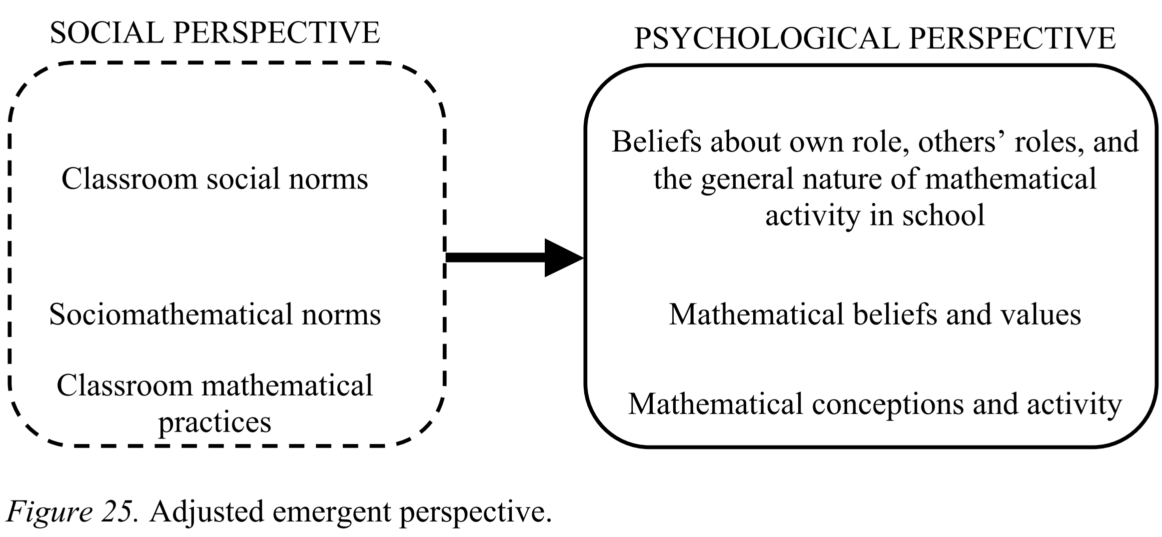


Figure 25. Adjusted emergent perspective.

For example, consider the nature of participants' understandings of Eva's method (Problem 4 from Interview 1). Nearly all of the participants understood that method for finding the GCF to be valid, and most of them explained that it was a geometric representation of the Euclidean Algorithm. Considering Eva's method is not a standard procedure for finding GCF, and without considering the classroom data, participants' validation and connection to the Euclidean Algorithm could be considered impressive. However, an analysis of my field notes and artifacts revealed that my participants had seen Eva's method before. They found the GCF on multiple occasions using the procedure, and their instructor emphasized its connection to the Euclidean Algorithm. Considering those collective classroom experiences partially explained participants' success with this task.

In another example, the microculture of the number theory classroom played a significant roll in participants' success with a task. In Problem 5 from Interview 1, participants were given the GCF and the LCM of one known number and one unknown number. I asked participants to find the unknown number, a task they had not seen in their number theory course. In response, participants demonstrated perseverance when problem solving and the ingenuity to draw from various experiences in their GCF/LCM repertoires. Nearly all participants arrived at a correct solution, but more than half first had failed attempts to solve the problem. Participants' persistence can be explained by the daily culture of the number theory classroom. Students spent a good portion of nearly every class day struggling through unfamiliar problems. Additionally, two participants (one, successfully) attempted to use the "lattice method" to solve the problem. Participants used the lattice method in their number theory class to relate the GCF and

LCM of two numbers. This was one example of an occasion when participants attempted to use their number theory course content to address the content of an interview task.

Implications

Teacher preparation programs are primarily responsible for providing preservice teachers with the mathematical content that they will need to teach effectively (Conference Board of Mathematical Sciences [CBMS], 2012). In light of recent research that suggests SCK has a positive influence on student achievement (Campbell et al., 2014), my results have potentially significant implications in teacher education concerning the development of preservice elementary teachers' development of number theory SCK. Additionally, the evidence suggesting that preservice elementary teachers can develop and possess PCK prior to having any significant teaching experiences has additional implications in preservice teacher education. In this section, I discuss these implications.

Story Problems

Accurately representing mathematical concepts via story problems and examining, or validating, story problems lies within the realm of SCK. Creating and critiquing story problems requires content knowledge specific to teachers, but as many of my participants acknowledged, story problems are also useful in "helping kids understand" concepts. Koedinger and Nathan (2004), who found that students were more successful in solving algebra story problems than the mathematical equations they represented, also support this idea. This suggests that teachers' use of story problems in the classroom may demonstrate pedagogical content knowledge or PCK (Shulman, 1986).

One of the Standards for Mathematical Practice put forth by the Common Core State Standards Initiative (CCSSI, 2011) is modeling real life situations with mathematics. At its most basic level, this implies that students should be able to solve word problems. At a deeper level, the CCSSI also suggests that mathematically proficient students are comfortable applying their mathematical understandings in practical situations. Liljedahl (2015) also suggested that “what is needed is not more abstraction, but more contextualization – and an increased ability to deal with this contextualization” (p. 625). Either as a pedagogical tool in the classroom or as a way for deepening one’s understanding of the mathematical content, my findings concerning GCF and LCM story problems hold implications for future teachers.

Both CBMS (2012) and Liljedahl (2015) support the notion of preparing preservice elementary teachers to understand mathematical concepts in context. My participants’ processes for developing GCF story problems suggest a way to scaffold preservice elementary teachers’ experiences with GCF to better help them understand and create GCF story problems. While there were many similarities, each of my participants demonstrated a unique understanding of GCF story problems. The analysis of these understandings culminates with the proposed model (see Figure 26) for preservice elementary teachers’ process for developing a robust understanding of GCF story problems. I describe this process as a set of stages, and I refer to specific examples from the data to justify the process. This process might be most appropriate for preservice elementary teachers, but in-service elementary teachers might also benefit from this experience as part of their professional development and in their classrooms.

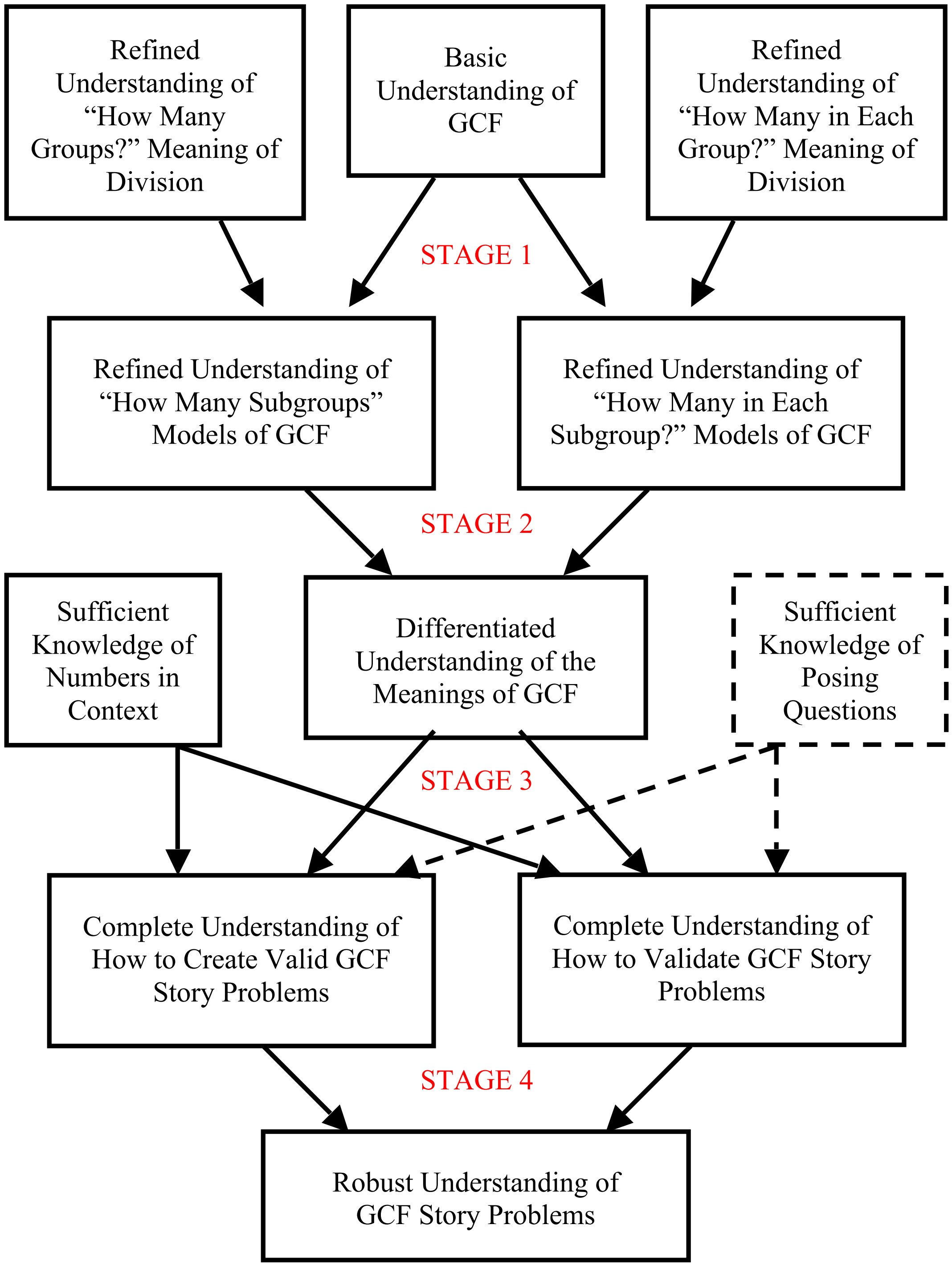


Figure 26. Preservice elementary teachers' process for developing a robust understanding of greatest common factor story problems. "GCF" refers to greatest common factor.

As mentioned in my answer to Research Question 1, the grouping structure of a "How many subgroups?" representation of the GCF of two numbers is similar to that of a

“How many groups?” representation of division. The same can be said for “How many in each subgroup?” and “How many in each group?” representations of GCF and division, respectively. Due to the similarities in their representations, an understanding of how to represent the GCF of two numbers is connected to an understanding of the two meanings of division. However, as evidenced by Isla and Lucy specifically, this connection is not always explicit, which can hinder preservice elementary teachers’ understandings of GCF. Stage 1 in the model highlights the importance of making explicit connections between the meanings of division and the basic definition of the GCF, which may help students develop a robust understanding of the different representations of GCF and their structures. Participants who used their representations to *find* the GCF of two numbers exhibited a better understanding of the representation than the participants who used a numerical method to find the GCF and then grouped objects to represent that value. This distinction proved to be important when participants created story problems, thus using their various representations to find GCF should be emphasized in scaffolding preservice elementary teachers’ understandings of GCF representations.

Most participants did not represent GCF using the two different meanings, and thus did not reconcile the differences and similarities between the two before attempting to create or validate a GCF story problem. As a result, these participants neglected necessary conditions of the GCF from their own story problems, struggled to validate story problems dissimilar from their own, and struggled to identify students’ mistakes in creating GCF story problems. This suggests that comparing and contrasting the two types of GCF representations, Stage 2, may facilitate keeping track of the various minutiae involved with GCF story problems.

Even though Cara demonstrated a relatively differentiated understanding of GCF, having successfully created and compared both types of representations, she had limited success in validating GCF story problems because she struggled to understand the GCF structure in context. Furthermore, Eden, Gwen, Isla, and Lucy all struggled, to varying degrees, contextualizing their GCF representations. This suggests that a differentiated understanding alone is insufficient, and that perhaps participants require some understanding of numbers in context to create and validate GCF story problems. This understanding, and its interaction with one's understanding of the meaning of GCF, is two-fold: (1) it can aid in contextualizing mathematical structures, as with creating story problems, and (2) it can aid in de-contextualizing or extracting mathematical structures from context, which may be helpful in validating story problems. Schwalbach and Dosemagen's (2000) study suggested that contextualizing mathematical concepts can deepen one's understanding of the concepts.

Half of the participants also posed vague questions in their story problems or incorrectly validated the questions posed in given story problems. It is unclear if this is due to a weak understanding of the meanings of GCF, or if preservice elementary teachers' understandings of GCF story problems might benefit from a general understanding of how to pose questions. Regardless, in Stage 3 of the model, it is important that preservice elementary teachers negotiate their understandings of GCF with their understandings of how numbers behave in context to gain more complete understandings of how to create and validate GCF story problems.

While participants simultaneously developed understandings pertaining to validating and creating GCF story problems in Stage 3, I propose that it is not until they have successfully done both of these things and reconciled the two types of experiences, Stage 4, that they will have a robust understanding of GCF story problems. It is even possible that an interplay between the two ideas is necessary before either one is robust. I also propose that this model may be extended to developing an understanding of story problems *related* to GCF, for example, problems that require a student to determine the GCF as a middle step to solving the problem.

Discussing GCF story problems and identifying which meaning they draw from can help to transition from modeling GCF visually to modeling GCF with a story problem. However, once one is exposed to a valid story problem, there is the temptation to copy the context and change the numbers in order to produce a "new" story problem. By attempting to create a GCF story problem prior to validating GCF story problems, as my participants did, it eliminates this temptation and forces the preservice elementary teacher to draw on their understandings of GCF in order to create the story problem.

I attribute participants' number theory course experiences with their success and ease in modeling LCM, and I do not have the same degree of evidence to suggest that participants developed their understandings of LCM story problems in much the same way they did for GCF story problems. However, my observations suggest a process for understanding LCM story problems that is similar to understanding GCF story problems (refer to Figure 27). Another study could confirm or inform this process.

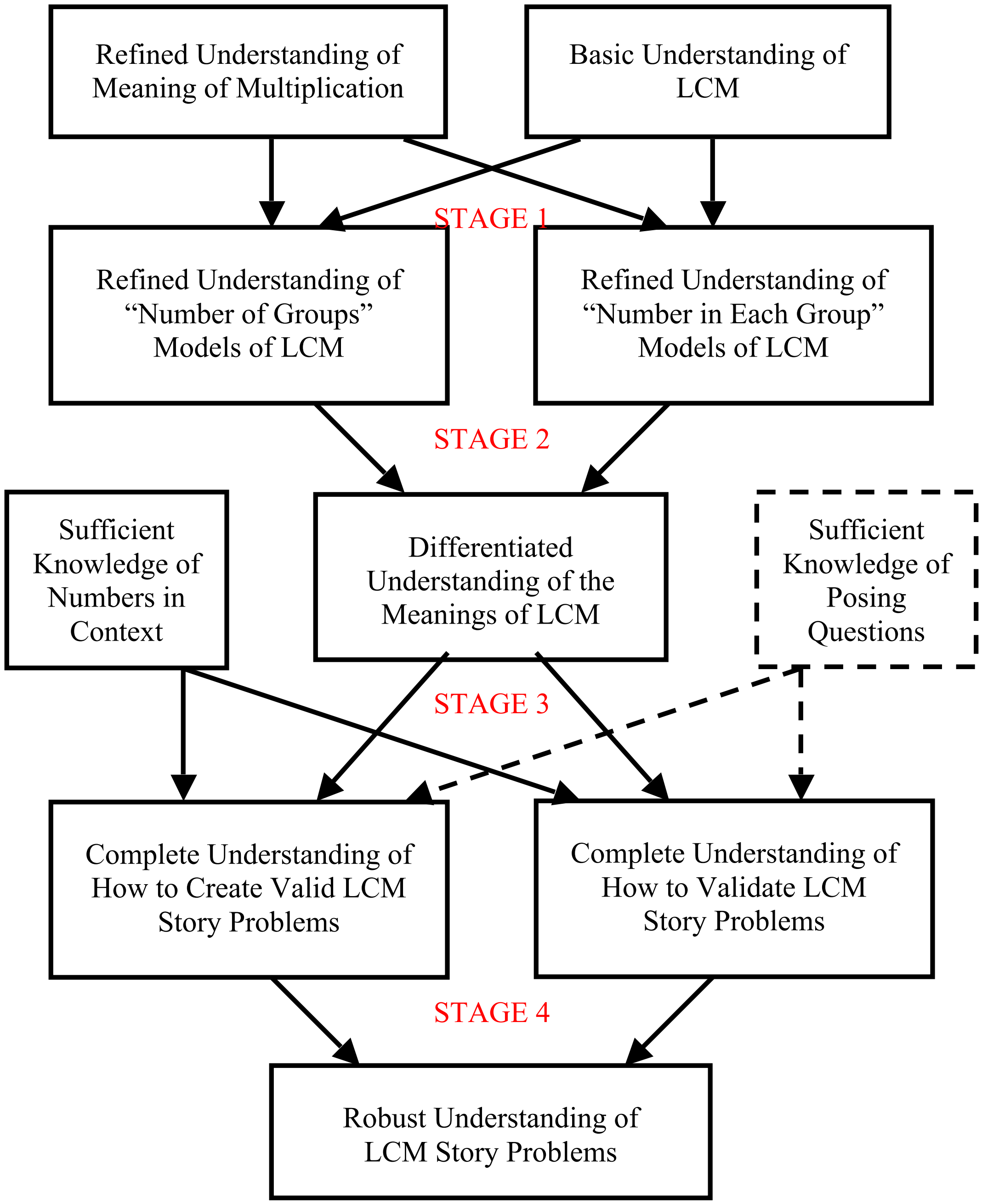


Figure 27. Preservice elementary teachers' process for developing a robust understanding of least common multiple story problems. "LCM" refers to least common multiple.

Number Theory Content

The nature of my participants' understandings of number theory content holds certain implications for practice concerning the mathematics education of future elementary teachers. For instance, participants' struggles in modeling the GCF of two numbers and using prime factorizations to find the LCM of two numbers suggests that more time should be spent or more emphasis should be made on developing preservice elementary teachers' understandings of these ideas in their number theory course. Similarly, my participants' distrust of the Fundamental Theorem of Arithmetic as it relates to divisibility by two warrants a discussion to reinforce the idea that in order for a number to be even it *must* have a two in its prime factorization.

All of my participants demonstrated success and understanding in determining the validity of number theory related claims and conjectures. They recognized that the product of two natural numbers is not equal to their LCM, the difference of two numbers is not equal to their GCF, and one is not prime. However, some participants struggled to explain why one cannot be prime, to identify the types of numbers whose LCM is their product, or to identify the relationship between the difference of two numbers and their GCF. This suggests that participants' understandings of these concepts may be improved upon.

Participants were exposed to the mathematics necessary to address all of these tasks in their number theory course, which suggests that perhaps participants struggled in the interview because this content presented in a different context: in the form of a validation task. According to CBMS (2012), "prospective teachers should examine the mathematics they will teach in depth, from a teachers' perspective" (p. 17). Preservice elementary teachers would benefit from learning number theory concepts in the context of validation tasks. According to Van Driel and Berry (2010), "in [teacher education] programs that were not deliberately structured to help promote integration of different types of knowledge, preservice teachers appeared to retain separate views of subject matter and pedagogy as opposed to the integrated knowledge base advocated by PCK" (p. 659). Validation tasks can not only situate the mathematics preservice teachers are learning in the context of teaching, but they can be used to segue into developing their PCK.

Number Theory Pedagogical Content Knowledge

My data suggest that preservice elementary teachers may have some opportunity to develop number theory PCK during their coursework experiences, but the data also suggest that the SCK they develop during their courses can play a significant role in informing that PCK. Van Driel and Berry (2010) suggested that, in general, "PCK can be promoted by addressing both preservice teachers' subject matter knowledge and their educational beliefs, in combination with providing them with opportunities to gain teaching experience, and in particular, to reflect on these experiences" (p. 659). Van Driel and Berry acknowledged, however, that there is little research to inform teacher education programs how to develop preservice teachers' PCK of specific topics. Traditionally, developing preservice elementary teachers' mathematical PCK might not be a goal outside of their mathematics education methods course. Even then, there may be more of an emphasis on lesson planning than on interpreting and reacting to student conceptions about mathematics. Instead, perhaps the best place to present preservice teachers with the opportunity to develop content specific PCK is in their content coursework.

While not all preservice elementary teachers are required to take a number theory course, the vast majority of them are required to take a mathematics course with a number theory unit, similar to the number and operations course that half of my participants took. During that time, preservice elementary teachers should be exposed to student scenarios that not only help them reflect critically on their number theory understandings (as suggested in a previous section), but these scenarios should help prepare them for similar interactions with students. By posing questions like "what concepts does the student understand?" and "what misconceptions might the student be exhibiting?", one can elicit KCS. By asking participants to respond to the hypothetical students in the scenarios, teacher educators can elicit potential KCT. This might be a more time-efficient alternative to something like "lesson play" (Zazkis, Liljedahl, & Sinclair, 2009), whose format requires enough time to allow for multiple revisions.

So that preservice elementary teachers may learn to consciously use their SCK to explain and identify student reasoning and then respond to students in ways that build on students' understanding of the material, instructors should scaffold validation tasks with additional prompts that draw attention to the content. For example, consider the Problem 3 from Interview 1, where Mark suggests that the product of two natural numbers is equal to their LCM. First, an instructor might ask preservice elementary teachers to answer the questions and discuss, "What mathematical content is at play in this scenario? Which concepts might a teacher need to know in order to teach this content?" If preservice elementary teachers address these prompts at a surface level, merely pointing out that "LCM" and "multiplication" are the pertinent concepts, the instructor should ask additional probing questions, such as "Which number theory concepts, if any, relate the LCM of two numbers and their product?"

By drawing attention to preservice elementary teachers' SCK prior to eliciting KCS and KCT, they may be more likely to draw from it. After posing KCS and KCT prompts, (e.g., "What does the student understand about these concepts?" and "How might you respond to the student to help him recognize his misconception(s) and build on his understanding?"), instructors should follow up with additional prompts relating back the content. For example, "How did you use your understanding of the content to respond to the student?" As my data suggested, preservice elementary teachers' SCK can strengthen or weaken their PCK. These prompts can also serve as additional opportunities to ensure that preservice elementary teachers do not have misconceptions about the content themselves.

Because of how heavily they rely on an understanding of specific mathematical content, validation tasks might best be used in content courses for preservice elementary teachers. However, if a mathematics education course covers specific mathematical content, a validation task best exemplify how to use that content in the classroom. Mathematics education methods courses might also afford preservice elementary teachers with more authentic validation experiences; many mathematics education methods courses require practicum experiences, where preservice teachers observe or interact with students and in-service teachers. Mathematics educators can instruct preservice teachers to write down their observations about student conjectures and claims, as well as the teacher's response. These authentic experiences could serve as topics of discussion, during which the mathematics educator could pose similar prompts to those mentioned above.

It may be more appropriate to encourage the development of GMP in a mathematics education course than in a mathematics content course, because of its general nature. Mathematics educators can introduce and discuss "canned" responses to an array of student scenarios without needing to discuss the content in much depth. However, the nature of GMP is still underexplored and requires additional investigation.

Limitations and Future Research

In this section, I discuss some of the limitations of my study. Limitations primarily stemmed from my methodology, but others arose over the course of the study. Inspired by the limitations, I also make suggestions for future research.

Participants

The participants I chose to investigate present a limitation of my study in multiple respects. My participants were preservice elementary teachers seeking a mathematics concentration, as opposed to preservice elementary teachers seeking a different concentration, preservice secondary mathematics teachers, or in-service teachers at any level. By excluding these other types of participants, I did not have a way to make an explicit comparison between my participants' understandings of number theory and the understanding of other preservice and in-service teachers. Thus, future studies may want to investigate the nature of these teachers' understanding of number theory content and PCK. In particular, an investigation into in-service elementary teachers' number theory content and PCK understandings would allow for a comparison between the nature of preservice elementary teachers' potential number theory PCK and in-service elementary teachers' number theory PCK.

An additional limitation arose when certain number theory students declined participation. When I solicited the participation of the number theory students, I approached pairs of students that worked together regularly during in-class groupwork. As discussed in Chapter III, this might have added another dimension to my analysis, one that drew heavily from the social lens of my theoretical perspective. However, due to availability and scheduling conflicts with select number theory students, I was unable to solicit the participation of three pairs of students.

While soliciting participation from pairs of students was not part of my original research design, my participant selection limited the use of my theoretical perspective. In a future study, I could solicit participation from students who worked together in class. As stated by the College Board Mathematical Academic Advisory Committee (2007), "learning occurs through social interactions among learners and teachers" (p. 6). A future study could focus more heavily on group interactions and the effects of these interactions on individual responses to the interview tasks.

Interview Tasks

Another limitation of my study arose from the interview tasks. Number theory content, even that which is most relevant to elementary school teachers, is fairly broad. I limited my investigation of preservice elementary teachers' understandings of number theory by limiting the scope of the content of my tasks. Most of my tasks pertained to GCF and LCM, some pertained to prime numbers, and even fewer pertained to divisibility and factoring. Future studies may build on my findings by investigating preservice elementary teachers' understandings of other topics in number theory.

The nature of a one on one task-based interview is itself a limitation, especially concerning the student scenario tasks. On these tasks, I asked participants to identify student conceptions and misconceptions and to respond to the student in a way that would help him or her better understand the concept at hand. Participants' responses were one-sided. They could not converse with the hypothetical student to ascertain more information about the students' understandings. Participants also could not adjust their instructional responses according to the student's reaction. In short, my interview tasks did not allow for the conversation that usually takes place in a classroom when a teacher is validating and responding to a student claim or conjecture. Thus, future studies may want to examine preservice teachers' potential PCK in more authentic settings.

I developed a model for how preservice elementary teachers' various experiences and types of knowledge influence their number theory PCK (see Figure 24) from participants' responses to those one-sided interview tasks. This was merely a preliminary model and it requires further investigation to confirm relationships and influences and determine additional influences. One could also investigate any interplay between the influences I have already identified. Investigating preservice elementary teachers' potential PCK in a more authentic setting may allow for this. For instance, I could observe preservice elementary teachers tutoring middle school students or peers, and I could conduct a follow-up interview to investigate their validation of student thinking and their reasoning for responding to the student in that way.

Additionally, while evidence for "canned" mathematical pedagogical knowledge exists (GMP), it was sparse. I did not design my tasks to elicit GMP, but further investigation is warranted. There appears to be a partition between the instances of KCT versus GMP. In some ways, GMP had more in common with general pedagogical knowledge than with KCT. Participants' beliefs on how students best learn was most influential on their instances of pedagogical knowledge and participants' beliefs on how students best learn *math* were most influential on their instances of GMP. In other words, participants' epistemological perspectives seemed to affect their pedagogical knowledge and GMP. In contrast, participants' specialized content knowledge and KCS seemed to be the strongest influences on KCT. The existence of such a partition in knowledge concerning responding to students suggests that teacher educators may need different strategies in aiding the development these types of knowledge.

Developing Number Theory Pedagogical Content Knowledge

When I designed this study, it was unclear whether or not my data would address part b of Research Question 2. An indirect answer to the question emerged from the data: participants' number theory course experiences contributed to their SCK, and my data suggested that participants' KCS and KCT drew on their SCK. Without having any expectations on how the number theory course might affect participants' PCK, I did not explicitly attempt to capture this influence, which is a limitation of my study. A future study is warranted to further investigate how precisely the number theory course provides preservice elementary teachers opportunities to develop their number theory PCK. Furthermore, one could account for the implications of my study and incorporate validating and responding to student scenarios into the number theory coursework. I suspect that this would provide a more direct opportunity for preservice elementary teachers to develop their number theory PCK.

Transfer of Learning

I frequently observed instances in which participants referred to the content from their number theory course when addressing the interview tasks. However, I also observed many instances where participants could have drawn from the course content and did not. This study was not a transfer study; my theoretical framework and task design could not account for why participants attended to certain content and not others. In the future someone could explicitly investigate preservice teachers' number theory content and PCK connection under a transfer framework. For instance, a new study could investigate preservice elementary teachers' understandings of number theory under Lobato, Hohensee, and Rhodehamel's (2013) mathematical noticing framework. Lobato and colleagues' framework "treats noticing as a complex phenomenon" (p. 809), an aspect of reasoning. It can account for which information the preservice elementary teachers attend to and what they do not. According to the researchers, "what students notice has significant ramifications for how they reason about [a concept]" (p. 810). This framework would allow for both a micro- and a macro-level analysis of the data.

Story Problems

Participants' in-class experiences modeling LCM with manipulatives and story problems also served as a limitation. To my knowledge, participants had not modeled GCF with manipulatives or story problems prior to their task-based interviews with me. This allowed me to observe how they built their understandings of GCF story problems during the interview, which resulted in the process depicted in Figure 25. I believe that participants' previous experiences modeling LCM kept me from observing their process in the same way that I observed their process for developing an understanding of GCF story problems.

