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Nearly Balanced and Resolvable Block Designs

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NEARLY BALANCED AND RESOLVABLE BLOCK DESIGNS

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ABSTRACT

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One of the fundamental principles of experimental design is the separation of heterogeneous experimental units into subsets of more homogeneous units or *blocks* in order to isolate identifiable, unwanted, but unavoidable, variation in measurements made from the units. Given v treatments to compare, and having available b blocks of k experimental units each, the thoughtful statistician asks, "What is the optimal allocation of the treatments to the units?" This is the basic block design problem. Let n_{ij} be the number of times treatment i is used in block j and let N be the $v \times b$ matrix $N = (n_{ij})$. There is now a considerable body of optimality theory for block design settings where binarity (all $n_{ij} \in \{0, 1\}$), and symmetry or near-symmetry of the concurrence matrix NN^T, are simultaneously achievable. Typically the same classes of designs are found to be best using any of the standard optimality criteria. Among these are the balanced incomplete block designs (BIBDs), many species of two-class partially balanced incomplete block designs, and regular graph designs.

However, there are triples (v, b, k) in which binarity *precludes* near-symmetry. For these combinatorially problematic settings, recent explorations have resulted in new optimality results and insight into the combinatorial issues involved. Of particular interest are the *irregular BIBD settings,* that is, triples (v, b, k) where the necessary conditions for a BIBD are fulfilled but no such design exists. A thorough study of the smallest such setting, (15,21,5), has produced some surprising optimal designs which will be presented in the first chapter of this document.

An incomplete block design is said to be *resolvable* if the blocks can be partitioned

into classes, or *replicates* such that each treatment appears in exactly one block of each replicate. Resolvable designs are indispensable in many industrial and agricultural experiments, especially when the entire experiment can not be completed at one time or when there is a risk that the experiment may be prematurely terminated. In chapters two and three we will investigate the classes of resolvable designs having five or fewer replications and two blocks of possibly unequal size per replicate. Theory for identifying the best designs with respect to important optimality criteria will be developed, and with the optimality theory in hand, optimal designs will be identified and constructions provided. We will conclude with a comment on the robustness of resolvable designs to the loss of a replicate.

Dedicated

to

Anne, Donald, and Jennifer Reck

whose unconditional love and support encourage me to pursue my dreams.

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TABLE OF CONTENTS

LIST OF TABLES

Table Page

LIST OF FIGURES

CHAPTER I

NEARLY AND VIRTUALLY BALANCED INCOMPLETE BLOCK DESIGNS

1.1 Introduction

A proper block design is the assignment of v treatments to $n = bk$ experimental units arranged in b blocks of identical size k, see figure 1.1. For these specified *setting*

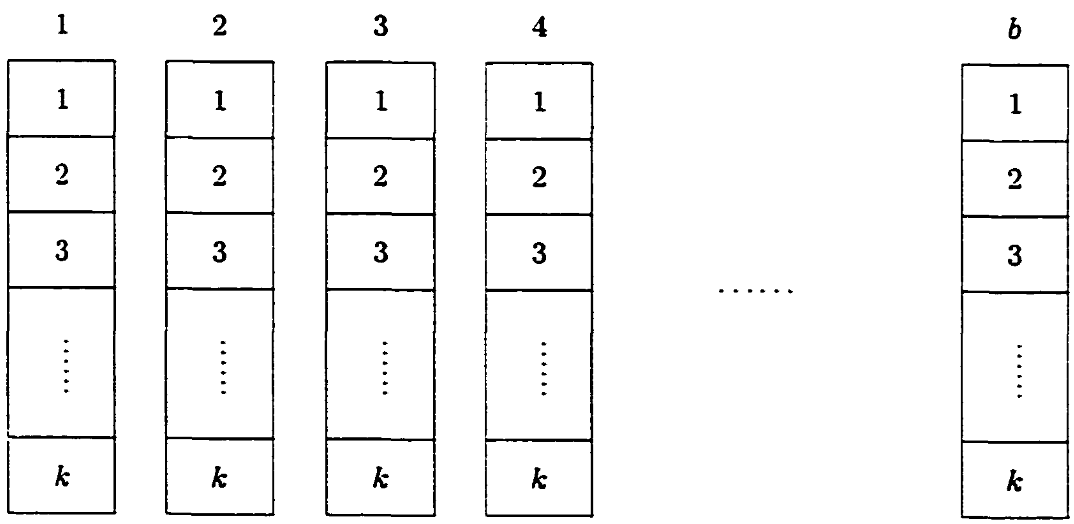


Figure 1.1: Proper Block Design Setting: b Blocks and k Plots Per Block

parameters (v, b, k), there is a potentially large, but always finite, set of feasible designs from which an experimenter much choose. Denoting this class of all possible designs by $D(v, b, k)$, the task at hand is to choose a design $d \in D(v, b, k)$ that is best,

The Model Journal used for this dissertation is *Statistica Sinica.*

that is, that in some sense (to be made rigorous below) maximizes the experimental information that will result. When $k < v$, (v, b, k) is referred to as an *incomplete block design setting*. For such settings, the *balanced incomplete block designs* (BIBDs) are known to be best with respect to all of the standard symmetric optimality criteria whenever they exist. Let n_{dij} be the number of units in block j assigned treatment i by design d. Then a BIBD is any design d for which

(i) $n_{dij} = 0$ or 1 for all i, j,

(ii) $\sum_j n_{dij} = r$ for all i,

(iii) $\sum_j n_{dij} n_{di'j} = \lambda$ for all $i \neq i'$.

Thus a BIBD is (i) *binary,* (ii) *equireplicate,* and (iii) *pairwise balanced.* The common replication for a BIBD is r, and the common pairwise concurrence is λ. These two integer-valued auxiliary parameters satisfy $r = \frac{bk}{v}$ and $\lambda = \frac{bk(k-1)}{v(v-1)}$, thereby identifying two necessary conditions for the existence of a BIBD:

$$v|bk \qquad \text{and} \qquad v(v-1)|bk(k-1). \tag{1.1}$$

When the necessary conditions (1.1) are satisfied, $D(v, b, k)$ is called a *BIBD setting.* That a BIBD need not exist in a BIBD setting (that is, the necessary conditions do not guarantee existence) has been long known; such a setting is called an *irregular* BIBD setting. Nandi (1945) proved that $D(15, 21, 5)$ is an irregular BIBD setting, and Hanani (1961) proved that (1.1) are sufficient for the existence of a BIBD for $k = 3$ and 4, establishing that the smallest block size for which a BIBD setting is irregular is $k = 5$. A comprehensive list of BIBD settings for $r \leq 41$ along with whether a BIBD exists, does not exist, or is not known, is given in Mathon and Rosa (1996). From this list we see that the minimum value of v for which an irregular BIBD setting exists is $v = 15$, and the unique setting is $D(15, 21, 5)$. The setting $D(22, 33, 12)$ has the minimum value of v for which the necessary conditions (1.1) are satisfied and for which it is not known whether a BIBD exists.

Again, if a BIBD exists, then it is optimal in a wide variety of senses. But what if a BIBD does not exist? That is, what is the best design in an irregular BIBD setting? W.G. Zang in his PhD. Thesis (1994) and Hedayat, Stufken, and Zhang (1995a, 1995b) employed a combinatorial approach to this problem, preserving the assignment properties (i) and (ii) while seeking a natural combinatorial approximation to the full balance (iii) of BIBDs. They show that the resulting designs are typically highly efficient under the commonly used optimality criteria.

Central to their approach are the concepts of *unfinished balanced incomplete block designs* and *virtually balanced incomplete block designs* (U-BIBDs and V-BIBDs, respectively). In any BIBD setting, a U-BIBD is an assignment of the first $v - w$ of the v total treatments so that BIBD properties (i)-(iii) hold for $i, i' \leq v - w$; the parameter w, called the *deficiency*, is the number of treatments yet to be assigned. Completing an U-BIBD by assigning the remaining w treatments to the blocks so that (i) and (ii) hold, and further requiring them to appear simultaneously in a block with any other treatment either $\lambda - 1$, λ, or $\lambda + 1$ times, results in a V-BIBD. For a given V-BIBD d define its *discrepancy* δ_d as the number of treatment pairs $i < i'$ occurring together in $\lambda - 1$ blocks. Then the approach of Zang (1994) and Hedayat, Stufken, and Zhang (1995a,1995b) is a two-stage search procedure:

- first find a U-BIBD with minimum w, then
- among all completions of the unfinished design(s) so determined find the d with minimal discrepancy δ_d.

