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Optimality and Construction of Designs with Generalized Group Divisible Structure

by

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ABSTRACT

Optimality and Construction of Designs with Generalized Group Divisible Structure

Sudesh K. Srivastav
Old Dominion University, 1996
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This thesis is an investigation of the optimality and construction problems attendant to the assignment of v treatments to experimental units in b blocks of size k , paying special attention to settings for which equal replication of the treatments is not possible. The model is that of one way elimination of heterogeneity, in which the expectation of an observation on treatment i in block j is $\tau_i + \beta_j$ (treatment effect + block effect), where τ_i and β_j are unknown constants, $1 \leq i \leq v$ and $1 \leq j \leq b$. All observations are assumed to be uncorrelated with same variance.

The generalized group divisible design with s groups, or $GGDD(s)$, is defined in terms of the elements of the information matrix, instead of in terms of the elements of the concurrence matrix as done by Adhikary (1965) and extended by Jacroux (1982). This definition extends the class of designs to include non-binary members,

and allows for broader optimality results. Some sufficient conditions are derived for $GGDD(s)$ s to be E - and MV - optimal. It is also shown how augmentation of additional blocks to certain $GGDD(s)$ s produces other nonbinary, unequally replicated E - and MV - optimal block designs. Where nonbinary designs are found, they are generally preferable to binary designs in terms of interpretability, and often in terms of one or more formal optimality criteria as well.

The class of generalized nearly balanced incomplete block designs with maximum concurrence range l , or $NBBD(l)$, is defined. This class extends the nearly balanced incomplete block designs as defined by Cheng & Wu (1981), and the semi-regular graph designs as defined by Jacroux (1985), to cases where off-diagonal entries of the concurrence matrix differ by at most the positive integer l . Sufficient conditions are derived for a $NBBD(2)$ to be optimal under a given type-I criterion. The conditions are used to establish the A - and D -optimality of an infinite series of $NBBD(2)$ s having unequal numbers of replicates. Also, a result from Jacroux (1985) is used to establish the A -optimality of a new series of $NBBD(1)$ s.

Several methods of construction of $GGDD(s)$ s are developed from which many infinite series of designs are derived. Generally these designs satisfy the obtained sufficient conditions for E - and MV -optimality.

Finally, in the nested row-column setting, the necessary conditions for existence

of 2×2 balanced incomplete block designs with nested rows and columns (*BIBRCs*) are found to be sufficient. It is also shown that, sufficient for a *BIBRC* with $p = q$ to be generally balanced, is that the row and column classifications together form a balanced incomplete block design, as does the block classification. All of the 2×2 *BIBRCs* are constructed to have this property.

Dedicated

to

My Parents

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Chapter 1

Introduction

The purpose of the class of experiments to be considered here is to compare a number of treatments on the basis of the responses they produce on a set of experimental units. The concept of blocking in statistically planned experiments, which originated in agricultural field experiments, is now applied in many areas of science and engineering. In agriculture the treatments may be varieties of wheat, barley, or some other crop, or even different fertilizers; in engineering they may be different metal alloys or temperature levels or some combination of the two; and in a chemistry experiment the intent may be to compare several catalysts, perhaps under several different pressures. Blocking is a method of increasing the sensitivity of an experiment by first grouping the experimental units (also called plots) so that the plots within a group or block are relatively homogeneous. This concept led to the birth

of the science of block designs, which is the study of which treatments should be used in which blocks. A block design is successful in so far as it is able to eliminate systematic variation in the experimental units from the treatment comparisons in which the experimenter is interested.

Suppose that N experimental units or plots are available for experimentation. Let there be v treatments labeled $1, 2, \dots, v$ and b blocks having k plots each, with $k \leq v$, and so that $bk = N$. A design d is a particular arrangement of the v treatments in the b blocks. A design can be displayed as a $k \times b$ array with varieties as entries and blocks as columns. Designs as defined here, with every block containing the same number of experimental units, are called proper.

Let $D(v, b, k)$ denote the class of all block designs which are available in such an experimental setting. The usual additive model specifies the expectation of an observation on treatment i in block j as $\tau_i + \beta_j$, where τ_i is effect of the i th treatment and β_j is the effect of the j th block. All observations are assumed to be uncorrelated and have the same variance σ^2 (usually unknown). Observe that each design $d \in D(v, b, k)$ has associated with it a $v \times b$ incidence matrix N_d whose entries n_{dij} are nonnegative integers indicating the number of times treatment i occurs in block j . Thus all designs in $D(v, b, k)$ can be identified with nonnegative integer matrices having column sums equal to k . The matrix $N_d N_d^T$ is referred to as the

concurrence matrix of d , and its entries are denoted by λ_{dij} . The reduced normal equations for estimating the treatment effects, when the design d is used, are known to be

$$C_d \hat{\tau} = Q = T_d - \frac{1}{k} N_d B_d,$$

where $C_d = \text{diag}(r_{d1}, \dots, r_{dv}) - \frac{1}{k} N_d N_d^T$, B_d denotes the $b \times 1$ vector of block totals in d , T_d is the $v \times 1$ vector of treatment totals, r_{di} represents the number of times treatment i is replicated by d , and $\text{diag}(r_{d1}, \dots, r_{dv})$ is a $v \times v$ diagonal matrix. C_d , the information matrix or C -matrix of the design, is known to be positive semi-definite with zero row sums for all $d \in D(v, b, k)$. Let x be any positive real number and define

$$T_{dx} = C_d + x J_{vv} \tag{1.1}$$

where J_{mn} is the $m \times n$ matrix of ones. Then for any x and any connected design d (defined below), $\hat{\tau} = T_{dx}^{-1} Q$ is a solution to equation (1.1) and the covariance matrix of $\hat{\tau}$ is $\text{cov}(\hat{\tau}) = T_{dx}^{-1}$.

A treatment contrast is any linear combination $l'\tau = \sum l_i \tau_i$ of the treatment effects, where $\mathbf{1}'l = \sum l_i = 0$. It is said to be an elementary contrast if l has only two nonzero elements 1 and -1. A block design is said to be variance balanced if all the elementary contrasts are estimated with the same precision. A block design in which all treatment contrasts are estimable is said to be connected, and

all competing designs in this dissertation are assumed to have this property; in particular, no member of $D(v, b, k)$ that is not connected need be considered. It is known that a block design is connected if and only if its C -matrix has rank $v - 1$.

A design d is called equireplicated if all varieties appear in the design the same number of times, that is, $r_{d1} = r_{d2} = \dots = r_{dv}$. A design d is called binary if all its blocks consist of distinct varieties, i.e., $n_{dij} = 0$ or 1 for all i and j . Let $\text{tr } A$ denote the trace of a given square matrix A . Let $M(v, b, k)$ denote the subclass of designs in $D(v, b, k)$ whose C -matrices have maximal $\text{tr } C_d$. It is easily seen that $M(v, b, k)$ consists of all the binary designs in $D(v, b, k)$.

For many parameter combinations v , b , and k there exist variance balanced designs. We mention only those that we shall deal with in this thesis. A design d is variance balanced if the information matrix C_d is completely symmetric, that is, of the form $\alpha I + \beta J$ for constants α and β ; we shall call designs with this property completely symmetric designs, or CSD s. Balanced incomplete block designs ($BIBD$ s) and the nonbinary designs satisfying Theorem 1 of Morgan & Uddin (1995) are CSD s with known optimality properties. In particular it is easy to see that the design

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 5 & 5 & 4 & 5 & 5 & 4 \\ 2 & 4 & 3 & 3 & 4 & 3 & 3 \end{array}$$

is a CSD with $v = 5$, $b = 7$ and $k = 3$. A more detailed account of the various

kinds of balanced designs can be found in Raghavarao (1971, pages 51-54).

An unequally replicated and binary design d in which each treatment occurs in either r or $r + 1$ blocks and each pair of treatments is contained in either λ or $\lambda + 1$ blocks, where r and λ to denote the greatest integers not exceeding bk/v and $r(k - 1)/(v - 1)$ respectively, is called a nearly balanced incomplete block design (*NBBD*) (Cheng and Wu, 1981) or a semi-regular graph design (*SRGD*) (Jacroux, 1985). The notion of an *NBBD* or *SRGD* is a generalization of the definition given by Mitchell and John (1977) for a regular graph design (*RGD*), and reduces to their definition when bk/v is an integer. If d is an *RGD* and its concurrence matrix has the additional property that all its off diagonal elements are equal, then d is called a *BIBD*.

Choice of a design d is usually based on some optimality criterion defined on the matrices $\{C_d : d \in D\}$. Let $z_{d0} = 0 < z_{d1} \leq \dots \leq z_{dv-1}$ denote the eigenvalues of C_d . The eigenvalues of C_d can be used in determining the optimality (in some sense) of a given design d . A design $d \in D(v, b, k)$ is said to be $\phi_f(C_d)$ -optimal provided

$$\sum_{i=1}^{v-1} f(z_{di}) \tag{1.2}$$

is minimal over all designs in $D(v, b, k)$ where f is a nonincreasing, convex, real valued function. The recent book by Shah & Sinha (1989) provides an excellent overview of the various criteria ϕ typically employed. In this dissertation we will be

interested in criteria of the form given in (1.2) corresponding to type-1 optimality, defined as follows:

DEFINITION 1.1 *A design $d \in D(v, b, k)$ is said to be type-1 optimal provided (1.2) is minimal over all designs in $D(v, b, k)$, where f is a convex, real valued function satisfying the following conditions:*

- (i) *f is continuously differentiable on $(0, \max_{d \in D(v, b, k)} \text{tr } C_d)$, and $f' < 0$, $f'' > 0$, $f''' < 0$ on $(0, \max_{d \in D(v, b, k)} \text{tr } C_d)$, and*
- (ii) *f is continuous at 0 or $\lim_{x \rightarrow 0} f(x) = f(0) = \infty$.*

Any particular criterion f falling in the type-1 framework is called a type-1 criterion. For instance, the well known A -, D -, and Φ_p -criteria (see Kiefer, 1975) are type-1 criteria which result from taking $f(x) = 1/x$, $-\log x$, and x^{-p} in the above definition, respectively. The E -criterion (Ehrenfeld, 1955), which is the maximum variance over all treatment contrasts, is also covered as a pointwise limit of the Φ_p -criteria as $p \rightarrow \infty$: $\lim_{p \rightarrow \infty} \Phi_p = \max_i (1/z_{di})$.

Another criterion of great importance, but which is not a function solely of the eigenvalues, is the MV -criterion introduced by Takeuchi (1961) and later given this name by Jacroux (1983). In many experiments the primary interest lies not in arbitrary treatment contrasts, but in the simple differences in the effects that the treatments under study have on the various experimental units. In this case the

MV -criterion is preferred. An MV -optimal design minimizes the maximum variance over all paired treatment contrasts $\tau_i - \tau_{i'}$, among all the designs in $D(v, b, k)$. The E - and MV -criteria are both minimax criteria.

In recent years, there have been some studies on E - and MV -optimal designs in design set-ups $D(v, b, k)$ where blocks are allowed to be nonbinary and bk/v is not an integer (see Shah & Das, 1988; Morgan & Uddin, 1995). Shah & Das showed in particular that the Bagchi (1988) design with $v = 6$, $b = 7$, $k = 3$ is E -better than any binary competitor. Morgan & Uddin (1995) established an infinite series of MV -optimal nonbinary block designs that are MV -superior to all binary block designs of the same parameters. Morgan & Uddin (1995) also found infinitely many designs which are Φ_p -superior for all sufficiently large p , to all binary designs. These results contradict the general belief that nonbinary designs are necessarily worse with respect to any reasonable optimality criteria. In the present research we find many optimality conditions and constructions for many settings where equally replicated designs are not possible. Most of the designs possess a widely applicable structure in the information matrix, and many happen to be nonbinary. For a variety of applications these nonbinary designs are not only reasonable, but are preferable to binary competitors.

Generalized group divisible designs with s groups, or $GGDD(s)$ s, are defined in

chapter 2. This definition is in terms of the elements of the information matrix, rather than in terms of the elements of the concurrence matrix as has been done previously. The new approach includes nonbinary members, allows for broader optimality results, and subsumes the *CSDs* as *GGDD(1)s*. Several sufficient conditions are derived for these designs to be *E*- and/or *MV*-optimal. Where nonbinary designs are found, they are generally preferable to binary designs in terms of interpretability, and often in terms of one or more formal optimality criteria. It is also shown how augmentation of additional blocks to certain *GGDD(s)s* produces infinite series of other nonbinary, unequally replicated *E*- and *MV*-optimal block designs. Many examples are given to show how the results obtained can be applied.

Chapter 3 deals with an investigation of the type-1 optimality of block designs, when the number of replications are unequal and the off-diagonal entries of the concurrence matrix differ by more than one. We elucidate this problem by defining a class of generalized nearly balanced incomplete block designs (*l*), or *NBBD(l)*. This extends the concept of nearly balanced incomplete block designs of Cheng (1981) or semi-regular graph designs of Jacroux (1985) to the case when off-diagonal entries of the concurrence matrix differ by an positive integer $l \geq 1$. Sufficient condition are found for a binary block design having unequal replication numbers and the off-diagonal entries of the concurrence matrix differing by two, to be optimal under

a given type-I criterion. Examples illustrating usage of the results are included. In particular, it is demonstrated how the sufficient conditions can be used to establish the existence of A - and D -optimal $NBBD(2)$ s within a given class $D(v, b, k)$, where no other optimality results known to the author are applicable.

The construction and existence of $GGDD$ s is the subject matter of study in chapter 4. Many general and simple methods of constructing CSD s and $GGDD(2)$ s with blocksize three are discussed. Several other constructions of completely symmetric designs are also presented.

Although a large number of block designs are available in the literature, there are some situations where there are more sources of variation than can be controlled for by ordinary blocking. In this context, the balanced incomplete block design with nested rows and columns ($BIBRC$ s) are introduced in chapter 5. Existence and construction of the 2×2 $BIBRC$ s is completely solved. A general balance property of $BIBRC$ s with numbers of rows equal to numbers of columns is also discussed.

In chapter 6, some important aspects of the results obtained in this dissertation are reiterated, with the intent of bringing the entirety of the work into perspective.

Finally examples are used to explain various results and constructions in all chapters of this dissertation.

Chapter 2

Optimality of Generalized Group Divisible Designs

2.1 INTRODUCTION

This chapter is an investigation of E - and MV -optimal designs within various classes of proper block designs in which treatments can not be replicated the same number of times. In section 2.2 we define a generalized group divisible design with s groups, or $GGDD(s)$, in terms of the elements of the information matrix C_d , instead of in terms of the elements of the concurrence matrix $N_d N_d^T$ as done by Adhikary (1965) and extended by Jacroux (1982). This definition generalizes the class of designs to include nonbinary members, and allows for broader optimality results. Several suffi-

cient conditions are derived for $GGDD(s)$ s to be E -optimal in $D(v, b, k)$ and several large classes of $GGDD(s)$ s are shown to satisfy these sufficient conditions. It is also shown how augmentation of additional blocks to certain $GGDD(s)$ s produces infinite series of other nonbinary, unequally replicated E -optimal block designs. Finally section 2.3 deals with the MV -optimality of certain types of $GGDD(s)$ s. Some sufficient conditions are derived for $GGDD(s)$ s to be MV -optimal in the class $D(v, b, k)$, typically with bk/v not being an integer. Several examples are also demonstrated to show how the results obtained can be applied.

Throughout this chapter the greatest integer not exceeding the number x will be denoted by $\text{int}(x)$. For the class $D(v, b, k)$, we use r and λ to denote $\text{int}(bk/v)$ and $\text{int}(r(k-1)/(v-1))$, respectively. Also, c will denote the number $\frac{r(k-1)}{k}$.

2.2 E -OPTIMALITY

DEFINITION 2.1 *The design $d \in D(v, b, k)$ is called a $GGDD(s)$ if the treatments in d can be divided into s mutually disjoint sets V_1, \dots, V_s of size v_1, \dots, v_s such that*

(i) *for $g = 1, \dots, s$ and all $i \in V_g$, $(C_d)_{ii} = r_{di} - \lambda_{dii}/k = c_g$, where c_g depends on the set g but not otherwise on the treatment i ,*

(ii) *for $g, h = 1, \dots, s$ and all $i \in V_g$ and $j \in V_h$, with $i \neq j$ if $g = h$, $\lambda_{dij} = \gamma_{gh}$, where γ_{gh} depends on the sets g and h but not otherwise on the treatments i*

and j .

Assume that the sets V_1, \dots, V_s in the above definition are arranged so that the c_g 's are in nonincreasing order: $c_1 \geq c_2 \geq \dots \geq c_s$.

Among designs satisfying the definition of a $GGDD(s)$ are the $GGDD(1)$ s, what we have called the completely symmetric designs in chapter 1. Group divisible partially balanced incomplete block designs with two associate classes (which are $GGDD(2)$) and concurrence parameters differing by one, and some previously studied binary $GGDD(s)$ s are known to be E -optimal in $D(v, b, k)$; see Takeuchi (1961) and Jacroux (1982). The goal in this section is to obtain a single result providing E -optimality conditions for members of the wider class of $GGDD(s)$ s as defined above. The following lemma will be needed in this regard.

LEMMA 2.1 (*Jacroux, 1982*) *Let $d \in D(v, b, k)$ have incidence matrix N_d and thus C -matrix $C_d = \text{diag}(c_{d11} + \lambda_{d11}/k, \dots, c_{dvv} + \lambda_{dvv}/k) - 1/k(N_d N_d^T)$. Also let M denote a set containing $m \leq v - 1$ subscripts corresponding to treatments whose c_{dii} 's are equal to $c = r(k - 1)/k$. Then*

(a) $z_{d1} \leq c_{dii}v/(v - 1)$ for all $i = 1, (1), v$.

(b) If $i, j \in M$, then $z_{d1} \leq (kc + \lambda_{dij})/k$.

(c) If for all $i, j \in M$ with $i \neq j$, $\lambda_{dij} \geq z$ for some $z \geq 0$, then $z_{d1} \leq \frac{[kc - (m-1)z]v}{(v-m)k}$.

Theorem 2.1 uses lemma 2.1 to generalize Theorem 2.3 of Jacroux (1982).

THEOREM 2.1 *Let $D(v, b, k)$ be a class of designs such that $bk = vr + p$ for some $0 \leq p < v$; $r(k - 1) = (v - 1)\lambda + q$ for some $0 \leq q < v - 1$; and $v \leq (v - p)(v - q)$.*

Now let $d^ \in D(v, b, k)$ be any GGDD(s) satisfying the following conditions:*

(i) $c_g \geq c$ for $g = 1, (1), s$,

(ii) if $c_g = c$ then $\gamma_{gg} = \lambda$,

(iii) if $c_g > c$ then $kc_g + \gamma_{gg} \geq kc + \lambda$, and

(iv) for $g \neq h$, $\gamma_{gh} = \lambda + e$ for some $e \geq 0$ such that $v(\lambda + e) \geq kc + \lambda$.

Then d^ is E -optimal in $D(v, b, k)$.*

PROOF For d^* satisfying the conditions of the theorem, write C_{d^*} in block form with matrices $(c_g + \gamma_{gg}/k)I_{v_g} - \gamma_{gg}/kJ_{v_g v_g}$ along the main diagonal, $g = 1, (1), s$, and matrices $[-(\lambda + e)/k]J_{v_g v_h}$ elsewhere, $g, h = 1, (1), s, g \neq h$. Inspection of $C_{d^*} + uJ_{vv}$, where $u = (\lambda + e)/k$, shows that C_{d^*} has $s - 1$ eigenvalues equal to vu and $v_g - 1$ eigenvalues equal to $c_g + \gamma_{gg}/k$, $g = 1, (1), s$. Also, since $c_g \geq c$ for $g = 1, (1), s$, and $bk = vr + p$, d^* has at least $v - p$ treatments replicated exactly r times. Hence $c_g = c$ is achieved and $z_{d^*1} = (c + \lambda/k) = (vr(k - 1) - q)/k(v - 1)$. We now show that d^* is E -optimal in $D(v, b, k)$.

Suppose $d \in D(v, b, k)$ is E -better than d^* . If d has some treatment replicated $r - z$ times for $z \geq 1$ then by lemma 2.1 (a),

$$z_{d1} \leq \frac{(r - z)(k - 1)v}{(v - 1)k} \leq \frac{(r - 1)(k - 1)v}{(v - 1)k} < \frac{r(k - 1) + \lambda}{k} = z_{d^*1}.$$

Thus d must have all treatments replicated at least r times. Hence at least $v - p$ treatments are replicated exactly r times, and if any of these treatments occur more than once in a block, then again by lemma 2.1 (a),

$$\begin{aligned} z_{d1} &\leq (c - 2/k)(v/(v - 1)) = (vr(k - 1) - 2v)/k(v - 1) \\ &\leq (vr(k - 1) - q)/k(v - 1) = z_{d^*1}. \end{aligned}$$

Thus we have at least $v - p$ treatments whose c_{dii} 's are exactly equal to c . Let M be the set of size $m \geq v - p$ containing the subscripts of treatments whose c_{dii} 's are exactly equal to c . If $m = 1$ (which implies that $p = v - 1$ and $q = 0$) then by lemma 2.1(a) $z_{d1} \leq (\lambda v/k) = ((r(k - 1) + \lambda)/k) = z_{d^*1}$. Otherwise for $i, j \in M$, $i \neq j$, from lemma 2.1(b), if $\lambda_{dij} \leq \lambda$ then $z_{d1} \leq z_{d^*1}$. Thus C_d can have $z_{d1} > z_{d^*1}$ only if $\lambda_{dij} \geq \lambda + 1$ for all $i, j \in M$, $i \neq j$, and $|M| = m \geq 2$. By using the fact that $(v - 1)(\lambda + 1) \geq r(k - 1)$ we have

$$(v - m)[(v - 1)(\lambda + 1) - r(k - 1)] \leq p[(v - 1)(\lambda + 1) - r(k - 1)],$$

that is,

$$p[r(k - 1) - (m - 1)(\lambda + 1)] \leq (v - m)[r(k - 1) - (v - p - 1)(\lambda + 1)].$$

Also, $v \leq (v - p)(v - q)$ gives

$$v[r(k - 1) - \lambda(v - 1)] - v^2 + v \leq p[r(k - 1) - \lambda(v - 1)] - pv$$

which can be rewritten

$$v[r(k - 1) - (v - p - 1)(\lambda + 1)] \leq p[r(k - 1) + \lambda].$$

Using these facts in conjunction with lemma 2.1 (c) gives

$$\begin{aligned} z_{d1} &\leq [r(k - 1) - (m - 1)(\lambda + 1)] \frac{v}{(v - m)k} \\ &\leq [r(k - 1) - (v - p - 1)(\lambda + 1)] \frac{v}{pk} \leq \frac{[r(k - 1) + \lambda]}{k} = z_{d^*1}, \end{aligned}$$

a contradiction. \square

EXAMPLE 1 Consider the design d^* for $v = 9$ and $k = 5$ with blocks given by the

$b = 12$ columns

1	1	1	1	1	8	8	8	9	9	9	1
1	1	8	8	8	2	2	3	2	2	3	2
2	5	9	9	9	3	4	4	3	4	4	3
3	6	2	3	4	6	5	5	5	6	5	4
4	7	5	6	7	7	6	7	7	7	6	5

It is easy to see that d^* is a $GGDD(2)$ with $V_1 = \{1, 2, 3, 4, 5\}$, $V_2 = \{6, 7, 8, 9\}$,

$\gamma_{11} = 4$, and $\gamma_{12} = \gamma_{22} = 3$, which satisfies all conditions of Theorem 2.1. So d^* is

E -optimal in $D(9, 12, 5)$.