I attribute participants' number theory course experiences with their success and ease in modeling LCM, and I do not have the same degree of evidence to suggest that participants developed their understandings of LCM story problems in much the same way they did for GCF story problems. This suggests an opportunity for future research. In another study, I could conduct the first interview task prior to when participants modeled LCM in class. Alternatively, my observations suggest a process for understanding LCM story problems that is similar to understanding GCF story problems (refer to Figure 27). Another study could confirm or inform this process.

In conclusion, the results of this study inform a particularly sparse area in the literature, suggest a variety of implications for teaching preservice elementary teachers number theory and assisting in the development of their PCK in number theory, and suggest direction for future research regarding preservice elementary teachers' understandings of number theory. According to the CBMS (2012), "prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach" (p. 17). This dissertation informs the development of such a number theory course for future elementary and middle school teachers. Additionally, CBMS suggests, "prospective teachers should examine the mathematics they will teach in depth, from a teachers' perspective" (p. 17). This recommendation stems from the findings that subject matter knowledge is a prerequisite for developing PCK, but that "a strong and well-integrated subject matter knowledge does not guarantee the smooth development of an individual's PCK" (Van Driel & Berry, 2010, p. 658). The results of this study clarify the relationship between subject matter knowledge and PCK, suggest ways in which to provide preservice elementary teachers opportunities to develop PCK in number theory, and inevitably they support the necessity of having a strong understanding of number theory concepts in order to develop strong PCK, which the literature suggests is important for teaching mathematics for understanding.

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APPENDIX A

PILOT STUDY

Pilot Study

According to Merriam (1998), case study researchers get good data by asking good questions, and piloting these questions is crucial to the process. “Not only do you get some practice in interviewing, you also quickly learn which questions are confusing and need rewording, which questions yield useless data, and which questions, suggested by your respondents, you should have thought to include in the first place” (p. 75-76). Thus, to inform and enrich my dissertation study, I conducted a pilot study, focusing on the interview portion of the investigation. I mimicked my proposed dissertation study through the participant selection as well as the data collection and analysis procedures. In the results and implications sections that follow, I state the rationale for any suggested changes to my methodology and discuss some possible implications of my dissertation.

Participants and Setting

I chose a university in the Rocky Mountain region of the United States, henceforth known as Mountain State University, as the site for this study because of convenience and because the university is known within the state for its teacher education programs. I am currently a PhD student at the university, which helped me access professors and their students. As a result, my already-established relationship with the professors made the data collection process easier than if I were not a student at the school.

Mountain State University elementary education majors with a mathematics concentration who recently (since Fall 2010) completed a number theory course designed for preservice elementary teachers, constituted the participants of this study. This shared experience constitutes the “bounded system” required of a case study (Creswell, 2007;

Merriam, 1998). As the literature has already documented much about how preservice elementary teachers *without* a concentration in mathematics understand number theory, I wished to round out the research by soliciting participants *with* a concentration in mathematics. Not only are these participants required to take more mathematics courses, but they are the only preservice elementary teachers required to take a course explicitly pertaining to number theory. This mathematics-intensive experience most likely contributed to participants' rich responses. Each participant was an adult, none of whom were vulnerable. I obtained all participants on a volunteer basis.

Soliciting participation. In Spring 2011, I solicited the participation of students who successfully completed the number theory course in Fall 2010 via email (Appendix E). The students I emailed were recommended to me by their number theory and mathematics methods instructors as students who might be interested in participating and who were skilled at communicating their understandings. I obtained the contact information from those instructors, both of whom are members of my dissertation committee. Based on the email responses, only four students were willing to participate, so they comprised my pilot study participants.

The pseudonyms I assigned to the participants of the pilot study were Amy, Jen, Mia, and Zoe. All four ladies successfully completed the number theory course in Fall 2010. All but Jen were also concurrently enrolled in a two-credit mathematics education course for preservice elementary teachers that same semester. Jen had not yet taken the mathematics education course. She had transferred to Mountain State University from a community college and believed that her prior coursework would replace that course.

However, it did not appear that Jen's prior coursework was at all similar to the mathematics education course at Mountain State University.

Ethics of participation. I informed all participants, verbally and in the consent letters, of their right to decline participation, of the procedures taken to ensure confidentiality, and of the benefits of participation. None of the participants declined participation. To protect their identities, I assigned each of the participants a pseudonym. In the results section, I refer to these participants as Amy, Jen, Mia, and Zoe.

I am storing signed Informed Consent forms in a locked file cabinet and the original audio files from interviews on my personal, password-protected computer. After storing audio files on my computer, I immediately deleted them from the recording device. I assigned all participants with a pseudonym, which I used to label all memos and interview transcriptions. I also used these pseudonyms to refer to participants during the interview. The pseudonyms only indicate the gender of my participants; they do not indicate ethnicity or any other identifiers. The only document linking the participants' names to their respective pseudonyms is stored on my personal, password-protected computer. This document will be destroyed immediately after all data has been recorded and cataloged according to participants' pseudonyms. I am the only one to have access to the original audio files and document linking participants to their pseudonyms. My dissertation committee may be allowed access to memos and interview transcriptions, all of which will refer to participants' pseudonyms rather than their actual names. My research advisor has access to the signed consent forms, because she is required to store them for a period of three years, after which she will destroy them. I will destroy the audio files after 5 years. I may retain de-identified data, including interview transcriptions and notes, indefinitely for future use and publication purposes.

Data Collection Procedures

For my pilot study, after soliciting four volunteers via email, I set up a time to meet with each participant at their convenience. Interviews took place in a neutral environment on campus. When I first met with each participant, I provided them with an Informed Consent to Participate in Research Form (similar to Appendix F) and reviewed it in detail with each of them. Upon the participant's consent and signature, I conducted and audio-recorded one-on-one task-based interviews that lasted 60-90 minutes each, depending on the depth of each participant's responses and the set of interview questions I asked.

I chose to conduct one-on-one, semi-structured interviews for a number of reasons. Conducting one-on-one interviews allowed me to investigate each student's understanding of number theory individually, which coincides with my constructivist perspective. In addition, the semi-structured nature of the interviews allowed me to use various prompts to enrich the participants' responses, meanwhile staying focused on my research purpose. Since many of the interview questions concern mathematical tasks, I provided the participants with scrap paper. As a result, I also had artifacts to analyze. In the data collection of case studies, Patton (1990) calls for multiple forms of data. From the types of interviews I have created, I had memos, interview transcriptions, and artifacts to contribute to my findings. These characteristics of the one-on-one semi-structured interview helped me provide thick, rich description and an illuminated understanding of the experience to my readers.

The tasks were broken into two sets of interviews, one consisted of tasks related to greatest common factor and least common multiple (Appendix I), while the other mostly related to prime number concepts (Appendix I). Both consist of elementary number theory problems, to ascertain the participant's content understanding of number theory, and questions pertaining to hypothetical student scenarios and modeling number theory ideas, to ascertain aspects of the participant's PCK. I also asked participants to reflect on how they responded to student scenarios to better understand the relationship between the participant's content knowledge and PCK, and to ascertain any impact the number theory instructor may have had on the participant's PCK. Throughout the interview, I used probing questions to help participants demonstrate the full range of their understanding. However, I neither confirmed nor denied their responses until after the interview was complete. At the completion of the interview, I answered any and all questions the participants had concerning the tasks and my research. I completed my pilot study interviews by May 2011.

Interview Tasks, Rationale, and Connections to the Literature

As evidenced by my review of several elementary school mathematics curriculums (Bell et al., 2007; Fuson, 2006; Willoughby et al., 2007), K-6 number theory is typically restricted to evens and odds, factors and multiples, primes and composites, greatest common factor and least common multiple, and divisibility. Research investigating preservice elementary teachers' understanding of number theory is limited to prime numbers (Zazkis & Liljedahl, 2004), prime decomposition (Zazkis & Campbell, 1996b), divisibility (Zazkis & Campbell, 1996a), even numbers (Zazkis, 1998), and least common multiple (Brown, Thomas, & Tolias, 2002), but their understanding of topics like greatest common factor has yet to be explored empirically.

Even within the mentioned topics, the research is not exhaustive. Many of these researchers suggest that participants' understanding of other areas of number theory, or even arithmetic, may be connected to their findings. For instance, Brown, Thomas, and Tolias (2002) suggest that preservice elementary teachers first need a flexible understanding of prime factorization and how it relates to factors, multiples, and divisibility to possess a conceptual understanding of LCM. However, these connections, as with other studies, were underexplored.

Perhaps the largest hole in the research related to preservice elementary teachers' understanding of number theory concerns their understanding as it relates to PCK, at least not explicitly. Also, none of the aforementioned studies explore connections between participants' content understanding and their anticipated experience with teaching elementary number theory topics to children. The closest any of the researchers came to putting number theory in the context of an elementary school classroom was Brown, Thomas, and Tolias (2002), when they asked their participants to create a story problem representing LCM.

It was in response to these needs that I, Dr. Soto-Johnson, and Dr. Karakok created the pilot study interview tasks. The tasks cover enough topics to allow me to investigate participants' understanding between number theory concepts, but I have also asked enough questions per topic to hopefully afford me the thick, rich description necessitated by an interpretive case study (Merriam, 1998). Most of the tasks are posed so that they will not only reveal participants' content understanding, but aspects of their

PCK as well. To accommodate for all of these goals, the interview task were split into two question sets: (1) GCF and LCM and (2) Prime numbers.

GCF/LCM question set. As preservice elementary teachers' understanding of GCF is an unexplored area in the literature, it was especially important that some of my tasks relate to this concept. However, as GCF is so related to the concept of LCM, and in an effort to achieve connectedness among concepts, I have also included tasks addressing LCM and the relationship between GCF and LCM. Brown, Thomas, and Tolias (2002) suggested that preservice elementary teachers require a connected understanding of LCM across representations, i.e., the various ways of representing and finding LCM, like creating story problems and using prime factorization to find LCM. As they are so connected, the same can be said for GCF. I posed most of the GCF/LCM tasks in an effort to determine the connections between participants' understandings of these topics. See Appendix I for a detailed account of the interview tasks from the GCF/LCM question set. In the following sections I will discuss each task, its connections to the existing research, and which research questions I hope to address with it.

Problems 1 and 2. Ball (1990) found that preservice elementary teachers struggled to create story problems representing division of fractions. One implicitly uses GCF and LCM while operating on fractions. Thus, I suggest that since GCF and LCM are as complex as division of fractions, and I anticipate preservice elementary teachers may struggle to put these concepts in context as well. In the first two tasks, I begin by asking participants to create story problems representing LCM and GCF, and then I asked them how to model the idea with diagrams or manipulatives. If the participant struggled with these tasks, I planned on asking a few prompting questions, like "what is GCF/LCM?", "How is it used?", and "Can you think of a context where this idea might be useful?"

Finally, at the end of each task, I presented participants with four story problems and asked them to identify which, if any, related to GCF or LCM. Ball (1990) used a similar technique in her study. Some of her participants correctly identified story problems even if they could not create one themselves. I posed this part towards the end of the task so that it was not suggestive when participants wrote their own problems.

The first two tasks will not only help me to establish participants' basic understanding of GCF and LCM, but the majority of the task pertains to their specialized content knowledge (SCK) (Ball, Thames, & Phelps, 2008). Knowing how to create and identify story problems, as well as representing mathematical ideas with diagrams and manipulatives, is knowledge fairly unique to teachers. Thus, these tasks are primarily geared toward addressing my first research question pertaining to the nature of participants' content knowledge. However, the data I obtain from these tasks may also contribute to answering my third research question; In order to investigate connections between participants' number theory PCK and their content knowledge, it is important that I pose questions like tasks one and two.

Problem 3. Another way participants can demonstrate their SCK is by validating students solutions and conjectures (Ball, Thames, & Phelps, 2008). In this third task, I posed a hypothetical student's conjecture that one can find the LCM of two numbers by multiplying them. Since this claim is only true when the two numbers are relatively prime, or their GCF is 1, this task may also reveal some of the connections participants may have made linking the concepts of GCF and LCM, addressing aspect of my first research question.

After participants validated the hypothetical students' claim, I asked them why they felt a student might believe this method is valid, which requires them to access their knowledge of content and students (KCS) (Ball, Thames, & Phelps, 2008). Next, I asked whether or not the conjecture is ever valid, addressing SCK again. Finally, to investigate participants' knowledge of content and teaching (KCT), I asked them how they might respond to the student to help him correct his misconceptions. I also asked how they knew to respond in that way to help determine any influences on participants' development of KCT. As this problem addresses aspects of PCK in addition to content knowledge, responses may contribute to my second research question as well. It is also possible that this task will address the connection between those two types of knowledge, which pertains to research question three.

Problem 4. In the fourth GCF/LCM task, I shared a hypothetical student's strategy for finding GCF using a diagram. I broke the student's work into four stages, presenting participants with an "unpacked" version of the solution strategy. Then, I proceeded to ask participants questions similar to those in the third task. I asked them to validate the student's strategy, to determine if it would always work, and to justify their answers, which requires the use of SCK. As this particular strategy for finding GCF always works, I also asked participants why a student might not be convinced that this is a valid method for finding GCF. Since anticipating student struggles is an aspect of KCS (Hill, Ball, & Schilling, 2008), participants may have used this type of knowledge when responding to this task.

While the hypothetical student's strategy may look relatively unpacked because I presented it in stages, it is actually quite complex. The diagram the student makes is actually a picture proof of the Euclidean algorithm for finding GCF. The algorithm and its proof rely on the fact that the GCF of two numbers, A and B, is equivalent to the GCF of A and $|A-B|$, as well as the GCF of B and $|A-B|$. Through recursion we can eventually find the GCF, and through transitivity we can show that this is also the GCF of the original two numbers. Explaining this idea requires an in-depth understanding of the Euclidean algorithm and its proof, which is part of SCK. I anticipate that this task will demonstrate the effect participants' SCK has on their PCK, as it will be challenging to convince students of a method that they themselves do not understand. Similarly to Problem 3, I anticipate that this task will help to answer all three research questions.

Problem 5. In a change of pace, the fifth task does not require number theory SCK to respond. I gave students the GCF and LCM of two numbers, told them what one of the numbers was, and asked them to determine the other. This task is related to one that Mason (2006) posed, requesting participants to multiply the LCM and GCF of two numbers and compare it to their product. Mason's research suggested that this was an exemplary task that would reveal a great deal of mathematical understanding. Since the pilot study participants already had the opportunity to acknowledge the relationship between these two products in Problem 3, I decided to extend this idea with this fifth task. While there are many solution strategies for this task, the most efficient ones will rely on knowing the product of two numbers is equal to the product of their GCF and LCM. This task will help me to determine whether or not participants know and understand this relationship, and will therefore help to address the first research question.

Problems 6 and 7. GCF and LCM are not only connected to each other, they are connected to other areas in elementary mathematics as well. I designed the last two tasks to investigate participants' understanding of these connections. I began the sixth task by asking participants if they could think of any other areas in mathematics where GCF or LCM might play a role and how. If participants could not think of any, I suggested adding and multiplying fractions and working with ratios in general. I also asked if using GCF or LCM was absolutely necessary in those cases and why or why not. This task will help to address the first research question, but it may also help to address the third when paired with the response to Problem 7.

The last task provided participants the opportunity to use number theory relationships to make a seemingly complicated problem much simpler. I posed a student scenario where a student resorted to using a calculator rather than add two fractions by hand. I then asked participants why they thought the student would have such a reaction or aversion to solving the problem by hand, which required participants to draw from their KCS (Hill, Ball, & Schilling, 2008). Next, I asked participants what questions they would ask the student to help guide him through the problem without a calculator. Not only does this question access participants' KCT, but their guiding questions may help me to investigate their own solution strategies. Since Problem 7 does not require participants to validate a claim, conjecture, or proof, it does not elicit SCK like Problems 3 and 4. Thus, Problem 7 most likely only addresses the third research question.

In Table A1, I summarized how each GCF and LCM task related to the literature and my proposed research questions. As the PCK-oriented tasks were designed to elicit both participants' SCK and PCK, I hope to address all three research questions with those tasks. However, I fully anticipate that other content-oriented tasks will contribute to establishing the connection between participants' content knowledge and PCK in number theory. Also note that none of these tasks addressed the second portion of the second research question regarding opportunities participants had to develop PCK in their number theory class. As any information about this that arose in interviews would have been purely anecdotal, and without classroom observation to triangulate participants' accounts the data is not trustworthy.

Table A1

How the greatest common factor and least common multiple tasks relate to the literature and research questions

Task	Connection to Literature	Connections to Research Questions
1	Ball (1990); Ball, Thames, & Phelps (2008)	Q1
2	Ball (1990); Ball, Thames, & Phelps (2008)	Q1
3	Ball, Thames & Phelps (2008)	Q1, Q2a, Q3
4	Hill, Ball, & Schilling (2008)	Q1, Q2a, Q3
5	Mason (2006)	Q1
6	N/A	Q1
7	Hill, Ball, & Schilling (2008)	Q3

Prime number question set. Zazkis and Liljedahl (2004) asserted that preservice elementary teachers should know a great deal about prime numbers: (1) the definition of a prime number; (2) all natural numbers greater than 1 are either prime or composite; (3) if one can represent a number as a product (where none of the factors are 1), then the number is composite; (4) composite numbers have a unique prime factorization; and (5) there are infinitely many prime numbers. They investigated preservice elementary teachers' understanding of some of these concepts, but not all, and certainly not in depth.

This question set attempts to address all of these ideas in one way or another as well as participants' PCK associated with primality. See Appendix I for a detailed account of the interview tasks from the prime number question set. In the following sections I will discuss each task, its connections to the existing research, and which research questions I hope to address with it.

Problem 1. Like Zazkis and Liljedahl (2004) did in their study of preservice elementary teachers' understanding of primes, I asked participants "What is a prime number?" to determine the rigor of their working definition of the concept. I asked about the importance and role of prime numbers to investigate whether participants had made any readily accessible connections between prime numbers and other areas in mathematics. This task will not only help me establish how each participant's understanding of "prime number" relates to the concept definition (Tall & Vinner, 1981), but it may provide some insight to the span, or breadth, of their concept image. This task primarily addresses the first research question.

Problems 2, 3, and 4. An important step in understanding the concept of prime numbers is being able to identify what is and what is not a prime number (Zazkis & Liljedahl, 2004). Composite numbers and the number "1" are among the numbers that are *not* prime. In my years teaching fundamental mathematics for preservice elementary teachers, I found that many of my students believed 1 to be prime. Those that knew 1 is neither prime nor composite rarely produced a convincing argument explaining why that is. As this evidence is merely anecdotal, the second interview task will allow me to investigate further.

The third interview task was inspired by a common misconception concerning composite numbers. Zazkis and Liljedahl (2004) found that more than one-sixth of their preservice elementary teacher participants (n = 116) incorrectly identified the product of two prime numbers as also being prime. The researchers suggested that this may be an indication of "a profound psychological inclination toward closure, that two of a kind produce a third of the same kind" (p. 175). This misconception most likely perpetuated from the participants' own school mathematics experience. The third interview task will allow me to investigate this idea with my own participants.

Problems 2 and 3 are two-fold; they provide insight to content as well as pedagogical content knowledge. In the context of interpreting student work and responding to the student, I can explore participants' SCK, KCS, and hints of KCT, as with many of the tasks in the GCF/LCM question set. Unpacking students' mathematical reasoning and determining its validity involves SCK, while determining what this infers about a student's understanding makes use of KCS, and knowing how to respond to the student requires KCT (Ball, Thames, & Phelps, 2008). I also asked participants how they knew to respond the way they did. This will help me to determine participants' PCK. As a result, both Problems 2 and 3 address all three research questions.

Along the same lines as the last two tasks, the fourth task asks participants to discuss different ways for determining the primality of large (n = 853) and small (n < 50) numbers. To determine the primality of large numbers, the least sophisticated method is testing n for divisibility by whole numbers less than n. Someone with a more developed understanding of prime factorization may recognize that by testing for divisibility by prime numbers, we eliminate the need to test for divisibility by multiples of prime numbers. Furthermore, knowing that factors come in pairs eliminates the need to test for divisibility by primes less than $\sqrt{n}$. For small numbers, there are other ways of determining primality through the use of manipulatives, which may reveal participants' SCK (Ball, Thames, & Phelps, 2008). This fourth task may reveal any number of connections that participants have made between primality, divisibility, factors, multiples, and manipulatives. I did not provide participants with manipulatives during the pilot study interview, but I will bring them to the dissertation interview to encourage responses about how to use them. Since this task focuses on content, it addresses aspects of the first research question.

Problem 5. In this task, I pose two student strategies for factoring 540. Both students decomposed the number into its prime factors, but insist that their own answer is the correct one. The first prompt asks participants to discuss why the students might be having this conflict. This may provide participants the opportunity to discuss the surface features of the students' strategies, like the ways they have organized their work, or participants could scratch beneath the surface and discuss the uniqueness of prime factorization. Zazkis and Campbell (1996b) found many of their participants struggled with this concept. For instance, when presented with a large number's prime factorization, some of the preservice elementary teachers were not convinced that the number was not divisible by primes that were not in the prime factorization.

I also prompted participants to compare, contrast, and determine the validity of the students' methods and answers. When Deon (2009) posed a similar task in her dissertation interviews, she found her participants more readily accepted the counting problem solution that was more "unpacked", i.e., the method that revealed more of the student's process, even though both students' solutions were correct. Zazkis (1998) and Zazkis and Campbell (1996b) found that many participants preferred to work with whole number representations to determine divisibility rather than the prime decomposition of a number. Through this fifth task, I may find that my participants prefer one of the students' solution strategies over the other, and I hope to gain some insight to that preference through participants' comparison of the two methods.

Since I also asked participants to explain the hypothetical students' dilemma, it is possible that this task may also reveal aspects of participants' KCS and KCT (Ball, Thames, & Phelps, 2008). Since validating student work also incorporates SCK, this task may help to address all three research questions.

Problem 6. Zazkis and Liljedahl (2004) claim it is important for preservice elementary teachers to recognize that there are infinitely many primes, yet their study does not address this. I agree with Zazkis and Liljedahl; it is important that preservice elementary teachers understand this idea, but it is also important that they know why there are infinitely many primes and that they can make sense of this idea to a middle school student. This last part of the task may incorporate KCS (Ball, Thames & Phelps, 2008). I created this interview task in response to these goals. Since the task asks participants to demonstrate content knowledge and PCK in number theory, it is possible that it addresses all three research questions.

Problem 7. Here I pose a number in its prime factorized form, $M = 3^3 \times 5^2 \times 7$, and ask participants if M is divisible by 2, 7, 9, 11, 15 and 63. I adopted this task from Zazkis and Campbell's (1996b) study, whose participants were in their first fundamental mathematics course for elementary teachers. Posing this task to a new population of preservice elementary teachers, those that are attempting a concentration in mathematics and are enrolled in a number theory course, may produce some very different results. For instance, about half of Zazkis and Campbell's participants insisted on calculating *M* and dividing by the divisors in question. Of those participants who attempted to reason through the task using the prime decomposition of *M*, half struggled to reason why *M* was not divisible by 11. I would not anticipate either response from the participants in this study.

In Table A2, I summarized how each prime number task related to the literature and my proposed research questions. As with the GCF and LCM tasks, the PCK-oriented tasks were designed to elicit both participants' SCK and PCK and may address all three research questions. However, I fully anticipate that other content-oriented tasks will contribute to establishing the connection between participants' content knowledge and PCK in number theory. Also note that, as before, none of these tasks addressed the second portion of the second research question regarding opportunities participants had to develop PCK in their number theory class. As any information about this that arose in interviews would have been purely anecdotal, and without classroom observation to triangulate participants' accounts the data is not trustworthy.

Table A2

How the prime number tasks relate to the literature and research questions

Task	Connection to Literature	Connections to Research Questions
1	Zazkis & Liljedahl (2004)	Q1
2	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3
3	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3
4	Ball, Thames & Phelps (2008)	Q1
5	Ball, Thames, & Phelps (2008); Deon (2009); Zazkis & Campbell (1996b)	Q1, Q2a, Q3
6	Ball, Thames, & Phelps (2008); Zazkis & Liljedahl (2004)	Q1, Q2a, Q3
7	Zazkis & Campbell (1996b)	Q3

Follow-up prompts. At the conclusion of both sets of interviews, I asked follow-up questions to enrich the data and inform my perceptions of participants' number theory and PCK understandings. After the GCF/LCM interview tasks, I followed up by asking about participant's experiences creating the GCF and LCM story problems. As Ball (1990) found, creating story problems can be a challenge for preservice elementary teachers. To gain more insight to this experience, I asked participants what prior knowledge a student might need to create GCF and LCM story problems. I also asked why a student might struggle with creating these types of story problems and how they, as the teacher, might alleviate this struggle. Not only will this enrich my observations of the participants' experiences creating GCF and LCM story problems, but it may inform my understanding of their PCK related to this activity.

For both interviews, I ask participants about any influences on their responses to tasks, such as coursework or experiences. I ask them to be specific about which experience(s) influenced which response(s), how, and why. While the findings from this data are inconclusive, it did suggest how participants may have developed their content and pedagogical content understandings. It may have also suggested an answer to the second part of Research Question 2.

With the final follow-up question, I asked participants to reflect on the tasks themselves. For example, if a participant struggled during any of the tasks, I asked questions about the phrasing of the task and if I could have phrased it differently to make it clearer. I also verbally explained what I meant to accomplish from the task so as to stay consistent with the rephrasing of the task. This helped me to revise tasks for when I collect data for my dissertation.

In the following section, I describe how I conducted my analysis of the interview data. I also establish connections to my theoretical framework, and I describe how this analysis will address the research questions.

Data Analysis Procedures and Codes

Although the case study framework does not claim specific data collection or analysis methods, qualitative case studies are meant to be thick with description and heuristic in nature, illuminating the reader's understanding of the phenomenon (Merriam, 1998). While it is customary for case study researchers to gather multiple forms of data to achieve this level of description, it is important to keep in mind that this was a pilot study, and I merely piloted part of the case study that I hope to conduct for my dissertation. Due to the nature of the data I collected for my pilot study, I also only drew from part of my theoretical perspective. The interview responses provide insight to individual constructions of number theory content and PCK, not the understandings of a collective. Thus, it is only appropriate that I analyze the data using a radical constructivist lens. I am, however, assuming that my participants' learning was somehow affected by social interaction. Implicit in this is the assumption that some form of internalization has occurred prior to the interview. I chose my data techniques according to descriptive nature of case studies and the psychological nature of my data.

In order to unpack the participants' understanding of number theory, I started by transcribing the audio recording from each interview. Within the transcription, I occasionally made notes concerning the participant's actions. For example, if the participant was pointing to something or writing on their scratch paper, I would refer to the paper I provided participants and write clarifications in my transcription. In a separate document, I also inserted the images of scratch work, written statements, or drawings that participants produced into the relevant parts of the transcription.

After completing each transcription, I read each interview to obtain a general sense of the data, and then I began coding. As I coded, I kept a record of my process, an audit trail (Merriam, 1998). First, I used open thematic coding (Corbin & Strauss, 2008). In my first round of coding, I categorized statements according to content knowledge and PCK so that I could best address my research questions later in the analysis. Any cognitive connections that participants may have made between number theory content and PCK was more implicit; this arose later in the analysis process.

In my next few rounds of coding, I clustered like statements within the content statements to create subcategories. Similarly, I produced subcategories within the PCK statements. Further analysis revealed subcodes within the subcategories, however, I eventually collapsed these categories due to lack of data. Since only two participants partook in each interview, I had some subcategories with only one subcode (if the two participants had similar responses to an interview task related to that subcategory) or only one statement of evidence for each of two subcodes (if their responses were different). However, I maintained the references to most of these subcodes in my code books (see Appendices J and K) in case they are helpful in my dissertation analysis.

I began the next stage of my analysis by building a model of how participants understand number theory. Once I created a map of the different types of content and PCK understandings, I organized the already-categorized evidence for each idea. I interpreted and extracted my own meaning from the categories and subcategories in order to establish any relationships between them. Not only did I find similarities and connections within each of the two main categories of data, I saw relationships between participants' content understandings and their number theory PCK. Finally, I attempted to reduce the models and suggest a general model with which to answer my research questions.

In the next two sections, I elaborate on how I developed the codes I used to categorize instances within each of the interview set's responses. I also cite sample responses as support for creating the codes and present models of the sets of codes and their hierarchy.