Essentially this approach seeks a design containing the largest possible "sub-BIBD" (the unfinished design with minimal deficiency), then controls the departure from the full balance (iii) of a BIBD by minimizing the discrepancy induced by the w deficient treatments. Although constructing V-BIBDs in this way is effective in finding highly efficient designs in various irregular BIBD settings (Hedayat, Stufken, and Zhang, 1995a,1995b), establishing exact optimality of designs in irregular BIBD settings remains elusive.

Morgan and Srivastav (2000) address this issue by determining sufficient conditions for a member of a certain design class to be optimal with respect to a type-1 optimality criterion in irregular BIBD settings. Though they did not search for any designs, they do note that for $D(22,33,8)$ the design found by Hedayat, Stufken, and Zhang(1995a,1995b) with deficiency 2 and discrepancy 4 implies that their optimality conditions are met for the A- and D-criteria (BIBD existence is still not settled for this setting). The interesting contrast is that the combinatorial implications of Morgan and Srivastav's (2000) optimality work differ from the approach described above in that discrepancy plays a key role while treatment deficiency is not of explicit concern.

In this document the optimality results for irregular BIBD settings given by Morgan and Srivastav (2000) are extended. E-optimality is investigated, and it is found that an E-optimal design need not have minimum discrepancy. For the irregular BIBD setting $(15,21,5)$, an enumerative search is described through which the A-, D-, and E-optimal designs are found. The optimal designs do not possess minimal deficiency, though the U-BIBD concept is very helpful in sorting through the possibilities in arriving at optimal designs.

1.2 Preliminaries

Consider a proper block design setting $D(v,b,k)$. The $v \times b$ incidence matrix N_d for a design $d \in D(v,b,k)$ has elements n_{dij} that are nonnegative integers representing the number of times treatment i appears in block j. The concurrence matrix is the $v \times v$ matrix $N_d N_d^T$ whose off-diagonal elements $\sum_{j=1}^{b} n_{dij} n_{di'j} = \lambda_{dii'}$, called concurrence parameters, are the number of times treatments i and i' simultaneously appear in the same block. Under the usual additive linear model, the least squares estimates of the treatment effects τ are found by solving the normal equations $C\tau = Q$ where $Q_{v\times 1}$ is a linear combination of the experimental measurements and $C_d = \text{diag}(r_{d1}, r_{d2}, \ldots, r_{dv}) - \frac{1}{k}N_d N_d^T$ is the $v \times v$ information matrix, also called the C-matrix for design d. Here $\text{diag}(r_{d1}, r_{d2}, \ldots, r_{dv})$ is the $v \times v$ diagonal matrix containing the treatment replications. The information matrix C_d is positive semi-definite with zero sum rows, and the Moore-Penrose inverse C_d^+ is an effective variance-covariance matrix for the treatment effect estimates. All contrasts of treatment effects are estimable using design d if and only if the rank of C_d is $v - 1$, in which case d is said to be connected. Since it is desirable for all treatment contrasts to be estimable, $D(v, b, k)$ is henceforth restricted to be the class of all connected block designs. As earlier mentioned, design d is binary if $n_{dij} = 0$ or 1 for all i and j, which is the condition for maximization of the trace of C_d over $d \in D(v, b, k)$. For a block design setting $D(v, b, k)$, define $M(v, b, k)$ as the binary subclass of $D(v, b, k)$ and $M_0(v, b, k)$ as the equireplicate subclass of $M(v, b, k)$.

Because of the relationship of the information matrix to estimate variances, design optimality conditions are usually defined in terms of non-increasing, real-valued functions f of the positive eigenvalues of C_d: $0 < z_{d1} \le z_{d2} \le \cdots \le z_{d,v-1}$. A design $d \in D(v, b, k)$ is said to be ϕ_f-optimal provided $\phi_f(C_d) = \sum_{i=1}^{v-1} f(z_{di})$ is minimal over all designs in D. The function f is frequently chosen as a member of the family of type-1 criteria defined by Cheng (1978).

Definition 1.2.1 $\phi_f(C_d) = \sum_{i=1}^{v-1} f(z_{di})$ is a type-1 criterion if f is a convex, real-valued function for which

(i) f is continuously differentiable on $(0, max_{d\in D(v,b,k)} \text{ tr } C_d)$, and $f' < 0$, $f'' > 0$, $f''' < 0$ on $(0, max_{d\in D(v,b,k)} \text{ tr } C_d)$, and

(ii) f is continuous at 0 or $lim_{x\to 0} f(x) = f(0) = \infty$.

Three commonly used type-1 criteria are the A-, D- and ϕ_p-criteria which are defined by taking $f(x) = x^{-1}$, $f(x) = -\log x$, and $f(x) = x^{-p}$, $0 < p < \infty$, respectively. Since C_d^+ is the variance-covariance matrix for the treatment effect estimates, then the average variance of all $v(v-1)$ elementary treatment contrast estimates is proportional to

$$\sum_{i=1}^{v-1} z_{di}^{-1}. \tag{1.2}$$

If a design $d^* \in D(v, b, k)$ minimizes the average variance of the treatment contrast estimates, hence minimizes (1.2), over all competing $d \in D$, then d^* is A-optimal. Equivalently, the A-optimal design will minimize $\mathrm{tr}C_d^+$. In linear models with fully estimable parameter vector $\boldsymbol{\theta}$ in which $\mathrm{var}(\widehat{\boldsymbol{\theta}})$ is nonsingular, the volume of the confidence ellipsoid for $\boldsymbol{\theta}$ is proportional to $|\mathrm{var}(\widehat{\boldsymbol{\theta}})|$ = product of the eigenvalues of $\mathrm{var}(\widehat{\boldsymbol{\theta}})$. The D-criterion in the block design setting is an analogous extension: since $\mathrm{var}(\widehat{\boldsymbol{\ell}^T \boldsymbol{\theta}}) = \sigma^2 \boldsymbol{\ell}^T C_d^+ \boldsymbol{\ell}$ for every estimable $\boldsymbol{\ell}^T \boldsymbol{\tau}$, we take as the relevant volume the product of the eigenvalues of C_d^+. Then the D-optimal design $d^* \in D$ minimizes

$$\prod_{i=1}^{v-1} z_{di}^{-1} \tag{1.3}$$

or, equivalently, minimizes

$$-\sum_{i=1}^{v-1} \log z_{di}. \tag{1.4}$$

Using the ϕ_p-criterion, which is a general class of optimality criteria given by

$$\phi_p = \left(\sum_{i=1}^{v-1} z_{di}^{-p} \right)^{(1/p)}, \tag{1.5}$$

a fourth widely used criterion, called the E-criterion, is defined by

$$\phi_\infty(C_d) = \lim_{p \to \infty} \phi_p(C_d) = \max_{1 \le i \le v-1} z_{di}^{-1}. \tag{1.6}$$

A design is E-optimal if it minimizes the maximum variance of normalized treatment contrast estimates over all competing designs in D. Furthermore, when $p = 1$, minimizing (1.5) is equivalent to minimizing (1.2), that is, ϕ_1-optimal designs are A-optimal. Various optimality criteria and their statistical significance are discussed in Kiefer (1958, 1974), Cheng (1978), Shah (1960), and Shah and Sinha (1989). In the subsequent discussion we will concentrate on designs that minimize the type-1 criteria:

$$\begin{aligned} \text{A-criterion:} \quad A_d &= \sum_i z_{di}^{-1} \\ \text{D-criterion:} \quad D_d &= -\sum_i \log(z_{di}) \\ \text{E-criterion:} \quad E_d &= z_{d1}^{-1}. \end{aligned} \tag{1.7}$$

Optimality criteria can also be used to compare two designs, d and $\bar{d}$ say, using the *relative efficiency* of design d compared to design $\bar{d}$.