As applications of Theorem 2.1, we will show that every block design meeting the MV -optimality conditions of Theorem 1 of Morgan and Uddin (1995) is E -optimal, and also how E -optimal designs with fewer blocks can be obtained from $CSDs$.

COROLLARY 2.1 *Suppose $d^* \in D(v, b, k)$ satisfies the conditions of Theorem 1 of Morgan & Uddin (1995). Then d^* is E -optimal.*

PROOF It is easy to see that necessary conditions for d^* to satisfy Theorem 1 of Morgan & Uddin (1995) are that $bk = vr + 1$, and that $v - 1$ treatments must be replicated binarily r times each. It follows that the quantity $\frac{r(k-1)}{v-1} = \lambda$ is an integer, and thus $p = 1$ and $q = 0$, which implies $v \leq (v - p)(v - q)$. Since C_{d^*} is completely symmetric, the conditions of Theorem 2.1 are immediate. \square

Morgan & Uddin (1995) had established the result of corollary 2.1 for $k = 3$ only. By using the upper bound for z_{d1} within the binary class, given in Theorem 3.2 of chapter 3, it can easily be seen that all of the designs are E -superior to binary competitors.

COROLLARY 2.2 *Suppose $d \in D(v, b, k)$ is a $GGDD(1)$ with $bk = v\tilde{r} + \tilde{p}$, for some $0 \leq \tilde{p} < v$; $\tilde{\lambda} = \tilde{r}(k - 1)/(v - 1)$ is an integer; and $(C_d)_{ii} = \tilde{c} = \tilde{r}(k - 1)/k$, for all $i = 1, (1), v$. Let w be an integer satisfying $(v + k\tilde{p})/k^2 \leq w$, and write $b^* = b - w$. If d^* is the design obtained from d by deleting w mutually disjoint binary blocks, then d^* is E -optimal in $D(v, b^*, k)$.*

PROOF Observe that r and λ for $D(v, b^*, k)$ are $\tilde{r} - 1$ and $\tilde{\lambda} - 1$ respectively. We also have $b^*k = vr + p$ where $p = v - wk + \tilde{p}$, and $r(k - 1) = (v - 1)\lambda + q$ where $q = v - k$. For $g = 2, (1), w + 1$, let V_g contain the subscripts in the g th block deleted from d , and let V_1 contain the remaining subscripts. Then d^* is a $GGDD(w + 1)$ with $c_g = c = r(k - 1)/k = \tilde{c} - (k - 1)/k$ and $\gamma_{gg} = \lambda$ for $g = 2, (1), w + 1$; $c_1 \geq c$; and $kc_1 + \gamma_{11} \geq kc + \lambda$ where $\gamma_{11} = \tilde{\lambda} = \lambda + 1$. Also $\gamma_{gh} = \tilde{\lambda} = \lambda + 1$ for all $g \neq h$. The result now follows from Theorem 2.1 since $(v + k\tilde{p})/k^2 \leq w$ is equivalently $v \leq (v - p)(v - q)$. \square

EXAMPLE 2 This design $d \in D(13, 16, 5)$ is a $GGDD(1)$. With $\tilde{p} = 2$ in corollary 2.2, deletion of $w \geq 1$ disjoint blocks will leave an E -optimal $d^* \in D(13, 16 - w, 5)$. Since no two blocks of d are disjoint, we are limited to $w = 1$.

10	10	10	10	1	4	7	1	2	3	1	2	3	1	2	3
10	10	10	10	2	5	8	4	5	6	5	6	4	6	4	5
1	3	6	11	3	6	9	7	8	9	9	7	8	8	9	7
2	4	7	12	11	11	11	11	11	12	12	12	12	13	13	13
9	5	8	13	6	9	3	12	13	13	8	5	2	4	7	1

On starting with the series of $GGDD(1)$ s with $v = 3t + 2$, $b = 3t^2 + 3t + 1$, and $k = 3$ for $t \geq 1$ from Morgan & Uddin (1995) and using corollary 2.2, we arrive at the new series of E -optimal designs with parameters $v = 3t + 2$, $b = 3t^2 + 3t + 1 - w$, and $k = 3$, for all $t \geq 1$ and $(3t + 5)/9 \leq w \leq t$ (the required w disjoint blocks can always be found). In a construction shown in chapter 4 we have also established the

existence of the series of $GGDD(1)$ s with parameters $v = 2t + 1$, $b = 4t - 1$, and $k = t + 1$, where $2t - 1$ is a prime or prime power. Deletion of a single block yields the new series of E -optimal designs with parameters $v = 2t + 1$, $b = 4t - 2$, and $k = t + 1$.

EXAMPLE 3 Let d be this $GGDD(1)$ with $v = 8$ and $k = 3$:

8	8	8	8	8	8	8	7	7	7	1	2	3	1	2	4	5	6	7
8	1	3	5	1	2	4	1	2	3	4	3	4	2	3	5	6	7	1
7	2	4	6	3	5	6	6	4	5	5	6	6	4	5	7	1	2	3

Any design d^* obtained from d by deleting two disjoint binary blocks is E -optimal.

An interesting question is whether, when corollary 2.2 requires $w > 1$, deletion of fewer blocks will still result in an E -optimal design. We suspect that the answer will often be yes, but unfortunately the result is not amenable to the current technique, for the condition $v \leq (v - p)(v - q)$ of Theorem 2.1 in this case fails.

We now show how other nonbinary, unequally replicated E -optimal designs can be obtained by adding blocks to certain $GGDD(s)$ s.

THEOREM 2.2 *Let $D(v, b, k)$ be a class of designs for which v, b, k, r and λ satisfy the conditions of Theorem 2.1, and let $d^* \in D(v, b, k)$ be a $GGDD(s)$ as described in Theorem 2.1. Now let $\hat{b} > 0$ be an integer such that $p + \hat{b}k \leq (v - 1)$ and $v \leq (v - p - \hat{b}k)(v - q)$. If \hat{d} is any design with members of the same treatment set*

arranged in \hat{b} blocks of size k , then the design \bar{d} having $N_{\bar{d}}=(N_{d^*}, N_{\hat{d}})$ is E -optimal in $D(v, \bar{b}, k)$, where $\bar{b} = b + \hat{b}$.

PROOF By Theorem 2.1, d^* is E -optimal in $D(v, b, k)$ with $z_{d^*1} = [r(k-1) + \lambda]/k$. Since $C_{\bar{d}}=C_{d^*}+C_{\hat{d}}$ and $C_{\hat{d}}$ is positive semi definite, $z_{\bar{d}1} \geq z_{d^*1}$. Now let $d \in D(v, \bar{b}, k)$ be arbitrary. Using lemma 2.1 (a), if $r_{di} < r$ for some i , then

$$z_{d1} \leq (r-1)(k-1)v/(v-1)k < z_{d^*1} \leq z_{\bar{d}1}.$$

Thus an E -optimal design in $D(v, \bar{b}, k)$ must have all treatments replicated at least r times. Hence at least $v - p - \hat{b}k$ treatments are replicated exactly r times, and if any of these treatments occur more than once in a block, then by lemma 2.1 (a)

$$\begin{aligned} z_{d1} &\leq (c - 2/k)(v/v - 1) = (vr(k-1) - 2v)/k(v-1) \\ &\leq (vr(k-1) - q)/k(v-1) = z_{d^*1} \leq z_{\bar{d}1}. \end{aligned}$$

Thus we have at least $v - p - \hat{b}k$ treatments whose c_{dii} 's are exactly equal to c . Let M be the set of size $m \geq v - p - \hat{b}k$ containing the subscripts of treatments whose c_{dii} 's are exactly equal to c . As in the proof of Theorem 2.1 for $m = 1$, $p = v - \hat{b}k - 1$ and $q = 0$, and using lemma 2.1 (a) $z_{d1} \leq z_{d^*1}$. Otherwise for $i \neq j$ and $\lambda_{dij} \leq \lambda$, we have $z_{d1} \leq z_{d^*1} \leq z_{\bar{d}1}$. Therefore C_d can have $z_{d1} > z_{\bar{d}1}$ only if $\lambda_{dij} \geq \lambda + 1$ for all $i, j \in M$ and $i \neq j$. By using the fact that $(v-1)(\lambda+1) \geq r(k-1)$ we have

$$(v-m)[(v-1)(\lambda+1) - r(k-1)] \leq (p + \hat{b}k)[(v-1)(\lambda+1) - r(k-1)]$$

that is,

$$(p + \hat{b}k)[r(k-1) - (m-1)(\lambda+1)] \leq (v-m)[r(k-1) - (v-p-\hat{b}k-1)(\lambda+1)].$$

Also $v \leq (v-p-\hat{b}k)(v-q)$ gives

$$v[r(k-1) - \lambda(v-1)] - v^2 + v \leq (p + \hat{b}k)[r(k-1) - \lambda(v-1)] - (p + \hat{b}k)v$$

which rearranges to

$$v[r(k-1) - (v-p-\hat{b}k-1)(\lambda+1)] \leq (p + \hat{b}k)[r(k-1) + \lambda].$$

Using lemma 2.1 (c) with these facts gives

$$\begin{aligned} z_{d1} &\leq [r(k-1) - (m-1)(\lambda+1)]v/(v-m)k \\ &\leq [r(k-1) - (v-p-\hat{b}k-1)(\lambda+1)]v/(p+\hat{b}k)k \\ &\leq [r(k-1) + \lambda]/k = z_{d^*1} \leq z_{\bar{d}1}. \end{aligned}$$

Hence $z_{d1} \leq z_{\bar{d}1}$, and since d was arbitrary in $D(v, \bar{b}, k)$, \bar{d} is E -optimal in $D(v, \bar{b}, k)$. \square

An interesting fact about designs satisfying Theorem 2.2 is that they need not be $GGDD$ s, for the blocks being added are completely arbitrary. However one would generally pay attention to other design criteria, including the structure of the C -matrix relative to the structure of the treatment set, in choosing the blocks to add.

EXAMPLE 4 This $GGDD(1)$ $d^* \in D(9, 15, 5)$ has $p = 3$, so addition of any one block of size 5, be it binary or not, produces an E -optimal design $\bar{d} \in D(9, 16, 5)$.

1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
1	3	4	5	6	7	8	9	3	4	5	6	7	8	9
1	6	7	8	9	3	4	5	6	7	8	9	3	4	5
1	9	3	4	5	6	7	8	9	3	4	5	6	7	8
2	8	9	3	4	5	6	7	8	9	3	4	5	6	7

The proof of the following corollary follows immediately from Theorem 2.2 upon using the facts stated in the proof of corollary 2.2.

COROLLARY 2.3 *Suppose $d^* \in D(v, b, k)$ satisfies the conditions of Theorem 1 of Morgan & Uddin (1995), and let $\hat{b} > 0$ be an integer such that $\hat{b}k \leq v - 2$. If \hat{d} is any design with members of the same treatment set arranged in \hat{b} blocks of size k , then the design \bar{d} having $N_{\bar{d}} = (N_{d^*}, N_{\hat{d}})$ is E -optimal in $D(v, \bar{b}, k)$, where $\bar{b} = b + \hat{b}$.*

Corollary 2.3 applied to the series of designs with parameters $v = 3t + 2$, $b = 3t^2 + 3t + 1$, and $k = 3$ from Morgan & Uddin (1995) produces the E -optimal series with parameters $v = 3t + 2$, $b = 3t^2 + 3t + 1 + \hat{b}$, and $k = 3$, for every $\hat{b} \leq t$ and $t \geq 1$. For instance, the starting design $d \in D(8, 3, 19)$ in example 3 can be augmented to give E -optimal designs in $D(8, 3, 20)$ or $D(8, 3, 21)$.

2.3 *MV*-OPTIMALITY

This section generalizes the idea behind Theorem 2.6 of Jacroux (1983) to accommodate the broader class of *GGDDs* as defined in section 2.2, resulting in the *MV*-optimality of a wider class of block designs, including some nonbinary, unequally replicated designs. But before doing so, an error in Jacroux's (1983) theorem needs to be pointed out. In the proof, it is stated that one can easily see that the maximum variance of an elementary contrast using the proposed design d^* is (in our notation) $\frac{2k}{r(k-1)+\gamma_{ss}} = m^*$. This is not always true under the conditions given there, and hence designs satisfying those conditions need not be *MV*-optimal, as shown by the following counterexample.

EXAMPLE 5 For $v = 9$, $b = 17$, and $k = 3$, consider the designs d_1 and d_2 :

$$\begin{array}{r}
 d_1 : \begin{array}{cccccccccccccccc}
 7 & 7 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 9 & 1 & 1 & 1 & 2 & 2 & 5 \\
 8 & 8 & 1 & 3 & 5 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 2 & 5 & 5 & 4 & 6 \\
 9 & 9 & 2 & 4 & 6 & 3 & 6 & 5 & 5 & 4 & 6 & 6 & 3 & 4 & 3 & 6 & 3
 \end{array} \\
 \\
 d_2 : \begin{array}{cccccccccccccccc}
 1 & 4 & 7 & 1 & 4 & 1 & 4 & 7 & 1 & 2 & 3 & 1 & 4 & 7 & 1 & 4 & 7 \\
 2 & 5 & 8 & 2 & 5 & 2 & 5 & 8 & 4 & 5 & 6 & 5 & 8 & 2 & 8 & 2 & 5 \\
 3 & 6 & 9 & 3 & 6 & 3 & 6 & 9 & 7 & 8 & 9 & 9 & 3 & 6 & 6 & 9 & 3
 \end{array}
 \end{array}$$

The *MV*-value of d_1 , which does not have generalized group divisible structure, is .5154. That of the *GGDD*(3) d_2 , which satisfies Jacroux's (1983) Theorem 2.6 and has $\gamma_{33} = 2$, is .5222. The corresponding value of m^* is 0.5.

To find the correct condition to make m^* the maximum variance, we have explicitly calculated the variances of elementary contrasts for the proposed d^* and used Jacroux's (1983) arguments to establish the main result of this section. With respect to the MV -optimality of designs in $D(v, b, k)$ where bk/v is not an integer, the only results known to the authors are those given by Jacroux (1983) and Morgan & Uddin (1995). The latter authors also establish an infinite series of MV -optimal nonbinary block designs that are MV -superior to all binary block designs of the same parameters. First we need the following lemma, which is one of the key tools in MV -optimality arguments.

LEMMA 2.2 (Takeuchi, 1961) *Let $d \in D(v, b, k)$ be arbitrary. Then for any i and j , $i \neq j$, the variance with which $\tau_i - \tau_j$ is estimated in d satisfies*

$$\text{Var}(\widehat{\tau_i - \tau_j}) \geq 4k/((r_{di} + r_{dj})(k - 1) + 2\lambda_{dij})$$

THEOREM 2.3 *Let $D(v, b, k)$ be a class of designs such that $bk = vr + p$ for some $0 \leq p \leq v - 2$, and $r(k - 1) = (v - 1)\lambda + q$ for some $0 \leq q < v - 1$. Let $d^* \in D(v, b, k)$ be any $GGDD(s)$ satisfying*

(i) $\gamma_{ss} \geq \text{int}(\frac{pk + 2r(k - 1)}{2(v - 1)}),$

(ii) $c_g \geq c$ for $1 \leq g \leq s - 1$, and $c_s = c$,

(iii) $\gamma_{gg} \geq \gamma_{ss}$ for $1 \leq g \leq s - 1$,

(iv) for $1 \leq g, h \leq s$ with $g \neq h$, $\gamma_{gh} = \gamma$,

(v) $\frac{(\gamma_{gg} + (v-1)\gamma_{1s})}{v\gamma_{1s}(kc_g + \gamma_{gg})} + \frac{(\gamma_{hh} + (v-1)\gamma_{1s})}{v\gamma_{1s}(kc_h + \gamma_{hh})} \leq \frac{2}{kc + \gamma_{ss}}$ for $1 \leq g, h \leq s$ with $g \neq h$.

Then d^* is MV-optimal in $D(v, b, k)$.

PROOF Let d^* be as described in the theorem and consider the matrix T_{d^*x} defined in chapter 1. Putting $x = \gamma_{1s}/k = \gamma/k$ gives $T_{d^*x} = \frac{1}{k} \text{diag}((kc_g + \gamma_{gg})I_{v_g} + (\gamma_{1s} - \gamma_{gg})J_{v_g})$ so that

$$T_{d^*x}^{-1} = k \text{diag} \left(\frac{1}{kc_g + \gamma_{gg}} I_{v_g} - \frac{(\gamma_{1s} - \gamma_{gg})}{(kc_g + \gamma_{gg})[(kc_g + \gamma_{gg}) + v_g(\gamma_{1s} - \gamma_{gg})]} J_{v_g} \right).$$

The relationship $C_{d^*}1 = 0$ implies $kc_g + \gamma_{gg} + v_g(\gamma_{1s} - \gamma_{gg}) = v\gamma_{1s}$ and thus

$$T_{d^*x}^{-1} = k \text{diag} \left(\frac{1}{(kc_g + \gamma_{gg})} [I_{v_g} + \frac{(\gamma_{gg} - \gamma_{1s})}{v\gamma_{1s}} J_{v_g}] \right).$$

Since $p \leq v - 2$, there exist at least two treatments which are replicated exactly r times, and these treatments has to occur in the same group. For suppose one of these treatments is alone in a group. Then without loss of generality, there exist two groups, say V_{s-1} and V_s , such that $|V_{s-1}| = v_1$ and $|V_s| = 1$. By inspection of the C -matrix of d^* , it follows that $-\lambda(v-1)/k + c = 0$, and $c - \gamma_{11}(v_1 - 1)/k - \lambda(v - v_1)/k = 0$. Together these imply that $\lambda = \gamma_{11}$, that is, every treatment which occurs exactly r times is in fact in the same group. Therefore, if $i, j \in V_g$, then conditions (ii) and (iii) imply

$$\text{Var}(\widehat{\tau_i} - \tau_j) = \frac{2k}{(kc_g + \gamma_{gg})} \leq \frac{2k}{(kc + \gamma_{ss})}$$

with equality for $g = s$.

If $i \in V_g$ and $j \in V_h$,

$$\text{Var}(\widehat{\tau_i} - \widehat{\tau_j}) = \frac{k}{(kc_g + \gamma_{gg})} + \frac{k}{(kc_h + \gamma_{hh})} + \frac{k(\gamma_{gg} - \gamma_{1s})}{(v\gamma_{1s}(kc_g + \gamma_{gg}))} + \frac{k(\gamma_{hh} - \gamma_{1s})}{(v\gamma_{1s}(kc_h + \gamma_{hh}))}.$$

If $m^* = 2k/(kc + \gamma_{ss})$ is to be the largest of the elementary contrast variances then

it must be greater than or equal to

$$\frac{k(\gamma_{gg} + (v-1)\gamma_{1s})}{v\gamma_{1s}(kc_g + \gamma_{gg})} + \frac{k(\gamma_{hh} + (v-1)\gamma_{1s})}{v\gamma_{1s}(kc_h + \gamma_{hh})}$$

which is condition (v).

With this in mind, let $d \in D(v, b, k)$ be arbitrary and $r_{d1} \geq r_{d2} \geq \dots \geq r_{dv}$. If $r_{dv} < r$ then

$$\text{Var}(\widehat{\tau_i} - \widehat{\tau_v}) \geq 4k/((r_{di} + r_{dv})(k-1) + 2\lambda_{div}) = 4k/A_{iv}$$

where $A_{iv} = (r_{di} + r_{dv})(k-1) + 2\lambda_{div}$. Consider

$$\begin{aligned} \sum_{i=1}^{v-1} A_{iv} &= (k-1)\{(v-1)r_{dv} + \sum_{i=1}^{v-1} r_{di}\} + 2 \sum_{i=1}^{v-1} \lambda_{div} \\ &\leq (k-1)\{(v-1)r_{dv} + bk - r_{dv}\} + 2r_{dv}(k-1) = bk(k-1) + vr_{dv}(k-1) \\ &\leq (k-1)(vr + p) + v(k-1)(r-1) = 2kc(v-1) + (2r + p - v)(k-1). \end{aligned}$$

Since each A_{iv} is an integer, and using condition (i),

$$\begin{aligned} \min_{1 \leq i \leq v-1} A_{iv} &\leq 2kc + \text{int}((2r + p - v)(k-1)/(v-1)) \\ &\leq 2(kc + \gamma_{ss}). \end{aligned}$$

Thus if $r_{dv} < r$, it follows from lemma 2.2 that for some $1 \leq i \leq v-1$

$$\max_{1 \leq i \leq v-1} \text{Var}(\widehat{\tau}_i - \widehat{\tau}_v) \geq 2k/(kc + \gamma_{ss}) = m^*.$$

So now assume $r_{dv} = r$ and observe that since $bk = vr + p$, d must have at least $v - p \geq 2$ treatments which are replicated exactly r times, that is, $r_{d,p+1} = \dots = r_{dv} = r$.