Greatest common factor and least common multiple interview codes. In my first round of coding, I determined whether or not each statement pertained to participants' content knowledge or pedagogical content knowledge to focus my analysis around answering my research questions about the nature of participants' content knowledge and PCK. Throughout the coding process, I also made notes as to which statements presented possible participant misconceptions. Secondly, I categorized each content statement according to topic. Unsurprisingly, the mathematics content of responses could be grouped into three categories: GCF, LCM, and "other". Due to the nature of some tasks, I occasionally coded responses with more than one topic. For instance, Problem 3 presents an opportunity for participants to discuss the relationship between GCF and LCM, or at least to acknowledge that the product of two relatively prime numbers is equal to their LCM. Since two numbers are "relatively prime" when their GCF is 1, a statement acknowledging this relationship would be coded into both GCF and LCM categories. For instances such as this, I also created the subcode "relationship". If participants merely referred to the concept of relatively prime, I coded those statements as "relatively prime", a subcode of "GCF".

Another round of coding content statements revealed even more subcodes. The first major subcode to arise under the "GCF" umbrella code was "personal definition". While none of the tasks explicitly asked participants to define GCF, both spontaneously referred to it as early as Problem 2, when they tried to create their GCF story problem. I coded responses with this code if the participant referred to what GCF "is" or "means". Another subcode that arose was "method for finding", which I used when a participant referred to or used a method for determining the GCF of two numbers. I found that this code overlapped with another subcode, "modeling", which I used when a participant used or referred to a non-numerical method for representing GCF, like through pictures or a story problem. The overlap between these two categories occurred when a participant attempted to use a model to determine the GCF of two numbers.

Another code, "validation", arose from participants' responses to student scenario problems like Problems 3 and 4. The type of knowledge necessary to validate a claim, conjecture, or proof is *content* knowledge, albeit *specialized* content knowledge, as opposed to PCK. In contrast, evaluating student reasoning or understanding and then proposing responses to that student fall under the realm of PCK (Ball, Thames, & Phelps, 2008). In determining the validity of a claim, participants would respond about the accuracy of it.

The major subcodes that arose under the "LCM" umbrella code were quite similar to the subcodes with the "GCF" category: "personal definition", "methods for finding", "modeling", and "validation". Lastly, I coded content statements as to whether or not they pertained to specialized content knowledge (SCK), or content knowledge that may be specific to the teaching profession (Hill, Ball, & Schilling, 2008). Many of the interview tasks presented opportunities for participants to demonstrate this type of content knowledge, like when I asked participants to create story problems representing the LCM and GCF of two numbers in Problems 1 and 2, respectively.

When I returned to the statements I coded as "PCK", I found that my codes clustered around Hill, Ball, and Schilling's (2008) three subconstructs of PCK: knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of curriculum. Due to the nature of the problems in this question set, most PCK statements were coded under KCS. This seems appropriate due to some researchers' concerns that one cannot demonstrate KCT through hypothetical situations or without teaching (e.g., Hill, Ball, & Schilling, 2008). Also, none of the tasks explicitly prompted participants to refer to elementary number theory curriculum, so participant responses that incorporated curricular knowledge were spontaneous and infrequent.

As there were fewer tasks that provoked PCK-related responses from participants than content-related responses, fewer PCK subcodes arose upon further rounds of coding. The subcodes that arose from under the "KCT" umbrella related mostly to ways in which participants suggested they would scaffold students' understanding of GCF and LCM ideas. Participants phrased most of these statements like "I would ask Eva to do [this]" or "I would ask Mark questions about [this]". Due to the nature of Amy and Jen's PCK, their KCT-related responses mostly suggested that they would pose questions or tasks to help the hypothetical students in the scenarios.

Four subcodes arose from under the "KCS" umbrella code: "student solution strategy", "student reasoning", "student challenge", and "prerequisite knowledge". I used the first subcode when participants described a typical student solution strategy. These statements referred to the *actions* a student might take to solve a problem rather than their *reasoning*, which I coded with "student reasoning". I also used this code when a participant referred to *why* a student might think a claim or conjecture is true or false. The "student challenge" code arose from participants' suggestions that a students might struggle with a certain procedure or idea, whether it was due to a misconception, common error, or physical difficulty. For instance, Amy suggested that students may have difficulty using a pictorial model to find LCM, because they may struggle to keep the bars the same length or get them to "line up" at the right length. A fourth subcode arose when participants volunteered information about what students would need to know before successfully being able to complete a procedure or understand a concept. Refer to Figure A1 for the structural model of the GCF and LCM interview codes. To better convey my coding methods, I have attached a portion of Amy's coded interview in Appendix L.

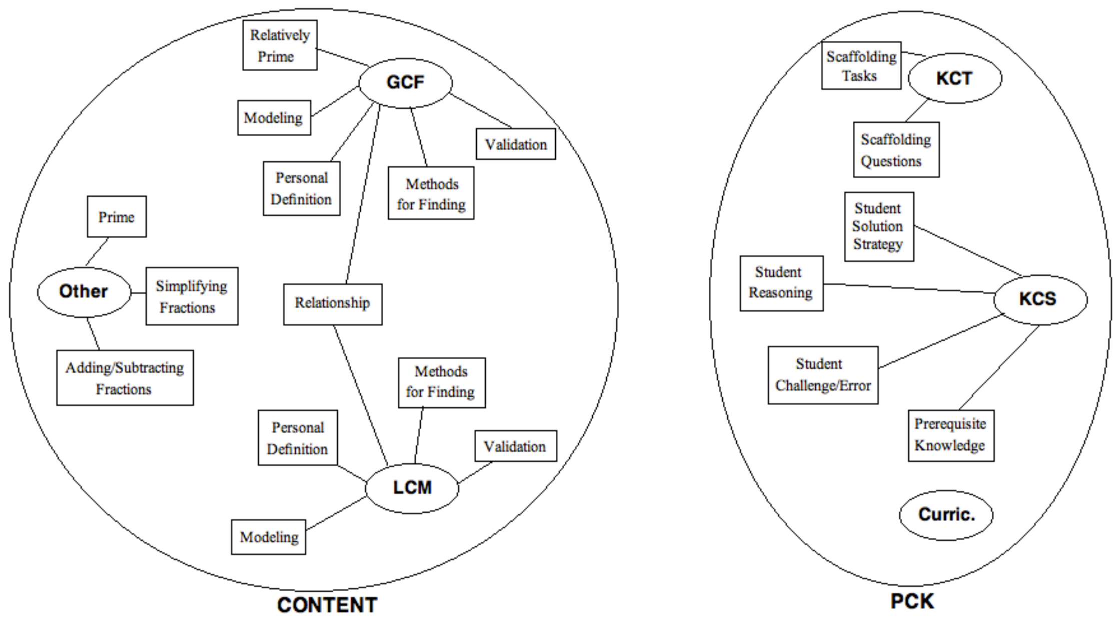


Figure A1. Greatest common factor and least common multiple interview codes and structure. "GCF" refers to greatest common factor, "LCM" refers to least common multiple, "KCT" refers to knowledge of content and teaching, and "KCS" refers to knowledge of content and students.

Prime number interview codes. As with the GCF and LCM interview codes, I first determined whether or not each statement pertained to participants' content knowledge or pedagogical content knowledge. Secondly, I categorized each content statement according to topic. I found that my content codes could be grouped into three areas: prime number, factoring, and divisibility. I then recoded participants' statements accordingly. I made the distinction between factoring and divisibility in that factoring statements specifically refer to the act of decomposing whole numbers into factors and divisibility statements refer to determining whether a whole number is a factor of another whole number. I feel justified in making this distinction and including statements concerning factors within the divisibility category because of the various ways one might say "*N* is divisible by *a*". For instance, "*a* is a factor of *N*" and "*a* divides evenly into *N*" equivalently refer to divisibility. Certainly, an understanding of "factoring" relies on an understanding of "divisibility", as my data shows, so these categories are related, but my distinction allowed me to code most statements as either one concept or another. For example, I coded statements like "six is 2 times 3" as "factoring" because the participant carried out the action of factoring, and I coded statements like "32 is divisible by 2 because it's even" as "divisibility" because the participant identified a factor rather than actually factoring the number. In spite of the interrelated nature of these concepts, I rarely coded a single statement with both topic codes.

The concept of prime numbers is also related to factoring and divisibility, but the distinction is easier to identify. I coded all comments that used the word 'prime' as statements pertaining to prime numbers. Occasionally, I coded some statements as 'prime' if participants did not use the word explicitly, but they were clearly referring to an aspect of prime numbers. For instance, in Problem 4, I asked participants to discuss how they might determine whether or not a number is prime. I coded entire sections of responses as 'prime' for this problem if the participant was actively working towards determining the primality of a number. However, many of these statements also pertained to divisibility and were coded as such. For example, while responding to Problem 4, participants frequently made comments like, "I would check to see if the number is divisible by 7", which clearly makes a reference to divisibility, or indivisibility.

Participants made very few comments relating primes to factoring, but most of them referenced an inability to factor or decompose prime numbers, almost as if participants acknowledged an anti-relationship between the two concepts. I also coded statements as "prime" and "divisibility" when participants discussed the divisibility of a number while referring to its prime factorization.

A third round of coding revealed various subcategories within each of the content areas. The first subcode under the umbrella content code of "prime" to emerge was "personal definition", which was prompted in Problem 1 when I asked participants to define "prime". However, participants frequently made reference to this definition throughout the interview, often using it as justification for their responses. Another code, "validation", arose from participants' responses to student scenario problems about prime numbers like Problems 2 and 3. As with the GCF and LCM question set, the type of knowledge necessary to validate a claim, conjecture, or proof is *content* knowledge, albeit *specialized* content knowledge, as opposed to PCK. In contrast, evaluating student reasoning or understanding and then proposing responses to that student fall under the realm of PCK (Ball, Thames, & Phelps, 2008). In determining the validity of a claim, participants would respond about the accuracy of it.

Two more major subcodes that arose under the "prime" umbrella code were "determining primality" and "cardinality", which were primarily isolated to Problems 4 and 6, respectively. I coded statements as "determining primality" if the participant used or referred to methods for determining whether a number is prime. One participant mistakenly interpreted this objective as *finding* or *generating* prime numbers. In this instance, the participant suggested using the Sieve of Eratosthenes, or something similar, to *find* prime numbers. I coded this type of response as “method for finding” as opposed to “determining primality”. Also within Problem 4, I prompted participants to suggest ways of representing prime numbers through pictures or manipulatives. While responses to this task were not rich, I did code them separately as “modeling”. Finally, in Problem 6, I coded statements as “cardinality” if the participant referred to the size of the set of prime numbers.

The major subcodes that arose under the “factoring” umbrella code were “personal definition”, “methods”, “modeling”, “validation”, and “Fundamental Theorem of Arithmetic”, which I used occasionally when coding statements as pertaining to both prime numbers and factoring. While I did not prompt participants to share their “personal definition” of factoring, they sometimes referred to what factoring “means” to them. I coded factoring statements as “methods” if a participant used or referred to a methods for factoring. This category had the potential to overlap with “modeling” if the participant referred to using a model to factor a number. Finally, the “validation” code arose during student scenario tasks and in similar ways as other validation subcodes.

Many of the “divisibility” subcodes that emerged were similar to earlier codes as well. Since the research suggested that preservice elementary teachers do not necessarily see words like “factor” and “divisor” as being synonymous (Zazkis, 1998b), I created two separate definition-related codes for the terms “factor” and “divisibility”. The “methods” code refers to methods for determining the divisibility of a whole number, which includes using divisibility tests, guess and check methods, or referring to a factored representation. This is different from the “factoring: methods” code, because that refers to the action of factoring, while this code may only refer to the final factored form of the number. Similar to other umbrella topics categories, this one also revealed subcodes for “modeling” and “validation”. A final subcode that arose was “evenness”. As with Zazkis’ (1998a) findings, I suspected that “evenness” might be a separate, albeit related, concept for participants than “divisibility”.

As with my analysis of the GCF and LCM question set, when I returned to the statements I coded as “PCK”, I found that my codes clustered around Hill, Ball, and Schilling’s (2008) three subconstructs of PCK: knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of curriculum. Due to the nature of the problems in the prime number question set, most PCK statements were coded under KCS. This seems appropriate due to some researchers’ concerns that one cannot demonstrate KCT without teaching (e.g., Hill, Ball, & Schilling, 2008). Also, none of the tasks explicitly prompts participants to refer to elementary number theory curriculum, so participant responses that incorporated curricular knowledge was spontaneous and infrequent.

Like the GCF and LCM interview question set, there were fewer tasks that provoked PCK-related responses from participants than content-related responses, which resulted in fewer PCK subcodes arose upon further rounds of coding. Like Amy and Jen, Mia and Zoe suggested that they would pose questions or tasks to the hypothetical students in the scenarios to scaffold their understandings, but they also suggested that they would take a more direct approach by “telling” or “explaining” ideas as well, resulting in a new KCT subcode, “explaining”. The subcodes under the “KCS” umbrella were the same as those that arose while coding the GCF and LCM interview responses. Refer to Figure A2 for the structural model of the prime number interview codes.

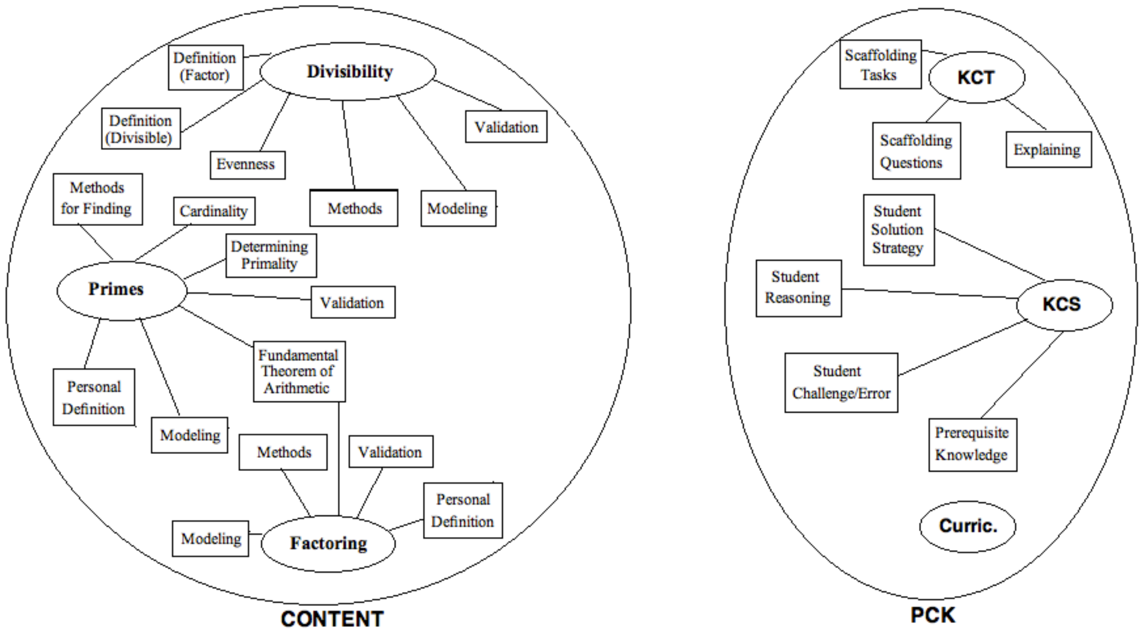


Figure A2. Prime number interview codes and structure. "KCT" refers to knowledge of content and teaching, "KCS" refers to knowledge of content and students, and "PCK" refers to pedagogical content knowledge.

In the following section, I relay the results of my findings. So as to achieve the thick, rich description required of a case study, I begin by summarizing participants' responses to the tasks. This also helps to orient the reader to the depth and variety of evidence associated with each of the response categories and subcategories.

Results

As I hope to use these pilot study interview questions for my dissertation, it is important that I demonstrate that each task reveals valuable information about the participants' understandings of number theory. To achieve this, I have summarized my findings for each problem within each question set. I have also selected quotes from the pilot study interviews to support my findings and to further orient readers to the categories I developed.

While it is important that each task provides me with information concerning the participants' understandings, it is equally important that these tasks help me to develop holistic participant profiles so that I can describe the nature of each participants' content and pedagogical content understandings related to number theory, as well as any connections between the two. Thus, I have also synthesized my findings for each participant.

Greatest common factor and least common multiple question set. From my four volunteers, I randomly chose to interview Amy and Jen using the GCF/LCM question set. Both of these interviews took the full 90 minutes and were rich with information about how the participants thought about GCF, LCM, the relationship between the two, and responding to students' work and conjectures related to the topics. As very little of the existing literature discusses preservice elementary teachers' understanding of these topics, most of my findings are new. The problems related to GCF, in particular, have revealed it to be an extremely sophisticated concept, one that Amy and Jen struggled to fully grasp. Recall that a complete list of the interview tasks from the GCF/LCM question set can be found in Appendix I.

Problem 1. Amy and Jen had very different approaches to writing their story problems. Amy immediately started writing a story problem when prompted, shown in Figure A3. While she chose a scenario that naturally fits an LCM problem, she neglected to establish the starting point for the timeline, i.e., it is unclear when Sarah and Andrew last shopped on the same day, information that would make her question unanswerable.

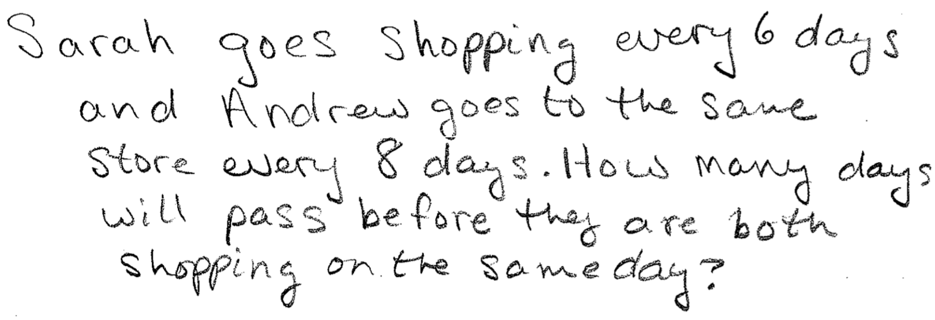


Figure A3. Amy's story problem for the least common multiple of 6 and 8.

In contrast, Jen requested to draw a picture before creating her story problem, see Figure A4, and afterwards admitted that her story problem would not give students enough information to solve the problem; they would need the picture that she drew to go along with it. Even when prompted, Jen was not sure how she could adjust her story problem to make it clearer.

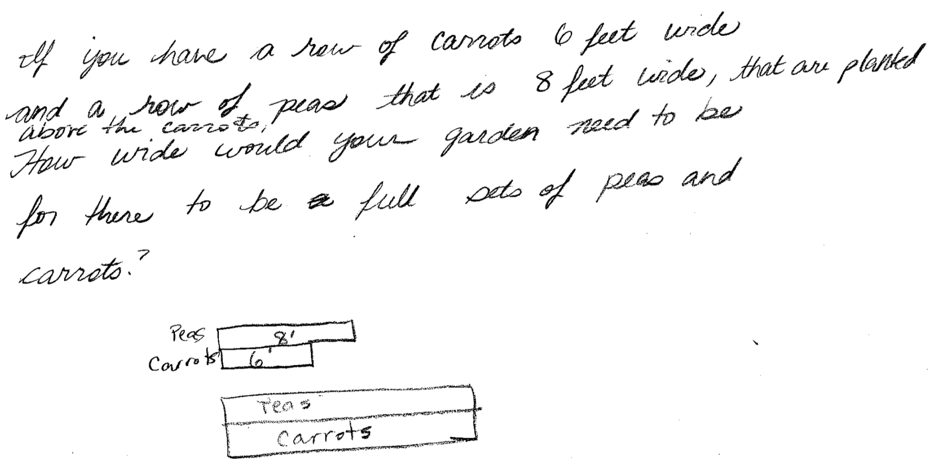


Figure A4. Jen's story problem for the least common multiple of 6 and 8.

Both participants drew similar strip diagrams to represent the LCM of 6 and 8, see Figure A5. Both claimed that when the strips "line up", you are done, and the total length will represent your LCM. When I asked the participants how they could represent this idea using manipulatives, Jen suggested that you could create stacks of 8 linking cubes and stacks of 6 linking cubes and determine when they would "meet up". Amy claimed that she would use Cuisenaire rods: “I would give students which ones represent specific numbers, like 6 or 8, so that they could line them up. Because it’s hard to draw a picture to the scale, but because those [Cuisenaire rods] are already in a set proportion, they could use them that way.” While this task does not require KSC, Amy went beyond the task and justified her choice in manipulatives with her anticipations about students’ potential struggles modeling LCM.

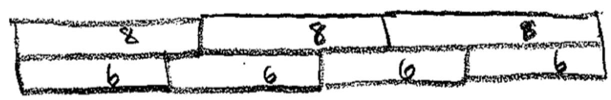


Figure A5. Jen’s strip diagram representing least common multiple of 6 and 8.

When presented with the story problems, both participants tried to identify which mathematical statement each story problem represented to justify whether or not the problem represented the LCM of 6 and 8. Both correctly identified that you need to multiply 6 and 8 to solve story problem (a). For part (b), Jen said, “that would be too many”, indicating that the answer was more than the LCM of 6 and 8. Amy described it as a “pairing problem”. When I asked her to clarify, she said that “most LCM problems are concerned with when two things will happen at the same time… because we want to find a multiple that has both 6 and 8 as a divisor, and I don’t think pairing them meets that criteria.”

Amy quickly identified story problem (c) as a GCF problem, but Jen was less certain that it was not an LCM problem. Initially, she thought it might be, but after carefully reading the problem a few times, and not being able to recognize how 6 and 8 could “meet up”, she decided story problem (c) did not relate to LCM. Both participants recognized that story problem (d) was set up like an LCM problem. However, only Jen saw that 6 and 8 had different units, keeping it from being a valid story problem for the LCM of 6 and 8.

In summary, Amy and Jen both understood the basics of LCM. They knew what a multiple is as well as what it means for two numbers to have common multiples. Amy and Jen also had visual understandings of the concept. They knew that when trains of two lengths first "line up", that total length is the LCM. While both recognize LCM in terms of time, as evidenced by their acknowledgement of story problem (d)'s relationship to LCM, Amy's understanding of LCM appeared to be more connected to the concept of time. Not only did she instinctively develop her story problem around the context of time, but she repeatedly used the phrase "happening at the same time" in reference to LCM. Since Jen required a visual aid to help her develop a story problem, which she admitted would not make sense without the accompanying visual, it's possible that her understanding of LCM is rooted in the visual representation of the concept.

Problem 2. While Problem 1 allowed me to identify a great deal about Amy and Jen's understanding of LCM, Problem 2 revealed much more about GCF, including the complex nature of GCF itself. Amy's interview brought to the forefront this idea that there are two interpretations to GCF, just like there are two interpretations to division (Ball, 1990). Recall that the first interpretation of division is the *measurement model*: when forming groups of a certain size, the number of groups is your quotient. The second is the *partitive model*: when forming a certain number of groups, the number of objects within each equal group represents the quotient. Beckmann (2008) refers to the former as the *"How many groups?"* interpretation and the latter as the *"How many in each* *group?"* interpretation, and suggests a flexible understanding of division requires facility with both meanings.

When Amy tried to model GCF with a diagram, she slightly stumbled through the task, but then suggested that she knew how to model the idea with manipulatives. She explained how she would use linking cubes to express the idea, and it became clear that Amy thought of GCF in two different ways.

> Amy: I would use the linking cubes and start out with 28 and 32 and then get the students to break them down into smaller and smaller sections until they got all their rows to be equal. So they'd first break their 28-rod in half so they'd have 14 and 14, and then they have their 32 [rod] where they would have 16 and 16, and just have them break them down until they get all of their stacks in the 28 and all of their stacks in the 32 to be the same height.
>
> Me: And in what way would the GCF be represented in that?
>
> Amy: The GCF would be represented by… I was thinking the height of the stacks would be the GCF but I could also see it where students could have the number of stacks being their GCF. So students would have to know what GCF means, where it means the number that goes into both… So if they have 4 stacks of 8 for their 32 and they have 4 stacks of 7, I'd be asking them, "what is the number they have in common here? Is it the number of rows or is it how high our rows are?" And they would say "the number of rows." But also, depending on how they manipulated their cubes, they could have 7 stacks that are 4 high and 8 stacks that are also 4 high… we're looking for the number that's in common.

Afterwards Amy tried to model her use of the manipulatives with a diagram. As one can see in Figure A6, when finding the GCF of 28 and 32, Amy tried to break up each length into as many equal sized lengths as possible so that the number of strips was the same. When finding the GCF of 6 and 9, she tried to break up the lengths into as few strips as possible (with as large a length as possible) so that the lengths of all the strips were the same. Amy's models clearly correspond to the two meanings of division. Since factoring involves division, it is no wonder that that there are two different ways to interpret factoring and thus two different ways to interpret GCF.

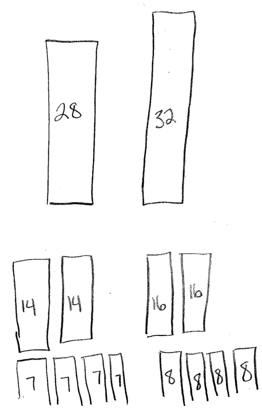


Figure A6. Amy's illustration of the two interpretations of greatest common factor.

After identifying the different meanings of GCF, it became much easier to interpret Amy and Jen's understandings of GCF. Both participants struggled to create an appropriate story problem. Amy created the story problem in Figure A7. Initially, she had a line that said, "We want to make as many bunches as possible" but erased it because she thought the statement was confusing GCF with LCM.

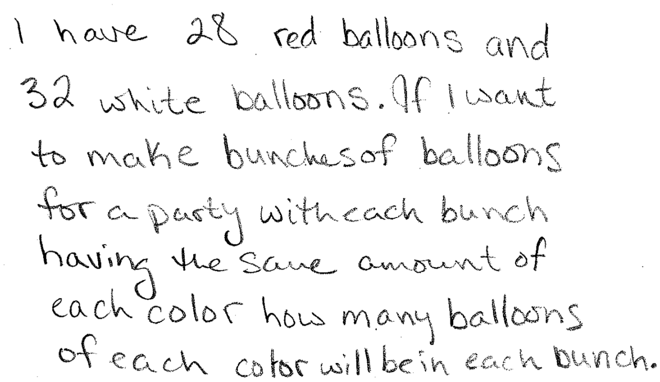


Figure A7. Amy's story problem for the greatest common factor of 28 and 32.

Amy wasn't confusing GCF and LCM with the statement she erased, but she would have been confusing the two interpretations of GCF. She clearly attempted to create a "*How many in each group?*" type of story problem, but by maximizing the number of bunches she would have been conflating this interpretation with the "*How many groups?"* interpretation of GCF. As her story problem stands, though, Amy needed clarification to make the problem answerable. For instance, her wording makes it unclear whether bunches have balloons of both colors or only one. She needs to specify whether she is using all of the balloons or is allowed to have leftovers. Also, in order to achieve the "greatest" common factor, she needs to specify that she wants each bunch to have as many balloons as possible. Amy did clarify verbally that she wants there to be as few bunches as possible to achieve as many balloons as possible in each bunch.

Jen also struggled to create a story problem for GCF of 28 and 32 and, similar to Problem 1, she requested to use other methods to model the concept prior to creating one. Jen insisted on "breaking it down mathematically", then drawing a diagram, and finally creating a story problem, because GCF does not come as easily to her. She also claimed that she tends to understand mathematics better through numerical models than visual models. However, Jen consciously attempted to be more versatile with her representations because of her anticipated career as an elementary educator.

> Jen: This is probably where it's bad that I'm doing elementary [education], because the math makes more sense to me than the picture part of it, so I'm trying to pull that part of it together so I can think both ways, but it's been a process, through school, I'm getting there. I'm getting better.

Jen used factor trees to produce the prime factorizations of 28 and 32, but then she struggled to remember how to use them to find the GCF. At first Jen was using the largest powers in the prime factorizations to help her find GCF, but eventually she realized she was finding LCM instead. After thinking about what the GCF actually represented, "the biggest thing that goes into both", she correctly found it. Abandoning her original strategy of illustrating the concept first, Jen attempted to create a story problem. After some deliberation, Jen came up with the following story problem: “If we have 28 bananas and 32 kiwis and I want to make bags with an equal number of both bananas and kiwi what is the biggest number I can take of both kinds of fruit?”

Jen then proceeded to model the problem with a picture. When she tried to draw a picture of bags with 4 kiwis and 4 bananas in each bag, she realized she would not end up using all the fruit. While it was unclear from the vague wording of her story problem, Jen was conflating the two interpretations of GCF. When creating bags that contain both types of fruit, if you were to create as many bags as possible, the number of bags (i.e., the number of groups) would represent the GCF. However, Jen represented the GCF as the number of objects in each group, replicated for each fruit.

After struggling to model her story problem pictorially, Jen abandoned this tack and decided to change it. After nearly 10 minutes of writing, drawing, and erasing, Jen finally came up with the following story problem (Figure A8). She clarified that each bag only contains one kind of fruit and that she wants to know how many pieces of fruit should go into each bag. With her clarification, Jen’s story problem is an appropriate representation of the “*How many in each group?*” interpretation of GCF.