Definition 1.2.2 The relative efficiency of a design $d \in D$ compared to another design $\bar{d} \in D$ with respect to the A-, D-, and E-optimality criteria are:

$$\text{A-efficiency} = \frac{A_{\bar{d}}}{A_d}, \quad \text{D-efficiency} = \frac{D_{\bar{d}}}{D_d}, \quad \text{and} \quad \text{E-efficiency} = \frac{E_{\bar{d}}}{E_d}.$$

When $D(v, b, k)$ is a BIBD setting, that is, when the necessary conditions (1.1) are satisfied, the average treatment concurrence λ is $\lambda = \frac{r(k-1)}{v-1}$ and a BIBD d achieves equality of treatment concurrences, that is, $\lambda_{dii'} = \lambda$ for all $i \neq i'$. If a BIBD exists, it is the universally optimal design in $D(v, b, k)$ (Kiefer, 1975), which includes optimality with respect to *all* type-1 criteria. Of concern here are the irregular BIBD settings, for which the conditions (1.1) hold but the combinatorics do not allow $\lambda_{dii'} = \lambda$ for all $i \neq i'$. What is the optimal or most efficient design in an irregular BIBD setting? After reviewing and extending some previously known results concerning irregular BIBD settings, we will observe some of their surprising consequences in $D(15, 21, 5)$.

1.3 Definitions and Results

We begin by formally defining some of the concepts and terms introduced above. Afterward we will develop optimality theorems and proofs. In the next section we will apply the results to the irregular BIBD setting $(v, b, k) = (15, 21, 5)$.

Definition 1.3.1 An *unfinished balanced incomplete block design* with deficiency w, denoted by U-BIBD$(v, b, k; w)$, is a block design containing $v-w$ of v total treatments in b blocks of size k such that

(i) each $n_{dij} = 0$ or 1, $i = 1, 2, \ldots, v-w$

(ii) each $r_{di} = r$, $i = 1, 2, \ldots, v-w$

(iii) $\lambda_{dii'} = \lambda$, $i \neq i' \in \{1, 2, \ldots, v-w\}$.

Definition 1.3.2 A *virtually balanced incomplete block design,* denoted V-BIBD$(v, b, k; w)$, for v treatments in b blocks of size k such that

(i) each $n_{dij} = 0$ or 1,

(ii) each $r_{di} = r$,

(ii) $\lambda_{dii'} = \lambda$, $i \neq i' \in \{1, 2, \ldots, v-w\}$, and

(iv) $\lambda_{dii'} \in \{\lambda - 1, \lambda, \lambda + 1\}$, $i > v - w$ or $i' > v - w$, $i \neq i'$.

Thus a V-BIBD$(v, b, k; w)$ contains a U-BIBD$(v, b, k; w)$, and the remaining w treatments have been assigned in such a way that all of their concurrences are within one of the ideal common concurrence λ.

Definition 1.3.3 The *concurrence range* of a block design $d \in D(v, b, k)$ is a measure of its maximum pairwise unbalance and is given by

$$l_d = \max_{i \neq i', j \neq j'} |\lambda_{dii'} - \lambda_{djj'}|.$$

Definition 1.3.4 A *nearly balanced incomplete block design* $d \in D(v,b,k)$ with concurrence range l, or NBBD(l), is an incomplete block design satisfying the following conditions:

(i) each $n_{dij} = 0$ or 1,

(ii) each $r_{di} = r$ or $r+1$,

(iii) $l_d = l$,

(iv) d minimizes tr C_d^2 over all designs satisfying $(i)-(iii)$.

Clearly in a BIBD setting, when $r_{di} = r$ for all i and $l = 0$, the definition of an NBBD(l) reduces to that of a BIBD. If for a design $d \in M(v,b,k)$, combinatorics force $\lambda_{dii'} \leq \lambda - 1$ for at least one treatment pair $i \neq i'$, then for some other treatment pair $s \neq s'$, $\lambda_{dss'} \geq \lambda + 1$ and the nonexistence of a NBBD(l) with $l \leq 1$ follows. Such settings were generally referred to as *category one* settings by Morgan and Srivastav (2000) and include irregular BIBD settings. In an irregular BIBD setting a NBBD(2) is the V-BIBD having minimum tr C_d^2. Thus in an irregular BIBD setting, NBBD(2)s are a subclass of V-BIBDs.

Definition 1.3.5 The *pairwise concurrence discrepancy,* for treatments i and i', $1 \leq i \neq i' \leq v$, of a design $d \in D(v,b,k)$ is the quantity

$$\delta_{dii'} = \lambda_{dii'} - \lambda.$$

The *concurrence discrepancy* for d is

$$\delta_d = \sum\sum_{i<i'} \max\{0, -\delta_{dii'}\}$$

and is a measure of the combinatorial asymmetry of the design. The *minimum discrepancy* over the binary subclass $M(v,b,k)$ is denoted by

$$\delta = \min_{d \in M} \delta_d.$$

If $d \in D(v,b,k)$ is a BIBD, then $\delta_d = 0$ and consequently $\delta = 0$. A BIBD setting is irregular if and only if $\delta \geq 2$. In the sequel, frequently the treatment concurrence discrepancy will be shortened to *treatment discrepancy* and the concurrence discrepancy to *discrepancy*. We now state a lemma relating the discrepancy of a design to the maximum treatment unbalance of the design.

Lemma 1.3.1 *(Morgan and Srivastav, 2000) Let d be a binary, equireplicate design in a BIBD setting $D(v,b,k)$. Then $\delta_d \geq 2\max_{i,i'} |\delta_{dii'}|$.*

Not much is known about optimality in irregular BIBD settings. Intuitively it is desirable to find a design with minimum discrepancy δ_d, i.e. the most balanced design, and evidence suggests the efficiency of a design improves as the design discrepancy decreases (Hedayat, Stufken, and Zhang; 1995a, 1995b), but determining the minimum discrepancy δ for a design setting can be combinatorially difficult. For the setting $D(15,21,5)$, Zhang (1994) and Hedayat, Stufken, and Zhang (1995a, 1995b) investigated A-, D-, and E-efficiency by constructing VBIBDs with minimum discrepancy δ_d for the smallest possible treatment deficiency w. They discovered that the smallest treatment deficiency for this setting was $w = 3$, and for $w = 3$ the minimum discrepancy design reported was the $\delta_d = 6$ shown in table 1.1. Although

Table 1.1: Zhang's (1994) Most Efficient $D(15,21,5)$ Design

1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5
2	2	3	4	5	7	8	3	4	5	6	8	4	5	6	7	5	7	8	6	6
3	6	9	6	9	10	11	11	7	10	11	9	6	7	8	12	8	9	10	7	9
4	7	10	12	13	12	15	12	9	13	13	10	10	8	9	14	11	11	13	10	12
5	8	11	13	15	14	14	13	15	14	14	12	15	13	14	15	12	14	15	11	15

this design was the most A-, D-, and E-efficient design they found in the class, having respective optimality values of 2.33781, -25.07125, and 0.18149, they did not claim that the minimum discrepancy for the class is $\delta = 6$ nor did they claim their design to be the A-, D-, or E-optimal design in the class.

Morgan and Srivastav (2000) addressed the optimality problem by describing sufficient conditions for a NBBD(2) to be optimal in a category one design setting $D(v,b,k)$. Conditions for optimal designs in an irregular BIBD setting are consequences of their main result and are given explicitly as a corollary. First we will review and extend their main result, and later we will use the result to state and prove a slightly more general corollary for irregular BIBD settings.

For a general design setting $D(v,b,k)$, let $\bar{d} \in D$ be a NBBD(l) with discrepancy value $\delta_{\bar{d}}$. Optimality arguments can be constructed around $\bar{d}$ as a function of the traces of its information matrix and its square, so define the quantities

$$A = \text{tr } C_{\bar{d}} \quad \text{and} \quad B_2 = \text{tr } C_{\bar{d}}^2 + \frac{4}{k^2}$$

where for a binary design d,

$$\text{tr } C_d^2 = \left(\frac{k-1}{k}\right)^2 \sum_{i=1}^{v} r_{di}^2 + \frac{2}{k^2} \sum_{i<i'}\sum \lambda_{dii'}^2. \tag{1.8}$$

Let z_1 and z_1^* be upper bounds for the minimum nonzero eigenvalues z_{d1} of designs in $M(v,b,k)$ and $D(v,b,k)$, respectively, which satisfy

$$(A-z_1)^2 \geq B_2 - z_1^2 \geq \frac{(A-z_1)^2}{(v-2)} \text{ and } (A-z_1^*)^2 \geq B_2 - z_1^{*2} \geq \frac{(A-z_1^*)^2}{(v-2)}.$$

Given z_1 and for $P = [(B_2 - z_1^2) - \frac{(A-z_1)^2}{(v-2)}]^{1/2}$, define z_2 and z_3 by

$$z_2 = [(A-z_1) - \sqrt{\frac{(v-2)}{(v-3)}}P_2]/(v-2) \text{ and } z_3 = [(A-z_1)+\sqrt{(v-2)(v-3)}P_2]/(v-2).$$

Let $z_4 = [A - (2/k) - z_1^*]/(v-2)$ be the common nonzero eigenvalue of completely a symmetric information matrix with trace equal to $A - (2/k)$. All of these quantities are integral to Morgan and Srivastav's (2000) main result, stated next, as well as the generalization for irregular BIBD settings to follow.