If $\lambda_{dij} \leq \gamma_{ss}$ for some $i \neq j$ and $p+1 \leq i, j \leq v$, then again by lemma 2.2, $\max_{i \neq j} \text{Var}(\widehat{\tau}_i - \widehat{\tau}_j) \geq m^*$. Thus the only way d can have $\text{Var}(\widehat{\tau}_i - \widehat{\tau}_j) < 2k/(kc + \gamma_{ss}) = m^*$ for all $i \neq j$ is if $\lambda_{dij} \geq \gamma_{ss} + 1$ for all $i \neq j$ and $p+1 \leq i, j \leq v$. However, if this happens and we let A_{iv} be as defined earlier, then

$$\begin{aligned} \sum_{i=1}^p A_{iv} &= \sum_{i=1}^p [(r_{di} + r_{dv})(k-1) + 2\lambda_{div}] \\ &= rp(k-1) + (k-1)(bk - r(v-p)) + 2[kc_{dvv} - \sum_{i=p+1}^{v-1} \lambda_{div}] \\ &\leq rp(k-1) + (k-1)(bk - r(v-p)) + 2k[c_{dvv} - (v-p-1)(\gamma_{ss} + 1)/k] \\ &= p[2(kc + \gamma_{ss}) + k + 1] + 2[r(k-1) - (v-1)(\gamma_{ss} + 1)] \end{aligned}$$

Again on using the fact that each A_{iv} is an integer,

$$\min_{1 \leq i \leq v-1} A_{iv} \leq \min_{1 \leq i \leq p} A_{iv} \leq \text{int} \left(2(kc + \gamma_{ss}) + k + 1 + \frac{2[r(k-1) - (v-1)(\gamma_{ss} + 1)]}{p} \right)$$

so that another application of lemma 2.2 will give the result provided the right-hand side of this inequality is no greater than $2(kc + \gamma_{ss})$. This will be true whenever

$k + 1 + \frac{2[r(k-1)-(v-1)(\gamma_{ss}+1)]}{p} < 1$, or equivalently, when $\gamma_{ss} > \frac{pk+2r(k-1)-2(v-1)}{2(v-1)}$. The

last inequality is implied by condition (i). \square

It is easy to observe that Theorem 1 of Morgan & Uddin (1995), when specialized to the proper block design setting, is a special case of Theorem 2.3. Incidentally, design d_2 of example 5 fails condition (v) of this theorem.

EXAMPLE 6 Consider this $d \in D(6, 11, 3)$:

2	3	4	5	6	1	2	1	1	1	2
3	4	5	6	2	2	3	4	5	6	3
4	5	6	2	3	4	1	6	5	3	5

Putting $V_1 = \{2, 3\}$ and $V_2 = \{1, 4, 5, 6\}$, d is a $GGDD(2)$ with $\gamma_{11} = 4$ and $\gamma_{12} = \gamma_{22} = 2$. All conditions of Theorem 2.3 are satisfied. So d is MV -optimal in $D(6, 11, 3)$.

EXAMPLE 7 This design is a $GGDD(2)$ with $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, 6, 7, 8\}$.

It satisfies all of the conditions of Theorem 2.3, so is MV -optimal in $D(8, 39, 3)$.

8	8	8	8	8	8	8	7	7	7	1	2	1	2	3	4	5	6	7	
8	1	2	3	4	5	6	1	2	3	3	4	2	3	4	5	6	7	1	
7	2	3	4	5	6	1	4	5	6	5	6	4	5	6	7	1	2	3	
7	7	7	7	7	7	7	6	6	6	1	2	1	2	3	4	5	8	6	1
7	1	2	3	4	5	8	1	2	3	3	4	2	3	4	5	8	6	1	2
6	2	3	4	5	8	1	4	5	8	5	8	4	5	8	6	1	2	3	3

COROLLARY 2.4 Suppose $d^* \in D(v, b, k)$ is a GGDD(1) with $bk = vr + \tilde{p}$ for some $0 \leq \tilde{p} < v$; $\lambda = r(k-1)/(v-1)$ is an integer; and $(C_d)_{ii} = c$ for all $i = 1, (1), v$. Let \bar{d} be any design obtained from d^* by adding \hat{b} binary blocks containing disjoint sets of treatments, where \hat{b} satisfies

$$\hat{b} \leq \frac{2(v-1) - \tilde{p}k}{k^2}.$$

Then \bar{d} is MV-optimal in $D(v, \bar{b}, k)$, where $\bar{b} = b + \hat{b}$.

PROOF Write $s = \hat{b} + 1$. For $g = 1, (1), s-1$, let V_g contain the treatments in the g^{th} added block; V_s contains the remaining treatments. Then $c_1 = \dots = c_{s-1} = c + \frac{k-1}{k}$, $c_s = c$, $\gamma_{gg} = \lambda + 1$ for $g \leq s-1$, $\gamma_{ss} = \lambda$, and $\gamma_{gh} = \lambda$ for all $g \neq h$. Also, $\bar{b}k = vr + p$ where $p = \hat{b}k + \tilde{p}$.

Condition (i) of Theorem 2.3 is $2\lambda \geq \text{int}\left(\frac{(\hat{b}k + \tilde{p})k}{2(v-1)} + 2\lambda\right)$, i.e. $\hat{b} < \frac{2(v-1) - \tilde{p}k}{k^2}$. Verification of the remaining conditions of the theorem are similarly straightforward. \square

EXAMPLE 8 Consider the design $d^* \in D(15, 53, 4)$ with blocks

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	5	6	7	8	9	10	11	12	13	14	15	3	4	
2	11	12	13	14	15	3	4	5	6	7	8	9	10	
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5	6	7	8	9	10	11	12	13	14	15	3	4		
8	9	10	11	12	13	14	15	3	4	5	6	7		
7	8	9	10	11	12	13	14	15	3	4	5	6		

along with two copies of the blocks

3	3	3	3	4	4	4	5	5	6	6	7	8
4	5	7	9	5	8	10	6	9	7	10	11	12
6	11	8	13	7	9	14	8	10	9	11	12	13
12	15	10	14	13	11	15	14	12	15	13	14	15

d^* is clearly a $GGDD(1)$ with $v = 15$, $k = 4$ and $\lambda = 3$, and for $\hat{b} = 1$ all conditions of corollary 2.4 are satisfied. So the design obtained by adding any $\hat{b} = 1$ binary block containing to d^* is MV -optimal in $D(15, 54, 4)$.

Putting $\tilde{p} = 1$ in corollary 2.4 specializes it for the designs considered by Morgan and Uddin (1995). For instance, the designs with $v = 3t + 2$, $b = 3t^2 + 3t + 1$, and $k = 3$ from Morgan & Uddin (1995) produce the new infinite series of MV -optimal designs with $v = 3t + 2$, $b = 3t^2 + 3t + 1 + \hat{b}$, and $k = 3$, where $\hat{b} < \frac{6t-1}{9}$. On using corollary 2.3, it is easy to verify that these MV -optimal designs are also E -optimal.

The following corollary is the corrected version of corollary 2.9 of Jacroux (1983).

COROLLARY 2.5 *Suppose $d^* \in D(v, b, k)$ is a BIBD having parameters v , b , r , $k = v/2$ and λ . Let \bar{d} be any design obtained from d^* by adding $2t - 1$ blocks, $t \geq 1$, t of which contain treatments $1, \dots, v/2$ and $t-1$ of which contain treatments $v/2+1, \dots, v$.*

If

(i) $(\lambda + t - 1) \geq \text{int}((v^2 + 4(r + t - 1)(v - 2))/8(v - 1))$, and

(ii) $((\lambda v + t)/((r + t)(v - 2) + 2(\lambda + t)) \leq ((\lambda v + 1 - t)/((r + t - 1)(v - 2) + 2(\lambda + t - 1)))$.

Then \bar{d} is *MV-optimal* in $D(v, b + 2t - 1, k)$.

PROOF Observe that \bar{d} is a *GGDD*(2) design with $V_1 = \{1, 2, \dots, v/2\}$ and $V_2 = \{v/2 + 1, v/2 + 2, \dots, v\}$. So in this case $\bar{b} = b + 2t - 1$, $\bar{r} = r + t - 1$, $p = v/2$, $\gamma_{11} = \lambda + t$, $\gamma_{22} = \lambda + t - 1$ and $\gamma_{12} = \lambda$ which satisfies the conditions of Theorem 2.3. Thus \bar{d} is *MV-optimal*. \square

EXAMPLE 9 Consider the *BIBD* d^* having parameters $v = 6, b = 10, r = 5, k = 3$ and $\lambda = 2$. Then the designs \bar{d} obtained from d^* by adding $2t - 1$ blocks as described in corollary (2.5) are *MV-optimal* in $D(6, 10 + 2t - 1, 3)$ for $t = 1, 2$.

Theorem 2.3 does not cover settings with $p = v - 1$. The method of its proof does not apply in this case, for it is possible that $|V_s| = 1$ and hence that the *MV-comparison* need not occur within V_s . To derive a result for $p = v - 1$, we will adapt some work from the test-treatment vs control literature.

Consider the experimental situation in which $v - 1 \geq 2$ test treatments are to be compared to some standard treatment in a block design consisting of b blocks of size k . Let $1, 2, \dots, v$ denote the v treatments being studied with v being used to denote the standard treatment and $1, 2, \dots, v - 1$ to index the test treatments. An excellent overview and many references on comparing test treatments with a standard treatment may be found in the survey paper of Hedayat, Jacroux and Majumdar (1988).

Jacroux (1987) gave this definition for a group divisible treatment design with $(s + 1)$ classes, or $GDTD(s + 1)$:

DEFINITION 2.2 *The design $d \in D(v, b, k)$ is a $GDTD(s + 1)$, if*

- (i) $V_{s+1} = \{v\}$ and $c_{s+1} = r(k - 1)/k$,
- (ii) V_1, \dots, V_s are the sets of size $(v - 1)/s = \bar{v}$,
- (iii) $c_1 = c_2 = \dots = c_s = (r + 1)(k - 1)/k = \bar{c}$,
- (iv) $\gamma_{gg} = \bar{\lambda}_1$ for $g = 1, \dots, s$,
- (v) $\gamma_{gh} = \bar{\lambda}_0$ for $g \neq h$ and $g, h \neq s + 1$,
- (vi) $\gamma_{g,s+1} = \bar{\lambda}_2$ for $g = 1, \dots, s$.

Clearly a $GDTD(s + 1)$ is also a $GGDD(s + 1)$. Hence optimality results for $GDTD(s + 1)$'s can be adopted for our setting. We will need the following notation.

For fixed values of v, b , and k and any positive integer r_0 between 1 and bk , define $N(r_0) = \text{int}(r_0/b)$, $\lambda(r_0) = (r_0 - bN(r_0))(N(r_0) + 1)^2 + (b - r_0 + bN(r_0))N^2(r_0)$, and $R(r_0) = \text{int}((bk - r_0)/v)$. Let $D_{r_0}(v, b, k)$ denote the subclass of $D(v, b, k)$ for which treatment v is replicated r_0 times.

Theorem 2.4, which addresses MV -optimality of $GDTD(s + 1)$'s within the subclass $D_{r_0}(v, b, k)$, is due to Jacroux (1987).

THEOREM 2.4 For a given value of r_0 , let $d^* \in D_{r_0}(v, b, k)$ be a $GDTD(s+1)$ such that

$$r_{d^*v}k - \lambda_{d^*vv} = r_0k - \lambda(r_0), \quad \bar{\lambda} = (r_0k - \lambda(r_0))/(v-1),$$

$$r_{d^*i}k - \lambda_{d^*ii} = R(r_0)(k-1) \quad \text{for } i = 1, \dots, v-1,$$

and

$$\bar{\lambda}_2 = \bar{\lambda}_1 + 1 \quad \text{where } \bar{\lambda}_1 = [R(r_0)(k-1) - \bar{\lambda}_0]/(v-2).$$

Also, let for positive integers x and y , $B(x, y) = (1 - [(1 - x\bar{m}(1 - y\bar{m})]^{1/2})/\bar{m}$, where

$$\begin{aligned} k\bar{m} &= (k/((v-1)\bar{\lambda}_0)) + (s(\bar{v}-1)k/((v-1)(\bar{v}(s-1)\bar{\lambda}_2 + (v-1)\bar{\lambda}_1 + \bar{\lambda}_0))) \\ &+ ((s-1)k/((v-1)(\bar{v}\bar{\lambda}_2 + \bar{\lambda}_0))). \end{aligned}$$

Now, if

$$r_0k - \lambda(r_0) - 2 < (v-1)B(r_0k - \lambda(r_0) - 2, R(r_0)(k-1)); \quad (2.1)$$

$1/(R(r_0) - 1)(k-1) > \bar{m}$ or

$$\begin{aligned} r_0k - \lambda(r_0) &< (v-3)B(r_0k - \lambda(r_0), R(r_0)(k-1)) \\ &+ B(r_0k - \lambda(r_0), (R(r_0) - 1)(k-1)) \\ &+ B(r_0k - \lambda(r_0), (R(r_0) + 1)(k-1)); \end{aligned} \quad (2.2)$$

$$\begin{aligned} k\bar{m} &< \min\{(\bar{c}_{dvv} + \bar{c}_{dii} - 2\bar{c}_{div})/(\bar{c}_{dvv}\bar{c}_{dii} - \bar{c}_{div}^2), \\ &(\bar{c}_{dvv} + \bar{c}_{djj} - 2\bar{c}_{djj})/(\bar{c}_{dvv}\bar{c}_{djj} - \bar{c}_{djj}^2)\}, \end{aligned} \quad (2.3)$$

where \bar{c}_{dvv} is defined as in (2.3),

$$\begin{aligned}\bar{c}_{dvv} &= (r_0k - \lambda(r_0))/k, & \bar{c}_{dii} &= R(r_0)(k - 1), \\ \bar{c}_{div} &= -(\bar{\lambda}_0 - 1)/k, & -\bar{c}_{djj} &= (R(r_0)(k - 1) - 2)/k & \bar{c}_{djv} &= -\bar{\lambda}_0/k;\end{aligned}$$

and

$$\begin{aligned}k\bar{m} &< \min\{\bar{c}_{dpp}(\bar{c}_{dvv} + \bar{c}_{dqq} - 2\bar{c}_{dpq}) - (\bar{c}_{dpv} + \bar{c}_{dpq})^2\} / & (2.4) \\ &\{\bar{c}_{dvv}\bar{c}_{dpp}\bar{c}_{dqq} - \bar{c}_{dvv}\bar{c}_{dpq}^2 - \bar{c}_{dvv}\bar{c}_{dqv}^2 - \bar{c}_{dqq}^2\bar{c}_{dpv} + 2\bar{c}_{dpv}\bar{c}_{dqv}\bar{c}_{dpq},\end{aligned}$$

where \bar{c}_{dvv} is defined as in (2.3),

$$\begin{aligned}\bar{c}_{dpp} &= \bar{c}_{dqq} = R(r_0)(k - 1), \\ \bar{c}_{dpv} &= \bar{c}_{dqv} = -\bar{\lambda}_0/k, & -\bar{c}_{dpq} &= (\bar{\lambda}_1 - 1)/k \quad \text{or} \quad (\bar{\lambda}_1 + 2)/k;\end{aligned}$$

then d^* is MV-optimal $D_{r_0}(v, b, k)$.

Now we can give a theorem for the case $p = v - 1$.

THEOREM 2.5 Let $d^* \in D(v, b, k)$, where $3 \leq k < v$, $bk = vr + (v - 1)$, and

$\lambda = r(k - 1)/(v - 1)$ is an integer, be a GGDD($s + 1$) for which

- (i) $V_{s+1} = \{v\}$ and $c_{s+1} = r(k - 1)/k$,
- (ii) V_1, \dots, V_s are the sets of size $(v - 1)/s = \bar{v}$,
- (iii) $c_1 = c_2 = \dots = c_s = (r + 1)(k - 1)/k$,

$$(iv) \quad \lambda_{gg} = \lambda \quad \text{for } g = 1, \dots, s$$

$$(v) \quad \lambda_{gh} = \lambda + 1 \quad \text{for } g \neq h \quad \text{and } g, h \neq s + 1$$

$$(vi) \quad \lambda_{g,s+1} = \lambda \quad \text{for } g = 1, \dots, s,$$

and at least one of the following two inequalities holds:

$$\lambda v[(k-2)v - (k^2 + 3k - 6)] \leq v^2 - v(k^2 + 4k - 6) + (k^3 - 8k + 9) \quad (2.5)$$

or

$$\lambda v[(k-5)v - (k^2 - 3k - 6)] \leq 4v^2 - v(k^2 + 2k + 9) + (k^3 + k + 6). \quad (2.6)$$

Then d^* is MV-optimal in $D(v, b, k)$.

PROOF First note that the conditions imply that d^* is binary, and that $r_{d^*v} = r$. Strip the last row and column from C_{d^*} and call the resulting $(v-1) \times (v-1)$ matrix C^* . Then $\text{diag}((C^*)^{-1}, 0)$ is a g -inverse for C_{d^*} where $(C^*)^{-1} = I_s \otimes [k/(\lambda v + k - 1)]I_{\bar{v}} - k/((\lambda v + k - 1)(\lambda v + v - 1))J_{\bar{v}} + ((\lambda + 1)/\lambda)(k/(\lambda v + v - 1))J_s \otimes J_{\bar{v}}$. Thus for d^* , elementary treatment contrasts are estimated with three variances: $\text{var}_1 = 2k/(\lambda v + k - 1)$, which is the “within group” variance; $\text{var}_2 = 2k/(\lambda v + k - 1) - 2k/((\lambda v + k - 1)(\lambda v + v - 1))$, which is the “between group” variance for the first s group; and $\text{var}_3 = k/(\lambda v + k - 1) - k/((\lambda v + v - 1)(\lambda v + v - 1)) + k(\lambda + 1)/(\lambda(\lambda v + v - 1))$, which is the “comparison to last group” variance. It is easy to

check that $var_2 \leq var_1 \leq var_3$, so var_3 is the MV -value of d^* . Write $var_3 = k\bar{m}$ where $\bar{m} = (2\lambda^2v + 2\lambda v - 3\lambda + k\lambda + k - 1)/(\lambda(\lambda v + k - 1)(\lambda v + v - 1))$.

Let $d \in D(v, b, k)$ and suppose that d is MV -better than d^* . Now d must have at least one treatment replicated at most r times, say $r_{dv} \leq r$. If $r_{dv} < r$ then $c_{dvv} \leq ((r-1)(k-1)/k) = (\lambda(v-1) - (k-1))/k$ and

$$\begin{aligned} \min_{1 \leq i \leq v-1} (c_{dii} + c_{dvv} - 2c_{div}) &\leq \sum_{i=1}^{v-1} ((c_{dii} + c_{dvv} - 2c_{div})/(v-1)) \\ &= (tr(C_d) + vc_{dvv})/(v-1) \\ &= (2\lambda v(v-1) - k + 1)/(k(v-1)). \end{aligned}$$

Thus if $r_{dv} < r$, it follows from lemma 2.2 that for some $1 \leq i \leq v-1$

$$\max_{1 \leq i \leq v-1} Var(\widehat{\tau_i} - \tau_v) \geq 4k(v-1)/(2\lambda v(v-1) - k + 1)/(k(v-1)) = m_d \text{ (say).}$$

Using the computer software MAPLE, we can establish that $f_1(\lambda, v, k) = m_d - k\bar{m} \geq 0$ for $\lambda \geq 1$ and $v \geq 4$, that is, $f_1(\lambda + 1, v + 4, k) \geq 0$ for $\lambda \geq 0$ and $v \geq 0$. By using MAPLE, $f_1(\lambda + 1, v + 4, k) = \lambda^2[v^2(2k^2 - 2k) + v(16k^2 - 16k) + (32k^2 - 32)] + \lambda[v^2(6k^2 - 6k) + v(44k^2 - 44k) + (k^3 + 80k^2 - 81k)] + [v^2(4k^2 - 4k) + v(28k^2 - 28k) + (k^3 + 46k^2 - 48k)]$, which is clearly a positive quantity.

Hence we may assume that d has all treatments replicated at least r times, and has at least one treatment replicated exactly r times, say $r_{dv} = r$. That is, we need only to prove that d^* is MV -optimal in the class of designs with $r_{dv} = r$. This

problem is addressed by Theorem 2.4 in the context of test-treatment vs control designs; here treatment v with $r_{dv} = r$ serves the role of the control treatment, and r is the r_0 of that theorem. If Theorem 2.4 is fulfilled, then for any d ,

$$\max_{i < j} \text{Var}_d(\widehat{\tau}_i - \widehat{\tau}_v) \geq k\bar{m} = \max_{i \neq j} \text{Var}_{d^*}(\widehat{\tau}_i - \widehat{\tau}_j),$$

proving d^* is MV -optimal. We need to verify that conditions (2.1)-(2.4) of Theorem 2.4 are satisfied.

