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Mathematics Teachers' Models of Quantitative Reasoning

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MATHEMATICS TEACHERS' MODELS OF QUANTITATIVE REASONING

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ABSTRACT

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Teacher education focuses on impacting teachers' practice in ways aligned with current reform efforts. One particular emphasis in mathematics education is ensuring teachers, and subsequently students, are able to reason quantitatively. The purpose of this study was to document mathematics teachers' models of quantitative reasoning as they participated in a Model-Eliciting Activity (MEA) grounded in their classroom practice. This MEA was designed and implemented in a master's course of 21 in-service mathematics teachers. The MEA asked teachers to construct and revise a quantitative reasoning task, along with supporting documents, intended for their middle and high school students. This MEA served as a method that simultaneously documented and developed teachers' models as they received feedback from the instructor, each other, undergraduate students, and in some cases teachers' own students. The documents produced by the teachers, along with observations and interview data, were analyzed using a models and modeling perspective to determine how teachers' models of quantitative reasoning developed through the MEA.

Findings from this study detail how teachers' models of quantitative reasoning were not fully communicated in terms of defining quantitative reasoning in settings not connected to their classroom. As teachers went through the course and the MEA iterations, they began grappling with quantities and quantitative relationships as aspects of quantitative reasoning. Teachers' attention to these aspects better positioned these teachers to reason covariationally about the mathematical content in their documents, thus promoting deep conceptual understanding of functions and more advanced mathematical topics. The development of these teachers' models, along with the MEA itself, extends prior work regarding how teacher MEAs can document teachers' models within teacher education efforts. This study also identifies generalizable methods for understanding and promoting the productive development of mathematics teacher thinking about quantitative reasoning through this teacher MEA.

Keywords: quantitative reasoning, mathematics teacher education, model-eliciting activity

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CHAPTER I

INTRODUCTION

"In theory there is no difference between theory and practice. In practice there is."

The quote above has been attributed to Jan L. A. van de Snepscheut, Albert Einstein, and Lawrence Peter "Yogi" Berra, each considered experts in their field of computer science, physics, and baseball, respectively (Chandler, 2014; Popik, 2010). Given the range of applications, it may be unsurprising how the same message relates to the field of mathematics teacher education. Experts have long noted a disconnect between the theories teachers learn in professional development and teachers' practices within their classrooms. This disconnect creates a problem called the theory-practice gap, where professional development efforts fail to change teacher practice in productive ways (National Research Council, 2002).

Efforts to bridge the theory-practice gap in teacher education aim to change classroom practices in ways that support educational reform goals (Da Ponte et al., 2009; Korthagen & Kessels, 1999). One of these efforts has come from Lesh and colleagues, who created the *models and modeling* perspective to help bridge the theory-practice gap in teacher education. This perspective provides mathematics teacher educators guidelines to structure teacher education experiences in ways that connect teacher education efforts to the complex environments that teachers navigate in everyday situations. This perspective provides researchers the tools to document how teachers interpret their practice and promote teachers' ways of thinking about mathematics in productive ways (Doerr & Lesh, 2003; Lesh et al., 2003; Lesh & Sriraman, 2010; Sriraman & English, 2010; Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008).

Some teacher education efforts include developing specific content knowledge for teachers in order to positively impact student thinking (Hill & Ball, 2004). In contrast, a models and modeling perspective incorporates many more components into teacher knowledge, including "how teachers interpret the complexity and the situated variability of the practical problems of the classroom, how those interpretations evolve over time and across settings, and how and when those interpretations influence decisions and actions in the classroom" (Doerr & Lesh, 2003, p. 127). Lesh's work defines a teacher's model of teaching, learning, and problem solving to be their ways of thinking about a complex, situated setting such as their conceptual system regarding a mathematical topic within the context of their classroom practice. Developing teachers' models is the main goal of a models and modeling perspective (Lesh et al., 2003). Using a models and modeling perspective, researchers have the tools to address aspects of the theory-practice gap in teacher education in the area of quantitative reasoning. Since this perspective was developed to investigate how teachers interpret educational settings, studies based in this perspective have the potential to reveal these ways of thinking and how they change in teacher education settings (Doerr & Lesh, 2003; Lesh, 2006; Zawojewski et al., 2008).

Little literature addresses what is known about in-service teachers' models of quantitative reasoning. Quantitative reasoning has longstanding importance in mathematics, and has been defined as attending to identifying quantities, constructing new quantities, and identifying and representing relationships between quantities (Moore, Carlson, & Oehrtman 2009; Thompson, 2011). Researchers have emphasized the importance of quantitative reasoning in promoting learners' understanding of mathematical content, and recent reform efforts have focused on developing K-12 students and teachers' ability to reason quantitatively (Common Core State Standards for Mathematics, 2010; Ellis, 2012, 2013; Johnson, 2012, 2013; Moore, 2012). Researchers have subsequently emphasized the need to understand how to improve teacher education and support the development of teachers' models of quantitative reasoning ("Closing the Expectations Gap", 2011; Conference Board of the Mathematical Sciences, 2012; Confrey & Krupa, 2010; Ellis, 2007; Garfunkel et al., 2011; Sztajn, Marrongelle, Smith, & Melton, 2012; Thompson, 2011).

The purpose of this study was to investigate teachers' models of quantitative reasoning and how these ways of thinking develop. I incorporated a models and modeling perspective to elicit and document teachers' models of quantitative reasoning in terms of their practice using a Model-Eliciting Activity (MEA). Teacher MEAs are realistic and complex problems that engage teachers in thinking about mathematics in a way embedded in their practice, in order to create and use models in documented ways (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). These MEAs are created by following Doerr and Lesh's (2003) reality, multilevel, multiple context, sharing, and self-evaluation principles. I upheld these principles to help ensure teachers' models were simultaneously documented and developed as teachers completed the MEA. This study was classified as a type of multi-tiered teaching experiment because I designed the study to focus on how teachers develop longitudinally after receiving feedback from students and other sources (English, 2003; Lesh & Doerr, 2003; Lesh & Kelly, 2000).

Context

The context of this study was a graduate-level course for in-service middle and secondary teachers. Teachers were in a two-year master's program in mathematics, where teachers took a combination of mathematics and mathematics education courses. The study focused on one of the newly developed summer mathematics education courses in the program, called Quantitative Reasoning in Secondary Mathematics. The 21 teachers in the course taught grades 6-12 mathematics, and constituted the primary participants for this study. The MEA in this study asked teachers to create multiple sets of documents throughout the course. These documents consisted of the bulk of the data for this study, although I also conducted interviews, made observations, and collected additional documents to further record teachers' models. The ultimate goal of the MEA and other data collection methods was to answer the inquiry statement below.

Inquiry Statement

In an effort to address the need to understand how to improve teacher education and support the development of teachers' models of quantitative reasoning, I used a models and modeling perspective to design and implement part of a multi-tiered teaching experiment. Over the course of the study, the guiding research question was:

Q1 How do mathematics teachers' models of quantitative reasoning develop through an MEA focused on quantitative reasoning in teachers' classroom practice?

This research question was answered using a models and modeling perspective because this perspective can capture teachers' ways of thinking through a well-designed MEA. This study incorporated an MEA that simultaneously documenting teachers' existing models and then documenting the development of these existing models throughout the teacher education setting. By documenting teachers' models, I could identify the factors that influenced change in teachers' models. These factors have the potential to be generalized to other teacher education and professional development settings (Adler, Ball, Krainer, Lin, & Novotna, 2005; Lesh, 2006). By promoting teachers to develop their models, the MEA encouraged improvements to teachers' classroom practice and student learning in productive ways that support the Common Core State Standards for Mathematics (CCSSM).

Significance

This section includes the anticipated significance of this study on mathematics education; the actualized significance is detailed in Chapter 5. This study was designed to contribute to the field of mathematics education in three main ways: by developing research and teaching methods that promote teacher growth in thinking about quantitative reasoning, by helping bridge the theory-practice gap, and describing teacher thinking about quantitative reasoning and how this thinking develops.

The first way this study was designed to contribute to the field of mathematics education is through the creation of an MEA as a research method that captured teachers' models of quantitative reasoning. The MEA was designed to engage teachers in creating and analyzing a quantitative reasoning task for their students and documenting the students' reasoning and teachers' own thinking. Creating this MEA could have contributed to both teaching and research methods because (a) the MEAs were designed to have teachers develop ways of thinking that productively impacted their practice, and (b) the MEA was designed to document teacher thinking in order to identify teachers' models and the sources of change that occur. Both these contributions could benefit other researchers and teacher educators, as the design of the activity could be shared and reused in similar contexts (Lesh & Doerr, 2003; Lesh & Lehrer, 2003).

This study's MEA had the potential to contribute to the field of mathematics teacher education as a teaching method because it focused on quantitative reasoning and was designed for a summer course taken online. At the time the study was implemented, MEAs focusing on quantitative reasoning did not exist despite the need for mathematics teachers to develop deeper ways of thinking about quantitative reasoning (CCSSM, 2010; Moore et al., 2009; Smith III & Thompson, 2008). Since an MEA of this type was both needed and unavailable, this study was designed to identify a way to advance teacher knowledge about quantitative reasoning. Additionally, this MEA took place during a time when teachers do not have access to their own students, which is atypical of teacher MEAs. Developing a teacher MEAs that addressed the difficulty of teachers not having their own students could have offered a unique instructional design for other mathematics teacher educators wishing to develop teachers' models of quantitative reasoning in summer professional development courses or programs.

Second, this study was designed to bridge the theory-practice gap in teacher education by identifying ways that teacher education efforts, such as teacher educators exposing teachers to theory about quantitative reasoning, can promote productive changes to teachers' classroom practices. Model-Eliciting Activities are constructed using a reality principle (Doerr & Lesh, 2003), meaning the MEA asks teachers to interpret quantitative reasoning in terms of their classroom context. One planned outcome of this study was findings that identified how teacher education efforts can impact teacher practice by examining how this study's MEA promoted teachers to think about quantitative reasoning in a way that was grounded in their classroom.

Third, this study was designed to advance theory about how teachers think about quantitative reasoning. A models and modeling perspective is used to promote the development of a researcher model of the phenomena being investigated by providing the tools for researchers to theorize about what it means for teachers' models to develop, how development occurs, and what factors might further development in ways that are transportable to other situations (Doerr & Lesh, 2003; Lesh et al., 2003; Zawojewski et al., 2008). Planning to use this perspective for this study allowed me the opportunity to develop a theory about how teachers develop models of quantitative reasoning. This theory could describe how teachers externalize their models in the MEA and how the models develop over time. Little literature investigated the way in-service mathematics teachers think about quantitative reasoning for their students despite increasing demands by educational reform documents and mathematics education researchers (CCSSM, 2010; Moore et al., 2009; Smith III & Thompson, 2008; Thompson, 2011). Therefore this study had the potential to significantly contribute to the field of mathematics education.

Delimitations

A delimitation of this study was my choice to conduct part of a multi-tiered teaching experiment rather than a full experiment. A full multi-tiered teaching experiment requires multiple researchers working over extended periods of time to investigate students, teachers, and researcher at multiple sites. Teacher MEAs incorporated in multi-tiered teaching experiments typically use the interaction between teachers and students as a driving force for the development of teachers' models. The conclusions from a multi-tiered teaching experiment discuss patterns in how students, teachers, and researchers think and how interactions between these populations contribute to changes in thinking (Lesh & Doerr, 2003; Lesh & Sriraman, 2010).

I chose to restrict this study to part of a multi-tiered teaching experiment for two reasons. First, access to the teachers' students was not available because the study's setting took place during the summer. Choosing to conduct the study in this setting restricted the data I could collect from K-12 students and data about teacher-student interactions. Second, I chose to restrict the study because I was the only researcher, which limited my ability to conduct research at multiple sites. This choice allowed me to concentrate my efforts on one research site to answer the research question.

The choice to conduct part of a multi-tiered teaching experiment choice narrowed the implications of the findings because I was unable to characterize the models held by the students. However, this delimitation did not hinder my ability to report significant findings about teachers' models of quantitative reasoning.

CHAPTER II

REVIEW OF LITERATURE

This chapter familiarizes readers with the critical terms and relevant literature about the study in order to frame the work that took place. This chapter is structured to contain the most important constructs related to the research question: How do mathematics teachers' models of quantitative reasoning develop through a Model-Eliciting Activity (MEA) focused on quantitative reasoning in teachers' classroom practice? This research question rests on four major constructs: (1) the "reasoning abstractly and quantitatively" Standard for Mathematical Practice given by the Common Core Standards for Mathematics (CCSSM, 2010), (2) quantitative reasoning as defined by mathematics education literature, and (3) teacher education within professional development. The fourth construct, a models and modeling perspective, is detailed in the theoretical perspective section of Chapter 3.

The first construct frames this study by identifying the current shifts in K – 12 standards in the United States and the subsequent needs for research about reasoning abstractly and quantitatively in the field of mathematics teacher education. The second construct defines quantitative reasoning and presents relevant literature referenced throughout this study. This construct also details what researchers know about how people think about quantitative reasoning and further highlights the need for this study to take place. The third construct describes the efforts of professional development for mathematics teachers and details successful strategies for changing teacher practice. The final construct of a models and modeling perspective provides information on the theoretical framework and methods used in this study.

Selection Process

The selection process for the information that follows began with a broad search in Academic Search Premier using the construct, or similar phrase, as the search entry. After skimming the titles and abstracts of at least the first 20 top results, I compiled an annotated bibliography of the sources that were relevant to my study, which came from peer-reviewed journals, international handbooks, and key conference proceedings where a peer-reviewed process was employed. Since some of the current research topics, such as the CCSSM, are extremely recent developments, summaries from conferences and other non-peer reviewed sources about these topics were included, as peer-reviewed sources are not currently available.

For each article, I examined the reference list, and used Google Scholar to evaluate relevant references. Then I used Google Scholar's "cited by" feature to view the articles that referenced the relevant work, and continued this process until I was unable to generate additional articles related to the four constructs. In addition, I also searched each construct in the last ten years of four major mathematics education journals: *Journal for Research in Mathematics Education, Journal of Mathematics Teacher Education, Educational Studies in Mathematics,* and *Journal of Teacher Education*.

The compiled list of literature was then read and coded for themes relevant to my study, as recommended by Foss and Waters (2007). The codes were put into categories, which were then compared to meta-analyses and literature reviews previously published to ensure main categories of literature were complete. The relevant categories are included in the appropriate constructs below.

The Common Core State Standards' "Reasoning Abstractly and Quantitatively" Standard for Mathematical Practice

This section overviews the Common Core State Standards in Mathematics (CCSSM) before detailing the standard of mathematical practice called "reasoning quantitatively and abstractly" before talking about how this standard impacts mathematics teacher education. The CCSSM and this standard relate to the study because the teachers in this study will be expected to understand and teach students how to reason quantitatively and abstractly because they reside in states implementing the CCCSM.

In 2010, a growing consensus for raising K-12 educational standards spurred the state-led Common Core State Standards in Mathematics (CCSSM) to be published (CCSSM, 2010). These standards were developed by educators to have the goals of (a) supporting all students in receiving a high quality education that prepares them for postsecondary education and the workforce, (b) providing an increased opportunity for states to efficiently share experiences, practices, and assessments in order to better serve student needs, (c) helping teachers meet these objectives through clear and focused goals for student learning, and (d) creating a clear set of expectations to support collaboration between educators, policy evaluators, parents, students, and other members of the education community (CCSS, 2010; "Closing the Expectations Gap," 2011; Garfunkel et al., 2011; Porter, McMaken, Hwang, & Yang, 2011). The CCSSM was designed to meet these goals by changing the role of the student from receptors to processors of information, increasing the use of instructional technology, as well as placing greater emphasis on number sense, operations, and basic algebra and geometry rather than advanced algebra and geometry at the high school level ("Closing the Expectations Gap," 2011; Porter et al., 2011).

The content standards detail the subject and grade level standards of the material, while the Standards for Mathematical Practice are intended to be applied across all content standards, and articulate expectations that mathematics educators of all levels should develop their students' expertise with these practices (CCSS, 2010). The Standards for Mathematical Practice rest on longstanding processes and proficiencies in mathematics education. Conley, Drummond, Gonzalez, Rooseboom, & Stout (2011) detail the design processes for creating these standards, which include input given by mathematics and science instructors and cross-sections of respondents from a variety of content-related fields (CCSS, 2010). From their work, the eight mathematical practices were rated as highly applicable across a wide range of content areas.

The CCSSM details what is meant by the Standard for Mathematical Practice called "reason abstractly and quantitatively" by stating:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (CCSSM, 2010, p. 6)

In the description of the high school Number and Quantity domain, the CCSSM (2010) states quantities are "numbers with units, which involves measurement" (p. 58). In the high school standards for Number and Quantity the CCSSM describes quantification as an important conceptual process for students to know. The CCSSM authors defined quantification as conceiving attributes of interest within novel situations.

The “reason abstractly and quantitatively” Standard for Mathematical Practice has similarities to other standards. For example, the “model with mathematics” Standard for Mathematical Practice indicates students should identifying quantities in a situation and modeling the relationships between them. Furthermore, the “attention to precision” standard says students should “clarify the correspondence with quantities in a problem.”

The Standards and Mathematics Teacher Education

The CCSSM has major influence on what students, and subsequently teachers, need to know. Researchers note that “the Standards for Mathematical Practice should be implemented and assessed across subject areas in a wide range of contexts and courses in secondary schools and in state and consortia assessments” (Conley et al., 2011, p. 98). Since the Standard for Mathematical Practice have been presented separate from the content standards, Confrey and Kupa (2010) and other researchers have expressed concerns that the Standards for Mathematical Practice may be under-emphasized and disconnected from content (Garfunkel et al., 2011; Neubrand et al., 2009). Instead these researchers argue that these practices need to act as a vehicle for learning the content standards, allowing students to build lasting knowledge that prepares them to learn new mathematics. New classroom expectations warrant changes that need to occur at the professional development level in order to support teachers’ ability to enact the CCSSM. Sztajn, Marrongelle, and Smith. (2011) indicate “such changes are likely to occur only through sustained and focused professional development opportunities for those who teach mathematics” (p. 3). Marrongelle, Sztajn, and Smith (2013) add that “professional development materials are needed that explicitly address the mathematics content and practices of the CCSSM and provide vivid images of teaching and learning that are consistent with CCSSM" (p. 206).

Prior literature provides overall guidelines for professional development supporting change in teacher practice, but researchers have also offered advice specific to the CCSSM (Sztajn et al., 2011). Recommendations for professional development opportunities to support the CCSSM mathematical aims include (a) helping teachers identify and interpret the mathematical standards in the CCSSM (Confrey & Krupa, 2010), (b) creating time and resources for teachers to develop instructional practices aligned with mathematical standards in the CCSSM (Krupa, 2011), (c) changing teacher perceptions about school mathematics (Nichols, 2010), and (d) assisting teachers in monitoring and understanding student learning in relation to the mathematics standards in the CCSSM (Garfunkel et al., 2011; Sztajn et al., 2011).

Sztajn et al. (2011) detailed recommendations for professional development related to the CCSSM. They stated professional development should focus on a few specific content standards and integrate the Standard for Mathematical Practice rather than many content standards all at once. Also educators should evaluate teacher growth in ways that are specifically tied to the CCSSM mathematical standards by collecting videos and teacher artifacts such as tasks and lesson plans. Sharing and improving the artifacts in professional development could then support and evaluate teacher growth related to standards outlined in the CCSSM.

In relation to the Standards for Mathematical Practice, and specifically quantitative reasoning, teachers need to understand the role of quantitative reasoning in order to foster student learning of this practice (Confrey & Krupa, 2010; Ellis, 2007; Usiskin, 2001). Usiskin (2001) described how incorporating quantitative reasoning into K-12 education will remain a challenge “until a generation of teachers has learned its mathematics with attention to quantitative literacy—a chicken-and-egg dilemma” (p. 85). More recent research (Sztajn et al., 2012) reiterates this challenge, recognizing that while good standards are important, more is needed to make changes in teacher practice and student learning. “States must now ensure that the higher expectations they have adopted in their standards are carried out in related policies such as graduation requirements, assessments and accountability systems that value college and career readiness and make sure teachers have the tools, time and professional development to teach effectively to the standards” ("Closing the Expectations Gap,” 2011, p. 22).

Implications

One question researchers have asked about the Standards for Mathematical Practice is how teachers will interpret these practices in their classroom (Heck, Weiss, Pasley, 2010; Marrongelle et al., 2013; Wiener, 2013). The “Common Core’s standards of mathematical practice need operational, shared definitions to support feedback and guidance to improve instruction…tools need to be designed to look for and develop these skills in teaching math content and assessing students’ mastery” (Wiener, 2013, p. 14). Another question researchers have asked is how the CCSSM reform movement influences teachers’ in thinking and practices in the classroom. Since this study addresses this need in terms of quantitative reasoning, the following section defines quantitative reasoning and relates this term to the CCSSM’s “reasoning abstractly and quantitatively” Standard for Mathematical Practice.

Quantitative Reasoning in Mathematics Education

This section first defines quantitative reasoning and one frequently-cited component of quantitative reasoning, called covariational reasoning. Then the connection between quantitative reasoning and K-12 mathematics education is discussed. Defining researcher-identified components of quantitative reasoning is valuable for this study because this information provides a mechanism for me to identify and contrast components in teachers' thinking about quantitative reasoning.

Defining Quantitative Reasoning

Even though quantitative reasoning is widely believed to be important, there is little agreement on what constitutes quantitative reasoning (Quantitative Literacy Design Team, 2001; Mayes, Bonilla, & Peterson, 2012). Literature depicts quantitative reasoning as a diverse and complex concept with strong ties to context. In their meta-analysis of the meaning of this term, Mayes et al. (2012) found quantitative reasoning definitions usually included the use of mathematics or statistics in a context in a way requiring advanced reasoning with elementary mathematics. Other definitions of quantitative reasoning reflect the CCSSM's Standard for Mathematical Practice of reasoning abstractly and quantitatively. For example, the National Numeracy Network (2011) defines quantitative reasoning to be a habit of mind of working with and critiquing quantitative information and being able to use "higher-order reasoning and critical thinking skills needed to understand and to create sophisticated arguments supported by quantitative data" (p. 1).

The work of Thompson (1990; 1993; 1994; 2011; 2012; 2013) and colleagues (Smith III & Thompson, 2008) offers a theory of quantitative reasoning, highlighting learners' construction of quantities and quantitative relationships. According to Thompson's theory, quantity results from a person completing an act of quantification, defined as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit" (Thompson, 2011, p. 8). Thus quantities are a cognitive object composed of four components: (a) an object, (b) a measureable attribute of the object, (c) a unit of measurement for the attribute, and (d) a conceivable numerical value, or values, associated through a proportional relationship with the unit of measurement. The interplay between quantities and the act of quantification make these two ideas intertwined, as quantities are created through quantification and quantification cannot be completed without creating a quantity. Learners quantify a quantity by deciding on a way to identify, measure, and interpret these components. A static quantity is a quantity in which a person conceives of a single numerical value associated with the unit of measurement. Alternatively, a quantity in which a person considers multiple numerical values associated with a unit is called a varying quantity.

Quantities can be related through a quantitative operation, which is the conception of two quantities being taken to produce a new quantity (Thompson, 1990). Quantitative operations differ from numerical operations, which deal only with numbers. "Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations children (and teachers) can confound them unproblematically" (Thompson, 2011, p. 15). Ellis (2011, p. 216) offered an example of this, saying "one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division [respectively].” When a person conceives of two quantities being joined through a quantitative operation to create a third quantity, Thompson calls this a quantitative relationship (Thompson, 1990).

A person’s mental network of quantities and quantitative relationships is their quantitative structure, according to Thompson (1990, 2011). This structure may contain multiple layers, all within the individual’s mind rather than in the world. Thompson views quantitative reasoning as the mental process where a person’s quantitative structure is used when the learners attempts to achieve a desired goal.

The work of Moore et al. (2009) have summarized quantitative reasoning, in light of Thompson’s theory, as attending to and identifying quantities, identifying and representing relationships between quantities, and constructing new quantities. For this study, I refer to both Thompson (1990, 2011) and Moore et al.’s (2009) definitions of quantity, quantification, and quantitative reasoning for a common reference point for what is meant by these terms.

Covariational Reasoning

Covariational reasoning is an essential component of quantitative reasoning, according to Thompson (2011). Covariational reasoning is defined as coordinating two quantities while attending to how they change in relation to each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Moore et al., 2009; Saldanha & Thompson, 1998). “The importance of covariational reasoning for modeling is that the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations” (Thompson, 2011, p. 22). Students’ difficulties with covariational reasoning has been established throughout literature (Carlson, 1997, 1998; Trigueros & Jacobs, 2008).

Carlson (1998) and colleagues (Carlson, Jacobs, & Larsen, 2001) developed a framework for studying students' covariational reasoning, which provides "a lens for analyzing and reporting students' covariational reasoning abilities when responding to dynamic function tasks" (Carlson et al., 2002, p. 8).

Researchers have examined how students engage in covariational reasoning, often focusing on how this type of reasoning relates to higher-level mathematics (McCoy, Barger, Barnett, & Combs, 2012; Oehrtman et al., 2008). For example, the work of Moore and Carlson (2012) highlights the importance of covariational reasoning in determining functions and graphs for pre-calculus undergraduate students (Carlson, Oehrtman, & Thompson, 2007). These authors had students engage in activities that required students to conceptualize varying quantities and how two quantities covary together. The authors found these activities supported students' ability to reason covariationally, and thus determine correct formulas and graphs in various problem contexts.

Quantitative Reasoning
Connection to K – 12
Mathematics Education

Current reform efforts suggest K-12 students need to reason quantitatively (CCSSM, 2010), but little information exists about what this means or how this should occur. The precise definitions of quantitative reasoning given in the previous section are not used in the CCSSM, as researchers have called for "operational, shared definitions" (Wiener, 2013) and more "vivid images" (Marrongelle et al., 2013) for the CCSSM Standards for Mathematical Practice (Garfunkel et al., 2011). One way to make the CCSSM Standards for Mathematical practice more operational and vivid is to include some components of the definitions from literature given in the previous section.

Some similarities exist between the CCSSM and quantitative reasoning, as defined by literature. The CCSSM mentions quantification in the Number and Quantity domain for high school. The CCSSM states quantification is a valuable skill that high school students need to use to understand science and business settings, though no explicit definition of this term is given other than when students "encounter novel situations in which they themselves must conceive the attributes of interest" (2010, p. 58). Having students understand how quantities covary within a relationship is also included in the CCSSM. For example Grade 8 domain 8.SP standard 3 asks students to interpret slope as an association between *x* and *y* quantities. Additionally the high school domain F-LE standard 1 asks students to recognize patterns between input quantities and output quantities in various functional relationships.

Researchers have begun examining how to foster students' quantitative reasoning skills (Castillo-Garsow, 2013; Moore, 2012; Thompson, 2011). Past work on this question has indicated that placing students in quantitatively-rich situations does not guarantee the development of meaningful mathematical concepts (Cuban, 2001; Lobato & Siebert, 2002; Noble, Nemirovsky, Wright, & Tierney, 2001). Ellis (2007) suggests the following ways to support students' quantitative reasoning:

> Students should therefore be confronted with problem situations that require them to explore the phenomenon in question; they should have opportunities to engage in activities such as (a) exploring how changing one or both initial quantities will affect the emergent quantity, (b) determining how to adjust the initial quantities while keeping the emergent quantity constant, and (c) determining how to double, halve, or otherwise manipulate the emergent quantity in relationship to the initial quantities. (p. 475)

Ellis' (2007; 2013) research focused on middle school algebra and pre-algebra students and corroborates others researchers' findings about the importance of using quantitative reasoning as a meaningful base for students' development of mathematical concepts such as algebraic reasoning, functions and ratios (Chazan, 2000; NCTM, 2000; Oehrtman et al., 2008). Ellis has found that reasoning directly with quantities and quantitative relationships can help students build conceptions of functions and even incorporate algebraic practices while in middle school. However, students traditionally have more difficulty reasoning conceptually when working with functions in calculus (Monk, 1992; Carlson, 1999).

Ellis' (2007) work has also highlighted how instruction focusing on quantitative reasoning supported middle school students to make generalizations about algebra, mathematical relationships, and patterns. In this work generalization is defined according to the work of Lobato and Siebert (2002) and Kaput (1995) as "(a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases" (Ellis, 2007, p. 444). Ellis found that when students constructed the quantity of a ratio from two initial quantities, their ability to make generalizations about ratio relationships improved. Researchers have recognized the importance of the mathematical process of generalization as part of algebraic thinking, and have called for more research investigating how quantitative reasoning can support generalization (Ellis, 2007; Ellis & Grinstead, 2008; Lobato & Siebert, 2002; Smith III & Thompson, 2006).

Implications

Further research is needed in multiple areas related to quantitative reasoning. Teachers need to be able to identify the generalizations students make about quantitative relationships, develop and incorporate assessments to measure students' quantitative reasoning skills, and identify practices and models that support students' quantitative reasoning (Confrey & Krupa, 2010; Ellis, 2007; Garfunkel et al., 2011; Lobato & Siebert, 2002). Currently little literature exists about how teachers are thinking about quantitative reasoning. Research in this area is needed in order to help teacher education be structured in ways that promote productive ways of thinking about quantitative reasoning.

Both the CCSSM's reasoning abstractly and quantitatively Standard for Mathematical Practice and literature defining quantitative reasoning include components of quantification, covariational reasoning, conceptual understanding, and flexibility in thinking about quantities and relationships of quantities in content (CCSS, 2010; Ellis, 2013; Thompson, 2011, 2013). While the CCSSM developers and supporters have expressed optimistic hopes for student achievement related to the reasoning quantitatively Standard for Mathematical Practice, concerns remain for the foundational work needed to attain these goals (Kilpatrick, 2011). "Closing the Expectations Gap" (2011) summarizes these challenges:

> Over the years, many commentators have correctly noted that the promise of standards-based education reform has not always been met. Changing policies such as standards, graduation requirements, assessments and accountability is a critical first step, but to fully meet the promise, careful and intentional implementation that provides teachers and students with the tools and support they need to successfully meet the standards is critical. The reform movement is at a critical precipice. The nearly universal adoption of college and career-ready standards and a majority of states engaged in the development of next-generation assessments are promising. State progress on the rest of the agenda, while more incremental, still suggests a commitment to college and career readiness for all. The next few years will be critical, testing the resolve of policymakers, states, districts, schools and the public. The results could be transformative if we continue to push together to create schools and classrooms in which students are able to reach their full potential over the course of their K–12 education and graduate prepared for the real world they will enter after high school, as well as if we support teachers and leaders in getting there. (p. 8)

This passage emphasizes another vital role in the implementation of any major change within the educational system: supporting teacher development and knowledge (Garfunkel et al., 2011; National Research Council, 2002; Sztajn, 2003; Usiskin, 2001).

Mathematics Teacher Knowledge and Education

The multitude of studies stemming from sources like the *Journal of Mathematics Teacher Education (JMTE)* and Rosa Leikin and Rina Zazkis' book *Learning Through Teaching Mathematics* (2010) highlight researchers' goal to understand the teaching and learning of mathematics and then improve the education of mathematics teachers through teacher education efforts (JMTE, 2012). An essential method for facilitating this improvement is through professional development, which is often defined as planned opportunities for teacher learning in postgraduate settings (Kelly, 2006). Professional development is aimed at increasing teacher expertise through these experiences and has been an essential component of realizing these goals of mathematics education (Arbaugh & Brown, 2005; Even & Ball, 2009; Kreiner, 2008; Da Ponte et al., 2009; Sullivan & Wood, 2008).

As Leikin (2011) summarized, "changing teachers' beliefs, advancing their knowledge and skills, increasing their self-confidence and developing more professional perspectives among practicing teachers for coping with challenges incorporated in their practice are among the central aims of teacher educational programs" (p. 995). For decades researchers have documented positive changes associated with professional development (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Frechtling, Sharp, Carey, & Vaden-Kiernan,1995; Guskey, 2000). However, researchers still call for additional empirically valid methods of studying professional development (Desimone, 2009; Simon, Tzur, Heinz, Kinzel, & Smith, 2000; Zaslavsky Chapman, & Leikin, 2003).

Ways Professional Development Attempts to Change Practice

Research on teacher learning has impacted the content, structure, and theory of professional development. This work has helped improve teachers and schools by providing information and ideas to educators, as well as promoting an environment for discussing teaching practices (Evans, 2002). The following sections summarize the advances in professional development for mathematics education, the popular theories of professional development, and the challenges of professional development.

In the past, professional development programs have attempted to teach mathematics education in ways similar to how mathematics was traditionally taught: through use of memorization, repeated skills, and mimicking techniques (Zaslavsky, et al., 2003). Experts would attempt to stimulate the transfer of knowledge, typically with lectures, to the teachers in order to promote teacher practices that exemplified the goals of teacher educators (Carlson, 1999; Clandinin, 1995; Korthagen & Kessels, 1999; Korthagen, Loughran, & Russell, 2006; Zaslavsky et al., 2003). This philosophy of teacher education was been called the *technical-rationality* model (Schön, 1987), and was criticized to be unsuccessful in meaningfully permeating teacher practice (Ball & Cohen, 1999). One reason for this criticism was that

> the knowledge domains of mathematics content, mathematics pedagogy, and student thinking tend to be treated separately. In particular, teachers often take some specific courses to learn mathematics, different ones to learn pedagogy, and others to gain information about how students learn. Moreover, the content of the mathematics courses is often provided apart from any deep consideration of its use in the work of teaching. One consequence of such a treatment of knowledge is that the learner is burdened with the responsibility for making the needed connections across domains and recognizing the settings in which the knowledge could be appropriately used. (Da Ponte et al., 2009, p. 191)

Researchers were not the only ones with problems with this method of professional development: “teachers have long perceived professional development, though well intentioned, to be fragmented, disconnected, and irrelevant to the real problems of their classroom practice” (Lieberman & Pointer Mace, 2010, p. 77). Furthermore, studies emphasizing the powerful role teacher preconceptions play in teacher learning, and their resistance to change, gave support to the idea that simple transfer of theory to practice was inadequate for teacher education programs (Ben-Peretz, 1995; Korthagen & Kessels, 1999; Scardamalia & Bereiter, 1989).

In response to criticism, professional development programs began to move towards reform-oriented perspectives of learning in the late 1990’s (Bullough & Kauchak, 1997; Darling-Hammond, 1994; Llinares & Krainer, 2006; Zaslavsky et al., 2003). Some of these theories, such as constructivism (von Glasersfeld, 1991; Fosnot, 1996), cognitive guided instruction (Carpenter et al., 1989), and situated cognition (Brown, Collins, & Duguid, 1989) began a trend to structure programs to have an emphasis on practice instead of theory, where teachers actively make sense of their prior knowledge, beliefs, and nature of teaching (Korthagen et al., 2006; Llinares & Krainer, 2006; Smith, 2001; Zaslavsky et al., 2003). In contrast to the *technical-rationality model* of teacher education, in these *practice-based models* knowledge was viewed as connected and tied to the educational setting in which the knowledge was used (Da Ponte et al., 2009). Ball and colleagues (Ball & Cohen, 1999; Ball & Bass, 2003) advocated that this highly contextualized and integrated nature of learning experiences made these professional development programs more suited to the authentic aspects of mathematics teaching.

Changes in teacher practice using the *practice-based model* were slow to emerge, but some positive results, such as teachers' increased acceptance of learning and teaching modes, were found (Brown & Borko, 1992; Carpenter et al., 1989). Even when these changes did emerge, "a basic problem was still not being addressed adequately, much less solved, namely, how to connect theory and practice in such a way that teachers would be able to handle the problems of everyday teaching through theory-guided action" (Korthagen et al., 2006, p. 1021). While this gap between theory and practice is not unique to the field of education, (Malara & Zan, 2002; Mason, 1994; Wideen, Mayer-Smith, & Moon, 1998), "the theory-practice issue seems intractable: telling new teachers what research shows about good teaching and sending them off to practice has failed to change, in any major way, what happens in our schools and universities" (Korthagen et al., 2006, p. 1038).