Theorem 1.3.2 (Morgan and Srivastav, 2000) *Let $D(v,b,k)$ be a setting with $\delta > 0$, let $\bar{d} \in D(v,b,k)$ be a NBBD(2) with information matrix $C_{\bar{d}}$ having nonzero* *eigenvalues* $z_{\bar{d}1} \leq z_{\bar{d}2} \leq \cdots \leq z_{\bar{d},v-1}$ *and having* $\delta_{\bar{d}} = \delta > 0$, *and let* f *be a convex, real-valued function satisfying the conditions of definition 1.2.1. Let* $z_1 = \frac{r(k-1)+\lambda-1}{k}$ *and* $z_1^* = \frac{r(k-1)v}{(v-1)k}$. *Then if* $z_1 \leq z_2$ *and*

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) < f(z_1) + (v-3)f(z_2) + f(z_3), \tag{1.9}$$

a ϕ_f*-optimal design in* $M(v,b,k)$ *must be a NBBD(2). If, moreover,* $z_1^* \leq z_4$ *and*

$$\sum_{i=1}^{v-1} f(z_{\bar{d}i}) < f(z_1^*) + (v-2)f(z_4), \tag{1.10}$$

then a ϕ_f*-optimal design in* $D(v,b,k)$ *must be an NBBD(2).*

Theorem 1.3.3 (Morgan and Srivastav, 2000) *Let* $D(v,b,k)$ *be an irregular BIBD setting, and let* $\bar{d} \in D$ *be a* $NBBD(2)$ *with* $\delta_{\bar{d}} \leq 4$. *Taking* $z_1 = z_1^* = \frac{\lambda v-1}{k}$, *if (1.9) and (1.10) of Theorem 1.3.2 hold, then a* ϕ_f*-optimal design must be a* $NBBD(2)$.

For irregular BIBD settings with $r \leq 41$, Morgan and Srivastav (2000) prove as a corollary to Theorem 1.3.3 that a NBBD(2) is A- and D-optimal, provided that such a design exists and that $\delta \leq 4$. We will extend their result to $\delta \leq 5$, but first, we state their corollary and prove a slightly more general version of Theorem 1.3.3.

Corollary 1.3.4 *Let* $D(v,b,k)$ *be an irregular BIBD setting in which* $r \leq 41$. *If there exists a a design* $\bar{d}$ *satisfying the first three conditions of definition 1.3.4 with* $l_d = 2$ *and* $\delta_{\bar{d}} \leq 4$, *then a A-optimal design must be a NBBD(2), and a D-optimal design must be a NBBD(2).*

The next lemma will be necessary for the proof of our generalization of Theorem 1.3.3.

Lemma 1.3.5 *Let* $D(v,b,k)$ *be an irregular BIBD setting, and suppose* $d \in D$ *has discrepancy* $\delta_d \geq 2$ *and concurrence range* $l_d \geq 2$. *If* $\gamma_{d\alpha}^+$ *and* $\gamma_{d\alpha}^-$ *are the number of* *times* $\lambda+\alpha$ *and* $\lambda-\alpha$, $\alpha = 1,2,\ldots,l_d-1$, *appear below the diagonal of the* $v\times v$ *concurrence matrix* $N_dN_d^T$, *respectively, then*

$$\sum\sum_{i<i'} \lambda_{dii'}^2 = \frac{v(v-1)}{2}\lambda^2 + 2\delta_d + \sum_{\alpha=2}^{l_d-1}\alpha(\alpha-1)\,\gamma_{d\alpha}^+ + \sum_{\alpha=2}^{l_d-1}\alpha(\alpha-1)\,\gamma_{d\alpha}^-. \qquad (1.11)$$

Proof Suppose $N_dN_d^T$ is the $v\times v$ concurrence matrix for a design $d\in D(v,b,k)$ having discrepancy δ_d and concurrence range l_d. If $N_dN_d^T$ has $\gamma_{d\alpha}^+$ occurrences of $\lambda+\alpha$ and $\gamma_{d\alpha}^-$ occurrences of $\lambda-\alpha$, $\gamma_{d\alpha}^+\ge 0$, $\gamma_{d\alpha}^-\ge 0$, and $\alpha = 2,3,\ldots,l_d-1$, below the diagonal, then there are $\delta_d - \sum_{\alpha=2}^{l_d-1}\alpha\,\gamma_{d\alpha}^+$ occurrences of $\lambda+1$, $\delta_d - \sum_{\alpha=2}^{l_d-1}\alpha\,\gamma_{d\alpha}^-$ occurrences of $\lambda-1$, and $[\frac{v(v-1)}{2} - 2\delta_d + \sum_{\alpha=2}^{l_d-1}(\alpha-1)\gamma_{d\alpha}^+ + \sum_{\alpha=2}^{l_d-1}(\alpha-1)\gamma_{d\alpha}^-]$ occurrences of λ below the diagonal. Therefore

$$\begin{aligned}\sum\sum_{i<i'}\lambda_{dii'}^2 \;=\;& \sum_{\alpha=2}^{l_d-1}\gamma_{d\alpha}^+(\lambda+\alpha)^2 + \left(\delta_d - \sum_{\alpha=2}^{l_d-1}\alpha\,\gamma_{d\alpha}^+\right)(\lambda+1)^2 + \\ & \sum_{\alpha=2}^{l_d-1}\gamma_{d\alpha}^-(\lambda-\alpha)^2 + \left(\delta_d - \sum_{\alpha=2}^{l_d-1}\alpha\,\gamma_{d\alpha}^-\right)(\lambda-1)^2 + \\ & \left[\frac{v(v-1)}{2} - 2\delta_d + \sum_{\alpha=2}^{l_d-1}(\alpha-1)\,\gamma_{d\alpha}^+ + \sum_{\alpha=2}^{l_d-1}(\alpha-1)\,\gamma_{d\alpha}^-\right]\lambda^2.\end{aligned}$$

The result follows by expanding the above expression and collecting on λ. □

Corollary 1.3.6 *Let* $D(v,b,k)$ *be an irregular BIBD setting. If* $\bar d\in D$ *has* $(\delta_{\bar d}, l_{\bar d}) = (5,2)$, *or* $(\delta_{\bar d}, l_{\bar d}) = (4,3)$ *with* $\delta_{\bar d2}^- + \delta_{\bar d2}^+ = 1$, *then*

$$\sum\sum_{i<i'}\lambda_{\bar dii'}^2 = \frac{v(v-1)}{2}\lambda^2 + 10.$$

Furthermore, if no design having $l_d = 2$ *has* $\delta_d \le 4$, *then any* $d\in D$ *not satisfying the conditions of* $\bar d$ *has*

$$\sum\sum_{i<i'}\lambda_{dii'}^2 \ge \frac{v(v-1)}{2}\lambda^2 + 12.$$

Theorem 1.3.7 *Let* $D(v,b,k)$ *be an irregular BIBD setting for which a NBBD(2) with* $\delta_d\le 4$ *does not exist. Let* $\bar d\in D$ *be a NBBD(2) with* $\delta_{\bar d} = 5$, *or a NBBD(3) with* $\delta_{\bar d} = 4$ *and* $\gamma_{\bar d2}^- + \gamma_{\bar d2}^+ = 1$. *For* $z_1 = z_1^* = \frac{\lambda v - 1}{k}$, *if (1.9) and (1.10) of Theorem 1.3.2 hold, then a* ϕ_f*-optimal design must be a NBBD(2) or a NBBD(3).*

Proof The bounds $z_1 = z_1^*$ for z_{d1} follow from lemma 2.2 of Morgan and Srivastav (2000) for unequally replicated d and from propositions 3.1 and 3.2 of Jacroux (1980b) for equireplicated d. The relations $z_1 \leq z_2$ and $z_1^* \leq z_4$ are easy to check. From the proof of Theorem 1.3.3 (Morgan and Srivastav, 2000, page 10), the ϕ_f-optimal design must be binary if condition (1.10) is satisfied for z_1^* and z_4.