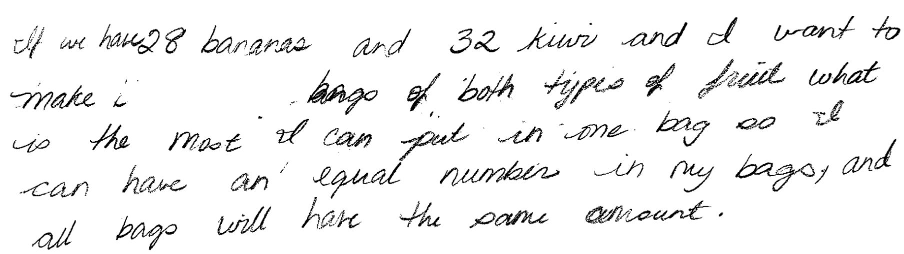

If we have 28 bananas and 32 kiwi and I want to make i bags of both types of fruit what is the most I can put in one bag so I can have an equal number in my bags, and all bags will have the same amount.

Figure A8. Jen’s second story problem for the greatest common factor of 28 and 32.

A little later, Jen came up with a strip diagram in which she broke up 28 and 32 into segment that were 4 units long (see Figure A9). Throughout this task, Jen consistently used the *“How many in each group?”* interpretation of GCF. When I asked

Jen to describe how students could solve the problem using manipulatives, she suggested that students could make a length of 28 linking cubes and a length of 32 linking cubes.

> Jen: You could say, "what would the length need to be for each of these pieces if you broke it into pieces." I don't like how that's worded… I guess if you had a length of 32 and a length of 28 and you lined them up, how much would you be able to put into each one… And I guess when you lined them up you could see that, 28 to 32, there's a difference of 4 so that may be enough of a clue to start with 4 anyway and then they could see that they could break it into equal segments. That's something I'd have to play with more because I'm not very clear on it myself.

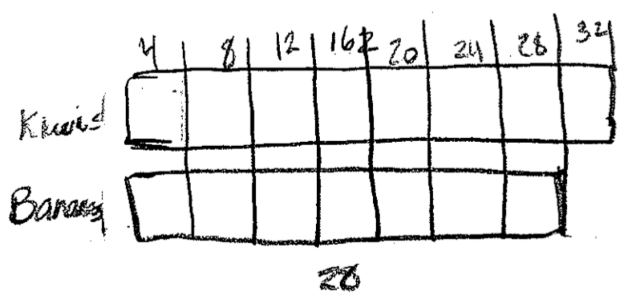


Figure A9. Jen's diagram of the greatest common factor of 28 and 32.

In her attempt to understand how to use the manipulatives, Jen hinted at an important connection between the numbers and their GCF, possibly without even realizing it. Jen's suggestion that students try to find GCF by somehow using the difference between the two numbers was a valid one, since the GCF is a factor of the difference, and in this case it is equal to the difference. When I asked Jen to clarify how the difference relates to GCF, she was unconvinced it always relates, but that in this case it was a good place to start.

When I presented both participants with the list of story problems and asked them to identify the ones related to GCF of 12 and 14, both eliminated story problems (b), (c), and (d) immediately after reading them, but spent more time considering story problem (a). While Amy appeared to clearly differentiate between the two interpretations of GCF with manipulatives, she did not appear as flexible in her understanding when it came to story problems. Story problem (a) uses the *"How many groups?"* interpretation of GCF, but Amy seemed to think that it couldn't represent GCF unless the question was asking *"How many in each group?"*.

> Amy: Part (a) involves that [students] are able to find the GCF, but the final result, asking how many goodie bags can she make, will not be the GCF… because the GCF is going to be the number 12 and 14 have in common… the greatest number they have in common… and that number will be in each bag. Yeah, that amount of candies will be in each bag… So it involves that they find the GCF, but they can't just stop there they have to take the number of candies divided by the GCF to get the number of bags.

As Amy tried to work through the problem though, she confused herself, because she ended up with leftover chocolates. While listening to Amy's interpretation of the story problem, I became aware that the wording may be slightly ambiguous; Amy thought that the number of dark chocolates in each goodie bag had to be the same as the number of milk chocolates in each goodie bag, similar to Jen initial scenario with the kiwis and bananas. As a result, I am unsure which confusion led her to believe that (a) did not represent that GCF of 12 and 14: the wording of the problem, or struggling to recognize a *"How many groups?"* interpretation of GCF in context.

Since Jen had a clear preference for the *"How many in each group?"* interpretation of GCF, she also struggled to determine whether or not story problem (a) represented GCF. She kept wavering between "maybe" and "no". When she finally settled on "no", I asked her to explain. I also clarified the problem for her since my interview with Amy led me to believe the problem was worded poorly. While Jen was trying to explain she stumbled onto the solution, but was still unconvinced that this method of portraying GCF always worked. Eventually, she conceded the validity of the problem, but still was not entirely comfortable with it.

At the conclusion of this interview task, I not only had a sense of how Amy and Jen understood GCF, but I recognized just how sophisticated a concept it is, especially as it relates to story problems. Prior to these interviews, my own understanding was fairly limited, and I tended towards a "*How many groups?*" understanding of GCF in contrast to Amy and Jen. Amy recognized the different interpretations, but only as they related visual and manipulative models. This versatile understanding was limited concerning story problems, as evidenced by her insistence that story problem (a) could not represent GCF. Jen's understanding of GCF appeared to be far less flexible. She tended towards the *"How many in each group?"* interpretation exclusively. Interestingly, Jen kept repeating the same phrase over and over to remind herself that GCF is "the biggest thing that goes into both", which lends itself much more to a *"How many in each group?"* interpretation rather than a *"How many groups?"* interpretation. The phrase that Amy kept repeating, "the number they have in common", was more versatile, like her understanding of GCF. However, the shear amount of time and energy spent on this task, as well as the participants' comments pertaining to the perceived challenge of the task, indicates that GCF is far more complex a concept than LCM, despite their relationship.

Problem 3. In Problem 3, I proposed a hypothetical student claim that the LCM of two numbers is equivalent to their product. Both participants immediately knew that Mark's method would not always work. Amy's first instinct was to come up with a counterexample to prove it. Without prompting, Amy volunteered that Mark's method worked whenever the two numbers were relatively prime or "when they don't have anything in common", making a clear connection to GCF. She used this idea to help her generate a counterexample.

> Amy: So it won't work when they have something in common, like 4 and 6. If you multiply them together, you get 24, but we know … the LCM is 12. For Mark, he would need to know that you could multiply the numbers together, but you would have to divide that number by the number that goes into both A and B, unless there's no number that goes into both.

Brown, Thomas, and Tolias (2002) found that many of their preservice elementary teacher participants treated "the smallest counting number that is a multiple of both" differently than "the least common multiple", suggesting that the way a concept is phrased triggers different connections. Upon prompting, Amy recognized that "the number that goes into both A and B" refers to GCF, but she may not have formalized that connection until that point.

Next, I asked the participants how they would respond to Mark to help him recognize his misconception. Amy's hypothetical response to Mark was rich with mathematical PCK. She chose to ask questions about Mark's understanding so that she could tailor her response to his learning needs. Amy wanted to know what Mark understood so that she could build on it. She then used her KCS to choose examples for him to investigate that would cause a cognitive conflict. To complete her scenario, Amy also phrased questions that would help guide Mark through his reconciliation of this conflict.

> Amy: I would start by asking Mark why he thought that he could just multiply A and B, and see what he was thinking about that. Then, depending on the numbers he was working with, if he was working with numbers that had a GCF of 1, he was seeing a pattern there, but if he were working with numbers like 6 and 4, I would have him write out multiples. I know when we were first doing the GCF and stuff, it was very helpful to write out the multiple of 4, the multiples of 6, and see where they first line up. So he

> could see, "Oh, this works in some cases, but not others." Then I could ask him what's the difference. We've already found the GCF of these two numbers, what do you notice? Then we would start with his observations about the numbers… their GCFs and their LCMs. Go from there, questioning and guiding.

Jen's response to Mark was far less sophisticated than Amy's. Jen felt that Mark's claim indicated that he does not understand LCM and that she would need to reteach the concept. While her suggestion that Mark misunderstood the meaning of LCM is valid, it is only one possible scenario, one that presumes that Mark has very little understanding of LCM.

> Jen: I think probably what I would do is give him two numbers that already have a multiple in it, like 12 and 6. And have him multiply them and [ask] what's your least common multiple. Oh and he actually doesn't know what a least common multiple is, then, is that what it's saying? So I have to teach him what a least common multiple is, because he doesn't even get that part.
>
> Me: How do you know?
>
> Jen: Because he's saying that to find the least common multiple, you just take 12 and 6 and multiply them together and then you have your answer and that's the least common multiple. You need to go back and say, well, no. Maybe I would even go back to the peas and carrots question and say it's when [the multiples] are the same, so it's not quite multiplying them together, because in this scenario, 6 can actually go into 12. So can you multiply 6 by something to get to 12? Well, yeah, you can multiply it by 2. So in this case the least common multiple of these is just 12 itself. So I think he needs to go back to the basics of what a least common multiple is.
>
> Me: Ok. Why do you think Mark might think this?
>
> Jen: It's just what it sounds like. If you tell someone to find the least common multiple, just the vernacular is multiply two numbers and find out what the number is that's your smallest number. I think it's a logical conclusion.

When I asked Jen if Mark's idea ever worked, she immediately responded that it worked when A and B were relatively prime to each other. Her quick response indicated that she had already been thinking about this, but it is curious that she did not consider the possibility that Mark's conjecture may have been a result of his own recognition of this idea. Also, it was unclear from Jen's response whether or not she made a clear connection between the GCF of two numbers and whether they were relatively prime. She did not use her standard phrase for GCF while working on the task, "the biggest thing that goes into both", and the way she talked about her example of 6 and 12 makes the LCM more transparent than the GCF.

When I asked Jen if she would do anything else (aside from bringing Mark back to the basics) to help correct his misconception, Jen acknowledged that not all students think alike and that while pictures may help some students to understand, factor trees may help others, which demonstrated some KCS. However, it was not until I asked Jen how she could find out how Mark learned that she suggested that she could ask him questions about his reasoning. She seemed to prefer to just observe Mark while he worked on problems to decipher his learning style and level of understanding.

At the end of the task, I asked both participants about the influences on their responses to Mark. Amy claimed that the mathematics and mathematics methods courses that she had taken stressed student investigation of problems.

> Amy: Being told the answer is less effective than being guided through the answer. Being asked questions… makes you think about your reasoning and your thought process and it really makes you center in on why [the math] works.

Similarly, Jen claimed that through being a student she recognized that if someone gave her the answer, it was "the easy way out", and that she did not necessarily understand the problem. She claimed that it is better to let students work through the problem themselves. Jen also suggested that knowing how the student thinks makes it easier to help them with their understanding. However, considering how quickly Jen determined that Mark did not understand LCM without investigating his understanding, "knowing how they think" may be limited to appealing to the student's learning style rather than establishing the student's actual understanding of the concept and building on it.

In summary, both participants recognized that Mark's claim was incorrect, but that it would work when the numbers in question were relatively prime. Amy made an obvious connection between LCM and GCF, while Jen's phrasing and example suggested that this connection may not have been obvious to her. Amy's response to the student demonstrated sophisticated KCS through her suggestion to build on what the student knew through scaffolding, while Jen believed that the student did not understand LCM and instead attempted to appeal to his learning style.

Problem 4. For this task, I proposed a diagrammatic method for finding GCF. Both Amy and Jen immediately cited its validity because they had seen the method in their number theory class. Amy noted that if Eva, the student in the scenario, were to "take this whole rectangle and fill it up with 6 by 6 squares it would fit evenly." At this point, Amy tiled the 18 by 12 rectangle, as shown in Figure A10. Jen also broke the Stage 4 diagram up into 6 by 6 square, but she made the added connection that whatever size square we ended up finding would tile the other large squares that we blocked off in earlier stages. However, when I asked Jen how she knew that we would always be able to tile the larger squares, she simply explained that it would work here because 6 is a factor of 18 and 12. This merely checks that 6 by 6 squares will tile 18 by 18 and 12 by 12 squares; it does not explain why Eva's method for finding the 6 by 6 square works.

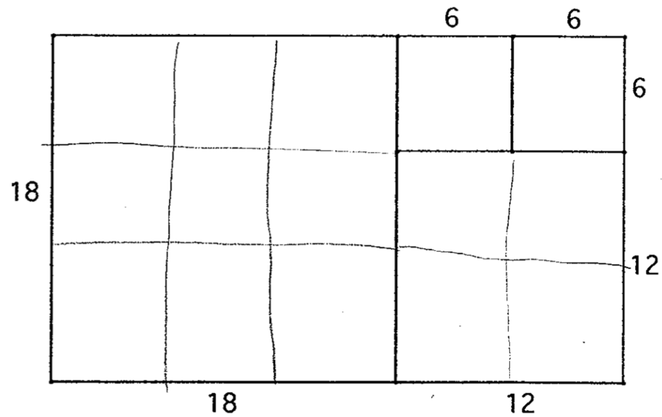


Figure A10. Amy's tiling of the rectangle with six by six squares.

Amy also struggled to explain *why* Eva's method worked. She made some valuable observations, like the fact that we need to find a square with which to tile the rectangle so that the length and width is the same, representing a common factor. Initially, Amy claimed that the only possible square had side lengths equal to the GCF. Upon prompting, Amy also recognized that any square whose sides are common factors would tile the rectangle. She claimed that we are looking for the largest square that does this. While this explains why the 6 by 6 square is indicative of the GCF, it does not explain why Eva's method for finding that square works.

When I asked the participants why Eva's classmates may not be convinced of her method, both cited how abstract the method was. Amy claimed, "it's hard to know the reasoning behind turning this rectangle into various squares, so I think her classmates would be leery about why she's splitting it into squares and how that was going to get her to the GCF." This parroted her own struggle to understand why Eva's method worked.

When I asked Amy how she could help her students to understand Eva's method, Amy finally alluded to, but never formalized, *why* Eva's method worked. While Amy may not have realized it, what she's left with after she's "blocked off" the 18 by 18 square is a rectangle whose dimensions are one of the original dimensions and the difference of the two dimensions. The difference of the two numbers will also have the GCF as a factor, so the GCF of 12 and 18 is the same as the GCF of 18 and 30. As we "block off" another square, we are left with another rectangle whose dimensions have the same GCF as 12 and 18, and 18 and 30. This recursive process maintains the GCF, but makes the dimensions themselves smaller so that eventually one of the dimensions is the GCF.

> Amy: I think I would describe to them that we're looking for the biggest number that goes into 18 and 30 and she was able to do that by creating squares, which are the same on both sides… The best way for her to go about doing that was block off the sections that she knew, I guess, wouldn't go in. Like 18, she knew wouldn't go in, so she made an 18 by 18, 12 by 12, and just kept making it smaller and smaller, until the squares would fit both the length and the width evenly… then we could illustrate that by showing them the picture with all the squares in it. "See, these 6 by 6 squares fit evenly, both across and up."

Jen had a different approach to helping her students understand Eva's method: have Eva explain it. She claimed that this might be more helpful to students than proclaiming to know what Eva was thinking. She also suggested that this may be a valuable experience for Eva herself, an opportunity to practice sharing her reasoning. Jen suggested that if Eva struggled, she could help her to clarify, but that the burden of explanation would mostly lie with Eva. While this strategy may have arisen out of Jen's own struggle to understand the method, it also demonstrates some KCS.

This task allowed me to investigate participants' understanding of certain connections to GCF, namely the relationship between the GCF of two numbers and their difference. Participants could have also cited the Euclidean algorithm, which can be represented visually using Eva's diagram. While neither participant fully explained why

Eva's method worked, Amy touched on some of the subtle but important connections. Jen also demonstrated KCS with her suggestion for explaining the method to her class. It's possible that this task will allow me to observe the depth and connectedness of a participants' understanding of GCF, especially if they have easily completed the previous tasks.

Problem 5. In Problem 5, I attempted to create a scenario that required participants to use the relationship between LCM and GCF. Until this problem, the tasks mostly isolated the two concepts. Here, I offered participants the LCM and GCF of two numbers. After I gave them one of those numbers, they needed to find the other. While a guess and check method would have sufficed, the numbers were large enough to deter this strategy, thus requiring some use of the relationship between GCF and LCM.

Amy started to work the problem using the values of A and the LCM of A and B. She reasoned that the LCM was a multiple of A and that the quotient may be able to tell her something about B. When she found her answer to be 7, she was slightly confused because B needed to be at least as large as the GCF, which was 42. Amy then made a connection back to Problem 3, "If the GCF was 1, then you could just multiply" to find the LCM. After being distracted by the fact that 7 is a factor of 42, Amy eventually recalled her other observation from Problem 3, that the LCM is the product of the two numbers divided by their GCF. She abandoned her first strategy and used this idea to create an equation, which she used to solve for B, see Figure A11. Amy clearly recognizes the relationship between A, B, and their LCM and GCF, but her understanding does not appear to be flexible. By manipulating the same formula she used to solve for B, she could have recognized that 7 times the GCF would have also given her B.

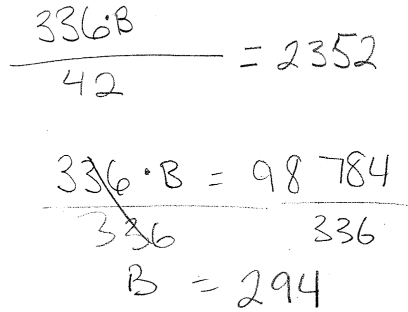


Figure A11. Amy’s equation for solving for B.

Jen had some vague recollection of the existence of a relationship between LCM and GCF, but she could not remember what it was. After more than ten minutes of creating factor trees and comparing prime factorizations, Jen decided that she would like to come back to this problem later on. At this point, she had identified the prime factorization of 336, and figured out that 336 is 8 times the GCF. She also recognized that 336 times 7 would give her the LCM. While all of these calculations and observations could help her to solve the problem, Jen was not making the necessary connections to discern *how* they could help her. At the end of the interview, Jen tried the problem again. She recognized that B has to be at least 42, but then went back to A and saw again that 336 is 8 times the GCF. While she could not explain why, and was not confident in her answer, Jen decided to divide the LCM by 8 to come up with the correct value of B.

While Problem 3 only allowed me to theorize about the connections that Jen had made between LCM and GCF, this task allowed me to fully investigate them. It was clear that she had some vague understanding of the relationship, but it was not formalized for her. In contrast, Amy’s understanding of the relationship was fairly clear, to the point where her strategy was much more efficient than Jen’s, albeit slightly inflexible.

Problem 6. In Problem 6, I switched tacks again. GCF and LCM are not only connected to one another, but to other areas in elementary mathematics. In this task, I hoped to explore the connections the participants had consciously made between the various topics. Without prompting, both participants recognized that finding common denominators for adding fractions requires number theory. Amy claimed, "any common multiple will work, but the least common multiple is most useful." Jen, however, insisted that to find common denominators, one needed to find the LCM in one way or another.

Jen could not recall how the GCF related to other areas of mathematics, but Amy claimed that when you simplify fractions, you are "taking out" the greatest common factor. She realized that this was not necessary, however, because you can divide numerator and denominator by any common factors until the fraction is in its simplest form. When I asked Amy if GCF or LCM played a role in multiplying fractions, she thought their role was limited to simplifying fractions, either before or after you multiply across, which indicates that she is not aware of other algorithms for multiplying fractions. For instance, another algorithm allows for simplifying across fractions prior to multiplying numerators and denominators. Amy connected fractions and ratios, claiming that they were "the same thing" and that you'd simplify ratios just like you would fractions.

While Amy appeared to have a fairly connected understanding of LCM, GCF, and other areas of mathematics, Jen struggled to make these connections even when prompted. She also frequently used number theory ideas in her examples without recognizing it. For instance, when I asked her if multiplying fractions made use of LCM or GCF, Jen came up with the example $\frac{2}{6} \times \frac{1}{12} = \frac{2}{72}$, which "reduces down" to $\frac{1}{36}$. Jen claimed that the LCM was 36, but later corrected herself and hypothesized that GCF could play a role, but she could not identify how.

Next, I asked Jen about ratios and whether GCF or LCM play a role in working with those. At first, it seemed like she may have seen the connection, but then it became obvious that she was thinking that, when you "reduce down", one of the numbers would be the GCF. After some prompting, she realized that you could divide both parts by the GCF to "reduce down", but never made the connection between fractions and ratios.

Me: Do LCM or GCF play a role in working with ratios at all?

Jen: Probably, because if you get up to a 2:6 ratio, that's the same as a 1:3 ratio. So you're finding the GCF of these.

Me: Of 2 and 6?

Jen: Yeah

Me: So that you can end up with the 1:3 ratio?

Jen: Yeah

Me: So what are you doing with the GCF?

Jen: Well, you're just finding the GCF of 2 and 6… Oh, the GCF isn't 3. I don't know… I just divided both by 2. So I don't know how those [GCF or LCM] play into ratios… Maybe if you had a 2:12, you could just divide both by 2 and have a 1:6.

Me: So, in both of these cases, the 2:6 and the 2:12 ratios, what is the GCF of each of those pairs of numbers?

Jen: Um, 2.

Me: And we're dividing by 2.

Jen: Oh, I see what you're saying. So if we had 4:12 and I to the GCF as being 4, then we could have a 1:3 ratio. Do you always divide by the GCF?

Me: That's a good question. Do you always need to use the GCF when you're working with ratios?

Jen: I guess you could just leave it, but if you wanted to reduce it down, you'd have to know the GCF… to reduce it to its lowest… You don't have to. You could just do it in steps. 16:28 to 8:14, and then see that 2 goes in there again, so 4:7. So you could do it down by steps, but it's a lot faster with the GCF.

With suggestive prompting, Jen started to grasp at a connection between ratios and GCF. However, her understanding was far from flexible or even formalized. Amy on the other hand had clearly thought of these types of relationships before and required little prompting to recall them. It was clear, however, that she lacked familiarity with some nonstandard algorithms, for multiplying fractions in particular.

Problem 7. Next, I gave participants the opportunity to demonstrate how they would use number theory ideas to add fractions. However, the numerators and denominators of these fractions were larger than one would see in an introductory lesson to adding fractions. These slightly large numbers appeared to confound participants. While both claimed to recognize the relationship between adding fractions and LCM in Problem 6, they felt that this task was too daunting a task. Both participants had particular trouble with the number 51. Amy described 51 is an "obscure" number; "It's not prime, but it almost feels like it is, to students." Jen initially identified 51 as prime, but later realized that it was not.

Amy suggested that students try and identify common factors between 51 and 34 so that they could use them to find the LCM. However, Amy could not identify any common factors for 51 and 34 herself, so she also suggested that students could multiply 51 and 34 to find a common denominator. If students struggled with any of these concepts, she claimed that she would review what a multiple was and how it is different from a factor. “Because personally, I have to really think about what the two mean, so I don’t confuse them. I think a lot of students have to do that as well.”

When I asked Jen why she felt a student would have such an aversion to working the problem by hand, she seemed to know the answer based off of experience: “I know for a fact, because I just worked with algebra 2 students, and they don’t know how to add fractions. They don’t know how to find the LCM. I don’t know when the breakdown is, but for whatever reason, they don’t like fractions.” However, as Jen herself admits, she did not know where the “breakdown” was. It may have been that the students she was working with knew how to add fractions, but they were just intimidated by them. For Jen, it appears that if a student does not do something correctly, they simply do not know how to do it. This is also clear from her interaction with Mark in Problem 3.

When I asked Jen how she could respond to Remi to help him see that the fraction addition problem was actually pretty doable by hand, Jen seemed concerned with how to get Remi excited about the problem, but could only think to go “back to the basics” of what LCM is and how to find it. She suggested potentially reminding Remi of LCM story problems to help him understand what LCM was. Then she suggested trying to factor the fractions’ denominators to find the LCM. Here it became obvious that Jen felt that 51 was prime, but she did not have an efficient way of determining that it was prime. While this problem set does not address prime numbers explicitly, Jen’s understanding of prime numbers may have influenced how she responded to Problem 7.

After Jen factored 34, and found that 17 was a factor, she thought that she should probably check 51 for divisibility by 17, because “that would make the problem so much easier.” She found that her LCM was $17\times3\times2$, and rather than multiply 34×3 or, easier yet, 51×2 to simplify the expression mentally, Jen decided to use the multiplication algorithm to multiply 17×6. Jen continued to go through the algorithm for adding fractions and ended up with an answer or 69/102, which she did not attempt to simplify.

With prompting, Jen recognized that Remi may have also incorrectly identified 51 as prime, like Jen did, and that perhaps this misunderstanding would lead to frustration with the problem. Next, I asked Jen how she could help Remi to recognize that 51 was not prime, to which she responded, "Well, if he has a calculator, you could ask, 'Is it prime? How do you know? What numbers are prime? Do they divide into that? Does anything divide into that?'"

While a simple test for divisibility by 3 would have made this problem much simpler, Jen's procedural approach does not allow for that connection. When I prompted her, she admitted that she knew that 34 was not prime because she could see that it was divisible by 2, because it was even. After I reminded her what the factors of 51 were, I asked her if there was an easier way to "see" that 51 was divisible by 3. She knew there was a trick, but couldn't remember what it was. "I never learned these when I was growing up, but everyone else [in number theory] knew them like that (snaps fingers)."

Jen eventually solved the problem efficiently, but neither participant made effective use of their number theory understandings to help Remi to make sense of the problem. This could indicate that any connections that they have made between number theory and adding fractions are superficial and limited to recognizing number theory procedures used in "easy" fraction addition problems. While this problem also provides a glimpse into participants' understanding of prime numbers, it is enlightening enough to clarify participants' perceptions of this problem.

Amy's understanding. Throughout the question set, Amy's understanding was the more connected of the two. Although there was still room for growth within the content spanned by these tasks, Amy's understanding was impressive. Amy understanding of LCM was extremely connected. From Problem 1, we saw that she linked the concept of LCM to both time (i.e., "happening at the same time") and length (i.e., "line up") and that she was proficient in modeling LCM via story problems, diagrams, and manipulatives. From Problem 3, we saw that Amy also connected LCM to GCF, a connection she formalized in Problem 5. Amy also connected LCM to fraction addition by finding the lowest common denominator. While this connection was partially stifled in Problem 7, possibly due to her understanding of prime numbers, it was generally accurate.

Amy also had an advanced understanding of GCF. She thought of it as "the number they have in common", which allowed for a certain flexibility in her understanding. She clearly interpreted GCF in two different ways via manipulatives and diagrams, but this flexibility in her understanding had yet to manifest itself concerning story problems, as seen in Problem 1. In Problem 4, Amy made subtle, but important connections between GCF and Eva's diagrammatic method for finding GCF, such as her justification for the need for squares. Amy was also quick to make connections between GCF and simplifying fractions and ratios in Problem 6.

Amy frequently demonstrated her pedagogical content knowledge in these tasks, often without prompting. She tended to interpret each task as it pertains to her role as a teacher. This PCK ranged from her anticipations about student struggles to her knowledge of how to scaffold student understanding. For instance, in Problem 1, Amy recognized students' struggle in drawing diagrams to scale; In Problem 7, she recognized students' potential struggle with the number 51 because of their perception about prime numbers; and Amy frequently discussed students potential struggle to differentiate between GCF and LCM. When Amy would respond to the student scenarios, she was primarily concerned with determining what the student understood and then designed plans for scaffolding from there. Her most impressive example of this was when Mark felt that he could find the LCM of two numbers by multiplying them, in Problem 3. Amy recognized that Mark could be seeing a pattern between relatively prime numbers and their LCM. From that starting point, she designed a series of questions and tasks to help Mark achieve cognitive conflict and develop his own understanding.

Jen's understanding. There were many similarities between Jen's understanding and Amy's, but there were also many differences. Jen struggled with many of the tasks, but her persistence frequently prevailed to achieve at least a basic understanding of each task. As Jen's understanding of LCM and GCF were not quite as flexible as Amy's, she tended to have different approaches to the tasks. Also, possibly due to the level of her understanding of LCM and GCF, Jen had not formalized many of the connections between LCM, GCF, and other areas of mathematics that Amy had.