Writing $p = \lambda(v - 1) - 2$ and $q = \lambda(v - 1) + (k - 1)$ and simplifying, condition (2.1) reduces to $\lambda(v - 1) - 2 < (v - 1)B(p, q)$, that is, $f_a(\lambda, v, k) = \bar{m}_t p[(v - 1)^2 q - p] - \bar{m}_b[(v - 1)^2 p + (v - 1)^2 q - 2p(v - 1)] < 0$ for $\lambda \geq 1$ and $v \geq 4$, where $\bar{m} = \frac{\bar{m}_t}{\bar{m}_b} = \frac{2\lambda^2 v + 2\lambda v - 3\lambda + k\lambda + k - 1}{\lambda(\lambda v + k - 1)(\lambda v + v - 1)}$. Equivalently, $f_a(\lambda + 1, v + 4, k) < 0$ for $\lambda \geq 0$, $v \geq 0$. Using MAPLE, we found $f_a(\lambda, v, k)$ reduces to $-\lambda^3[2v^4 + 28v^3 + 142v^2 + 308v + 240] - \lambda^2[8v^4 + v^3(3k + 107) + v^2(k^2 + 32k + 565) + v(6k^2 + 111k + 1061) + (9k^2 + 126k + 797)] - \lambda[10v^4 + v^3(10k + 126 + v^2(4k^2 + 100k + 566) + v(24k^2 + 326k + 1090) + (36k^2 + 352k + 780))] - [4v^4 + v^3(7k + 7) + v^2(5k^2 + 64k + 195) + v(30k^2 + 191k + 349) + (45k^2 + 194k + 237)]$, which is a negative quantity, proving that (2.1) holds. Similarly condition (2.2) reduces to $f_b(\lambda, v, k) = [\bar{m}_b^2(v - 2)^2 - [(v - 3)^2 + 1](\bar{m}_b - \bar{m}_t p)^2 + [(v - 3)^2 + 2]\bar{m}_t(k - 1)(\bar{m}_b - \bar{m}_t p)]^2 - [2(\bar{m}_b - \bar{m}_t p)^2(v - 3)[\bar{m}_b - \bar{m}_t p - \bar{m}_t(k - 1)]^{1/2}[\bar{m}_b - \bar{m}_t p - 2\bar{m}_t(k - 1)]^{1/2}]^2 > 0$, which on using MAPLE, is found to be positive—the result is a straight forward but very messy expression, being a multivariate polynomial in λ , v , and k with degrees

12, 10 and 6 respectively. This same approach using MAPLE to expand polynomials in λ , v , and k verifies that (2.3) holds, and that (2.4) reduces to the conditions (2.5) and (2.6). Hence by Theorem 2.4, d^* is *MV*-optimal. \square

The conditions given by the inequalities of our theorem are not always met but are often so. For instance, they always hold if $k \leq 5$ or $k \geq (v - 5)$, if $k = 6$ for $\lambda \leq 3$, and for all $v \leq 12$.

EXAMPLE 10 Consider this design $d \in D(4, 5, 3)$:

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 & 2 \\ 3 & 4 & 4 & 4 & 3 \end{array}$$

This d is a *GGDD*(2) with $V_1 = \{1, 2, 3\}$ and $V_2 = \{4\}$, and for $s = 1$ all conditions of Theorem 2.5 are satisfied. Hence d is *MV*-optimal.

Chapter 3

Type-1 optimality of block designs

3.1 INTRODUCTION

This chapter deals with type-1 optimality criteria as defined in chapter 1. A number of results are already known regarding the type-1 optimality of block designs in $D(v, b, k)$. One example is the celebrated result that a *BIBD* is optimal in $D(v, b, k)$ under all type-1 optimality criteria. Various types of block designs which are not *BIBDs* have also been shown to be optimal under different type-1 criteria in a number of classes and subclasses of $D(v, b, k)$ where bk/v is an integer (see Kiefer (1975); Conniffe and Stone (1975); Shah, Ragavarao, and Khatri (1976); William, Patter-

son, and John (1977); Cheng (1978, 1979); and Jacroux (1985, 1989)). Jacroux (1985, 1991) has also derived some sufficient conditions which can be applied to establish the existence of type-1 optimal *NBBDs* in classes $D(v, b, k)$ where bk/v is not an integer. However, those designs which are type-1 optimal in a vast majority of classes $D(v, b, k)$ remain unknown.

In section 3.2 we define the class of generalized nearly balanced incomplete block designs with maximum concurrence range l , or *NBBD*(l). This class extends the nearly balanced incomplete block designs as defined by Cheng & Wu (1981), and the semi-regular graph designs as defined by Jacroux (1985), to cases where off-diagonal entries of the concurrence matrix differ by at most the positive integer l . A result related to the nonexistence of *NBBD*(1)s and an upper bound for a minimum eigenvalue of a particular class $D(v, b, k)$ are derived. In particular, they are used in section 3.3 to derive sufficient conditions for the optimality under a specific type-1 criterion of some particular types of *NBBD*(2)s having unequal numbers of replicates. Finally section 3.4 deals with an infinite series of *NBBD*(2)s having unequal numbers of replicates that satisfy the derived sufficient conditions for the *A*- and the *D*-optimality criteria. Also, a result from Jacroux (1985) is used to establish the existence of *A*-optimal *NBBD*(1)s for a new class of designs.

3.2 NON-EXISTENCE OF NBBD(1)s

DEFINITION 3.1 A generalized nearly balanced incomplete block design d with maximum concurrence range l , or $NBBD(l)$, with v varieties and b blocks of size k is an incomplete block design satisfying the following conditions:

- (i) each variety appears in each block at most once,
- (ii) each $r_{di} = r$ or $r + 1$, where $r = [bk/v]$ is the integral part of bk/v , and
- (iii) for each $i \neq i'$ and $j \neq j'$, $|\lambda_{dii'} - \lambda_{djj'}| \leq l$.

This definition of $NBBD(l)$ is a generalization of the definition given by Mitchell and John (1977) for a regular graph design (RGD) and Cheng & Wu (1981) for a nearly balanced incomplete block design ($NBBD$) or Jacroux (1985) for semi regular graph designs ($SRGD$). It reduces to the definition of an RGD if bk/v is an integer and $l = 1$, and to the definition of an $NBBD$ ($SRGD$) if $l = 1$.

For $l = 1$, if s is the number of varieties i with $r_{di} = r$, then

$$s = v - (bk - vr) \tag{3.1}$$

Note that $1 \leq s \leq v$ and, when $s = v$, the design is a regular graph design. It is easily seen from the definition that if $r_{di_0} = r$, then variety i_0 appears together with any other variety λ or $\lambda + 1$ times, where λ is the integral part of $r(k - 1)/(v - 1)$.

In this case, the number of varieties each of which appears together with variety i_0 just λ times is

$$n = v - 1 - \{(k - 1)r - \lambda(v - 1)\}. \quad (3.2)$$

The proof of the following lemma is given in Cheng & Wu (1981).

LEMMA 3.1 *For any given positive integers b, v, r, k with $k \geq 2$ and $k < v$, let s and n be defined by (3.1) and (3.2). If $n \geq k - 1$ and there exists a $NBBD(1)$ with v varieties and b blocks of size k , then*

$$s(n - s + 1) \leq (v - s)(n - k + 1). \quad (3.3)$$

The next theorem shows that for some sets of parameters, no $NBBD(1)$ exists.

THEOREM 3.1 *For any given positive integers b, v, r, k with $k \geq 2$ and $k < v$, such that $bk = vr + p$ for some $1 \leq p < k$, and $\lambda = r(k - 1)/(v - 1)$ is an integer, let s and n be defined by (3.1) and (3.2). If $n \geq k - 1$ then no $NBBD(1)$ exists for these parameters.*

PROOF Suppose for the given parameters a $NBBD(1)$ exists, so by lemma 3.1 the condition (3.3) has to be satisfied. Since $s = v - (bk - vr) = v - p$ and $n = (v - 1) - \{(k - 1)r - \lambda(v - 1)\} = v - 1$, condition (3.3) becomes

$$(v - p)p \leq p(v - k).$$

This implies that $k \leq p$, which is a contradiction. \square

In the following theorem we have generalized lemma 2 of Morgan & Uddin (1995) to obtain an upper bound for the minimum nonzero eigenvalue of a certain class of block designs.

THEOREM 3.2 *A binary block design d for v treatments in b blocks of size $2 \leq k < v$ such that $bk = vr + 1$ and $\lambda = r(k - 1)/(v - 1)$ is an integer has*

$$z_{d1} \leq \frac{(k - 1)r + \lambda - 1}{k}.$$

PROOF Suppose some treatment is replicated $r_{dp} < r$ times, then by using the fact that $v(k - 2) + 1 \geq 0$ and Theorem 3.1 from Jacroux (1980),

$$z_{d1} \leq \frac{v(k - 1)r_{dp}}{k(v - 1)} \leq \frac{v(k - 1)(r - 1)}{k(v - 1)} \leq \frac{(k - 1)r + \lambda - 1}{k}.$$

So assume $r_{d1} = r_{d2} = \dots = r_{d,v-1} = r$ and $r_{dv} = r + 1$. Since the design is binary there are $r + 1$ blocks containing the v th treatment. So the total number of ordered pairs containing the v th treatment is $(r + 1)(k - 1) = \lambda(v - 1) + (k - 1)$, which implies that there is at least one treatment, say i_0 , which occurs exactly $(\lambda + l)$ times in these blocks for some $1 \leq l \leq k - 1$. Thus $(\lambda + l)(k - 2)$ ordered pairs are possible in these $r + 1$ blocks, and $(r - \lambda - l)(k - 1)$ are in the other blocks, involving treatment i_0 and treatments other than v . So the total number of possible ordered pairs with treatment i_0 but not with treatment v , is

$$(\lambda + l)(k - 2) + (r - \lambda - l)(k - 1) < \lambda(v - 2),$$

which clearly shows that there exists at least one pair (i, i_0) for some $i \in \{1, 2, \dots, v - 1\}$, $i \neq i_0$, such that $\lambda_{dii_0} \leq \lambda - 1$. So we may assume without loss of generality that for $i_0 = 1$ and $i = 2$, $\lambda_{d12} \leq \lambda - 1$. Write $h' = (1, -1, 0, \dots, 0)$ and $T_{dx} = kC_d - x(I - (1/v)11')$. The spectral decomposition of T_{dx} is $\sum_{i=1}^{v-1} (kz_{di} - x)e_i e_i'$ where $e_i' 1 = 0$ for all i and the z_{di} are the eigenvalues of C_d . If there exists x such that $h' T_{dx} h \leq 0$ then certainly $z_{d1} \leq x/k$; $x = r(k-1) + \lambda_{d12}$ satisfies the inequality and the result is established. \square

3.3 TYPE-1 OPTIMALITY

In this section we apply the results derived in section 3.2 to derive a new method which can be used to establish the type-1 optimality of $NBBD(2)$ s in various classes $D(v, b, k)$ where bk/v is not an integer. First we state some preliminary results from Jacroux (1985) on minimization of type-1 optimality functions subject to various constraints, which are useful for deriving the main results of this chapter. So let $n \geq 3$ be an integer and let C and D be positive constants such that $C^2 \geq D \geq C^2/n$ and $C \leq \max_{d \in D(v,b,k)} \text{tr } C_d$. Now let $f(x)$ satisfy definition (1.1). We wish to find x_1, x_2, \dots, x_n which

$$\text{minimize } \sum_{i=1}^n f(x_i) \tag{3.4}$$

subject to the constraints

(i) $x_i \geq 0$ for $i = 1, 2, \dots, n$,

(ii) $\sum_{i=1}^n x_i = C$,

(iii) $\sum_{i=1}^n x_i^2 = D'$ for any D' such that $C^2 \geq D' \geq D$,

(iv) $x_i \leq F$ for some i and a number F such that

$$(a) F \leq (C - [n/(n-1)]^{1/2}P)/n \text{ where } P = [D - (C^2/n)]^{1/2}$$

$$(b) (C - F)^2 \geq D - F^2 \geq (C - F)^2/(n - 1).$$

It is easily seen that $C^2 \geq D' \geq C^2/n$ is the necessary and sufficient condition for the existence of n nonnegative numbers x_1, x_2, \dots, x_n such that $\sum_{i=1}^n x_i = C$ and $\sum_{i=1}^n x_i^2 = D'$. Thus (3.4)(iv)(b) assures that if some $x_i = F$, then there exists $n - 1$ other nonnegative numbers such that together with F they satisfy the rest of the conditions (3.4). Since the solutions to (3.4) subject to constraints (i)-(iv) are permutation invariant i.e. the order of the values of solutions for the x_i are irrelevant, we can assume without loss of generality that $0 \leq x_1 \leq x_2 \leq \dots \leq x_n$.

The following lemmas yield the solution to two related minimization problems.

The proof of these lemmas are given in Jacroux (1985).

LEMMA 3.2 *Let $P_F = [(D - F^2) - ((C - F)^2/(n - 1))]^{1/2}$. Then the solution to (3.4) subject to the constraints (i)-(iv) occurs when*

$$x_1 = F,$$

$$x_i = \{(C - F) - [(n - 1)/(n - 2)]^{1/2} P_F\} / (n - 1) \quad \text{for } i = 2, \dots, n - 1,$$

$$x_n = \{(C - F) + [(n - 1)(n - 2)]^{1/2} P_F\} / (n - 1).$$

Note that the solution in lemma 3.2 is found at a set of x_i 's for which $\sum_{i=1}^n x_i^2 = D$, that is, the quantity $\sum_{i=1}^n x_i^2$ is made as small as possible. As the bound D for $\sum_{i=1}^n x_i^2$ is made smaller, the solution for x_n moves to that of x_2, \dots, x_{n-1} . When the constraint is dropped altogether, one gets $x_2 = x_3 = \dots = x_n$, which is the solution found in lemma 3.3.

LEMMA 3.3 *The minimal value of the function given in (3.4) subject to constraints (i) and (iv) and the additional constraint that*

$$\sum_{i=1}^n x_i \leq C$$

occurs when

$$x_1 = F$$

$$x_i = (C - F) / (n - 1) \quad \text{for } i = 2, \dots, n.$$

We now apply these minimization results to derive sufficient conditions for the type-1 optimality of $NBBD(2)$ s within various classes $D(v, b, k)$. The ideas are very similar to those used by Jacroux (1985) in deriving sufficient conditions for his Theorem 3.2

and Theorem 3.7. We use upper bounds given in section (3.2), Jacroux (1980), and Constantine (1981) for z_{d1} corresponding to $d \in D(v, b, k)$ which are not $NBBD(2)$ s, and then apply Theorem 3.1 and lemmas 3.2 and 3.3 to show that such designs cannot be type-1 optimal. It is assumed in the rest of this section that $v \geq 4$ and r and λ denote the greatest integers not exceeding bk/v and $r(k-1)/(v-1)$ respectively. Let $\bar{d} \in D$ be a generalized $NBBD(l)$ and define the following quantities, used in the remainder of this chapter for $\bar{d} \in D$:

$$A = \text{tr } C_{\bar{d}},$$

$$B = \text{tr } C_{\bar{d}}^2 + \min(1/2; 4/k^2),$$

$$(A - m_1)^2 \geq B - m_1^2 \geq (A - m_1)^2/(v - 2), \quad \text{and}$$

$$(A - m_1^*)^2 \geq B - m_1^{*2} \geq (A - m_1^*)^2/(v - 2),$$

where m_1 and m_1^* are nonnegative constants representing the maximum upper bounds for nonzero minimum eigenvalues of designs in the subclasses of nonbinary and binary designs in $D(v, b, k)$, respectively. Let $m_4 = [A - (2/k) - m_1^*]/(v - 2)$ be a nonzero eigenvalue of completely symmetric matrix C^* with $\text{tr } C^* = A - (2/k) - m_1^*$. For $P_1 = [(B - m_1^2) - ((A - m_1)^2/(v - 2))]^{1/2}$ we defined the constants m_2 and m_3 as follows:

$$m_2 = \{(A - m_1) - [(v - 2)/(v - 3)]^{1/2} P_1\}/(v - 2)$$

$$m_3 = \{(A - m_1) + [(v - 2)(v - 3)]^{1/2} P_1\} / (v - 2).$$

The proof of the following theorem can be found in Jacroux (1985).

THEOREM 3.3 *Let $D(v, b, k)$ be such that bk/v is not an integer and let \bar{d} be an NBBD(1) with C-matrix $C_{\bar{d}}$ having nonzero eigenvalues $z_{\bar{d}1} \leq z_{\bar{d}2} \leq \dots \leq z_{\bar{d},v-1}$.*

Now let

$$m_1 = m_1^* = \frac{r(k-1)v}{(v-1)k}.$$

If $m_1 \leq m_2$, $m_1^* \leq m_4$ and

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) < \min\{f(m_1) + (v-3)f(m_2) + f(m_3), f(m_1^*) + (v-2)f(m_4)\},$$

then a type-1 optimal design in $D(v, b, k)$ must be an NBBD(1).

Now we set about deriving the main result of this section.

THEOREM 3.4 *Let $D(v, b, k)$ be such that $bk = vr + 1$ and $\lambda = r(k-1)/(v-1)$ is an integer, and let \bar{d} be a NBBD(2) with C-matrix $C_{\bar{d}}$ having nonzero eigenvalues $z_{\bar{d}1} \leq z_{\bar{d}2} \leq \dots \leq z_{\bar{d},v-1}$. Now let*

$$m_1 = \frac{r(k-1) + \lambda - 1}{k}$$

$$m_1^* = \frac{r(k-1)v}{(v-1)k}.$$

If $m_1 \leq m_2$, $m_1^* \leq m_4$ and

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) < \min\{f(m_1) + (v-3)f(m_2) + f(m_3), f(m_1^*) + (v-2)f(m_4)\}$$

Then a type-1 optimal design in $D(v, b, k)$ must be an $NBBD(2)$.

PROOF Suppose the conditions of the theorem hold and let $d \in D(v, b, k)$ have C -matrix C_d having nonzero eigenvalues $z_{d1} \leq z_{d2} \leq \dots \leq z_{dv}$. If $d \notin M(v, b, k)$ then N_d has $|n_{dip} - n_{djg}| > 1$ for some i, j and $p \neq q$, thus $\text{tr } C_d \leq \text{tr } C_{\bar{d}} - (2/k)$. Using this fact and the results of Jacroux (1980(b), Theorem 3.1) or Constantine (1981, Theorem 3.2), the z_{di} are seen to satisfy the following constraints:

$$(i) \quad z_{di} > 0, \text{ for } i = 1, \dots, v-1,$$

$$(ii) \quad \text{tr } C_d = \sum_{i=1}^{v-1} z_{di} \leq A - (2/k),$$

$$(iii) \quad z_{d1} \leq r(k-1)v/(v-1)k.$$

It is easy to observe that these constraints for the z_{di} are the exactly the same as those considered in lemma 3.3 for a corresponding set of x_i having $n = v-1$, $C = A - 2/k$, $D = (A - 2/k)^2/(v-1)$, and $F = (rv(k-1))/(v-1)k$. Thus lemma 3.3 is applicable, and we must have

$$\sum_{i=1}^{v-1} f(z_{di}) \geq f(m_1^*) + (v-2)f(m_4) > \sum_{i=1}^{v-1} f(z_{\bar{d}i}).$$

Hence, all type-1 optimal designs in $D(v, b, k)$ must be in $M(v, b, k)$. So now suppose d is in $M(v, b, k)$ but is not an $NBBD(2)$. Since by Theorem 3.1, no $NBBD(1)$ exists, it must be true that either $|r_{di} - r_{dj}| > 1$ for some $i \neq j$ or $|\lambda_{dij} - \lambda_{dkl}| > 2$ for

some fixed values of i , j , k , and l , $i \neq j, k \neq l$. In either case, we have $\text{tr } C_d^2 \geq B$.

Using this fact and Theorem 3.2, the z_{di} are seen to satisfy

$$(i) \quad z_{d1} > 0,$$

$$(ii) \quad \text{tr } C_d = \sum_{i=1}^{v-1} z_{di} = A,$$

$$(iii) \quad A^2 \geq \text{tr } C_d^2 = \sum_{i=1}^{v-1} z_{di}^2 \geq B \geq A^2/(v-1),$$

$$(iv) \quad z_{d1} \leq (r(k-1) + \lambda - 1)/k.$$

Observe now that the restrictions given above for the z_{di} are the same as those given in lemma 3.2 for a corresponding set of x_i with $n = v - 1$, $C = A$, $D = A^2/(v - 1)$, and $F = (r(k - 1) + \lambda - 1)/k$. Thus by lemma 3.2, we have that

$$\sum_{i=1}^{v-1} f(z_{di}) \geq f(m_1) + (v-3)f(m_2) + f(m_3) > \sum_{i=1}^{v-1} f(z_{\bar{d}i}).$$

and we have the result. \square

The following two useful remarks are translations to our setting of comments made by Jacroux (1985, pages 393-395) for his Theorem 3.2 and Theorem 3.7 (Theorem 3.3 of this chapter).

Firstly, finding a D -optimal design in $D(v, b, k)$ is equivalent to finding $d \in D(v, b, k)$ such that $\prod_{i=1}^{v-1} z_{di}$ is maximal. Thus from Theorem 3.4 it follows that if

there exists an $NBBD(2)$ $\bar{d} \in D(v, b, k)$ such that

$$\prod_{i=1}^{v-1} z_{\bar{d}_i} \geq \max\{m_1^* m_4^{(v-2)}, m_1 m_2^{(v-3)} m_3\}, \quad (3.5)$$

then a D -optimal design in $D(v, b, k)$ must be an $NBBD(2)$. Secondly, in many classes $D(v, b, k)$ where $NBBD(2)$ s exist, they are not unique. Thus Theorem 3.4 can only be used to reduce the search for type-1 optimal designs to the sets of $NBBD(2)$ s contained within their classes. However in certain situations, the C -matrices corresponding to $NBBD(2)$ s in $D(v, b, k)$ can all be put in exactly the same form. When this happens and the associated eigenvalues satisfy the conditions of (3.4) then the unique $NBBD(2)$ in $D(v, b, k)$ is type-1 optimal. For example it is easy to verify that the C -matrix of $NBBD(2)$ with $bk = vr + 1$ and $\lambda = r(k-1)/(v-1)$ being an integer, where $v = 3t+1$, $b = 3t^2+3t+1$, and $k = 3$, obtained from a completely symmetric design of Morgan & Uddin (1995) by replacing one copy of the the multiply-replicated treatment in a nonbinary block with any other treatment so that the nonbinary block becomes a binary, is unique.