Suppose binary $d \in M(v, b, k)$ is not a NBBD(2) or a NBBD(3) as described in the theorem. Then it must be true that either (i) d is not equireplicate; (ii) d is in M_0, has $l_d \geq l_{\bar{d}}$, and $\delta_d > \delta_{\bar{d}}$; (iii) d in M_0, has $l_d > l_{\bar{d}}$, and $\delta_d \geq \delta_{\bar{d}}$; or (iv) d is in M_0, has $(\delta_d, l_d) = (4, 3)$, and $\gamma_{d2}^- + \gamma_{d2}^+ \geq 2$. It will be established that for each of these cases, tr $C_d^2 \geq B_2$.

Case (i). If d is not equireplicate, then $\delta_d \geq 4$ (Morgan and Srivastav, 2000, page 18) which implies $l_d \geq 2$, and, from lemma 1.3.5, $\sum\sum_{i<i'} \lambda_{dii'}^2 \geq \frac{v(v-1)}{2}\lambda + 8$. Thus, by corollary 1.3.6,

$$\sum\sum_{i<i'} \lambda_{dii'}^2 - \sum\sum_{i<i'} \lambda_{\bar{d}ii'}^2 \geq -2.$$

Furthermore, from the proof of Theorem 1.3.2 (Morgan and Srivastav, 2000, page 10),

$$\sum_{i=1}^{v} r_{di}^2 - \sum_{i=1}^{v} r_{\bar{d}i}^2 \geq 2.$$

Therefore, from (1.8),

$$\text{tr } C_d^2 - \text{tr } C_{\bar{d}}^2 \geq 2\left(\frac{k-1}{k}\right)^2 + \frac{2}{k^2}(-2) = \frac{2(k^2-2k-1)}{k^2} \geq \frac{4}{k^2}$$

for $k \geq 3$.

Case (ii). Suppose d is in M_0, has discrepancy $\delta_d > \delta_{\bar{d}}$, and concurrence range $l_d \geq l_{\bar{d}}$. Then, from corollary 1.3.6,

$$\sum\sum_{i<i'} \lambda_{dii'}^2 - \sum\sum_{i<i'} \lambda_{\bar{d}ii'}^2 \geq 2,$$

and, from (1.8),

$$\text{tr } C_d^2 - \text{tr } C_{\bar{d}}^2 \geq \frac{4}{k^2}.$$

Case (iii). Suppose d is in M_0, has discrepancy $\delta_d \geq \delta_{\bar{d}}$, and concurrence range $l_d > l_{\bar{d}}$. Then, from corollary 1.3.6,

$$\sum_{i<i'}\sum \lambda_{dii'}^2 - \sum_{i<i'}\sum \lambda_{\bar{d}ii'}^2 \geq 2,$$

and, from (1.8),

$$\text{tr } C_d^2 - \text{tr } C_{\bar{d}}^2 \geq \frac{4}{k^2}.$$

Case (iv). Suppose d is in M_0, has $(\delta_d, l_d) = (4, 3)$, and $\delta_{d2}^- + \delta_{d2}^+ = 2$. Then, again from corollary 1.3.6,

$$\sum_{i<i'}\sum \lambda_{dii'}^2 - \sum_{i<i'}\sum \lambda_{\bar{d}ii'}^2 = 2,$$

and

$$\text{tr } C_d^2 - \text{tr } C_{\bar{d}}^2 = \frac{4}{k^2}.$$

The result follows from Theorem 1.3.2. □

The information matrix for a design $d \in M_0(v, b, k)$ can be written as

$$C_d = \frac{\lambda v}{k}(I - \frac{1}{v}J) - \frac{1}{k}\Delta_d, \tag{1.12}$$

where Δ_d is the $v \times v$, possibly null, *discrepancy matrix* for the design and has elements

$$(\Delta_d)_{ii'} = \begin{cases} \delta_{dii'}, & \text{for } i \neq i' \\ 0, & \text{for } i = i'. \end{cases}$$

Equation (1.12) says that the information matrix for any design in M_0 is completely described by the discrepancy matrix Δ_d, which depends on the discrepancy δ_d and concurrence range l_d of the design. Moreover, with an appropriate labeling, the treatments $i \neq i'$ having $\lambda_{dii'} \leq \lambda - 1$ can, for some $s \leq v$, be restricted to the first s members of the treatment set, and hence, the nonzero elements of Δ_d can be restricted to the first s rows and columns. Furthermore, $C_d\mathbf{1} = 0$ implies that $\Delta_d\mathbf{1} = 0$; consequently, any $s \times s$ integer-valued matrix having zeros on the diagonal and zero-sum rows and columns is a principal minor for discrepancy matrices of designs in

$M_0(v, b, k)$ for all $v \geq s$. Therefore, by enumerating a complete list of nonisomorphic discrepancy matrices for fixed values of δ_d and l_d, optimality competitors for large classes of designs are characterized, and in some cases, as will seen in corollaries 1.3.4 and 1.3.8 below, conditions for optimality in irregular BIBD settings with respect to various criteria can be derived. The 11 discrepancy matrices having $\delta_d \leq 4$ and $l_d = 2$ are provided by Morgan and Srivastav (2000, page 19), and we have enumerated the 40 discrepancy matrices having $(\delta_d, l_d) = (5, 2)$, or $(\delta_d, l_d) = (4, 3)$ and $\gamma_{d2}^- + \gamma_{d2}^+ = 1$. The complete list of the principal minors of all 51 discrepancy matrices can be found in Appendix A.

Corollary 1.3.8 *Let $D(v, b, k)$ be an irregular BIBD setting in which $r \leq 41$ and for which a desing with $l_d = 2$ and $\delta_d \leq 4$ does not exist. If there exists a design $\bar{d}$ satisfying the first three conditions of the NBBD(l) definition and having $(\delta_{\bar{d}}, l_{\bar{d}}) = (5, 2)$, or $(\delta_{\bar{d}}, l_{\bar{d}}) = (4, 3)$ with $\gamma_{\bar{d}2}^- + \gamma_{\bar{d}2}^+ = 1$, then an A-optimal design d must be a NBBD(2) or a NBBD(3), and a D-optimal design d must be a NBBD(2) or a NBBD(3).*

Proof The corollary amounts to saying that conditions (1.9) and (1.10) of Theorem 1.3.2 hold for all equireplicate, binary designs $\bar{d}$ having $(\delta_{\bar{d}}, l_{\bar{d}}) = (5, 2)$, or $(\delta_{\bar{d}}, l_{\bar{d}}) = (4, 3)$ with $\gamma_{\bar{d}2}^- + \gamma_{\bar{d}2}^+ = 1$, in all irregular BIBD settings with $r \leq 41$. The list of settings $D(v, b, k)$ satisfying the necessary conditions for the existence of a BIBD with $r \leq 41$ for which either a BIBD does not exist or for which existence is not known found in Mathon and Rosa (1996) has 497 cases when complements are included. Since the proof of Theorem 1.3.7 establishes that designs d not satisfying the conditions of $\bar{d}$ will have tr $C_d^2 \geq B_2 \geq$ tr $C_{\bar{d}}^2 + \frac{4}{k^2}$, following a procedure analogous to the one used by Morgan and Srivastav (2000, pages 18-20) in their proof of corollary 1.3.4, the result can be established for all designs in irregular BIBD settings with $r \leq 41$, by checking (1.9) and (1.10) for each of the 51 conceivable information matrices listed in Appendix A in each of the 497 potentially irregular BIBD design settings. A computer program written to accomplish this task found that conditions (1.9) and (1.10) do in fact always hold. Therefore, the theorem is established for essentially all of the cases of practical interest. □

With corollaries 1.3.4 and 1.3.8 in hand, we return to the irregular BIBD setting $D(15, 21, 5)$. The discrepancy matrices in Appendix A are listed in A- and D-value order from smallest or optimal to largest for this setting (the order is the same with respect to both criteria). This ranking is not the same for E-values, nor necessarily maintained for different parameter sets (v, b, k). We can, however, make a few useful observations from the list. First, as explained by Morgan and Srivastav's (2000) corollary 1.3.4, designs $d \in D$ having a discrepancy matrix with $\delta_d \leq 4$ and $l_d = 2$ are A- and D-superior to designs having any other discrepancy matrix in the list; however, designs with $(\delta_d, l_d) = (5, 2)$ may either be A- and D-superior or inferior to $(\delta_d, l_d) = (4, 3)$ designs. For example, design D12 is A- and D-superior to design D13 while design D13 is A- and D-superior to design D18. Also observe that minimum deficiency does not imply minimum discrepancy, and A- and D-value rank and design deficiency are not related. These facts are evident in the $(\delta_d, l_d) = (4, 2)$ group. According to corollaries 1.3.4 and 1.3.8, Zhang's (1994) design $d \in D(15, 21, 5)$ having $(\delta_d, l_d, w) = (6, 2, 3)$ given in table 1.1 is A- and D-inferior to a design having any of the 51 discrepancy matrices shown in the appendix, whenever they exist. Furthermore, since the first step of Zhang's search for efficient designs was to minimize deficiency, thereby restricting the search to designs with $w = 3$, there are 35 discrepancy candidates in the list with $w > 3$ that, if a design in $D(15, 21, 5)$ exists corresponding to one of these candidates, is A- and D-superior to Zhang's design shown in table 1.1. We will use these observations in section 1.4 to construct the A-optimal and D-optimal design in $D(15, 21, 5)$, and in section 1.5 we will address the issue of finding the E-optimal design.