A variety of theoretical concepts have been incorporated into teacher education in recent years in order to overcome the theory-practice gap as well as other challenges in mathematics teacher education. For instance, "Piagetian notions of assimilation and accommodation, social constructivism, noticing/awareness, critical colleagueship, actor-network theory, communities of practice, psychoanalytical and post-modern theories" (Goos & Geiger, 2010, p. 500) have all been used to study teacher learning (Sánchez, 2011; Sriraman & English, 2010). However, researchers admit that only preliminary conjectures have been made about what and how teachers learn in professional development settings, and how this development contributes to educational settings and students' outcomes (Borko, 2004). There is continued effort to expand the knowledge base concerning professional development programs. The three sections below describe developing areas of mathematics teacher education. These sections include developing meaningful mathematical tasks in professional development; focusing on the social components of teacher learning through the use of communities of practice; attempting to understand and advance teacher knowledge, and integrating the CCSSM into professional development.

Productive Tasks for Mathematics Teacher Education. Researchers such as Zaslavsky et al. (2003) have emphasized the important role mathematical and pedagogical tasks play in professional development. These and other researchers have found that tasks combining mathematics and pedagogy are integral to teachers learning the material and subsequently teaching their students (Llinares & Krainer, 2006). Leikin expands upon his previous work with Zaslvasky and others (Jaworski, 1994; Zaslavsky & Leikin, 2004), saying effective tasks promote this development by being designed to be "powerful, from the mathematical and pedagogical points of view and encourage teachers to explore situations from the perspective of mathematical challenge, [and] sensitivity to students and management of learning" (2011, p. 995). Given the sometimes fragmented depiction of teacher knowledge within professional development programs (Korthagen & Kessels, 1999), productive tasks can also be a way for developing connectedness in mathematical knowledge and in teaching that knowledge; therefore incorporating these types of tasks in professional development for mathematics teachers has been promoted by researchers (Adler, 2005; Da Ponte et al., 2009; Leikin, 2011).

Communities of Practice. A focus on professional development supporting teachers' construction of individual and social knowledge, rather than transferring knowledge, has created the need for researchers to examine the environment in which teachers interact (Zawojewski et al., 2003). As learning in professional development becomes conceptualized in terms of a social process, communities of practice have become incorporated into professional development (Matos, Powell, Sztajn, Ejersbø, & Hovermill, 2009). Defined as a group of people who share an interest or a profession, the collaboration between teachers has contributed to the success of new professional development initiatives (Lave & Wenger, 1991; Llinares & Krainer, 2006; Peter-Koop, Santos-Wagner, Breen & Begg, 2003).

Advances in Teacher Knowledge. The learning of mathematics continues to be an important goal in the professional development for mathematics teachers (Zawojewski et al., 2003). Improving mathematics teacher knowledge requires attention to both content and pedagogical content knowledge (Mohr, 2006; Sánchez, 2011), and professional development is a main mechanism to promote development of teacher knowledge (Ball, 1991; Hill & Ball, 2004). Specifically, Ball and colleagues stress the need for professional development to improve mathematics teachers' range of knowledge in focused ways (Ball, Thames, & Phelps, 2008). However, issues in developing quantitative instruments to measure such learning in professional development programs seem to have deterred research in this area (Adler, 2005; Hill & Ball, 2004).

Implications

A number of important questions remain about how teacher education and professional development can support teachers to make productive changes to their practice in ways that reflect the CCSSM and other reform efforts (Adler et al., 2005; Arbaugh & Brown, 2005; "Closing the Expectations Gap," 2011; Sztajn et al., 2012). Researchers claim literature surrounding professional development is lacking, which is problematic because this research base is needed to ground professional development efforts (Heck et al., 2010; Marrongelle et al., 2013; Wiener, 2013). Questions about teachers' systems of interpretation, also called conceptual systems, for teaching and learning have arisen (Doerr & Lesh, 2003). Researchers cannot directly see how teachers are thinking, nor do they have the ability to completely describe the multidimensional components of these ways of thinking. However, the goal of professional development is to influence these patterns of thought in ways that support productive classroom practice and student learning (Zawojewski et al., 2008). Thus further research has been recommended to investigate teachers' interpretive systems in order to help students, teachers, and educators understand the nature of changes that need to take place (Lesh, 2006; Lesh, Middleton, Caylor, & Gupta, 2008; Lesh & Zawojewski, 2007; Zawojewski et al., 2008).

In reference to investigating teachers' interpretive systems, it remains to be seen how professional development can productively alter these ways of thinking, especially since this has been deemed difficult to do in practice (Schorr & Koellner-Clark, 2003; Stigler & Hiebert, 1999). Zawojewski et al. (2008) elaborate that "if the goal is to study teachers' interpretive systems as teachers develop, then professional development experiences need to be designed to make teachers' interpretive systems grow and to trace those changes" (p. 227). This need to improve professional development has spurred new research paradigms attempting to advance teacher education and provide ways for research to be conducted in ways that support construction of effective professional development (English, 2003; Lesh, 2006). This study's theoretical perspective section discusses one perspective developed by Lesh and colleagues aimed at addressing this research need.

Summary of Literature Review

In the United States the CCSSM attempts to provide students a curriculum that is aligned with skills students need to succeed in the real world and is expected to bring changes in the next years for K-12 mathematics education. Quantitative reasoning is one aspect of mathematical reasoning with long-standing importance in mathematics education, with reform efforts putting increased attention on this type of reasoning. However, “the success of any plan for improving educational outcomes depends on the teachers who carry it out and thus on the abilities of those attracted to the field and their preparation” (National Research Council, 2010, p. 1)

Research is needed about how teacher education can identify and improve teachers’ ways of thinking about quantitative reasoning. By including quantitative reasoning into professional development, teachers are more likely to be able to help their students focus on quantities and the language of quantitative relationships, and thus more aptly shape classroom discussion, and pose new problems and information in ways that support students’ development of powerful quantitative reasoning skills (Ellis, 2007). Many questions remain about how teachers think about quantitative reasoning, as well as how professional development can advance these ways of thinking (Kilpatrick, 2011; National Research Council, 2010). Intelligent strategies are needed to simultaneously investigate and improve teachers’ thinking about mathematics (Llinares & Krainer, 2006). The following chapter details one strategy this study used to attempt to address these questions.

CHAPTER III

METHODS

This chapter details the methods used to investigate the research question about how mathematics teachers' models of quantitative reasoning develop through a Model-Eliciting Activity (MEA) grounded in their classroom practice. The initial two subsections of this chapter justify why I classified the study as a multi-tiered teaching experiment and why a models and modeling perspective was the best choice of theoretical perspective to investigate this study's research question. The following subsections detail the research setting, participants, and methods of data collection and analysis.

Classification of the Current Study as a Multi-Tiered Teaching Experiment

In this study I used qualitative methods that included aspects of a design study and a multi-tiered teaching experiment. These elements guided the decisions I made regarding how the data collection instruments were created and how the data collection and analysis was structured. Qualitative methods best fit this study's research questions because of the focus on the meanings teachers made of their experiences (Ernest, 1997). Qualitative methods facilitated this study because these methods "typically produce a wealth of detailed information about a much smaller number of people and cases" (Patton, 2002, p. 14), and have the potential to make significant contributions to the field of mathematics education (Adler et al., 2005). As detailed in the literature review, little is known about teachers' constructs and their systems of interpretation about their practice. Predetermined categories of analysis and standardized instruments have not been developed in this research area, making quantitative methods an unfit choice for the study (Korthagen et al., 2006). Furthermore, "random assignment and quasi-experimental designs tend to be based on a variety of assumptions that are inconsistent [with] the kind of complex, dynamic, interacting, and continually adapting systems that are of greatest interest to mathematics educators" (Lesh & Sriraman, 2010, p. 133).

This study is best classified as a design study because the research question focused on how the evolution of learning that occurred in an educational setting (Kelly, Lesh, & Baek, 2007; Shavelson, Phillips, Towne, & Feuer, 2003), specifically teachers' learning about quantitative reasoning in a graduate-level course. A design study approach best fit this study because this approach aims to "trace both an individual's (or group's…) learning by understanding successive patterns in the reasoning and thinking displayed and the impact of instructional artifacts on that reasoning and learning (Shavelson et al., 2003, p. 26). This approach aligned with this study's theoretical perspective, since design studies have been imported using a models and modeling perspective and allow for flexibility in simultaneously designing and studying a complex situation (Zawojewski et al., 2008).

This study employed a design study approach by having iterative refinement cycles occurring as teachers completed an MEA. These cycles documented and promoted development of teachers' models. However, this study did not take on all aspects of a design study since this study was not longitudinal in nature nor did it occur over varying contexts due to limitations in time and resources (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Design-Based Research Collective, 2003).

This design study was specifically classified as a multi-tiered teaching experiment (English, 2003) because the goal was to investigate how teacher thinking developed over time. One of the advantages to using aspects of design studies and specifically a multi-tiered teaching experiment was that these choices aligned with the theoretical perspective and research question of this study. Mathematics educators have put extensive efforts into the development of guidelines for how to develop and implement data collection instruments that align with these design studies and multi-tiered teaching experiments, including teacher MEAs (Lesh & Lehrer, 2003; English & Lesh, 2003; English, 2003; Koeller-Clark & Lesh, 2001; Schorr & Koellner-Clark, 2003). The MEA designed for this study aligned with the guidelines of design studies and multi-tiered teaching experiments, and constituted the main data collection method in the study.

As stated in the delimitations section, this study was not a full multi-tiered teaching experiment because the research question did not include any investigations about students or how the teachers and students interacted (Doerr & Lesh, 2003). This study had delimitations in light of these requirements, as only one researcher investigated the question at one site over a short period of time, rather than having multiple researchers at multiple sites as is common for full multi-tiered teaching experiments. Another way this study was not a full multi-tiered teaching experiment was because I did not investigate the researcher model. Researcher models are usually incorporated in multi-tiered teaching experiments because they contribute to theory building about how student and teacher interactions contribute to development in students' and teachers' thinking (Doerr & Lesh, 2003). I departed from a multi-tiered teaching experiment approach because the research question focused on the investigation of teachers' models of quantitative reasoning. As detailed in the following sections, I incorporated qualitative methods to analyze this data and produce findings that can be used to generate theory about how teachers consider quantitative reasoning.

A models and models perspective served as the methodological framework for this study because this perspective was used to design the study, particularly by creating and implementing and MEA that served as the data collection method. This perspective is detailed in the next section because I also used a models and modeling perspective as the theoretical framework of this study.

Theoretical Perspective

The theoretical perspective used for this study was a models and modeling perspective, as described by Lesh (2003, 2006) and colleagues (Doerr & Lesh, 2003; Lesh & Yoon, 2006; Sriraman & English, 2010; Zawojewski et al., 2008). A models and modeling perspective's main focus is to document and promote learners' development of their conceptual systems, called models. For the purpose of this study, a models and modeling perspective was used to investigate teachers' models of quantitative reasoning. A central tenet of a models and modeling perspective is that teachers' model development occurs through teachers representing their ways of thinking for specific purposes, then testing and revising these ways of thinking. Model-Eliciting Activities are central to researchers who use a models and modeling perspective. These activities are carefully designed to act as a research tool by requiring teachers to document their ways of thinking, allowing teachers to test then revise their ways of thinking, and giving researchers the opportunity to observe how teachers' ways of thinking develop throughout the revisions. An MEA was used in this study as a research tool to document teacher thinking about quantitative reasoning and thus answer the research question.

This section details the major components of the theory and the rationale explaining why a models and modeling perspective was the best choice for the research question of this study. While also a learning theory in education, here the primary focus lies on the potential this perspective offers as a research framework.

Components of a Models and Modeling Perspective

A models and modeling perspective is a qualitative research perspective designed to be a method for educational research (Ernest, 1997; Lesh & Doerr, 2000, 2003). A models and modeling perspective was developed to have the power to explain conceptual systems within realistically complex problem-solving situations, particularly within educational contexts. The focus of this perspective is on the interaction between students, teachers, and researchers in order to provide contexts for development to occur for each group, or tier, or individuals (Lesh, 2006). "For example, in cases where model-eliciting activities are used, students develop models of mathematical problem solving situations; teachers develop models of students modeling abilities; and, researchers develop models of interactions among teachers and students" (Lesh, 2006, p. 10).

The following subsections focus on how a models and modeling perspective can inform researcher practice in a way that provides meaningful information about teachers' models. Each subsection provides essential information used to design and implement the methods of this study, including the nature of teachers' models, the nature of developing teachers' models, the how researchers can document the development of teachers' models.

Nature of Teachers' Models. Models for both teachers and students need to be purposeful, shareable, and reusable in other situations (English, 2003). Doerr and Lesh (2003) describe teachers' models as broader in scope in comparison to student models. Teachers' models must be able to evaluate student thinking, to implement mechanisms that promote further thinking along a multitude of directions, "to differentiate the nuances of particular contexts and situations, to see principles and more generalized understandings that cut across contexts and situations, and to support the continual revision of their own interpretations in light of evidence from experiences" (p. 131). These authors go on to say:

> A modeling perspective on teachers' knowledge allows us to go beyond the limits of constructs such as pedagogical content knowledge or knowledge of the development of children's ideas…teachers' models are not single models that conform to some predetermined standard of excellence, but rather they too are models that develop along many dimensions…hence, teachers' models are less likely to have simple names that encompass and convey the meaning of a significant portion of a teachers' knowledge about [a specific mathematical topic] (p. 132)

Developing Teachers' Models. The process of a teacher developing representation descriptions for specific purposes is called modeling, and "usually involves a series [of] iterative testing and revision cycles in which competing interpretations are gradually sorted out or integrated or both – and in which promising trial descriptions and explanations are gradually revised, refined, or rejected" (Lesh & Lehrer, 2003, p. 109). A teacher evaluates his or her model based on the model's ability to be powerful in explaining or predicting the behavior of some complex system, reusable in other situations, and sharable with others (Greeno, 1991; English, 2003; Lesh & Lehrer, 2003). These criteria are similar to model development in other fields such as engineering and architecture, where even incorrect models are useful in their rejection as they serve to advance the scientific field of research (Yildirim, Besterfield-Sacre, & Shuman, 2010). According to this perspective teachers are viewed as continually changing, with conceptual systems that are developed to make sense of new information; "therefore, as soon as we understand a system, we tend to change it; and, when we change it, our understandings also generally need to evolve" (Lesh & Sriraman, 2010, p. 126).

Two main factors constitute the criteria for whether models should be accepted or rejected: usefulness and generalizability. Useful models are ones that are simple and clearly understood from given assumptions and that generate conclusions that are not obvious yet still useful for the learner and others; generalizable models are ones that the learner evaluates as useful within contexts differing from the original context (Lesh et al., 2003).

Teachers develop their models through cycles where the assumptions, goals, and solutions can be reevaluated. This modeling cycle has been characterized in Figure 1 (Lesh & Lehrer, 2003). Each cycle requires learners to make adaptations to their ways of thinking as well as document their thinking in the product of the activity. This product can be shared with others to promote growth, and can also be used by researchers to directly observe development as well as the factors contributing to the development (Lesh, 2006, Lesh et al., 2008). Researchers such as English, Lesh, and Fennewald (2008) add that observations are also vital to researchers' work in documenting the development of learners' conceptual systems, since "after going through multiple cycles in the problem solving process and resolving model mismatches, the finished product represents a more complete, complex solution than the students' first way of thinking about the problem"

(Lesh et al., 2003, p. 226). This same line of reasoning holds for teachers' first way of thinking about a problem grounded in their classroom practice (Doerr & Lesh, 2003).

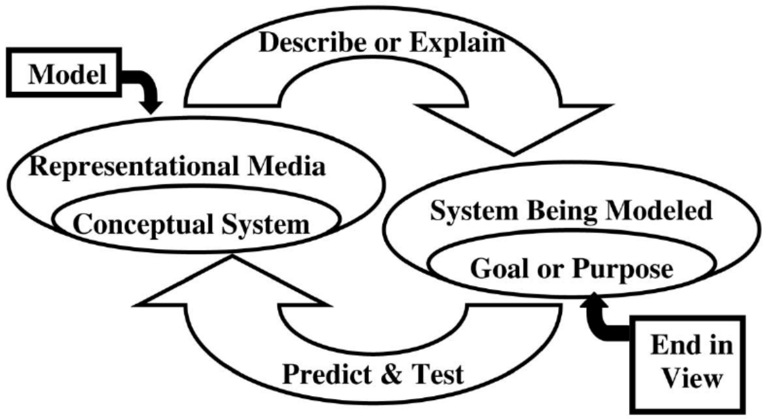

Figure 1. A modeling cycle, as depicted by Lesh and Lehrer (2003).

Documenting Teachers' Model Development. The work on developing methods to document teachers' model development occurred first by trying to capture K-12 student's model development. Lesh, Hoover, Hole, Kelly, and Post (2000) investigated how learners' understanding of a problem context can be examined. They first tried to use clinical interviews to investigate learner's understanding, but found the amount of productive student thinking, the number of interventions required, and the ability of learners to reveal explicitly their evolving ways of thinking was unsatisfactory; the high cost and time intensity for both learners, interviewers, and data analysis made this approach unfeasible for model development. As a consequence, a new type of problem was designed that would be:

(a) self-adapting ("students would be able to interpret them meaningfully using different levels of mathematical knowledge and ability, as well as using a variety of different types of mathematical descriptions or explanations"),
(b) self-documenting ("responses that students produce would reveal explicitly how they are thinking about the problem situation"), and
(c) self-monitoring ("students themselves would have a basis for monitoring their own thinking and would continue thinking in productive ways without

> continually needing to depend on adjustments by interviewers”). (Lesh et al., 2000, p. 605)

These activities are called Model-Eliciting Activities (MEAs), and are “realistic, complex problems that engage students in mathematical thinking beyond traditional school mathematics where the solution involves the creation of conceptual tools or models that can be used to communicate, makes sense of, and resolve realistic situations” (Thomas & Hart, 2010, p. 533). Model-Eliciting Activities have been used with teachers before, and have been used with students to develop quantitative reasoning skills (Doerr & English, 2003; Doerr & Lesh, 2003). Lesh and colleagues have determined that MEAs are useful for instruction, assessment, research, and investigating learning development. Lesh et al. (2000) say this is due to several reasons;

> First, to learn more about the nature of students’ (or teachers’) developing knowledge, it is productive to focus on tasks in which the products that are generated reveal significant information about the ways of thinking that produced them. Second, if intermediate solutions steps are externalized in forms that can be examined (by researchers, teachers, or students themselves), then the by-products of these learning or problem solving activities often generate trails of documentation that go beyond providing information about final results; they also reveal important information about the processes that contributed to these results. Third, when a series of trial ways of thinking is externalized, tested, and refined or extended repeatedly, such thought revealing activities often go beyond providing documentation about completed learning or problem solving experiences; they also support the productivity of ongoing learning or problem solving experiences. (p. 593-594)

While some characteristics of successful MEAs, such as being purposeful, sharable, and reusable, have already been described, researchers have developed other criteria for activities for teachers that elicit and foster the development of models. These principles intend to guide the design of MEAs in ways that promote teachers’ ability (a) to “reveal their current ways of thinking; (b) to test, revise, and refine those ways of thinking for some particular purpose; (c) to share with colleagues for replication; and (d) to reuse their ways of thinking in multiple contexts" (Doerr and Lesh, 2003, p. 133). Doerr and Lesh (2003) describe five principles, summarized in Table 1, that aim to develop activities that fit these goals, while also promoting teacher development of systems models in ways that more closely align with the "goals, purposes, and contexts that are explicit and shared with colleagues and the larger community of schooling" (p. 134). They add that establishing communities where teachers' interpretations can be seen in multiple ways, providing the potential for disturbing teachers' current ways of thinking and for resolving mismatches between the interpretation and the experienced realities of other teachers. Such interpretations include having teachers view others' interpretations from multiple perspectives (such as on the details or on the big picture), from multiple levels (such as the mathematical or the pedagogical), or for multiple purposes (such as for introducing a concept or extending it).

In addition to being successful for research purposes, MEAs for teachers also have the potential for beneficial effects on teacher learning, teacher practice, and assessment purposes in teacher education environments. Model-Eliciting Activities contribute to teacher development because they promote teachers to think more deeply about student thinking, engage in mathematics, and reflect on prior held beliefs about problem solving (Lesh, 2006; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003). Some of these changes in teachers' models have been documented to influence changes in the adoption of new strategies in teachers' classrooms; thus a models and modeling perspective "is useful in considering the conditions that are necessary for generating fundamental changes in practice" (Schorr & Koellner-Clark, 2003, p. 208).

Table 1

Teacher MEA Principles, abbreviated from Doerr and Lesh (2003, p. 133)

Teacher MEA Principle	Description of how the principles guide MEA design
Reality Principle	The situations which we ask teachers to interpret must be in the context of their actual practice. Asking teachers to interpret student work from their own classroom, or to analyze student's thinking over time on a concept they teach, or to develop an assessment task that would reveal how students are thinking would engage teachers in activities embedded in their own practice. Pulling teachers out of their classrooms to develop local curriculum standards would violate this principle.
Multilevel Principle	Tasks for teachers should address the multiple levels or aspects of the teaching and learning environment. Teachers most often need to simultaneously address mathematical content, pedagogical strategies, and psychological aspects of a teaching and learning situation. Simply addressing student thinking is not enough.
Multiple Contexts Principle	The variability in the settings, the students, and the mathematical contexts of teaching needs to be accounted for. Teachers' knowledge varies across contexts and the multiple dimensions of those contexts, so a model-eliciting activity for teachers should include variations in context that require interpretation and analysis. This leads to thinking in ways that are increasingly generalizable.
Sharing Principle	Ideas about teaching and learning need to be shared among multiple teachers and reused by those other teachers. Particularly powerful tasks for teachers are those that come from other teachers and can be used by other teachers. This in turn leads to revision and refinement.
Self-evaluation Principle	Are the purposes against which success can be evaluated sufficiently clear? Fuzzy statements of educational goals can preclude effective judgements about teachers' actions or interpretations. More importantly, teachers need to [be] able to judge for themselves whether their interpretations and consequent actions (such as teacher plans or assessment strategies) are moving towards desired ends in particular contexts.

Researchers can use MEAs as powerful tools when investigating teachers' models. Model-Eliciting Activities that require teachers to test, revise, and refine the ideas they use in the classroom can reveal teachers' models as well as the mismatches between the teachers' interpretation and the experienced realities of other teachers. The factors influencing the mismatches can then be used as the driving force for promoting development of teacher knowledge, allowing the researcher to document this development (Doerr & Lesh, 2003; Schorr & Koellner-Clark, 2003).

Relation to Study

Research on professional development and other settings that encourage teacher learning have employed a variety of theoretical perspectives (Llinares & Krainer, 2006; Mellone, 2011), but in this section I argue a models and modeling perspective was the best choice for this study's theoretical perspective. I first explain two benefits this perspective offered to the study: providing a method to examine the ways teachers interpret their practice, and (2) providing a method to structure the data collection process.

The first advantage a models and modeling perspective offered the study was a mechanism to examine teachers' models of quantitative reasoning in a way relating to their practice. Unlike other frameworks that have been used to examine teacher education, a models and modeling perspective was created specifically to have the power to explain conceptual systems within realistically complex problem-solving situations, including how teachers think about their practice. Some methods to investigate teacher thinking using a models and modeling perspective include (a) promoting teachers to think about how student learn mathematical ideas, selecting, (b) asking teachers to select curricular materials that support student learning of these mathematical ideas, and (c) having teachers evaluate the effectiveness of these curricular materials. These methods generate data that researchers can use to make conclusions about teachers' models and teachers' model development (Lesh & Doerr, 2003).

The second advantage of using a models and modeling perspective is that this perspective provided guidelines for the methods of the current study in ways that supported the potential for significant findings given the current research question. Design principles for MEAs specify how teachers' models can be elicited in observable ways (Lesh, 2006). I incorporated these principles in this study to develop an MEA aligned with the goals of a models and modeling perspective, allowing the documentation of teachers' models. A models and modeling perceptive also offers frameworks for understanding teachers' models and their development by focusing the researchers' attention to how teachers' model develop through iterations of revisions (Koellner-Clark & Lesh, 2003; Hiebert & Grouws, 2007; Hjalmarson, 2008; Silver & Herbst, 2007; Sriraman & English, 2010). In this study I adapted Hjalmarson's (2008) framework to analyze the MEA data.

Answering my research question in terms of a models and modeling perspective involved indicating how teachers' models of quantitative reasoning developed in the course. Since sharing this information with other mathematics educators and those involved with professional development is a goal of this perspective (Doerr & Lesh, 2003; English, 2003; Lesh & Doerr, 2000), a models and modeling perspective offered me a way to produce significant findings with regards to theory, MEA development, and practical implications that support teacher learning.

Finally, a models and modeling perspective allowed me to answer the research question in ways that impact the practices of teacher and teacher educators. Using MEAs to reveal teacher interpretive systems allows researchers to evaluate the usefulness of teachers' models (Lesh et al., 2003; Schorr and Koellner-Clark, 2003; Thomas & Hart, 2010) and "potentially support the development of more refined and integrated models of teaching" (Doerr & Lesh, 2003, p. 132). A models and modeling perspective offers researchers a way to conduct research that will address this need in the field of mathematics teacher education, though additional research is still needed (Carlson, Larsen, & Lesh, 2003; Lesh, Hamilton, & Kaput, 2007; Llinares & Krainer, 2006). Specifically researchers have recommended the development of additional descriptions, artifacts, and models depicting how teachers understand quantitative reasoning and other Standards for Mathematical Practice in order to support collaborative implementation efforts of the Common Core State Standards for Mathematics (CCSSM) across the United States; these models could impact how future professional development is structured and the design of CCSSM instructional materials and assessments (Confrey & Krupa, 2010; Garfunkel et al., 2011).

One way this study did not incorporate a models and modeling perspective was by my choice to use other techniques of qualitative analysis to generate the findings rather than documenting the researcher model as part of the study. A goal of a models and modeling perspective is to eliminate bias views from researchers about learners' models (Doerr & Lesh, 2003). Documenting the researcher's model and removing researcher interpretations from the analysis as the MEA progresses is a main way that a models and modeling perspective reduces biases in the data analysis and findings of a study. I departed from this perspective's approach to removing bias because I used qualitative techniques that worked to eliminate researcher biases from influencing results. These techniques included keeping a research journal, creating an audit trail, and incorporating a coding scheme (Hjalmarson, 2008) aligned with a models and modeling perspective. These techniques are detailed in the Trustworthiness section that follows.

Research Setting

The research setting includes details relevant for this study, such as the setting of the university, the participants involved in the study, the researcher's position within the study, and the basic procedures that describe how data was collected and analyzed. Each of these details is included in the subsections below.

University Setting

This research took place within a university with approximately 10,000 undergraduate students and 2,300 graduate students, with a student to faculty ratio of 19:1. This study took place within two settings in a mathematics department: one at the graduate level and one undergraduate level. The university reported 88% of undergraduate students come from the state of the institution, with 60% of these students female, 40% male. With a first year retention rate of 70%, 19% of students identify with an ethnic minority, with the top two ethnicity categories being White (62%) and Hispanic (13%). The largest portion of undergraduate students (30%) select majors in the College of Natural and Health Sciences, where the School of Mathematical Sciences was located.

The graduate setting was within a grant-funded program for teachers attaining a master's degree in mathematics with a teaching emphasis. A National Science Foundation grant funded the program, which worked through two mid-sized doctoral granting universities in the Rocky Mountain region to offer courses to mathematics teachers in the surrounding states. This two-year program was designed for middle and high school mathematics teachers, where courses were offered year-round through online and blended educational settings. The self-described goals of this program included developing culturally competent teachers and improving these teachers' practice and student achievement within the local, regional, and national levels. Specific emphasis was placed on advancing teachers' content knowledge, pedagogical content knowledge, reflective practice, and cultural knowledge of both educational settings and mathematics. The program incorporated a variety of online technologies during the school year and combined distance and face-to-face instruction in the summer. The software used to facilitate the online courses is Blackboard Collaborate, formerly called Elluminate, where weekly synchronous video conferencing took place using computers connected to the internet.

This Master of Arts in Mathematics for Secondary Teachers was a two to three year, 30-credit program where teachers completed 18 credit hours of mathematics and 12 credit hours of mathematics education coursework. The mathematics content courses were designed to reflect secondary mathematics content, and included Abstract Algebra, Number Theory, Applied Probability and Statistics, Modern Geometry, and Continuous Mathematics. The mathematics education courses were usually paired with the content courses, and offered an adjacent semester, if not concurrently during the same summer, as the corresponding content course. Examples of these courses included Teaching Geometry, Teaching Applied Probability and Statistics, Teaching Algebra and Trigonometry, Teaching Discrete Mathematics, and Quantitative Reasoning in Secondary Mathematics.

This study focused on the Quantitative Reasoning in Secondary Mathematics course offered in June. The Quantitative Reasoning in Secondary Mathematics course met synchronously online four times a week for four weeks. During these meetings live audio and video feed were used for interaction, and a whiteboard was used as a tool for sharing written texts, such as PowerPoint slides, between the instructor and the teachers. Additionally, virtual spaces where small groups of teachers can interact, called breakout rooms, were used to facilitate small group discussions between the 21 in-service mathematics teachers.

The instructor of the course was a mathematics educator who had designed and taught numerous secondary mathematics and science courses for pre- and in-service teachers. This was the instructor's first time teaching a pedagogy course on a quantitative reasoning topic for teachers, and his first time teaching an online course for teachers. A graduate student facilitated the use of technology and assisted in administrative tasks during the course.

The course used aspects of a models and modeling perspective to promote teacher development of quantitative reasoning and other mathematics concepts. An MEA constituted 50% of the course grade and task analyses constituted the other 50%. While the MEA assignments are detailed in the data collection section, the task analyses were not used in the data collection. As indicated in the course syllabus (Appendix A), a main course objective included teachers understanding ideas such as the meaning of quantities, quantitative relationships, and quantitative reasoning. Additional goals included teachers being able to identify these ideas in secondary mathematics curriculum and deepen their understanding of secondary mathematics content involving quantities and quantitative reasoning. A final goal was for teachers to develop MEAs that support and document the development of student understanding and reasoning. The course reading list was comprised of articles focused on MEA development and quantitative and mathematical reasoning for students.

While the main portion of this study took place at the graduate level, participants of this study also came from two undergraduate courses. These students were asked to complete tasks that the teachers created in the Quantitative Reasoning course. The first undergraduate course, called Business Calculus, was offered in June and was for undergraduates who had passed college algebra or two years of high school algebra. The focus was on concepts of calculus with emphasis on the applications of economics and business in differentiation and integration. This course, designed specifically for students majoring in the areas of business and social sciences, used the Applied Calculus (4th edition) by Hughes-Hallet et al. (2010). Course objectives included developing skills in discussing quantitative relations using multiple representations, acquiring an intuitive interpretation of local and instantaneous rates of change (derivatives) using the multiple representations, developing computational skills that permit the efficient determination of rates of change, and applying calculus theory to analyze significant problems in the social sciences. The course was 3 credit-hours, and met for 100 minutes, four days a week, for six weeks. This course was required for economics majors, and students majoring in business have the option of taking Business Calculus or the standard calculus course, though Business Calculus was recommended.

The other undergraduate course was called Liberal Arts Math, and was designed for students with at least one year of high school algebra. Course objectives included developing understanding of the techniques involved in constructing mathematical models using mathematical problem solving strategies, as well as evaluating, proposing a solution method, and solving real life problems. The course used the book *Using & Understanding Mathematics: A Quantitative Reasoning Approach* (5th edition) by Bennett and Briggs (2011). This 3 credit-hour course met for 100 minutes, four days a week, for six weeks.

Participants

Two groups of people are the participants of this study: undergraduates in the summer Business Calculus and Liberal Arts Math courses and teachers in the summer Quantitative Reasoning in Secondary Mathematics course. I describe each of these populations below.

Nine students were in the Business Calculus course and seven were in the Liberal Arts Mathematics course. All 16 undergraduates agreed to participate and all were over the age of 18. Ten men and six women comprised the undergraduate population. Other demographic information was not collected, and was assumed to reflect that of the undergraduate university population described above.

The graduate population consisted of all 21 mathematics teachers enrolled in the Quantitative Reasoning in Secondary Mathematics course. The teachers in this population were in the master's program for at least one year prior to taking the course. This population was similar to previous cohorts in the program, who, based on prior research, had a strong sense of community among individuals (Glassmeyer, 2012; Glassmeyer, Dibbs, & Jensen., 2011; Glassmeyer & Goss, 2011). The teachers had experienced the online software and were familiar with their peers in the program. The requirements for admittance into the program ensured all teachers have taught for at least two years, and were currently teaching mathematics between grades 6 and 12.

The 21 participants had taught a mean of 8.5 years, with a range of 3 to 20 years of experience teaching K-12 mathematics. Eleven women and 10 men participated in the study, with 15 of them teaching high school grades (9-12) and six of them teaching middle school grades (6-8). These teachers worked in schools across two western states. Of the 19 teachers who provided demographic information, 18 were White and one was Hispanic. Eight worked in remote town or rural school districts, based on a United States Department of Education's urban-centric coding of the school location[1]. This group was my accessible population, and the target population I considered was mathematics teachers within similar focused graduate-level coursework in the United States.

Researcher Stance

My own experience comes into the setting of the study, particularly since in qualitative research the researcher is a data instrument (Guba & Lincoln, 1989; Patton, 2002). Here I explain my experience with the master's program for teachers and the research question.

I had worked for two and a half years in various positions in assisting this grant in the construction, delivery, assessment, and teaching of the courses. Being a co-instructor the prior summer allowed me to know some of the teachers quite well, and most of the individuals were also in the Quantitative Reasoning in Secondary Mathematics course. I assisted in the development of the Quantitative Reasoning in Secondary Mathematics course, which offered additional information about teachers' experience prior to the

[1] A rural district was classified as having an urban-centric coding of 33, 41, 42, or 43.

study's start. In addition to my teaching and assistant roles in the program, I also had a previous researcher role within the master's program, as I had conducted studies about the impact technology-delivered instruction had on these teachers (Ku, Akarasriworn, Rice, Glassmeyer, Mendoza, 2011; Powers, Glassmeyer, & Ku, 2011), the role of formative assessment in these courses (Glassmeyer et al., 2011), and the importance of a sense of community these teachers have (Glassmeyer, 2013). I have also published a methodological article on how data can ethically be collected in these virtual settings (Glassmeyer & Dibbs, 2012).

I have been through a master's program for teachers, though in education rather than mathematics. Throughout the program I was faced with the challenge of applying information in coursework to the classroom I was teaching; this experience shaped my opinion that professional development should be tied to teachers' needs in ways that were accessible to them. During the design of the study I continued to have this perspective. Having worked with this program and these teachers, I believe my experiences helped the design and implementation of this study.

I positioned myself as a researcher within the Quantitative Reasoning in Secondary Mathematics course. I occasionally assisted with the administration and organization of the MEA documents, though this role did not interfere with my duties as a researcher. These roles reinforced my previous researcher role with the teachers in the course. More importantly, these roles are distinct from an evaluator or co-teaching role, since I aimed only to capture teacher thinking rather than evaluate it. My position with the students in the undergraduate courses was as a researcher and partial designer of the MEA task. I introduced myself near the beginning of each course and was a sporadic participant observer to establish my role as a researcher wishing to see how the students think about mathematics. I observed a portion of each class every other week prior to implementing the tasks in the classroom. Having the undergraduate students in the course interact with me prior to the MEA implementation in the course helped facilitate quality data collection.