EXAMPLE 11 Consider the design \bar{d} for $v = 8$ and $k = 3$ with blocks given by the $b = 19$ columns

6	8	8	8	8	8	8	7	7	7	1	2	3	1	2	4	5	6	7
8	1	3	5	1	2	4	1	2	3	4	3	4	2	3	5	6	7	1
7	2	4	6	3	5	6	6	4	5	5	6	6	4	5	7	1	2	3

The C -matrix of \bar{d} has $\text{tr } C_{\bar{d}} = 38$, $\text{tr } C_{\bar{d}}^2 + \min(1/2; 4/k^2) = 207.776$ and nonzero eigenvalues 5.3333, 6.3333, and 5 with multiplicities of 5, 1, and 1 respectively. The sum of reciprocals of these nonzeros eigen values is 1.2954. If m_1 , m^*_1 , m_2 , m_3 , and m_4 are computed as described in Theorem 3.4 it is seen that $m_1 = 5$, $m^*_1 = 5.3333$, $m_2 = 5.294$, $m_3 = 6.532$, $m_4 = 5.3333$, $(1/m^*) + (6/m_4) = 1.3125$ and $(1/m_1) + (5/m_2) + (1/m_3) = 1.2976$. It now follows again from Theorem 3.4 and the preceding remark that \bar{d} is A -optimal in $D(8, 19, 3)$.

EXAMPLE 12 Consider the $NBBD(2)$ \bar{d} having parameters $v=9$, $b=11$, and $k=5$ whose blocks are the columns

1	1	1	1	1	8	8	8	9	1	9
5	9	8	8	8	2	2	3	2	2	3
2	5	9	9	9	3	4	4	3	4	4
3	6	2	3	4	6	5	5	5	6	5
4	7	5	6	7	7	6	7	7	7	6

Then $\text{tr } C_{\bar{d}}=44$, $\text{tr } C_{\bar{d}}^2 + \min(1/2; 4/k^2) = 243.36109$, $\prod_{i=1}^8 z_{\bar{d}i} = 821789.9779$ and $\sum_{i=1}^{v-1} f(z_{\bar{d}i}) = 1.46120$. If m_1 , m^*_1 , m_3 , and m_4 are calculated as outlined in Theorem 3.4, we get $m_1 = 5.2$, $m^*_1 = 5.4$, $m_2 = 5.36977$, $m_3 = 6.58136$, $m_4 = 5.45714$. Since $m_1 \leq m_2$, $m^*_1 \leq m_4$, $m^*_1 m_4^7 = 749478.2124$, $m_1 m_2^6 m_3 = 820451.8457$, $(1/m_1) + (6/m_2) + (1/m_3) = 1.461624$ and $(1/m^*) + (7/m_4) = 1.467908$, it follows from Theorem 3.4 that A - or D -optimal in $D(9, 11, 5)$ must be an $NBBD(2)$.

3.4 APPLICATIONS

In this section we first use Theorem 3.4 to establish the A - and D -optimality of an infinite series of $NBBD(2)$ s in $D(v, b, k)$ for which $v = 3t + 1$, $b = 3t^2 + 3t + 1$, and $k = 3$. It is easy to see by using Theorem 3.1 that $NBBD(1)$ s do not exist for these parameters. The construction and uniqueness of C -matrices of the $NBBD(2)$ s for these parameters is shown in the remark preceding example (11). Let $\bar{d} \in D(v, b, k)$ be an $NBBD(2)$. The C -matrix of \bar{d} has $tr C_{\bar{d}} = 6t^2 + 6t + 2$, $tr C_{\bar{d}}^2 + \min(1/2; 4/k^2) = 12t^3 + 20t^2 + 40/3t + 46/9$ and nonzero eigenvalues $2t + 4/3$, $2t + 7/3$, and $2t + 1$ with multiplicities of $(3t - 1)$, 1, and 1 respectively.

For A -optimality, consider $f(x) = 1/x$ in Theorem 3.4, so that

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) = (108t^3 + 216t^2 + 111t + 19)/(72t^3 + 168t^2 + 122t + 28)$$

Computing m_1, m_1^*, m_2, m_3 , and m_4 as described in Theorem 3.4, it is seen that $m_1 = 2t + 1$, $m_1^* = 2t + 4/3$, $m_2 = 2t + 4/3 + (1/3t) - (1/3t)\sqrt{(13t - 3)/(9t - 3)}$, $m_3 = 2t + 4/3 + (1/3t) + (1/3t)\sqrt{(3t - 1)(13t - 3)/3}$ and $m_4 = 2t + 4/3$. Since $m_1 \leq m_2$, and $m_1^* \leq m_4$, it remain to show that

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) < \min\{f(m_1) + (v - 3)f(m_2) + f(m_3), f(m_1^*) + (v - 2)f(m_4)\}$$

where $f(m_1) + (v - 3)f(m_2) + f(m_3) = 1/(2t + 1) + (3t - 1)/(2t + 4/3 + (1/3t) - (1/3t)\sqrt{(13t - 3)/(9t - 3)}) + 1/(2t + 4/3 + (1/3t) + (1/3t)\sqrt{(13t - 3)(3t - 1)/3})$ and

$f(m_1^*) + (v-2)f(m_4) = (9t+3)/(6t+4)$. For this it is sufficient to show that

$$\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) < f(m_1) + (v-3)f(m_2) + f(m_3), \quad (3.6)$$

since trivial calculations establish that $\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) < (9t+3)/(6t+4)$. Now look at expression (3.6), which is equivalent to

$$\begin{aligned} 1/(2t+7/3) - 1/(2t+4/3 + (1/3t) + (1/3t)\sqrt{(13t-3)(3t-1)/3}) &\leq (3t-1) \\ [1/(2t+4/3 - 1/3t\sqrt{(13t-3)/(9t-3)}) - 1/(2t+4/3)] & \end{aligned}$$

that is,

$$\begin{aligned} 1 \leq (1 + 1/(2t+4/3))(1 + ((1/3t)\sqrt{(13t-3)(3t-1)/3} + \\ (1/3t)\sqrt{(13t-3)/(9t-3)})/((2t+4/3 + (1/3t) - (1/3t)\sqrt{(13t-3)/(9t-3)})) \end{aligned}$$

which is clearly true for all $t \geq 1$. Hence it follows from Theorem 3.4 and uniqueness of C -matrices, these designs are in fact A -optimal in $D(3t+2, 3t^2+3t+1, 3)$.

To show \bar{d} is also D -optimal it will be enough to establish that

$$\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) < \min\{f(m_1) + (v-3)f(m_2) + f(m_3), f(m_1^*) + (v-2)f(m_4)\}$$

for $f(x) = -\log(x)$, where $\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) = -\log((2t+7/3)(2t+1)(2t+4/3)^{(3t-1)})$,

$f(m_1) + (v-3)f(m_2) + f(m_3) = -\log((2t+1)(2t+4/3 + (1/3t) - (1/3t)\sqrt{\frac{13t-3}{9t-3}})^{(3t-1)}(2t+4/3 + (1/3t) - (1/3t)\sqrt{(13t-3)(3t-1)/3}))$, and $f(m_1^*) + (v-2)f(m_4) = -\log(2t +$

$4/3)^{(3t+1)}$. It is easy to verify that $\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) \leq f(m_1^*) + (v-2)f(m_4)$. To show $\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) \leq f(m_1) + (v-3)f(m_2) + f(m_3)$ is equivalent to proving the following inequality:

$$(3t-1)\log\left[\frac{1}{1 - \frac{(1/3t)(\sqrt{(13t-3)/(9t-3)}-1)}{(2t+4/3)}}}\right] - \log\left[\frac{1}{1 - \frac{(1/3t)(\sqrt{(13t-3)(3t-1)/3+1-3t})}{(2t+4/3+(1/3t)+(1/3t)\sqrt{(13t-3)(3t-1)/3})}}}\right] > 0$$

which on rewriting is

$$\log(1 - B) - (3t - 1)\log(1 - A) > 0. \quad (3.7)$$

where $A = \frac{(1/3t)(\sqrt{(13t-3)/(9t-3)}-1)}{(2t+4/3)}$ and $B = \frac{(1/3t)(\sqrt{(13t-3)(3t-1)/3+1-3t})}{(2t+4/3+(1/3t)+(1/3t)\sqrt{(13t-3)(3t-1)/3})}$. Observe that $0 < A < 1$ and $0 < B < 1$ for all $t \geq 1$. Thus for showing 3.7 it is sufficient that for some constant A^* and B^* satisfying $A^* < A$ and $B^* > B$ that

$$\log(1 - B^*) - (3t - 1)\log(1 - A^*) > 0.$$

Let $A^*(t) = \frac{a}{2t(3t+2)}$ where $a = \sqrt{13/9} - 1$ is a constant and $B^*(t) = (3t - 1.5)A^*$. Define $h(t) = \log(1 - B^*) - (3t - 1)\log(1 - A^*)$. Calculations for showing $A^* < A$ and $B^* > B$, $t \geq 5$ are straight forward.

Now left to show that $h(t) > 0$ for all $t \geq 5$. This fact will be established by showing

(i) $\lim_{t \rightarrow \infty} h(t) = 0$,

(ii) $h'(t) < 0$ for all $t \geq 5$.

Showing (i) is easy. For (ii) we have explicitly calculated the first derivative of the $h(t)$ whose numerator and denominator terms are respectively $-a(t-1)[12(36-54a)t^4 + (144+424a)t^3 + (3112+1002a+54a^2)t^2 + (232+786a-13a^2)t + (232+692a-89a^2+9a^2)] - a(232+692a-65a^2+4.5a^2)$ and $2(6t^2+4t)(6t^2+4t-a)[36t^4 + (48-18a)t^3 + (16-9a)t^2 + (6-4a+3a^2)t - 1.5a^2]$. Clearly we can see that the numerator is always negative and the denominator is always positive for $t \geq 5$. Hence the result.

Finally in this section we shall use Theorem 3.3 to establish the existence of A -optimality of $NBBD(1)$ s in $D(v, b, k)$ for which $v = 3t + 2$, $b = 6t^2 + 6t + 2$, and $k = 3$. Construction of $NBBD(1)$ designs for these parameters is as follows:

Consider the CSD with $v = 3t + 2$, $b = 3t^2 + 3t + 1$, and $k = 3$ as given by Morgan & Uddin (1995). Add the block $(\infty_2 \ 1 \ 2)'$ and replace the blocks $(\infty_1 \ \infty_1 \ \infty_2)'$ and $(\infty_2 \ 1 \ 2)'$ by the blocks $(\infty_1 \ 2 \ \infty_2)'$ and $(\infty_1 \ 1 \ \infty_2)'$. Then it is easy to verify that these designs are $NBBD(1)$ s and that the C -matrices of these designs are unique. Let \bar{d} be one of these designs. Then the C -matrix of \bar{d} has $tr C_{\bar{d}} = 6t^2 + 6t + 4$, $tr C_{\bar{d}}^2 + \min(1/2; 4/k^2) = 12t^3 + 20t^2 + 64/3t + 12$ and nonzero eigenvalues $2t + 4/3$, $2t + 2$, and $2t + 8/3$ with multiplicities of $(3t - 2)$, 2 , and 1 respectively. For A -optimality, consider $f(x) = 1/x$ in Theorem 3.3, so that $\sum_{i=1}^{v-1} f(z_{\bar{d}i}) = (27t^3 + 72t^2 + 45t - 2)/(2(3t + 2)(t + 1)(3t + 4))$. The values m_1 , m^*_1 , m_2 , m_3 , and m_4

as described in Theorem 3.3 are $m_1 = m_1^* = 2t + 4/3$, $m_2 = 2t + 4/3 + (8/9t) - (1/\sqrt{3t(3t-1)})\sqrt{\frac{84t-64}{27t}}$, $m_3 = 2t + 4/3 + (8/9t) + (2/9t)\sqrt{(3t-1)(21t-16)}$ and $m_4 = 2t + 4/3 + (2/3t)$. It is easy to verify that $m_1 \leq m_2$, $m_1^* \leq m_4$, and

$$\sum_{i=1}^{v-1} f(z_{\bar{d}_i}) < \min\{f(m_1) + (v-3)f(m_2) + f(m_3), f(m_1^*) + (v-2)f(m_4)\}$$

where $f(m_1) + (v-3)f(m_2) + f(m_3) = \frac{1}{2t+4/3} + \frac{3t-1}{2t+4/3+(8/9t)-(2/9t)\sqrt{(21t-16)/(3t-1)}} + \frac{1}{2t+4/3+(8/9t)+(2/9t)\sqrt{(3t-1)(21t-16)}}$ and $f(m_1^*) + (v-2)f(m_4) = \frac{1}{2t+4/3} + \frac{3t}{2t+4/3+(2/3t)}$.

Hence, it follows from Theorem 3.3 that \bar{d} is an A -optimal design in $D(v, b, k)$

Chapter 4

Generalized Group Divisible

Designs: Existence and

Construction results

4.1 INTRODUCTION

In chapter 2 we defined the $GGDD(s)$ s with s groups and discussed their E - and MV -optimality. This chapter is devoted to methods of construction and existence of $GGDD$ s which are often E - and MV -optimal. The parameters v , b , k , r , and λ , where r and λ are $\text{int}(bk/v)$ and $\text{int}(r(k-1)/(v-1))$ respectively, for any block

design satisfy the following relations:

$$bk = vr + p \quad \text{and} \quad r(k - 1) = \lambda(v - 1) + q \quad (4.1)$$

for some $0 \leq p, q \leq v - 1$, where p is the number of plots available for use in the particular design setting over and above the $bk - p$ that would be used to replicate each treatment with equal frequency r . It is assumed throughout this chapter that $q = 0$, that is, $\lambda = r(k - 1)/(v - 1)$ is an integer. All the constructions are chosen such that the c_{dii} 's are at least equal to $c = r(k - 1)/k$. Section 4.2 discusses the construction of the most useful classes of such $GGDD(1)$ s, or CSD s, for $p = 1, 2, 3$ and $k = 3$. Necessary conditions for the existence of these designs are found to be sufficient. Several methods for constructing $GGDD(2)$ s are given in section 4.3. It is shown that no $GGDD(2)$ satisfying the above conditions for $p = 1$ or 2 and $k = 3$ exists. All of the designs constructed in section 4.2 and 4.3 satisfy the E -optimality conditions of Theorem 2.1, while only the designs with $p = 3$ for four and five treatments and with $p = 4$ for five and six treatments fail the additional condition of Theorem 2.3 for MV -optimality. Moreover, depending on the design parameters, other E - and/or MV -optimal designs can be found, by appending blocks to, or deleting blocks from, CSD s and $GGDD(2)$ s, using the many different augmentation theorems from chapter 2. Finally in section 4.4 some other constructions of CSD s are given. A $BIBD$ or CSD with parameters (v, b, r, k, λ) is usually referred to as

$BIBD(v, k, \lambda)$ or $CSD(v, k, \lambda)$, respectively.

4.2 CONSTRUCTION AND EXISTENCE OF CSDs WITH BLOCKSIZE 3

The subsections that follow cover the constructions of $CSDs$ with $k = 3$ for $p = 1$, $p = 2$, and $p = 3$. The maximum conceivable trace for a CSD is $\frac{vr(k-1)}{k}$, and the necessary conditions for the existence of a CSD with this trace from (4.1) are

$$2|\lambda(v-1) \quad \text{and} \quad 6|[\lambda v(v-1) + 2p]. \quad (4.2)$$

Conditions (4.2) implies that $p \equiv 0 \pmod{3}$ whenever $v \not\equiv 2 \pmod{3}$. Thus to cover all v , one must proceed at least to $p = 3$. All blocks are given as columns of 3.

4.2.1 Settings with $p = 1$

For $p = 1$ the necessary conditions (4.2) lead to the two parameter series

$$(I) \quad v = 3t + 2 \quad r = (3m + 1)(v - 1) \quad \lambda = 6m + 2 \quad t \geq 1 \quad m \geq 0$$

$$(II) \quad v = 6t + 5 \quad r = (6m + 5)\frac{(v-1)}{2} \quad \lambda = 6m + 5 \quad t \geq 0 \quad m \geq 0.$$

The general solutions for these series are found by adding m copies of a $BIBD(v, 3, 6)$ to the designs with $m = 0$. Since $BIBD(v, 3, 6)$ exists for all $v > 3$ (Hanani, 1961; the same reference is relevant for all $BIBDs$ mentioned in this and the next sec-

tion), one need only solve the problem for $m = 0$. Series (I) with $m = 0$ is solved in Morgan and Uddin (1995), and so series (I) is completely solved.

A series (II) design for $v = 5$ and $m = 0$ is displayed as example 13.

EXAMPLE 13 A $CSD(5,3,5)$.

1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	3
1	2	2	2	3	3	4	3	3	4	3	3	4	3	3	4	4
2	3	4	5	4	5	5	4	5	5	4	5	5	4	5	5	5

To construct a $CSD(11,3,5)$, start with the blocks of example 13. To these, append five copies of a $BIBD(6,2,1)$ on the treatment symbols $\{6,7,8,9,10,11\}$. Finally, for $i = 1, (1), 5$, add treatment symbol i to every block in the i^{th} copy of the $BIBD(6,2,1)$.

For $t \geq 2$, $CSD(6t+5,3,5)$ can be constructed by using Bose's (1939) method of differences, embedding $CSD(5,3,5)$ in a larger design generated from initial blocks ($\text{mod } 6t$). First, form the design of example 13 on the symbols $\infty_1, \infty_2, \dots, \infty_5$, which are invariant to the group operation. To these, add the $15t$ blocks generated by the initial blocks

∞_1	∞_2	∞_3	∞_4	∞_5
0	0	0	0	0
$3t$	$3t$	$3t$	$3t$	$3t$

each taken through a $\frac{1}{2}$ -cycle. Another $30t(t-1)$ blocks are generated from the

$5(t - 1)$ initial blocks

$$\begin{array}{cccc}
 0 & 0 & 0 & 0 \\
 3t + i & i & 2i & 2i - 1 \\
 3t - i & -i - 1 & 3t + i & t + i + 1 \\
 i = 1, (1), \text{int}(\frac{3t-1}{2}) & i = 1, (1), \text{int}(\frac{3t-2}{2}) & i = 1, (1), t - 2 & i = 1, (1), t - 1 \\
 i = 1, (1), \text{int}(\frac{3t-1}{2}) & i = 1, (1), \text{int}(\frac{3t-2}{2}) & i = 1, (1), t - 2 & i = 1, (1), t - 1
 \end{array}$$

The remaining $60t$ blocks are generated from the ten initial blocks

$$\begin{array}{cccccccccc}
 \infty_1 & \infty_1 & \infty_2 & \infty_2 & \infty_3 & \infty_3 & \infty_4 & \infty_4 & \infty_5 & \infty_5 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 2 & 1 & 2t - 2 & 1 & 2t - 1 & 2t & 2t + 1 & 2t + 1 & \text{int}(\frac{3t}{2})
 \end{array}$$

Here, as elsewhere in the chapter, the tedious routine of checking that the differences are correct is omitted.

These constructions for series I and II establish the following result.

THEOREM 4.1 *A maximal trace CSD for $k = 3$ and $p = 1$ exists if and only if*

$$\lambda(v - 1) \equiv 0 \pmod{2} \text{ and } \lambda v(v - 1) \equiv 4 \pmod{6}.$$

4.2.2 Settings with $p = 2$

For $p = 2$ the necessary conditions (4.2) lead to the two parameter series

$$\begin{array}{ll}
 \text{(III)} & v = 3t + 2 \quad r = (3m - 1)(v - 1) \quad \lambda = 6m - 2 \quad t \geq 1 \quad m \geq 1 \\
 \text{(IV)} & v = 6t + 5 \quad r = (6m + 1)\frac{(v-1)}{2} \quad \lambda = 6m + 1 \quad t \geq 0 \quad m \geq 1.
 \end{array}$$

The solutions are simple. For series (III), use two copies of a $CSD(v, 3, 2)$ from section 4.2.1, with $m - 1$ copies of a $BIBD(v, 3, 6)$. This construction replicates one treatment $(3m - 1)(v - 1) + 2$ times, and the others $(3m - 1)(v - 1)$ times. Though

it makes no difference for the bottom stratum analysis, the analysis with recovery of interblock information is more efficient if the treatment replication numbers are made as equal as possible. From the mathematician's, as opposed to the statistician's, viewpoint, having treatments replicated as equally as possible is also more desirable, in that it makes for a more elegant combinatorial structure. To do this, the first and second copies of $CSD(v, 3, 2)$ should have different treatments with the excess replication of $r + 1$. However, the statistician must also keep in mind that for the full analysis, these two CSD s admit different decompositions of the treatment space, so that treatment structure could potentially dictate the choice.

For series (IV), combine a $CSD(v, 3, 2)$ and a $CSD(v, 3, 5)$, both from section 4.2.1, with $m - 1$ copies of a $BIBD(v, 3, 6)$. The comments of the previous paragraph concerning replication numbers, efficiency in the full analysis, and the general balance treatment decomposition, apply here also. The two CSD s used in the construction can be chosen to have the same, or two different, treatments with excess replication. Having constructed series III and IV, we can write the following theorem.

THEOREM 4.2 *A maximal trace CSD for $k = 3$ and $p = 2$ exists if and only if*

$$\lambda(v - 1) \equiv 0 \pmod{2} \text{ and } \lambda v(v - 1) \equiv 2 \pmod{6}.$$

4.2.3 Settings with $p = 3$

For $p = 3$ the necessary conditions (4.2) allow the possibility of CSDs for any $v > 3$.

Six distinct parameter series can be written, depending on the ($\text{mod } 6$) value of v :

$$\begin{array}{llllll}
 \text{(V)} & v = 6t & r = m(v-1) & \lambda = 2m & t \geq 1 & m \geq 1 \\
 \text{(VI)} & v = 6t + 1 & r = \frac{m(v-1)}{2} & \lambda = m & t \geq 1 & m \geq 2 \\
 \text{(VII)} & v = 6t + 2 & r = 3(v-1) & \lambda = 6m & t \geq 1 & m \geq 1 \\
 \text{(VIII)} & v = 6t + 3 & r = \frac{m(v-1)}{2} & \lambda = m & t \geq 1 & m \geq 2 \\
 \text{(IX)} & v = 6t + 4 & r = v-1 & \lambda = 2m & t \geq 0 & m \geq 1 \\
 \text{(X)} & v = 6t + 5 & r = \frac{3m(v-1)}{2} & \lambda = 3m & t \geq 0 & m \geq 1
 \end{array}$$

These designs can generally be built with any of several different patterns for the replication numbers (see example 14). Here they will all be constructed with the replication numbers r_{di} as equal as possible. All of the previously discussed issues regarding the full analysis apply here also. Each series will be constructed with the smallest possible m . Larger values of m are obtained by adjoining the appropriate *BIBD*.