1.4 Search for the A- and D-optimal design

Now the theory of section 1.3 will be turned to the problem of finding optimal designs in $D(15,21,5)$. If we can construct a design $d \in D$ having one of the discrepancy matrices listed in Appendix A, then corollaries 1.3.4 and 1.3.8 guarantee the A- and D-optimal design exists and is either d itself or a design having one of the discrepancy matrices appearing sooner in the list than the discrepancy matrix contained in d. Thus our initial universe is possible designs $d \in D(15,21,5)$ having $\delta_d \leq 5$ and $l_d = 2$, or $(\delta_d, l_d) = (4,3)$ and $\gamma_{d2}^- + \gamma_{d2}^+ = 1$. Moreover, the treatment deficiency for this class satisfies $2 \leq w \leq 5$.

In order to make our initial attempt at constructing the A- and D-optimal design more manageable, we will search for $\delta_d \leq 4$, and consequently, impose the limit $2 \leq w \leq 4$. These restrictions imply that a successful search will result in a design d containing a U-BIBD$(15,21,5;w)$ for $w = 4$ (the existence of U-BIBDs with $w = 4$ is guaranteed by the fact that Zhang's design (1994) in table 1.1 has $w = 3$). Therefore, our search will first concentrate on constructing an exhaustive list of nonisomorphic U-BIBDs for the smallest $w \geq 4$ that can be managed, say w^*. The list will be exhaustive because all possible placements of the first $v - w^*$ treatments into the blocks will be accounted for, and each U-BIBD will be nonisomorphic in that it will be unique with respect to all possible treatment relabelings and block relabelings. Once we have an exhaustive and nonisomorphic list of U-BIBDs for $w = w^*$, if $w^* > 4$, we will enumerate all possible extensions of each design in the list to U-BIBDs with $w = 4$. Finally, all possible completions of each U-BIBD(15,21,5;4) in the list, by addition of the w^* missing treatments, will be enumerated taking into account the discrepancy and concurrence range restrictions described above.

In order to get a handle on the search, there are two lemmas concerning admissible block sizes and treatment placements that will be very useful to the process. Before we state and prove the lemmas, we will review two sets of equations given by Zhang

(1994).

If n_i is the number of blocks of size i, $0 \leq i \leq k$, then the block sizes of a U-BIBD(v,b,k;w) must satisfy the following *block size* equations:

$$\begin{aligned} \sum_{i=0}^{k} n_i &= b \\ \sum_{i=1}^{k} i n_i &= r(v-w) \\ \sum_{i=2}^{k} \binom{i}{2} n_i &= \lambda \binom{v-w}{2}. \end{aligned} \tag{1.13}$$

If θ_{ti} is the number of blocks of size i containing treatment t, then any treatment t in a U-BIBD(v,b,k;w) design must satisfy the following *theta pattern equations*:

$$\begin{aligned} \sum_{i=1}^{k} \theta_{ti} &= r \\ \sum_{i=2}^{k} (i-1)\theta_{ti} &= \lambda(v-w-1). \end{aligned} \tag{1.14}$$

From equations (1.13) and (1.14), the theoretically possible block sizes for a U-BIBD(15,21,5;4) are given in table 1.2, and from (1.14) the possible theta patterns

Table 1.2: U-BIBD(15,21,5;4) Theoretical Block Sizes

n_1	n_2	n_3	n_4	n_5
0	0	12	4	5
0	1	9	7	4
0	2	6	10	3
1	0	6	12	2
0	3	3	13	2
1	1	3	15	1
0	4	0	16	1
1	2	0	18	0

are given in table 1.3. Using table 1.3 we can reduce the theoretical block size list, table 1.2, by use of the following lemma.

Table 1.3: U-BIBD(15,21,5;4) Theoretical Theta Pattern

θ_{t1}	θ_{t2}	θ_{t3}	θ_{t4}	θ_{t5}
2	0	0	0	5
1	1	0	1	4
1	0	2	0	4
0	2	1	0	4
1	0	1	2	3
0	2	0	2	3
0	1	2	1	3
0	0	4	0	3
1	0	0	4	2
0	1	1	3	2
0	0	3	2	2
0	1	0	5	1
0	0	2	4	1
0	0	1	6	0

Lemma 1.4.1 *The number of blocks of size five in a U-BIBD(15,21,5;4) is necessarily greater than one.*

Proof Suppose $n_5 = 0$. Then from table 1.2 $\mathbf{n} = (n_1, n_2, n_3, n_4, n_5) = (1, 2, 0, 18, 0)$, and since $\theta_{t5} = 0 \ \forall \ t$, from table 1.3 $\boldsymbol{\theta}_t = (\theta_{t1}, \theta_{t2}, \theta_{t3}, \theta_{t4}, \theta_{t5}) = (0, 0, 1, 6, 0) \ \forall \ t$. This is a contradiction because it is clearly not simultaneously possible for all $\theta_{t1} = 0$ and $n_1 = 1$. Now suppose $n_5 = 1$. Then $\mathbf{n} = (1, 1, 3, 15, 1)$ or $(0, 4, 0, 16, 1)$ and the possible $\boldsymbol{\theta}_t$ are $\boldsymbol{\theta}_{t(1)} = (0, 0, 1, 6, 0)$, $\boldsymbol{\theta}_{t(2)} = (0, 0, 2, 4, 1)$, or $\boldsymbol{\theta}_{t(3)} = (0, 1, 0, 5, 1)$. Let x_j be the number of treatments with theta pattern $\boldsymbol{\theta}_{t(j)}$, $j = 1, 2, 3$. Then

$$\sum_{j=1}^{3} x_j \boldsymbol{\theta}_{t(j)} = (n_1, 2n_2, 3n_3, 4n_4, 5n_5).$$

Thus

$$x_1(0, 0, 1, 6, 0) + x_2(0, 0, 2, 4, 1) + x_3(0, 1, 0, 5, 1) = (1, 2, 9, 60, 5) \text{ or } (0, 8, 0, 64, 5) \tag{1.15}$$

for the two respective values of $\mathbf{n}$. The first system in (1.15) gives us the equations

$$x_3 = 2$$

$$\begin{aligned} x_1 + 2x_2 &= 9 \\ 6x_1 + 4x_2 + 5x_3 &= 60 \\ x_2 + x_3 &= 5. \end{aligned}$$

These equations are inconsistent. The second system in (1.15) yields the equations

$$\begin{aligned} x_3 &= 8 \\ x_1 + 2x_2 &= 0 \\ 6x_1 + 4x_2 + 5x_3 &= 60 \\ x_2 + x_3 &= 5. \end{aligned}$$

These equations are also inconsistent. Therefore $n_5 \neq 1$, and $n_5 \geq 2$. □

The reduced list of possible block sizes for U-BIBD(15,21,5;4) is shown in table 1.4.

Table 1.4: U-BIBD(15,21,5;4) Theoretical Block Sizes - Reduced list

n_1	n_2	n_3	n_4	n_5
0	0	12	4	5
0	1	9	7	4
0	2	6	10	3
1	0	6	12	2
0	3	3	13	2

We can assume the first five treatments of all U-BIBD(15,21,5;4)s occur in the first block, for otherwise, we can rename treatments and blocks so that this is the case. The placement of the first five treatments, requiring each treatment to be present in the first block, results in exactly one U-BIBD(15,21,5;10). The design is shown in table 1.5. Notice that a U-BIBD(15,21,5;10) must use all 21 blocks.