I believe I am qualified to conduct this research because I have designed and implemented studies on this population and program in the past, publishing and presenting the results in peer reviewed conferences and journals. This study took place in a setting similar to my previous work, allowing me to anticipate some of the issues that can occur in teacher education settings as well as technology-delivered instruction. In addition to completing all required coursework and examinations, I have also elected to obtain a doctoral minor in statistical research methods, focusing on qualitative methods. This decision allowed me to work with an expert qualitative methodologist to design, conduct, and write up a number of qualitative research studies. Finally, I have participated in discussions, presentations, and discussion groups at national conferences in topics related to the present study. In particular, the *Association of Mathematics Teacher Educators* annual conference gave me perspective on the need for helping teachers adopt to the Common Core State Standards. Also, my involvement in the Models and Modeling working group at the *Psychology of Mathematics Education-North American Chapter* annual conference allowed me to do hands-on work with MEAs and to discuss a models and modeling perspective with experts on this research approach, such as Lyn English. I therefore feel I have the expertise needed to conduct this study, which is overviewed in the following section.

Basic Procedures

Working with the instructor of the course, I helped design the MEA and the implementation procedures for the MEA. The instructor selected the course readings and activities and then engaged the 6 middle school mathematics teachers and 15 high school mathematics teachers in (a) readings from the CCSSM and educational research related to quantitative reasoning, (b) mathematical tasks drawn from the Pathways to Calculus (Carlson & Oehrtman, 2011) materials, (c) analysis of the tasks using the perspectives developed from the readings, and (d) an MEA. The instructor and TA of the course evaluated all assignments but met with me regularly to discuss the progress of the course.

The MEA was designed to engage the teachers in thinking about realistic and complex problems embedded in their practice at multiple levels (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). At the first level, the teachers were to design a task that engaged their students in a complex mathematical task involving quantitative reasoning and requiring students to produce a model that is to be powerful and testable in explaining or predicting the behavior of the complex system, reusable in other situations, and sharable with others (Lesh & Lehrer, 2003). The task allowed for self-documentation of the students' quantitative reasoning in a way that could be shared with other students and the teacher. At a second level, the teachers were asked to document their task design in a way that made it powerful and testable for student learning and assessment, reusable in other classes, and sharable with other teachers. This documentation was a part of the MEA requirements, and encouraged teachers to reveal aspects of their model of quantitative reasoning. Finally, my analysis was focused on producing a characterization of teachers' models of quantitative reasoning. My documentation was intended to produce a powerful, testable, reusable, and sharable model for how teachers think about quantitative reasoning.

In the MEA, the teachers went through four rounds of revision by incorporating feedback from the instructor, peers, and student work. Throughout the MEA, teachers documented their models by generating four documents. First, a Quantitative Reasoning Task documented what teachers envisioned quantitative reasoning looking like in their classrooms. Second, the Facilitator Instructions documented teacher thinking about how the Quantitative Reasoning Task would unfold and what they would attend and respond to in the process. Third, the Assessment Guidelines documented the reasoning teachers expected students to develop and what teachers would count as evidence of that reasoning. Fourth, the Decision Log documented the reasons teachers made each changes to the other three documents throughout their multiple revisions.

Data collection consisted mostly of collecting the iterations of documents generated by the MEA, with observations of in-class time for the group to work on creating these documents. Using content analysis on the documents, I identified patterns in the ways teachers' thinking about quantitative reasoning developed through this process.

Data Collection

Three types of data answered the research question. First, the documents from the MEA comprised the majority of the data. Second, in-class observations of teachers working on the MEA provided supplemental data about how teachers' models developed. Third, two interviews were conducted at the conclusion of the course provided additional data for the development of one group's model of quantitative reasoning. This section outlines the three data sources chronologically, since each iteration of the MEA influenced the subsequent data collection processes.

Documents from the Model-Eliciting Activity

Developing an MEA for this study was a significant undertaking, which I completed with the help of multiple individuals and after going through several iterations of development. This study's MEA was created and implemented as the main data source in this study, and was tailored to both the setting of this research study as well as the research question. This type of MEA was unprecedented in the mathematics education literature, as no other MEAs existed that (a) addressed mathematics teachers in a summer professional development course or (b) documented learners' ways of thinking about quantitative reasoning. This section describes the construction of the MEA before focusing on the documents teachers created as they completed the MEA.

Construction of the Model-Eliciting Activity. The construction of the MEA for this study followed prescribed guidelines given by a models and modeling perspective to ensure this tool would be suitable for data collection purposes. As indicated in Chapter 3, careful planning is needed on the researcher's part in order to ensure an MEA for teachers is purposeful, sharable, and reusable, and also upholds the five principles given to guide the design of activities within teachers' practice (English, 2003). Table 1 summarized Doerr and Lesh's (2003) five principles for MEAs for teachers, which include the (1) reality, (2) multilevel, (3) multiple contexts, (4) sharing, and (5) self-evaluation principles. This section details how the construction of this study's MEA upheld each of these principles before describing why this MEA provides the data needed to answer this study's research question.

The first MEA principle, the reality principle, was addressed in this study by having teachers select, adapt, or create a Quantitative Reasoning Task for their future students. Since the course did not take place during the school year, one of the first challenges in developing an MEA that upheld the reality principle was to create a situation similar to teachers' everyday practice that revealed teacher thinking about quantitative reasoning (Doerr & Lesh, 2003; Koeller-Clark & Lesh, 2001, 2003; Schorr & Koellner-Clark, 2003). Resolution to this challenge came when the context was framed in terms of developing a task that would be implemented in the teachers' future classroom. The main crux of the MEA was to create or modify a task that captured and evaluated students' thinking about quantitative reasoning, where the students were indicated to be the students that teachers anticipated teaching in the fall. Examining what the teachers choose as an important quantitative reasoning task would reveal their ways of thinking about quantitative reasoning (Chamberlin, 2004).

Including Assessment Guidelines into the MEA also supported the reality principle. The Assessment Guidelines asked teachers to provide expectations for their Quantitative Reasoning Task and a way to evaluate student thinking. These requirements supported the reality principle by having teachers interpret and assess student thinking (Doerr & Lesh, 2003). To facilitate the need for assessment guidelines, the MEA included a undergraduate student feedback iteration during the summer course. This iteration gave the teachers a chance to see student thinking and evaluate the students' work based on the assessment guide they developed. The challenge of designing this iteration, however, was that teachers did not have access to their own students during the summer. Thus the incorporation of the Business Calculus and Liberal Arts Math courses offered a mechanism for providing teachers feedback on their task in order to test their Assessment Guidelines.

The second MEA principle, the multilevel principle, was upheld by requiring teachers to create Facilitator Instructions that accompanied the Quantitative Reasoning Task. The Facilitator Instructions document had teachers detail how their task should be implemented and discuss potential challenges and anticipated student thinking. One design feature in creating the Facilitator Guidelines was that this document needed enough detail for me, the researcher, to implement the Quantitative Reasoning Task without the teachers present. Having the teachers implement the task themselves was not an option, given the physical distance between teachers in the online summer course. Thus Facilitator Instructions were framed in terms of how another educator should implement the task, including the appropriate pedagogical strategies related to the task. These MEA requirements prompted teachers to simultaneously address the mathematical content in the task and pedagogical strategies and student thinking in the Facilitator Instructions, and thus supported multilevel principle (Doerr and Lesh, 2003).

The third MEA principle of multiple context was supported by having teachers work in groups and by providing peer feedback iteration in the MEA. Groups of three to four teachers worked together, where the teachers in each group worked in common grand bands. Together, the group completed the MEA for a setting similar to their classrooms, and using the variety of experiences from each member to contribute to the way the Quantitative Reasoning Task is developed. The MEA thus allowed teachers to select their student population anticipated in the fall. To facilitate between group interactions, a peer feedback process constituted one of the iterations of the MEA. By swapping the Quantitative Reasoning Task between two groups, the teachers were exposed to other settings and quantitative reasoning tasks. Asking the teachers to analyze another group's Quantitative Reasoning Task required teachers to interpret and analyze different activities they could implement in their classroom. This activity helped teachers account for "variability in the settings, the students, and the mathematical contexts of teaching needs" (Doerr and Lesh, 2003, p. 133). In addition to promoting the multiple context principle, these requirements also supported generalizable thinking through exposing teachers to different ways of thinking about quantitative reasoning through group work and the peer feedback process (Chamberlin, 2004).

The fourth MEA principle, the sharing principle, was supported in several ways. The setup of this study's MEA upheld this principle by asking teachers to create Quantitative Reasoning Tasks that can be implemented by another educator. Additionally this principle was incorporated by requiring teachers to create the Facilitator Instructions for another educator who wishes to implement their Quantitative Reasoning Task, and by asking the Assessment Guidelines to be able to be used by another educator. Furthermore, a Decision Log was also required of the teachers. This document asked teachers to develop a running log of decisions as they developed their Quantitative Reasoning Task, and was framed in terms of being able to share this information with other teachers in order for them to develop their own quantitative reasoning tasks. These requirements supported the sharing principle because ideas about the teaching and learning of quantitative reasoning were shared between teachers with the possibility of being reused by those teachers (Doerr and Lesh, 2003).

The fifth principle of self-evaluation was supported by the MEA's Decision Log and the peer feedback iteration. The prompts in the Decision Log were structured to promote teachers' reflection on their actions in the other documents. The peer feedback iteration also supported reflection by having teachers swap documents with another group of teachers and evaluate the documents in light of the requirements stated by their Quantitative Reasoning Task. This evaluation process was designed to promote reflective thinking both in the assessment of another group's work as well as in receiving the feedback from others. Creating the MEA in this way promoted teachers to judge whether "their interpretations and consequent actions (such as teacher plans or assessment strategies) are moving towards desired ends in particular contexts" (Doerr & Lesh, 2003, p. 133). Thus the Decision Log and peer feedback process supported the self-evaluation principle.

In addition to satisfying the five teacher MEA principles, this study's MEA was designed to answer the research question regarding how teachers' models of quantitative reasoning develop. This MEA was designed to provide the data to do this by having teachers document multiple ways of thinking about quantitative reasoning in the various tasks. The Quantitative Reasoning Task was designed to capture teachers' ways of thinking about quantitative reasoning in terms of how they thought their students should engage in quantitative reasoning. Teachers' ways of thinking about quantitative reasoning are further shown by the Assessment Guidelines developed to evaluate student work as well as the when teachers completed the actual assessments of student work. Teachers' thinking about quantitative reasoning in their classroom can be seen in the Facilitator Guidelines in the form of pedagogical decisions and anticipated student response. Finally the Decision Log was structured to ask teachers to define quantitative reasoning and how it relates to their task. These multiple perspectives provided data to infer teachers' models of quantitative reasoning.

To capture the development of teachers' models, multiple iterations of these documents were prompted by five interaction cycles constructed in the MEA. These iterations gave teachers the opportunity to test, revise, and refine their models of quantitative reasoning in ways that were documentable. The first iteration occurred after the assignment was given during the first week of class. A second iteration occurred at the end of the first week, in response to feedback from the instructor. The third iteration occurred at beginning of the second week, prompted by peer feedback as one group of teachers swapped all documents with another group and provided feedback. The fourth iteration occurred during the fourth week of the course in response to undergraduate student work on teachers' tasks. The fifth iteration occurred only for some teachers after the course had ended, prompted by feedback from the teachers' implementation of the Quantitative Reasoning Task with their own students. Details for these documents and the feedback cycles are provided in the following section.

Description of Documents Created Through the Model-Eliciting Activity. This section describes in detail the documents teachers created because of the MEA. A brief summary is given of each document that teachers generated (Table 2), followed by the chronological view of the documents as they occur in the course (Figure 2). Since the MEA document, located in Appendix B, will be referenced frequently in this section, it may be helpful to read this appendix prior to the explanations that follow the table and figure.

Table 2

Overview of MEA documents and iteration cycles.

Document Name	Brief summary of the document
Pre-Assignment	Assignment requesting teachers' initial models of quantitative reasoning, quantitative reasoning tasks for students, and the relation of these models to the upcoming course
Version 1	Assignment requesting (a) a Quantitative Reasoning Task that captures deep thinking about students' quantitative reasoning skills; (b) Facilitator Instructions suitable for other educators to implement the task and foresee potential challenges; (c) Assessment Guidelines suitable for someone else to evaluate the task; and (d) Decision Guidelines that articulate your decisions, changes, and refinement of the above three documents
Instructor's Feedback	Instructor's comments and suggestions to Version 1
Version 2	Updated Version 1 documents in response to the instructor's feedback
Teachers' Feedback	A group of teachers' comments and suggestions to another group's Version 2
Version 3	Updated Version 2 documents in response to the teachers' feedback
Undergraduate Student Feedback	Tasks (part (a) of Version 3) completed by a small group of undergraduate students
Version 4	Updated Version 3 in response to the student work, plus the actual evaluation of the student work
K12 Student Feedback	Tasks (part (a) of Version 4) completed by five of the K-12 students in the teacher's fall courses
Version 5	Updated Version 4 in response to the feedback received from implementation, plus the actual evaluation of the K-12 student work.

All documents except for the Implementation Feedback and Version 5 occurred during the Quantitative Reasoning Course. Specific due dates are given in the course calendar (Figure 2). This calendar was constructed with the instructor of the course.

June 2012				
Monday	Tuesday	Wednesday	Thursday	Friday
4 *1st Day of Class*	5 Pre-Assignment Due	6	7	8 Version 1 Due
11 Instructor Feedback	12	13 Version 2 Due	14	15 Peer Feedback
18 Version 3 Due	19	20	21 Undergraduate Student Feedback	22
25	26 Version 4 Due	27	28 *Last Day of Class*	29

Figure 2. Calendar of the course schedule and MEA feedback cycles.

I detail the data collection events that occur in a chronological fashion, indicating what actions are completed by me, the teaching assistant (TA) of the course, and the instructor of the course. I made sure to obtain Institutional Review Board approval (Appendix C) and collect all signed consent forms (Appendix D) prior to data collection.

Pre-Course Data Collection. The Pre-Assignment was given in the form of an email from the TA and instructor of the course, and began by welcoming the teachers to the course. The teachers were asked to read the information about the Pre-Assignment and respond to the questions given in Appendix B by Tuesday, June 5 at 8am. One purpose of assigning the Pre-Assignment early was to give teachers ample time to work on the task in order to promote quality responses. If we had assigned the assignment the first night of class, teachers would have had less time to complete the assignment by the deadline we requested.

The format of the assignment was purposefully left open ended in order to capture teachers' initial models of quantitative reasoning, quantitative reasoning tasks for students, and the relation of these tasks to the upcoming course. The TA of the course sent a reminder of this assignment on Friday, June 1 to all teachers who did not complete the Pre-Assignment. The teachers were asked to email their responses the TA, the instructor of the course, and me by the beginning of the second day of class, Tuesday June 5. This date and time was selected so the instructor could form groups of teachers for the MEA based on what students the teachers anticipated and preferences listed in their responses. All but one teacher turned in their Pre-Assignment.

Week 1 Data Collection. One the first day of the course, Monday June 4, the instructor discussed the syllabus and introduced the MEA document. The instructor indicated the MEA would constitute 50% of the course grade. Pages one through three of the MEA document (Appendix B) were distributed to the teachers and the instructor introduced the MEA as a way to guide the teachers towards the ultimate goal of implementing a quantitative reasoning task in their classroom the following fall. An overview of models, a models and modeling perspective, and MEAs was also incorporated into the first day introduction discussion. During the first class, I indicated how I was conducting research about the process each group goes through by looking at documents created by the Pre-Assignment, the MEA documents, and from classroom observations. I also articulated how my role was not as an evaluator, but as an observer. The teachers had the opportunities to ask any questions or concerns about the project or to request additional information.

Before the second day of class, the instructor, TA, another teacher educator familiar with the teachers, and I formed the groups based on the 20 teachers who had submitted the Pre-Assignment. Six groups, described in Appendix E, were eventually decided upon after taking teacher preferences into consideration. These groups were announced on the second day of the course and were given instructions to create a Dropbox folder shared with the instructor, TA, and me. The instructor indicated that all documents associated with the MEA were to be placed in this folder. Then class time was allocated for these new groups to brainstorm ideas about how to create the Version 1 documents. These documents, outlined in the MEA document given to the teachers the previous day, included the Quantitative Reasoning Task, the Facilitator Guidelines, the Assessment Guidelines, and the Decision Log.

During subsequent days of week 1, I selected Group 1 to focus my observations on, based on their initial observation patterns of clear communicating between the group members. During the group work time, I observed the conversations that occurred and video recorded these conversations using Camtasia. Using the observation protocol in Appendix F based on recommendations by Creswell (2007), I documented any notions of quantitative reasoning during this work time in my researcher journal.

On Thursday, June 7, I made a copy of every group's Version 1 folder in a private, secure folder for safekeeping. Each group submitted the requested files, and all of the documents were in Word 2003/2007 format. Also, I conducted a preliminary analysis of this first iteration, with special focus on the Group 1, since I had more data on this group from class observations. The instructor and TA evaluated Version 1, and I offered some input regarding data collection. For instance, I suggested we ask groups who did not provide definitions of quantitative reasoning to address this prompt in Version 2. Instructor feedback was returned to each group's Dropbox folder on Friday June 8. I made a copy of the instructor's feedback for analysis.

Week 2 Data Collection. Similar to the first week, work time was given for groups to think about the instructor feedback, ask questions, and revise their documents. Again I recorded Group 1, answering the occasional question about MEA deadlines. On Monday, June 11, the instructor gave the Version 2 prompt to the teachers (Appendix B), which was completed by all groups on Wednesday June 13. I made a copy of every group's Version 2 folder in a private, secure folder for safekeeping. On June 13 the instructor gave instructions on the peer feedback process, including which groups would switch documents. The instructor asked each teacher to individually read through another group's Version 2 documents as homework for the following day. Teachers were expected to come into the next class with an understanding of the documents so that productive conversations would occur in the groups, and that the evaluation questions, given in the Peer Feedback section of the MEA document (Appendix B), could be answered. The teachers were asked to make a copy of the group's Version 2 folder. Within this folder the teachers offered feedback using track changes to mark thoughts, correct errors, or ask questions. I made copies of these folders for data analysis.

The remainder of the peer feedback process occurred during the last 40 minutes of class on Thursday, June 14. At the end of this work time the instructor reminded all teachers to complete the Peer Feedback form in the MEA document. These instructions directed the teachers to discuss and evaluate the other group's documents based on the questions provided, and to write up this feedback as a document that was placed in the other group's Dropbox folder. Each of these subfolders had the edited documents with more micro-level feedback from teachers as well as the group responses from the prompt. The instructor announced that these folders and documents needed to be finalized by the following day, Friday, June 15, so that Version 3 could be completed by Monday, June 18. I observed the peer feedback process using the observation protocol and recorded all Group 1's interactions using Camtasia. The TA of the course used Camtasia to record Group 2 interactions since they evaluated Group 1's documents. I retrieved this file and used it as part of my data analysis in order to have recordings of the peer feedback that Group 1 gave and received.

At the end of week 2, I decided Groups 1, 4, and 5 focused on secondary content, and thus would be implemented in the Business Calculus course. The middle school content groups, Groups 2, 3, and 6, were decided to be implemented in the Liberal Arts Mathematics course. I also made a copy of all the Teacher Feedback folders in a secure private folder. All teachers submitted individual peer feedback, and all groups submitted group feedback as responses to the questions in the MEA.

Week 3 Data Collection. All six groups submitted Version 3 by Monday June 18. After making a secure copy of these folders, I prepared to implement the tasks with the undergraduate classes. As scheduled, I implemented Groups 1, 4, and 5 in the Business Calculus course, and Groups 2, 3, and 6 in the Liberal Arts Mathematics course. Prior to implementing the tasks, I talked to the instructors of each course to discuss details of the process and to decide on which students would work on each task. We kept the groupings the undergraduates had been using previously, and assigned a quantitative reasoning task that fit the students' mathematics mastery level evidenced in their coursework. For example, Group 5's task was intended for students with a basic trigonometry understanding; since trigonometry is not a prerequisite for Business Calculus, the instructor assigned the task to a group of students who had previously shown a relatively higher understanding of the subject in comparison to other groups in the class. All students in both classes agreed to participate, and after collecting the IRB forms, the remaining 90 minutes of class time was used for the activity.

My role as a facilitator was to deliver printouts of the task and any manipulatives to each undergraduate group of students. After giving the basic instructions to complete the task in the next 90 minutes, I followed the Facilitator Guidelines each group provided. I implemented four of the groups' tasks, two in the Business Calculus course and two in the Liberal Arts Mathematics course. The instructor of each course implemented one group's task, and followed the same protocol as me. During the implementation, I observed the groups using the observation protocol in Appendix G. Students worked in groups of two or three, and had between 90 and 100 minutes to complete the task. At the end of the implementation, I collected the tasks from each group and made a blinded copy of student work. The original work was returned back to the instructor. Since the quantitative reasoning tasks aligned with some of the course goals, the students were expected to complete the tasks as part of their course requirements. I organized the blinded student work by naming the files in accordance with the teachers' last names who created the task. The blinded student work was uploaded to the Dropbox folders on

Tuesday June 19, and I made secure copies of the data in my own folder. I adhered to the same protocol for observing in-class time devoted towards groups working on their MEA in response to the undergraduate feedback[2].

At the end of week 3, the instructor announced Version 4 was due on Tuesday, June 26. Part of Version 4 included a new document, called the Student Evaluation document, that explained how the teachers implemented the Assessment Guidelines and the framework the teachers used to understand the student work. This document, along with updates of the previous documents in response to the student work, is detailed in the MEA document.

Week 4 Data Collection. All six groups completed Version 4 by the requested deadline. I made copies of each group's Version 4 document and placed them in a secure private folder. Monday was the last day of in-class time devoted toward the MEA, and I observed the same protocols as mentioned earlier. On the day before the course ended, the instructor overviewed the details and expectations for Version 5. On the last day of the course, I arranged to interview two of the teachers in Group 1: Nicholas and Percy. Plans were made for me to also interview Joyce, but these intentions did not come to fruition.

Post-Course Data Collection. I conducted interviews with two teachers and received four teachers' Version 5 documents after the course ended. Nicholas and Percy were interviewed approximately a month after the class ended. I followed the interview protocol and interview questions indicated in Appendix H. The interviews took between 25 and 35 minutes to complete and were audio recorded and transcribed.

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[2] By undergraduate feedback, I refer to the third iteration of the MEA where undergraduate students completed the task. Student work was returned to the groups for analysis.

The four teachers (Joyce, Tiffany, Penny, and Allie) who submitted Version 5 all followed the instructions indicated in the MEA. First, they updated their Decision Log and emailed it to the instructor and me by Monday, September 17. The Decision Log detailed how the teacher plans to implement the task in his or her classroom, including any adjustments to the Version 4 task. These changes tailored the activity to the students and school restrictions of time, technology, and subject matter, or due to the final feedback the instructor gave at the end of the course. I made secure copies of these documents and clarified questions on the next step of the assignment.

The requirement of having the Decision Log turned in prior to implementing the activity served three purposes. First, it required the teachers to think about and express their model of quantitative reasoning in terms of their specific practice rather than in a group setting; this serves mainly a research purpose. Second, by having the teachers write out this process, it encouraged them to take time to think about the details of giving this task to their students and what changes might need to be made; this promoted a practical purpose. Third, having the teachers inform us of their intent added a sense of accountability for the teachers to finish the project; this served as an administrative and motivational purpose.

All four teachers completed the second part of Version 5 by Saturday December 1. As detailed in the MEA, this part asked teachers to implement their Quantitative Reasoning Task with at least five of their students. Some teachers chose to implement the task with a single class or all of their classes. Once the task was implemented, the teachers blinded the names and scanned five students' responses and put them in a Version 5 folder on Dropbox. The Assessment Guidelines, Facilitator Instructions, Student Evaluation, and Decision Log were also updated by this date. I made secure copies of all these files. These four teachers were then reimbursed for the time and effort in completing Version 5.

Instrumentation

The validity, reliability, standardization, and practicality of data collection instruments are important components to consider in any study (Gall, Gall, & Borg, 2007; Thorndike & Thorndike, 2011). While some of these components are less critical to qualitative studies such as this one, this section details how I made an effort to support these components in the design and implementation of the data collection.

Validity in this study's data collection was supported in two main ways: through designing the MEA according to established protocol and through supporting face validity. As described in the Construction of the MEA section previously, the MEA was designed in a way aligned with the research question and theoretical perspective of this study. Face validity (Gall et al., 2007) of the MEA was supported two ways. First, Dr. Michelle Chamberlin, a veteran MEA developer for pre-service teachers, offered feedback on the MEA at various stages of development. Second, I consulted with Drs. Jodie Novak, Michael Oehrtman and Steven Leth during the development to ensure the tasks being asked in the MEA aligned with the setting of a master's level course focusing on calculus pedagogy and its relation to the Quantitative Reasoning in Secondary Mathematics course.

Reliability was supported through the clearly detailed methods. In particular the theoretical perspective was detailed to indicate how data collection would proceed. While standardization in qualitative research is not nearly as important a characteristic of quality research as it is in quantitative research (Gall et al., 2007), the observational protocol used in this study supported standardization in this observations that occurred.

Finally, much care was taken into making this study practical. As indicated in the literature and my selected theoretical perspective, teacher education needs to be grounded in practice, and the MEA was designed to be doable in a way similar to a realistic situation. The workload, while intensive, was suitable for a three credit-hour, 4-week course during the summer. The requirements outside of the class, Version 5, are aligned with the work teachers perform on a regular basis and the goals of the field of mathematics education. The MEA also did not put an overwhelming amount of demands on the instructor. The flexible design of this study allowed for adaptations that could increase practicality if the demands of the current project had become overwhelming for the teachers, instructor, or researcher.

Data Analysis

Like the data collection instruments and procedures, the analysis of the data must also be aligned to the research question and theoretical framework. The analysis procedures presented in this section were adapted from the proposal, following the guidelines of the holistic-deductive paradigm that governs qualitative research (Lincoln & Guba, 1985). Given this study's research question, analyzing the data in terms of a models and modeling perspective required indicating how teachers' models of quantitative reasoning developed in the course. The following sections detail the document analysis process, then coding of the observations and interviews.

Document Analysis

I used content analysis on all documents created by the MEA because this method allowed me to reduce a large amount of data into manageable bits that allowed me to identify patterns in data in ways answering this study's research question. Content analysis refers to "any qualitative data reduction and sense making effort that takes a volume of qualitative material and attempts to identify core consistencies and meanings" (Patton, 2002, p. 453). To prepare for conducting the content analysis I organized the documents into suitable folders. For example, all documents from the Group 1's Version 1 were stored in a folder, while documents from Group 1, Version 2 were stored in another folder, etc. for all groups and versions. Then I read each document twice, the first time abstaining from writing or analyzing the information, and the second time reading in order to gain a general impression of the information. I wrote up this general sense in a memo that described my initial thoughts about teachers' models of quantitative reasoning. These memos were my first attempt to make meaning of the data by identifying patterns of similarities and differences across teachers and groups as well as development within teachers and groups across time. These memos went into a single document called a research journal that also incorporated my field notes and overall reflections about the study.

Content analysis was appropriate for me to use on the MEA data because this analysis allows the researcher to classify textual material and from these classifications make inferences from the text (Hodder, 1994; Merriam, 1998; Weber, 1990). For this study, I drew inferences about how the teacher was thinking about quantitative reasoning from statements teachers made in their MEA documents, and how their ways of thinking changed by examining changes in statements across iterations of the MEA documents. I did this by building on my initial memos and going back to reanalyze data to evaluate the patterns I had identified based on a broader view of the data. I continued writing memos throughout this process, detailing how my thinking about teachers' models of quantitative reasoning was changing.

To detail the types of changes occurring in teacher thinking, I adopted a coding scheme during the data analysis process. I used Hjalmarson's (2008) analytical tool as a coding scheme, which aligned with the theoretical perspective and research questions of this study. Hjalmarson's (2008) analytical tool was developed to describe "models of curriculum that are part of mathematics teaching and learning initiatives" (p. 259). This definition of model referred to a models and modeling perspective, and thus this framework aligned with the theoretical perspective of this study. Hjalmarson designed the tool for researchers to analyze curricular systems, and is thus appropriate for me to use with the data collected in this study because the MEA had teachers develop a small portion of a curriculum in the form of a 90-minute task for their students. By including Assessment Guidelines, Facilitator Instructions, and Decision Log, teachers documented information about how they select and implement materials for their students. These documents provided data that could appropriately be analyzed using Hjalmarson's analytical tool. Using an established coding scheme that aligns with the research questions and theoretical perspective of the study supported me in inferring accurate conclusions from the data (Guba & Lincoln, 1981; Lesh et al., 2003; Merriam, 1998).

I incorporated the work of Hjalmarson's by analyzing each group's MEA documents based on the four components in her analysis tool: conceptual systems, purpose and goals, pedagogical framework, and mathematical content. This tool allowed me to identify multiple ways teachers communicated quantitative reasoning in the documents by adapting Hjalmarson's definitions for each component based on this study's research question. The first adapted definition of conceptual systems was defined to be the theoretical frameworks teachers indicated were part of quantitative reasoning. Second, purpose and goals were defined as the expectations in a teacher's documents, as well as justifications surrounding these expectations. Third, pedagogical frameworks were the pedagogical strategies in a teacher's documents. Finally, mathematical content was the content, skills, and topics associated with quantitative reasoning within a teacher's documents. In addition to each group's submission of MEA iterations, the same definitions applied to an individual teacher's documents, such as the Pre-Assignment, the individual portions of Version 4, and Version 5 (Appendix B). Thus this method offered an organizer for analyzing all teacher-supplied documents in the study.

After coding the components of each document, I conducted two analyses: a between-group comparative analysis and a within-group holistic analysis. These analyses reflect prior analyses using a models and modeling perspective (Doerr & English, 2003). Both analyses incorporated a constant comparison method (Corbin & Strauss, 2007; Creswell, 2007) to compare two sets of documents. For the between-group comparative analysis, I compared each document with documents of the same type from other groups to develop themes about how teachers' models differed at various cycles in the MEA. For example, I compared Group A, Version N to the documents in Group B, Version N. Using the identified components of these documents from Hjalmarson's analysis tool, I compared each set of documents' components in order to identify similarities and differences across different groups and teachers (Corbin & Strauss, 2007). These comparisons provided insight into how teachers' models of quantitative reasoning were developing in relation to the other groups (English et al., 2008; Lesh, 2006; Merriam, 1998; Thomas & Hart, 2010). I then coded the comparisons in an inductive process (Corbin & Strauss, 2007) until comprehensive and represented themes were developed based on the codes.

I used a similar process to conduct the within-group holistic analysis. I compared a group's documents to the same group's documents in the next iteration. For example, I compared Group A's documents from Version N to Group A's documents from Version N+1. Again I used the identified components of these documents from Hjalmarson's coding scheme to compare each set of documents' components, identifying similarities and differences between the versions (Corbin & Strauss, 2007). These comparisons provided insight into how teachers' models of quantitative reasoning were developing over time within a single group (English et al., 2008; Thomas & Hart, 2010). Themes emerged as I coded the comparisons until comprehensive and represented themes were developed based on the codes (Corbin & Strauss, 2007). While these steps are presented linearly, it is important to remember the iterative nature of the constant comparison method (Lincoln & Guba, 1985); cycles between steps occurred when I constructed the overall themes about teachers' models of quantitative reasoning. The themes emerging from these data provided the basis of answering the research question.

I also used the observation and interview data within the constant comparison analysis. To prepare these data for analysis, transcripts of all conversation in the observations and interviews were made. Then I coded the observation protocol document, my field notes, and the interview transcripts using the constant comparison analysis described above. This additional data allowed me to detail the kinds of thinking that were taking place with each individual in Group 1 and the reason these teachers identified for their shifts in thinking (Doerr & English, 2003). I incorporated this data into the themes by corroborating patterns evident in the Group 1's documents or adding details to the teachers' models of quantitative reasoning. In the rare occasions where interview and observation contrasted data from the MEA documents, I noted these differences when describing how Group 1 developed their models of quantitative reasoning. Overall the data from interviews, observations, and documents allowed me to create a more fine-grain description Group1's teachers' models of quantitative reasoning.

To create the language used in the themes, I first attempted to use the teachers' language to describe their ways of thinking about quantitative reasoning. I found this decision problematic due to the inconsistencies between teachers (teacher A used "quantity" in a way different that teacher B) and inconsistencies between the teacher and external sources (teacher A used "quantity" in a way different than Thompson, 2011 and the common vernacular in mathematics education literature). Thus I incorporated language from literature to describe patterns in teacher thinking of quantitative reasoning. In particular, I used Thompson's (2011) definitions of quantity, quantitative relationship, and numerical relationship. To contrast quantity, I developed the language of pseudo-quantity, which is not a common term in quantitative reasoning literature. The use of pseudo is similar to Sfard's (1992) use of the term pseudo-structural object to indicate a learners' conception of something with only partial components of an object. Specifically Sfard defines an object as an outcome of a result of processes that is a way of thinking that includes internal, mental, processes about relationships to do mathematics, and a pseudostructural object as an object with no internal structure (Sfard, 1992; Sfard & Linchevski, 1994). Zandieh (2001) abbreviated this language as pseudo-object, and notes that a “pseudo-object is not meant to have a negative connotation, rather it merely denotes that the object a person is using does not refer to the underlying process…of the true object” (p. 107). Similarly, my use of pseudo-quantity denotes a person who does not refer to all components of Thompson’s (2001) notion of a true quantity. Pseudo-objects in Sfard’s framework serves as building blocks for higher-level processes, even if these building blocks are somewhat crippled due to the lack of full structure. Similarly, teachers and students can build operations based on pseudo-quantities, though the lack of underlying structure may limit these operations.

Trustworthiness

All research aims to produce knowledge that is reliable and valid, and in qualitative studies this idea is framed in terms of trustworthiness (Merriam, 1998). In this section I consider Guba’s four criteria of trustworthiness in terms of this study in order to provide evidence quality research took place. These criteria have been used extensively in qualitative research over an extended period of time, and consist of credibility, transferability, dependability, and confirmability (Guba, 1981; Lincoln & Guba, 1985).

Credibility. Credibility is concerned with how congruent the findings are with reality, and is used in preference to internal validity (Merriam, 1998; Shenton, 2004). Credibility is one of the most important factors in establishing trustworthiness, and in this section I elaborate on how my choices of setting, study design, and future plans of this study supports credibility (Lincoln & Guba, 1985; Shenton, 2004).

The choice of setting within the master’s program for teachers as well as the undergraduate calculus course was chosen because this setting provided a population that could provide data I could use to answer the research question, and because this research setting was accessible and familiar to me. As described earlier in the chapter, this research setting included in-service mathematics teachers taking a course in quantitative reasoning, and allowed me to document teachers' models through the use of an MEA to answer my research question. The research setting was accessible and familiar to me because of my prior work with this teacher population, my actions of observing portions of the class, and my choice to interview two teachers. In addition, my established rapport with the population can be considered a factor in supporting teachers to give their honest opinion during data collection (Shenton, 2004). Thus this choice of research setting supported the credibility of the study because being knowledgeable and immersed in data are important factors when gaining an understanding of the organization and population (Lincoln & Guba, 1985; Merriam, 1998).

Credibility is supported in the design of the study through well-established research methods, such as peer checking, researcher reflexivity, and triangulation. This study was designed with these criteria in mind, beginning with research methods that were incorporated by distinguished researchers in the field of mathematics education working with similar populations (Doerr & Lesh, 2003; English, 2003). Specifically, multi-tiered teaching experiments and other design experiments using a models and modeling perspective are established research methods, as argued in the beginning of this chapter.