EXAMPLE 14 Four $CSD(6, 3, 2)$'s, arranged in order of increasing dispersion of the r_{di} 's.

$$\begin{array}{cccccccc}
 1 & 2 & 3 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 4 \\
 1 & 2 & 3 & 2 & 2 & 3 & 3 & 3 & 3 & 5 & 5 \\
 4 & 5 & 6 & 6 & 6 & 5 & 5 & 4 & 4 & 6 & 6 \\
 \\
 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
 1 & 1 & 2 & 2 & 2 & 5 & 4 & 4 & 4 & 4 & 5 \\
 3 & 4 & 3 & 5 & 6 & 6 & 5 & 6 & 5 & 6 & 6
 \end{array}$$

1 1 1 1 1 2 2 3 2 2 3
 1 1 1 5 5 3 4 4 3 4 4
 2 3 4 6 6 5 5 5 6 6 6
 1 1 1 1 1 1 2 2 2 3 4
 1 2 2 3 3 4 3 3 5 4 5
 1 4 5 5 6 6 4 6 6 5 6

Series (V) for $v = 6$ is covered by example 14, so assume $t \geq 2$. The designs will be constructed using the integers ($\text{mod } 6t - 3$) with the three invariants $\infty_1, \infty_2, \infty_3$.

There are three blocks

$$\begin{array}{ccc}
 \infty_1 & \infty_2 & \infty_3 \\
 \infty_1 & \infty_2 & \infty_3 \\
 \infty_2 & \infty_3 & \infty_1
 \end{array} \tag{4.3}$$

and another $2(3t + 1)(2t - 1)$ blocks generated by the $2t + 1$ initial blocks

$$\begin{array}{ccccccc}
 \infty_1 & \infty_2 & \infty_3 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 3t - i - 1 & t + 1 - i & 2t - 2 & 2t - 1 \\
 1 & 2t - 3 & 3t - 2 & 3t + i - 1 & t + i & 2t & 4t - 2 \\
 & & & i = 1, (1), t - 2 & i = 1, (1), t - 2 & &
 \end{array}$$

the last of which is taken through a $\frac{2}{3}$ -cycle. To these blocks, add $(m - 1)$ $BIBD(v, 3, 2)$'s.

For series (VI), start with two copies of a $BIBD(v, 3, 1)$. Delete both occurrences of any one block, say $(1\ 2\ 3)'$, and replace those two blocks by the three blocks $1\ 2\ 3$, $1\ 2\ 3$, $2\ 3\ 1$. The result is $CSD(v, 3, 2)$, to which $m - 2$ copies of a $BIBD(v, 3, 1)$ may be appended. The same technique solves series (VIII), since $BIBD(v, 3, 1)$ also exists for $v = 6t + 3$.

Series (VII) for $m = 1$ is constructed with three $CSD(v, 3, 2)$'s from section 4.2.1, each having a different treatment replicated nonbinarily. To this basic design,

$(m - 1)$ copies of a $BIBD(v, 3, 6)$ can be added.

The construction for series (IX) is similar to that of series (V). Three invariants are used with the integers $(\text{mod } 6t + 1)$. For $m = 1$, begin with the three blocks (4.3), and to these add the blocks generated by the $2(t + 1)$ initial blocks

$$\begin{array}{cccccc} \infty_1 & \infty_2 & \infty_3 & 0 & & 0 \\ 0 & 0 & 0 & 3t + 1 - i & & t + 1 - i \\ 1 & 2t - 1 & 3t & 3t + 1 + i & & t + i \\ & & & i = 1, (1), t & & i = 1, (1), t - 1 \end{array}$$

Series (X) is also constructed using three invariants, and now the initial blocks are $(\text{mod } 6t + 2)$. The blocks involving the invariants are (4.3) and, if t is even, the six initial blocks

$$\begin{array}{cccccc} \infty_1 & \infty_2 & \infty_3 & \infty_1 & \infty_2 & \infty_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3t + 1 & 3t + 1 & 3t + 1 & 2t & 3t & \frac{5t}{2} \end{array}$$

or if t is odd, the six initial blocks

$$\begin{array}{cccccc} \infty_1 & \infty_2 & \infty_3 & \infty_1 & \infty_2 & \infty_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3t + 1 & 3t + 1 & 3t + 1 & 2t & 3t & \frac{3t-1}{2} \end{array}$$

In either case, the first three of the initial blocks are developed through a $\frac{1}{2}$ -cycle.

The remaining $3t - 1$ initial blocks are

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ t + 1 - i & 3t + 1 - i & t - 1 + i & 2t + i \\ t + i & 3t + 1 + i & 2t - i & 3t - i \\ i = 1, (1), t & i = 1, (1), t & i = 1, (1), \text{int}(\frac{t}{2}) & i = 1, (1), \text{int}(\frac{t-1}{2}) \end{array}$$

The following theorem is immediate.

THEOREM 4.3 *A maximal trace CSD for $k = 3$ and $p = 3$ exists if and only if*

$$\lambda(v - 1) \equiv 0 \pmod{2} \text{ and } \lambda v(v - 1) \equiv 0 \pmod{6}.$$

4.3 CONSTRUCTION OF GGDD(2)S

In this section the nonexistence of $GGDD(2)$'s satisfying the conditions laid out in section 4.1 for $p = 1$ or 2 and $k = 3$ is shown. Constructions of certain $GGDD(2)$'s for $k = 3$ with $p = 3, 4$ are also given with all c_{dii} values of each constructed design being at least $r(k - 1)/k = c$.

It is easy to see from (4.1) that preliminary necessary conditions for the existence of $GGDD(2)$'s are as given in (4.2). We remind the reader that we are considering only settings with $q = 0$, and seek only designs with $\min_i\{c_{dii}\} \geq c$. The restriction on $\min_i\{c_{dii}\}$ is required by the optimality results of chapter 2. Indeed, within the range of parameters allowed by (4.2), we will typically seek only $GGDD(2)$ s that meet one of the sets of optimality conditions derived earlier. These designs can also be built with several different patterns for the replication numbers, similar to subsection 4.2.3. Here all designs will be constructed with the replication numbers r_{di} as equal as possible. All of the previously discussed issues in section 4.2 regarding the full analysis apply in this section also, since these different $GGDD(2)$ s admit different decompositions of the treatment space.

4.3.1 Settings with $p = 1$

In this case the necessary conditions (4.2) lead to the same series of parameters as in subsection (4.2.1). Thus to construct $GGDD(2)$ s for these series we need to divide the treatments into two groups. Since $p = 1$, $v - 1$ treatments must be replicated r times each and occur in the same group. Therefore only one treatment, v (say), can occur in the other group, and it must occur binarily in all blocks, for otherwise $c_{dvv} \leq c$. Now it is easy to verify by counting concurrences that no binary $GGDD(2)$ is possible with $k = 3$, $p = 1$, $q = 0$, and so no $GGDD(2)$ meeting our conditions exists for this case.

4.3.2 Settings with $p = 2$

For $p = 2$ the necessary conditions (4.2) also lead to the same series of parameters as in subsection (4.2.2). Thus to construct $GGDD(2)$ s for these series again we need to divide the treatments into two groups. Since $p = 2$, at least $v - 2$ treatments must be replicated r times each and occur in the same group. Therefore at most two treatments can occur in the other group. If exactly two treatments occur in that group, each treatment must occur binarily in blocks, for otherwise some $c_{dii} \leq c$. Now it is easy to verify by counting that no binary $GGDD(2)$ exists in this case. When only one treatment v (say) occurs in the other group, it must occur $r + 2$

times, and its $c_{dvv} > c$ implies that exactly two of its $r + 2$ replicates must occur in the same block, for otherwise either $c_{dvv} \leq c$, or the design is binary in which case a $GGDD(2)$ with $k = 3$, $p = 2$, $q = 0$ is known not to exist. On again using elementary counting of concurrences, no $GGDD(2)$ exists for the nonbinary case. Hence the result.

4.3.3 Settings with $p = 3$

For $p = 3$ the necessary condition (4.2) allows the following six possibilities of $GGDD(2)$'s for $v \geq 4$. These six parameter series are the same as mentioned in (4.2.3), but in a different format:

$$\begin{array}{llllll}
 \text{(I)} & v = 3t & r = m(v - 1) & \lambda = 2m & t \geq 2 & m \geq 1 \\
 \text{(II)} & v = 3t + 1 & r = m(v - 1) & \lambda = 2m & t \geq 1 & m \geq 1 \\
 \text{(III)} & v = 3t + 2 & r = 3m(v - 1) & \lambda = 6m & t \geq 1 & m \geq 1 \\
 \text{(IV)} & v = 6t + 1 & r = (2m + 1)\frac{(v-1)}{2} & \lambda = 2m + 1 & t \geq 1 & m \geq 1 \\
 \text{(V)} & v = 6t + 3 & r = (2m + 1)\frac{(v-1)}{2} & \lambda = 2m + 1 & t \geq 1 & m \geq 1 \\
 \text{(VI)} & v = 6t + 5 & r = (12m + 3)\frac{(v-1)}{2} & \lambda = 12m + 3 & t \geq 1 & m \geq 1
 \end{array}$$

We shall display the constructions of $GGDD(2)$ s, for certain sets of γ values, for each of these series of parameters. All of the designs will satisfy the E - and MV -optimality (except for $v = 4$ or 5 and $p = 3$) conditions of theorems (2.1) and (2.3). Construction (I): For this series of parameters take $\gamma_{11} = 2(m - 1) + 4$ and $\gamma_{12} = \gamma_{22} = 2(m - 1) + 2$. To construct these designs it is sufficient to construct $GGDD(2)$ s with $\gamma_{11} = 4$ and $\gamma_{12} = \gamma_{22} = 2$ for $m = 1$, since then the general series of designs

can be obtained just by adding $(m - 1)$ copies of $BIBD(3t, 3, 2)$.

The construction of $GGDD(2)$ with parameters

$$v = 3t, \quad b = 3t^2 - t + 1, \quad r = 3t - 1, \quad \gamma_{11} = 4, \quad \text{and} \quad \gamma_{12} = \gamma_{22} = 2 \quad (4.4)$$

can be accomplished via a symmetric idempotent Latin square of order v . First we shall define a quasigroup (see Street & Street, 1987).

DEFINITION 4.1 *A set Q which is closed under an operation \star and has left and right cancellation laws as follows:*

$$aa_1 = aa_2 \quad \text{implies that} \quad a_1 = a_2$$

and

$$a_1a = a_2a \quad \text{implies that} \quad a_1 = a_2,$$

is a quasigroup under \star , denoted by (Q, \star) .

As we know a Latin square L_n of order n in which $l_{ii} = i, i = 1, (1), n$ is said to be *idempotent*. It is also a well known fact that a Latin square which is idempotent and symmetric must be of odd order. A more detailed account on the various kinds of Latin squares and their properties can be found in Street & Street (1987, pages 109-125).

Now divide the series (4.4) into two cases as t is odd or even.

Case 4.4.(a): $v = 6s + 3, b = 12s^2 + 10s + 3, r = 6s + 2, \gamma_{11} = 4, \gamma_{12} = 2 = \gamma_{22}$. Let

L be the symmetric idempotent Latin square of order $(2s + 1)$ given by $L = (l_{ij})$ where $l_{ij} = (s + 1)(i + j) \pmod{2s + 1}$, $1 \leq l_{ij} \leq 2s + 1$. Notice here that $2(s + 1) \equiv 1 \pmod{2}$, so that $s + 1 = 2^{-1}$ in the ring Z_{2s+1} . In other words, we are effectively finding l_{ij} by averaging i and j in Z_{2s+1} . Thus L defines a symmetric, idempotent quasigroup (Q, \star) where $Q = \{1, 2, \dots, 2s + 1\}$. For each x , $1 \leq x \leq 2s + 1$, take three symbols x_i , $i = 1, 2, 3$. For $s \geq 1$, consider these blocks of a $BIBD(6s + 3, 3, 2)$ design (Stinson & Wallis, 1983a):

$$\begin{array}{cc} x_1 & x_1 \\ x_2 & x_2 \\ x_3 & (x + 1)_3 \end{array}$$

for $1 \leq x \leq 2s + 1$, and

$$\begin{array}{cccccc} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 \\ (x \star y)_2 & ((x \star y) + 1)_2 & (x \star y)_3 & (x \star y)_1 & (x \star y)_1 & ((x \star y) - 1)_3 \end{array}$$

for $1 \leq x < y \leq 2s + 1$.

All addition and subtraction is taken modulo $2s + 1$. To get the required series, for some fixed $z \in \{1, 2, \dots, s + 1\}$, add one more block $(z_2 \ z_3 \ (z + 1)_3)'$ and replace $(z_1 \ z_2 \ z_3)'$ and $(z_1 \ z_2 \ (z + 1)_3)'$ by $(z_1 \ z_2 \ z_2)'$ and $(z_1 \ z_3 \ (z + 1)_3)'$. It is easy to verify that modified design with $V_1 = \{z_3, (z + 1)_3\}$ and $V_2 = \{\text{all other treatments except } z_3 \text{ and } (z + 1)_3\}$ is the required design.

EXAMPLE 15 Take $s = 1$ so that $t = 3$ and $v = 9$. The idempotent Latin square

matrix is

$$\begin{array}{ccc} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{array}$$

The resulting $BIBD(9, 3, 2)$ is

$$\begin{array}{cccccccccccc} 1_1 & 1_1 & 1_1 & 2_1 & 1_1 & 1_1 & 1_1 & 2_1 & 2_1 & 1_2 & 1_2 & 2_2 \\ 1_2 & 2_1 & 3_1 & 3_1 & 1_2 & 2_1 & 3_1 & 3_1 & 2_2 & 2_2 & 3_2 & 3_2 \\ 1_3 & 3_2 & 2_2 & 1_2 & 2_3 & 1_3 & 3_3 & 2_3 & 2_3 & 3_3 & 2_3 & 1_3 \\ \\ 2_1 & 1_2 & 1_2 & 2_2 & 3_1 & 1_3 & 1_3 & 2_3 & 3_1 & 1_3 & 1_3 & 2_3 \\ 2_2 & 2_2 & 3_2 & 3_2 & 3_2 & 2_3 & 3_3 & 3_3 & 3_2 & 2_3 & 3_3 & 3_3 \\ 3_3 & 3_1 & 2_1 & 1_1 & 3_3 & 3_1 & 2_1 & 1_1 & 1_3 & 2_2 & 1_2 & 3_2 \end{array}$$

Now add $(1_2 \ 1_3 \ 2_3)'$ and replace $(1_1 \ 1_2 \ 1_3)'$ and $(1_1 \ 1_2 \ 2_3)'$ by $(1_1 \ 1_2 \ 1_2)'$ and $(1_1 \ 1_3 \ 2_3)'$.

Also if we assign $1_1 \rightarrow 1, 1_2 \rightarrow 2, 1_3 \rightarrow 3, 2_1 \rightarrow 4, 2_2 \rightarrow 5, 2_3 \rightarrow 6, 3_1 \rightarrow 7, 3_2 \rightarrow 8,$

$3_3 \rightarrow 9$, we have the required design:

$$\begin{array}{cccccccccccccccccccc} 2 & 1 & 1 & 4 & 4 & 7 & 7 & 1 & 1 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ 3 & 2 & 3 & 5 & 5 & 8 & 8 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 7 & 7 & 8 & 8 & 9 & 9 \\ 6 & 2 & 6 & 6 & 9 & 9 & 3 & 8 & 3 & 9 & 7 & 7 & 5 & 5 & 9 & 6 & 4 & 4 & 2 & 2 & 6 & 3 & 1 & 1 & 8 \end{array}$$

Case 4.4.(b): $v = 6s, b = 2s(6s - 1) + 1, r = 6s - 1, \gamma_{11} = 4, \gamma_{12} = 2 = \gamma_{22}$. For

$s \geq 3$ we start from an idempotent Latin square of order s . For each $x, 1 \leq x \leq s,$

we take six symbols $x_i, 0 \leq i \leq 5$ and construct a simple $BIBD(6, 3, 2)$ design based

on the set $x_0, x_1, x_2, x_3, x_4, x_5$; this gives altogether $10s$ blocks. Also $12s(s-1)$ blocks

are defined in terms of the Latin square. The blocks are listed as follows:

$$\begin{array}{cccccccccc} x_1 & x_2 & x_1 & x_2 & x_1 & x_4 & x_1 & x_3 & x_1 & x_3 \\ x_2 & x_3 & x_3 & x_4 & x_4 & x_4 & x_5 & x_4 & x_2 & x_5 \\ x_3 & x_5 & x_4 & x_5 & x_5 & x_0 & x_0 & x_0 & x_0 & x_0 \end{array}$$

where $1 \leq x \leq s$, and

$$\begin{array}{cccc} x_i & x_i & x_i & x_i \\ y_i & y_i & y_{i+1} & x_{i+1} \\ (x \star y)_{i+2} & (x \star y)_{i+3} & (x \star y)_{i+2} & (x \star y)_{i+3} \end{array}$$

$$\begin{array}{cccc} x_{i+1} & x_{i+1} & x_{i+1} & x_{i+1} \\ y_i & y_i & y_{i+1} & y_{i+1} \\ (x \star y)_{i+2} & (x \star y)_{i+3} & (x \star y)_{i+2} & (x \star y)_{i+3} \end{array}$$

where $1 \leq x < y \leq s$, $i = 0, 2, 4$, and also subscripts are reduced to modulo 6.

The above construction follows from Stinson & Wallis (1983b). Now for some fixed $z \in \{1, 2, \dots, s\}$, add the block $(z_2 \ z_3 \ z_4)'$, and replace $(z_1 \ z_2 \ z_3)'$ and $(z_1 \ z_3 \ z_4)'$ by $(z_1 \ z_3 \ z_3)'$ and $(z_1 \ z_2 \ z_4)'$. It is easy to see that this gives the required design with $V_1 = \{z_2, z_4\}$ and $V_2 = \{\text{all other treatments except } z_2 \text{ and } z_4\}$ for $s \geq 3$. For $s = 1$

and $s = 2$ the designs are as follows:

$$\begin{array}{cccccccccccc} 2 & 3 & 4 & 5 & 6 & 1 & 2 & 1 & 1 & 1 & 2 \\ 3 & 4 & 5 & 6 & 2 & 2 & 3 & 4 & 5 & 6 & 3 \\ 4 & 5 & 6 & 2 & 3 & 4 & 1 & 6 & 5 & 3 & 5 \end{array}$$

with parameters $v = 6, b = 11, r = 5, k = 3, p = 3, \gamma_{11} = 4, \gamma_{12} = \gamma_{22} = 2$ and

$$\begin{array}{cccccccccccccccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 5 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 4 & 9 & 10 & 11 & 12 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 6 \end{array}$$

$$\begin{array}{cccccccccccccccccccc} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 6 & 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 3 \\ 7 & 8 & 9 & 10 & 11 & 12 & 2 & 3 & 4 & 5 & 7 & 2 & 9 & 10 & 11 & 12 & 8 & 3 & 4 & 5 & 6 & 8 \end{array}$$

with parameters $v = 12, b = 45, r = 11, k = 3, p = 3, \gamma_{11} = 4, \gamma_{12} = \gamma_{22} = 2$.

This completes the construction of designs of series 4.4, and thus the construction

of $GGDD(2)$ s for parameter series (I) with $\gamma_{11} = 2(m - 1) + 4$, and $\gamma_{12} = \gamma_{22} = 2(m - 1) + 2$.

Construction (II): The general construction of $GGDD(2)$ s with $\gamma_{11} = 2(m - 1) + 4$, $\gamma_{12} = \gamma_{22} = 2(m - 1) + 2$ for parameter series (II) can be obtained by adding $(m - 1)$ copies of $BIBD(3t + 1, 3, 2)$ to the $GGDD(2)$ for $m = 1$ with $\gamma_{11} = 4$, $\gamma_{12} = \gamma_{22} = 2$.

Before we shall show the construction of $GGDD(2)$ s of parameter series

$$v = 3t + 1, \quad b = 3t^2 + t + 1, \quad r = 3t, \quad \gamma_{11} = 4, \quad \gamma_{12} = \gamma_{22} = 2, \quad (4.5)$$

we need the fact from Street & Street (1987) that a skew idempotent Latin square (that is, an idempotent Latin square $L = [l_{ij}]$ such that $l_{ij} \neq l_{ji}$ unless $i = j$) exists for all orders except 2 and 3.

For $t \geq 4$ take a skew idempotent Latin square of order t . Consider the symbols $x_1, x_2, x_3, 1 \leq x \leq t$ together with one additional symbol ∞ . A $BIBD(3t + 1, 3, 2)$ design on these $v = 3t + 1$ treatments is given by

$$\begin{array}{ccccccc} x_1 & x_1 & \infty & x_2 & \infty & x_3 & \infty \\ x_2 & y_1 & x_1 & y_2 & x_2 & y_3 & x_3 \\ x_3 & (x \star y)_2 & x_2 & (x \star y)_3 & x_3 & (x \star y)_1 & x_1 \end{array}$$

where $1 \leq x \leq t$, $1 \leq y \leq t$ and $x \neq y$ (see Stinson & Wallis, 1983a). For some z and $w \in \{1, 2, \dots, t\}$, add the block $(z_1 \ (z \star w)_2 \ (w \star z)_2)'$, and replace $(z_1 \ w_1 \ (z \star w)_2)'$ and $(w_1 \ z_1 \ (w \star z)_2)'$ by $(w_1 \ (z \star w)_2 \ (w \star z)_2)'$ and $(z_1 \ z_1 \ w_1)'$.