When we extend table 1.5 to a U-BIBD(15,21,5;w), $w \leq 10$ we can use the following useful lemma.

Table 1.5: A U-BIBD(15,21,5;10) Design

1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5
2	2	3	4	5			3	4	5			4	5			5				
3																				
4																				
5																				

Lemma 1.4.2 *For any U-BIBD(15,21,5;w),* $w \leq 10$ *the following are true:*

1. *A block of size five can have at most two treatments in common with any other block, and*
2. *It is not possible for the design to contain two identical blocks of size four.*

Proof The first statement follows immediately from the uniqueness of UBIBD(15,21,5;10). If there are two identical blocks of size four, say

1	1
2	2
3	3
4	4,

then none of treatments 1-4 can occur again in a common block, but each must occur 5 more times. Hence 20 more blocks are required, a contradiction. □

Since U-BIBD(15,21,5;4)s must have at least two blocks of size five, we will extend our U-BIBD(15,21,5;10) to a U-BIBD(15,21,5;w) containing two blocks of size five with the largest possible value of w (i.e. the maximum number of missing treatments), depending on the structure of the size five blocks. Since the treatments in a block of size five can have only one structure throughout the design (table 1.5) in addition to lemma 1.4.2, we can take advantage of this structure when adding treatments to the design. Thus, any two blocks of size five must have at least one and at most two treatments in common, and we need only investigate these two cases.

Since our U-BIBD(15,21,5;10) (table 1.5) is symmetric in all five treatments (i.e. any renaming of treatments will result in an identical design), we can assume in a design where the two size-five blocks have one treatment in common, that each size-five block contains treatment 1 (*one common case*), and in a design where the two size-five blocks have two treatments in common, that each size-five block contains treatments 1 and 2 (*the two common case*).

1.4.1 One Common Case

If we extend our U-BIBD(15,21,5;10) to a U-BIBD(15,21,5;w) with exactly two blocks of size five having one treatment in common and having maximum w subject to lemma 1.4.2, then $w = 6$ and $v - w = 9$. The two blocks of size five are

$$\begin{array}{cc} 1 & 1 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \\ 5 & 9 \end{array},$$

and we begin with the structure shown in table 1.6.

Table 1.6: One-common Starter

1	1	1	1	1	1	1	2	2	2	3	3	4	2	2	3	3	4	4	5	5
2	6	2	3	4	5		3	4	5	4	5	5								
3	7																			
4	8																			
5	9																			
section one							section two						section three							

Since $\lambda = 2$, the sub-block candidates that must be added, all in separate blocks, to table 1.6 are shown in table 1.7.

For convenience, as can be seen in tables 1.7 and 1.6, treatment pairs will be referred to as *doubles* and single treatments as *singles*, and the seven blocks containing treatment 1 are referred to as *section one*, the remaining six blocks of size two as *section two*, and the other eight blocks of size one as *section three* in the following discussion.

Table 1.7: Assignment Candidates - One-common Starter

doubles						singles											
6	6	6	7	7	8	6	6	6	7	7	7	8	8	8	9	9	9
7	8	9	8	9	9												

Since removing treatments 2 to 5 from the resulting U-BIBD(15,21,5;6) will give a U-BIBD(15,21,5;10), then treatment 1 with treatments 6 to 9 must have the same structure, for some ordering of the blocks, as the U-BIBD(15,21,5;10) in table 1.5. From tables 1.6 and 1.7, since 18 assignment candidates must be placed in 19 blocks, we know $n_1 \leq 1$. Furthermore, from the block size equations (1.13) we know the possible block sizes for the U-BIBD(15,21,5;6) with exactly two blocks of size five and either one or zero blocks of size one are

n_1	n_2	n_3	n_4	n_5
0	7	9	3	2
1	4	12	2	2.

(1.16)

Clearly one replication of each of treatments 6 to 9 must be placed in section one, and the remaining replications in sections two and three. Since no block can have more than two treatments in common with a block of size five, blocks of section one can only receive singles from the candidate list. Once treatments 6 to 9 are added to section one, 13 singles and doubles will remain in the candidate list to be placed in the 13 blocks of sections two and three. Thus, if the U-BIBD(15,21,5;6) has a block of size one, it must be in section one, and placement of treatments in section one will determine whether the design has zero or one block of size one. Using this observation and the theta pattern equations (1.14) given above, we have the following admissible theta-pattern list

θ_{t1}	θ_{t2}	θ_{t3}	θ_{t4}	θ_{t5}
0	3	0	3	1
0	2	2	2	1
0	1	4	1	1
0	0	6	0	1
1	0	4	0	2
0	2	3	0	2.

(1.17)

Since section one is symmetric in treatments 2 to 5, there are only two nonisomorphic ways to place treatments 6 to 9 in section one. They are shown in table 1.8.

Table 1.8: Section One Arrangements - One-common Design

1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	6	2	3	4	5	9	2	6	2	3	4	5	
3	7	6	7	8			3	7	6	7	8	9	
4	8						4	8					
5	9						5	9					
zero size one							one size one						

As can be seen in table 1.8, we will refer to designs having these section one arrangements as *zero size one* and *one size one* designs respectively. Given one of these arrangements, the admissible block sizes (1.16) and the possible theta patterns (1.17) determine the number of singles and doubles from the candidate list (table 1.7) that must be placed in sections two and three.

Zero Size One Designs

First we will investigate zero size one designs. In this case, treatments 6 to 8 must be placed in section two twice each, and treatment nine must be placed there three times, in order to have two concurrences with each of treatments 2 to 5. Hence section two gets three doubles and three singles from the candidate list (table 1.6), and section three gets three doubles and five singles. Since treatment five needs to gain a total of eight treatment concurrences (two with each of treatments 6 to 9), and there are five occurrences of treatment 5 in sections two and three, then treatment five must go with three doubles and two singles from the candidate set. Furthermore, from (1.17), since $\theta_{92} \leq 3$ we know that up to three doubles containing treatment 9 can be placed in section two, and that treatment 9 is required to be a part of at least one section two double candidate.

What, then, are the distinct ways of choosing three double candidates for section two? There are 20 ways to choose three doubles from the six doubles in the candidate set. Immediately we can eliminate the candidate doubles containing three 6s, three 7s, and three 8s and the candidate containing zero 9s. Since any permutation of treatments 2,3,4 does not change sections 2 and 3, and permutations of treatments 6,7,8 combined with the same permutation of treatments 2,3,4, does not change section one, we can reduce the remaining 16 ways of choosing three doubles from the candidate list to just four nonisomorphic double sets. Each double set determines a corresponding set of singles for adding to section two. The section two candidate collections are:

$$\text{Case 1:} \quad \begin{array}{cccccc} 6 & 7 & 8 & 6 & 7 & 8 \\ 9 & 9 & 9 & & & \end{array},$$

$$\text{Case 2:} \quad \begin{array}{cccccc} 6 & 6 & 7 & 8 & 9 & 9 \\ 7 & 8 & 9 & & & \end{array},$$

$$\text{Case 3:} \quad \begin{array}{cccccc} 6 & 6 & 7 & 7 & 8 & 9 \\ 8 & 9 & 9 & & & \end{array}, \text{ and}$$

$$\text{Case 4:} \quad \begin{array}{cccccc} 6 & 6 & 7 & 8 & 8 & 9 \\ 7 & 9 & 9 & & & \end{array}.$$

Suppose the first candidate collection above is placed in section two. Then treatment 9 must be placed in two blocks containing treatment 5 in section two. Otherwise, since each section two double candidate contains treatment 9 and no section two single candidate contains treatment 9, fewer than two doubles from the candidate collection would be placed in a block containing treatment 5 in section two, and a treatment 9 single would be forced to go in a block containing treatment 5 at least once in section 3. This makes it impossible for three double candidates to be placed in a block containing treatment 5 in sections 2 and 3.