This study was designed to incorporate the researcher's reflexivity in being self-questioning and self-understanding in order to continually examine how knowledge is being drawn from the data (Patton, 2002; Shenton, 2004). The researcher journal I kept provides evidence of my main mechanism of being attentive to my perspective, its inherent biases, and the perspective of my participants. In my researcher journal I documented my personal experience of the research process before, during, and following data collection and analysis. I supported an awareness of my perspective, biases, and view of participants in ways that supported informed professional decision making (Janesick, 1998). As the research study moved forwards I reread earlier entries to recall my thinking behind key decisions and to question earlier assumptions I had made about aspects of the study (Borg, 2001). These actions promoted the study's credibility.

Secondly, the researcher stance included earlier indicated my background and qualifications contribute to the credibility of a study; this information provided evidence that the researcher can take "seriously the responsibility to communicate authentically the perspectives of those we encounter during our inquiry" (Patton, 2002, p. 65). Furthermore, reflexivity was encouraged by the peer checking of the project design and findings through my dissertation committee. My co-chairs and committee members offered fresh perspectives that challenged the assumptions I made. This feedback supported changes to the study that strengthened my arguments and supported the credibility of the study (Shenton, 2004).

Triangulation provided some of the most important evidence of credibility; triangulation can occur primarily in four ways: triangulation of data, investigators, theory, and methods (Guba, 1981). These characteristics strengthen a study by combining multiple components in order to have an "arsenal of method[s] that have nonoverlapping weaknesses in addition to their complementary strengths" (Brewer & Hunter, 1989, p. 17). My study included the first three types of triangulation. Data triangulation occurred through the iteration of documents combined with observations and interviews of the participants. Investigator triangulation occurred by the incorporation of my co-chairs in the peer-checking of the findings chapter. Finally, a models and modeling perspective incorporates several theories (constructivism, social constructivism, and situated cognition, for instance), that provided strong evidence of theory triangulation, and provide evidenced that supports credibility of the study (Lincoln & Guba, 1985). Aspects of member checking supported the credibility of the study. These aspects included cycles of questioning that supported participant honesty and allowed teachers to alter their MEA documents in later iterations. My incorporation of a detailed description of the phenomenon also supported credibility of the study (Guba, & Lincoln, 1989; Shenton, 2004).

This section detailed evidence that supported the credibility of the study. Just like any other study, it is typically not possible for one study to provide evidence of every facet of quality; for example, this study does not use random samples, frequent debriefing sessions, or methodological triangulation to support credibility. However, I do address most of the indicators of credibility in way that provides strong evidence that this study produced findings that are congruent with reality.

Transferability. Transferability "is concerned with the extent to which the findings of one study can be applied to other situations" (Merriam, 2009, p. 223), and is in preference to external validity or generalizability (Merriam, 1998). While this aspect of qualitative research has been heavily debated, here I take the perspective that naturalistic generalization is the goal of this study. Naturalistic generalization is when "people look for patterns that explain their own experience as well as events in the world around them" (Merriam, 1998, p. 211) by recognizing the similarities of contextualized situations (Stake, 1978, 1995). This aligns with the goals of a models and modeling perspective, as the models presented in this study aim not to be global theories, but explanations of how people view contextualized educational settings (Lesh, 2006; Lesh & Sriraman, 2010). These ways of thinking, however, are generalizable to other educational settings which the reader may deem appropriate, which is why naturalistic generalization fits this study well given the effort to include significant contextual information for the reader to be able to make such a generalization (Firestone, 1993; Lincoln & Guba, 1985; Shenton, 2004). These efforts include the detailed description of the setting, participants, and data collection procedures.

Dependability. This aspect of trustworthiness asks if the findings are consistent with the data; in other words, if the same methods, participants, and context were used, would similar findings be produced (Merriam, 1998). Dependability is supported by detailing the study's processes, communicated in ways that would "enable a future researcher to repeat the work, if not necessarily to gain the same results…[and] allows the reader to assess the extent to which proper research practices have been followed" (Shenton, 2004, p. 71). This concern was addressed by creating an audit trail, which is a trail of how decisions were made in the study (Merriam, 1998). I included an audit trail as part of my research journal I kept through the study. This running document included my thoughts and decisions in a chronological format that includes rationale and explanations. This dissertation conveys parts of this audit trail by including the major decisions related to how findings were produced. For example, I have provided rationale behind decisions such as selecting the study's theoretical perspective, data collection and analysis methods. Additionally Chapter 2 included details regarding how the literature review was compiled (Gall et al., 2007).

Confirmability. Confirmability, which is used instead of objectivity, is the extent to which "findings are the result of the experiences and ideas of the informants, rather than the characteristics and preferences of the researcher" (Shenton, 2004, p. 73). Confirmability was supported through many of the actions detailed in the previous sections, including triangulation, the use of negative cases, and audit trails. Triangulation helped eliminate biases in this study through the use of multiple data sources, theories, and through the collaboration of my dissertation committee. Negative cases were considered in order to challenge the researcher's model in each cycle of data collection. For example, Gary did not exhibit the pattern of development most other teachers went through, so I detailed this teacher individually throughout Chapter 4's themes. Finally, the audit trail allowed the trace of each decision in order to indicate the findings come from the data rather than anywhere else (Patton, 2002).

Two additional aspects supported confirmability, the first being the theoretical perspective and analysis methods selected for this study. The models and modeling perspective aims to collect and analyze data that captures teachers' ways of thinking, and thus the findings of this type of study are grounded in these data (Doerr & Lesh, 2003). The last way I supported confirmability was by recognizing the shortcomings of the study's methods (Shenton, 2004), which I include in the limitations section.

CHAPTER IV

FINDINGS

The chapter is structured around six themes that support arguments appearing in Chapter 5. As the Model-Eliciting Activity (MEA) progressed, I found three patterns in the features of quantitative reasoning that teachers recognized, which I called the aspects of *identifying quantities*, *relating quantities*, and *coordinating relationships of quantities*. Themes 1, 2, and 3 define each respective aspect and detail how teachers made statements referring to this aspect in their MEA documents. Theme 4 discusses how teachers characterized features of quantitative reasoning more similarly across teaching and non-teaching settings as the MEA progressed. Theme 5 presents evidence that teachers became more certain in their ability to develop students' quantitative reasoning as they completed the MEA. Theme 6 discusses factors that teachers did not identify as being influential to how they thought about quantitative reasoning. The remainder of this section outlines data sources for individual teachers, teacher groupings, and initial factors influencing teacher thinking about quantitative reasoning.

Each teacher was asked to respond to several questions about quantitative reasoning, which allowed me to infer how individuals thought about quantitative reasoning. Evidence of individual thinking came from teacher responses to the Pre-Assignment, Version 4, and Version 5 prompts. The Pre-Assignment asked teachers to describe (a) what the phrase "quantitative reasoning" meant with respect to secondary mathematics, (b) what quantitative reasoning looked like in their classroom, (c) a task that measured students' quantitative reasoning skills, and (d) what the teachers wanted to get out of the course. I analyzed teachers' responses to all these questions.

Twenty of the 21 teachers submitted the Pre-Assignment. Nine teachers submitted the Pre-Assignment before the first course meeting, while the remaining 11 teachers submitted between the first course meeting and the second course meeting. Although the first course meeting may have influenced these 11 teachers' responses on the Pre-Assignment, two teachers stated that they attempted to consider their responses from their perspective prior to the start of the class. For example, Joyce said, "I am probably influenced by what we discussed in class today, but I will do my best to give a response that reflects my previous thoughts on the subject." The other teacher who indicated her responses reflected her thoughts on quantitative reasoning prior to the course beginning was Carol. Carol wrote at the end of her Pre-Assignment that she had responded to the questions "before our first class…After class today, I feel like [quantitative reasoning] is the opposite of identifying the quantities–it is more the act of simply working with and abstractly manipulating the quantities–to arrive at the final answer."

The only teacher with a response suggesting the first course meeting influenced his Pre-Assignment was Gary. When interpreting the Common Core State Standards for Mathematics (CCSSM), Gary's response in the Pre-Assignment was that the CCSSM "break down the idea of quantity a little more, including the idea of units and objects, although they didn't define it as clearly and completely as we did in class." I interpreted Gary's response as him applying the four components of quantity (Thompson, 2011) that were presented in the first course meeting to the CCSSM reasoning abstractly and quantitatively Standard for Mathematical Practice.

All groups completed Versions 1, 2, 3, and 4, and I analyzed all documents included in these versions. Group responses from the MEA documents provided evidence for each group's thinking about quantitative reasoning as well as the factors that developed this thinking. The MEA asked groups to document their thinking about quantitative reasoning in four ways, each prompted by the guidelines of the MEA (Appendix B): by creating their own questions that would capture students' quantitative reasoning skills in the Quantitative Reasoning Task; by establishing criteria to assess student responses to the task in the Assessment Guidelines and Student Evaluation; by recording pedagogical decisions to support the task objectives in the Facilitator Guidelines; and by stating how the group thinks about quantitative reasoning and any changes in their thinking in the Decision Log.

The teachers in each group were roughly clustered by the content they taught and were comprised of either three or four teachers. Teachers in Groups 1, 4, and 5 taught high school courses such as algebra 2, pre-calculus, and trigonometry, and these teachers were classified as high school teachers. Teachers in Groups 2, 3, and 6 taught courses at middle school or entry-level high school courses such as algebra 1 or pre-algebra, and I call these teachers middle school teachers. Details on the individual teachers and the groups can be found in Appendix E.

Version 4 also asked each individual teacher to record how they thought about quantitative reasoning, how this thinking has changed, and how they thought their group's task related to quantitative reasoning. All 21 teachers completed this part of

Version 4. Additional individual information was obtained from the four teachers (Joyce, Tiffany, Penny, and Allie) who implemented Version 5 and the two interview participants (Nicholas and Percy). Versions 4 and 5 comprise the final MEA documents I received from the teachers.

Of the four teachers who implemented Version 5, Joyce and Penny were specific about how the K12 student feedback influenced their thinking about quantitative reasoning. For example, Joyce made statements referring to how the K12 feedback influenced her thinking about *identifying quantities* as an aspect of quantitative reasoning. Details of both Joyce and Penny are provided in the themes below. Tiffany and Allie gave statements in their Version 5 documents that echoed earlier statements from the undergraduate student feedback. These two teachers' Version 5 documents did not provide much additional information about their model of quantitative reasoning beyond statements made in Version 4, and thus are not included as a main data source of the following themes.

The only missing piece of data from the entire MEA was one teacher who did not submit a Pre-Assignment. Therefore when reporting on teacher responses to the Pre-Assignment, I reference the total number of teachers as 20. When reporting on the other documents I reference the 21 total teachers who provided data.

Theme 1: Teachers' Attention to Identifying Quantities

Identifying quantities was the first aspect of quantitative reasoning I identified in the statements teachers made about quantitative reasoning in their MEA documents. *Identifying quantities* refers to the act of identifying components of a contextual problem as part of quantitative reasoning. Teachers made statements referring to the aspect *identifying quantities* by using words such as "variables," "unknowns," or "quantities" in their MEA documents, though the way they used these words shifted as the MEA progressed. Initially, most teachers made statements about the aspect *identifying quantities* by referring to what I called pseudo-quantities. Pseudo-quantities are numerical values, unknowns, or other features of a contextual setting where the teachers did not fully distinguished the object, attribute of the object, and units of the attribute being considered.

By the MEA conclusion, most middle school teachers made statements referring to the aspect *identifying quantities* by making statements referring to quantities as defined by Thompson (2011). Quantities are the conceptual objects created as the interplay of one's attention to an object, a measurable attribute of the object, a way to assign values to this measure, and an accompanying unit such that the measure entails a proportional relationship with its unit. Most high school teachers continued to make statements I coded as referring to pseudo-quantities as part of quantitative reasoning in their MEA documents. The following subsections provide evidence about how teachers attended to *identifying quantities* as an aspect of quantitative reasoning along with teacher reflections on how they developed their thinking about *identifying quantities*. Rather than detail the development of each teacher or group, I describe initial patterns in responses, final patterns in responses, and then evidence about why teacher responses shifted and when these shifts occurred.

Teachers' Attention to Pseudo-Quantities

When first creating the MEA documents, 16 of the 20 teachers made statements I coded as pseudo-quantities. These 16 teachers made statements about quantitative reasoning being solutions, numbers, or amounts that were important to consider within a problem. For example, Allie said “I think quantitative reasoning applies to real world application problems and if when the problem is solved if the ‘answer’ makes sense and why.” Here Allie’s response about quantitative reasoning focused on solutions to contextual problems, but did not attend to how this solution consisted of an object, measureable attribute of the object, or accompanying unit of this attribute. Thus Allie’s response was coded as referring to pseudo-quantities as an aspect of quantitative reasoning.

In addition to making statements about attending to solutions, numbers, or amounts, 7 of the 16 teachers made statements about units being a part of quantitative reasoning. For example, Tiffany said quantitative reasoning “requires students to be able to interpret quantitative data (tables, charts, graphs) in context and apply meaning to the data sets. This includes…reading these data displays, using formulas to make predictions, and determining scales and units.” While Tiffany included units as something to consider when working with numerical information, she did not distinguish the objects or attributes of the objects associated with the information. Thus her response was coded as referring to pseudo-quantities as an aspect of quantitative reasoning.

Seven of these 16 teachers used the word “quantity” in their Pre-Assignment responses in ways that were either synonymous with “solution,” “number,” or “amount,” or used this word in vague ways. For example, Penny gave the response that quantitative reasoning was “giving students a problem involving quantities where they have to determine a strategy for solving the problem,” with no further statements about what was meant by “quantities.” Since her use of this word was vague and had no evidence of attending to an object, a measurable attribute of the object, or an accompanying unit, her response was coded as referring to pseudo-quantities. Aside from the 16 teachers' responses that were coded as pseudo-quantities, two teachers made comments that were coded as quantities and the remaining two teachers' responses were coded as neither pseudo-quantities nor quantities. One of the teachers who gave responses coded as neither pseudo-quantities nor quantities instead focused her responses on composition of functions; the other teacher gave responses focusing only on mathematical proof.

All six groups made statements in their Version 1 documents that I coded as referring to pseudo-quantities as an aspect of quantitative reasoning. Similar to the individual responses, groups' responses often used the word "quantity" as synonymous with 'variable' or used the word in vague ways. For example, Group 1 made statements about pseudo-quantities in their Version 1 Decision Log by saying "Richter scale and energy" were the "quantities" in their task. This group's explanation was, "our task involves the concept of logarithms. We have all taught the subject, however the students demonstrate poor or inadequate understanding of what logarithms are, and more importantly, what the quantities associated to a logarithmic function represent." Despite acknowledging students' inadequate understanding of what quantities are being represented, the group did not describe what the "quantities" Richter scale and energy represented in any of their Version 1 documents. Groups 1's statements about these vague features of the context provided evidence that pseudo-quantities were an aspect of quantitative reasoning for this group.

Similar to Group 1's use of the word "quantity", Groups 2, 3, 5, and 6 included the word "quantity" in their Version 1 in ways either synonymous with "variable" or in vague ways. For instance, Group 3's Quantitative Reasoning Task asked students to label both axes and interpret various graphical features such as slope and *y*-intercept. Their expectations in their Assessment Guidelines were that students identified the *y*-intercept as "the distance between the towns." In some of their expectations, Group 3 identified a unit associated with the graphical feature. However, Group 3 did not give attention to what object or attribute was being considered when interpreting these graphical features in their Quantitative Reasoning Task, Assessment Guidelines, or Facilitator Instructions, and thus I coded these documents as pseudo-quantities. Group 3 was unique among the groups because they defined quantity in their Decision Log using Thompson's (2011) definition. Since their Quantitative Reasoning Task, Assessment Guidelines, and Facilitator Instructions had statements coded as pseudo-quantities, I did not consider their definition evidence that they were thinking about identifying quantities in ways that aligned with Thompson's definition of quantity.

Group 4 did not mention the word "quantity" in their Version 1 but instead asked students to explain the process of solving for a missing side length of a right triangle. They also asked students "what units are you measuring in this figure?" No attention was given to the measure of the side lengths in other documents. They also stated in their Decision Log "quantitative reasoning can take many forms. It can simply be looking at numbers that have some meaning with respect to measurement, value, or even perspective." Since only numbers and their units were considered in their Version 1 documents, I coded Group 4 as having pseudo-quantities as part of quantitative reasoning.

By the conclusion of the MEA, the high school groups continued to make statements about quantities that were coded as pseudo-quantities while the middle school groups were making statements that were coded as quantities. Group 4 did not change how pseudo-quantities were incorporated in their MEA documents, while Groups 1 and 5 referred to pseudo-quantities as an aspect of quantitative reasoning but also added some evidence of quantities also being an aspect of quantitative reasoning. The additions Groups 1 and 5 gave in their documents are detailed in the following section, along with Groups 2, 3, and 6 shift to include quantities in the MEA documents.

Nine teachers individually made statements in their Version 4 or 5 documents that were coded as pseudo-quantities being an aspect of quantitative reasoning; the remaining 12 teachers made comments coded as quantities, and are detailed in the following section. Of the nine individual teachers, eight were teachers who had already made statements referring to pseudo-quantities at the onset of the MEA, while one teacher had originally not made statements coded as pseudo quantities. An example of one teacher continuing to make statements coded as pseudo-quantities was Nicholas. He said in an interview after the conclusion of the MEA:

> I used to be okay with kids writing v = volume…Yep you identified the variable let's move on. But now going through that quantitative class I really started appreciating what does that really mean in context to the problem. Do you understand what units the volume is in, and how does it relate to the beginning of the problem?

This interview quote indicated Nicholas began considering attention to variables as part of quantitative reasoning. He provides evidence variables are an attribute (volume) with associated units, but does not attend to the object (water in a container) or a way to assign measures to the attribute (height of water from the bottom of the container). Because of this lack of evidence, I coded his final response as referring to pseudo-quantities as an aspect of quantitative reasoning.

Other responses from these nine teachers included using the word “quantity” in vague ways or using this word synonymously with numerical values, reflecting these teacher’s initial responses that were coded as pseudo-quantities. For example, Allie said in her Version 4 reflection that quantitative reasoning related to her task because “students need to identify quantities from a written situation. After they have defined their quantities they were required to explain their thought process for how they determined their quantities.” Since Allie did not define quantities elsewhere in her Version 4 or Version 5, Allie’s response is vague regarding what she means by quantities. Since I could not determine if quantities included an object, attribute of the object, or unit of the attribute, her response was coded as pseudo-quantity.

Teachers’ Attention to Quantities

When first creating the MEA documents, only two teachers made statements that were coded as quantities being an aspect of quantitative reasoning. The responses of Gary and Rose were unique in that they were the only ones to explicitly attend to quantities in their responses to the Pre-Assignment prompts. Gary did this by expanding on his interpretation of the CCSSM, saying:

> [The CCSSM] break down the idea of quantity a little more, including the idea of units and objects, although they didn’t define it as clearly and completely as we did in class. I like their definition, but it is definitely written in heavy academic jargon with an emphasis on buzzwords.

Recall Gary was the only teacher to reference information from the first course meeting in his Pre-Assignment, and the course’s impact is seen in this response. Gary’s statement provides evidence he was referring to quantities, since he attended to an object, how to measure the object, and units to measure the object. While he did not explicitly mention attributes of the object or a way to assign values to the attributes, his statement about the

CCSSM definition lacking clarity and completeness suggests he might have been referring to these missing portions of his definition of quantity. Thus Gary's responses to the Pre-Assignment were coded as referring to quantities as an aspect of quantitative reasoning. Gary acknowledged the course materials that he "did in class" as influencing his ways of thinking about quantitative reasoning; he was most likely referring to the presentation on Thompson's components of quantity given by the instructor on the first course meeting.

Rose was the other teacher who explicitly attended to quantities in her Pre-Assignment responses. In question (d) of the Pre-Assignment, Rose stated that in her classroom:

> Having very little experience with studying what quantitative reasoning looks like, [explaining what quantitative reasoning looks like in my classroom] is the question I am most unsure about. However, the second part to the standard for mathematical practice that involves quantitative reasoning seems to give me the biggest clue about what I should be looking for…quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of the quantities, not just how to compute them…using symbols to represent different quantities in a problem and understanding exactly what the meaning of those quantities are *throughout* the problem, not just in the answer at the end. (emphasis in original)

Here Rose described the role of symbols representing different quantities, and that students must understand quantities in the context of the problem and recognize some unit associated with the quantity. She referenced the units involved and attended to the meaning of the quantities, suggesting Rose initially referred to quantities as an aspect of quantitative reasoning. Rose's statement about the CCSSM standard for mathematical practice suggests reading this document gave her an idea about how to think about quantities.

Group 5 was the only group to provide evidence of quantities being an aspect of quantitative reasoning in their Version 1 documents. This group described quantities in their Quantitative Reasoning Task by asking students to identify quantities relevant to the problem context, to explain why these quantities are important to the problem, and to identify how the quantity was represented. These goals were reflected in the Version 4 Facilitator Instructions when they asked the facilitator to:

> begin by asking the students what quantities they see in the problem. Once you have a list, ask them what object each quantity is connected with, what attribute of the object the quantity is measuring, what units will be used, and what values they can expect to see for the quantity… make sure they include vertical distance from the ground to the seat, horizontal position…… make sure the idea of rotation comes up in the discussion on quantities. If no one brings it up, ask how they will know where each seat is located, and try to lead them into the idea that they will need to know an angle of rotation (although they are not likely to use that terminology, and you don't need to give them that vocabulary yet)…[make sure] they are aware of these three quantities.

Group 5 gave details for how students should measure the quantities of vertical and horizontal distance. Group 5 also made statements about how the values of these quantities change with respect to the rotation angle. While vertical and horizontal distance contained some characteristics of quantities, rotation angle did not. Rotation angle was described to have "degree measurements," but did not have an object, attribute, or unit associated with this variable[3]. Furthermore, Group 5 said in the task that the Ferris wheel "turns counter-clockwise at a rate of one revolution every two minutes." While Group 5 mentions the fixed quantity "rate of revolution" and its influence on rotation angle and hence vertical and horizontal distance, they do not mention in any of their

[3] An example of rotation angle being stated as a quantity would be to define the object to be an angle, the attribute to be openness, unit of measure to be the fractional amount of a circle's circumference subtended by an angle (computed according to how time the ride has been turning since loading the last seat in minutes), and units to be radians.

documents the role of elapsed time influencing these quantities. Both rate of revolution and the elapsed time were coded as being pseudo-quantities given the lack of description accompanying these terms. Thus Group 5 made statements referring to both quantities and pseudo-quantities as aspects of quantitative reasoning.

As Group 5 went through the MEA, they made more statements referring to quantities as an aspect of quantitative reasoning. These statements occurred mainly in the Facilitator Instructions as Group 5 added more details about the vertical and horizontal distance quantities as well as expectations for the rotation angle quantity. For example, Group 5 added to their Facilitator Guidelines that:

> A good way to get [students] started if they are totally stuck is to have them actually begin measuring the distances with a ruler and making a table. Then when they come up with ideas for calculating the values, they can check their answers against those measurements.

Here Group 5 detailed they wanted students to consider the vertical and horizontal distance by knowing what these measurements represented: distances of the seat from a certain reference point. These expectations were also reflected in the Assessment Guidelines. Increased focus on the attributes and units of the vertical and horizontal distance quantities indicated Group 5 was referring to quantities as the MEA progressed.

Group 5's earlier passage also provides some evidence that Group 5 considered rotation angle as a quantity by the MEA conclusion. Group 5 indicated they wanted students to attend to rotation angle as a quantity because they asked the facilitator to "make sure the idea of rotation comes up in the discussion of quantities." Since Group 5 also told instructors to "ask [students] what object each quantity is connected with, what attribute of the object the quantity is measuring, what units will be used, and what values they can expect to see for the quantity," these statements indicated Group 5 wanted students thinking about rotation angle as a quantity. Group 5 did not specify what the object, attribute of the object, or units of the attributes for rotation angle in any of their MEA documents, which would have provided stronger evidence that Group 5 considered rotation angle a quantity. Finally, Group 5 never defined or discussed “elapsed time” in any way that differed from their initial statements, so Group 5 was coded as referring to “elapsed time” as a pseudo-quantity.

Group 1 also added statements referring to quantities as an aspect of quantitative reasoning as the MEA progressed. Specifically, Group 1 gave instructions for the facilitator to “be careful not to use the variables x and y, rather focus on the quantities, time in years, and amount of stock value.” Here Group 1 considered attributes of objects with units when working with quantities. Evidence of quantities being a part of quantitative reasoning was also supported in Group 1’s Assessment Guidelines, which had expectations for students that I coded as conceptualizing objects, attributes of objects, and units. Thus Group 1 made statements about both quantities and pseudo-quantities throughout their MEA documents.

The three middle school groups (2, 3, 6) all made statements in their final MEA documents that referred to quantities as an aspect of quantitative reasoning. Group 2 incorporated a table “designed to help [students] think critically about what quantities would be present in fundraising situations.” This table was in the Quantitative Reasoning Task and had accompanying expectations in the Assessment Guidelines that asked students to identify the object, attribute, unit, for “all of the varying and unvarying quantities that are present in a fundraising situation.” These expectations indicated Group 2 referred to quantities as an aspect of quantitative reasoning by the MEA conclusion.

Groups 3 and 6 also made statements about quantities in their final versions of the MEA documents. These groups asked students to identify quantities in a scenario by determining what the important characteristics of the scenario were and assigning units to these characteristics. Group 3 said in their final Assessment Guidelines that:

> Students should explain how they think about variables and determining appropriate labels for their axes. They might mention how they list all the quantities in the scenario and then discuss which numerical values are constant and which ones change. While describing their variables, make sure students understand they need to label their axes specifically (i.e., distance from the starting point) and with units (i.e., yards).

This passage indicated Group 3 was considering variables as quantities that have attributes, units, and can change. Group 6 also depicted quantities as an object, measurable attribute, units, and the numeric value of the measurable attribute. Both groups made statements referring to quantities in this way across the task and supporting documents, indicating these groups referred to quantities as an aspect of quantitative reasoning.

Twelve of the 21 teachers made statements in their Version 4 or 5 documents referring to quantities as an aspect of quantitative reasoning. These 12 teachers made statements depicting quantities as specifying objects, attributes, and units. For example, Byron said:

> I understand quantitative reasoning to be sorting through a situation to identify measurable attributes, how they relate to each other, which are appropriate to work within a given task, and how to work with them…As we have worked through this project, I have shifted away from looking at the values of the measurements and looking more at the attributes themselves…the students must look for patterns between the quantities using actual values that will help them transition to looking at the general behavior of the quantities in relation to each other which should help the students see them as actual attributes as opposed to specific values at specific points in time.

Byron's description says quantities have measureable attributes that vary in accordance to the context students are using. The statements "actual attributes" and "actual values" suggest Byron considered quantities as attributes of an object, and that the measureable values of the attribute most likely had units to make them meaningful in the context. Thus, I coded Byron's response as referring to quantities. Byron also indicated working in Group 5 influenced him to consider attributes of quantities and how they vary within the context of the problem. Byron's group mates, Gary and Ken, also expressed quantities in their final reflections in similar ways that related to their group's task.

Besides these three teachers in Group 5, the only other high school teacher to make statements about quantities in their final reflections was Joyce in Group 1. Joyce was one of the teachers to complete Version 5, where she commented on the role of quantities in her own students' work. She said that by looking over her students' work on her Quantitative Reasoning Task, she learned "when I discuss quantities in class, I need to move beyond saying, for example, '*x* represents time,' and say, '*x* represents the time in years since money was first invested in the account.'" In addition to identifying objects and attributes, Joyce clarified how quantities were a part of quantitative reasoning in her Student Evaluation document, where she said students needed to include a way to assign values to attributes as well as units associated with this attribute. Her statements were coded as referring to quantities as an aspect of quantitative reasoning by the MEA conclusion.

Eight middle school teachers individually made comments in their Version 4 or 5 documents that were coded as having quantities as an aspect of quantitative reasoning. These responses all identified quantities as having an object, measureable attribute of the object, and units associated with the measurable attribute. For example, Glen described his shift of what he thought a quantity was:

> I understand quantitative reasoning to be the ability to not only recognize what a quantity is in terms of measurement and units, but also use that quantity in problem solving. One of the biggest misconceptions that I had was thinking about a quantity as simply a number with units.

Glen's statement reflects how his initial MEA responses were coded as containing pseudo-quantities while his final MEA responses were coded as quantities as an aspect of quantitative reasoning. Penny, who completed Version 5, indicated the K12 student feedback influenced her thinking about quantities. Penny stated in her Version 5 documents, "with just the first week of classes under my belt [and after implementing the Quantitative Reasoning Task], I find myself already having much more clarity about quantities, and I can pass this clarity on to my students." Penny references quantities in ways that reflected her group's (Group 3's) statements in Version 4, but did not give details for how the K12 student feedback gave her clarity about quantities."

Teachers' Reflections on Quantities

All groups commented on factors that influenced their ways of thinking about quantitative reasoning, and my coding identified three factors that influenced teachers to consider quantities as an aspect of quantitative reasoning. Undergraduate student feedback was the most commonly identified influence on the aspect of quantities, followed by course materials, then finally the peer feedback process. Groups 1 and 6 said that undergraduate student feedback prompted them to be more explicit about how quantities were included in the MEA. For example, Group 6 said one of the undergraduate students "used the word 'quantity' a few times but never said what that quantity was. (Perhaps we should include a more explicit definition of what 'quantity' means in terms of what we have talked about in class in the facilitator instructions?)." Similarly, Group 1 responded to student performance on their task by saying, "students articulated the general sense of the variables, but none of the students spent much time defining the variables and their units of measure. Certainly a point that needs to be addressed for Version 4 is the articulation of what we want the students to produce." Both groups made changes in their Version 4 that aligned with the problems they identified from the undergraduate student feedback. These changes reflect the groups' statements about quantities, rather than pseudo-quantities, being an aspect of quantitative reasoning. Three of the four teachers who completed Version 5 said K12 student feedback influenced their ways of thinking about quantitative reasoning, which was evidenced in Joyce's passage at the conclusion of the previous section

One group and two teachers identified course materials as a contributing factor to how they thought about quantities. Group 2 remarked on the Pathways to Calculus materials (Carlson & Oehrtman, 2011) in influencing them to incorporate quantities in the task. For example, Group 2 stated in their Version 2 Decision Log:

> After doing our homework 6 we decided to offer the students a table to fill out to help organize their work. This table is designed to help them think critically about what quantities would be present in fundraising situations and how they might affect any decisions they'll need to make.

The instructions for Homework 6, which was due the day before Version 2 was due, asked teachers to read three worksheets in the Pathways to Calculus materials (Carlson & Oehrtman, 2011, Module 2, Worksheets 1-3, Appendix I). Group 2 continued to include that table in subsequent Versions 3 and 4 by adding scaffolding, additional questions, and expectations related to quantities. Group 2 did not comment that instructor feedback was influential in their decision to incorporate the materials even though after Version 1 the instructor asked Group 2 to consider how students were "thinking about proportional reasoning and quantities based on their product."

Gary mentioned the course in general influenced his ways of thinking, saying:

> My understanding of quantitative reasoning has evolved a great deal over the course of this class. Before this class I don't think I would have made a distinction between mathematical/arithmetic reasoning and quantitative reasoning. I probably equated the word "quantity: with the words "number" and "amount" and didn't stop to think that these are only part of the idea of "quantitity" (sic). One of the greatest insights I developed was the idea that there are four parts to quantity: object, measurable attribute, unit and number. Although I think I was aware of all of these aspects, I didn't always stop to consider them for each quantity, and I didn't realize how much that could help avoid mistakes and deepen understanding. I know that I will be focusing on these ideas in my teaching in the coming year.

While Gary was not specific in what part of the course influenced his thinking, the similarities between his definition and the definition of quantity given in the Thompson article (1990) presented in the first week of the course may be one connection referenced here, especially since he referenced this first course meeting in his Pre-Assignment. Similarly, Darium referenced the Moore, Carlson, and Oehrtman (2009) article in his final reflection as influencing his ways of thinking about quantities, but did not give further details about how or why this occurred. Rose was the only teacher to make a statement suggesting the CCSSM being impactful on how she considered quantities in her Pre-Assignment.

Another contributing factor to how teachers thought about quantities was through peer feedback. Group 1 acknowledged the receiving and giving of peer feedback influenced their thinking, which I interpreted as a possible factor that promoted them to consider quantities as an aspect of quantitative reasoning. Group 2, who gave feedback to Group 1, stated:

> The task asks for students to explain ideas to another student but does not explicitly imply the use of quantities…instead of just identifying variables, have them look at all of the quantities more in depth and how it will relate to the situation and the formula they're supposed to come up with.

Group 2 challenged Group 1 to consider quantities rather than pseudo-quantities in the task. This excerpt indicated how the peer feedback provided motivation for groups to consider an object, a measurable attribute of the object, a way to assign values to this measure, and an accompanying unit. In their Version 3 Decision Log, Group 1 made a comment about the impact of the peer feedback process:

> We also received feedback from our peers. They had some excellent suggestions concerning the quantitative reasoning task. In particular, they suggested questions that ask students to analyze the quantities involved with the stock problem in more detail. We added a little more to the directions in order to give the students an idea of what we wanted them to explore.

In this example, Group 1 acknowledged the influence of peer feedback on their ways of thinking about quantitative reasoning, particularly by being more specific and identifying the attributes and units involved in the problem. This change in thinking suggests Group 1's shift towards including quantities as an aspect of quantitative reasoning was promoted through the peer feedback process.

Group 1 was also influenced to consider quantities as an aspect of quantitative reasoning by providing peer feedback. In their feedback to Group 6, Group 1 commented on an "awesome list of four prompting questions…[for] investigating quantitative reasoning." Three of these questions referred to Group 6's questions about quantities, including: "What quantities should be represented in your explanation? How will you measure each of the quantities (i.e., what kind of units?). What quantities are important to the situation?" Group 1 incorporated these questions into the following Version 3 documents. While Group 1 did not directly acknowledge the impact Group 6 had on their thinking, the implication of Group 1's comment in the peer feedback process suggests the origin of the added questions came from Group 6. Compared to Group 1 initial statements about pseudo-quantities, evidence from the peer feedback suggests they attended to quantities because they were exposed to another group's statements about quantities.

Theme 2: Teachers' Attention to Relating Quantities

The second aspect I identified in the statements teachers made about quantitative reasoning in their MEA documents was *relating quantities*. *Relating quantities* refers to attending to interactions between components of a contextual problem as part of quantitative reasoning. Teachers made two types of statements that I coded as *relating quantities*: numerical and quantitative relationships. Numerical relationships relate two pseudo-quantities through arithmetic or algebraic operations to compute a new pseudo-quantity. In other words, numerical relationships use arithmetic or algebraic operations between numbers, variables, or unknowns to create or compute a new number, variable, or unknown in a problem context. I did not find any evidence of teachers combining pseudo-quantities with quantities in numerical relationships, nor did I find evidence of teachers using quantities in numerical relationships.

An example of a numerical relationship is Group 1's statement in their Version 1 Quantitative Reasoning Task, "Write an equation that relates the variables from the table on the previous page. What type of equation is this?" Recall from Theme 1 that Group 1's task was coded as including the pseudo-quantities Richter scale and relative intensity. By using a table with numerical values relating these two pseudo-quantities, Group 1 asked students to create a new algebraic representation of this existing exponential relationship. Group 1's Assessment Guidelines stated the expected response from students was, "Let $y = 10^x$; this is an exponential equation." Teachers expected students to create this equation by raising the number 10 to the power of the input pseudo-quantity, Richter scale. This equation generated the output pseudo-quantity of relative intensity. These statements were coded as a numerical relationship because students were asked to combine a pseudo-quantity (the number 10) with another pseudo-quantity (Richter scale) using an algebraic operation (exponentiation) in order to create a new pseudo-quantity (relative intensity).