Then we have a design of series (4.5) with $V_1 = \{(z \star w)_2, (w \star z)_2\}$ and $V_2 = \{\text{all}$

treatments except $(z \star w)_2$ and $(w \star z)_2$ for $t \geq 4$. For $t = 2$ and $t = 3$, the designs with parameters $v = 7, b = 15, r = 6, p = 3, \gamma_{11} = 4, \gamma_{12} = \gamma_{22} = 2$ and $v = 10, b = 31, r = 9, k = 3, p = 3, \gamma_{11} = 4, \gamma_{12} = \gamma_{22} = 2$ are given as follows:

1	1	1	1	1	1	1	1	2	2	2	7	7	7	7	3
3	3	4	2	2	2	2	3	3	4	4	7	4	5	5	
5	4	6	7	7	5	6	4	6	5	5	3	6	6	6	

and

2	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
3	3	2	3	4	5	6	7	8	9	3	4	5	5	6	7
4	4	2	5	6	7	8	9	10	10	6	10	8	9	7	9
2	3	3	3	3	3	3	4	4	4	4	5	5	6	6	
8	4	4	5	7	8	9	5	5	6	7	6	7	7	8	
10	7	8	6	10	9	10	9	10	9	8	10	8	10	9	

This completes the construction of parameter series (II) for the given set of γ values.

Construction (III): We shall divide this series into two parts according as t is odd or even.

Case III.(a): When t is even, the $GGDD(2)$ parameters are

$$v = 6s + 2, \quad r = 3m(6s + 1), \quad \gamma_{11} = 6m + 8, \quad \gamma_{12} = \gamma_{22} = 6m + 6. \quad (4.6)$$

The general solutions for this series is found by adding $(m - 1)$ copies of $BIBD(6s + 2, 3, 6)$ to the design with $m = 1$.

For the construction of a design with parameters $v = 6s + 2, r = 18s + 3, \gamma_{11} = 8,$ and $\gamma_{12} = \gamma_{22} = 6,$ let $L = (l_{ij})$ be an idempotent Latin square of order $2s$ and

consider the symbols $x_1 x_2 x_3, 1 \leq x \leq 2s$ together with two additional symbol ∞_1 and ∞_2 . Start with three sets of blocks. First are the $6s(2s - 1)$ blocks

$$\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ (x \star y)_2 & (x \star y)_3 & (x \star y)_1 \end{matrix}$$

where $1 \leq x \leq 2s, 1 \leq y \leq 2s, x \neq y$, and each block is taken three times. Second are the $6(2s - 1)$ blocks

$$\begin{matrix} \infty_1 & \infty_2 & \infty_1 & \infty_2 & \infty_1 & \infty_2 \\ x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ x_2 & x_2 & x_3 & x_3 & x_1 & x_1 \end{matrix}$$

where $2 \leq x \leq 2s$, and again each block is taken three times. Third are the 10 blocks

$$\begin{matrix} \infty_1 & \infty_1 & \infty_1 & \infty_1 & \infty_1 & \infty_1 & \infty_2 & \infty_2 & \infty_2 & 1_1 \\ \infty_2 & \infty_2 & \infty_2 & 1_1 & 1_2 & 1_3 & 1_2 & 1_2 & 1_3 & 1_2 \\ 1_1 & 1_2 & 1_3 & 1_2 & 1_3 & 1_1 & 1_2 & 1_3 & 1_1 & 1_3 \end{matrix}$$

This is a *BIBD*(3, 6; v) from Stinson & Wallis (1983a) and covers all the cases with $v \equiv 2(mod 6)$ except for $v = 8$. To have a designs of case (4.6) add one more block $(1_1 1_2 \infty_1)'$ and replace the two blocks $(1_1 \infty_1 \infty_2)'$ and $(1_2 \infty_1 \infty_2)'$ by $(1_1 1_2 \infty_2)'$ and $(\infty_1 \infty_1 \infty_3)'$. It is easy to verify that these designs are the required designs with $V_1 = \{1_1, 1_2\}$ and $V_2 = \{\text{all other treatments except } 1_1 \text{ and } 1_2\}$ for all $t \geq 2$. For $t = 2$, the design is

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 8 & 4 & 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 & 6 & 7 & 8 & 7 \end{matrix}$$

1	1	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3
6	7	3	3	3	3	3	4	4	4	4	5	5	5	6	6	7	4	4	
8	8	4	5	6	7	8	5	6	7	8	6	7	8	7	8	8	5	6	
3	3	3	3	3	3	3	3	4	4	4	4	4	4	5	5	5	6	1	
4	4	5	5	5	6	6	7	5	5	5	6	6	7	6	6	7	7	2	
7	8	6	7	8	7	8	8	6	7	8	7	8	8	7	8	8	8	3	

with parameters $v = 8$, $r = 21$, $b = 56$, $\gamma_{11} = 8$, $\gamma_{12} = \gamma_{22} = 6$.

Case III.(b): When t is odd we shall show the constructions of $GGDD(2)$'s with $v = 6s + 5$, $r = 3m(6s + 4)$, $b = m(6s + 4)(6s + 5) + 1$, $\gamma_{11} = 3m + 5$, $\gamma_{12} = \gamma_{22} = 3m + 3$, $s \geq 2$, $m \geq 1$, by adding $(m - 1)$ copies of $BIBD(6s + 5, 3, 6)$ to the corresponding design with $m = 1$.

For the constructions of designs of the parameter series $v = 6s + 5$, $b = 18t^2 + 27s + 11$, $r = 9s + 6$, $\gamma_{11} = 5$, $\gamma_{12} = \gamma_{22} = 3$ consider the $BIBD(6s + 5, 3, 3)$ (Stinson & Wallis, 1983a), which is given by the initial blocks $(y, x + y, 2x + y)'$, where $0 < x < v/2$, $0 \leq y < v$ and all the additions are reduced modulo v . Add $(2\ 3\ 5)'$ and replace $(1\ 2\ 3)'$ and $(1\ 3\ 5)'$ by $(1\ 3\ 3)'$ and $(1\ 2\ 5)'$. We have the required series of designs with $V_1 = \{2, 5\}$ and $V_2 = \{\text{all other treatments except } 2 \text{ and } 5\}$, completing the construction of parameter series (III).

Constructions of parameter series (IV), (V), and (VI) are obtained by using the constructions of parameter series (II) for even values of t , of parameter series (I) for odd values of t , and of parameter series (III) for odd values of t , and adding a copy of $BIBD(6t + 1, 3, 1)$, of $BIBD(6t + 3, 3, 1)$, and of $BIBD(6t + 5, 3, 3)$,

respectively.

4.3.4 Settings with $p = 4$

The necessary conditions from (4.2) for the existence of $GDDD(2)$ s for $k = 3$ and $p = 4$ lead to the following three parameter series

$$(VII) \quad v = 3t + 2 \quad r = (3m + 1)(v - 1) \quad \lambda = 6m + 2 \quad t \geq 2 \quad m \geq 0$$

$$(VIII) \quad v = 6t + 5 \quad r = (12m + 5)\frac{(v-1)}{2} \quad \lambda = 12m + 5 \quad t \geq 1 \quad m \geq 2$$

$$(IX) \quad v = 6t + 5 \quad r = (12m + 11)\frac{(v-1)}{2} \quad \lambda = 12m + 11 \quad t \geq 1 \quad m \geq 1.$$

For each of these parameter series we will construct the $GGDD(2)$ with certain sets of γ values. All of the designs constructed in this subsection satisfy the E -optimality conditions of Theorem 2.1, while only the designs with $p = 4$ for five and six treatments fails the additional condition of Theorem 2.3 for MV optimality.

Construction (VII): For this series, it is sufficient to show the construction of $GGDD(2)$ with parameters $v = 3t + 2$, $r = 3t + 1$, $\gamma_{11} = 3$, $\gamma_{12} = \gamma_{22} = 2$, $t \geq 2$.

The design of the desired series can then be derived by adding $(m - 1)$ copies of a $BIBD(3t + 2, 3, 6)$ to this design. This construction follows from the construction given by Morgan & Uddin (1995) for CSD s with parameters $v = 3t + 2$, $b = 3t^2 + 3t + 1$, $r = 3t + 1$, $k = 3$, $\lambda = 2$ by adding one more block of size 3 of any three distinct treatments.

Construction (VIII): Construction of $GGDD(2)$'s with $\gamma_{11} = 12m + 7$, $\gamma_{12} = \gamma_{22} =$

$12m + 5$ for this series follows by adding $(4m + 1)$ copies of a $BIBD(6t + 1, 3, 3)$ to a design of $GGDD(2)$ with parameters $v = 6t + 5, r = 6t + 4, b = 12t^2 + 18t + 8, \gamma_{11} = 4, \gamma_{12} = 2 = \gamma_{22}, t \geq 1$.

The solution of parameter series $v = 6t + 5, b = 12t^2 + 18t + 8, r = 6t + 4, \gamma_{11} = 4, \gamma_{12} = \gamma_{22} = 2$ are found by considering the construction of $CSD(6t + 5, 3, 2)$ s given by Morgan & Uddin (1995). There are $6(t + 1)$ initial blocks mod $(2t + 1)$, given by

$$\begin{matrix} 1_1 & 2_1 & \dots & t_1 & 1_2 & 2_2 & \dots & t_2 \\ 2t_1 & (2t - 1)_1 & \dots & (t + 1)_1 & 2t_2 & (2t - 1)_2 & \dots & (t + 1)_2 \\ 0_2 & 0_2 & \dots & 0_2 & 0_3 & 0_3 & \dots & 0_3 \\ & & & 1_3 & 2_3 & \dots & t_3 \\ & & & 2t_3 & (2t - 1)_3 & \dots & (t + 1)_3 \\ & & & 0_1 & 0_1 & \dots & 0_1 \end{matrix}$$

and the 7 blocks

$$\begin{matrix} \infty_1 & \infty_1 & \infty_1 & \infty_2 & \infty_2 & \infty_2 & \infty_1 \\ 0_1 & 0_1 & 0_2 & 0_1 & 0_1 & 0_2 & \infty_1 \\ 0_2 & 0_3 & 0_3 & 0_2 & 0_3 & 0_3 & \infty_2. \end{matrix}$$

To obtain the required design, add one more block $(0_1 0_2 0_3)'$ and replace $(\infty_1 0_1 0_2)'$ and $(\infty_1 0_1 0_3)'$ by $(\infty_1 0_1 0_1)'$ and $(\infty_1 0_2 0_3)'$. It is easy to verify that the above design is $GGDD(2)$ with $V_1 = \{0_2, 0_3\}$ and $V_2 = \{\text{all other treatments except } 0_2 \text{ and } 0_3\}$.

Construction (IX): Construction of this series follows by adding 2 more copies of a $BIBD(6t + 5, 3, 3)$ to a design of parameter series (VIII).

4.4 SOME OTHER CONSTRUCTION OF *CSDs*

In this section we derive constructions of *CSDs* in the following two parameter series

$$(I) \quad v = 8t + 7 \quad r = 8t + 6 \quad k = 2t + 2 \quad \lambda = 2t + 1 \quad p = 2t \quad t \geq 0,$$

$$(II) \quad v = 2t + 1 \quad r = 4t - 1 \quad k = t + 1 \quad \lambda = t \quad p = t - 1 \quad t \geq 1.$$

Before doing so we need the following two theorems due to Sprott (1954).

THEOREM 4.4 *Suppose $v = 2m(2\lambda + 1) + 1$ is a prime or prime power and x is a primitive element of $GF(v)$. Then the design with parameters $v = 2m(2\lambda + 1) + 1$, $b = mv$, $r = m(2\lambda + 1)$, $k = 2\lambda + 1$, and λ can be constructed via the initial blocks $(x^i x^{i+2m} \dots x^{i+4m\lambda})'$, $i = 0, 1, \dots, m - 1$.*

THEOREM 4.5 *Suppose $v = 2m(2\lambda - 1) + 1$ is a prime or prime power and x is a primitive element of $GF(v)$. Then the design with parameters $v = 2m(2\lambda - 1) + 1$, $b = mv$, $r = 2m\lambda$, $k = 2\lambda$, and λ can be constructed via the initial blocks $(0 x^i x^{i+2m} \dots x^{i+4m(\lambda-1)})'$, $i = 0, 1, \dots, m - 1$.*

These two constructions for *BIBDs* can be modified to yield the construction for a subset of designs of series (I), as follows:

THEOREM 4.6 *Suppose $v = 8t + 5$ is a prime or prime power, where $t \geq 0$. Then there exists a *CSD* with parameters $v = 8t + 7$, $b = 32t + 21$, $r = 8t + 6$, $k = 2t + 2$, $\lambda = 2t + 1$, and $p = 2t$.*

PROOF Let x denote a primitive element of $\text{GF}(8t+5)$, and let the v treatments be the elements of $\text{mod}(8t+5)$ and ∞_1, ∞_2 . Using Theorem 4.4 for $m=2$ and $\lambda=t$, and Theorem 4.5 for $m=2$ and $\lambda=t+1$, we have the construction of *BIBDs* with parameters

$$v = 8t + 5, b = 16t + 10, r = 4t + 2, k = 2t + 1, \lambda = t \quad (4.7)$$

and

$$v = 8t + 5, b = 16 + 10, r = 4t + 4, k = 2t + 2, \lambda = t + 1 \quad (4.8)$$

respectively. For the *CSD*, first take the initial blocks $(0 \ 1 \ x^4 \ \dots \ x^{8s})'$ and $(0 \ x \ x^5 \ \dots \ x^{8s+1})'$ of *BIBD* $(8t+5, 2t+2, t+1)$ from (4.8). Next take the initial blocks $(\infty_1 \ 1 \ x^4 \ \dots \ x^{8s})'$ and $(\infty_2 \ x \ x^5 \ \dots \ x^{8s+1})'$ found by adding ∞_1 and ∞_2 to the initial blocks $(1 \ x^4 \ \dots \ x^{8s})'$ and $(x \ x^5 \ \dots \ x^{8s+1})'$ of *B* $(8t+5, 2t+1, t)$ given in (4.7). The development of these blocks along with the single block containing $(\infty_1 \ \infty_1 \ \dots \ \infty_1 \ \infty_2)'$ completes the construction. \square

Observe that the $t=0$ design is the trivial *BIBD* with parameters $v=7, k=2$ and $\lambda=1$. So consider the case $t=1$. It is easy to verify that the initial blocks $(\infty_1 \ 1 \ x^4 \ x^8)'$, $(\infty_2 \ x \ x^5 \ x^9)'$, $(0 \ 1 \ x^4 \ x^8)'$, $(0 \ x \ x^5 \ x^9)'$ (*mod* 13), and a single block containing $(\infty_1 \ \infty_1 \ \infty_1 \ \infty_2)'$, where x is a primitive element of $\text{GF}(13)$, provide a *CSD* $(15, 4, 3)$.

The next theorem gives some designs in parameter series (II).

THEOREM 4.7 *Let $2t - 1$ be an prime or prime power, $t \geq 1$. Then there exist $CSDs$ with parameters $v = 2t + 1$, $b = 4t - 1$, $r = 2t$, $k = t + 1$, $\lambda = t$, $p = t - 1$, $t \geq 1$.*

PROOF Let x denote a primitive element of $GF(2t - 1)$ and the v treatments are all the elements of $GF(2t - 1)$ and ∞_1, ∞_2 . Now divide the given series into two cases according as t is odd or even.

Case (a). When t is odd, the construction of a CSD with parameter $v = 4s + 3$, $b = 8s + 3$, $r = 4s + 2$, $k = 2s + 2$, $\lambda = 2s + 1$, and $p = 2s$ can be obtained by developing the initial blocks $(\infty_1 0 x^1 x^3 \dots x^{4s-1})'$ and $(\infty_2 1 x^2 x^4 \dots x^{4s})'$, and taking a single block containing $(\infty_1 \infty_1 \dots \infty_1 \infty_2)'$.

Case (b). For t even, develop the initial blocks $(\infty_1 0 x^1 x^3 \dots x^{4s-1})'$ and $(\infty_2 x^1 x^3 \dots x^{4s-1})'$, and take a single block containing $(\infty_1 \infty_1 \dots \infty_1 \infty_2)$, to obtain $CSD(4s + 1, 2s + 1, 2s)$. \square

EXAMPLE 16 For $t = 3$, initial blocks $(\infty_1 0 2 3)'$ and $(\infty_2 1 4 0)' \pmod{5}$, and a single blocks $(\infty_1 \infty_1 \infty_1 \infty_2)'$ provides a $CSD(7, 4, 3)$.

EXAMPLE 17 For $t = 5$, construction of $CSD(11, 6, 4)$ is given by one block $(\infty_1 \infty_1 \infty_1 \infty_1 \infty_1 \infty_2)'$ and blocks obtained by developing the initial blocks $(\infty_1 0 x^1 x^3 x^5 x^7)'$ and $(\infty_2 1 x^2 x^4 x^6 0)'$ over $GF(3^2)$.

EXAMPLE 18 Construction of $CSD(5,3,2)$ for $t = 2$ is given by the initial blocks $(\infty_1 0 2)'$ and $(\infty_2 0 2)' \pmod{3}$, and a single block $(\infty_1 \infty_1 \infty_2)'$.

It can easily be seen that all designs constructed in this section are E -optimal (Theorem 2.1). Designs obtained from Theorem 4.6 for $t = 1, 2, 3$, and from Theorem 4.6 for $t = 1, 2, 3, 4$, are the only designs satisfying Theorem 2.3 for MV -optimality. All other designs fail the condition (i) of Theorem 2.3 for MV -optimality.

Chapter 5

Balanced incomplete block designs with nested rows and columns

5.1 INTRODUCTION

In the earlier chapters we have shown some very important results related to optimality and construction of incomplete block designs for controlling one directional heterogeneity in the experimental material. In many experiments, however, there may be more sources of variation than can be eliminated by ordinary block designs. Such sources of variation may make a significant contribution to the variability among

the experimental units. In the present chapter, we introduce the block designs with nested rows and columns BIBRCs, which are useful for controlling heterogeneity in two directions. We study a general balance property, and solve the construction of 2×2 (BIBRC)s.

Consider experimental designs composed of incomplete blocks with nested rows and columns, such that within each block the row \times column classification is orthogonal. These designs were introduced and discussed by Srivastava (1978) and Singh & Dey (1979) for eliminating heterogeneity in two directions within each block. Let there be v treatments and b blocks each containing k treatments in p rows and q columns, $k = pq \leq v$. Then a *balanced incomplete block design with nested rows and columns* is defined to be a design for which

- (i) each treatment appears r times,
- (ii) each treatment appears in each block at most once, and
- (iii) each pair of treatments (α_1, α_2) satisfies the relationship

$$(p-1)\lambda_{r(\alpha_1, \alpha_2)} + (q-1)\lambda_{c(\alpha_1, \alpha_2)} - \lambda_{e(\alpha_1, \alpha_2)} = \lambda. \quad (5.1)$$

Here $\lambda_{r(\alpha_1, \alpha_2)}$, $\lambda_{c(\alpha_1, \alpha_2)}$ and $\lambda_{e(\alpha_1, \alpha_2)}$ denote the number of blocks in which α_1 and α_2 occur in the same row, the same column, and elsewhere respectively, and λ is a constant independent of the pair of treatments chosen. It follows that $\lambda =$

$r(p-1)(q-1)/(v-1)$. These designs will be denoted by $\text{BIBRC}(v, b, r, p, q, \lambda)$.

Let A , Z_1 , Z_2 , and Z_3 be respectively the plot-treatment, plot-block, plot-row and plot-column incidence matrices. Our model for the $bpq \times 1$ vector of yields Y is

$$Y = A\tau + Z_1\beta + Z_2\rho + Z_3\gamma + e \quad (5.2)$$

where $\tau(v \times 1)$, $\beta(b \times 1)$, $\rho(bp \times 1)$, and $\gamma(bq \times 1)$ are the vectors of treatment, block, row and column effects, $E(e) = 0$ and $\text{Var}(e) = \sigma^2 I$. The blocking effects β , ρ , and γ are assumed in this section, and τ is taken as such throughout the paper, to be fixed vectors of unknown parameters.

Now write $N_i = A^T Z_i$; N_1 , N_2 , and N_3 are the treatment-block, treatment-row, and treatment-column incidence matrices. The usual fixed effects (or within-rows-and-columns) analysis for τ leads to the reduced normal equations $C\hat{\tau} = Q$ with

$$C = A^T A - \frac{1}{q} N_2 N_2^T - \frac{1}{p} N_3 N_3^T + \frac{1}{pq} N_1 N_1^T. \quad (5.3)$$

The conditions (i)-(iii) above make the information matrix C of (5.3) completely symmetric, meaning that BIBRCs are variance balanced for the bottom stratum analysis. Constructions for BIBRCs are given by Street (1981), Agrawal and Prasad (1982, 1983), Jimbo and Kuriki (1983), Cheng (1986), Sreenath (1989), Uddin and Morgan (1990, 1991), and Uddin (1992). It is clear that $vr = bpq$ and $\lambda(v-1) = r(p-1)(q-1)$ are necessary conditions for the existence of BIBRCs. In section 5.2, we find that the necessary conditions for a BIBRC with $p = q = 2$ are also sufficient.

Cheng (1986) discusses general balance of BIBRCs, and shows that if N_1 , N_2 and N_3 are each incidence matrices of BIBDs, then the BIBRC is generally balanced. Section 5.3 establishes the general balance of BIBRCs with square blocks when $N_{23}=(N_2:N_3)$ and N_1 are each incidence matrices of BIBDs. All of the 2×2 BIBRCs constructed in section 5.2 satisfy this condition, which has important optimality implications.

5.2 Construction and existence of the 2×2 BIBRC

For $p = q = 2$, the necessary conditions reduce to

$$4b = vr \quad \text{and} \quad \lambda(v - 1) = r.$$

Hence 4 must divide $\lambda v(v - 1)$, and the corresponding restrictions on v are conveniently stated as follows, where (x, y) denotes the greatest common divisor of x and y .

$$\text{If } \lambda \equiv 1 \pmod{2}, \text{ then } v \equiv 0 \text{ or } 1 \pmod{4}. \quad (5.4)$$

$$\text{If } \lambda \equiv 0 \pmod{2}, \text{ then } v \text{ is unrestricted.} \quad (5.5)$$

Also we need $v \geq 4$.