Jen appeared to be comfortable with two different representations of LCM: visual and numerical. In Problem 1, she recognized LCM as when the two numbers "line up" visually, but in Problems 5 and 7, Jen relied on prime factorizations to determine LCM. Jen did not appear to connect these methods to each other, but she certainly connected her visual model to modeling LCM with manipulatives and story problems. Jen recognized the relationship between fraction addition and LCM, and fully demonstrated this relationship in Problem 7. However, Jen did not appear to fully recognize the relationship between GCF and LCM. In Problem 3, she recognized that the LCM of two numbers is their product if they are relatively prime, but she did not connect this to the GCF or suggest how to adjust the product if the two numbers were not relatively prime. This lack of connection further presented itself in Problem 5 when Jen struggled to use the LCM and GCF to determine the value of B. While she arrived at the correct answer, she could not explain how or why she got it. In Problem 7, she found that 51 and 34 had a common factor of 17, which made finding the LCM "easier". So it is clear that some connection between the two concepts exists for Jen, it just was not formalized.

As she admitted herself, GCF was more of a struggle for Jen. She insisted on calculating the GCF and drawing diagrams prior to creating a story problem. She also tended towards one interpretation of GCF, the *"How many in each group?"* interpretation, over the other. The only time she acknowledged the possibility of another interpretation was at the end of Problem 1, where she begrudgingly conceded that story problem (a) represented GCF. Also in Problem 1, Jen hinted at the relationship between the GCF of two numbers and their difference. However, this connection was not strong enough to be of use to her in her explanation of Eva's method in Problem 4. She did recognize in Problem 5 that both numbers need to be larger than their GCF, but could not reason through how to use this idea. Jen also struggled to connect GCF to other areas in mathematics. While she routinely used GCF to "reduce down" fractions, she did not recognize it. Only through suggestive prompting did she realize that simplifying ratios related to GCF, and she did not extend this connection to simplifying fractions, which may infer that these two concepts are unrelated to her.

While Jen's understanding of GCF and LCM were limited, she still exhibited PCK in the area. Jen frequently anticipated students' struggles with different representations or ideas. For instance, she knew that her story problem was not clear enough for students to understand LCM and that they would need an accompanying diagram. She also recognized the abstraction of Eva's method and suggested that students might struggle to fully understand it. Jen also acknowledged that students may confuse GCF and LCM. Some of Jen's responses to the students in the scenarios demonstrated intuition about how children learn, while others appeared to be faired canned. For instance, Jen knew from experience and coursework that telling a student the answer was less likely to help them understand that creating opportunities to correct themselves. In contrast, Jen insisted that Mark in Problem 3 did not understand anything about LCM because his conjecture was invalid, and suggested that she go "back to the basics". Jen voiced this same strategy again in Problem 7 when she described her experiences with Algebra 2 students. This type of response indicates that Jen may not have recognized that a student can understand aspects of a concept without achieving proficiency with it. Jen did however recognize the value of learning styles, and identifying how students learn. She frequently suggested appealing to these learning styles when she went "back to the basics."

Problem 7 may have been the only task on which Jen made more progress than Amy, but Jen's comments concerning prime numbers and divisibility raised even more questions about her understanding of other number theory ideas. Initially, Jen believed 51 to be prime and did not have an efficient or complete way of determining its primality with confidence. Even once she realized that 51 is the product of 3 and 17, she was unsure how she could have “seen” that earlier. While prime numbers and divisibility were not the focus of this set of interview questions, it is clear that they relate to ones understanding of GCF and LCM. Thus, I piloted a second set of interview questions related to these topics.

Prime number question set. I randomly chose to interview Mia and Zoe using the prime number question set. These interviews, while still rich with data, took much less time than the GCF/LCM question set at about 40 minutes per interview. As Mia and Zoe are in elementary education programs with mathematics emphases, and have successfully completed a semester long course in number theory, I did not anticipate that they would have any of the misconceptions about prime numbers that researchers like Zazkis and Liljedahl (2004) found true of their participants. To my surprise, I found remnants of similar misconceptions, but to my excitement, I found evidence of a great deal more about what Mia and Zoe *do* know and understand about prime numbers. The tasks also proved useful in revealing Mia and Zoe’s methods and philosophies on responding to students, in general and with regards to number theory ideas. Recall that a complete list of the interview tasks from the prime number question set can be found in Appendix I.

Problem 1. To establish their working definitions of “prime”, I asked participants what they thought it meant for a number to be prime. Both Mia and Zoe quoted a version of the grade school definition, claiming that a prime number is “only divisible by 1 and itself.” When I asked Mia whether prime numbers were valuable in mathematics, she claimed that they are important for understanding “basically everything about number theory”, for example, “factoring non-prime numbers down”. When I asked Mia to clarify what she meant by "non-prime" numbers, she said that they were numbers that had other factors other than 1 and itself, like the number 15. It appeared that Mia classified numbers in two categories, either prime or not prime. While this was merely a curiosity in her response to Problem 1, it proved to be a problematic classification system later on in the interview.

While Mia struggled to connect prime numbers to topics outside of number theory, Zoe made a wide array of valuable connections. Similarly, she saw the relationship between prime numbers and other number theory concepts, like LCM and GCF. In relation to GCF, Zoe made connections to simplifying fractions whose numerators and denominators were composite, as opposed to fractions whose numerators and denominators were prime, which "can't reduce." It is possible that Zoe was thinking about fractions whose numerators and denominators are *relatively* prime, as opposed to prime, which would have been more accurate, but she did not elaborate. Zoe also connected prime numbers to more advanced areas in mathematics like cryptology.

Problem 2. Mia and Zoe's definition of "prime" proved problematic for them when addressing the second task of the interview. Here, I asked the participants to validate a student's conjecture that 1 is prime since its only factors are 1 and itself. Since Shayna, the student, cited the same definition of "prime" that Mia and Zoe did, this gave participants the opportunity to clarify their definitions to account for exactly two distinct factors, "1 and itself." However, neither participant did this.

Mia said that Shayna was incorrect, but then she wavered. It appeared that she was trying to categorize 1 as either prime or "non-prime" and decided that it best fit the definition of "prime". This decision may have been influenced by both her definition of prime and the duality of her categorizations.

> Mia: Well, I don't think [Shayna is correct], because you could take the square root of 1 and it's a whole number, but you can't take the square root of a prime number and have it be a whole number. Maybe 1 is a prime number… I think it is a prime number. I think it is.

> Me: Ok. So what you're saying is that Shayna is correct… that she has a correct understanding…

> Mia: Um, I feel like it isn't a prime number… OK, I'm going to say that she is correct, because its only factors are 1 and itself, and that's the definition of prime number, so that's why she's correct.

Zoe correctly identified 1 as being neither prime nor composite, but she claimed that 1 is an exception to the rule, a "special case." Determining 1 to be an exception is not only unnecessary, but it is confusing. Mathematical definitions are meant to be valid within their conditions, without exception. While Zoe recognized that "1 and itself" reduces down to one factor, she does not seem to realize that the definition of prime only allows for two *distinct* factors. Also, claiming 1 is a special case does not really explain why 1 is *not* prime.

> Zoe: I would tell her that she's on the right logic path, that her idea is correct, that normally a prime number does fit that definition, but that 1 is a special case, because itself is also 1. So you have to be careful when you put it into that category, because it's neither composite nor prime. Just kind of explain that. The concept of 0 is similar. They're special cases.

After asking participants to validate Shayna's claim, I had planned to prompt the participants to respond to Shayna in a way that would help to correct her misconception. Since Mia believed Shayna to be correct, I could not share this prompt with her without correcting her own misconception mid-interview. Zoe at least believed Shayna's claim to be incorrect, so I did ask her to respond to the student. She liked the idea of giving

Shayna a task that would allow her to investigate the idea, but she could not think of anything during the interview. From her own experience as a student, Zoe also suggested the importance of positively reinforcing students and not discouraging them from participating. Zoe believed it was important to start any response to students with encouragement and gradually get to what was wrong with what they said. Part of this process is acknowledging what the student *does* know about a concept, which is similar to Amy's strategy in responding to students. Zoe continued to insist, however, that Shayna understood the definition of prime but that 1 is just a "special case."

Problem 3. In the next task, I posed another student scenario that portrayed a different misconception about prime numbers: Magnus claimed that 6 is prime because it is the product of two prime numbers. In this case, both participants correctly determined Magnus's claim to be invalid. Mia reasoned that 6 is not prime because the only even prime is 2. Also, she identified two additional factors of 6, 1 and itself, for a total of four factors.

Both participants recognized that Magnus has at least a basic understanding of "prime". Zoe thought that Magnus might understand what a prime is, but not what a composite is. "I think he might understand the beginning of it, but he's taking the concept of prime too far." This could relate to what Zazkis and Liljedahl (2004) described as a desire to achieve closure with prime numbers. Mia's reasonably correct response contradicts her earlier reasoning about why 1 is prime. Also, If 1 were prime, then every prime number would be the product of two prime factors. Mia did not appear to recognize these contradicting ideas.

> Mia: He understands that a prime number only has two factors, but he doesn't understand that 1 times 6 also give you 6, and he doesn't understand…

> Well, he also understands that 6 is the product of 2 prime factors. But he doesn’t understand that the product of 2 prime factors doesn’t give you a prime number.

Afterwards, I asked each participant to respond to Magnus so that they could correct his misconception. Mia suggested that she would give Magnus tasks to work through, with a partner so that he did not get frustrated, and ask him to find relationships between the tasks and come up with a generalization. Some of the tasks she suggested involved practicing determining whether a number was prime or “non-prime” by listing its factors. She also suggested taking ”prime numbers and multiply them by each other, and take prime and non-prime numbers and multiply them by each other, and take non-prime numbers and multiply them by each other and give him that so he can kind of make his own results from that too.” While Mia’s understanding of prime numbers had holes in it, she appeared to demonstrate fairly sophisticated PCK. She knew from experience, like her mathematics methods course, that Magnus would understand the concept better if he investigated it through activity rather than being told the answer.

Zoe also demonstrated a fair amount of PCK. She suggested that she investigate Magnus’s understanding of prime before she responded to him. For instance, she would ask Magnus what the definition of prime was, and if he answered correctly, she would ask questions to help him recognize that 6 did not meet the necessary criteria. Zoe recognized the importance of acknowledging what students know and building on it. She also felt strongly about reinforcing ideas through a visual or something like a factor tree.

Even though Mia and Zoe appear to have valuable strategies for helping their students understand prime numbers, their own misconceptions about the content could confound their students’ confusion. For instance, a couple times during this task, Zoe claimed she would tell Magnus that, "if we can break it up into equal groups, 2 groups of 3, 3 groups of 2, then it's not a prime number." However, 1 group of 5 and 5 groups of 1 are still "even" groups, and 5 is a prime number.

Problem 4. Prior to Problem 4, Mia and Zoe both mentioned the value in determining whether a number was prime, but these comments were in reference to small, familiar numbers. In Problem 4, I asked them about more general techniques for determining the primality of a number. Both participants initially suggested using divisibility tests. Mia suggested that she would check for divisibility by 2 and 3, and she recited the divisibility rules. She also said that she would try to determine if the number in question was divisible by other numbers that she knew. By this, Mia seemed to mean that if she recognized the number as being the product of two numbers, she would know it was not prime.

Since her method did not appear to be exhaustive or convincing, I prompted Mia further by suggesting she tell me how to determine whether 71 was prime. She elaborated on a brute force method, dividing 71 by numbers 2 through 35 to see if they were divisors of 71. She explained that since 70 divided by 2 was 35, if she divided 71 by any whole number larger than 35, she would get a decimal. While this method demonstrates some intuition about the factors of a number, Mia did not appear to recognize that if a number is *not* divisible by a prime, then it is also *not* divisible by multiples of that prime.

Zoe brought up "tricks" for knowing if a number is divisible by 9, 3, 6, 10. "And if it's even, you know it's not prime." It was unclear whether Zoe equated evenness with divisibility by 2. She could not recall what the divisibility tests were, but she did say that you only really use divisibility tests with odd numbers, "because even numbers aren't prime." As with her grouping comment in Problem 3, she neglected to discuss the trivial case, where the number itself is 2, which is prime. While later comments indicate that she knows that even numbers are divisible by 2, her comments about divisibility tests suggest that she does not think of evenness as a test for divisibility.

Neither participant could recall an efficient method for determining the primality of large numbers like 853. Mia's first instinct was to make a factor tree. After little success, since 853 is prime, she started plugging numbers into the calculator. When I asked her to describe what she was doing, she told me that she tried to divide 853 by 9, 21, and 23 and kept getting decimals. "So those aren't factors, obviously. And it's not an even number, and the digits don't add up to be divisible by 3. It's not divisible by 5… So it would be a long process if I continued to do it that way." Mia did not appear to have a systematic way for determining that 853 was prime. At least her method for determining 71 was prime was systematic. Perhaps because it was a brute force method, and due to the much larger scale, Mia abandoned her method in hopes of finding that 853 was "non-prime" by guessing its factors. There was a moment when I thought she was on the right track, when she suggested dividing by primes. But when she already knew that it was not divisible by 3 and she checked 853 for divisibility by 9 and 21, I knew that she was grasping at straws.

Zoe's failed attempts at determining 853 to be prime suggested that she had misconceptions about divisibility as well. Zoe acknowledged that once you tested for divisibility by any prime, it eliminated the need to test for divisibility by any power of that prime. But for composite numbers, she felt she needed to test for divisibility by all of their prime factors before eliminating it as a potential factor. Throughout the interview, Zoe repeatedly uses the term ‘factor’ when she was describing ‘multiple’. Here, however, it appeared that she meant ‘power’ rather than ‘factor’.

> Me: We’re given a big number like 853. Would we need to test it for divisibility by all of the numbers up to 853?
>
> Zoe: No, because if we know it is not divisible by 2, it isn’t divisible by a factor of 2. We don’t need to see if it’s divisible by 4, because it’s not divisible by 2. And same with 3. If it isn’t divisible by 3, it won’t be divisible by 9 or 27. So by knocking out 2, 3, or 5. If it’s not divisible by 5, it’s not divisible by 10 or 25. Or any factor of 5.

With some forceful probing, Zoe acknowledged that if we check 853 for divisibility by primes that would be sufficient to determine it was prime. When I asked if we would need to check it for divisibility by all primes less than 853, she suggested that we would only need to check the primes less than halfway to 853, because there will not be any factor pairs in the second half of the numbers since it is not divisible by 2. Zoe shortly revised this statement and suggested that we would only need to check primes less than one-third of the way to 853, because 853 was not divisible by 3.

For the last prompt of the task, I asked participants how to determine the primality of smaller numbers using manipulatives or diagrams. Mia suggested that she could use Cuisenaire rods and make trains to determine if the number is prime. After thinking about the task for a minute, Mia came up with another way for determining if a number is prime, however her way (the Sieve of Eratosthenes) determines all of the primes less than a certain number rather than just considering one number.

> Mia: You could have a diagram and have 1 through 50 on it… so you would have 1 through 10, and then 11 through 20, 21 through 30… and then you would go through and you would take, well, the 2 for example, and cross off every multiple of 2 up through 50. And for the 3, you would do the exact same thing, and cross off all the multiples of 3, and then 4, and then 5, and so on. And you will eventually be left with all the prime numbers that will not be crossed off.

Me: So how does that process leave us with all the prime numbers?

Mia: It eliminates all of the numbers that are divisible by 4, 5, 6 and all of those, which therefore makes them non-prime. The prime numbers will be left because they aren't divisible by anything but 1 and themselves.

Mia described an extremely inefficient version of the Sieve of Eratosthenes. She continues to demonstrate a disconnected understanding of prime, factor, multiple, and divisibility. She does not seem to recognize that she does not need to cross off all the multiples of 4 and 6, because they are also multiples of their factors, 2 and 3, which have already been crossed off earlier in the process. Even though Mia acknowledged that she was crossing off multiples, she also did not make the connection that all whole numbers are multiples of 1, which she claimed was prime, and that according to her reasoning she should have crossed out all whole numbers. Mia also seemed to be skirting anything that would cause her disequilibrium. For instance, she started at 2 and crossing off the multiples of 2 so that she would not have to discuss what happens to 1.

While Mia's method was not well reasoned or efficient, it was a valid method for determining prime numbers, thus demonstrating SCK. Zoe's response, however, did not appear to demonstrate SCK. Zoe suggested that she could give students a certain number of manipulatives and ask them if they could split it up into equal groups. If they can, then the number is not prime. She also suggested that students could use a factor tree to determine whether or not a number is prime. Zoe's suggestion about using the manipulatives is inaccurate, because you can always make 1 even group of p objects or p even groups of 1 object. For whatever reason, Zoe continued to ignore the trivial cases. As for Zoe's second suggestion, creating a factor tree is only helpful when the number we are attempting to factor is composite, as seen in Problem 5.

Problem 5. For this problem, I showed each participant two students' equally valid methods for factoring the same number. I asked Mia and Zoe why the students may not recognize the validity of each others' methods. Both participants suggested that the surface features of the methods may have been the problem, because the factoring methods "look different". Mia and Zoe correctly made sense of both diagrams, acknowledging that the students found the same prime factors. Zoe commented that since Tom's method kept the factors on the right and divided by the smallest factors first, it may be a "good method for students to start with." While the factor tree method that Talisa used could also be used to do this, it is possible that Zoe was demonstrating SCK with this comment.

Tom and Talisa's conflict quite possibly could have been that their methods "looked different," but another possibility may have been that their answers looked different as well. This conflict may relate to confusion about the uniqueness of prime factorization, which is not a trivial concept. Mia also saw that the students factored in different orders, and suggested that this may have contributed to their confusion. She started to address this possible conflict by refactoring Talisa's work to get the same order as Tom's factors, but quickly discarded this activity and reaffirmed her belief that the students' conflict was due to their differing methods.

After asking them to determine the validity of Talisa and Tom's methods, I asked Mia and Zoe how they might respond to the students to resolve their conflict. Both participants suggested that Talisa and Tom explain their own methods to each other. According to Zoe, "The kids need to learn how to work cooperatively together and learn from each other too. They can learn from each other's mistakes… So now they can understand how both concepts work, and they worked it out together." Mia added that she would use this opportunity to start a class discussion and see if anyone else had a different way to factor. Zoe felt that encouraging Talisa and Tom to critique each others' methods may also help them to see that "there isn't any math-magic going on." She also wanted to spin this into a discussion about there is more than one correct way to solve mathematics problems. When I asked Mia how she knew to respond this way, she stressed the importance of encouraging students to explain "why".

> Mia: Well, because it's not a huge conflict. They're arguing over which way is better. I think if you give them the chance to explain their reasoning… a lot of times students don't get a chance to discuss why they did something in a certain way, they're just told, 'oh, you did it the wrong way.' But then they don't know why they did it… If they were both explaining it, the other person would see that they were doing it a different way but they would also see that's why they did it, and it's also a right way to do it.

Both participants' reactions to the students' conflict were fairly intuitive. Even if their evaluations of the conflict may have been incomplete, having Tom and Talisa to share their reasoning would surely allow Mia and Zoe to verify their suspicions. Also, recognizing that student reasoning is valuable for other students' learning as well demonstrates PCK.

Problem 6. This next problem investigates participants' understanding of the cardinality of prime numbers. Both participants recalled that the set of prime numbers is infinite, but they struggled to explain how they knew this. Mia acknowledged that even in the millions you can identify prime numbers, but she seemed conflicted with this idea, commenting, "you would think after a while [numbers] would be divisible by something… other than one and themselves." When I asked Mia to explain what she meant, she admitted that she had never been convinced that there are infinitely many primes and that it seemed counter-intuitive to her. She also admitted that she felt more convinced that there are infinitely many primes after she took the number theory course, because it caused her to question a lot of her understanding. Then Mia changed her mind.

> Mia: I don't know. I guess I do believe that there's infinitely many… I remember someone talking about the biggest gap – a straight number of non-primes – can't remember how many. But there's always a prime number somewhere. I figure if that happens within the first million numbers, it's going to keep happening. There's just patterns… and it's going to continue. I guess I'm pretty convinced.

Mia later acknowledged that the "pattern" she referred to is not consistent. However, coupled with "the fact that I don't know how to disprove it," the idea there exists a biggest gap between prime numbers convinced Mia that there are infinitely many primes. While she did not specifically say so, it is possible that knowing the biggest gap itself is not infinitely long is what convinced her.

Zoe appeared convinced of the cardinality of prime numbers for other reasons. She claimed that "there's an infinite number of composite numbers, because there's an infinite number of even numbers, because there's an infinite number of numbers." Zoe reasoned that there are infinitely many primes because of the cardinality of the number system itself. Since her reasoning was inconclusive, I asked Zoe to elaborate on how she knew this.

> Zoe: Um, computer programs? We know our number system is infinite. There's not an end to numbers. So there are computer programs that predict how far out primes go and I don't think it can ever end because… there's always one more prime? I know I did a proof for this, with the number of primes. I don't remember it.

Even though Zoe could not remember the proof that there are infinitely many primes, when I asked her if she remembered being convinced by it, she said yes. From what she could remember, it was a proof by contradiction and the assumptions were that there are finitely many primes and that our number system was finite. While not exactly an assumption of the proof, the contradiction to our number system being infinite seemed especially compelling for Zoe.

Mia could not remember if there was a proof that there are infinitely many primes, but she did remember there was a proof concerning the biggest gap of "non-primes". She said that she would convince students that there are infinitely prime numbers by finding a proof and using that. She did not make any caveats about making it grade level appropriate. Zoe also suggested that she would find a proof, but acknowledged that it may not be grade level appropriate. Instead, she would try and convey the basic concepts behind the proof. For instance, if the proof relies on the fact that there are infinitely many numbers in our number system, Zoe would ask the class scaffolding questions so that they could develop and understanding of how large the number system is.

> Zoe: How many numbers are there? I'm sure we're get some guesses, 'We have an infinite number of numbers.' OK, well, if we have an infinite number of numbers, we can always add one more to get the next number. And if we have a number that's divisible by 2 and we add one more, and it's not divisible by anything else, then you have one more prime. You can always have one more prime. So there is also an infinite number of primes.

While she may not be aware of it, the idea we "can always have one more prime" is a major component in Euclid's proof that there are infinitely many primes. Zoe decided that either way she attempted to convince her students of this idea, she would also let them know that this is a difficult concept to understand and that "mathematicians spent hundreds of years trying to figure this out." Zoe's suggestions for disseminating the proof by grade level appropriateness demonstrated PCK.

Problem 7. In the final task, I asked participants to consider the prime factorization of a number, *M*, and use it to determine whether or not *M* was divisible by certain numbers. Mia immediately expanded the prime factorization so that it was written as a product of prime rather than a product of powers of primes, when I asked her to explain why she did that, she claimed that help it helped her ”visualize” the problem better. “When I see 3^3, I see 27, and when I see 5^2, I see 25. So this just help me visualize it better.” Afterwards, Mia used sound reasoning to quickly determine the divisibility of *M*.

> Mia: M is not divisible by 2, because if it was divisible by 2, it would have 2 in the prime factorization. It’s divisible by 7, because 7 is in the prime factorization. It’s divisible by 9, because you could just multiply two of the 3s together to get 9. It would not be divisible by 11, because 11 is a prime number, and if M were divisible by 11, 11 would be included in the prime factorization. It would be divisible by 15, because we could multiply 3×5 in the prime factorization and that gives us 15, which means that it would be divisible by it. And 63… it would not be divisible by 63. Because 63… is not a prime factor, but its factors are 9 and… oh, it would be, just kidding. Because you could take 3×3×7 to get 63.

Mia’s response demonstrated a fairly developed understanding of the connections between factors, divisibility, and prime factorization. Zoe’s response, however, indicated that her understanding of these connections had some holes.

> Me: So, is *M* divisible by 2?
>
> Zoe: No, I would say that *M* is not divisible by 2, because it has no even factors, and no even factor pairs, I don’t think. Because 25 times 7 is not even… Yeah, so there are no even factor pairs. Like 3 isn’t even, neither is 9 or 27. 5 isn’t [even], or 25, nor is 7 or any of the factor pairs. So it won’t go into any of the factors, so it won’t go into the number as a whole.

I asked Zoe how she was looking at this factorization of *M*; was she seeing a product of factors or something different? She said that she was seeing it as the prime factorization of *M*, because 3, 5, and 7 are all prime. However, her need to determine whether the product of any of these prime numbers was even was curious. As she had already acknowledged, primes are only divisible by 1 and themselves, and 2 is prime. Since 3, 5, and 7 are all prime, none of them can be divisible by 2. Also, since they are prime, when you multiply any number of them together, that product would represent a prime factorization itself, and 2 would never be in it. As with many of Zazkis's (1998) participants, Zoe seemed to have a disconnected understanding of evenness. She seemed to understand that a number that is divisible by 2 is even, and that 2 is prime, but she did not make the connection that all even numbers have a 2 in their prime factorization.

As with Mia, Zoe was confident that 7, 9, 15, and 63 were factors of *M*, because she could identify them in the prime factorization. She was less confident in identifying whether or not 11 was a factor. Eventually, she settled on "no". Zoe appeared to be using the same reasoning for divisibility by 11 as divisibility by 2, but it was subtler. When questioned her reasoning, she seemed to be talking about factorizations in general, as opposed to just prime factorizations.

Zoe: In order for a number to be divisible by 11, it would have to have 11 as a multiple here in what *M* is listed as, or it's numbers that pair up to give you 11 or a multiple of 11, and it doesn't have any of that.

Me: So what kind of a number is 11?

Zoe: A prime number?

Me: So, can it be a multiple of two other numbers?

Zoe: No, I'm talking about, there's no 22, or 33, or anything like that.

Me: Ok. And would we find something like that in a prime factorization?

Zoe: No.

Zoe may have some misconceptions about primes themselves as well as about prime factorizations. Her understanding seems inconsistent, as if she believes that different rules apply in different cases, “special cases”.

Mia and Zoe demonstrated a great deal about their understandings of number theory through these interview tasks. Much of their content understandings related to their conception of prime numbers, determining primality, factoring, and divisibility. Both participants also demonstrated number theory PCK. While I detected some connections between the two, my understanding of these connections is incomplete due to insufficient data. In the next two sections, the nature of each individual participant’s understanding of the topics in number theory are addressed in this question set, with a focus on how the understood the content, the PCK they demonstrated, and any connections between the two.

Mia’s understanding. It was clear from Mia’s consistent comments that she classified natural numbers as either prime or “non-prime”. To her, a prime number is “only divisible by 1 and itself” and a non-prime number is divisible by at least one other number. Her dichotomous understanding proved problematic for her in Problem 2, where she incorrectly identified 1 as a prime number “because its only factors are 1 and itself.” While Mia seemed convinced by this idea, it appeared to be separate from her other conceptions about primes, factoring, and divisibility. For instance, Mia frequently made comments that, while correct, contradicted her claim that 1 is prime. An example of this lies in her response to Problem 3. While validating the student’s claim that 6 is prime, because it is the product of two primes, 2 and 3, Mia stated that the product of two prime numbers is not prime itself. However, if 1 were prime, as Mia suggested, it would be possible for the product of two primes to be prime.

Since Mia appeared to have mentally cordoned off the idea that 1 is prime from her other conceptions about primes and factoring, many of her other responses to the interview tasks demonstrated a fairly strong understanding. For instance, her repertoire with respect to identifying factors and factoring "non-prime" numbers was substantial. While these particular tasks did not provide Mia an opportunity to factor a composite number herself, she immediately recognized the validity of two methods for factoring in Problem 5. She also recalled the tests for divisibility by 2 and 3, which would aid in the factoring process. While Mia did not explicitly address the importance of uniqueness, up to order of the factors, in prime factorization, she clearly relied on some understanding of this concept in her response to Problem 7. She recalled that if a number *M* were divisible by a prime, that prime would appear in its prime factorization. She also recognized that if *M* were divisible by a composite number, that you will be able to multiply prime factors of *M* to find the composite number. To help her with this strategy, Mia immediately expanded the prime factorization of *M* so that she could "visualize" the partial products within the prime factorization. An equally valid method would have been to identify the prime factorization of the composite number in the prime factorization of *M*. It is possible that Mia did not recognize this method, but her understanding of divisibility with respect to the prime factorization was connected enough that her response to Problem 7 was well reasoned and efficient.

While connected to factoring composite numbers, determining the primality of a number was more of a struggle for Mia. For small numbers and numbers whose factors she knew, she suggested she could make trains with Cuisenaire rods or list the factors to determine whether or not they were only divisible by 1 and itself. For other numbers, Mia did not have an efficient method for determining primality. For large numbers *M* less than 100, she suggested trying to divide *M* by all natural numbers less than half of *M*. This indicated at least some understanding of the relationship between factor pairs – one number in each pair will always be less than half the number itself. More precisely, one number in each pair will always be less than the square root of *M*, but Mia did not appear to recognize this connection. Mia also did not recognize that once she determined that a prime does not divide *M*, none of the multiples of that prime will divide *M*. This was obvious from her inefficient description of the Sieve of Eratosthenes and her attempts to check a large number, 853, for divisibility by 9 and 21 even though she knew from her divisibility test that 853 is not divisible by 3.