The second type of statement that I coded as *relating quantities* was quantitative relationships, where two quantities are conceived and used to produce a new quantity. The conception of two quantities being taken to produce a third quantity is called a quantitative operation, based on the work of Thompson (2011) and Moore et al. (2009). According to Thompson (2011), quantitative relationships relate already-conceived quantities based on mutual constraints on the measurable attributes involved. "Mutual constraints on the measurable attributes" means considering how the quantities covary together in the relationship. I used these definitions to help distinguish numerical relationships from quantitative relationships. Since quantitative relationships require quantities, it is not possible to have quantitative relationships between pseudo-quantities. I coded statements as quantitative relationships if the teachers attending to two quantities being related with a quantitative operation to produce a new quantity. If the teacher coordinated two quantities while attending to how they change in relation to each other within a quantitative relationship, I said the teacher attended to covariation within the quantitative relationship (Carlson et al., 2001; Oehrtman et al., 2008; Moore et al., 2009; Saldanha & Thompson, 1998).

An example of quantitative relationships came from Group 6, whose Version 4 documents I coded as quantitative relationships because they asked students in their Quantitative Reasoning Task, “How is the price in dollars related to the number of pounds of tomatoes?” As mentioned in Theme 1, Group 6 made statements in their Version 4 coded as quantities, which were the price and the number of pounds of tomatoes. In their Version 4 documents, Group 6 focused on ensuring students understood the importance and meaning of the unit rate. For example, Group 6 said in their Facilitator Instructions that “we can relate the two quantities [price and the number of pounds of tomatoes] in a rate of $1.50 per pound. The rate found can be used as a common multiplier to find the cost given any number of pounds of tomatoes.” Here Group 6 provides evidence the unit rate is more than just a number that, when multiplied by the pounds of tomatoes, yields the price. Instead, these statements indicate the unit rate was being considered as a quantity itself because the unit rate was presented as an attribute of the relationship between two quantities, with units in dollars per pound. These statements were coded as a quantitative relationship because one quantity (pounds of tomatoes) was being taken (through multiplication) with a second quantity (unit rate of price per pound of tomatoes) to produce a new quantity (price). Here multiplication was considered a quantitative operation since it combined two quantities to produce a new quantity. If pseudo-quantities were being combined to produce a new pseudo-quantity, multiplication would have been coded as an algebraic or arithmetic operation.

Almost every teacher made statements that were coded as *relating quantities* as an aspect of quantitative reasoning throughout all the MEA iterations. Initially, almost every teacher made statements that were coded as referring to numerical relationships. By the

MEA conclusion, middle school teachers had shifted from making statements about numerical relationships to making statements about quantitative relationships. Most high school teachers continued to refer to numerical relationships in their MEA documents even at the MEA conclusion. The following three subsections provide evidence about how teachers attended to *relating quantities* as an aspect of quantitative reasoning by summarizing teacher statements about numerical relationships, quantitative relationships, and teacher reflections on their development in thinking about *relating quantities*. Similar to Theme 1, I describe initial patterns in responses, final patterns in responses, then evidence about why teacher responses shifted and when these shifts occurred.

Teachers' Attention to Numerical Relationships

When first creating the MEA documents, most teachers made statements coded as referring to numerical relationships as an aspect of quantitative reasoning. Initially 17 of the 20 teachers gave Pre-Assignment responses that were coded as numerical relationships. Of the three remaining teachers, two teachers gave responses that were coded as referring to quantitative relationships, while the final teacher included no statements coded as referring to *relating quantities*.

One example of these 17 teachers who were coded as having referred to numerical relationships in their Pre-Assignment responses was Charles. Charles said quantitative reasoning is when students understand "how to write equations and functions" that model situations. He added:

> A simple task could be some sort of money saving problem. If you have \$100, and make \$40 per week mowing lawns this summer, define your variables and write a function modeling this situation. How long will it take you to have saved \$500?

In this statement Charles focused on writing a function and then using algebra to solve the function for given a specific amount, \$500. The components of the contextual problem included the initial amount of money, amount of money increasing each week, the number of weeks, and the final total amount of money. These components were not clearly defined because Charles did not attend to what object, attribute, or in some cases what units were associated with each component. Thus Charles' response was coded as pseudo-quantities because the components of the contextual problem were not indicated. The type of interactions Charles described in this statement were arithmetic operations because after setting up an equation, algebraic operations were needed to solve for the number of weeks it takes to save \$500. Thus Charles' responses were coded as referring to numerical relationships because he made statements about algebraic operations (subtractions, division) between pseudo-quantities (the initial amount of money, amount of money increasing each week, the total amount of money saved) to calculate a new pseudo-quantity (the number of weeks).

Version 1 for all six groups had statements that were coded as numerical relationships because these statements asked students to solve for a pseudo-quantity within an equation or function. For example, Group 4's Quantitative Reasoning Task had questions such as, "Given the right triangle below, EXPLAIN in complete sentences the process for solving for each of the remaining unknowns." Expectations for these questions were that students apply the Pythagorean Theorem for missing side lengths and recall the complementary angle relationship for missing angle measures. Group 4's Version 1 was coded as having only pseudo-quantities as part of quantitative reasoning, as detailed in Theme 1. Given these questions and expectations, Group 4 wanted students to identify an equation using known facts and formulas and to use algebra to solve for a desired answer. Thus, Group 4's Version 1 was coded as having statements referring to numerical relationships because they asked students to identify and perform algebraic operations between pseudo quantities (two side lengths of a right triangle) in order to compute a new pseudo-quantity (the missing side length of the triangle).

In addition to having students solve for a pseudo-quantity within an equation or functions, each group made statements coded as referring to *relating quantities* by having students identify or create an equation or function between pseudo-quantities. Groups 1, 4, 5, and 6 were coded as making statements referring to numerical relationships because they asked students to identify an existing equation or function before solving for specific numbers or variables. Examples already presented in this section include Group 1 asking students to identify an exponential function given tabular data and Group 4 asking students to apply the Pythagorean Theorem given partial measurements of right triangle side lengths.

Another example of a group asking students to identify a numerical relationship comes from Group 5. Recall that Theme 1 detailed Group 5's initial statements about how the vertical and horizontal positions of the Ferris wheel seats were coded as referring to quantities while revolution rate was coded as referring to a pseudo-quantity. In their Version 1 Quantitative Reasoning Task, Group 5 asked students to use the elapsed time and revolution rate "to determine where to position the rescue ladder." However, no direct statements were made about how students were supposed to create equations that allowed them to solve for the horizontal and vertical positions. One indirect statement relating to how students should create and solve these equations was in the Facilitator

Instructions, which said, “Students completing this activity should already have some experience with right triangle trigonometry, specifically with using sine and cosine to find missing lengths in right triangles.” This statement suggested Group 5 wanted students to calculate the seat positions based on the rotation angle, but no information was included in their Version 1 about how the rotation angle should be computed from the revolution rate. Since the revolution rate was coded as a pseudo-quantity, Group 5 was coded as having numerical relationship as part of their Version1 Quantitative Reasoning Task.

Instead of identifying existing equations or functions, Groups 2 and 3 asked students to create new equations or functions within a problem context before asking students to solve for specific numbers or variables. For example, Group 3 asked students to “draw a sketch of each situation” where a number of different kinds of linear functions could be considered by the student. When creating the graph of the total cost of a plumber’s time versus the time a job takes, Group 3 made statements in their Assessment Guidelines and Facilitator Guidelines that the student should create a sensible y-intercept and slope for this context, and recognized these could be different values among students. As mentioned in Theme 1, Group 3’s Version 1 was coded as pseudo-quantities because details about components of the context, such as the total cost of a plumber’s time, time a job takes, y-intercept, and slope, were not given in regards to the object, attribute, or units associated with each component. Group 3 allowed students to create their own linear function that related these pseudo-quantities through algebraic operations such as multiplication and addition. Therefore Group 3 was coded as numerical relationships by having students create their own functions in a various contexts. By asking questions such as “how much does the plumber charges for a 3-hour job?” Group 3 also attended to numerical relationships by asking students to solve for specific numbers within the linear function they created.

In addition to identifying, creating, and solving equations or functions between pseudo-quantities, Groups 1, 4, 5, and 6 initially attended to numerical relationships by having students create new representations of numerical relationships. An example of this was given at the beginning of the section, where Group 1 asked students to create a new algebraic representation of an existing exponential relationship.

I found the three high school groups and 10 individuals continued to make statements coded as referring to numerical relationships as an aspect of quantitative reasoning through Versions 4 and 5. For example, Group 1 continued to have students create and solve exponential functions, thus not changing how relationships were included in their MEA documents. Group 4 still asked students to apply known facts and formulas and use algebra to solve for a desired answer, thus attending to numerical relationships. Group 5 never added detail about how students were to relate elapsed time and revolution rate, thus continuing to attend to these numerical relationships. In these ways the high school groups were coded as continuing to make statements referring to numerical relationships as an aspect of quantitative reasoning. As detailed in the following section, Group 5 added statements referring to quantitative relationships between rotation angle and seat positions, and the middle school groups also added statements referring to quantitative relationships in final MEA documents.

Ten individual teachers made statements in Version 4 or 5 coded as numerical relationships. These statements were similar to statements these teachers had made in their Pre-Assignment. Five of the 10 teachers were in groups that had made statements coded as quantitative relationships. While these teachers' group responses were coded as quantitative relationships, I interpreted the teachers' individual reflections as indicative of how the teacher was thinking about the aspect of *relating quantities* as part of quantitative reasoning. Thus, while the group may have made statements coded as referring to quantitative relationships, I did not consider these five teachers as providing evidence that they shared their group's view that quantitative relationships were an aspect of quantitative reasoning.

Teachers' Attention to Quantitative Relationships

Only two of the 20 teachers gave Pre-Assignment responses that were coded as quantitative relationships: Gary and Rose. Theme 1 detailed how Gary made statements coded as quantities in his Pre-Assignment, and that he was the only teacher to indicate the first course meeting influenced his responses. In addition to describing quantities, Gary's Pre-Assignment included the passage:

> [For quantitative reasoning] in the secondary classroom, I would expect students to be able to learn how different quantities relate to each other; in a slope, for example, they should learn how to relate change in y to change in x. In an applied problem, they should be able to see the slope as a rate of change for the quantities involved.

In this statement Gary indicates two quantities (*x* and *y*) that are taken to produce the slope. Gary provides evidence he thought about slope as a quantity because he described slope as a rate of change of the quantities, suggesting he considered slope an attribute of the relationship between *x* and *y* with units of "change in y to change in x." Since two quantities are being related to produce a new quantity, I coded Gary's statement as referring to a quantitative relationship.

Rose was the only other teacher to make statements in her Pre-Assignment coded as quantities and quantitative relationships. Like Gary, Rose submitted her Pre-Assignment after the first day of class, and stated:

> Quantitative reasoning is not about rushing through a problem just to get to the solution. It is about using symbols to represent different quantities in a problem and understanding exactly what the meaning of those quantities are *throughout* the problem, not just in the answer at the end…Too often students just want you to give them a formula so they can quickly find a solution. Hopefully the introduction of the standards for mathematical practice, including quantitative reasoning, helps us steer away from that type of thinking in teachers and students.

In this passage, Rose considered how multiple quantities are taken together to find a solution to a problem and how understanding that solution within the problem context is part of quantitative reasoning. I interpreted her statement about "understanding exactly what the meaning of those quantities are *throughout* the problem, not just in the answer at the end" as indicating how the solution to the problem is also a quantity that needed to be understood within the problem context. I coded Rose's passage as referring to a quantitative relationship because these statements attend to taking existing quantities to create a new quantity. I also coded Rose's last sentence in the passage as referencing the CCSSM as being influential to her thinking about quantitative relationships because she references reason abstractly and quantitatively Standard for Mathematical Practice.

None of the groups made statements in their Version 1 that were coded as referring to quantitative relationships, but Groups 2, 3, 5, and 6 added such statements to their MEA documents by the end of the study while Group 1 and 4 did not add such statements. For example, Group 3 made statements that were coded as quantitative relationships when they directed their students to think about slope as the result of a quantitative operation. In their Version 4 Quantitative Reasoning Task, Group 3 asked students to "describe what slope is in terms of the given quantities." These teachers said in their Assessment Guidelines that an appropriate response was that “slope is the ratio of the change of the dependent variable (the y-axis quantity) to a corresponding change in the independent variable (the x-axis quantity).” Recall from Theme 1 that Group 3 provided evidence that the *x*- and *y*-quantities for each scenario were coded as referring to quantities

Group 3 provided evidence they also considered slope a quantity by adding detail to their expectations in the Assessment Guidelines. Group 3 asked students the question “how much does the plumber charge per hour?” and expected them to say “The slope of the graph would provide this information as the charge per hour.” This statement was coded as attending to slope as a quantity because a measurable attribute of a graph was defined as well as units of that measureable attribute. Thus Group 3 was coded as making statements referring to a quantitative relationship because they described two quantities (*x* and *y*) being taken together to create a new quantity (slope). Additionally, attending to the “corresponding change” between these quantities was coded as Group 3 thinking about covariation within this quantitative relationship.

Groups 2 and 6 provided evidence of quantitative relationships being an aspect of quantitative reasoning by having the students model profits in business settings. By the MEA conclusion, both groups made statements coded as quantities when attending to the unit price per item, the number of items sold, and the profit generated from selling that many items. Each group asked students to create an equation that combined the quantities unit price per item and number of items sold in order to create a new quantity, the profit. Thus Groups 2 and 6 made statements coded as quantitative relationships because quantities were being taken together to form new quantities.

While all three middle school groups made statements coded as quantitative relationships, Group 5 was the only high school group to provide evidence of quantitative relationships being an aspect of quantitative reasoning in their final MEA documents. Recall the previous theme detailed how Group 5 made statements about vertical and horizontal positions of the Ferris wheel seat and rotation angle that were coded as being quantities while "revolution rate" was coded as a pseudo-quantity. Group 5 alluded to relationships between the vertical and horizontal distance quantities in their Quantitative Reasoning Task:

> Devise a way to determine how far above or below the ground each seat [on a Ferris wheel] will be and the horizontal position of each seat along the ground. The only information that will be consistently available to a ride operator is how long the Ferris wheel has been turning since loading the last seat.

Group 5 detailed in their Version 4 Facilitator Instructions and Assessment Guidelines how students should develop trigonometric relationships between the rotation angle and vertical and horizontal seat positions in order to answer the questions posed in their Quantitative Reasoning Task. Group 5 added expectations that "student(s) are able to explain how the Cosine relates to the horizontal position of a seat, and Sine relates to the vertical position in terms of the coordinate plane" and that "student(s) are able to graph the vertical positions with respect to angle measure and the horizontal positions with respect to angle measure that will match, with some degree of accuracy, the sine and cosine waves." Group 5 stated in their Student Evaluation Document, "Remember, the most important thing is to obtain insight into how students understand the relationship between the quantities in this scenario. Do they clearly see how the central angle is related to the position of a point on a circle? ... Does the student recognize the role that the radius plays as the hypotenuse and that the center plays as a vertex?" These statements described trigonometric functions as quantitative operations because the teachers attended to taking the quantity of rotation angle and the fixed quantity of the Ferris wheel radius and computing the quantity vertical or horizontal position of the Ferris wheel seat. Thus these statements were coded as Group 5 attending to quantitative relationships in their final MEA documents.

Additionally, Group 5 made statements about this quantitative relationship that highlighted covariation between the quantity rotation angle and the quantity horizontal or vertical position of the Ferris wheel seat. Group 5 added questions to Facilitator Instructions for the facilitator to ask students, "How will you know where a given seat is located? What if the ride turns on for just a little longer? What will change? What is changing as the ride operates?" Group 5 did not provide answers for these questions, but said they "hope [that] through [students'] struggle to find a general method for finding the seat position, they will come to the idea that if they know the angle of rotation, they can use trig functions to find the coordinates of the corresponding point on the circle." These questions and anticipated responses suggest Group 5 encouraged students to think about how the input quantity of rotation angle influences the output quantity of vertical or horizontal seat position. By asking questions such as how the seat positions change as the rotation angle changes, Group 5 provided evidence they wanted students to think covariationally about these quantities with attention to the problem context.

Nine of the 21 teachers made statements coded as referring to quantitative relationships as an aspect of quantitative reasoning by the MEA conclusion. Recall 10 of the teachers made statements coded as referring to numerical relationships by the MEA conclusion; the remaining two teachers did not make any statements coded as *relating* *quantities* as an aspect of quantitative reasoning in their final MEA documents. Eight of the nine teachers who made statements coded as quantitative relationships came from Groups 2, 3, 5, and 6. These teachers made statements similar to the respective group's statements. An example of one teacher doing this was Charlotte, when she said in her Version 4 final reflection:

> It's essential for students to focus on recognizing relationships and having them write or explain their thought processes in how quantities relate to one another and showing they work together in a process not individually, as well as, constructing new quantities that are not given to form a conclusion …Our groups MEA relates to quantitative reasoning when we have students…creating visuals to identify relationships, having students explain what it means to have quantities co-vary, constructing general equations through these discoveries, and presenting their work to peers and teachers.

Charlotte was in Group 2, and the MEA to which she referred had questions that asked students to "identify two co-varying quantities in your fundraising situation and explain in detail how they are related to each other." In Group 2's MEA documents, the quantities "cost" and "income" were related in a linear equation to create the new quantity "profit." Charlotte's statement was coded as referring to quantitative relationships because she referenced her group's activity in a way conveying a quantitative relationship and covariation within that relationship.

Joyce was the only teacher in Group 1 making a statement that referred to quantitative relationships at the end of the MEA. Her Version 5 final reflection provided evidence for her change in thinking:

> The linear reasoning activity that we did in class really made this evident to me – there was so much more going on than I had ever realized. As far as quantitative reasoning in my classroom, I still see it as something that helps students understand math concepts better. I need to discuss the ways that quantities affect each other so that students can move beyond superficial, symbolic understanding of problems. As far as what I have learned from looking over my students' work on this activity…I need to provide my students with opportunity for discussion about differences in how quantities vary/relate depending on what kind of

> function we are using. I need to make it more evident to my students that they can use their prior knowledge to support their conjectures about the way certain quantities vary and relate to each other.

In her reflection, Joyce stated quantitative reasoning was when students do more than symbolically understand problems, which I interpreted as having students attend to more than just algebraic or arithmetic operations when solving a problem. Instead, Joyce stated students should consider how quantities covary within a function. I interpreted her reference to varying quantities within functions as attending to how the input quantity relates to the output quantity. In this way Joyce considered the input quantities affecting the output quantity through covariation. I coded Joyce's statements as quantitative relationships because Joyce described how quantities (such as function inputs) are taken to create a new quantity (function output) within a problem context.

Like Joyce, Penny was another teacher who made statements coded as covariation only after implementing her group's Quantitative Reasoning Task with her students. Penny echoed her group's statements about covariation being part of the task, saying, "“Students have to define quantities that relate in given scenarios and graph the relationships.This requires that they think about how the quantities covary." I coded her statements at attending to covariation within quantitative relationships she goes on to reference her group's task and provides evidence similar to that presented in Group 3 given above. Along with Joyce, Charlotte, and the teachers in Group 5, Penny was one of the few teachers to make statements attending to covariation as a part of her Quantitative Reasoning Task.

Teachers' Reflections on Relating Quantities

Teachers made few statements coded as reflections about what influenced them to consider quantitative relationships. Five teachers all individually mentioned aspects that influenced their thinking regarding quantitative relationships, but in vague ways. For instance, Rose gave a vague comment in her Version 4 final reflection about her Group 2's development throughout the MEA:

> We also decided to add a series of questions about the quantities so that the students can really think about how they are related to each other. My favorite question we added came in Version 4. It explains what a co-varying quantity is and asks them to identify any co-varying quantities within the task. Throughout the revisions of this task I have also learned (besides the importance of identifying quantities) that in order to really reason quantitatively you have to be able to recognize the relationship between different varying quantities and how they will vary together.

For Rose, the MEA revisions seemed to have influenced her group to incorporate quantitative relationships in their MEA documents, but the process of this development was not specified.

The most specific influence on teacher ways of thinking about quantitative relationships was suggested in Joyce's earlier quote. She said, "the linear reasoning activity that we did in class" influenced her to think about quantitative relationships. She made no other references to this activity, and given the multiple activities that incorporated linear reasoning from the Pathways to Calculus materials, I was unclear which specific activity she was referring to here. Joyce's completion of Version 5 may have given her another opportunity to consider and make statements regarding quantitative relationships, since this additional iteration of the task was applied with her own students.

Both Gary and Rose made individual statements at the beginning of the MEA that I coded as reflecting on factors influencing them to attend to quantitative relationships. Recall from Theme 1 Gary made statements indicating the first course meeting influenced his view of quantities. Thus the first course meeting could have influenced his view of quantitative relationships because quantitative reasoning was defined as identifying and representing relationships between quantities and constructing new quantities using the Moore et al. (2011) definition of quantitative reasoning.

Rose initially made a statement about the CCSSM that was coded as a factor influencing her view on quantitative relationships. She stated in her Pre-Assignment that the reasoning abstractly and quantitatively standard for mathematical practice influenced her to clarify quantitative reasoning. Specifically, she stated that "too often students just want you to give them a formula so they can quickly find a solution," and that "hopefully the introduction of the standards for mathematical practice, including quantitative reasoning, helps us steer away from that type of thinking in teachers and students." I interpreted students "quickly finding a solution" as using algebraic and arithmetic operations on formula to solve a problem. Thus the CCSSM may have influenced Rose to think about other kinds of operations, such as quantitative operations, that led her to include statements coded as quantitative relationships in her Pre-Assignment responses as well as her subsequent MEA documents.

Theme 3: Teachers' Attention to Coordinating Relationships of Quantities

The third aspect of quantitative reasoning I identified in the statements teachers made about quantitative reasoning in their MEA documents was *coordinating relationships of quantities*. In the data I identified two ways teachers *coordinated* *relationships of quantities*: by attending to multiple numerical relationships or by attending to multiple quantitative relationships within a problem context. Teachers coordinated numerical relationships when they compared features of multiple numerical relationships or created a new numerical relationship from other relationships within a problem context. Teachers coordinated quantitative relationships when they compared features of multiple quantitative relationships within a problem context.

All statements made in Pre-Assignment referring to coordinating quantitative relationships were coded as coordinating numerical relationships. Teachers who made such statements coordinated numerical relationships by comparing features of numerical relationships or by creating new numerical relationships from existing relationships. Most teachers who made these statements continued to do so until the MEA conclusion, while a few teachers made statements coded as coordinating quantitative relationships by the MEA conclusion. The following sections detail how teachers attended to coordinating numerical relationships, coordinating quantitative relationships, and the factors teachers attributed to their ways of thinking about *coordinating relationships of quantities* as aspect of quantitative reasoning.

Teachers' Attention to Coordinating Numerical Relationships

Seven of the 20 teachers initially made statements coded as coordinating numerical relationships. Six of these teachers were coded this way because they compared features of numerical relationships in their Pre-Assignment, while the other teachers' Pre-Assignment responses attended to creating a new numerical relationship from existing numerical relationships through the operation of function composition while the remaining 13 teachers did not make any statements coded as *coordinating* *quantitative relationships*. One of the six teachers who gave responses that compared numerical relationships came from Jack. Jack described the following task as one that demonstrated quantitative reasoning in his classroom, “Bank of Trig offers 4.5% interest compounded continuously while Bank of Calc offers 4.75% interest compounded quarterly. Assuming you deposit the same initial principal, which bank will provide more interest after 8 years?” Jack gave no other details regarding how students should compare the different interest rates or the implications of these rates on the amount invested in this problem context. Given the lack of details, I had to infer that the goal was for students to create equations between the principle, the interest rate, and the final amount accrued after eight years. Creating these relationships and solving them for static values involves arithmetic operations, and thus Jack’s task was coded as referring to numerical relationships as an aspect of quantitative reasoning. Since the task also had students comparing the result of different interest rates within these numerical relationships, Jack’s response to the Pre-Assignment provides evidence he was thinking about comparing features of multiple numerical relationships at the beginning of the MEA.

Charlotte made statements in her Pre-Assignment coded as coordinating relationships by creating new numerical relationships. Charlotte’s example of a quantitative reasoning task had students working with a function, $f(x)$, that modeled the location of a playing card during a perfect shuffle[4]. She asked students to “shuffle 16 index cards and write the data in a table where the first column is the order of cards before the shuffle and the second column is the position of the card after the shuffle…Take the data from the table and write it as two linear piece-wise functions.”

[4] A perfect shuffle is when the card pile is cut in half then cards are interwoven perfectly by alternating the top card from one pile followed by the top card from the other pile.

Charlotte did not indicate what these two linear piece-wise functions would be, but I presume she expected a function of the form: $f(x) = \begin{cases} 2x - 1, & \text{if } x \leq 8 \\ 2x - 16, & \text{if } x > 8 \end{cases}$ where $f(x)$ gives the position of the card x after the cards are shuffled. Charlotte did not detail the object, attribute of the object, or units for the location of a playing card before the shuffle and the location of a card after the shuffle, and thus her statements were coded as pseudo-quantities. Charlotte did not make any statements about what operations were needed to relate these pseudo-quantities. I interpreted her vague description of these piece-wise functions as requiring algebraic operations, such as if the card location is in the first half of the deck, add two to the card position to determine the ending card position. The function $f(x)$ was coded as a numerical relationship because it required an algebraic operation between pseudo-quantities (starting card position, the number 2) to create a new pseudo-quantity (ending card position).

Charlotte attended to coordinating numerical quantities by asking students to "create a table with first column as $f(x)$, second column as $f(f(x))$, third column as $f(f(f(x)))$, and fourth column as $f(f(f(f(x))))$. What do you predict will happen? Fill in the columns and explain your data." This task asked students to algebraically compute the functions $(f \circ f)(x)$, $(f \circ f \circ f)(x)$, and $(f \circ f \circ f \circ f)(x)$. These functions that result from the act of composition represent numerical relationships for the same reasoning $f(x)$ was coded as a numerical relationship: they require algebraic operations between pseudo-quantities to create a new pseudo-quantity (ending card position). Since these composition functions were created from multiple numerical relationships, Charlotte's task was coded as coordinating numerical relationships.

Initially three of the six groups (1, 4, and 6) made statements in their Version 1 documents referring to coordinating numerical relationships as an aspect of quantitative reasoning. These groups referred to this aspect by making statements about comparing features of numerical relationships in their Version 1 documents. For example, Group 6 said their main objective of their Quantitative Reasoning Task was for students to "make a pricing system, so that you know what to charge a customer no matter how many pounds of tomatoes they buy." The expectations listed in the Assessment Guidelines were that "students will find the unit rate of the price per tomato." No other information was given on how students should compute or consider this relationship between the pseudo-quantities "*x* pounds of tomatoes" and "cost for *x* pounds." I coded Group 6 as referring to numerical relationships in their Version 1 documents because finding the unit rate involved the arithmetic operation of division: the "cost for *x* pounds" divided by "*x* pounds of tomatoes" that would yield the unit rate of price per pound of tomatoes.

Group 6 originally made statements coded as coordinating numerical relationships by asking students to make comparisons between different representations of this numerical relationship. In their Version 1 Facilitator Guidelines, Group 6 asked students to work in small groups of three or four, where each group was told in the task to "work with another group that has a different explanation then (sic) your group. Convince your boss that your two explanations are the same or alter your explanations so the representations are the same." Group 6 anticipated that a "typical response is that groups will combine or alter their representations and create a write-up explanation for the boss that convinces that the representations are the same [or groups would] alter their representations." Since I coded cost and pounds as pseudo-quantities and the unit rate as a numerical relationship, this statement asked students to make comparisons between representations of possibly different numerical relationships. No other expectations were given for what this comparison between numerical relationships might have looked like, and thus I coded Group 6 as initially making statements referring to coordinating numerical relationships as an aspect of quantitative reasoning.

By the MEA conclusion, teacher statements largely did not change much in regards to including the aspect of *coordinating relationships of quantities* as quantitative reasoning. The same three groups (1, 4, and 6) continued to make statements coded as coordinating numerical relationships, with Groups 4 and 6 not significantly altering their initial statements. Group 1's task increased the number of questions that referred to comparing numerical relationships, and had a question about creating new numerical relationships in their Quantitative Reasoning Task in Versions 2 and 3. The question and matching expectations, for creating new numerical relationships from existing numerical relationships, came from considering the composition of functions. In their Quantitative Reasoning Task Versions 2 and 3, Group 1 asked students to "write two rule for logarithmic functions by composing *f* with *g* and *g* with *f*", where $f(x) = 3^x$ and $g(x) = \log_3 x$." Group 1 expected students to reason "that a logarithm is an inverse of an exponential function, the composition of the two functions should equal the identity function *x* (prior knowledge)." This question and expectation refer to numerical relationships, since *f(x)* and *g(x)* rely on algebraic operation of exponentiation and taking the logarithm without attention to the quantities involved. No other details were provided regarding how the composition function was to be determined other than use of prior knowledge. Thus students were probably expected to use symbolic manipulation to compose logarithmic and exponential functions. This algebraic operation created a new numerical relationship, the identify function, from two existing numerical relationships, the logarithmic and exponential functions. Thus Group 1 referred to *coordinating relationships of quantities* by creating new numerical relationships at some points during the MEA, but removed this question and expectation by their final MEA documents. This final section in this theme details Group 1's stated rationale for their decision to add and remove these statements in their MEA documents.

Ten teachers individually made comments coded as coordinating numerical relationships by the MEA conclusion. Of these 10 teachers, five were teachers who had already made statements referring to coordinating numerical relationships at the onset of the MEA, while the other five had not made these statements until later in the MEA. These 10 teachers made statements that resembled earlier statements about coordinating numerical relationships or followed from their group's statements coded as coordinating numerical relationships as an aspect of quantitative reasoning.

Teachers' Attention to Coordinating Quantitative Relationships

No teachers or groups made statements coded as coordinating quantitative relationships in the Pre-Assignment or Version 1 documents. By the MEA conclusion, one group and two teachers made statements coded as coordinating quantitative relationships. As the previous themes detailed, by the conclusion of the MEA Group 6 made statements about "x pounds of tomatoes" and "cost for x pounds" in ways coded as quantities rather than pseudo-quantities. Additionally, computing the unit rate as a relationship between these quantities was coded as a quantitative relationship rather than numerical relationship. Group 6's Version 4 Quantitative Reasoning Task told students:

> Your boss is confused and frustrated. He is not convinced that the collection of pricing systems from the employees is consistent to his requests. Work with another group that has a different explanation then (sic) your group. Convince your boss that your two explanations are the same and that both your representations are correct.

Group 6 expected the students to "combine or alter their representations and create a write-up explanation for the boss that convinces that the representations are the same. If students alter their representations, then the explanation should include why changes were made." These statements were coded as coordinating quantitative relationships because Group 6 indicated students should compare representations of a quantitative relationship and convince others that the representation appropriately models the situation.

Two teachers made statements referring to coordinating quantitative relationships in their final MEA documents. Gary and Ken did this by comparing quantitative relationships that appear in their group's task. Recall from Theme 2 that the final version of Group 5's MEA documents were coded as quantitative relationships by having students create quantitative relationships between the quantity of rotation angle and the fixed quantity of the Ferris wheel radius and computing the quantity vertical or horizontal position of the Ferris wheel seat. Gary and Ken, both members of Group 5, elaborated in their final reflection about these relationships. For example, Gary said:

> I now think of quantitative reasoning as looking carefully at all aspects of quantities involved in a problem or context, and examining how they interact and co-vary, with the purpose of using the resulting knowledge to answer questions, make predictions, and deepen understanding of the given problem or context…Our task will help students connect the idea of trigonometric ratios and the unit circle with the quantities of side lengths of triangles, coordinates on a unit circle, angles of a right triangle, and angles of rotation. We hope it will give them a deeper understanding of the power of trigonometry and the unit circle, and how they can use their understanding of circles and triangles to develop an understanding of cyclical functions, in particular the functions of sine and cosine. We hope that giving the measurements a physical context will help them bring in

> their own experience and develop a deeper, more connected understanding of these concepts.

Since these statements reference quantitative relationships given in Group 5's MEA documents, this excerpt was coded as referencing the quantitative relationships between rotation angle and the horizontal and vertical Ferris wheel seat positions.

In particular, Gary makes the statement about connecting right triangle trigonometry to the unit circle.

I interpreted the connections Gary referenced as comparing two quantitative relationships: one between an angle of a right triangle and the side length of that right triangle, and the other relationship between the angle of rotation (from the *x*-axis and a radius of a unit circle) and the coordinate points where the radius connects to the unit circle. Presumably the comparison, or connection as he put it, Gary wants students to make is that understanding right triangle trigonometry can help understand angles of rotation and coordinates on a unit circle. Thus these statements were coded as Gary coordinating quantitative relationships as an aspect of quantitative reasoning.

Ken described similar comparisons between the unit circle and right triangle trigonometry, perhaps because both Ken and Gary were in Group 5 and had shared their ways of thinking about quantitative reasoning. These two teachers were the only ones with statements that referring to coordinating quantitative relationships in their MEA documents.

Teachers' Reflections on Coordinating Relationships of Quantities

The only evidence of teachers reflecting on the aspect *coordinating relationships of quantities* came from Group 1 and individual comments from teachers in Group 1. These teachers made statements suggesting that their thinking about *coordinating* *relationships of quantities* was influenced by instructor, peer, and undergraduate and K12 student feedback.

Group 1 acknowledged the role the instructor had in their thinking, which seemed to support them to include the aspect *coordinating relationships of quantities* in their MEA documents. In their Version 2 Decision Log, Group 1 said:

> [The instructor] had mentioned that we could try using tabular relationships to help students model exponential and logarithmic functions…[and] to learn about the logarithm using a simple logarithmic function. (We adapted this activity from the logarithms worksheet in [the] Pre-Calculus materials.) They are given the notation and must work on modeling the function with a table and graph, based on the corresponding exponential function… We hope that this problem will encourage students to think quantitatively (about how quantities relate to each other in different scenarios – exponential vs. logarithmic) and also develop an understanding of logarithms that they will carry throughout the rest of their study on the topic.

The Pathways to Calculus modules (Carlson & Oehrtman, 2011, Module 4, Worksheet 8, Appendix J) included questions that asked students to construct exponential and logarithmic functions on the same graph and to state observations about the relationship between these two functions. These same questions were added to Group 1's Version 2 documents, indicating the course materials influenced them to add additional questions that referred to quantitative reasoning as *coordinating relationships of quantities*. Additionally, the interactions with the instructor referenced in this passage were probably referring to the instructor's comment that Group 1 "might want to add to the task where they have to talk about relationships between various quantities…so that they actually engage in the structure of an exponential or logarithmic function." This comment indicated Group 1 was encouraged to continue including the aspect *coordinating relationships of quantities* as part of their task.

Peer feedback was acknowledged by Group1 as influencing them to continue thinking about quantitative reasoning in these ways. Group 2 said to Group 1:

> We really like that your overall theme is focused around inverses. It's a really great idea to have the students think about what exactly the relationship is between exponential functions and log functions so they understand that logarithms will be the inverse, or "undo" step, for solving exponential functions. We think the task has a reasonable difficulty level and will really challenge the students in a positive way. However, we're wondering if it will be too difficult for the student to understand that the second table in should be the inverse of the exponential function table they'll fill out first. Could you give some ideas to the teacher for how they could lead the students thought that?