If $(\lambda, 4) = d$ (say) and $\lambda = md$ for some integer m then we can certainly form a BIBRC($v, b, r, 2, 2, \lambda$) simply by taking an m -multiple of a 2×2 BIBRC with v treatments in b/m blocks, that is, by taking m copies of each block. Thus to

show that the necessary conditions (5.4) and (5.5) are also sufficient, it suffices to demonstrate, with $v \geq 4$, the existence of $\text{BIBRC}(v, v(v-1)/4, (v-1), 2, 2, 1)$ for every $v \equiv 0$ or $1 \pmod{4}$, and of $\text{BIBRC}(v, v(v-1)/2, 2(v-1), 2, 2, 2)$ for every $v \equiv 2$ or $3 \pmod{4}$.

Let a set E having v treatments be given. Further let $K = \{k_1, k_2, \dots, k_n\}$ be a finite set of integers with $3 \leq k_i, i = 1, \dots, n$. Let λ be a positive integer. If it is possible to form a design of binary blocks (subsets of E) in such a way that the number of elements in each block is some $k_i \in K$, and every (unordered) pair of elements of E is contained in exactly λ blocks, then we shall call such a design a *binary pairwise balanced design* and denote it by $B[K, \lambda, v]$. These designs were introduced by Hanani (1961) under the name B-systems. The class of all numbers v for which designs $B[K, \lambda, v]$ exist will be denoted by $B(K, \lambda)$. The proof of the following lemma is given in Hanani (1961).

LEMMA 5.1 *If $v \equiv 0$ or $1 \pmod{4}$ and $v \geq 4$, then $v \in B[K, 1]$, where $K = \{4, 5, 8, 9, 12\}$.*

LEMMA 5.2 *BIBRCs with the following parameters all exist: $(4, 3, 3, 2, 2, 1)$, $(5, 5, 4, 2, 2, 1)$, $(8, 14, 7, 2, 2, 1)$, $(9, 18, 8, 2, 2, 1)$, and $(12, 33, 11, 2, 2, 1)$.*

PROOF The designs are

BIBRC(4,3,3,2,2,1):

1 2	1 3	1 4
3 4	4 2	2 3

BIBRC(5,5,4,2,2,1):

1 5	1 4	1 3	1 2	2 3
3 2	5 3	2 4	4 5	4 5

BIBRC(8,14,7,2,2,1):

1 4	1 6	1 7	1 8	1 2	1 5	1 3
3 2	5 3	6 4	4 5	8 6	2 7	7 8
2 5	2 8	2 4	2 6	3 4	3 6	5 8
8 3	7 4	5 6	3 7	7 5	4 8	6 7

BIBRC(9,18,8,2,2,1):

2 6	3 4	1 5	5 9	8 3	6 7	7 2	4 8	9 1
8 3	9 1	7 2	2 6	5 9	3 4	4 8	1 5	6 7
4 5	5 6	6 4	7 8	1 2	8 9	3 1	9 7	2 3
9 7	7 8	8 9	3 1	6 4	1 2	5 6	2 3	4 5

BIBRC(12,33,11,2,2,1):

1 C	1 B	1 2	1 7	1 9	1 A	1 3	1 5	1 8	2 C	2 3
6 2	5 3	9 4	4 5	3 A	8 7	B 8	C B	7 C	7 3	A 5
2 5	2 5	2 B	2 9	3 9	4 9	4 7	4 7	4 A	5 A	1 6
8 6	3 7	9 8	8 C	A C	C 5	A 8	5 9	B C	C 6	A 9
1 4	2 7	2 A	5 8	5 8	6 B	6 7	3 4	3 8	3 4	3 6
2 6	B A	4 B	B 9	6 A	C 7	9 B	6 8	C 4	B 6	9 7

□

Designs with parameters as in lemma 5.2 can be found among the papers referenced in section 5.1. The reason for listing them here is for an additional property these particular designs possess. It is easy to verify that, when the row and column component designs are combined, each design in lemma 5.2 produces a BIBD with parameters $b = v(v - 1)$ and $k = 2$, and that the nesting blocks of these designs form BIBDs with parameters $b = v(v - 1)/4$ and $k = 4$ (taking for our purposes the BIBDs to include the complete block designs for $v = 4$). There are designs, as shown in section 3, that do not have this property.

The next lemma generalizes a construction found by several authors in various forms (Singh & Dey, 1979, Theorem 2; Jimbo & Kuriki, 1983, Theorem 2; Agrawal & Prasad, 1984, Theorem 2.1; Sreenath, 1989, Theorem 2.1).

LEMMA 5.3 *The existence of a binary pairwise balanced design B_0 , a $B[K, \lambda_0, v_0]$ with $K = \{k_1, k_2, \dots, k_n\}$, and of B_i , a $BIBRC(v_i = k_i, \bar{b}_i, \bar{r}_i, p, q, \lambda_1)$ for $i = 1, \dots, n$, implies the existence of a $BIBRC(v_0, b, r, p, q, \lambda)$ for $\lambda = \lambda_0 \lambda_1$ and appropriate b and r .*

PROOF Let b_i be the number of blocks of size k_i in B_0 , and let r_{ij} be the number of times that treatment j occurs in the b_i blocks of size k_i , for $i = 1, \dots, n$. Also let $\lambda_{(\alpha_1, \alpha_2)}^i$ denote the number of times that treatments α_1 and α_2 occur together

among the blocks of size k_i . Then $\lambda_0(v_0 - 1) = \sum_i r_{ij}(k_i - 1)$, and $\sum_i \lambda_{(\alpha_1, \alpha_2)}^i = \lambda_0$.

For the BIBRCs B_i , the relationship $\frac{\bar{r}_i}{v_i - 1} = \frac{\lambda_1}{(p-1)(q-1)}$ holds for each i .

For each block of size k_i in B_0 , use its treatments to construct the BIBRC B_i , producing $\bar{b}_i b_i$ blocks of the new design D . Repeating this for each i gives $b = \sum(\bar{b}_i b_i)$

blocks in which treatment j has replication

$$\sum_i (\bar{r}_i r_{ij}) = \sum_i \frac{\bar{r}_i}{k_i - 1} (k_i - 1) r_{ij} = \frac{\lambda_1}{(p-1)(q-1)} \sum_i (k_i - 1) r_{ij} = \frac{\lambda_1 \lambda_0 (v_0 - 1)}{(p-1)(q-1)}$$

not depending on j . And for D , it is clear that (5.1) holds with

$$(p-1)\lambda_{r(\alpha_1, \alpha_2)} + (q-1)\lambda_{c(\alpha_1, \alpha_2)} - \lambda_{e(\alpha_1, \alpha_2)} = \sum_i \lambda_{(\alpha_1, \alpha_2)}^i \lambda_1 = \lambda_0 \lambda_1.$$

□

Combining lemmas 5.1, 5.2, and 5.3 solves the problem presented by (5.4) for $\lambda=1$.

THEOREM 5.1 *BIBRC($v, \frac{v(v-1)}{4}, v-1, 2, 2, 1$) exists for $v \equiv 0$ or $1 \pmod{4}$, where $v \geq 4$.*

When $p = q$ it is clear that if the designs B_i of lemma 5.3 each have the property that the nesting block component design is a BIBD, as does the combined row and column component design, then the newly constructed BIBRC will also have this property. It follows from lemma 5.2 and the remark immediately following it that

every design constructed for Theorem 5.1 has this property, the importance of which will be made clear in section 5.3.

Now we consider construction of $\text{BIBRC}(v, \frac{v(v-1)}{2}, 2(v-1), 2, 2, 2)$. These designs will be obtained using the *strongly equineighbourbed designs* as defined in Martin & Eccleston (1991) and Street (1992). Strongly equineighbourbed designs have v treatments and $b = \frac{1}{2}sv(v-1)$ ordered blocks, each of size k . When written as the columns of a $k \times b$ array, the blocks satisfy

- (i) Each treatment occurs $s(v-1)$ times in rows j and $k+1-j$ together, $j \neq \frac{1}{2}(k+1)$, and $\frac{1}{2}s(v-1)$ times in row $\frac{1}{2}(k+1)$ if k is odd.
- (ii) The columns in the $2 \times b$ submatrices obtained from rows j and $k+1-j$ contain each unordered pair of distinct treatments exactly s times.
- (iii) The columns in the $2 \times b$ submatrices obtained from rows j and m , and from rows $k+1-j$ and $k+1-m$ ($j+m \neq k+1$), together contain each unordered pair of distinct treatments exactly $2s$ times.

Such designs will be denoted $\text{SEN}(k, s; v)$.

LEMMA 5.4 $\text{SEN}(4, 1; v)$ exists for all $v \geq 4$.

The proof of lemma 5.4 is in Street (1992).

LEMMA 5.5 *The columns of $SEN(4,1,v)$, where $v \geq 4$, form a BIBD with $k = 4$ and $\lambda = 6$.*

PROOF On writing the blocks of a $SEN(4,1,v)$ as the columns of a $4 \times b$ array, the SEN conditions translate thusly: (i) each treatment occurs $(v - 1)$ times in rows 1 and 4, and $(v - 1)$ times in rows 2 and 3, that is, each treatment occurs $2(v - 1)$ times; (ii) the two sets of columns in the $2 \times b$ submatrices obtained from rows 1 and 4, and those from rows 2 and 3, each contain each unordered pair of distinct treatments exactly once; (iii) the columns in the $2 \times b$ submatrices obtained from rows 1 and 2, and from rows 3 and 4, collectively contain each unordered pair of distinct treatments exactly twice, and likewise this holds for rows 1 and 3 combined with rows 2 and 4. From (ii) and (iii), each unordered pair of distinct treatments occurs a total of exactly six times in this arrangement, and so it is a BIBD. \square

THEOREM 5.2 *$BIBRC(v, v(v - 1)/2, 2(v - 1), 2, 2, 2)$ exists for all $v \geq 4$.*

PROOF We shall use $SEN(4,1;v)$ designs to form the required BIBRCs. Write the ordered column $(\delta_1 \ \delta_2 \ \delta_3 \ \delta_4)'$ of an $SEN(4,1;v)$ in the form

$$\begin{array}{cc} \delta_1 & \delta_2 \\ \delta_3 & \delta_4. \end{array}$$

By SEN property (ii), every unordered pair of treatments (α_1, α_2) occurs $\lambda_{e(\alpha_1, \alpha_2)} = 2$ times in diagonals, and by SEN property (iii), occurs $\lambda_{r(\alpha_1, \alpha_2)} = 2$ times in rows and $\lambda_{c(\alpha_1, \alpha_2)} = 2$ times in columns. Hence (5.1) holds. \square

EXAMPLE 19 For $v=6$, a $SEN(4,1;6)$ design is

1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
3	2	2	4	3	4	6	3	5	6	6	2	3	3	1
4	5	6	2	5	5	5	6	1	1	4	1	1	2	4
2	3	4	5	6	3	4	5	6	4	5	6	5	6	6

which gives this $BIBRC(6,15,10,2,2,2)$:

1	3	1	2	1	2	1	4	1	3	2	4	2	6	2	3
4	2	5	3	6	4	2	5	5	6	5	3	5	4	6	5
2	5	3	6	3	6	3	2	4	3	4	3	5	1		
1	6	1	4	4	5	1	6	1	5	2	6	4	6		

Clearly, all of designs constructed in Theorem 5.2 give BIBDs with blocksize 2 when their row and column component designs are combined, and lemma 5.5 says that the nesting block component designs are also BIBDs.

5.3 General Balance of BIBRCs with $p=q$

In this section, a condition for general balance of BIBRCs will be obtained, and shown to hold for the 2×2 BIBRCs constructed in section 5.2. The terminology used will be that of Houtman and Speed (1983), where a general discussion of this concept can be found. To see the idea in the nested row and column setting, first define the strata projectors $S_0 = G$, $S_1 = B - G$, $S_2 = R - B$, $S_3 = G - B$, and $S_4 = I - R - C + B$, where for the data vector Y of any equireplicate nested

row and column design, the averaging operators are $B = (\frac{1}{pq})Z_1Z_1^T$ for blocks, $R = (\frac{1}{q})Z_2Z_2^T$ for rows, $C = (\frac{1}{p})Z_3Z_3^T$ for columns, $G = (\frac{1}{bpg})11^T$ for the grand mean, and also $T = (\frac{1}{r})AA^T$ for treatments. Strata $S_0, S_1, S_2, S_3,$ and S_4 are called the mean, block, row, column, and within-rows-and-columns strata respectively. Analysis in stratum i is estimation of treatment contrasts using the data S_iY . A design is balanced in stratum i if the information matrix for analysis in stratum i is completely symmetric, and the strong form of general balance we employ requires this to hold for each i . General balance of a BIBRC thus implies that information on treatment contrasts is not only easily obtained from each stratum, but is easily combined, greatly simplifying optimality considerations for the full analysis. To recover information from the other strata it is assumed that the nuisance parameters $\beta, \rho,$ and γ in (5.2) are mutually uncorrelated, mean zero random vectors, each with a variance matrix which is a scalar multiple of the identity. More explicit technical details can be found in Morgan and Uddin (1993).

Cheng (1986) gave this necessary and sufficient condition for a BIBRC to be generally balanced:

LEMMA 5.6 *A BIBRC is generally balanced iff $TRTCT = TCTRT$.*

He further showed that if for a BIBRC, N_1, N_2 and N_3 are each incidence matrices of BIBDs, then the BIBRC is generally balanced. Theorem 3.2 eases this condition

when $p = q$.

THEOREM 5.3 *If N_{23} and N_1 are incidence matrices for BIBDs, then the BIBRC(v, b, r, p, p, λ) is generally balanced.*

PROOF To prove that the given BIBRC is generally balanced, it is sufficient by lemma 3.1 to show that $A[N_2N_2^TN_3N_3^T - N_3N_3^TN_2N_2^T]A^T = 0$. Since N_{23} is the incidence matrix of a BIBD, for some scalar λ_{23} ,

$$N_{23}N_{23}^T = N_2N_2^T + N_3N_3^T = (2r - \lambda_{23})I_v + \lambda_{23}11^T. \quad (5.6)$$

Using (5.6) and the fact that $N_2N_2^T11^T = pr11^T = 11^TN_2N_2^T$, gives

$$\begin{aligned} A[N_2N_2^TN_3N_3^T - N_3N_3^TN_2N_2^T]A^T &= A[N_2N_2^T(N_3N_3^T + N_2N_2^T) - \\ &\quad (N_2N_2^T + N_3N_3^T)N_2N_2^T]A^T \\ &= A[N_2N_2^TN_3N_3^T - N_{23}N_{23}^TN_2N_2^T]A^T \\ &= A[pr\lambda_{23}(11^T - 11^T)]A^T \\ &= 0. \quad \square \end{aligned}$$

An immediate consequence of Theorem 5.3 is that all of the designs constructed in section 5.2 are generally balanced. This is of considerable importance, because BIBRCs with $p = q = 2$ are grossly inefficient in the bottom stratum (Morgan and Uddin, 1993, pg. 83), with the consequence that in many fields of application it

is routine to recover the information found in higher strata. For the full analysis, depending on the stratum variances, BIBRCs *can* be universally optimum (Morgan and Uddin, 1993, pg. 91), and for the 2×2 's of section 5.2, an exact condition for this optimality is failure of the inequality (for $m = 1$) that appears in Theorem 11 of Morgan and Uddin (1993).

We conclude by showing that not all 2×2 BIBRCs are generally balanced. The design

1 8	1 4	1 7	1 6	1 3	1 5	1 4
6 4	5 3	2 4	4 5	7 6	8 7	3 8
2 8	2 3	2 7	2 4	3 2	3 6	2 7
7 3	6 4	5 6	8 5	7 5	5 8	6 8

is a BIBRC(8, 14, 7, 2, 2, 1). Unlike the BIBRCs of section 5.2, neither the blocks, nor the rows and columns combined, form a BIBD. The design does not satisfy lemma 5.6, so is not generally balanced. Hence Cheng's (1986, pg. 700) observation that balance in the bottom stratum of a row-column design implies general balance, does not extend to nested row and column designs.

Chapter 6

Concluding Remarks

The main contribution of this research is to provide results dealing with the optimality and construction of block designs in many settings where equally replicated designs are not possible. We have found that binarity is not a necessary condition for a good design. It is shown that relaxing the binarity condition can lead to a very important symmetrical structure in the information matrix. In some cases it even becomes necessary to sacrifice maximal trace in favour of symmetry for E - and MV -optimality. This research has established infinitely many designs which are both E - and MV -superior to all binary designs. Almost all of the results for E - and MV -optimality (discussed in chapter 2) are very general, covering both binarity and nonbinarity, although most of the examples given are of nonbinary designs. Indeed, demonstrating the optimality of nonbinary block designs is the main contribution,

in that many of the results are already known when restricted to the binary class.

In that nonbinarity has essentially been considered anathema in the past twenty years of optimality-based design theory, some defense of our approach is certainly required. It is our contention that the arguments for these nonbinary designs can be in fact *compelling*, as we shall try to make clear by two examples, the first of which is perhaps less clear cut than the second. Consider the design d^* of example 1. The nonzero eigenvalues of d^* are (6.4, 6.4, 6.4, 6.4, 5.4, 5.4, 5.4, 5.4); the A -value is .170; and the MV -value is .370. If one wishes to use a binary design for the same setting, the proper choice is not immediately clear, but keeping the range of the r_{di} 's and of the λ_{dij} 's at their minimum values of 1 is certainly reasonable. MV -efficiency will require $\lambda_{dij} = 3$ if r_{di} or $r_{dj} = 6$ (see the proof of Theorem 2.3). Trial and error then led us to the following design d_1 , which we suspect is A -optimal:

1	4	1	2	1	3	7	7	7	1	2	1
2	5	4	3	6	4	8	8	8	3	5	5
3	6	5	6	2	5	9	9	9	4	6	6
7	7	7	8	8	8	2	3	4	9	9	9
6	2	3	4	5	1	1	5	6	2	3	4

For d_1 , the eigenvalues are (6.8, 6.4, 6.4, 6.2, 6.0, 5.4, 5.4, 5.4); the A -value is .168; and the MV -value is .370. Thus d_1 is the equal of d^* in also having optimal E and MV values, and the A -efficiency of d^* relative to d_1 is .983. In terms of any eigenvalue-based criterion, the comparison is essentially that of three eigenvalues, two of which

are superior for d_1 , and one of which is superior for d^* . The argument for d^* is the argument for the group divisible pattern in the information matrix and hence in the estimated variances. If there is a natural structure on the treatment set corresponding to the partition V_1 and V_2 shown in example 1 (e.g. in a drug study, five generic products and four name brands), then d preserves that structure in the estimated variances, which no efficient binary design can do. With six distinct eigenvalues rather than two, d_1 has a much messier structure that is unlikely to correspond to the structure of realistic treatments sets, and which lends itself less readily to interpretation of results. Our compensation for the 1.7% loss in A -efficiency is in the structure of d^* .

For the second example, form $d_2 \in D(9, 11, 5)$ by deleting the last block from the display of d^* in example 1. Then d_2 is nonbinary but variance balanced: C_{d_2} is completely symmetric. For a binary competitor, it will usually be reasonable to demand that treatment replications vary by no more than one, though this forces the concurrence numbers λ_{dij} to vary by at least two. A reasonable and simply constructed choice is d_3 , the cyclic design (12459) ($\text{mod } 9$) with the two additional blocks (13579) and (12468). The constant nonzero eigenvalue for d_2 is 5.4. The eigenvalues for d_3 are (6.386, 5.793, 5.753, 5.545, 5.471, 5.115, 5.086, 4.850); the A, E and MV efficiencies of d_2 relative to d_3 are 0.988, 1.113, and 1.044. It is our assertion

that the 1.2% loss in the A-value is more than compensated for, not just by gains in the other two criteria, but as importantly or more so, by the symmetric structure of C_{d_2} that is consistent with the structure of virtually any treatment set. Unless the A-criterion is the only concern, d_2 will usually be the better choice.

The issues being raised here are not unlike those tackled by Bailey (1994, section 5) in the context of a discussion of the desirability of general balance in block designs, wherein she states: "... for structured treatments I would sacrifice 10% on the overall mean harmonic efficiency factor to have a design more easily interpretable in terms of that treatment structure...". The group divisible structure, which has been studied here, is after the completely symmetric, the most widely applicable, and the most easily interpretable, of all of the structures.

The example based on d^* and d_1 above illustrates another gain offered by the generalized approach: once the existence of an optimal nonbinary design is determined, the precise criterion bound is also established. It will then sometimes be possible, as was just done for d_1 , to prove the optimality of a binary competitor simply by calculating its criterion value. General analytic proofs of optimality, in situations where GGD-structure is combinatorially impossible for binary designs, are likely to be difficult.

If the A-criterion is judged to be the *only* criterion of relevance in choosing a

design, we have largely failed. We have found no nonbinary designs in this dissertation that cannot be at least slightly improved upon in the A -sense by a careful conversion to binarity. However in chapter 3 we have found sufficient conditions for establishing the type-1 optimality of $NBBD(2)$ s having unequal numbers of replicates and three different concurrence values, with examples coming from just such a conversion, further validating the study of nonbinary designs. This result opens new avenues in the A - and D -optimality problems when regular graph designs do not exist. We hope that further research along this line be carried out in the future.

Several simple and general methods for the construction of $GGDD(1)$ s and $GGDD(2)$ s for specific sets of parameters have been developed and discussed in chapter 4. It is found that the $GGDD(s)$ s of maximal conceivable trace are often E - and MV -optimal. In spite of the construction of these designs, questions concerning the existence theorem of $GGDD(s)$ s for many different combinations of parameters are still very much open.

There are no general necessary and sufficient conditions known for the existence of $BIBRC$ s. However we have succeeded in establishing the necessary and sufficient conditions for $BIBRC$ s with blocks of size 2×2 (see chapter 5). We have also shown that, unlike for row-column designs, balance in the bottom stratum of a nested row and column designs does not imply general balance.

In summary, we have investigated a very useful structure in the information matrix for block designs, that offers protection against poor behavior of individual estimates (E-optimality and MV-optimality) in addition to useful interpretability of patterns in the estimated contrast variances. The cost is sometimes a very minor loss in terms of the A-criterion. If this cost is judged too high, then in some cases the structure can be sacrificed in converting to an *A*-optimal *NBBD*(2). For settings requiring two-way elimination of heterogeneity, new results concerning *BIBRC*s have also been established.

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