In Problems 2 through 6, Mia had an opportunity to demonstrate number theory PCK. While she did not capitalize on all of these opportunities, I did find some evidence of her mathematical PCK. In Problem 2, since Mia herself believed 1 to be prime, I did not ask her how she would correct Shayna's misconception. In Problem 4, she suggested lower level strategies for determining primality, but did not connect it to student learning or strategies she would use to teach these techniques. As a result, Mia's response pertained more to SCK than PCK. Her response to Problem 6 also lacked PCK. Mia demonstrated a vague understanding of the cardinality of prime numbers; she recognized that there are infinitely many primes, but could not recall why. She did recall that a proof concerning the largest gap between primes had convinced her in the past. Possibly due to her insufficient understanding of the cardinality of primes, Mia merely suggested that she would find a proof and use it to help her students understand the concept. Again, since Mia did not address student understanding, methods for teaching, or even how she might adapt the proof for grade-level appropriateness, Mia's response did not demonstrate PCK.

In contrast, Mia did demonstrate fairly sophisticated PCK in her response to Problem 3. She suggested that Magnus would understand the concept better if he investigated it himself, so Mia described how she would scaffold tasks for him to complete so that he could recognize on his own that the product of two prime numbers is not prime itself. First, she would have him list the factors of multiple numbers and identify which numbers only had factors of 1 and themselves, reminding Magnus of the definition of "prime". Then she would have Magnus multiply pairs of numbers, both prime and composite, to determine if he could ever have a product that was prime.

In her response to the students' conflict in Problem 5, Mia suggested that Talisa and Tom share their reasoning for their methods. She also claimed that she would open it up for class discussion to see if anyone else had a different way to factor. Intuitively, Mia believed that if her students shared their reasoning with each other, they would not only understand the different ways of factoring better, but by sharing they would also have the opportunity to formalize their own understandings.

In Problems 2, 4, and 6, Mia demonstrated misconceptions about prime numbers and an incomplete understanding of determining primality and the cardinality of primes. It is possible that her responses to these tasks did not demonstrate PCK as a result of her poor understanding of the concepts. It is also possible that Mia's refined understanding of prime factorization and divisibility contributed to the sophisticated PCK she demonstrated in her response to Problem 3. The PCK Mia demonstrated in Problem 5 seemed to pertain to how students learn and understand mathematics in general; it did not appear to be connected to Mia's understanding of number theory specifically. These connections are merely conjecture, however. With more data, I may have been able to find more evidence to these claims.

Zoe's understanding. In general, Zoe's understanding of number theory was far less connected that Mia's, and as a result Zoe had developed some evident misconceptions about factors and multiples. Zoe had the same working definition of "prime" that Mia did, but it was less problematic. Zoe firmly believed that 1 was neither prime nor composite, but she kept referring to it as a "special case". This may be evidence that Zoe has a weak understanding of what it means to be a mathematical definition.

On the many occasions that Zoe had to demonstrate her understanding of how to determine primality, I found that Zoe had many other holes in her understanding, especially concerning the primality of small numbers. For instance, she suggested that students could determine whether a number was prime by creating a factor tree, which is not possible for prime numbers. She also suggested that students could try making equal groups from a set number of manipulatives. If unsuccessful, Zoe claimed that number is prime. Zoe suggested this strategy multiple times throughout the interview and ignored the trivial cases each time. Even with p manipulatives, one can still create p equal groups of 1 and 1 group of p.

Zoe demonstrated a slightly stronger understanding of how to determine whether a large number, e.g. 853, is prime. She knew of the divisibility tests for 3, 6, 9, and 10, but could not recall the tests for 3, 6, and 9 explicitly. She also knew that even numbers are not prime, again ignoring the trivial case of $p = 2$. It later became evident that Zoe recognized that even numbers are divisible by 2, but she did not seem to make the connection that determining whether a number is even is the divisibility test for 2. Similar to Mia, Zoe also did not recognize that once you eliminate p as a factor of, say, 853, you know that multiples of p also cannot be factors of 853. Interestingly, Zoe did recognize that this does imply powers of p cannot be factors of 853. Zoe demonstrated on multiple occasions that she believed in order to eliminate a composite number as a possible factor, one must first determine that none of its prime factors are factors of 853. When I used suggestive probing towards the end of the interview, Zoe appeared to acknowledge, that if we checked 853 for divisibility by all primes less than one-third of 853 that would be sufficient to determine that 853 itself was prime. While Zoe wavered concerning only testing divisibility by primes, she seemed confident that she only needed to test 853 for divisibility by numbers less than one-third of 853 because of the relationship between factor pairs.

Zoe appeared to have demonstrated an accurate, albeit incomplete, understanding of factor pairs in problems like Problem 4, but in her response to Problem 7, it became evident that Zoe may have had some deeply rooted misconceptions about factor pairs and prime factorizations, especially concerning even numbers. While Zoe could easily identify the prime factorizations of 7, 9, 15, and 63 in the prime factorization of M, she wavered on whether M was divisible by 2 and 11. At some point during the interview, Zoe had acknowledged that both 2 and 11 were prime numbers, but she had not made the connection that, as prime numbers, they would be present in prime factorizations if they were factors of M. To determine whether M was divisible by 2, Zoe checked the partial products of all the factors of *M* to make sure none of them were even numbers. She seemed to do the same for 11, but when I questioned her about it she seemed more conflicted.

Due to Zoe’s evidence misconceptions about prime numbers and factoring, I recognized some of the connections between Zoe’s understanding of number theory and her PCK related to number theory. For instance, in Zoe’s response to Problem 2, although she wanted to design a task for Shayna to explore the idea that 1 is not a prime number, she did not know of one. Thus, she resorted to suggesting to Shayna that 1 is just a “special case”. Here, Zoe’s insufficient and faulty understanding of the number theory concept resulted in a lack of PCK. In Problem 4, Zoe’s suggestion for students to use groupings or factor trees to determine primality also resulted in a lack of PCK due to her own misconceptions.

In Problem 3, Zoe claimed that she would ask Magnus questions about his definition of “prime” and help him to see that 6 does not meet the criteria of the definition. This technique stresses the importance of establishing what a student *does* understand and building on it, evidence of general pedagogy. While this is also evidence of mathematical PCK, because navigating mathematical definitions is a general mathematical skill, it may not be evidence of PCK in number theory specifically. Zoe then described how she would have Magnus use manipulatives to make equal groups to show that 6 is not prime. While suggesting a task for Magnus to explore the concept shows PCK to some degree, Zoe’s inaccurate understanding of using groupings to determine primality interfered with her PCK.

Zoe's response to Problems 5 and 6 did demonstrate some substantial PCK without the interference of personal misconceptions. In Problem 5, Zoe claimed that she would ask students to explain their own factoring methods and critique each other so that they would recognize the validity of each other's method. This may be evidence of more general mathematical PCK. In Problem 6, Zoe was more specific about how she could convince students that there are infinitely many primes. She felt that she could appeal to students' understanding of the cardinality of the natural numbers to help them understand. She also posed a series of questions that she could ask students to think about this idea. This may have demonstrated more number theory PCK than general mathematical PCK as it clearly connected to her own understanding of the number theory concept.

Answers to the research questions. The interview questions sets revealed an abundance of evidence concerning Amy, Jen, Mia, and Zoe's understanding of number theory and how it related to their number theory PCK, but a more complete study may have revealed a great deal more about these connections. For instance, Zoe repeatedly confused factors and multiples. Had she also participated in the GCF/LCM interview, I may have been able to reveal more about this phenomenon. This may have also helped me to decipher her questionable suggestions to students, concerning factoring in particular. So, while these question sets have revealed a great deal of rich and interesting data, I may be able to divine more definitive results from a larger study with more data.

However, the data has suggested much about the nature of participants' number theory content and pedagogical content understandings and the relationship between the two. While the sets of tasks are limited to certain number theory topics, it was clear that participants' content understanding was much more robust when it was flexible and connected. It was also evident that participants' number theory PCK was informed heavily by their content understandings, but also by their experiences with that content. In fact, participants' PCK was most robust when it was informed by a strong understanding of number theory accompanied by a related experience that resonated with them.

There were occasions where participants' number theory understandings did not inform their PCK, but that was most evident in cases where participants were demonstrating what I referred to as *general mathematical PCK*, or PCK that was easily transferable to other mathematical situations. For instance, Zoe suggested that she would ask Magnus to recall the definition of prime and determine whether or not the number 6 met the definition. This strategy for responding to students could just as easily apply to another mathematical context.

My findings also suggest that participants' KCT may be informed their interpretations of student understanding, i.e., their KCS. However, this suggestion is only based off few instances per interview. It is likely that more data would strengthen this argument rather than invalidate it. In contrast, I did not find much evidence that participants' KCT informed their KCS. However, I would anticipate, as Deon (2009) did, that this may occur frequently in the classroom. One instance where I did see this relationship was in Amy's response to Mark. She suggested that she would ask Mark to justify his claim that the LCM of two numbers is equal to their product (KCT). Amy then listed a couple possible responses from Mark, determined Mark's understanding of LCM from those responses (KCS), and then proposed a couple of appropriate scaffolding questions and tasks according to Mark's potential understandings (KCT).

Thus I propose the general model for the nature of preservice elementary teachers understanding of number theory, as seen in Figure A12. It is far more generalized and simplistic than what the actual data suggests, but the data were limited by the tasks that I posed. I assert that given the opportunity, i.e., the appropriate tasks, there is potential for this model to be representative of preservice elementary teachers' understandings. However, for now, I can only support this model with evidence from the limited domain resulting from my interview tasks.

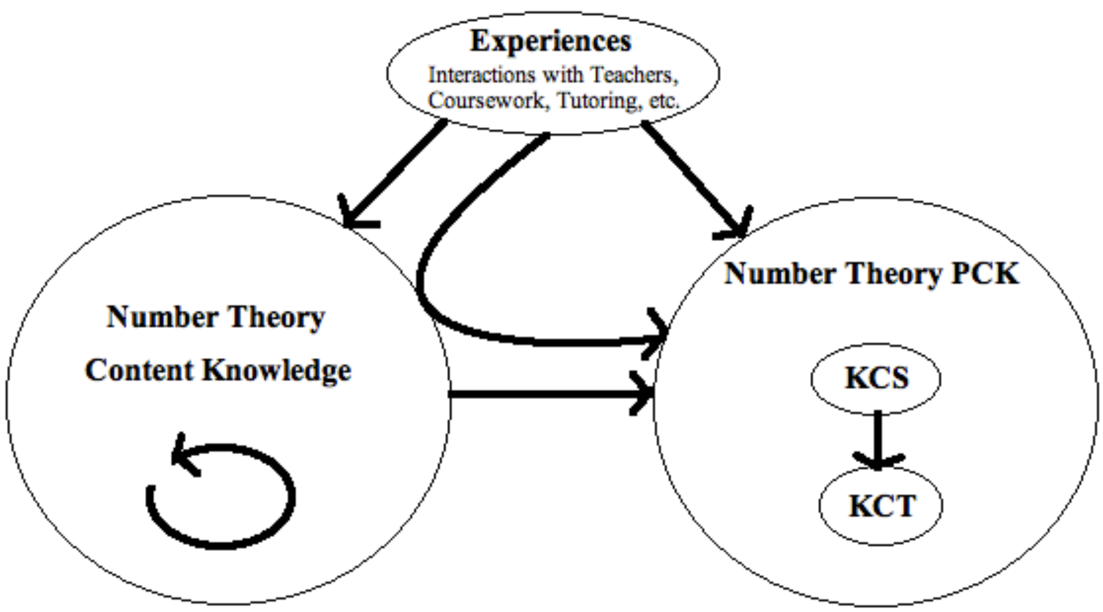


Figure A12. General model for preservice elementary teachers' understanding of number theory. The circles represent knowledge domains, while the arrows represent the relationships between the domains. "PCK" refers to pedagogical content knowledge, "KCS" refers to knowledge of content and students, and "KCT" refers to knowledge of content and teaching.

Implications

While valuable, my pilot study data is hardly conclusive. With so few participants per interview question set, this pilot study does not have the desired breadth of a dissertation. With so few data per participants, this pilot study hardly accomplishes the depth of a case study either. Also, while I found evidence of number theory PCK and content knowledge for each participant, it was challenging to establish relationships between the two without more data. However, I feel that my results are strongly suggestive of what I might find in a larger and deeper study. To assure the richest data possible, I have described alterations to the existing interview questions, complete with supporting documentation. I have also outlined potential findings in my dissertation study, citing examples from my pilot study and the literature.

APPENDIX B

INSTITUTIONAL REVIEW BOARD APPROVAL

Request for IRB Change

UNIVERSITY of NORTHERN COLORADO

Submit this request and all attachments to Sherry May, IRB Administrator, Office of Sponsored Programs, Kepner Hall, Suite #25

Date of Original UNC IRB Approval: 04/21/2011

Project Title: Preservice Elementary Teachers' Understanding of Number Theory

Lead Investigator
Name: Kristin Noblet
School: Mathematical Sciences
Email: kristin.noblet@unco.edu
Phone: (970) 351-2235

Research Advisor (if applicable)
Name: Dr. Hortensia Soto-Johnson
School: Mathematical Sciences
Email: hortensia.soto@unco.edu
Phone: (970) 351-2425

On a separate page, describe and provide justification for the changes being proposed. Be concise and specific in describing methodological changes that affect the experience of participants and/or relate to the risks/benefits of participation. Explain why these changes are necessary.

⦿ Yes ○ No The proposed changes in protocol will necessitate changes in documents such as recruitment flyers, consent forms, debriefing forms, or other project-related documents.

⦿ Yes ○ No If yes, copies of the revised documents with changes highlighted are attached to this request.

CERTIFICATION OF LEAD INVESTIGATOR
I certify that information contained in this request is complete and accurate.

[signature] 12/12/11
Signature of Lead Investigator Date

CERTIFICATION OF RESEARCH ADVISOR (If Lead Investigator is a Student)
I certify that information contained in this request is complete and accurate.

Hortensia Soto-Johnson 12/12/11
Signature of Research Advisor Date

emailed 12-21 & 1-17

Approved by: [signature] 1-17-12

Chairperson, Institutional Review Board Date

emailed 12-21-11
Clear Form

Date Request Received by OSP: 12/15/11

UNIVERSITY of
NORTHERN COLORADO
Institutional Review Board (IRB)

March 28, 2011

TO: Megan Babkes Stellino
School of Sport and Exercise Science

FROM: The Office of Sponsored Programs

RE: Exempt Review of *Preservice Elementary Teachers' Understanding of Number Theory,* submitted by Kristin Noblet (Research Advisor: Hortensia Soto-Johnson)

The above proposal is being submitted to you for exemption review. When approved, return the proposal to Sherry May in the Office of Sponsored Programs.

I recommend approval.

Megan B Stellino 4/21/11
Signature of Co-Chair Date

The above referenced prospectus has been reviewed for compliance with HHS guidelines for ethical principles in human subjects research. The decision of the Institutional Review Board is that the project is exempt from further review.

IT IS THE ADVISOR'S RESPONSIBILITY TO NOTIFY THE STUDENT OF THIS STATUS.

Comments:
- email 4/19
- slight modification to consent form(s)

25 Kepner Hall ~ Campus Box #143
Greeley, Colorado 80639
Ph: 970.351.1907 ~ Fax: 970.351.1934

APPENDIX C

INSTRUCTOR CONSENT TO PARTICIPATE IN RESEARCH FORM

INSTRUCTOR CONSENT FOR PARTICIPATION IN RESEARCH

Project Title: Preservice Elementary Teachers' Understanding of Number Theory
Researcher: Kristin Noblet, School of Mathematical Sciences, kristin.noblet@unco.edu
Research Supervisor: Dr. Hortensia Soto-Johnson, hortensia.soto@unco.edu, 970-351-2425

For my dissertation, I am investigating preservice elementary teachers' content and pedagogical content knowledge of number theory topics taught at the elementary level. As part of my data collection, I would like to observe your Math 391 course. I would focus my field notes on the content of the course, the methods with which it is taught, the opportunities you provide students to develop pedagogical content knowledge, and the ways in which you and your students communicate your understanding of number theory.

I also wish to solicit participation from the preservice elementary teachers in your class. At the beginning of the semester, I would like to ask for volunteers to allow me to photocopy their exams and other assignments. To ensure confidentiality, I would need to collect all student work prior to scoring, photocopy the participants' work, then turn in all student work to you. Towards the end of the semester, I will also ask 6 students to participate in one-on-one task-based interviews, but this will occur outside of class. Interview tasks will consist of elementary number theory problems, hypothetical student scenarios, modeling number theory ideas, and reflective questions about learning and understanding number theory. I foresee no risk to you or your students beyond those you normally encounter in a college classroom.

I am asking for your permission to observe your Math 391 in Fall 2011 and to collect student work from you. As a benefit to you, I am willing to help answer students' questions, develop coursework materials, etc., with the exception of being involved in grading student work. You will also receive a copy of my dissertation after its completion. By signing this letter, you are agreeing to allow me to observe your course and collect student work.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

Sincerely,

Kristin Noblet

______________________________ ______________________________ ________
Instructor's Name (please print) Instructor's Signature Date

______________________________ __________
Researcher's Signature Date

____________________________ __________
Research Supervisor's Signature Date

APPENDIX D

INFORMED CONSENT TO PARTICIPATE IN RESEARCH FORM A

INFORMED CONSENT FOR PARTICIPATION IN RESEARCH A

Project Title: Preservice Elementary Teachers' Understanding of Number Theory
Researcher: Kristin Noblet, School of Mathematical Sciences, kristin.noblet@unco.edu
Research Supervisor: Dr. Hortensia Soto-Johnson, hortensia.soto@unco.edu, 970-351-2425

For my dissertation, I am investigating future elementary teachers' understanding of number theory topics taught at the elementary level. As part of my data collection, I will be observing your Math 391 course. I will focus my notes on the content of the course, the methods with which it is taught, the opportunities you have to develop knowledge for teaching number theory, and the ways in which you, your classmates, and your instructor communicate your understandings of number theory.

I would like volunteers to allow me to photocopy their exams and other assignments throughout the semester. To ensure confidentiality, I will collect all student work prior to scoring, photocopy participants' work, then turn in your work to your instructor. I will blackout participants' names and replace them with pseudonyms. The key matching participants' names to their pseudonyms will be available to me alone, and it will be destroyed at the end of the semester. I will store hardcopies of participants' work in a locked filing cabinet, and I may retain this de-identified data indefinitely for future use. Towards the end of the semester, I will ask 6 of the students who have allowed me to photocopy their work to participate in one-on-one task-based interviews. Interviews will take 60-90 minutes, depending on the depth of participants' responses, and interview tasks will consist of elementary number theory problems, hypothetical student scenarios, modeling number theory ideas, and reflective questions about learning and understanding number theory. I foresee no risks to participants beyond those that are normally encountered in a classroom setting.

I am asking for your permission to collect and photocopy your Math 391 coursework in Fall 2011. As a benefit to you, you have the chance to participate in an interview later on this semester, for which you will be compensated with your choice of either a $15 gift certificate to Starbucks or a $15 gift certificate to Barnes & Noble. If you are not selected to participate in an interview, you will be compensated with your choice of either a $5 gift certificate to Starbucks or a $5 gift certificate to Barnes & Noble. Upon request, I will also debrief you with my findings after the completion of my research. By signing this letter, you are agreeing to allow me to collect your Math 391 coursework.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

Sincerely,

Kristin Noblet

____________________________ ____________________________ __________
Participant's Name (please print) Participant's Signature Date

____________________________ _________
Researcher's Signature Date

__________________________ _________
Research Supervisor's Signature Date

APPENDIX E

SAMPLE EMAIL: INTERVIEW PARTICIPANT SOLICITATION

SAMPLE EMAIL: INTERVIEW PARTICIPANT SOLICITATION

Hello,

My name is Kristin Noblet, and I am a graduate student at UNC in the School of Mathematical Sciences. You are receiving this email because Dr. Leth (your Math 391 instructor) informed me that you successfully completed Math 391 in Fall 2010 and that you might be interested in participating in a study that I am conducting.

For my dissertation, I am investigating future elementary teachers' understanding of number theory topics taught at the elementary level. To do so, I am looking for past Math 391 students to participate in one-on-one task-based interviews. Interview tasks will consist of elementary number theory problems, hypothetical student scenarios, modeling number theory ideas, and reflective questions about learning and understanding number theory. The interviews will take between 60 and 90 minutes, they will be audio-taped, I will conduct them on campus, and the time at which I conduct the interview is entirely up to the participant.

To compensate you for your time, participants will receive their choice of either a $15 gift certificate to Starbucks or a $15 gift certificate to Barnes & Noble. Upon request, I will also debrief participants of my findings when my study is complete. Participation is voluntary. You may decide not to participate in this study and if you begin participation, you may still decide to stop and withdraw at any time. Your participation in this study would be confidential; I will assign you a pseudonym and use it in my documentation.

So, if you are willing to participate, please respond within the week so that we can schedule an interview before the end of the semester. Thank you for your consideration,

Kristin Noblet

APPENDIX F

INFORMED CONSENT TO PARTICIPATE IN RESEARCH FORM B

INFORMED CONSENT FOR PARTICIPATION IN RESEARCH B

Project Title: Preservice Elementary Teachers' Understanding of Number Theory
Researcher: Kristin Noblet, School of Mathematical Sciences, kristin.noblet@unco.edu
Research Supervisor: Dr. Hortensia Soto-Johnson, hortensia.soto@unco.edu, 970-351-2425

For my dissertation, I am investigating future elementary teachers' understanding of number theory topics taught at the elementary level. I would like volunteers to participate in one-on-one task-based interviews. Interviews will take place during times that are convenient to participants, and they will be conducted in a conference room or classroom on campus. Interview tasks will consist of elementary number theory problems, hypothetical student scenarios, modeling number theory ideas, and reflective questions about learning and understanding number theory. Depending on the depth of participants' responses, these interviews will take between 60 and 90 minutes. So that I may transcribe the interviews at a later time, I will need to audio-record them. To ensure confidentiality, I will only refer to participants with pseudonyms on the audio-recording and in the interview transcription. The key matching participants' names to their pseudonyms will be available to me alone, and it will be destroyed at the end of the semester. Audio-recordings will be stored on my personal, password-protected computer and they will be destroyed within 5 years. I may retain de-identified data, like transcriptions and notes, indefinitely for future use. I foresee no risks to participants beyond those that are normally encountered in a classroom setting.

I am asking for your participation in an audio-recorded one-on-one task based interview that will take between 60 and 90 minutes to complete. As a benefit to you, you will be compensated with your choice of either a $15 gift certificate to Starbucks or a $15 gift certificate to Barnes & Noble. Upon request, I will also debrief you with my findings after the completion of my research. By signing this letter, you are agreeing to allow me to interview you and to audio-record that interview.

Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, 25 Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

Sincerely,

Kristin Noblet

______________________________ ______________________________ __________

Participant's Name (please print) Participant's Signature Date

______________________________ __________

Researcher's Signature Date

______________________________ __________

Research Supervisor's Signature Date

APPENDIX G

FIRST INTERVIEW QUESTION SET

FIRST INTERVIEW QUESTION SET

This interview will last 60-90 minutes, depending on how much you have to say about a task. With your permission, I will audio-tape it so that I can transcribe the interview later on. The tasks will consist of elementary number theory problems, questions pertaining to hypothetical student scenarios and modeling number theory ideas, and questions that ask you to reflect on your responses. Throughout the interview, I might ask you questions that get you to clarify or elaborate on your responses. If you get stuck, we can move on and come back or skip questions. I will neither confirm nor deny any of your responses until after the interview is complete. But once we're done, we can go through and talk about any of the problems you'd like. Try not to use a calculator. If you must, I would like you to really articulate what you're doing and why.

Before we start… I have a few quick background questions.
Can you tell me which Math, Math Ed, and Education classes you've taken or are currently taking?
Can you tell me a little about any math tutoring or teaching experiences you have had?

1. a. Create a story problem that would require one to compute the least common multiple of 6 and 8.
   b. What is your reasoning for phrasing your story problem like this?
   c. Use a picture or diagram to help illustrate this idea. How does this diagram represent the idea of LCM?
   d. How might you use manipulatives to model this idea? What made you think of this?
   e. Which of the following story problems represents the least common multiple of 6 and 8? Why or why not?
      a. Mario has 6 bags of 8 marbles each. How many total marbles does he have?
      b. Brandon and Matteo are building train tracks. Brandon is using 6-inch sections of track while Matteo is using 8-inch sections. What is the shortest length of track that both boys can make?
      c. Janet has 6 skirts and 8 blouses. How many different ways can she wear a skirt with a blouse?
      d. Light A blinks every 6 minutes, while light B blinks every 8 seconds. If both lights just blinked simultaneously, in how many seconds will they blink together again?

   Note: If the participant has trouble creating a story problem, ask the following prompts: What is LCM? How does one use it? Can you think of a context in which it might be useful to find the LCM? Etc. Ask (a), (b), (c), & (d) verbally. Have (e) typed out on a separate piece of paper.

2. a. Create a story problem that would require one to compute the greatest common divisor of 28 and 32.

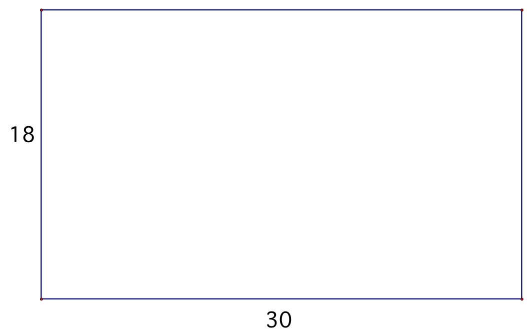
b. What is your reasoning for phrasing your story problem like this?
c. Use a picture or diagram to help illustrate this idea. How does this diagram represent the idea of GCF?
d. How might you use manipulatives to model this idea? What made you think of this?
e. Which of the following story problems represents the greatest common factor of 12 and 14? Why or why not?
    a. Maria has 12 dark chocolates and 14 milk chocolates. She wants to make goodie bags so that each bag has the same number of dark chocolates and each bag has the same number of milk chocolates. If Maria wants to use all of the chocolates, what is the most number of goodie bags that she can make?
    b. Lee goes grocery shopping every 12 days and Andrew goes grocery shopping every 14 days. If they both went shopping today, how many days will it be before they are shopping on the same day again?
    c. Carys is making boxes of oatmeal raisin cookies and boxes of chocolate chip cookies to sell at a bake sale. She wants each box to contain the same number of cookies without any cookies left over, and she wants each box to have as many cookies possible. If Carys has 12 oatmeal raisin and 14 chocolate chip cookies, how many cookies should go in each box?
    d. Carys is making boxes of oatmeal raisin cookies and boxes of chocolate chip cookies to sell at a bake sale. She wants each box to contain the same number of cookies without any cookies left over, and she wants to make as few boxes as possible. If Carys has 12 oatmeal raisin and 14 chocolate chip cookies, how many cookies should go in each box?

Note: If the participant has trouble creating a story problem, ask the following prompts: What is GCF? How does one use it? Can you think of a context in which it might be useful to find the GCF? Etc. Ask (a), (b), (c), & (d) verbally. Have (e) typed out on a separate piece of paper.

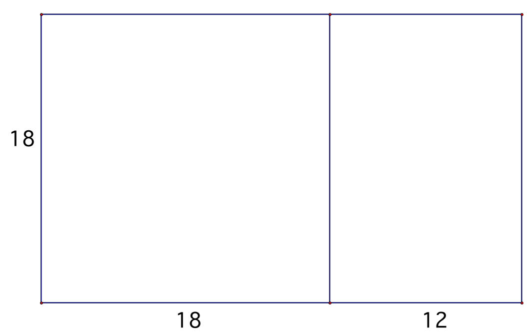
3. Mark claims that to find the least common multiple of any two numbers, A and B, all you have to do is multiply the numbers together.
    a. Will Mark's method always work? How do you know?
    b. Why might a student believe this method to be valid?
    c. Under which conditions might Mark's idea work? How do you know?
    d. How might you respond to Mark to help him recognize his misconception(s)?
    e. How did you know to respond to Mark in this way?

4. Eva claims that she has found a new method for finding GCF. Using GCF(18, 30) as an example, she draws the following diagrams, shown in stages, and claims the GCF is 6.

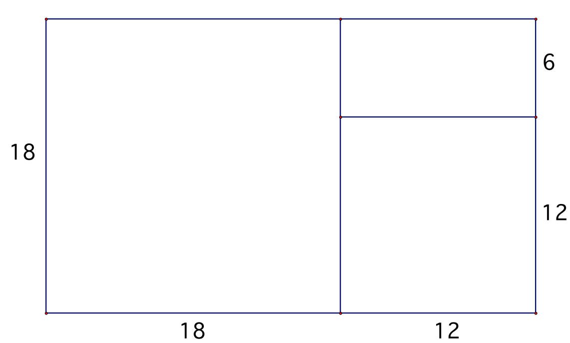
Stage 1: Eva drew an 18 by 30 rectangle.