Group 1 chose not to make changes to the question Group 2 referenced, but did offer this statement in regards to the peer feedback:

> An interesting comment was concerned the ability of students to make connections to an inverse using the tables. We don't know the answer to this, except at this point have students look at the exercise and we will see how the connections, or lack thereof, unfold.

While Group 1 did not change any of their documents in ways relating to *coordinating relationships of quantities*, these comments indicated Group 1 was encouraged to continue incorporating this aspect in their MEA documents.

Group 1's comments about the undergraduate student feedback provided evidence that the student work prompted Group 1 to reduce their focus on *coordinating relationships of quantities* within the MEA task. When evaluating one of the questions about *relationships of relationships of quantities*, Group 1 said:

> Three students demonstrated knowledge of the quantitative relationship between exponents and logarithms. This leads us to believe that the students think of logarithms as a noun instead of a function. In other words, they can point at an exponential function, and explain the base, exponent, logarithm relationship. However, number two presents the evidence the students don't know the functional relationship between exponential and logarithmic functions, and their inverse relationship. We added that students needed to demonstrate using an example; however we feel that the inquiry belongs on the functional relationships for future investigations.

This response indicated undergraduate student feedback seemed to discourage the group from including the aspect *comparing relationships of quantities* in their MEA documents. Individual comments from Nicholas and Joyce echoed these comments, indicating their thinking about *coordinating relationships of quantities* was influenced by instructor, peer, and undergraduate student feedback.

Theme 4: Teachers' Characterization of Quantitative Reasoning Across Teaching and Non-Teaching Settings

For this theme I make a distinction between how teachers provided information about quantitative reasoning in non-teaching settings and the information teachers provided in teaching settings, such as designing, implementing, and evaluating actual instructional activities. For example, when providing a task in her Pre-Assignment, Joyce stated "When I teach lessons, my goal is to help students think quantitatively as we work through problems. I want them to make sense of what they are doing, not to just do it." Thus Joyce provided information that, in a teaching setting, one feature of quantitative reasoning is the ability to make sense of a problem. An example of Joyce describing quantitative reasoning in a non-teaching setting is when she defined the term in a theoretical way, saying quantitative reasoning is "strongly associated with number sense and the ability to visualize." Thus Joyce indicated number sense and visualization are features of quantitative reasoning in a non-teaching setting. Information from how teachers thought about quantitative reasoning in non-teaching settings came primarily from the group Decision Logs, since this document asked teachers to describe their thinking about quantitative reasoning.

In this section I detail how teachers characterized features of quantitative reasoning in these two settings. Teachers' characterizations at the beginning of the MEA were different across the two settings. By the MEA conclusion, teachers characterized features of quantitative reasoning more similarly across teaching and non-teaching settings. Teacher reflections about their development suggested peer feedback, undergraduate student feedback, and course materials influenced them to characterize features of quantitative reasoning more similarly across these settings.

The following subsection details my inferences about the features teachers referred to in their MEA documents with attention to how teachers characterized these features in teaching and non-teaching settings. The first subsection details the features teachers initially characterized in individual Pre-Assignments and group Version 1's. The next subsection discusses group and individual final statements from Versions 4 and 5. The last subsection includes the factors teachers attributed to influencing how they characterized features of quantitative reasoning in teaching and non-teaching settings. The structure of these sections reflects the chronological development of teacher thinking as the MEA progressed. These sections reference the aspects *identifying quantities, relating quantities*, and *coordinating relationships of quantities* defined in the previous three themes; these aspects encompass some of the features teachers described in their MEA documents.

Teachers' Characterization of Quantitative Reasoning Initially

At the beginning of the MEA, teachers' characterizations of quantitative reasoning were different across teaching and non-teaching settings. Either teachers did not make statements about quantitative reasoning in non-teaching settings or teachers made statements in non-teaching settings that differed from the features they stated in teaching settings. Initially, three individual teachers and three groups did not provide information about quantitative reasoning in non-teaching settings. Jack and Darium stated directly they were unsure or unable to offer an explicit definition of quantitative reasoning in the Pre-Assignment; for instance, Jack stated, "I cannot honestly say I know exactly what the phrase [quantitative reasoning] means." Byron did not complete the Pre-Assignment, and thus offered no information about features he considered as part of quantitative reasoning. Groups 2, 5, 6 did not provide information in their Version 1 in response to the question asking "how you think about quantitative reasoning and how your thinking has changed?" Thus at the MEA's onset these teachers did not characterize features of quantitative reasoning in non-teaching settings.

Twelve individual teachers and the other three groups made statements about quantitative reasoning in non-teaching settings, but the features they identified did not relate to the features they identified in teaching settings of the Pre-Assignment and Version 1 documents. For example, Joyce's Pre-Assignment responses referred to different features of quantitative reasoning between teaching and non-teaching settings. In a non-teaching setting, Joyce defined quantitative reasoning as being "strongly associated with number sense and the ability to visualize (or conceptualize in some way) certain amounts." When describing quantitative reasoning in teaching settings, such as what this looks like in her classroom, she stated:

> When I teach lessons, my goal is to help students think quantitatively as we work through problems. I want them to make sense of what they are doing, not to just do it…when my students and I work with logarithms, I spend a lot of time discussing what a particular problem means. In general (overall), I do not constantly give lengthy explanations so as not to cause algebraic processes to become tedious and disjointed, but these explanations are necessary at the appropriate times.

Joyce did not mention number sense or visualization of amounts as features of quantitative reasoning in this setting, but instead focused on sense making when working with functions in a problem. While number sense could have been included in sense making, Joyce's responses did not provide evidence of this, and thus statements about features of quantitative reasoning in these two settings were not similar in a way I could observe.

Joyce's response to another teaching question reiterated the differences between the features she described in the two settings. When discussing quantitative reasoning in terms of a specific task she would implement with her own students, Joyce said she would ask students to:

> evaluate several logarithmic expressions of common base (with consecutive whole number answers, but mixed up); Order these expressions and their evaluations ('answers'); Ask how much different their answers are from each other as they progress through the order; Ask how much difference they actually represent; [and] Tie into a real-world logarithmic scale (sound, earthquakes, etc.).

Neither the task presented here nor the earlier teaching explanation included references to visualization as mentioned earlier. One could claim Joyce's teaching responses contained features of comparing values, which could be related to Joyce's notion of quantitative reasoning as "number sense." However, little evidence supported this claim because Joyce did not detail what she meant by "number sense." Similarly, the answers Joyce described in this statement could relate to the "amounts" she mentioned in her non-teaching description earlier, but little evidence supports this similarity because Joyce never defined "amounts." Overall Joyce's Pre-Assignment responses highlighted different features of quantitative reasoning in teaching and non-teaching settings.

Similar to these patterns in individual responses, Groups 1, 3, and 4 made statements referring to features of quantitative reasoning that differed across teaching and non-teaching settings. For example, Group 1 described two features of quantitative reasoning in their Version 1 that differed across teaching and non-teaching settings. The first feature of quantitative reasoning they gave in a non-teaching setting was seen in the following passage of their Decision Log:

> Our discussion on quantitative reasoning illuminated the fact that sometimes we take the variables and the quantities they represent for granted. For example, students arbitrarily adding the units on at the end or giving answers that don't make sense to the problem. By not concentrating on how the quantities are represented, the ability to create relationships mathematically could be reduced significantly.

This statement indicated Group 1 thought of quantitative reasoning as focusing on more than just variables with units, but did not detail what other components a quantity needed to be represented in a problem. Thus I interpreted Group 1's first feature of quantitative reasoning as the aspect of *identifying quantities* because these teachers described attended to pseudo-quantities by including contextual components of the problem as something beyond a variable with an arbitrary unit. This aspect of quantitative reasoning was not included the same way in their other Version 1 documents. The only statements Group 1 made about variables or quantities was in the Quantitative Reasoning Task question asking students to "identify the variables in this [exponential] relationship." Group 1 expected students to say "Let x = Richter scale number, Let y = Relative intensity of the earthquake." This expectation indicated the only expectations Group 1 had for attending to pseudo-quantities were for students to label the variables in the problem. Nowhere in their documents were other requirements for variables described, not even units. I interpreted this question and expectations in a teaching setting that did not align with the aspect of quantitative reasoning described earlier in a non-teaching setting because students were not expected to attend to pseudo-quantities as anything beyond a variable label.

Group 1 indicated a second feature of quantitative reasoning in a non-teaching setting in their Version 1 Decision Log. In addition to mentioning how students need to create relationships in the previous quote, Group 1 went on to say:

> The connections in mathematics can be made by quantifying the objects under consideration. Quantitative reasoning is highlighting the fact that math is doing something with objects. Every manipulation we make as mathematicians and students is creating and manipulating relationships to answer questions.

I found this passage difficult to follow because Group 1 referenced quantifying objects and then doing something with objects as part of quantitative reasoning, but the following sentence does not include objects or quantification. In their other Version 1 documents, Group 1 did not attend to objects or quantification. The only similarity I found between their task and this statement was that Group 1 attended to manipulating numerical relationships to solve for unknown pseudo-quantities, which was detailed in Theme 2. If the manipulations referenced in the earlier passage were meant to serve as an example of "doing something with objects" then this non-teaching feature would have some similarities with the teaching feature seen in the task. However, Group 1 did not indicate relationships were objects, thus I interpreted this non-teaching feature as differing from the teaching feature found in their Version 1 documents.

Group 1 described a third feature of quantitative reasoning in their Version 1 documents that did have some similarities across teaching and non-teaching settings. This group stated in their Decision Log that:

> Another aspect of quantitative reasoning is the importance of comparison. How big is big? How small is small? How do the quantities represent big changes vs. small? Our MEA should and eventually will tackle these seemingly simple questions in a way they should give insight on the mathematics involved with these objects.

In this statement Group 1 described quantitative reasoning as making comparisons between pseudo-quantities because, as detailed previously, Group 1 was unclear in how they were using the term "quantity" in a way different than "variable" or "unknown." I interpreted the questions about "how small is small" and how "quantities represent big changes vs. small" as examples of the comparisons Group 1 considered part of quantitative reasoning. I was unclear if this comparison occurred between one pseudo-quantity or between two pseudo-quantities. The final sentence in this passage suggested this feature of quantitative reasoning may not be addressed in their Version 1 documents.

I found two questions in Group 1's Quantitative Reasoning Task that partially attended to comparing pseudo-quantities. These questions asked students to determine how much greater one earthquake is than a second earthquake. Group 1 stated in their Assessment Guidelines that they wanted students to compute the relative intensities of the two earthquakes and to use division to reach conclusions of "about 50 times larger" and "a million times greater." These statements indicated Group 1 thought about quantitative reasoning as comparing pseudo-quantities in a teaching setting, but did not attend to quantitative reasoning by asking questions about how the pseudo-quantities represent "big changes vs. small" or how small or large these pseudo-quantities are. Thus Group 1 made statements indicating some similarities and some differences of how this feature of quantitative reasoning was presented in teaching and non-teaching settings. Overall I found little evidence Group 1's Version 1 that indicated non -teaching features of quantitative reasoning were similar to the features given in teaching settings. Like Group 1, Groups 3 and 4 initially made statements about features of quantitative reasoning that had few similarities across teaching and non-teaching settings.

In contrast to the 15 teachers and all six groups who were categorized either as initially not providing information or providing information about features of quantitative reasoning that differed across teaching and non-teaching settings, six teachers were exceptions. These six teachers provided non-teaching statements about quantitative reasoning that had similarities between teaching statements given in either their description of quantitative reasoning in their classroom or in their example of a quantitative reasoning task. For example, Allie's Pre-Assignment was prototypical of these six teachers, as her responses had similarities between her non-teaching description of quantitative reasoning, her teaching description of quantitative reasoning in her classroom, and her example of an actual quantitative reasoning task. Allie stated her definition of quantitative reasoning as the following:

> To me, quantitative reasoning means taking a value and interpreting how it applies to a specific problem or situation… I think quantitative reasoning applies to real world application problems and if when the problem is solved if the 'answer' makes sense and why.

Allie responded similarly in the Pre-Assignment question about what quantitative reasoning looked like in teaching settings, such as her classroom:

> In my classroom quantitative reasoning looks like students tackling problems with teacher support but not teacher lecture. Students attempt to solve problems using skills they have acquired but are challenged to think outside of the box and come up [with] their own interpretations of problems and ways to solve them. As a teacher, to help students reason quantitatively it is important to monitor students but never just tell them how to do a problem, they should be allowed to try a problem in any manner and allowed to struggle. I believe quantitative reasoning is developed with a lot of why questions and teacher facilitation but not direct lecture.

In both her teaching and non-teaching responses, Allie specified that quantitative reasoning included the features of conceptual emphasis in problem solving and attending to the problem context. When providing an example of a quantitative reasoning task, Allie stated the task would:

> ask students a lot of 'why' questions at specific parts of their presentation to give them the opportunity to express why they solved the problem the way they did. At the end they would need to explain their answer in context of the original word problem.

These questions in the example task indicated the same features of quantitative reasoning Allie stated in her earlier teaching and non-teaching responses. Thus Allie's initial responses referred to these features of quantitative reasoning in similar ways across teaching and non-teaching settings.

Teachers' Characterization of Quantitative Reasoning at the Model-Eliciting Activity Conclusion

By the MEA conclusion, teachers made statements highlighting similar features of quantitative reasoning in teaching and non-teaching settings. This section provides evidence from groups' Version 4 and 5 and the individual final reflections.

All six groups made statements referring to at least one feature of quantitative reasoning in similar ways across teaching and non-teaching settings. Groups 2, 3, 5, and 6 all made statements about the aspect *identifying quantities* in all final MEA documents. All six groups recognized the aspect *relating quantities* in all final MEA documents. Groups 4 and 6 all recognized the aspect *coordinating relationships of quantities* in all final MEA documents. These groups all made statements in teaching and non-teaching settings that aligned.

The responses from Group 1 offer an example of how the groups referred to the aspect *relating quantities* in all of their final MEA documents. In their Decision Log, Group 1 wrote:

> We chose this topic [of logarithms] because we feel that students typically don't reason quantitatively about logarithms much at all (many of them are successful only because they follow procedures). We have a quantitative understanding of logarithms (like "the answer to the logarithm is what exponent I would need to use on this base to make it into this number [the argument]"; or our visualization of the behavior and characteristics of logarithmic graphs; etc.).

This statement indicated Group 1 considered solving missing for variables within numerical relationships and representing numerical relationships as aspects of quantitative reasoning. These aspects were given in a non-teaching setting because only the mathematical topic of logarithms was discussed with no mention of teaching contexts. Group 1 also stated both of these aspects in teaching settings. Theme 2 detailed questions in this group's Quantitative Reasoning Task that attended to solving variables within numerical relationships because they asked students to identify an existing equation or function before solving for specific numbers or variables. Additionally, Group 1 asked students to "construct the graphs of $f(x) = 3^x$ and $g(x) = \log_3 x$ by making a table of values and plotting coordinate points." Thus Group 1 included the aspect of representing numerical relationships in a teaching setting as well. In this way other groups characterized aspects of quantitative reasoning seen in their final MEA documents in ways that were similar in both teaching and non-teaching settings.

Similar to the group responses, individual teachers characterized aspects of quantitative reasoning similarly in teaching and non-teaching settings. By the MEA conclusion all teachers recognized the aspect *identifying quantities*, 19 teachers recognized the aspect *relating quantities*, and seven teachers recognized the aspect *coordinating relationships of quantities* in ways that aspects aligned across teaching and non-teaching settings. For example, Charlotte recognized the aspects of *identifying* *quantities* and *relating quantities* in her final MEA documents, first by saying in her Decision Log that quantitative reasoning was:

> making sense of a problem by trying to visualize in your mind a model, interpreting data by breaking it down so one can identify relevant quantities and their meanings, representing relationships between quantities using graphs, tables, and algorithms then trying to create a formula through that reasoning. It's essential for students to focus on recognizing relationships and having them write or explain their thought processes in how quantities relate to one another and showing they work together in a process not individually, as well as, constructing new quantities that are not given to form a conclusion.

Here she identified that quantities were an aspect of quantitative reasoning, and referred to quantitative relationships by considering how quantities covary in relationships and how these relationships create new quantities. Thus Charlotte made statements that referred to the aspects of *identifying quantities* and *relating quantities* in this non-teaching setting. These aspects are seen in the next paragraph when she referred to the teaching setting in the context of her group's task:

> Our group's MEA relates to quantitative reasoning when we have students reason about which would be the best fundraiser for their school and explaining why it would be the best choice, identifying quantities (varying and not), determining what quantities mean and how they relate to each other, creating visuals to identify relationships, having students explain what it means to have quantities co-vary, constructing general equations through these discoveries, and presenting their work to peers and teachers.

Since quantities and quantitative relationships were in her group's task, in this paragraph Charlotte identified the aspects of quantitative reasoning in her group's task. The similarities between her two statements about *identifying quantities* and *relating quantities* in both teaching and non-teaching settings indicated Charlotte characterized these aspects of quantitative reasoning similarly across teaching and non-teaching settings.

Teachers' Reflections on Their Characterization of Quantitative Reasoning

Teachers made statements about factors that influenced how they thought about quantitative reasoning in their final reflections. I found three factors influenced teachers to characterize quantitative reasoning similarly across teaching and non-teaching settings: peer feedback, undergraduate feedback, and the course materials. Statements from the four teachers completing Version 5 indicated K-12 student feedback also influenced these teachers to characterize features of quantitative reasoning in teaching and non-teaching settings in ways that align. Groups did not comment on the instructor feedback as influencing them to align features of quantitative reasoning across these settings even though the instructor's feedback to each group advised for a more clear connection to how quantitative reasoning was being defined and related to the task.

One stereotypical example of how groups made statements indicating the factors that influenced them to characterize features of quantitative reasoning more similarly across settings comes from Group 2. Group 2 made comments about peer feedback and undergraduate feedback that were echoed by other groups, but also went into more detail as to how the course materials influenced them.

Group 2 acknowledged that peer feedback influenced how they thought about quantitative reasoning. Group 2 stated in their Version 3 decision log:

> After receiving feedback from our peers as well as reading the task of another group, we decided that our feedback should be more detailed so that it would be easily understood by another teacher implementing the task in their classroom or a substitute teacher. *We* knew all the places within our task where we were looking for quantitative reasoning, but it wasn't as clear to the people who had been reading through our tasks. By creating a more detailed description in the assessment guidelines were able to describe exactly how and where we'd like the students to show their quantitative reasoning. (emphasis in original)

This quote indicated that the peer feedback process influenced Group 2 to be more explicit about characterizing quantitative reasoning in teaching settings. Additionally Group 2 stated here that seeing additional examples from other groups helped Group 2 think about ways to characterize quantitative reasoning in teaching settings. Here Group 2 commented on how receiving feedback from peers provided a reason for Group 2 to recognize features of quantitative reasoning in their subsequent versions of the MEA. Group 2 was supported by the peer feedback to align features of quantitative reasoning across teaching and non-teaching settings because Group 2's Version 4 MEA documents included statements about features of quantitative reasoning in both teaching and non-teaching settings and these features aligned across these settings.

Group 2 made statements about undergraduate feedback that indicated this feedback influenced them to consider how features of quantitative reasoning appeared in teaching settings. Group 2 wrote the following passage in their Version 4 Student Evaluation document:

> Where we would like to see more quantitative reasoning would be in the recognition that profit and number of items sold are co-varying quantities as well as more specifics on the relationship between these two quantities as they vary together…What was made the most clear from the student feedback we received is that any classroom using our task will have to address the first aspect of quantitative reasoning, "Attending to and identifying quantities", - *before* giving the students this task. From the different tasks we've worked through and articles that we've read in this class, we've learned that the most important start to successful quantitative reasoning is first being able to identify the quantities present in the problem as well as how they are related to each other. The fact that these students weren't able to fill out the original quantities limited the evaluations we could make about their quantitative reasoning in general.

This passage suggests the teachers in Group 2 were prompted to think more deeply about how features of quantitative reasoning were considered in teaching and non-teaching settings. Here Group 2 referenced Thompson's definition of quantitative reasoning that was presented in class, indicating "attending to and identifying quantities" was a feature of quantitative reasoning given in a non-teaching setting. Group 2 was prompted to align this feature of quantitative reasoning in teaching and non-teaching settings because the statement also emphasizes how the undergraduate feedback prompted these teachers to think about how this feature of quantitative reasoning appeared in their Quantitative Reasoning Task.

As suggested in the previous passage, Group 2 made specific comments about how the course materials influenced their thinking about features of quantitative reasoning. Theme 1 detailed Group 2's comments regarding how the Pathways to Calculus materials influenced them to think about *identifying quantities* as an aspect of quantitative reasoning in their Quantitative Reasoning Task. These comments suggested the course materials influenced Group 2 to incorporate the aspect of *identifying quantities* in a teaching setting. In their Decision Log Group 2 gave a definition of quantity that matched Thompson's (2011) definition given during the first week of class, and thus incorporated the aspect of identifying quantities in a non-teaching setting. Since the Pathways to Calculus materials incorporated Thompson's definition of quantities I coded Group 2's alignment of the aspect *identifying quantities* to be due to the course materials such as the Pathways to Calculus materials and Thompson's (2011) definition of quantitative reasoning presented in class.

Individual teachers also acknowledged these three factors as influencing their ways of thinking about quantitative reasoning. For example, Julie's final reflection stated:

> When looking back at my pre-assignment, I realize I really had no idea what quantitative reasoning was…[we] did not ask a lot of questions about how different quantities are related or have them specifically look at all the attributes of the many quantities involved in doing a fundraiser. As our knowledge of

> quantities developed, and with the help of peer evaluations, we brought the MEA much further. We decided to ask students to specifically look at quantities and gave them a chart to fill in to do so. After the college students worked through our MEA, we chose to give an example in the first column of the chart to help 'show' the students how to list the different objects. We learned by expecting students to list all the objects, varying or unvarying, it would help them get a mental picture of the relationships between the different quantities. This is something I personally have never done before but plan to do in the future.

Julie indicated the peer and undergraduate student feedback helped her include *identifying quantities* as an aspect of quantitative reasoning across different settings such as creating, assessing, and scaffolding questions, and implementing these strategies in her own classroom. She thus reported her ability to align these aspects of quantitative reasoning in these teaching settings to her non-teaching notion of quantities was supported by the peer and undergraduate student feedback iterations.

Theme 5: Teachers' Confidence in Their Ability to Develop Students' Quantitative Reasoning

Another pattern in teacher responses was that as they completed the MEA, they expressed more confidence in their ability to develop their students' quantitative reasoning skills. Evidence for this theme came from unprompted teacher comments both at the beginning and conclusion of the MEA. These comments revealed teacher thoughts about their own confidence regarding how quantitative reasoning was incorporated in their classroom.

Nine teachers made statements in their Pre-Assignment expressing uncertainty about the amount of quantitative reasoning occurring in their classroom. The Pre-Assignment did not ask teachers about the level of quantitative reasoning occurring in their classrooms, but teacher comments about quantitative reasoning provided evidence they lacked confidence about how to incorporate quantitative reasoning in their classroom. For example, Nicholas stated:

> I have always had a basic understanding of how to reference State Standards, but I don't feel like I am correlating them as [effectively] in my classroom as I should. I would like to gain a better understanding of what exactly quantitative reasoning is and how it applies to my teaching and student learning.

Nicholas' and others' comments indicated these teachers were uncertain about how they could develop their students' quantitative reasoning.

At the conclusion of the MEA, 13 teachers commented on how they intended to implement quantitative reasoning in their future classroom practices, providing evidence of how teachers were thinking about quantitative reasoning in terms of their future classroom practice. The four teachers who submitted Version 5 were part of the 13 teachers who commented on how they thought about quantitative reasoning in their classroom practice. For example, Joyce made the following comment after evaluating her own students' work on her task:

> None of the students noted that the quantities represented amounts 'since the first investment was made.' Some students did not provide units for their quantities…I see that I need to help my students develop a more thorough understanding of quantities (by what I say and model when dealing with quantities)… When I discuss quantities in class, I need to move beyond saying, for example, '*x* represents time,' and say, '*x* represents the time in years since money was first invested in the account.'… I need to provide my students with opportunity for discussion about differences in how quantities vary/relate depending on what kind of function we are using.

Here Joyce identified having students work with quantities as one area she now knows how to incorporate into her classroom, specifically by adjusting student expectations and future classroom pedagogy to incorporate this aspect of quantitative reasoning. Teachers who did not submit Version 5 did not make as specific comments on how their practice would change. The MEA did not prompt teachers to provide information on how they planned to change their classroom practices to support students' quantitative reasoning. Therefore these 13 teachers made unprovoked statements, suggesting other teachers may have had also had more confident opinions about their ability to promote their students' quantitative reasoning. The teachers did not specify any reasons for the increased confidence.

Theme 6: Factors Not Promoting Development in How Teachers Thought About Quantitative Reasoning

The MEA documented two factors that did not influence teachers to develop their thinking about quantitative reasoning. First, teachers made statements indicating their prior experience with quantitative reasoning did not influence their thinking. Second, teacher exposure to the CCSSM definition of quantitative reasoning during the Pre-Assignment was documented to have little effect on how teachers thought about quantitative reasoning. Each of these factors is detailed in the remainder of this section.

Overall teachers did not communicate the impact of prior experience on their thinking about quantitative reasoning. Teachers were asked to report their prior experience with quantitative reasoning in the first part of the Pre-Assignment. Four teachers said they had such experiences, all indicating they encountered quantitative reasoning through their schools' efforts to introduce and incorporate the term in their classes. Three of these teachers, Samantha, Rose, and Brandon, indicated these experiences had limited impact on their thinking about quantitative reasoning. For instance, Brandon said he had "seen this phrase before through workshops focused on the Common Core Standards of Mathematical Practice,' but said "overall, I am not certain of how quantitative reasoning looks like on a macro scale of the content I teach." Similarly Rose made comments in her Pre-Assessment such as, "our district curriculum coordinator…has been doing a great job at introducing the math teachers to the new standards for mathematical practice, so we've briefly discussed [quantitative reasoning] in our classrooms over last summer and throughout this school year." However, when asked what quantitative reasoning looked like in her classroom, she responded, "Having very little experience with studying what quantitative reasoning looks like, this is the question I am most unsure about." The comments from Rose, Brandon, and Samantha indicated professional development did not impact how these teachers thought about quantitative reasoning.

Tiffany was the only teacher who made a comment suggesting prior experience influenced her ways of thinking about quantitative reasoning. She said:

> I have heard this phrase before in my school district and in my classes last summer. I have spent quite some time with the new standards, and although I am by no means an expert, I am familiar with them and their implications in the classroom.

Tiffany did not say specific instances of how these experiences shaped her views, nor did she reference the experience elsewhere in her documents.

Teachers did not make comments about the impact of the CCSSM on how they thought about quantitative reasoning. The Pre-Assignment asked teachers to interpret the CCSSM "reasoning quantitatively" standard for mathematical practice. Teacher responses included restatements of this standard for mathematical practice; four teachers directly referenced the CCSSM' definition of "reasoning quantitatively" when defining quantitative reasoning in other parts of the Pre-Assignment. For example, Ken gave an example task for his classroom "based on the description given in the Common Core." The CCSSM was brought up by five teachers in their responses to "what do you expect to get out of this Quantitative Reasoning in Secondary Mathematics course?", where teachers made responses such as "I really need to take this opportunity to become more familiar with the Common Core State Standards" (Nicholas). No groups or teachers indicated the CCSSM as being influential in their ways of thinking about quantitative reasoning in any of the following documents, providing limited evidence about the impact of reading the CCSSM on how teachers thought about quantitative reasoning.

CHAPTER V

DISCUSSION

Mathematics education literature suggests that teachers need to reason quantitatively (Conference Board of the Mathematical Sciences, 2012; Confrey & Krupa, 2010; Moore, 2012; Thompson, 1994, 2011). However, little literature exists about how teachers think about quantitative reasoning or how to develop teachers' thinking about quantitative reasoning in ways that impact teachers' practice (Heck et al., 2010; Marrongelle et al., 2013; Sztajn et al., 2012; Wiener, 2013). Given that quantitative reasoning has been defined as attending to and identifying quantities, identifying relationships between quantities, and constructing new quantities (Moore et al., 2009), this study was designed to document how in-service mathematics teachers thought about quantitative reasoning and how their thinking developed within a graduate course by using a Model-Eliciting Activity (MEA) to document teacher thinking throughout the course. This study incorporated a models and modeling perspective to answer the research question: How do mathematics teachers' models of quantitative reasoning develop through an MEA grounded in their classroom practice? This chapter answers the research question by building on the findings in the previous chapter and then discussing the significance, implications, limitations, and recommendations stemming from the answer to the research question.

Answering the Research Question

Overall, teachers' initial models of quantitative reasoning were not fully communicated in terms of defining quantitative reasoning in settings not connected to their classroom. When teachers did communicate features of quantitative reasoning, they described aspects of pseudo-quantities and numerical relationships. As teachers went through the course and the MEA iterations, they began grappling with quantities and quantitative relationships as aspects of quantitative reasoning instead of pseudo-quantities and numerical relationships. Additionally, the features of quantitative reasoning that teachers documented in different settings became more aligned across these settings as the MEA progressed. This section details the answer to the research question by describing patterns in teachers' models chronologically. The first part of this section summarizes the chronological development of each group, referencing results from the previous chapter. The second part of this section details the overall patterns of development in teachers' models of quantitative reasoning.

Model Development by Group

Group 1 was comprised of three high school teachers. These teachers all provided evidence that their initial models of quantitative reasoning included the aspects pseudo-quantities and numerical relationships, as evidenced by statements in their Pre-Assignment and Version 1 documents. Group 1's Version 1 (their initial MEA documents) included statements asking students to identify vague features of the context for variables, thus indicating this group attended to pseudo-quantities as an aspect of quantitative reasoning. Their Version 1 documents also focused on the aspect of numerical relationships by identifying quantitative reasoning as when students write and solve exponential and logarithmic equations. The third aspect Group 1 initially indicated was in their model of quantitative reasoning was coordinating numerical relationships, indicated by their focus on performing the composition of exponential and logarithmic functions in their Version1 documents. Group 1 described these three aspects of quantitative reasoning in their Version 1 differently across teaching settings (evidenced in their Quantitative Reasoning Task and supporting documents), and non-teaching settings (such as how this group defined quantitative reasoning in their Decision Log). As the MEA progressed, Group 1 added statements coded as coordinating numerical relationships in Versions 2 and 3 of the MEA, but they removed these statements in their Version 4 documents.

By the MEA conclusion, Group 1's model of quantitative reasoning shifted to include aspects of quantities rather than pseudo-quantities, but continued to include aspects of numerical relationships and coordinating numerical relationships (Table 3). This group attended to numerical relationships in all final MEA documents in ways that aligned across teaching and non-teaching settings by making similar statements about creating and solving exponential functions in both settings. This group continued to attend to coordinating numerical relationships by having students examine composition of exponential and logarithmic functions.

Table 3

Summary of Aspects of Quantitative Reasoning in Group 1's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities	Quantities
Relating Quantities	Numerical Relationships	Numerical Relationships
Coordinating Relationships of Quantities	Coordinating Numerical Relationships	Coordinating Numerical Relationships

Two of these teachers' individual reflections at the MEA conclusion echoed the group documents by including aspects of pseudo-quantities, numerical relationships, and coordinating numerical relationships in their model of quantitative reasoning. One of the group members, Joyce, implemented Version 5 and provided evidence that her final model of quantitative reasoning included aspects of quantities and quantitative relationships by describing how quantities (such as function inputs) are taken to create a new quantity (function output) within a problem context. This evidence suggests Joyce was influenced to develop her thinking about quantitative reasoning by implementing her group's Quantitative Reasoning Task with her own students; specifically this iteration of the MEA influenced her to include aspects of quantities and quantitative relationships in her model of quantitative reasoning. Similar to Joyce's K12 student feedback, Group 1 acknowledged the undergraduate feedback in developing their thinking about quantitative reasoning, in particular by prompting Group 1 to be more explicit about their expectations for what a quantity is and how students should think about quantities within a problem context. Other statements from the teachers in Group 1 indicated peer feedback played a large role in the development of their thinking about quantities as an aspect of quantitative reasoning by providing the teachers examples of how quantities can be incorporated in tasks.

Group 2 was comprised of four teachers who focused on middle school content in their MEA documents. While Rose's Pre-Assignment provided evidence her model initially included the aspect of quantity as a component of quantitative reasoning, the other three teachers in Group 2 had models that included aspects of pseudo-quantities and numerical relationships in their Pre-Assignments. Charlotte and Samantha's Pre-

Assignments also had students coordinate numerical relationships, suggesting these teachers considered these aspects as part of their model of quantitative reasoning. Group 2's initial MEA documents focused on attending to cost and revenue amounts in a fundraising scenario and representing these amounts in relation to profit. Initially this task had students create functions using these amounts and solve for specific profit values. Thus Group 2's model of quantitative reasoning included the aspects of pseudo-quantities and numerical relationships, but this group did not characterize features of quantitative reasoning in non-teaching settings such as defining quantitative reasoning in their Decision Log.

As the MEA progressed, Group 2 transitioned to making statements about quantities and quantitative relationships in ways that aligned across teaching and non-teaching settings. By the MEA conclusion, this group's model of quantitative reasoning included aspects of quantities by attending to the unit price per item, the number of items sold, and the profit generated from selling that many items. Group 2's model also incorporated quantitative relationships by asking students to create an equation that combined the quantities unit price per item and number of items sold in order to create a new quantity, the profit (Table 4). Like their group's model, both Charlotte and Rose's final models contained aspects of quantities and quantitative relationships. Charlotte also considered covariation within quantitative relationships as part of quantitative reasoning. The other two teachers in this group made statements indicating pseudo-quantities or numerical relationships were components of their final models. Comments from the teachers in Group 2 indicated the Pathways to Calculus materials (Carlson & Oehrtman, 2011) influenced them to incorporate quantities in their MEA documents. Group 2 also made comments about peer feedback and undergraduate feedback being impactful in their thinking about quantities. Rose was the only teacher in the entire class to make a statement suggesting the Common Core State Standards for Mathematics (CCSSM) impacted her thinking, which may have influenced Rose to include statements about quantities and quantitative relationships in her MEA documents.

Table 4

Summary of Aspects of Quantitative Reasoning in Group 2's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities	Quantities
Relating Quantities	Numerical Relationships	Quantitative Relationships
Coordinating relationships of quantities	Not Evidenced	Not Evidenced

The four teachers in Group 3 focused on middle school content, and all initially indicated in their Pre-Assignment that their model of quantitative reasoning included aspects pseudo-quantities and numerical relationships. Additionally, one teacher made a statement suggesting her model included as coordinating numerical relationships by attending to composition of functions. This group's Version 1 asked students to create new equations or functions within six different problem contexts, to solve for specific numbers or variables, and compare representations of these functions. Thus Group 2's initial model of quantitative reasoning included the aspects of pseudo-quantities, numerical relationships, and coordinating numerical relationships. The statements about these aspects of quantitative reasoning had few similarities across teaching and non-teaching settings.

Group 3's model shifted from including the aspects of pseudo-quantities and numerical relationships to including the aspects of quantities and quantitative relationships (Table 5). This group recognized quantitative relationships in all final MEA documents by having students think about slope as the result of a quantitative operation. Additionally, this group attended to quantities and had students consider the corresponding change between quantities, and thus evidenced Group 3's model included covariation. The group as a whole made a comment about the undergraduate feedback impacted their thinking about quantitative reasoning but was not specific in their response.

Three teachers in this group implemented their Quantitative Reasoning Task with their own students and submitted Version 5. Penny made statements suggesting the K12 student feedback helped her clarify the expectations of quantities to her students and made her realize she needed to provide her students more opportunities to reason covariationally. The other two teachers who completed Version 5 made statements indicating the K12 student feedback provided similar information as the undergraduate student feedback, and did not make statements about how this iteration impacted their thinking about quantitative reasoning.

Each teacher in Group 3 gave varied final responses about the aspects of quantitative reasoning. Penny made statements indicating her model of quantitative reasoning included quantities, quantitative relationships, and covariational reasoning. The other three teachers in the group provided evidence their models incorporated pseudo-quantities, and one teacher also included numerical quantities as an aspect of quantitative reasoning in her final reflection.

Table 5

Summary of Aspects of Quantitative Reasoning in Group 3's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities	Quantities
Relating Quantities	Numerical Relationships	Quantitative Relationships
Coordinating Relationships of Quantities	Not Evidenced	Not Evidenced

The three teachers in Group 4 focused on high school content, specifically trigonometry. Initially all three teachers gave responses in their Pre-Assignment indicating pseudo-quantities was part of their model of quantitative reasoning, and two teachers' models included aspects of numerical relationships and comparing numerical relationships. Group 4 initially made statements in their Version 1 that attended to pseudo-quantities, numerical relationships, and coordinating numerical relationships by focusing on numbers and their units of right triangle side lengths and angles and how existing equations, such as the Pythagorean Theorem, could be used to solve missing side lengths and angles. These initial comments had few similarities across teaching and non-teaching settings.

Overall Group 4's model did not change in terms of including aspects of pseudo-quantities, numerical relationships, or coordinating relationships of quantities (Table 6). This group still recognized numerical relationships and coordinating numerical relationships in all final MEA documents but provided evidence they thought about these aspects of quantitative reasoning similarly across teaching and non-teaching settings. Final comments from all three individual teacher documents reflected the group's model that quantitative reasoning includes the aspects of pseudo-quantities, numerical relationships, and coordinating numerical relationships. Group 4 was not clear in what influenced their thinking throughout the MEA iterations. .

Table 6

Summary of Aspects of Quantitative Reasoning in Group 4's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities	Pseudo-Quantities
Relating Quantities	Numerical Relationships	Numerical Relationships
Coordinating Relationships of Quantities	Coordinating Numerical Relationships	Coordinating Numerical Relationships

The three teachers in Group 5 focused on high school content and initially had a variety of responses in their Pre-Assignment. Byron did not submit a Pre-Assignment, Ken's model included the aspects of pseudo-quantities, numerical relationships, and coordinating numerical relationships, while Gary's model included the aspects quantities and quantitative relationships. Group 5's initial model attended to the aspects of quantities and pseudo-quantities as well as numerical relationships by asking students to solve for specific numbers or variables in right triangle relationships situated in a Ferris wheel problem context. Group 5 did not provide information about quantitative reasoning in non-teaching settings by not providing a definition in their Version 1 Decision Log.

As Group 5 went through the MEA, they added quantitative relationships to their model of quantitative reasoning. Group 5 was the only high school group to provide evidence of quantitative relationships being an aspect of quantitative reasoning in their final MEA documents by describing trigonometric functions as quantitative operations

(Table 7). Group 5 made statements referencing quantities and quantitative relationships in ways that aligned across teaching and non-teaching settings. Additionally, Group 5 made statements about this quantitative relationship that highlighted covariation between the quantities in the Ferris wheel context. This group continued to make statements indicating numerical relationships were part of their model because they asked students to identify an existing equation or function before solving for specific numbers or variables or creating representations of trigonometric functions. All three Group 5 teachers expressed quantities and quantitative relationships in their final reflections in similar ways that related to their group's statements. Two of the teachers' models also included coordinating quantitative relationships by comparing quantitative relationships that appear in their group's task. The only evidence for what influenced these teachers' thinking came from Gary, who indicated his thinking changed after he was exposed to Thompson's (1990) article presented in the first week of the course.

Table 7

Summary of Aspects of Quantitative Reasoning in Group 5's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities and Quantities	Pseudo-Quantities and Quantities
Relating Quantities	Numerical Relationships	Numerical Relationships and Quantitative Relationships
Coordinating Relationships of Quantities	Not Evidenced	Not Evidenced

The four teachers in Group 6 focused on middle school content, and gave descriptions indicating pseudo-quantities and numerical quantities were aspects in their model of quantitative reasoning. Similar to these individual responses, this group's

Version 1 contained statements suggesting pseudo-quantities and numerical relationships were aspects of the group's model of quantitative reasoning. Additionally, this group's model also included coordinating numerical relationships by asking students to make comparisons between different representations of the unit rate. These teachers did not characterize features of quantitative reasoning in non-teaching settings.

Group 6's model transitioned from attending to pseudo-quantities and numerical relationships to attending to quantities, quantitative relationships, and coordinating quantitative relationships, and characterized these aspects in ways that aligned across teaching and non-teaching settings (Table 8). Group 6 made this transition by making statements about pseudo-quantities and numerical relationships to making statements coded as quantities and quantitative relationships by attending to how one quantity (pounds of tomatoes) could being taken with a second quantity (unit rate of price per pound of tomatoes) to produce a new quantity (price). Group 6 indicated students should compare representations of this quantitative relationship and convince others that the representation appropriately models the situation. In their final reflections, all four teachers' models attended to quantities and two of these teachers attended to quantitative relationships. No individual teachers provided evidence that their models included the aspect of coordinating quantitative relationships, but two teachers' made statements indicating coordinating numerical relationships were a part of their final model of quantitative reasoning. Group 6 was prompted by undergraduate student feedback to be more explicit about how quantities were included in the MEA, but gave little other evidence of what influenced their thinking.

Table 8

Summary of Aspects of Quantitative Reasoning in Group 6's Model

Aspect	Initial	Final
Identifying Quantities	Pseudo-Quantities	Quantities
Relating Quantities	Numerical Relationships	Quantitative Relationships
Coordinating Relationships of Quantities	Coordinating Numerical Relationships	Coordinating Quantitative Relationships

Overall Patterns in the Development of Teachers' Models

In the Pre-Assignment and Version 1 stages of the MEA, teachers' models of quantitative reasoning were not fully communicated in terms of defining quantitative reasoning in settings not connected to their classroom. In the Pre-Assignment, some teachers said directly they did not know what quantitative reasoning was or how to effectively communicate their thinking about quantitative reasoning. Evidence of teachers' limited understanding of quantitative reasoning came from the fact that three of the six groups did not provide information about quantitative reasoning in a non-teaching setting in their Version 1. Even though these groups were prompted to provide information about quantitative reasoning in non-teaching settings, these three groups did not attempt to articulate quantitative reasoning in their initial MEA documents. When groups did communicate their thinking about quantitative reasoning in a non-teaching setting, they did so in ways that communicated different features of quantitative reasoning across teaching and non-teaching settings. Teachers reported few prior experiences with quantitative reasoning which may have contributed to their limited knowledge or communication about quantitative reasoning. For the few teachers who did say they received previous professional development addressing quantitative reasoning, the impacts of these experiences were minimal in their thinking about quantitative reasoning, as evidenced by their statements about the experiences. Additionally, comments on teachers' Pre-Assignment often expressed uncertainty about the amount of quantitative reasoning occurring in their classroom, thus limiting their communication about quantitative reasoning.

Teachers' models of quantitative reasoning developed during the MEA when teachers changed the ways they thought about aspects of quantitative reasoning. As teachers went through the course and the MEA iterations, they began grappling with the idea of a quantity. During the MEA teachers made statements that distinguished the object, attribute of the object, and unit of the attribute being considered. Teachers also began considering quantitative relationships, rather than numerical relationships, by considering quantitative operations between multiple quantities, rather than arithmetic or algebraic operations. Only a few teachers indicated they thought about quantitative reasoning as coordinating numerical relationships. Most of the teachers who indicated they initially thought about quantitative reasoning this way continued to do so throughout the MEA, most likely because they did not have a clear understanding of relationships that were not numerical. One middle school group and two high school teachers transitioned from this thinking about numerical relationships to consider coordinating quantitative relationships. As the MEA progressed this group and these two high school teachers analyzed and compared the quantitative relationships they had created, but gave no indication for why this transition occurred. Thus a small number of teachers' models developed to include the aspect of coordinating quantitative relationships, while other teachers' models continued to attend to coordinating numerical relationships throughout the MEA.

By the MEA conclusion, teachers' models of quantitative reasoning were more fully communicated in terms of describing features of quantitative reasoning in both teaching and non-teaching settings and having features of quantitative reasoning that aligned across these settings. Researchers have shown misalignments between theoretical notion of mathematics and the curricular materials that a teacher implements to support those ideas may have negative impacts on student learning (Confrey & Stohl, 2004; Mullis, Martin, Gonzalez, & Chrostowski, 2004). This suggests teachers' alignment of these features of quantitative reasoning was beneficial for teachers, and their students, because teachers developed ways of thinking about quantitative reasoning that they could better articulate and connect to their classroom. Peer feedback, undergraduate feedback, and course materials influenced teachers to communicate their thinking about quantitative reasoning more similarly across teaching and non-teaching settings. Additionally, comments on teachers' final MEA documents indicated they had increased confidence about how to support quantitative reasoning in their classroom, supporting the notion that their initial models of quantitative reasoning were more fully communicated in comparison to their initial models.

The factors influencing the development of teachers' models were K-12 and undergraduate and K12 student feedback, course materials, peer feedback, and to a lesser extent instructor feedback. Two factors that I found to not significantly impact teachers' models of quantitative reasoning were teachers reading the CCSSM and having professional development prior to the study.

This study did not focus on comparing middle school teachers' models with high school teachers' models, but I did notice, and subsequently document, a difference between these two teachers' models. In comparison to the high school teachers, middle school teachers were more likely to shift from considering pseudo-quantities to quantities. Teachers working with middle school content, where the quantities were less complex, transitioned from thinking about pseudo-quantities to thinking about quantities in part because undergraduate student feedback prompted teachers to be more explicit about quantities. Peer feedback and course materials, such as Pathways to Calculus (Carlson & Oehrtman, 2011), also gave middle school teachers examples of how quantities could be incorporated in their MEA. Joyce and the three teachers in Group 5 were the only high school teachers who made statements indicating they considered quantities in their final MEA documents. Like the middle school teachers, Joyce was influenced to consider quantities by implementing Version 5 with her own students. The other high school teachers in Groups 1 and 4 continued thinking about pseudo-quantities, perhaps because the quantities in high school contexts were more complex or less familiar than the quantities in middle school contexts. For example, quantities that involve measuring units of a logarithmic scale or units of radians are more difficult and unfamiliar to conceive than quantities with units of dollars or the number of movies rented. Thus middle school teachers' models of quantitative reasoning developed to include the aspect of quantities, while most high school teachers' models continued to include pseudo-quantities throughout the MEA.

Another pattern in the data was that in comparison to the high school teachers, middle school teachers were more likely to shift from considering numerical relationships to quantitative relationships as an aspect of their model of quantitative reasoning. Most of the middle school teachers transitioned from including numerical relationships in their model to including quantitative relationships, but gave only vague reasons for this transition. Most of the high school teachers did not demonstrate a clear understanding of quantitative relationships, perhaps because they did not demonstrate a clear understanding of quantities or the more complex nature of relationships on which high school teachers focused. Thus middle school teachers' models of quantitative reasoning developed to include the aspect of quantitative relationships, while most high school teachers continued to include numerical relationships throughout the MEA. Attending to quantities allowed Groups 3 and 5, as well as Joyce and Charlotte, to consider the attribute component of the quantity and how this attribute covaried within a quantitative relationship. No other groups provided evidence of covariational reasoning within relationships even though Groups 2 and 6 also made statements coded as quantitative relationships.

My conjecture is that the difference in how middle and high school teachers developed models of quantitative reasoning is linked to the complexity of the mathematics on which the teachers focused in their MEA documents and the familiarity of this material to the teachers. More complex, in this context, means a person must possess a more sophisticated conceptual structure to consider how quantities were being related in the problem (Thompson, 1992, 1993). The middle school teachers, focusing primarily on linear relationships and concrete, directly experienced quantities, were able to ground the mathematics in their MEA documents in contexts in which the quantities and quantitative relationships were fairly accessible and familiar, such as focusing on "number of dollars" as a quantity and proportional and linear functions as quantitative relationships. The high school teachers focused on more complex and unfamiliar quantities, such as angle measure, energy, and the Richter scale. The relationships among these quantities were also more complex than the quantitative relationships conceptualized by the middle school teachers, such as exponential relationships requiring repeated multiplication rather than linear relationships requiring repeated addition (Confrey, 1991).

An example of a more complex relationship comes from Group 4's conception of an angle's variation within right triangle trigonometry. Moore (2010) also found teachers had difficulty quantifying angle measure and conceiving how variation in angle measure influences trigonometric functions. Additionally, Moore argued that the learners' failure to attend to quantities can hinder learners' understanding of trigonometric functions as covarying relationships. While part of the course focused on developing quantities and quantitative relationships involving angle measures, this effort may have been too limited to make a large impact on these teachers' models, particularly because learners who are placed in quantitatively-rich situations do not always develop meaningful mathematical concepts from these experiences (Cuban, 2001; Lobato & Siebert, 2002; Noble et al., 2001).

In contrast to the high school groups, the middle school groups selected topics that incorporated quantities whose measures were more directly comprehensible and topics that required less prerequisite knowledge for students to think about the quantitative relationships involved. This difference in complexity could have contributed to Group 4 not providing evidence for transitioning from pseudo-quantities to quantities and Groups 1 and 4 not transitioning from numerical relationships to quantitative relationships. Group 4 also may have been deterred from considering quantitative relationship because these relationships depend on teachers' conception of quantities.

A final piece of evidence supporting the conjecture that the high school teachers did not reason covariationally due to the complexity of the material comes from Group 1, who stated prerequisite knowledge impacted their students' understanding of relationships. At the conclusion of the MEA, Group 1 recognized students' difficulty in thinking about functional relationships: "the students don't know the functional relationship between exponential and logarithmic functions, and their inverse relationship. We added that students needed to demonstrate using an example; however we feel that the inquiry belongs on the functional relationships for future investigations." In this comment Group 1 acknowledged that students' lack of prerequisite knowledge was impeding some of their task goals, and Group 1 even altered the task goals to accommodate these challenges. Since these are functional relationships from secondary content depending on many prerequisite skills, this comment suggests the complexity of the material may have influenced teachers' and students' ability to think about quantitative relationships in these settings.

Significance in Model Development

The course materials and the MEA iterations influenced teachers' models of quantitative reasoning to develop in two significant ways. First, by focusing on quantities and quantitative relationships, the teachers attended to sense making within problem solving and thus supported the broad goal of quantitative reasoning. Second, by including the aspects of quantities and quantitative relationships in their models of quantitative reasoning, teachers were better situated to reason covariationally about the mathematical content in their MEA documents and promote this type of reasoning from students. The remainder of this section details why these developments in teachers' models of quantitative reasoning are significant and what this development looked like for teachers in this study.

Teachers' Models Developed to Include Sense Making

The first significant development of teachers' models of quantitative reasoning was that sense making became a part of their models by the MEA conclusion. Researchers of mathematics education have summarized the many depictions of quantitative reasoning and have found that a broad purpose of quantitative reasoning is for people to make sense of and solve a problem using mathematics (Langkamp & Hull, 2007; Mayes, Peterson, & Bonilla, 2013). In terms of Thompson's (2011) theory of quantitative reasoning, sense making is seen in a learner's attention to both quantities and quantitative relationships. When creating a quantity, a learner must construct a quantity by considering the components of a quantity rather than taking these components as obvious. When working with quantitative relationships, a learner must consider how quantities relate to each other within a problem and understand how to solve the situation rather than only incorporating arithmetic or algebraic operations to solve the problem.

Teachers in this study increased their attention to sense making as part of quantitative reasoning, as evidenced by their transition from models of quantitative reasoning that included pseudo-quantities and numerical relationships to including quantities and quantitative relationships. Teachers promoted sense making by attending to quantities because teachers became explicit about the components of quantities within a context. Considering the components encouraged students to think deeply about a problem rather than assume these components in ways that could result in conflation between the components. Teachers also promoted the broad goal of sense making because teachers encouraged deep conceptual thinking about how quantities were related rather than having students rely on memorized procedures to solve problems. Five of the six groups' models developed in ways that did promoted this broad goal of sense making.

This shift towards sense making as part of teachers' models of quantitative reasoning was a significant development for these teachers, according to mathematics education researchers and reform documents. Sense making of a problem is a process with longstanding importance in mathematics education (CCSSM, 2010; Confrey & Kupta, 2010; Ma, 1999; Thompson, 2011). Teachers' attention to procedures rather than sense making has negative consequences for students, including students developing procedural knowledge that has only limited use in novel situations and lower performance levels on standardized tests in comparison to students taught in ways that promote sense making (Boaler, 1998, 2013; Even & Lappan, 1994; Riordan & Noyce, 2001). Additionally the CCSSM includes "make sense of problems and persevere in solving them" as the first Standard for Mathematical Practice, indicating teachers should promote this goal in their classrooms. These research findings and reform documents indicate the teachers in this study benefited from developing models of quantitative reasoning that included sense making.

Teachers' Models Developed to Become Better Positioned to Reason Covariationally

The second significant development of teachers' models of quantitative reasoning was that teachers became better positioned to reason covariationally by the MEA conclusion. Becoming better positioned to think about covariational reasoning is important for teachers because covariational reasoning is fundamental to deep conceptual understanding of functions and more advanced mathematical topics (Carlson, 1998, Carlson et al., 2002; Carson & Oehrtman, 2005; Oehrtman et al., 2008). As detailed in Chapter 2, a person reasons covariationally when he or she coordinates two quantities while attending to how they change in relation to each other (Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008; Moore et al., 2009; Saldanha & Thompson, 1998). Covariational reasoning is a foundation for a learners' understanding of function (Carlson, 1998; Oehrtman et al., 2008; Thompson, 2011), and a strong understanding of function is "central to undergraduate mathematics, foundational to modern mathematics, and…essential for any student hoping to understand calculus" (Oehrtman et al., 2008, p. 27). A strong understanding is when students comprehend a general mapping of a set of input values to a set of output values where these values change continuously, as opposed to symbolic manipulations and procedural techniques (Carlson, 1998; Monk & Nemirovski, 1994; Thompson, 1994).

In this study teachers became better positioned to engage in covariational reasoning because they developed models of quantitative reasoning that included the aspects of quantities and quantitative relationships. Teachers need to consider these aspects to reason covariationally because this type of reasoning requires the teachers to attend to the attribute of the quantity and how variation in this attribute influences, through a quantitative operation, the attribute of another quantity. By the MEA conclusion Groups 2, 3, 5, and 6 made statements coded as the aspects of quantities and quantitative relationships and were thus better positioned to reason covariationally.

While Groups 1, 2, 4 and 6 did not provide evidence that they considered covariation in their MEA documents, teachers in Groups 3 and 5, Joyce (Group 1), and Charlotte (Group 2) made statements attending to the attribute of quantities and how this attribute covaried within a quantitative relationship. Group 5's focus on covariation within trigonometric functions was especially noteworthy because researchers have documented how learners' attention to quantities and covariation of quantities can enable learners to build coherent meanings of trigonometric functions (Castillo-Garsow, 2010; Moore, 2010). Group 5's inclusion of covariation within their task indicated these teachers were thinking more deeply about trigonometric functions and encouraged students to engage in similar types of thinking. Besides Joyce, the high school teachers in Groups 1 and 4 did not consider quantitative relationships in their MEA documents. Given the need for teachers to reason quantitatively and support this type of reasoning in their classrooms, the limited covariational reasoning skills evidenced by the teachers should be a concern for mathematics teacher educators.

Implications

The answer to this study's research question can inform mathematics education research and mathematics teacher education. The implications to research and teacher education are detailed in the following sections.

Implications for Research

This study has two consequences for research. First, this study supports and extends prior work regarding how teacher MEAs can document teachers' models within teacher education settings. Second, the answer to this study's research question can be used by researchers to understand in-service teacher thinking about quantitative reasoning beyond the context of this study. The following subsections detail each of these implications.

Supporting and Extending Prior Research Findings. This study supports prior research indicating teacher MEAs can document teachers' models and how these models develop. Previous studies incorporated MEAs to investigate in-service teachers' models of teaching mathematics (Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003) and undergraduate student models of quantitative reasoning (Carlson et al., 2003). The current study also extends this work by incorporating an MEA with in-service mathematics teacher education to investigate their model of quantitative reasoning.

This study provided an answer to the research question regarding mathematics teachers' models of quantitative reasoning. Some particularly effective components of this MEA for documenting teacher thinking were: teachers explicitly defining quantitative reasoning in their Decision Logs; teachers' questions in both the task and the Facilitator Guidelines; teachers identifying approaches to promote quantitative reasoning in Facilitator Guidelines; and teachers specifying goals and expectations of students' quantitative reasoning in the Assessment Guidelines. These components prompted teachers to document their ways of thinking about quantitative reasoning in different settings, thus providing the data for this study. Combined with Hjalmarson's (2008) framework for analysis of curricular innovations, these components provided a method to effectively answer this study's research question.

This study's MEA, or the components within the MEA, provides researchers a method to use in other teacher education settings. The MEA in this study (Appendix B) was designed and implemented using a models and modeling perspective and was meant to be shared with and reused by other researchers. Other standards for mathematical practice could be examined by altering the MEA to focus on practices other than quantitative reasoning.

Understanding In-service Teacher Thinking about Quantitative Reasoning. The answer to this study's research question offers researchers a way to better understand in-service teacher thinking about quantitative reasoning beyond the context of this study. Researchers need ways to understand teacher thinking, including how teachers' think about quantitative reasoning and other CCSSM standards for mathematical practice (Confrey & Krupa, 2010; Marrongelle et al., 2013; Sztajn et al., 2012; Wiener, 2013; Thompson, 2013). This study was designed to address these needs, and does so by providing novel findings about how one group of teachers thought about quantitative reasoning and how those ways of thinking developed through an MEA.

The findings generated in this study are generalizable to teacher thinking in other settings. A models and modeling perspective guided how the MEA was created and implemented, and how the resulting data were analyzed. This perspective supports the naturalistic generalization described in Chapter 3 (Lesh et al., 2003; Lesh & Sriraman, 2010; Merriam, 1998), thus making the findings applicable to teachers in similar educational settings. The ability to generalize how teachers think about quantitative reasoning, and develop these ways of thinking, is significant because researchers can use these results as a lens for studying teacher ways of thinking about quantitative reasoning in other settings. Such settings would include middle or high school in-service teachers working in the United States without prior teacher education efforts addressing quantitative reasoning. This study also provides researchers the language to communicate teachers' models of quantitative reasoning by incorporating the language of pseudo-quantities to contrast quantities.

Implications for Mathematics Teacher Education

This study has consequences for the practice of mathematics teacher education in two ways. First, this study supports prior calls for teacher education to attend to thinking about quantitative reasoning (Castillo-Garsow, 2010; Ellis, 2007; Moore, 2012; Thompson, 1994, 2011). Second, this study established sharable practices other teacher educators can use to develop ways of thinking about quantitative reasoning that are connected to practice. The remainder of this section discusses these two implications in detail.

This study found teachers' models of quantitative reasoning initially did not include the aspects of quantities, quantitative relationships, or covariational reasoning, but many teachers developed these aspects as the MEA progressed. Researchers of mathematics education, quantitative reasoning, and mathematics teachers education recommend that students and teachers consider quantities, quantitative relationships, and covariation reasoning (Confrey & Kupta, 2010; Even & Ball, 2009; Garfunkel et al., 2011; Leikin, 2011; Marrongelle et al., 2013; Thompson, 2011, 2013). Thus this study identifies one way teacher educators can promote teachers' interpretation of quantitative reasoning and the standards for mathematical practice in ways align with the recommendations of mathematics education literature.

This study identified two practices that teacher educators can use to support teacher thinking about quantitative reasoning in ways that connect literature and their practice. First, teacher educators can have mathematics teachers develop a task and supporting documents for their own classroom practice and provide the opportunity for teachers to revise their documents after receiving various forms of feedback. For this study, teachers received feedback from the instructor, provided peer feedback, and received feedback from students similar to their own, and in some cases implemented their Quantitative Reasoning Task to acquire feedback from their own students. The feedback iterations influenced teacher ways of thinking consider the aspects of quantities and quantitative relationships, thus promoting sense making and covariational reasoning as part of quantitative reasoning. In addition to promoting these important parts of quantitative reasoning, these feedback iterations prompted teachers to address inconsistencies when thinking about quantitative reasoning across different settings. Addressing these inconsistencies supports changes in practice, continued design and assessment of their own curricula materials, and the CCSSM goals (Confrey & Stohl, 2004; Marrongelle et al., 2013; Mullis et al., 2004; Sztajn et al., 2012;).

Second, teacher educators can have teachers read selected articles and engage in carefully crafted activities in order to prompt revisions to the quantitative reasoning tasks they create. These readings and activities provide alternative ways of thinking about quantitative reasoning and give teachers examples of how to connect quantitative reasoning to their classrooms. This study found that exposing teachers to the work of Moore et al. (2011), Thompson (2011), and Carlson and Oehrtman (2011) was helpful for the development of teachers' models of quantitative reasoning. As suggested in the future research section, teacher educators could look for ways to provide extra support for secondary teachers' thinking about quantities and quantitative relationships.

These two practices were both incorporated in the MEA designed for this study, which can also be incorporated into other teacher education settings to develop teacher ways of thinking about quantitative reasoning. As discussed in Chapter 3, the construction of this MEA using a models and modeling perspective makes this entire activity sharable and applicable to other teacher populations, such as other mathematics teachers taking coursework or continuing professional development. Teacher education efforts taking place in the summer can use this MEA by incorporating undergraduate student feedback from universities or possible summer school student feedback to facilitate actual responses from learners. This type of alteration to an activity was not documented in models and modeling literature, but offers one way to overcome obstacles in teacher education settings where teachers do not have access to their own students. By a similar argument, this task could also be implemented with pre-service mathematics teachers who do not yet have access to their own students.

Limitations

Three limitations were encountered in the study. First, the setting of the study caused some difficulties in providing feedback to the teachers because the course took place in both an online environment and because the course occurred during the summer. Second, data from individual teachers were limited. Third, the stability of research findings could have been related to the way data was collected. Each of these limitations is discussed in this section.

The setting of the study was the first limitation, as teachers were engaged in the MEA online during the summer. The MEA should have teachers incorporating their own students, but that was impossible for this study because during the summer teachers did not have access to their students. Instead we decided to use undergraduate students as a proxy to generate learner feedback. According to a models and modeling perspective, teachers should have tested activities with their own students in order to promote the reality principle. This means teachers should have been given the chance to interpret feedback within the context of their actual practice, using their own students, rather than college-level students. All comments from groups about the undergraduate feedback indicated teachers thought the feedback was realistic and comparable to the teachers' K-12 students. Overall my decision to have undergraduates provide feedback was driven by the need to accommodate this course occurring in the summer, and while I did not observe many detrimental effects, this may have reduced the realistic nature of the activity for some teachers. I would not change this decision for any a future study that took place in a similar setting over the summer, since the undergraduate feedback influenced teachers to develop their thinking in positive ways.

Additionally, the online nature of the course limited the amount of feedback teachers received during the undergraduate feedback iteration of the MEA. While the technology allowed groups to collaborate even though separated geographically, teachers were unable to implement or observe their own Quantitative Reasoning Task with the undergraduate students. Having teachers implement the task themselves would have been preferable because this would further support the reality principle of the MEA; instead they had to rely on others to implement the task and had little information from observations of the students and thus limited the type of data teachers received from the undergraduate student feedback iteration. While five groups made comments indicating the amount of data they received was acceptable to make informed revisions of the task, Group 5 indicated they would have implemented the task differently than the facilitator, thus limiting the type of feedback and changes during this iteration. In future studies taking place in a similar online setting, I would try to promote some teachers to travel and implement the task themselves in order to generate even more realistic feedback from learners.

A second limitation of the study was the MEA could have been structured to better document individual teachers' models. Evidence of individual teachers' models came from the Pre-Assignment, Version 4 reflections[5], and from the teachers who completed Version 5. I was able to collect data from the three teachers in Group 1 by observing their group work time and conducting interviews at the conclusion of the course. No individual data came from Versions 1, 2, or 3. In hindsight, an individual component to each of these versions would have been a valuable addition because individual teachers' models could be determined from the extra detail. For instance, clearer conclusions could have been made regarding what and how course activities and readings influenced teachers' models had there been requirements throughout the MEA by having teachers state what they, individually, felt was impactful on their ways of thinking about quantitative reasoning. In the future, I would modify the MEA's Version 1, 2, and 3 to have the individual components seen in Version 4. This would allow me to track each teacher's ways of thinking, and the factors developing these ways of thinking, throughout the entire MEA.

The third limitation of the study was that allowing the teachers to choose the mathematical content in the MEA introduced instability in the data. By design, the MEA

[5] Recall Version 4 required teachers to include an individual entry from each group member that addressed how they understood quantitative reasoning, how this changed during the revisions, and how they believed their task related to quantitative reasoning.

allowed teachers to select the mathematical content they wanted to focus on in their Quantitative Reasoning Task and supporting documents. Therefore how teachers communicated quantitative reasoning differed because of the mathematical content they selected, introducing instability in the data and thus the findings of this study. For example, the middle school teachers' selection of linear functions might have provided these teachers opportunities to communicate quantities and quantitative relationships within this content. Instead, the high school teachers focused on more complex mathematical content, such as exponential functions, thus providing different opportunities to communicate quantities and quantitative relationships. The findings may have been different if the high school teachers had also selected linear functions.

Recommendations for Future Research

Given the need to develop teacher thinking about quantitative reasoning, future studies are recommended based on the preceding sections. First, extending this study to other teacher education settings would provide a way to expand this study's conclusions about teachers' models of quantitative reasoning. Using a models and modeling perspective, such a qualitative follow-up study would engage teachers with this MEA in a setting where all teachers could implement a task in their own classroom. The impact of Version 5 in this study suggests that future work grounded in the teachers' own classroom might have a more powerful impact on teacher practice, as evidenced from the four teachers in this study who implemented the task, and would circumvent the setting limitation described earlier. Alternatively, the MEA could remain unchanged and still be implemented with other online mathematics teacher courses, or with pre-service teachers who do not yet have access to their own students. The success of using undergraduate students in this study indicated work with pre-service teachers would also be fruitful, especially since research on their ways of thinking are also needed (Carlson et al., 2003).

The second recommendation is for researchers to conduct studies examining how teacher educators can support the development of high school in-service teachers' models of quantitative reasoning in ways that promote covariational reasoning. This study found teachers working with secondary content did not refer to quantities or quantitative relationships as aspects of quantitative reasoning as much as their peers focusing on middle school content. Future research can continue the work of Oehrtman et al. (2008) in determining how teachers think about functions in secondary content and the role of covariation. This study's methods show promise as a way for future research investigating teachers' models of quantitative reasoning and other mathematical ideas. The MEA in this study could be used to investigate questions such as whether additional iterations in the MEA support covariational reasoning or if additional instructor feedback impacts teachers' models. Another question raised by this study's limitations was how the mathematical content within an MEA might be fixed so that middle and high school teachers document their thinking about quantitative reasoning in similar content areas as they complete the MEA. Future research could address any of these questions, as the findings from this study indicate productive development in the field of teacher education and quantitative reasoning could result from such work.

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APPENDIX A

QUANTITATIVE REASONING IN SECONDARY
MATHEMATICS COURSE SYLLABUS

Summer 2012

Instructor: Dr. James (pseudonym)

Contact Information: james@university.edu , 555-555-5555
Class Times: MTWTh 9:00-10:15, 10:30-11:45
Please get on Elluminate 15 minutes before class starts to check your audio and video.

Location: Online
Credits: 3 semester credits
Co-Requisites/Prerequisites: Graduates only.
Course Description**:** We will analyze the mathematical and conceptual structure of quantities and relationships between quantities in secondary mathematics courses.
Course Objectives: Successful students will

- understand the meaning of quantities, quantitative relationships, and quantitative reasoning and be able to identify each in secondary mathematics curriculum
- deepen their understanding of secondary mathematics content involving quantities and quantitative reasoning
- understand research-based frameworks for quantitative reasoning, covariational reasoning, proportional reasoning, understanding of functions, and problem-solving and be able to apply these frameworks in analyses of student reasoning
- be able to develop model-eliciting activities to support and document the development of student understanding and reasoning

Outline of Course Content: The following topics will form the mathematical focus of these analyses:

- Quantities and quantitative reasoning
- Proportional relationships
- Constant rate and linearity
- Formalizing relationships between quantities with functions
- Exponential functions
- Polynomials and rational functions
- Angle measure and trigonometric functions

In each of these areas, we will discuss research-based conceptual frameworks that will serve as the foundation of our analyses. We will read a small number of articles about these frameworks.

Most of our work in this course will involve analyzing concrete examples of secondary mathematics classroom activities addressing quantitative reasoning in the content areas listed above. These analyses will follow a three-stage process. First, we will engage in the activities as students ourselves. Second, we will reflect on the development of our own mathematical understanding and reasoning throughout the activity. Finally, we will discuss students' reasoning in these activities and ways of supporting strong conceptual development.

Course Requirements: We will assign two types of homework throughout the course, task analyses and model-eliciting activities.

Task Analyses. At the end of each class day, we will assign extensions of the task analyses that we conducted in class as well as some analyses that you will conduct entirely on your own. You will write up a formal report of your analysis consisting of sections reflecting the three stages of our in-class analyses: your solutions to the tasks, characterization of important concepts and reasoning, and discussion of student reasoning and support. Grades will be determined based on completeness of your analysis and the accuracy and effectiveness of your application of the conceptual frameworks discussed in class.

Model-Eliciting Activities. You will work in groups to construct, evaluate and refine a model-eliciting activity (MEA). An MEA is an open-ended activity that, through careful design, engages students in developing important mathematical concepts and which asks students to produce a generalized and sharable description of their own understanding. This provides a teacher or researcher insight into student reasoning that can be difficult or time-consuming to obtain otherwise. In class, we will discuss principles of creating effective MEAs. You will refine your own MEA through five iterations. We will give you feedback on your version. In the second iteration, you will analyze another group's revised MEA and provide feedback to them, while receiving feedback from two other teachers on your own MEA. In the third and fourth iterations, we will pilot your MEA with students and you will analyze the results while again refining your MEA. The final version of the MEA and your rationale for its refinement will be submitted at the end of the course.

Method of Evaluation: Each category of assignments, task analyses and model-eliciting activities, will constitute half of your grade in the class. Letter grades will be assigned as follows:

A 90% - 100%, B 80% - 90%, C 70% - 80%, D 60% - 70%, F 0% - 60%.

Required Reading List: Journal articles and sample curricular materials from the following list and similar resources will be made available.

Carlson, M. & Oehrtman, M. (2011). *Precalculus, Pathways to Calculus: A Problem Solving Approach*, First Edition. Phoenix, AZ: Rational Reasoning.

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem solving framework. *Educational Studies in Mathematics*, *58*, 45–75.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Lesh, R., Hoover, M. et al. (2000). Principles for Developing Thought Revealing Activities for Students and Teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education*. Mahweh, NJ: Lawrence Erlbaum Associates, 591-6.

Moore, K., Carlson, M., & Oehrtman, M. (2009). The Role of Quantitative Reasoning in Solving Applied Precalculus Problems, *Proceedings of the Twelfth Conference on Research in Undergraduate Mathematics Education*, 26 pages, Web publication at http://rume.org/crume2009/Moore1_LONG.pdf.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational Reasoning Abilities that Promote Coherence in Students' Function Understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics, MAA Notes Volume, 73,* 27-41. Washington, DC: Mathematical Association of America.