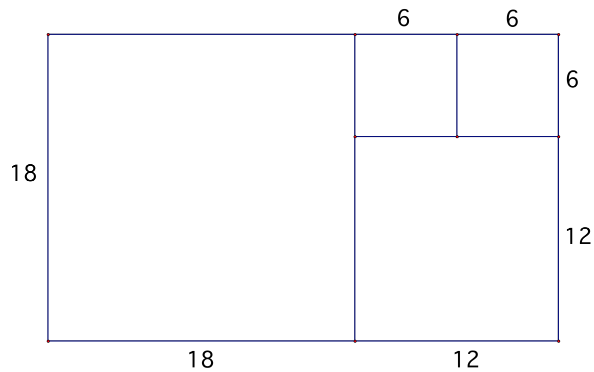
Stage 2: Eva broke up the rectangle
into a square and a 12 by 18 rectangle.



Stage 3: Eva broke up the *new* rectangle
into a square and a 6 by 12 rectangle.



Stage 4: Eva broke up the *new* rectangle into two 6 by 6 squares. Then, she announced that the GCF(18,30)=6.



Does Eva's method always work? How do you know? Under which conditions might her idea work? How do you know? Why would a student believe that it might not work?

5. The greatest common factor of A and B is 42, and the least common multiple of A and B is 2352. If A = 336, what is B? How did you arrive at this answer?

   Note: Participants may need a calculator for this problem.

6. Certain concepts in number theory appear in many other areas of mathematics. In which topics do you think GCF and LCM might play a role? How?

   a. Do GCF and LCM play a role in adding fractions? Do students NEED to use GCF/LCM when they add fractions, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when adding fractions?
   b. Do GCF and LCM play a role in multiplying fractions? Do students NEED to use GCF/LCM when they multiply fractions, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when multiplying fractions?
   c. Do GCF and LCM play a role in working with ratios? Do students NEED to use GCF/LCM when they work with ratios, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when working with ratios?
   d. Do GCF and LCM play a role in dividing fractions? Do students NEED to use GCF/LCM when they divide fractions, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when dividing fractions?

7. Remi, one of you students sees the following question and immediately pulls out his calculator to solve it:

$$\frac{18}{51}+\frac{11}{34}$$

Why do you think Remi might have that kind of reaction/aversion to solving this problem by hand? What kind of questions could you ask Remi to help guide him through this problem without a calculator? How did you know to respond to Remi in this way?

Follow-up #1: What prior knowledge would students need to create story problems about LCM and GCF? Why might they struggle to create story problems about LCM and GCF? How might you, as a teacher, help them to overcome that struggle?

Follow-up #2: In answering these questions, have you drawn any ideas from your experience in Math 391? Which ones and how so? Has any of your other coursework helped you in answering these questions? Which courses, which problems, and how so?

APPENDIX H

SECOND INTERVIEW QUESTION SET

SECOND INTERVIEW QUESTION SET

This interview will last 60-90 minutes, depending on how much you have to say about a task. With your permission, I will audio-tape it so that I can transcribe the interview later on. The tasks will consist of elementary number theory problems, questions pertaining to hypothetical student scenarios and modeling number theory ideas, and questions that ask you to reflect on your responses. Throughout the interview, I might ask you questions that get you to clarify or elaborate on your responses. If you get stuck, we can move on and come back or skip questions. I will neither confirm nor deny any of your responses until after the interview is complete. But once we're done, we can go through and talk about any of the problems you'd like. Try not to use a calculator. If you must, I would like you to really articulate what you're doing and why.

1. (a) Create a story problem that would require one to divide 12 by 4.
   (b) What is your reasoning for phrasing your story problem like this?
   (c) Use a diagram or picture to illustrate this idea. How does the diagram represent 12 divided by 4?
   (d) How might you use manipulatives to model this idea? What made you think of this?
   (e) Which of the following is a story problem for 12 divided by 4?
      i. Melanie had 12 apples, and she feeds her horse 4 apples per day. In how many days will Melanie run out of apples?
      ii. Leah had 12 dolls and she split them up evenly between 4 chairs. How many dolls did she place at each chair?
      iii. Nolan was creating teams for a 4-mile relay race. If each team member needed to run 12 kilometers of the race, how many people should there be on each team?
      iv. Bethany is making cakes for a bake sale, and each cake requires 4 cups of flour. If she only has 12 cups of flour, how many cakes can Bethany make?

2. You have asked your students to create LCM story problems about light A and light B that blink every 4 seconds and every 6 seconds, respectively. Two of your students' story problems are provided below:

   (a) Light A blinks every 4 seconds and Light B blinks every 6 seconds. When will they blink at the same time?
   (b) Two lights, one that blinks every 4 seconds and one that blinks every 6 seconds, have just blinked at the same time. How many more times will each light blink before they blink together again?

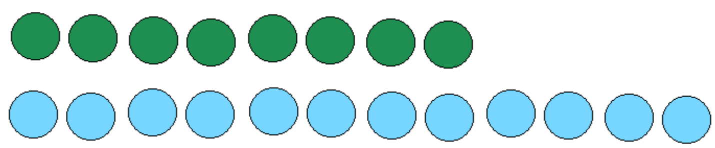
   Critique each story problem. Does is accurately represent the LCM of 6 and 8? Why or why not?

3. You have asked your students to create GCF story problems about making single-colored bunches of balloons with 8 red balloons and 12 white balloons. Two of your students' story problems are provided below:

   (a) Jenny has 8 red balloons and 12 white balloons. She wants to make bunches of balloons so that each bunch is all one color and each bunch has the same number of balloons. If Jenny wants to use up all of her balloons, how many balloons should be in each bunch?
   (b) The Party Store wants to sell bunches of balloons that are all one color and have the same number of balloons per bunch. An employee is creating bunches using 8 red balloons and 12 white balloons. What is the most number of bunches he can make?

   Critique each story problem. Does is accurately represent the GCF of 8 and 12? Why or why not?

4. You have given each of your students 8 green chips and 12 blue chips and asked them to use the chips to find the GCF of 8 and 12. When she paired up the green and blue chips, Maria, noticed that there were 4 blue chips left.



   As a result of this observation, Maria then tried making groups of 4 green chips and 4 blue chips. When she was successful, Maria conjectured that the difference between any two numbers is also their GCF.

   (a) Is Maria's conjecture valid? How do you know?
   (b) Is the GCF of two numbers related to their difference at all? How do you know?

5. At a factory, the production of a certain part takes n hours (where n is a whole number). Production starts at 12 o'clock on opening day and continuously puts out one part after another. For what values of n will a part ever be completed at exactly 1 o'clock?

   Follow-Up: Suppose this factory is relocated to a planet where clocks are split up into m hour-intervals. If production starts at the m^{th} hour on opening day, for what values of n will a part ever be completed exactly when the clock strikes the 1^{st} hour?

   Follow-Follow-Up: Prove it.

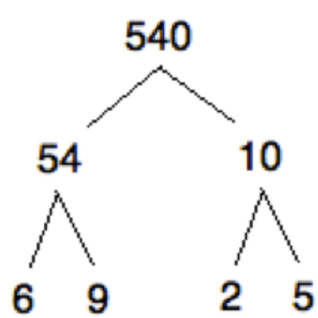
6. What is a prime number? Are they important? Why are they important? What role do they play in mathematics?

7. Shayna, one of your 4th grade students, insists that the number 1 is a prime number because its only factors are 1 and itself.
   a. Is Shayna correct? Why or why not?
   b. Describe the concepts that she understands.
   c. If Shayna is incorrect, how could you respond to help her recognize her misconception(s)?
   d. How did you know to respond to Shayna in this way?

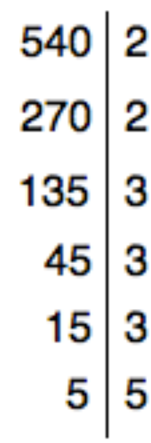
8. Magnus, one of your 4th grade students, claims that the number 6 is prime because it only has two factors, 2 and 3, which are also prime.
   a. What mathematical ideas does Magnus understand?
   b. What mathematical misconceptions does Magnus have?
   c. How might you help Magnus to resolve his conceptions?
   d. How did you know to respond to Magnus in this way?

9. Talisa and Tom factored 540 separately. There work is shown below. They both argue that their own method is the correct one.
   a. Why might they be having this conflict?
   b. As a teacher, how might you help them resolve this?
   c. How did you know to respond to Talisa and Tom in this way?
   d. Are both methods valid? Why or why not?
   e. How are these methods similar? How are they different?

Talisa's Work



Tom's Work



10. Consider the number $M = 3^3 \times 5^2 \times 7$. Is M divisible by 2, 7, 9, 11, 14, 15, 26 or 63? For each number, explain how you know that it is or is not a divisor of M.

    Follow-Up: Now consider $N = 2 \times 3^2 \times 5^3 \times 13$. What is the GCF of M and N? LCM? (If they use a procedure, ask about rationale)

11. How can you determine whether a large whole number N is prime?
    a. What's the most efficient way to determine whether N is prime?
    b. Is N = 853 prime? How do you know?
    c. What manipulatives, diagrams, etc. could you use to help students determine whether a large number (less than 100) is prime?

12. a. How many primes are there? How do you know?
    b. Are you convinced of that? Why?
    c. How might you convince a 5/6th grade class that there are infinitely many primes?

Follow-up #1: In answering these questions, have you drawn any ideas from your experience in Math 391? Which ones and how so? Has any of your other coursework helped you in answering these questions? Which courses, which problems, and how so?

APPENDIX I

PILOT STUDY INTERVIEW TASKS

PILOT STUDY INTERVIEW TASKS

GCF/LCM QUESTION SET

1. a. Create a story problem that would require one to compute the least common multiple of 6 and 8.
   b. Use a picture or diagram to help illustrate this idea.
   c. How might you use manipulatives to model this idea?
   d. Which of the following story problems represents the least common multiple of 6 and 8? Why or why not?
      e. Mario has 6 bags of 8 marbles each. How many total marbles does he have?
      f. Janet has 6 skirts and 8 blouses. How many different ways can she wear a skirt with a blouse?
      g. Beth has 6 red apples and 8 green apples. She is making fruit baskets, and each one needs an equal number of each type of apple. What is the most number of fruit baskets that Beth can make?
      h. Light A blinks every 6 minutes, while light B blinks every 8 seconds. If both lights just blinked simultaneously, in how many seconds will they blink together again?

Note: If the participant has trouble creating a story problem, ask the following prompts: What is LCM? How does one use it? Can you think of a context in which it might be useful to find the LCM? Etc. Ask (a), (b), & (c) verbally. Have (d) typed out on a separate piece of paper.

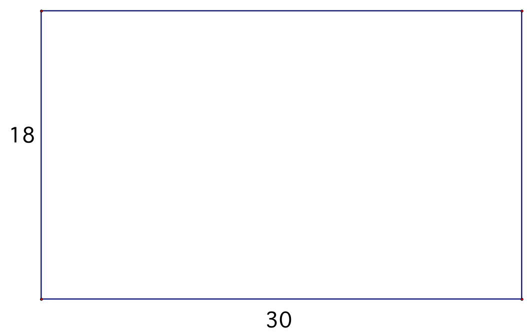
2. a. Create a story problem that would require one to compute the greatest common divisor of 28 and 32.
   b. Use a picture or diagram to help illustrate this idea.
   c. How might you use manipulatives to model this idea?
   d. Which of the following story problems represents the greatest common factor of 12 and 14? Why or why not?
      d. Maria has 12 dark chocolates and 14 milk chocolates. She wants to make goodie bags so that each bag has the same number of dark chocolates and the same number of milk chocolates. If Maria wants to use all of the chocolates, what is the most number of goodie bags that she can make?
      e. If Lee had 14 crackers and ate 12 of them, how many does he have left?
      f. Sarah is cutting paper to make fliers for the school dance. If each flier is 12 inches by 14 inches, how many square inches does Sarah need?
      g. The aquarium has 12 angelfish and 14 puffer fish. How many total fish did the aquarium have?

Note: If the participant has trouble creating a story problem, ask the following prompts: What is GCF? How does one use it? Can you think of a context in which it might be useful to find the GCF? Etc. Ask (a), (b), & (c) verbally. Have (d) typed out on a separate piece of paper.

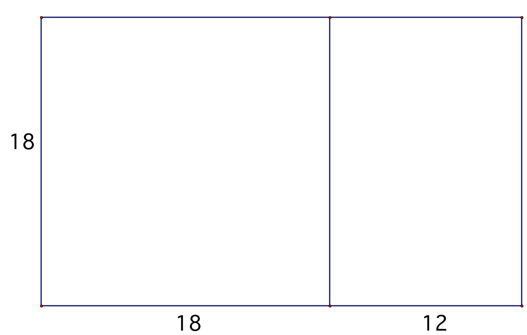
3. Mark claims that to find the least common multiple of any two numbers, A and B, all you have to do is multiply the numbers together.
   a. Will Mark's method always work? How do you know?
   b. Why might a student believe this method to be valid?
   c. Under which conditions might Mark's idea work? How do you know?
   d. How might you respond to Mark to help him recognize his misconception(s)?
   e. How did you know to respond to Mark in this way?

4. Eva claims that she has found a new method for finding GCF. Using GCF(18, 30) as an example, she draws the following diagrams, shown in stages, and claims the GCF is 6.

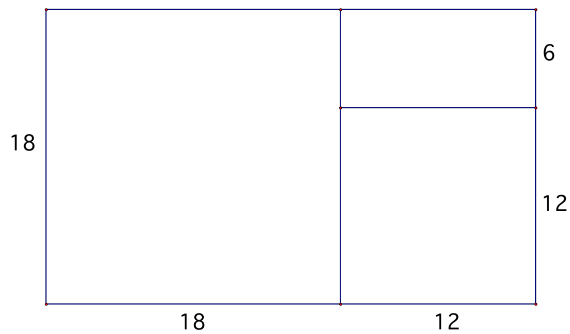
Stage 1: Eva drew an 18 by 30 rectangle.



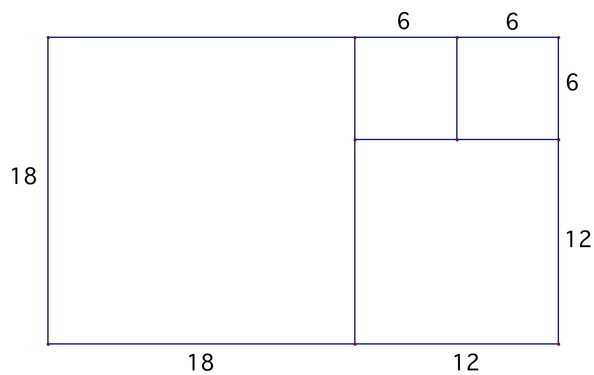
Stage 2: Eva broke up the rectangle into a square and a 12 by 18 rectangle.



Stage 3: Eva broke up the *new* rectangle into a square and a 6 by 12 rectangle.



Stage 4: Eva broke up the *new* rectangle into two 6 by 6 squares. Then, she announced that the GCF(18,30)=6.



Does Eva's method always work? How do you know? Under which conditions might her idea work? How do you know? Why would a student believe that it might not work?

5. The greatest common factor of A and B is 42, and the least common multiple of A and B is 2352. If A = 336, what is B? How did you arrive at this answer?

   Note: Participants may need a calculator for this problem.

6. Certain concepts in number theory appear in many other areas of mathematics. In which topics do you think GCF and LCM might play a role? How?

   a. Do GCF and LCM play a role in adding fractions? Do students NEED to use GCF/LCM when they add fractions, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when adding fractions?
   b. Do GCF and LCM play a role in multiplying fractions? Do students NEED to use GCF/LCM when they multiply fractions, or are there other ways of going about

it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when multiplying fractions?

c. Do GCF and LCM play a role in working with ratios? Do students NEED to use GCF/LCM when they work with ratios, or are there other ways of going about it? What are those other ways? Are there advantages/disadvantages to using GCF/LCM when working with ratios?

7. Remi, one of you students sees the following question and immediately pulls out his calculator to solve it:

$$\frac{18}{51}+\frac{11}{34}$$

Why do you think Remi might have that kind of reaction/aversion to solving this problem by hand? What kind of questions could you ask Remi to help guide him through this problem without a calculator? How did you know to respond to Remi in this way?

Follow-up #1: What prior knowledge would students need to create story problems about LCM and GCF? Why might they struggle to create story problems about LCM and GCF? How might you, as a teacher, help them to overcome that struggle?

Follow-up #2: In answering these questions, have you drawn any ideas from your experience in Math 391? Which ones and how so? Has any of your other coursework helped you in answering these questions? Which courses, which problems, and how so?

(Optional) Follow-up #3: Problem ___ did not go as well as I was hoping. Do you have any ideas as to how I might rephrase the prompts so that I could be clearer? So that you could have gotten further? Etc.

PRIME QUESTION SET

1. What is a prime number? Are they important? Why are they important? What role do they play in mathematics?

2. Shayna, one of your 4^{th} grade students, insists that the number 1 is a prime number because its only factors are 1 and itself.
   a. Is Shayna correct? Why or why not?
   b. Describe the concepts that she understands.
   c. If Shayna is incorrect, how could you respond to help her recognize her misconception(s)?
   d. How did you know to respond to Shayna in this way?

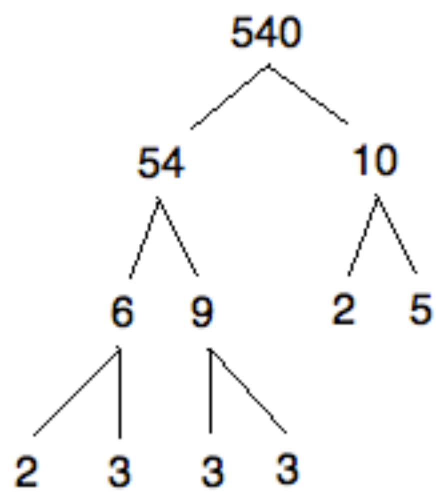
3. Magnus, one of your 4^{th} grade students, claims that the number 6 is prime because it only has two factors, 2 and 3, which are also prime.

a. What mathematical ideas does Magnus understand?
b. What mathematical misconceptions does Magnus have?
c. How might you help Magnus to resolve his conceptions?
d. How did you know to respond to Magnus in this way?

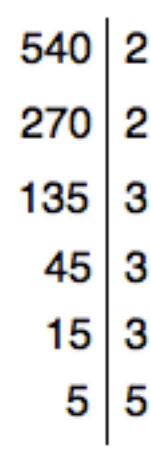
4. How can you determine whether a large whole number N is prime?
   a. What's the most efficient way to determine whether N is prime?
   b. Is N = 853 prime? How do you know?
   c. What manipulatives, diagrams, etc. could you use to help students determine whether a large number (less than 100) is prime?

5. Talisa and Tom factored 540 separately. There work is shown below. They both argue that their own method is the correct one.
   a. Why might they be having this conflict?
   b. As a teacher, how might you help them resolve this?
   c. How did you know to respond to Talisa and Tom in this way?
   d. Are both methods valid? Why or why not?
   e. How are these methods similar? How are they different?

Talisa's Work



Tom's Work



6. a. How many primes are there? How do you know?
   b. Are you convinced of that? Why?
   c. How might you convince a 5/6th grade class that there are infinitely many primes?

7. Consider the number $M = 3^3 \times 5^2 \times 7$. Is M divisible by 2, 7, 9, 11, 15 or 63? For each number, explain how you know that it is or is not a divisor of M.

Follow-up #1: In answering these questions, have you drawn any ideas from your experience in Math 391? Which ones and how so? Has any of your other coursework helped you in answering these questions? Which courses, which problems, and how so?

(Optional) Follow-up #2: Problem ___ did not go as well as I was hoping. Do you have any ideas as to how I might rephrase the prompts so that I could be clearer? So that you could have gotten further? Etc.

APPENDIX J

PILOT STUDY GREATEST COMMON FACTOR

AND LEAST COMMON MULTIPLE

INTERVIEW CODES

PILOT STUDY GCF/LCM INTERVIEW CODES

CONTENT – A statement was coded as "Content" if the participant referred to mathematical ideas such as definitions, rules, procedures, and examples, or ways of representing these ideas.

GCF – A statement was coded as "GCF" if the content statement specifically pertained to the greatest common factor of two numbers.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what GCF "is" or "means".

Method for Finding – A statement was coded as "method for finding" if the participant referred to or used a method for determining the GCF of two numbers.

Comparing Lists – A statement was coded "comparing lists" if the participant referred to using or compared lists of factors to determine the largest factor that two numbers have in common.

Prime Factorization – A statement was coded "prime factorization" if the participant referred to using or found the prime factorization of two numbers to determine their GCF.

Recognition – A statement was coded "recognition" if the participant declared the GCF of two numbers, as if by recall.

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing the GCF of two numbers. This category overlaps with "method for finding" since some participants used one or more of the models to find the GCF.

Story Problem – A statement was coded as "story problem" if a participant created, attempted to create, identified, or referred to a GCF story problem.

Pictorial – A statement was coded as "pictorial" if a participant created, attempted to create, identified, or referred to a pictorial representation of GCF. Some participants used a picture to determine the GCF of two numbers. As a result, this code also falls under the "method for finding" category.

Manipulatives – A statement was coded as "manipulatives" if a participant created, attempted to create, identified, or referred to a concrete representation of GCF. Some participants used or described how to use manipulatives to determine the GCF of two numbers. As a result, this code also falls under the “method for finding” category.

How Many Groups – A reference to a model was coded as “how many groups” if the divisor in the model determined the size of or number of objects within a group, thus making the quotient the number of groups that one could make from the dividend. Any of the models listed above could take the form of a “how many groups” model.

How Many in Each Group - A reference to a model was coded as “how many in each group” if the divisor in the model determined the number of equal groups one should make from the dividend, thus making the quotient the size of or the number of objects within a group. Any of the models listed above could take the form of a “how many groups” model.

Validation – A statement was coded “validation” if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to GCF.

Counterexample – A validation statement was coded “counterexample” if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded “verification” if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Relatively Prime – A statement was coded “relatively prime” if the participant referred to the case where the GCF of two numbers is 1.

LCM - A statement was coded as “LCM” if the content statement specifically pertained to the least common multiple of two numbers.

Personal Definition – A statement was coded as “personal definition” if the participant referred to what LCM “is” or “means”.

Methods for Finding – A statement was coded as “method for finding” if the participant referred to or used a method for determining the LCM of two numbers.

Comparing Lists – A statement was coded "comparing lists" if the participant referred to using or compared lists of multiples to determine the smallest multiple that two numbers have in common.

Prime Factorization – A statement was coded "prime factorization" if the participant referred to using or found the prime factorization of two numbers to determine their LCM.

Recognition – A statement was coded "recognition" if the participant declared the LCM of two numbers, as if by recall.

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing the LCM of two numbers. This category overlaps with "method for finding" since some participants used one or more of the models to find the LCM.

Story Problems – A statement was coded as "story problem" if a participant created, attempted to create, identified, or referred to a LCM story problem.

Pictorial – A statement was coded as "pictorial" if a participant created, attempted to create, identified, or referred to a pictorial representation of LCM. Some participants used a picture to determine the LCM of two numbers. As a result, this code also falls under the "method for finding" category.

Manipulatives – A statement was coded as "manipulatives" if a participant created, attempted to create, identified, or referred to a concrete representation of LCM. Some participants used or described how to use manipulatives to determine the LCM of two numbers. As a result, this code also falls under the "method for finding" category.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to LCM.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Relationship – A statement was coded "relationship" if the participant referred to or used the relationship between LCM and GCF in their reasoning or calculations.

OTHER – The content of a statement was coded "other" if the statement specifically pertained to a topic in mathematics other than GCF or LCM.

Fraction Addition/Subtraction – A statement was coded "fraction addition/subtraction" if the statement specifically referred to the topic of adding and subtracting fractions.

Simplifying Fractions – A statement was coded "simplifying fractions" if the statement specifically referred to the topic of simplifying fractions to lowest terms.

Prime – A statement was coded "prime" if the statement specifically referred to the topic of prime numbers.

SCK – Any "content" statement was also coded as "SCK", for "specialized content knowledge", if the participant demonstrated "mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Ball, Thames, & Phelps, 2008, p. 377-8). This differs from other mathematical knowledge in that it appears to be unique to the ways in which mathematics arises in the classroom, as opposed to other occupational environments.

PEDAGOGICAL CONTENT – A statement was coded as "pedagogical content" if the participant demonstrated or made reference to mathematical knowledge for teaching, i.e., pedagogical content knowledge (Shulman, 1986). This includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9).

KCS – A statement was coded "KCS" if it pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). This differs from SCK in that SCK plays a role in determining the mathematical accuracy of student work, while KCS is necessary for determining student understanding and depth of knowledge.

Student Solution Strategy – A statement was coded "student solution strategy" if the participant referred to *how* a student might solve a number theory problem.

Student Reasoning – A statement was coded "student reasoning" if the participant referred to *why* a student might '[believe a statement, claim, or conjecture about number theory is true or false.

Student Challenge – A statement was coded "student challenge" if the participant acknowledged a specific difficulty or misconception that a student might have related to a certain task or concept.

Prerequisite Knowledge – A statement was coded "prerequisite knowledge" if the participant referred to what is necessary for a student to know and understand before being able to successfully engage in a task or concept.

KCT – A statement was coded "KCT" if a participant demonstrated a "knowledge of content and teaching", as described by Ball, Thames, and Phelps, 2008. This includes knowing how to sequence the content for instruction, like which examples to use when introducing a topic versus when attempting to deepen students' understanding. KCT also includes being able to weigh the advantages and disadvantages of the different ways with which to represent mathematical concepts as well being able to make in-the-moment decisions about whether or not to pause for clarification or pose a new task to further student learning.

Scaffolding – A statement was coded "scaffolding" if the participant described how she or he might scaffold or guide a student's understanding by posing tasks that built on one another or through questioning techniques.

Curriculum – A statement was coded as "curriculum" if the participant demonstrated a knowledge of programs developed for the teaching of a mathematics, mathematical concepts covered at a given level, and instructional materials available. This includes knowledge of vertical curriculum, i.e., the knowledge of mathematics taught across grade levels.

Misconception – A statement was coded "misconception" if the participant demonstrated confusion or a misunderstanding about a concept. This code could have been used simultaneously with any other code.

APPENDIX K

PILOT STUDY PRIME NUMBER INTERVIEW CODES

PILOT STUDY PRIME NUMBER INTERVIEW CODES

CONTENT – A statement was coded as "Content" if the participant referred to mathematical ideas such as definitions, rules, procedures, and examples, or ways of representing these ideas.

Prime – A statement was coded as "prime" if the content statement specifically pertained to prime numbers.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what prime "is" or "means".

Determining Primality – A statement was coded as "determining primality" if the participant used or referred to a method for determining whether a number is prime.

Incomplete – A statement was coded "incomplete" if the participant used or referred to using a method for determining if a number is prime that would not definitively determine whether a number is prime.

Complete – A statement was coded "complete" if the participant used or referred to using a method for determining if a number is prime that was exhaustive and would definitively determine whether a number is prime.

Efficient – A complete method for determining the primality of a number was coded as "efficient" if the participant only suggested checking for divisibility by primes less than the square root of the number.

Inefficient – A complete method was coded as "inefficient" if the participant did not limit testing divisibility by primes or insisted on testing for divisibility by numbers larger than the square root of the number.

Method for Finding – A statement was coded as "method for finding" if a participant used or referred to a method for finding prime numbers.

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing a prime number.

Cardinality – A statement was coded as "cardinality" if a participant referred to the size of the set of prime numbers.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to prime numbers.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Factoring – A statement was coded as "factoring" if the content statement specifically pertained to factoring a whole number.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what factoring "is" or "means".

Methods – A statement was coded as "methods" if the participant referred to or used a method for factoring a whole number.

Tree – A statement was coded "tree" if the participant used or referred to using a factor tree to factor a whole number.

Pair – A statement was coded "pair" if the participant declared a factor pair of a whole number, as if by recall.

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for factoring whole numbers. This category overlaps with "methods" since some participants used models to factor.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to factoring.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Divisibility – A statement was coded “divisibility” if the content statement specifically pertained to whether a number was “divisible” by another, or whether a number is a factor of another number.

Personal Definition/Divisible – A statement was coded as “personal definition/divisible” if the participant referred to what divisible “is” or “means”.

Personal Definition/Factor – A statement was coded as “personal definition/factor” if the participant referred to what a factor “is” or “means”.