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Working Assumptions: We will strive to develop a vibrant intellectual community in this class. In particular,

Depth is favored over breadth. This is not a survey course. There is no attempt to “cover” all of anything. In general, we will limit our discussion of a topic to a few readings, a book, chapter, or journal article. However, we will have an opportunity to read much more than we discuss in our [online synchronous] sessions.

Ideas, not individuals, are open to challenge. The nature of the course should produce a diversity of ideas. To insure that multiple voices are heard, the course must foster safe participation. Given that “tone” and other aspects of personal interaction are invisible in online interactions, we must be especially careful to clarify assumptions, understandings, and misunderstandings with one another. You should feel comfortable being your own advocate concerning ideas and scholarly arguments. You should also feel comfortable challenging the ideas and thinking of others. However, the challenge cannot disparage the personhood of others. We are here, in part, to learn with, from, and about each other.

Questions present an opportunity to learn. Students sometimes feel that they should not ask questions because they may "sound dumb." On the contrary, questions can be an indication of one's engagement with the subject matter. Please do not self-censor because your questions may in fact lead to clearer understanding for us all.

The role of the instructors is to facilitate your learning, not dictate it. There is no one correct way for you to interpret or implement the material in this course. Therefore, in order for you to make meaning of this course, you will need to actively engage with the readings, your fellow students, and in implementing what you have learned into your teaching. Instruction that dictates meaning interferes with these learning activities. Consequently, the instructors will serve as guides and facilitators rather than "the sage on the stage."

Active participation is vital to this class. Participation means more than logging on to Elluminate. Participation means contributing to the discussion and making meaningful comments. Participation means asking questions, actively encouraging other class members to contribute, and making sure not to monopolize discussions. For us to have a strong intellectual community, we need everyone to complete readings before our [online synchronous] sessions and be ready to engage in activities and online discussions thoughtfully. Dig deeply into authors' arguments before responding.

Distance Based Learning: This course is distance learning based. Distance learning is self-directed and requires a high level of responsibility, dedication and self-discipline on the part of the student. In order to succeed you need to log in to the course regularly to check announcements, participate in discussions and access course content. At a minimum, you must attend all on-line course activities, participate in weekly threaded discussions, and submit assignments in a timely manner.

Attendance: *<u>Attendance is mandatory</u>.* We understand that emergencies come up and that you may have to miss a class activity or threaded discussion. However, failure to attend or participate in assigned discussions will influence the class participation portion of your grade. In the event of an absence, you are responsible for catching up on any missed material, and you may be assigned extra work to make up the missed activity or discussion. Absence is not an excuse to miss assignment deadlines.

Submission Requirement and Deadlines: All course assignments should be submitted to in the designated manner on or before the due date. If for some reason an assignment is submitted by email, then on the Subject header put the following: <name of assignment><your last name>. This is important because it will help us in tracking assignments. Conflicts with an assignment deadline should be discussed and resolved *before* the assignment's due date. Late assignments are only accepted if you have contacted an instructor in advance of the due date, and we agree to accept the late work. If you are not online on the day an assignment is due, and do not make other arrangements to get the assignment submitted, it will be considered late. Late assignments might be evaluated at a higher standard because of the additional time available to work on it.

Communication: All members of the class are expected to follow the rules of common courtesy in all email messages, threaded discussions, and chats. Failure to do so will result in a warning from us for the first offense and additional actions up to removal from the course for additional violations.

You will need the following technology requirements for this course:

1. High speed internet access either at your home or school
2. Headset with microphone and speakers for online discussions
3. Webcam so we can put faces to names
4. Writing tablet for written interaction on the white board
5. Access to Elluminate for synchronous sessions and Blackboard for discussion threads.
6. Word Processor (prefer *Microsoft Word*) and *Acrobat Reader* (download from www. adobe.com/products/acrobat/readstep2.html).

When sending an email other than assignments, please identify yourself fully by name and class along with a reason/subject for the email, not simply by an email address. I will check email regularly and respond to course related questions within 48 hours, excluding weekends. Comments on formal assignments may take up to two weeks.

Disability Support Services: Any student requesting disability accommodation for this class must inform the instructor giving appropriate notice. Students are encouraged to contact disability support services to certify documentation of disability and to ensure appropriate accommodations are implemented in a timely manner.

APPENDIX B

MODEL-ELICITING ACTIVITY ASSIGNMENT

MODEL ELICITING ACTIVITY (MEA)

Objective: Develop your thinking about quantitative reasoning through iterations of constructing a quantitative reasoning task that you implement with your students.

Overview: To help guide you to the ultimate goal of implementing a quantitative reasoning task in your classroom this fall, we have constructed a major course assignment, called a Model Eliciting Activity (MEA) to help develop your ways of thinking about quantitative reasoning tasks. In this sense, we use the word model to refer to sharable ways of thinking, or systems of interpretation that you use to understand your practice. MEAs are a realistic and complex problem that engages teachers in thinking about mathematics in a way embedded in their practice in order to develop ways of thinking that can be used to communicate and make sense of realistic situations. In this course, the problem you are given is to design a quantitative reasoning task that you implement in your classroom this fall, and our hope is that you will be able to generalize your model in ways that support the implementation of other standards for mathematical practice into your classroom.

The components below have been developed to promote the goals of an MEA. This includes having you create an assignment that: (a) reveals your current ways of thinking about quantitative reasoning; (b) promotes you to test, revise, and refine those ways of thinking for implementing a quantitative reasoning task in your classroom; (c) has you share your ways of thinking with colleagues for replication; and (d) encourages you to reuse your ways of thinking in multiple contexts. The project components are overviewed below, followed by a timeline and detailed description of each component.

This assignment will constitute 50% of your course grade, and you will have some in-class time to work on the MEA. In addition to being a course assignment, Dave plans to conduct research about this process by examining the patterns each group goes through on the MEA. While Dave is not evaluating any of the documents you create, he will use them and class observations for research purposes to investigate how teachers' ways of thinking change during and after the course you take. If you have questions are concerns please let Dave or James know. Anyone wanting to see the results of the study is also welcome to contact Dave for details.

PROJECT COMPONENTS

Pre-Assignment: Clarify thoughts

After completing the **Pre-Assignment**, you should have a basic idea about quantitative reasoning and its relation to K-12 mathematics. You should keep in mind the students you anticipate having in the fall when completing the components below. Based on this information, the instructor will soon put everyone into groups; you will be working in pairs or groups of three for all of the components below.

Version 1: Construct a quantitative reasoning task and supporting documents

Your goal is to develop a quantitative reasoning task that you can use in your classroom this fall as well as construct supporting documents that would allow other mathematics educators to implement your task and go through the same process. You will be asked to create, as a group, four documents that we call **Version 1**:

a) A **Quantitative Reasoning Task** that captures deeper thinking about students' quantitative reasoning skills
b) **Facilitator Instructions** suitable for other educators to implement the task and foresee potential challenges
c) **Assessment Guidelines** suitable for someone else to evaluate the task
d) **Decision Log** that articulate your decisions, changes, and refinement of the above three documents

Versions 2, 3, & 4: Update your documents based on feedback during class

During the course you will have your initial documents (called Version 1), and three chances to revise these documents based on feedback your receive (Versions, 2 ,3, and 4). Following your submission of Version 1, the process will go as follows:

- **Instructor Feedback**: given back to you on based on your group's Version 1
- **Version 2**: your updated documents in response to the feedback the instructor gives
- **Peer Feedback**: you will evaluate another group's Version 2 and offer feedback
- **Version 3**: your updated documents in response to the feedback your peers give on Version 2
- **Student Feedback**: we will give your task to undergraduate students to complete and will return their work to you
- **Version 4**: your updated documents in response to the students' work, plus evaluation of the student work

Version 5: Update your documents based on implementing the task this fall

For the final part of the MEA, we are asking you to implement the task you developed in your classroom this fall. We have been fortunate enough to acquire funding from the grant to support you in this effort, provided you submit at least 5 blinded copies of student work and a **Version 5** that includes the final updated documents of the MEA.

PROJECT TIMELINE

Monday	Tuesday	Wednesday	Thursday	Friday
4 *1st Day of Class*	5 Pre-Assignment Due	6	7	8 Version 1 Due
11 Instructor Feedback	12	13 Version 2 Due	14	15 Peer Feedback
18 Version 3 Due	19	20	21 Student Feedback	22
25	26 Version 4 Due	27	28 *Last Day of Class*	29

Note: All due dates are at 8am on the indicated dates. The feedback will occur by 8am on the designated dates. Version 5 is due October 1. Orange = instructor's task

PROJECT DETAILS

PRE-ASSIGNMENT

The email below will be sent out to all registered teachers in the course by Wednesday, May 16.

From: Course TA
To: All teachers in the course
CC: James, Dave

<message written by James or the TA welcoming teachers to the course>

The beginning of the semester can get busy, so we wanted to give everyone extra time to work on the first assignment of the course, which is due by the second day of class. Please read the information below and answer the questions in a word document titled LastName_PreAssignment.docx. Email this document to us by Tuesday, June 5 at 8am.

The introduction of the Common Core State Standards in most states (including CO and WY) means that new expectations are coming in mathematics for K-12 students. This also means that administrators will soon be looking for evidence that teachers support these standards. One of the increased emphases is on quantitative reasoning, one of the eight standards for mathematical practice, which is the focus of this upcoming course. Give each question below some thought before writing out your honest response for each question in about a paragraph.

a. We would like to know how people interpret the phrase "quantitative reasoning." Without looking up the definition from any source, write a few sentences about what the phrase "quantitative reasoning" means to you with respect to secondary mathematics. If you have seen the phrase used before, indicate where.

b. We would also like to know how you interpret others' definition of "quantitative reasoning." Look at the Common Core State Standards for mathematical practice (link here), and read the quantitative reasoning standard. After familiarizing yourself with the content standards (try reading a grade in an area you are familiar with), write a few sentences about how you interpret quantitative reasoning within a secondary mathematics context.

c. What do you think quantitative reasoning looks like in your classroom?

d. Sketch a rough outline of what a task might look like that measures and develops students' quantitative reasoning skills.

e. What do you expect to get out of this Quantitative Reasoning in Secondary Mathematics course you will take?

f. What grade band(s) and subject area(s) do you expect to teach in the fall?

g. In this course, a major assignment will be to develop a quantitative reasoning task for your students. You will be asked to work with one or two others in the class to develop this task as a group, and go through iterations of refinement to improve the task throughout the course. The final aspect of the assignment will be to implement the task in your own classroom this September. More will be said about this in class, but for now indicate who you would like to work with on this assignment. Keep in mind groups will be mainly formed based on similar grade bands and subject areas since you will be asked to implement these tasks in the fall.

<conclusion written by James or the TA>

Sincerely,
James and the TA

VERSION 1

Given your responses to the Pre-Assignment, the instructor should now have placed you into groups where everyone will be teaching similar grades or subject matter in the fall. Your first task for one of you to create a Dropbox Folder called LastName1_LastName2 (or LastName1_LastName2_LastName3). Share this folder with everyone in your group and with James, the TA, and David (david.glassmeyer@unco.edu). You will be using this folder to store all of your MEA documents in the course; this is the way you will submit MEA assignments and where feedback will be given back to you.

Version 1 has four components. Create each component in a separate document, and name them the **bolded** titles below. Place each document in your Version 1 folder on Dropbox by Friday, June 8 at 8am. You will have a chance to edit these materials after receiving feedback from the instructor on Monday.

a) **Quantitative Reasoning Task**, which should aim to:
   - capture some deeper thinking about students' quantitative reasoning skills
   - provide evidence about how students think about quantitative reasoning
   - be tailored to your grade and subject choice
   - have students work in small groups (between 2 and 4 students)
   - be completed by these types of students in two 45 minute sessions (90 minutes total)
     - the first 45 minutes should be for the students to engage with the task
     - the second 45 minutes should be for the group of student to write up their findings
   - be able to be implemented with minimal input from you or another educator
     - Think of this as if a substitute teacher will be implementing the task for you
     - Assume the sub will check on each group of students every 10-15 minutes but will only be able to offer minimal assistance

b) **Facilitator Instructions** created for a substitute teacher implementing your task. The instructions should:
   - assist any educator wishing to implement your task
     - explicitly state all materials required for the task (technology, manipulatives, etc.)
     - explain how to implement the task (the teacher role should be minimal)
     - indicate what types of information or answers the educator is allowed to give the students
     - include what types of prompting questions the facilitator can use to help students
   - indicate your anticipated student responses to your activity
     - include what population of students (grade, subject, course) you intend this activity for
     - clearly indicate what types of responses you expect these students to have
     - provide information about students in a way that might help another educator implement the task

c) **Assessment Guidelines** suitable for someone else to evaluate student responses to the task. The guidelines should:
   - establish some kind of criteria for assessing student responses to the task
   - be detailed enough for another educator to use for assessment purposes

d) **Decision Log** documenting this process. The log should:

- include a reflection about how you think about quantitative reasoning and how your thinking has changed
- help other teachers in your school understand what you were thinking when you created documents (a), (b), (c)
- articulate the refinements you have made as you designed the initial version of the documents (a), (b), (c)
- explain to other teachers in your school what to expect during this process if they were to try this

Your folder should be structured as follows:

- Folder called LastName1_LastName2 (shared with all group members, the instructor, TA, and David)
    - Version 1
        - V1_QR_Task.docx
        - V1_Facilitator_Instructions.docx
        - V1_Assessment_Guidelines.docx
        - V1_Decision_Log.docx

VERSION 2

Version 2 asks you to update all of your Version 1 documents in response to the feedback you received from the instructor. To do this, first copy your Version 1 folder and paste it into Dropbox, giving the copy a name Version 2.

After examining the instructor feedback, incorporate the suggestions you think are worthwhile into Version 2, and update all your actions in your Decision Log. Have your Version 2 folders complete by Wednesday June 13 at 8am.

Thus the folder structure should look like below:

- Folder called LastName1_LastName2
    - Version 1 (this folder should remain untouched since submission on June 8)
    - Instructor Feedback (this folder should remain untouched since submission on June 11)
        - Instructor Feedback.docx
    - Version 2 (the new folder you will need to create with your updated documents)
        - V2_QR_Task.docx
        - V2_Facilitator_Instructions.docx
        - V2_Assessment_Guidelines.docx
        - V2_Decision_Log.docx

PEER FEEDBACK

After submitting Version 2, the next iteration of the MEA is prompted by Peer Feedback, which has you and another group swapping Version 2 and evaluating each other. This evaluation is a portion of your grade on the MEA task, and the directions below will guide you through this process.

Your first task is to determine from the instructor which group will be swapping with (a list will be shared of the swaps during class on Wednesday, June 13). After receiving this information, you should share your LastName1_LastName2 (or LastName1_LastName2_LastName3) Dropbox folder with all members of the group that will be evaluating you. This should occur by the end of class on Wednesday, June 13.

Part of your homework to be completed before class on Thursday June 14 is to read through the other' groups Version 2 documents. Please make a copy of their Version 2 folder and call it V2_LastName_Feedback. Within this folder you can offer feedback you would like using track changes and comments to mark thoughts, correct errors, or ask questions. This micro feedback will be helpful to the other group, but your group will be evaluating together their Version 2 during class on Thursday, June 14.

Since you will have read through the other group's Version 2 before class on Thursday, June 14, your group will be ready to meet and discuss and evaluate the other group's task on a holistic level. Use the questions below to guide your thinking, and develop a document (called Peer_Feedback.docx) that answers these questions, as well as any other concerns that are brought up during your discussion. Place this document in a folder called Peer Feedback in the other group's Dropbox folder, and put all V2_LastName_Feedback folders in this folder. Thus the folder structure should look like below:

- Other group's MEA (the folder called LastName1_LastName2 that was shared with you)
    - Version 1 (this folder should remain untouched since submission on June 8)
    - Instructor Feedback (this folder should remain untouched since submission on June 11)
    - Version 2 (this folder should remain untouched since submission on June 13)
    - Peer Feedback (a person in your group needs to create this folder)
        - Peer_Feedback.docx
        - V2_LastNameA_Feedback (Teacher A created this folder)
            - V2_QR_Task.docx (document w/ track changes from Teacher A)
            - V2_Facilitator_Instructions.docx (document w/ track changes from Teacher A)
            - V2_Assessment_Guidelines.docx (document w/ track changes from Teacher A)
            - V2_Decision_Log.docx (document w/ track changes from Teacher A)
        - Version2_LastNameB_Feedback (Teacher B created this folder)
            - V2_QR_Task.docx (document w/ track changes from Teacher B)
            - V2_Facilitator_Instructions.docx (document w/ track changes from Teacher B)
            - V2_Assessment_Guidelines.docx (document w/ track changes from Teacher B)
            - V2_Decision_Log.docx (document w/ track changes from Teacher B)

The other group's folder should be formatted in this manner with all completed documents, including the Peer_Feedback.docx by Friday, June 15 at 8am. Use the questions below to help structure your written evaluation in the Peer Feedback.docx.

- **Quantitative Reasoning Task**
    - How sharable is the task for other teachers? What does/does not make it possible for the task to be implemented in another educator's classroom?
    - How does the task capture deeper thinking about students' quantitative reasoning skills? If the task does not capture deeper thinking, how could it be changed to do so?
    - How will the task provide evidence about how students think about quantitative reasoning?
    - What level of difficulty do you think the task will be for these types of students?
    - How reasonable is this activity to be completed by these students in two 45 minute sessions?
    - How does the intended grade level and subject relate to the task?
    - How clear is the task in explaining the grouping and expectations of the students?
- **Facilitator Instructions**
    - How easy would it be for a substitute teacher to implement the task based on these instructions?
    - What role must the substitute teacher take in implementing the task? Would you consider this role minimal?
    - Do you agree with the amount of information the educator is allowed to give the students? What concerns do you have about these protocols?
    - Do you agree with the types of questions the educator is allowed to give the students? What questions might you suggest adding, altering, or deleting?
    - What responses do you anticipate to this activity? What would you add, delete, and modify from the anticipated student responses given?
    - How do the anticipated responses help another educator implement the task?
    - How clear is the task in explaining the required materials needed? Are there materials you think are missing?
- **Assessment Guidelines**
    - How clear are these guidelines to you? What is unclear?
    - Do you think these guidelines could be used to evaluate the task? What concerns or suggestions do you have?
- **Decision Log**
    - How coherent was the log? Comment on your ability to follow the thought processes that went into the development of the above documents.
    - How helpful would the log be for developing a quantitative task if you were unfamiliar with the process? What additions or edits would improve the helpfulness of the log?
    - What changes in thinking appear to have occurred? How clearly are these changes stated?
    - What aspects are unclear about the decision making process the other group went through? What questions do you still have?
    - What did you glean from reading this log that you hadn't considered before?

VERSION 3

Notice your folder should be updated to now include a Peer Feedback Folder (this should have occurred by 8am on Friday June 15). Your next task is to update all your documents in response to the peer feedback you received. To do this, make a copy of the Version 2 folder, and name it Version 3. Make all updates and be sure to record all changes you do and do not make in your Decision Log. Version 3 is due at 8am sharp on Monday June 18th in order to implement your task in an undergraduate setting that day.

- Folder called LastName1_LastName2
    - Version 1 (this folder should remain untouched since submission on June 8)
    - Instructor Feedback (this folder should remain untouched since submission on June 11)
    - Version 2 (this folder should remain untouched since submission on June 13)
    - Peer Feedback (this folder should remain untouched since the other group created it on June 14)
    - Version 3
        - V3_QR_Task.docx
        - V3_Facilitator_Instructions.docx
        - V3_Assessment_Guidelines.docx
        - V3_Decision_Log.docx

VERSION 4

By Thursday, June 21, you should have received a scanned copy of the work that was done by undergraduates on your Version 3 (a). Two or three students will have completed your task, and we have scanned the documents they created and put them in a folder called Student Feedback in your Dropbox folder. To create Version 4, first make a copy of your Version 3 folder. You will be adding a new file, called the Student Evaluation document, to your Version 4 folder. Also, the decision log should include individual components addressing each group member's ways of thinking about quantitative reasoning.

Your task for Version 4 is to:

- Update all Version 3 documents based on the feedback you receive
    - especially the Decision Log, which should now include an individual entry from each group member that addresses:
        - How you understand quantitative reasoning
        - How your thinking about quantitative reasoning has changed during the revisions
        - How you believe the task your group relates to quantitative reasoning.
- Create a Student Evaluation document, which
    - explains how you implemented your Assessment Guidelines
    - indicates a conclusion on the evaluative measure you assign the work and what this means
    - details your rationale and the framework you used for understanding students' quantitative reasoning

Thus your MEA folder should now have the following hierarchy:

- Version 1 (this folder should remain untouched since submission on June 8)
- Version 2 (this folder should remain untouched since submission on June 13)
- Peer Feedback (this folder should remain untouched since the other group created it on June 14)
- Version 3 (this folder should remain untouched since submission on June 18)
- Student Feedback (we will have created this folder)
    - Student 1 Work.pdf
    - Student 2 Work.pdf
- Version 4 (you need to create this folder with the updated documents)
    - V4_QR_Task.docx
    - V4_Facilitator_Instructions.docx
    - V4_Assessment_Guidelines.docx
    - V4_Decision_Log.docx
    - V4_Student_Evaluation_Document.docx

VERSION 5

As we indicated at the start of the project, the main goal of the MEA is to have you implement the task your group creates into your classroom this fall. Since the course ends at the end of June, we have arranged for you to receive compensation for completion of Version 5, as this must occur outside the parameters of the course. Please follow the guidelines below to fulfill the requirements for Version 5.

At the end of the course, the instructor will do a final evaluation of your work. This feedback will be called End of Course Feedback. We will also create a folder in your Dropbox called Version 5; your first task will be to create a subfolder called LastName_V5. This is where you will be placing all of your Version 5 documents. Note that other group members will have other subfolders in the Version 5 folder.

The first document we are asking you to update is the Decision Log in response to how you plan to implement your task in your classroom. The adjustment from the Version 4 task to your classroom may be substantial, depending on the schedule and students that you teach in the fall. You should document all these changes in your Decision Log, and update this Log before you implement the activity. You should email this document to James and Dave by Monday, September 17 at 8am to let us know you are planning to implement Version 5, and we will begin the compensation process. You are encouraged to do this earlier than this date, of course.

After updating the Decision Log, you should implement your activity with a number of your students. We are only requesting evidence that 5 students complete the activity, but you are welcome to do this with one or all of your classes where students match the population aimed for by the task. Once you implement your task, we ask that you make copies of 5 students' work, blocking out the name and using an alias instead (similar to how we shared the undergraduate student feedback with you). We will ask you to scan the student work and submit it on Dropbox in a folder called Student Feedback. You need to implement the activity and submit the student work by Monday, October 1 at 8am. You are encouraged to do this earlier than this date, of course.

Finally, you should repeat the steps outlined in Version 4: create a Student Evaluation document that evaluates these 5 students' work, as well as update the other four documents (now called V5_QR_Task.docx, V5_Facilitator_Instructions.docx, V5_Assessment_Guidelines.docx, and V5_Decision_Log.docx). The Decision Log should include a final reflection on the process of implementing the task in your classroom, and how the process has influenced your view what quantitative reasoning is and how it relates to your classroom. You may want to look at your pre-assignment you submitted (via email) on June 5 to comment on how that thinking has changed during this time.

The final MEA folder needs to be completed by Monday, October 15 at 8am. Please email James and David to confirm your submission so we can make sure you are compensated for your work. You will receive $200 for completing Version 5 by the deadlines indicated.

Your final MEA folder should now have the following hierarchy:

- Version 1 (this folder should remain untouched since submission on June 8)
- Instructor Feedback (this folder should remain untouched since submission on June 11)
- Version 2 (this folder should remain untouched since submission on June 13)
- Peer Feedback (this folder should remain untouched since the other group created it on June 14)

  - Version 3 (this folder should remain untouched since submission on June 18)
  - Student Feedback (this folder should remain untouched since we created it on June 21)
  - Version 4 (this folder should remain untouched since submission on June 26)
  - End of Course Feedback
  - Version 5
    - LastName_Version5 (folder you create by Sept 17)
      - V5_Decision Log.docx (you must update this before you implement your task, and again after your task is implemented)
      - V5_QR_Task.docx
      - V5_Facilitator_Instructions.docx
      - V5_Assessment_Guidelines.docx
      - V5_Student_Evaluation_Document.docx
      - Student Feedback (folder you create by October 1)
        - Student1_Work.pdf
        - Student2_Work.pdf
        - Student3_Work.pdf
        - Student4_Work.pdf
        - Student5_Work.pdf

VERSION 5 TIMELINE

Monday	Tuesday	Wednesday	Thursday	Friday
Sept 17 Last day to email Decision Log	18	19	20	21
24	25	16	27	28
Oct 1 Last day to submit student work	2	3	4	5
8	9	10	11	12
15 Last day to submit Version 5	16	17	18	19

*Note: If you need additional time to implement the task in your classroom, please indicate this in the email you send to us that contains the Decision Log (due by Sept 17). Indicate the week you plan to implement the activity, and when you will have Version 5 completed.

APPENDIX C

INSTITUTIONAL REVIEW BOARD APPROVAL

UNIVERSITY *of*
NORTHERN COLORADO

Institutional Review Board

May 16, 2012

TO: Wendy Highby
University Libraries

FROM: The Office of Sponsored Programs

RE: Exempt Review of *Teachers' Development of Quantitative Reasoning Tasks for Students,* submitted by David Glassmeyer (Research Advisor: Michael Oehrtman)

The above proposal is being submitted to you for exemption review. When approved, return the proposal to Sherry May in the Office of Sponsored Programs.

I recommend approval.

W Highby 5-29-12
Wendy Highby Date

The above referenced prospectus has been reviewed for compliance with HHS guidelines for ethical principles in human subjects research. The decision of the Institutional Review Board is that the project is exempt from further review.

IT IS THE ADVISOR'S RESPONSIBILITY TO NOTIFY THE STUDENT OF THIS STATUS.

Comments:

APPENDIX D

INFORMED CONSENT FOR PARTICIPATION IN
RESEARCH

Project Title: *Teachers' Development of Quantitative Reasoning Tasks for Students*

Researchers: David Glassmeyer, School of Mathematical Sciences
david.glassmeyer@unco.edu, (970) 351-2229

Research Supervisors: Dr. Michael Oehrtman, School of Mathematical Sciences
michael.oehrtman@unco.edu, (970) 351-2344

Dr. Jodie Novak, School of Mathematical Sciences
jodie.novak@unco.edu, (970) 351-2463

We are conducting research to help improve the types of activities students are given in mathematics classes. This research focuses on examining students' work about quantitative reasoning, which is a focus of the class you are taking. Quantitative reasoning is the ability to make sense of quantities and their relationships in problem situations, and we hope to improve the types of tasks given in mathematics classes by supporting teachers' development of these activities.

To do this, we want to ask you if you would be willing to allow me (Dave) to make a copy of your work and share it with high school teachers working on their own degree. I would replace your name with an alternative label so nobody but the researchers would know it was your work. You will not be identifiable by the teachers or in the final report of this study. While you may be required to complete the task for classroom purposes, your participation in this study will not affect your course grade in any way. I also plan to keep observation notes of your class working on these tasks and to observe you at two additional class sessions; if you do not want to be a participant, you will not be included in these observations. If you do not wish to be present at all in the classroom during the observed sessions, the instructor will make reasonable accommodations for you to make up those sessions.

I foresee no risks to anyone wishing to participate beyond those normally associated with educational settings. This study could benefit the teachers by giving them a chance to see how students, such as yourselves, think about quantitative reasoning. The teachers would have a chance to modify the tasks in ways that would help their high school students learn these ideas.

Having read this above, please indicate your decision of participating in the study below and sign it. Participation is voluntary. You may decide not to participate in this study and if you begin participation you may still decide to stop and withdraw at any time. Your decision will be respected and will not result in loss of benefits to which you are otherwise entitled. I will store identifying information (such as signed Informed Consent forms) in a locked cabinet and in password protected computer files and that it will be destroyed after three years. Having read the above and having had an opportunity to ask any questions, please sign below if you would like to participate in this research. A copy of this form will be given to you to retain for future reference. If you have any concerns about your selection or treatment as a research participant, please contact the Office of Sponsored Programs, Kepner Hall, University of Northern Colorado Greeley, CO 80639; 970-351-2161.

☐ I wish to participate in this study, and allow the researcher to share my work (anonymously) with teachers.

Do you have a preferred alias? If so, write your preferred name here:

____

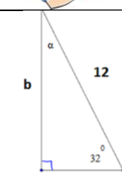
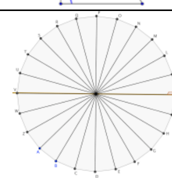
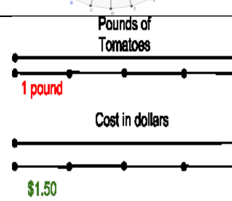
☐ I wish to abstain from the study, and am willing to be present in the observed classroom sessions.

☐ I wish to abstain from the study, and am not willing to be present in the observed classroom sessions.

Participant's Name (please print)	Participant's Signature	Date
Researcher's Name	Researcher's Signature	Date

APPENDIX E

SUMMARY OF EACH GROUP PARTICIPATING IN
THE STUDY

Group	Participant Pseudonym	Task Title	Task Picture	Grade Level Teachers Intended for Their Task
1	Nicholas Joyce Percy	Introduction to Logarithms		High school - algebra 2 - college algebra - pre-calculus
2	Julie Charlotte Samantha Rose	Fundraiser Profit Presentation		Middle school - algebra 1
3	Tiffany Alice Allie Penny	Modeling Scenarios Graphically		Middle or high school - algebra 1 - introduction to algebra 2
4	Jack Darium Dylan	Right Triangle Problem Classification		High school - geometry - trigonometry
5	Gary Ken Byron	Ferris Wheel Rescue Plan		High school - algebra 2 - trigonometry - pre-calculus
6	Brandon Charles Carol Glen	Proportional Reasoning Across Multiple Representations		Middle school - pre-algebra

APPENDIX F

OBSERVATION PROTOCOL USED ON TEACHERS

Time / Context (demographic information)	Event (evidence of model, change in model, rationale for decisions in MEA)	Teacher (name of teacher making statement)	Description (details about the event)	Reflective Notes (personal thoughts, speculations, impressions)

APPENDIX G

OBSERVATION PROTOCOL USED ON
UNDERGRADUATE STUDENTS

Group Members (names of students)	Task Description (overview of task & teachers who designed it)	Questions / Problems Encountered	Overall Impressions (went smoothly, some issues, significant issues)

APPENDIX H

INTERVIEW PROTOCOL AND QUESTIONS

Interview Protocol: Researcher may probe during the interview and ask new questions based on the participant's responses.

Interviewer initial statement: *Thanks for taking the time to participate in this interview The purpose of this interview is to explore your perceptions of Continuous Mathematics course you are taking. I will ask you some questions regarding your experience with the course and the relation you see between this course and the one you took last month. If at any time you feel uncomfortable with the interview I will stop the interview. Are you ready to begin?*

1. How is the Continuous Mathematics course going for you?
2. What connections do you see between this course and your classroom? [mention specific activities if possible]
3. What connections does this course have to the Quantitative Reasoning course you took last month?
4. Have you given any thought to the MEA project from last month? How have your ideas developed because of the Continuous Mathematics course?
5. How do you understand quantitative reasoning?
6. How has your thinking about quantitative reasoning changed during the revision process?
7. How do you believe the task your group created relates to quantitative reasoning?
8. Are there any other comments you would like to make?

Thank you for your time.

APPENDIX I

PATHWAYS TO CALCULUS MATERIALS
REFERENCED BY GROUP 2

Module 2: Worksheet 1, Problem 2 from Carlson & Oehrtman (2012, p. 2).

2. For the following situations, identify the quantities whose values vary and quantities whose values are constant. State possible units for measuring each of these quantities, then define variables for representing the values of each varying quantity.

   a. Monica runs 5 miles around a $\frac{1}{4}$ mile track beginning at a slow jog.

      i. Identify at least three varying quantities and state the units of measurement.

      ii. Identify at least two constant quantities.

      iii. Define variables to represent the values of at least two of the varying quantities.

   b. A cylindrical bottle is being filled with water. For every 1 cubic inch of water added to the bottle, the water level rises 0.5 inches.

      i. Arrange the following terms into the table below to define two primary attributes of the water in the bottle the object). (Recall that a quantity is an attribute of some object that you can imagine measuring in some units.) Note that terms can be used more than once or not at all.

      Inch   Volume   Height   Cubic Inch   Changing   Not Changing   *h*   *V*

Object	Water in the bottle	Water in the bottle
Attribute (Quantity)		
Unit		
Changing or Not Changing		
Variable		

      ii. If you wanted to fill the bottle to a height of 9 inches, and every 1 cubic inch of water added to the bottle raises the water level 0.5 inches, how many cubic inches of water are needed to fill the bottle to the desired height? Provide a rationale for your calculation.

      iii. Write a formula that expresses the height of the water in the bottle h measured in inches, in terms of V, the volume of water in the bottle measured in cubic inches.

      iv. Write a formula that expresses the volume of water in the bottle V measured in cubic inches, in terms of h, the value of the height of the water in the bottle measured in inches.

APPENDIX J

PATHWAYS TO CALCULUS MATERIALS
REFERENCED BY GROUP 1

Module 4: Worksheet 8, Problems 1 and 8 from Carlson & Oehrtman (2012, p. 161-163).

1. The population of Indianapolis, IN was 781,870 in the year 2000 and increased at a continuous percent rate of 5.45% per year.
   a. Define a function *f* that determines the population *P* of Indianapolis in terms of the number of years since 2000.

   b. Determine the annual growth factor and the Annual Percent Yield (APY) of the city's population.

   c. In what year does your model estimate the population to reach 1,000,000 people?

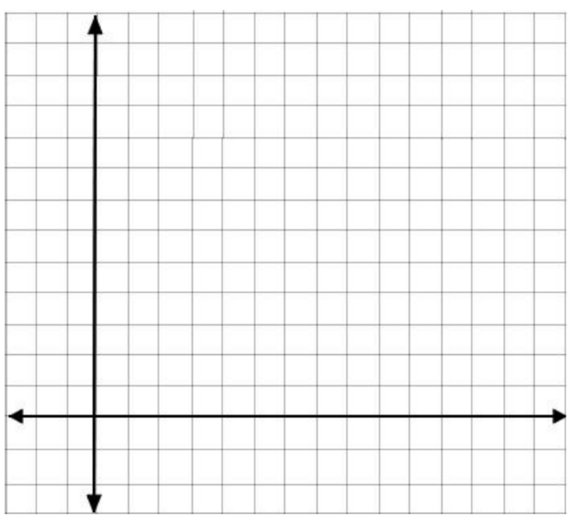
Until this point, we have been unable to solve equations with the unknown variable in the exponent without using graphs. Algebraically solving such equations necessitates the use of logarithms.

As an example, when solving equations like $5^x = 125$, we need to be able to write the equations in the form "$x =$ ". The logarithmic function is described in detail in your textbook, but essentially a logarithm references an exponent on a specified base. It is not surprising then that the laws of logarithms follow similar patterns as the laws of exponents.

Note: $y = \log_b x$ **is equivalent to** $b^y = x$**, for values of** $b > 0$, $b \neq 1$ **and** $x > 0$

So, $\log_5 125 = x$ is read "log base 5 of 125 equals *x*" and asks the question "5 raised to *what power* results in 125?" The answer to this question is 3, because $5^3 = 125$. This logarithmic expressions can be written in an equivalent exponential form: $\log_5 125 = x$ → $5^x = 125$ or vice versa.

8 a. Construct graphs of $f(x) = 2^x$, graph $g(x) = \log_2 x$ by making a table of values and plotting points.



a. What do you notice about the relationship between the functions *f* and *g*?

b. For values of $x > 0$, what is $f(g(x))$? What is $g(f(x))$?