Methods – A statement was coded as “methods” if the participant referred to or used a method for determining the divisibility (i.e., identifying factors) of a whole number.

Guess and Check – A statement was coded “guess and check” if the participant used or referred to using a guess and check method for determining the divisibility of a whole number. The “checking” part of this method is conducted through division calculations, either by hand or via calculator. If the quotient is a whole number, the divisor is a factor.

Test – A statement was coded “test” if the participant used or referred to a divisibility test for determining factors of a whole number.

Factorization – A statement was coded “factorization” if the participant used or referred to the factorization (prime or partial) of a number to determine the divisibility of that number.

Recognition – A statement was coded as “recognition” if the participant identified factors of a number, as if by recall.

Modeling – A statement was coded as “modeling” if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for factoring whole numbers. This category overlaps with “methods” since some participants used models to factor.

Evenness – A statement was coded as ‘evenness” if the participant referred to a whole number as being even.

Last Digit – An evenness statement was coded as “last digit” if the participant referred to a characteristic of even numbers as having an even one’s digit.

Factor of 2 – An evenness statement was coded as “factor of 2” if the participant referred to a characteristic of even numbers as having a factor of 2 or being divisible by 2.

Validation – A statement was coded “validation” if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to LCM.

Counterexample – A validation statement was coded “counterexample” if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded “verification” if the participant verified the case(s) in which the claim, conjecture, or proof are true.

OTHER – The content of a statement was coded “other” if the statement specifically pertained to a topic in mathematics other than prime numbers, factoring or divisibility.

SCK – Any “content” statement was also coded as “SCK”, for “specialized content knowledge”, if the participant demonstrated “mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Ball, Thames, & Phelps, 2008, p. 377-8). This differs from other mathematical knowledge in that it appears to be unique to the ways in which mathematics arises in the classroom, as opposed to other occupational environments.

PEDAGOGICAL CONTENT – A statement was coded as “pedagogical content” if the participant demonstrated or made reference to mathematical knowledge for teaching, i.e., pedagogical content knowledge (Shulman, 1986). This includes “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching” (p. 9).

KCS – A statement was coded “KCS” if it pertained to “students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems” (Hill, Schilling, & Ball, 2004). This differs from SCK in that SCK plays a role in determining the mathematical accuracy of student work, while KCS is necessary for determining student understanding and depth of knowledge.

Student Solution Strategy – A statement was coded "student solution strategy" if the participant referred to *how* a student might solve a number theory problem.

Student Reasoning – A statement was coded "student reasoning" if the participant referred to *why* a student might believe a statement, claim, or conjecture about number theory is true or false.

Student Challenge/Error – A statement was coded "student challenge/error" if the participant acknowledged a specific difficulty or misconception that a student might have related to a certain task or concept.

Prerequisite Knowledge – A statement was coded "prerequisite knowledge" if the participant referred to what is necessary for a student to know and understand before being able to successfully engage in a task or concept.

KCT – A statement was coded "KCT" if a participant demonstrated a "knowledge of content and teaching", as described by Ball, Thames, and Phelps, 2008. This includes knowing how to sequence the content for instruction, like which examples to use when introducing a topic versus when attempting to deepen students' understanding. KCT also includes being able to weigh the advantages and disadvantages of the different ways with which to represent mathematical concepts as well being able to make in-the-moment decisions about whether or not to pause for clarification or pose a new task to further student learning.

Scaffolding – A statement was coded "scaffolding" if the participant described how she or he might scaffold or guide a student's understanding by posing tasks that built on one another or through questioning techniques.

Curriculum – A statement was coded as "curriculum" if the participant demonstrated a knowledge of programs developed for the teaching of a mathematics, mathematical concepts covered at a given level, and instructional materials available. This includes knowledge of vertical curriculum, i.e., the knowledge of mathematics taught across grade levels.

Age Appropriate – A statement was coded as "age appropriate" if the participant made a reference to the age or grade level appropriateness of a particular concept or task. This pertains to curricular PCK, because it is a veiled reference to the number theory curriculum at a given grade level.

Misconception – A statement was coded "misconception" if the participant demonstrated confusion or a misunderstanding about a concept. This code could have been used simultaneously with any other code.

APPENDIX L

SAMPLE PILOT STUDY CODING: AMY'S CODED INTERVIEW

SAMPLE PILOT STUDY CODING: AMY'S CODED INTERVIEW

Problem 3

Who	What	Content				PCK			Subcode
		GCF	LCM	Oth.	SCK	KCS	KCT	Curr	
Amy	I know it doesn't always work. I'm trying to think of a scenario when it doesn't.		X		X				validation
Amy	It always works when the two numbers are relatively prime and they have no other factors in common. It works to just multiply the numbers together.	X	X		X				validation, relatively prime, relationship
Amy	But when the numbers have something in common, say like 4 and 6. If you multiply them together, you get 24,		X		X				validation
Amy	but we know… 4, 8, 12… 6, 12… the LCM is 12.		X						method for finding
Amy	For Mark, he would need to know that you could multiply the numbers together, but you would have to divide that number by the number that goes into both A and B, unless there's no number that goes into both.	X	X		X				relationship
Me	And that number that goes into both? What's that called?								
Amy	That's going to be your greatest common factor.	X							personal definition
Me	And when two numbers are relatively prime, what's the greatest common factor of those numbers?								
Amy	One.	X							rel. prime
Me	How might you respond to Mark to help him figure out that this doesn't always work?								
Amy	I would ask Mark to… I would probably start by asking him why he thought that he could just multiply A and B, and see what he was thinking about that.		X				X		scaffolding
Amy	Then, depending on the numbers he was working with, if he was working with numbers had a GCF of 1, then I would see that he's seeing something there – like he was seeing a	X	X			X			student reasoning, relationship

	pattern there.								
Amy	But if he were working with numbers like 6 and 4, I would have him write out multiples.	X	X				X		scaffolding, relationship
Amy	I know when we were first doing the GCF and stuff, it was very helpful to write out the multiple of 4, the multiples of 6, and see where they first line up.		X			X			student solution strategy, personal definition
Amy	So he could see, 'Oh, this works in some cases, but not others.' Then I could ask him what's the difference. We've already found the GCF of these two numbers, what do you notice? Then we would start with his observations about the numbers… their GCFs and their LCMs. Go from there, questioning and guiding. And giving him numbers that will give him… not the product of A and B as their LCM. Because if he's just picking two random numbers… it's plausible that he could pick two relatively prime numbers.	X	X				X		scaffolding, relationship

APPENDIX M

DISSERTATION CODE SPREADSHEET

DISSERTATION CODE SPREADSHEET

INTERVIEW 1, PROBLEM 1

CODE	B	C	E	G	I	L
LCM - Modeling (Story Problem)						
Valid story problem	X					
*Appropriate context	X	X	X	X		X
Invalid story problem		X	X	X	X	X
*No starting point			X		X	X
*Inappropriate question		X	X	X	X	X
LCM - Modeling (Visual)						
Valid visual model	X	X	X	X	X	X
Visual model contributed to story problem	X	X		X		
LCM - Validation						
Story Problem A: Not valid	X	X	X	X	X	X
Story Problem B: Valid	X	X	X	X	X	X
Story Problem C: Not valid	X	X	X	X	X	X
Story Problem D: Not valid		X	X	X		X
Story Problem D: Valid, incorrect	X				X	

INTERVIEW 1, PROBLEM 2

CODE	B	C	E	G	I	L
GCF - Modeling (Story Problems)						
Invalid: "How many subgroups?"		X				X
Invalid: "How many in each subgroup?"	X	X		X	X	
* Did not maximize		X		X		
* Contextualized	X	X				
* Not contextualized				X	X	X
* Appropriate question	X					
* Inappropriate question		X		X	X	X
GCF - Modeling (Visual)						
Valid: "How many subgroups?"		X				X
Valid: "How many in each subgroup?"	X	X		X	X	
Conflated Representation			X			
Used to create story problem		X		X		X
Used to find GCF	X			X	X	X
GCF - Personal Definition						
Referred to definition of GCF to create story problem or model		X	X		X	
GCF - Validation						
Story Problem A: Valid		X		X	X	X
Story Problem A: Not valid (Incorrect)	X					
Story Problem B: Not valid	X	X	X	X	X	X
Story Problem C: Valid	X		X	X	X	
Story Problem C: Not valid (Incorrect)		X				X

Story Problem D: Valid			X	X	X	
Story Problem D: Not Valid (Incorrect) - should be like C	X	X				X

INTERVIEW 1, PROBLEM 3

CODE	B	C	E	G	I	L
LCM - Validation						
Invalid	X	X	X	X	X	X
LCM - Validation (Counterexample)						
Accurate Counterexample	X	X	X	X	X	X
LCM - Validation (Verification)						
Sometimes Valid	X	X	X	X	X	X
*Conjecture works when A and B are consecutive (weak SCK)				X		
* Conjecture works when A and B are prime (weak SCK)			X	X	X	
* Conjecture works when A and B are relatively prime (strong SCK)	X	X				X
LCM/GCF - Validation/Relationship						
* Conjecture works when A and B are relatively prime (strong SCK)	X	X				X
KCS - Student Reasoning						
* It works sometimes		X	X	X	X	
* Kids start with small #s which are usually relatively prime		X		X		
* The product of two numbers is called a multiple	X				X	X
* Easier to multiply/Finding LCM takes too many steps	X					
* Confusing: multiply by a different number to get LCM	X					
KCT						
Response to Mark: (KCT)						
* Acknowledge that his conjecture is true sometimes		X	X	X		
* Have Mark investigate his conjecture w/ rods & discover counterexamples (draws on SCK about manipulatives)	X		X	X		X
* Tell Mark that the product is a multiple but not LCM (draws on SCK and perhaps KCS??)	X	X				
* Draw attention to common factors & how that affects the LCM (draws on SCK about relationship b/w GCF and LCM)	X			X		
* Give Mark a counterexample to explore	X	X	X	X	X	X
* Tell Mark that his conjecture only works for___ (bad KCT drawing on bad SCK)			X			
* Does NOT mention alternative method for finding LCM		X			X	

Response to Mark: (General Mathematical Pedagogy - GMP)						
* Tell Mark to find examples & counterexamples		X				
How'd you know to respond this way? (insight to KCT)						
* Experience w/ kids (tutoring experience)	X		X		X	
* Visual methods help students understand (learning strategy)	X		X	X		X
* Build on what they already know (constructivism)	X	X		X		
* Don't say "wrong, do it it this way"	X	X			X	X
* Number Theory course (experience as a student)	X	X				
* MED course (experience as a student & theory)		X		X		X
* Important that they understand that they're wrong			X			
* Better to "see" it themselves - discover/cognitive conflict		X		X		X

INTERVIEW 1, PROBLEM 4

CODE	**B**	**C**	**E**	**G**	**I**	**L**
GCF - Validation						
Valid	X	X	X	X		X
Misconception (Invalid)					X	
GCF - Validation (Verification)						
Geometric representation of the Euclidean Algorithm	X			X	X	X
Equal sides of square take off "common" amount	X	X				
"Square off" what's left, smallest square gives GCF	X	X	X	X		X
Tiled rectangle	X	X	X	X		X
Course Reference						
* "We did this in number theory"		X	X	X	X	X
* "We proved it in class"		X		X		

INTERVIEW 1, PROBLEM 5

CODE	**B**	**C**	**E**	**G**	**I**	**L**
GCF/LCM - Relationship						
CONTENT (SCK)						
Correct Solution: B = 294	X		X	X	X	X
Strategies						
a × b = GCF(a, b) × LCM(a, b)	X					
Worked backwards from prime factorization of LCM				X	X	
Check LCM for divisibility by multiples of GCF and double-check GCF		X				

Pick multiple of GCF for b so that GCF(a, b)=42 then check LCM for divisibility by b			X			
Graphical Lattice Method						X
Incorrect/inconclusive attempts	X	X		X		

INTERVIEW 1, PROBLEM 6

CODE	B	C	E	G	I	L
Fractions and LCM						
Any common multiple gives a common denominator	X	X				X
LCM gives the least common denominator	X	X	X	X	X	X
The product gives a common denominator			X	X	X	
Fractions and GCF						
Dividing numerator and denominator by common factor gives a simplified fraction	X	X	X	X	X	X
Dividing numerator and denominator by GCF gives the most simplified fraction	X	X		X	X	X
Nonstandard multiplication algorithm uses common factors	X			X		
Misconceptions						
GCF/LCM do not play roles in multiplying fractions		X			X	
GCF/LCM do not play roles in dividing fractions		X	X		X	

INTERVIEW 1, PROBLEM 7

CODE	B	C	E	G	I	L
Misconceptions						
51 is prime			X		X	X
Can not simplify 18/51			X	X	X	X
Fractions and GCF						
Simplified 18/51			X	X		
Fractions and LCM						
Found lowest common denominator/LCM		X		X	X	
The product gives a common denominator	X	X	X			X

INTERVIEW 2, PROBLEM 1

CODE	B	C	E	G	I	L
Division - Modeling						
Created valid "How many in each group?" story problem	X	X	X			
Did not equally distribute objects amongst groups				X		X
Technicality: Did not specify to use all objects	X	X	X	X		X
Accurately modeled "How many in each group?" division	X	X	X	X	X	X

Created valid "How many groups?" story problem					X	
Accurately modeled "How many groups?" division						
Discussed reasoning for context	X					
Remembers seeing these representations somewhere	X	X	X	X	X	X
Division - Validation						
Story Problem A: Yes	X	X	X	X	X	X
Story Problem A: How many groups?	X				X	
Story Problem B: Yes	X	X	X	X	X	X
Story Problem B: How many in each group?	X				X	
Story Problem C: No	X	X	X	X	X	X
Story Problem C: Inconsistent units	X	X	X	X	X	
Story Problem C: Yes division	X	X			X	
Story Problem D: Yes	X	X	X	X	X	X

INTERVIEW 2, PROBLEM 2

CODE	**B**	**C**	**E**	**G**	**I**	**L**
LCM - Validation						
Story Problem A: Not valid	X		X			X
Story Problem A: Needs starting point	X		X			X
Story Problem A: When they blink together = common	X	X		X	X	
Story Problem A: Asks for common multiple, but not LCM	X					
Story Problem A: Needs **1st** time they blink together again	X					
Story Problem A: Valid (Incorrect)		X		X	X	
Story Problem B: Not valid	X	X	X	X	X	X
Story Problem B: Has starting point	X		X			X
Story Problem B: When they blink together = common	X			X	X	
Story Problem B: Needs how long before they blink again	X	X	X	X		X
Story Problem B: # of times they blink ≠ LCM	X	X	X	X	X	X
Story Problem B: # of times they blink related to LCM	X	X		X		

INTERVIEW 2, PROBLEM 3

CODE	**B**	**C**	**E**	**G**	**I**	**L**
GCF - Validation						
Story Problem A: Not valid, valid reasoning	X					
Story Problem A: Not valid, invalid reasoning						X
Story Problem A: Valid (Incorrect)		X	X	X	X	
Story Problem B: Not valid, incomplete reasoning	X	X	X	X	X	
Story Problem B: Valid (Incorrect)						X

INTERVIEW 2, PROBLEM 4

CODE	B	C	E	G	I	L
GCF - Validation						
Invalid	X	X	X	X	X	X
GCF - Validation (Counterexample)						
Accurate counterexample	X	X	X	X	X	X
* Prime counterexample						X
* Relatively prime counterexample	X	X	X	X	X	
* Counterexample with non-relatively prime and composite numbers		X			X	X
GCF - Validation (Verification)						
Sometimes Valid	X	X	X	X	X	X
GCF/Difference - relationship						
* The difference is divisible by the GCF						X
* The difference is bigger or equal to the GCF		X				
* There is no clear relationship	X		X	X	X	

INTERVIEW 2, PROBLEM 5

CODE	B	C	E	G	I	L
Modular Arithmetic and Congruences						
Explicitly used mods	X					X
Implicitly used mods		X		X	X	
Efficient investigations	X					X
Brute force investigation		X	X	X	X	
-> n= 1 mod 12, and 5 and 7		X				
-> n=1, 5, 7, 11			X			
-> n=1 mod 12, and 5, 7, 11,17				X		
-> n=1, 5, 7, 11 mod 12					X	
-> n is **relatively prime** to 12	X				X	X
* n is **relatively prime** to m	X				X	X

INTERVIEW 2, PROBLEM 6

CODE	B	C	E	G	I	L
Prime - Personal Definition						
Its only factors are one and itself	X	X	X	X	X	X
Prime - Importance						
Prime Factorization	X	X		X	X	X
Primes are part of the natural numbers			X			

INTERVIEW 2, PROBLEM 7

CODE	B	C	E	G	I	L
Prime - Validation						
Invalid	X	X	X	X	X	X

Prime - Personal Definition						
prime numbers have two distinct factors				X	X	
Prime - Validation (Counterexample) Why isn't 1 prime?						
Square numbers cannot be prime	X					
changes understanding of prime	X	X				
Factor trees would never end	X		X	X		
KCS - Student Reasoning						
What concepts does Shayna understand? (KCS)						
* Her understanding of the definition is correct	X	X	X			X
* She understands part of the definition				X	X	
Shayna doesn't understand… (KCS)						
* 1 is an exception to the rule	X	X	X			X
* 1 doesn't fit the definition				X	X	
KCT						
How could you lead her to see 1 is not prime? (KCT)						
* Use factors trees to show that if 1 were prime, they would never end	X	X	X			
* Multiplying a number by itself is not prime		X				
* The factors of prime numbers are different				X	X	
* Alludes to the Fundamental Theorem of Arithmetic						X
Why did you respond this way? (insight into KCT)						
* Number theory course (experience)	X		X		X	X
* It was convincing (personal experience)	X		X	X	X	
* Factor trees are a concept that 4th graders understand (curricular content knowledge)	X	X				X
* Build on what she already knows (constructivism)		X				X
* Better to see it themselves - discover/cognitive conflict		X				

INTERVIEW 2, PROBLEM 9

CODE	**B**	**C**	**E**	**G**	**I**	**L**
Factoring - Validation						
Both valid	X	X	X	X	X	X
Talisa did not factor completely	X	X	X	X	X	X
Unique factorization	X		X	X	X	X
Response to students (**KCT**):						
* Explain the different methods to each of the students	X					
* Help Talisa recognize that she's not done		X				
Response to students (GMP):						
* Have them each try another example using the other's method	X					
* Have them explain their methods to one another		X	X	X	X	X

Reason for response:						
* They should get the same prime factorization no matter what method they use (SCK influence)	X					
* Collaborative learning (epistemological influence)		X	X		X	X
* Explaining your understanding can strengthen it (epistemological influence)		X		X		
* This helps me better understand				X		
* MED experience						X

INTERVIEW 2, PROBLEM 10

CODE	**B**	**C**	**E**	**G**	**I**	**L**
Divisibility - Method: Factorization						
Not divisible by 2, 11, 14, or 26	X	X	X	X	X	X
2 is not in the prime factorization		X		X	X	
None of the factors are even	X		X			X
Divisible by 7, 9, and 15	X	X	X	X	X	X
Divisible by 63	X	X		X	X	X
GCF - Method for Finding (Prime Factorization)						
Correctly determined GCF	X	X	X	X	X	X
Conceptual reasoning	X	X	X	X	X	X
LCM - Method for Finding (Prime Factorization)						
Unsuccessful attempt at determining LCM	X	X			X	X
Correctly determined GCF	X	X	X	X		
Conceptual reasoning	X					

APPENDIX N

DISSERTATION CODEBOOK

DISSERTATION CODEBOOK

STORY PROBLEM CODES

GCF – A statement was coded as "GCF" if the content statement specifically pertained to the greatest common factor of two numbers.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what GCF "is" or "means".

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing the GCF of two numbers.

Story Problem – A statement was coded as "story problem" if a participant created, attempted to create, identified, or referred to a GCF story problem.

Visual – A statement was coded as "visual" if a participant created, attempted to create, identified, or referred to a visual model of GCF. Participants' pictures and demonstrated use of manipulatives were similar enough to collapse pilot study codes "pictorial" and "manipulatives" and create the code "visual".

How Many Subgroups – A reference to a GCF model was coded as "how many subgroups" if each group (A or B, the numbers whose GCF is to be represented) is broken up into the same number of subgroups, which gives the GCF. Both GCF "story problem" and "visual" models can have a "how many subgroups" structure.

How Many in Each Subgroup - A reference to a GCF model was coded as "how many in each subgroup" if each group (A or B, the numbers whose GCF is to be represented) is broken up into subgroups of the same size or number of objects, which gives the GCF. Both GCF "story problem" and "visual" models can have a "how many in each subgroup" structure.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a GCF model.

LCM - A statement was coded as "LCM" if the content statement specifically pertained to the least common multiple of two numbers.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what LCM "is" or "means".

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing the LCM of two numbers. This category overlaps with "method for finding" since some participants used one or more of the models to find the LCM.

Story Problem – A statement was coded as "story problem" if a participant created, attempted to create, identified, or referred to a LCM story problem.

Visual – A statement was coded as "visual" if a participant created, attempted to create, identified, or referred to a visual model of LCM. Participants' pictures and demonstrated use of manipulatives were similar enough to collapse pilot study codes "pictorial" and "manipulatives" and create the code "visual".

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a LCM model.

Division – A statement was coded as "Division" if the content statement specifically pertained to the division of two numbers.

Modeling – A statement was coded as "modeling" if a participant used or referred to a non-numerical (e.g., contextual, pictorial, or concrete) method for representing the division of two numbers.

Story Problem – A statement was coded as "story problem" if a participant created, attempted to create, identified, or referred to a division story problem.

Visual – A statement was coded as "visual" if a participant created, attempted to create, identified, or referred to a visual model of division. Participants' pictures and demonstrated use of manipulatives were similar.

How Many Groups – A reference to a division model was coded as "how many groups" if the divisor in the model determined the size of or number of objects within a group, thus making the quotient the number of groups that one could make from the dividend. Both division "story problem" and "visual" models can have a "how many groups" structure.

How Many in Each Group - A reference to a division model was coded as "how many in each group" if the divisor in the model determined the number of equal groups one should make from the dividend, thus making the quotient the size of or the number of objects within a group. Both division "story problem" and "visual" models can have a "how many in each group" structure.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a division model.

SCK – Any "content" statement was also coded as "SCK", for "specialized content knowledge", if the participant demonstrated "mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Ball, Thames, & Phelps, 2008, p. 377-8). This differs from other mathematical knowledge in that it appears to be unique to the ways in which mathematics arises in the classroom, as opposed to other occupational environments. I do not claim that participants demonstrated the more well-developed and robust SCK of an in-service teacher, but rather the developing or potential SCK of a preservice teacher.

Misconception – A statement was coded "misconception" if the participant demonstrated confusion or a misunderstanding about a concept. This code could have been used simultaneously with any other code.

NUMBER THEORY CONTENT CODES

GCF – A statement was coded as "GCF" if the content statement specifically pertained to the greatest common factor of two numbers.

Method for Finding – A statement was coded as "method for finding" if the participant referred to or used a method for determining the GCF of two numbers.

Comparing Lists – A statement was coded "comparing lists" if the participant referred to using or compared lists of factors to determine the largest factor that two numbers have in common.

Prime Factorization – A statement was coded "prime factorization" if the participant referred to using or found the prime factorization of two numbers to determine their GCF.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to GCF.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Relatively Prime – A statement was coded "relatively prime" if the participant referred to the case where the GCF of two numbers is 1.

LCM - A statement was coded as "LCM" if the content statement specifically pertained to the least common multiple of two numbers.

Methods for Finding – A statement was coded as "method for finding" if the participant referred to or used a method for determining the LCM of two numbers.

Comparing Lists – A statement was coded "comparing lists" if the participant referred to using or compared lists of multiples to determine the smallest multiple that two numbers have in common.

Prime Factorization – A statement was coded "prime factorization" if the participant referred to using or found the prime factorization of two numbers to determine their LCM.

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to LCM.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

Relationship – A statement was coded "relationship" if the participant referred to or used the relationship between LCM and GCF in their reasoning or calculations.

PRIME – A statement was coded as "prime" if the content statement specifically pertained to prime numbers.

Personal Definition – A statement was coded as "personal definition" if the participant referred to what prime "is" or "means".

Validation – A statement was coded "validation" if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to prime numbers.

Counterexample – A validation statement was coded "counterexample" if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded "verification" if the participant verified the case(s) in which the claim, conjecture, or proof are true.

FACTORING – A statement was coded as "factoring" if the content statement specifically pertained to factoring a whole number.

Methods – A statement was coded as "methods" if the participant referred to or used a method for factoring a whole number.

Tree – A statement was coded “tree” if the participant used or referred to using a factor tree to factor a whole number.

Pair – A statement was coded “pair” if the participant declared a factor pair of a whole number, as if by recall.

Validation – A statement was coded “validation” if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to factoring.

Counterexample – A validation statement was coded “counterexample” if the participant produced or referred to a counterexample that invalidated the claim, conjecture, or proof in question.

Verification – A validation statement was coded “verification” if the participant verified the case(s) in which the claim, conjecture, or proof are true.

DIVISIBILITY – A statement was coded “divisibility” if the content statement specifically pertained to whether a number was “divisible” by another, or whether a number is a factor of another number.

Methods – A statement was coded as “methods” if the participant referred to or used a method for determining the divisibility (i.e., identifying factors) of a whole number.

Factorization – A statement was coded “factorization” if the participant used or referred to the factorization (prime or partial) of a number to determine the divisibility of that number.

Evenness – A statement was coded as ‘evenness” if the participant referred to a whole number as being even.

Factor of 2 – An evenness statement was coded as “factor of 2” if the participant referred to a characteristic of even numbers as having a factor of 2 or being divisible by 2.

Validation – A statement was coded “validation” if the participant attempted to determine or referred to the validity of a claim, conjecture, or proof pertaining to divisibility.

Verification – A validation statement was coded “verification” if the participant verified the case(s) in which the claim, conjecture, or proof are true.

OTHER – The content of a statement was coded “other” if the statement specifically pertained to a topic in mathematics other than GCF or LCM.

Fraction Addition/Subtraction – A statement was coded “fraction addition/subtraction” if the statement specifically referred to the topic of adding and subtracting fractions.

Simplifying Fractions – A statement was coded “simplifying fractions” if the statement specifically referred to the topic of simplifying fractions to lowest terms.

SCK – Any “content” statement was also coded as “SCK”, for “specialized content knowledge”, if the participant demonstrated “mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Ball, Thames, & Phelps, 2008, p. 377-8). This differs from other mathematical knowledge in that it appears to be unique to the ways in which mathematics arises in the classroom, as opposed to other occupational environments. I do not claim that participants demonstrated the more well-developed and robust SCK of an in-service teacher, but rather the developing or potential SCK of a preservice teacher.

Misconception – A statement was coded “misconception” if the participant demonstrated confusion or a misunderstanding about a concept. This code could have been used simultaneously with any other code.

PEDAGOGICAL CONTENT KNOWLEDGE CODES

PEDAGOGICAL CONTENT – A statement was coded as "pedagogical content" if the participant demonstrated or made reference to mathematical knowledge for teaching, i.e., pedagogical content knowledge (Shulman, 1986). This includes "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9).

KCS – A statement was coded "KCS" if it pertained to "students and their ways of thinking about mathematics – typical errors, reasons for those errors, developmental sequences, strategies for solving problems" (Hill, Schilling, & Ball, 2004). This differs from SCK in that SCK plays a role in determining the mathematical accuracy of student work, while KCS is necessary for determining student understanding and depth of knowledge.

Student Reasoning – A statement was coded "student reasoning" if the participant referred to *why* a student might believe a statement, claim, or conjecture about number theory is true or false.

Student Challenge – A statement was coded "student challenge" if the participant acknowledged a specific difficulty or misconception that a student might have related to a certain task or concept.

Student Conceptions – A statement was coded "student conceptions" if the participant appropriately identified the hypothetical student's valid mathematical conceptions.

KCT – A statement was coded "KCT" if a participant demonstrated a "knowledge of content and teaching", as described by Ball, Thames, and Phelps, 2008. This includes knowing how to sequence the content for instruction, like which examples to use when introducing a topic versus when attempting to deepen students' understanding. KCT also includes being able to weigh the advantages and disadvantages of the different ways with which to represent mathematical concepts as well being able to make in-the-moment decisions about whether or not to pause for clarification or pose a new task to further student learning.

Curriculum Content Knowledge – A statement was coded as "curriculum content knowledge" if the participant demonstrated a knowledge of programs developed for the teaching of a mathematics, mathematical concepts covered at a given level, and instructional materials available. This includes knowledge of vertical curriculum, i.e., the knowledge of mathematics taught across grade levels.

Misconception – A statement was coded “misconception” if the participant demonstrated confusion or a misunderstanding about a concept. This code could have been used simultaneously with any other code.