Once the design is completed using the first candidate collection above in section two, we can apply the permutation

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{pmatrix} \tag{1.18}$$

(which preserves section one). Doing so transforms double candidates containing treatment 9 to blocks of size two containing treatment 5, and blocks of size two containing treatment 5 to double candidates containing treatment 9. Since two double candidates with treatment 9 go in a block containing treatment 5 in section two, and the third double candidate containing treatment 9 goes in a block not containing treatment 5 in section two, the permutation results in two double candidates containing treatment 9 and one double candidate without treatment 9 being placed in section two. This is clearly a case of candidate collection three or four above, thus we can eliminate the first collection. For example, the U-BIBD(15,21,5,6)

1	1	1	1	1	1	1	2	2	2	3	3	4	2	2	3	3	4	4	5	5
2	6	2	3	4	5	9	3	4	5	4	5	5	7	8	6	9	6	9	7	6
3	7	6	7	8			8	7	6	6	8	7			7		8		8	
4	8						9		9			9								
5	9																			

is transformed to

1	1	1	1	1	1	1	2	2	2	3	3	4	2	2	3	3	4	4	5	5
2	6	2	3	4	5	9	3	4	5	4	5	5	7	9	6	6	7	6	7	8
3	7	6	7	8			7	8	6	9	8	6	8		8		9			
4	8								9		9	7								
5	9																			

by the permutation.

Suppose the second candidate collection is placed in section two. Three candidate doubles can not be placed in blocks containing treatment 5. If so, in order for treatment 9 to have two concurrences with treatment 5, a double candidate containing treatment 9 would be forced to be placed in a block containing treatment 5 in section three causing treatment 5 to have more than two concurrences with treatment 6, 7, or 8. Assignment of two candidate doubles to blocks containing treatment 5 in section two will be transformed under the permutation (1.18) to two candidate doubles containing treatment 9 being placed in section two. This is a case of collection three or four. If one candidate double is placed in a block containing treatment five, then under the same permutation, the resulting design would be isomorphic to another case of collection two. Thus the assignments using collection two for section two may be restricted to those with one double assigned to a block containing treatment 5. An exhaustive search of the remaining possibilities for designs using collection two in section two revealed no possible U-BIBD(15,21,5;6)s.

Now consider placing the third candidate collection in section two. Placement of the section two candidate doubles into blocks having the form

a	a	b
b	5	5

will be transformed under the permutation (1.18) to the placement of candidate doubles having the form

a'	a'	b'
b'	9	9

in section two. This double candidate form is isomorphic to the double candidates in candidate collection four.

An exhaustive search for designs with collection three in section two revealed 42 possible U-BIBD(15,21,5;6)s, and three designs have the section two structure mentioned above. Thus there are 39 designs that may not be isomorphic. An exhaustive search for designs with collection four in section two resulted in 20 U-BIBD(15,21,5;6)s. Therefore, there are 59 possible nonisomorphic zero size one U-BIBD(15,21,5;6)s.

One Size One Designs

Now we will investigate one size one U-BIBD(21,15,5;6)s. In order to have two concurrences with treatments 2 to 5, treatments 6 to 9 must be placed twice in section two and three times in section three. Thus section two gets two doubles and four singles from the candidate list in table 1.7 and section three gets four doubles and four singles from the candidate list. Of the 15 ways to select two doubles from the six double candidates, $\begin{smallmatrix}6 & 6\\ 7 & 8\end{smallmatrix}$ and $\begin{smallmatrix}6 & 8\\ 7 & 9\end{smallmatrix}$ are the only nonisomorphic pairs under all permutations of treatments 6,7,8,9 with the same permutation of treatments 2,3,4,5 (thus preserving section one). Therefore we have two nonisomorphic section two candidate collections. They are

1. $\begin{matrix}6 & 6 & 7 & 8 & 9 & 9\\ 7 & 8 & & & & \end{matrix}$ and

2. $\begin{matrix}6 & 8 & 6 & 7 & 8 & 9\\ 7 & 9 & & & & \end{matrix}$.

Designs resulting from placing collection two in section two in such a way that the two double candidates are placed in blocks with one treatment in common are isomorphic under the permutation (1.18) to designs resulting from placing collection one candidates in section two. That is, if the placement of the collection two candidate doubles in section two has the form

$$\begin{matrix}n & n\\ a & b\\ 6 & 8\\ 7 & 9,\end{matrix}$$

then under the permutation (1.18), the section two doubles have the new form

$$\begin{matrix}2 & 4\\ 3 & 5\\ n' & n'\\ a' & b'.\end{matrix}$$

This new form will result in a design that is isomorphic to a design that results from placing collection one in section two.

An exhaustive computer search using collection two in section two resulted in 106 designs, but after eliminating the designs that are isomorphic to collection one designs, ten possibly nonisomorphic designs remain. An exhaustive computer search using collection one candidates in section two resulted in 27 possibly nonisomorphic designs. Therefore, there are at most 37 nonisomorphic one size one U-BIBD(21,15,5;6)s. Therefore, there are at most 96 nonisomorphic U-BIBD(21,15,5;6) in the one common case.

1.4.2 Two Common Case

If we build our U-BIBD(15,21,5;10) into a U-BIBD(15,21,5;w) with exactly two blocks of size five having two treatments in common and having maximum w, then $w = 7$ and $v - w = 8$. The two blocks of size five are

$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{array},$$

and we begin with the structure shown in table 1.9.

Table 1.9: Two-common Starter

1	1	1	1	1	1	1	2	2	2	2	2	3	3	4	3	3	4	4	5	5
2	2	3	4	5			3	4	5			4	5	5						
3	6																			
4	7																			
5	8																			
section one												section two			section three					

Since removing treatments 3 to 5 from the resulting U-BIBD(15,21,5;7) will give a U-BIBD(15,21,5;10), then treatments 1,2,6,7,8 must have the same structure as in the U-BIBD(15,21,5;10) given above in table 1.5. From the block size equations (1.13) we know the possible block sizes for the U-BIBD(15,21,5;7) with two blocks of size five having two common treatments are

n_1	n_2	n_3	n_4	n_5
0	12	6	1	2
1	9	9	0	2.

(1.19)

From (1.14) and using (1.19), we have the admissible theta pattern set

θ_{t1}	θ_{t2}	θ_{t3}	θ_{t4}	θ_{t5}
1	2	2	0	2
0	4	1	0	2
0	3	2	1	1
0	2	4	0	1.

(1.20)

Since $\lambda = 2$ the candidate list containing the sub-blocks that must be added, all in separate blocks, to the U-BIBD(15,21,5;10) of table 1.9 is shown in table 1.10.

Table 1.10: Assignment Candidates - Two-common Starter

6	6	7	6	6	6	6	7	7	7	7	8	8	8	8
7	8	8												
doubles			singles											

As before, we will refer to candidate sub-blocks in table 1.10 consisting of two treatments as *doubles* and those consisting of a single treatment as *singles*. As is shown in table 1.9, the 12 blocks containing treatments 1 and/or 2 are referred to as *section one*, the remaining three blocks of size two as *section two*, and the other six blocks of size one as *section three*.

Since treatments 1 and 2 need one concurrence with treatments 6 to 8, then two replications of treatments 6 to 8 must be placed in section one. From lemma 1.4.2, we conclude that only singles of treatments 6 to 8 can be placed in section one. There are nine nonisomorphic ways one more replication of treatments 6 to 8 can be placed in section one. They are:

	1	1	1	1	1	1	1	2	2	2	2	2
	2	2	3	4	5	7	8	3	4	5	7	8
1.	3	6	6					6				,
	4	7										
	5	8										

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	7	8	3	4	5	7	8	
2.	3	6	6						6				,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	7	8	3	4	5	6	8	
3.	3	6	6					7					,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	7	8	3	4	5	6	8	
4.	3	6	6						7				,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	8		3	4	5	7	8	
5.	3	6	6	7				6					,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	8		3	4	5	7	8	
6.	3	6	6	7					6				,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	8		3	4	5	7	8	
7.	3	6	6	7						6			,
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	8		3	4	5	6	7	
8.	3	6	6	7				8					, and
	4	7											
	5	8											

	1	1	1	1	1	1	1	2	2	2	2	2	
	2	2	3	4	5	8		3	4	5	6	7	
9.	3	6	6	7						8			.
	4	7											
	5	8											

Section two candidates are determined by a particular section one arrangement and treatment replications. For example, consider the first section one arrangement. Since treatment 6 has two concurrences with treatments 1 to 3 in section one and treatments 7 and 8 have two concurrences with treatments 1 and 2 only, then treatment 6 must have a total of four concurrences (two with treatments 4 and 5) in sections two and three, and treatments 7 and 8 require a total of six concurrences (two with treatments 3 to 6). Since there are a total of four occurrences of treatments 6 to 8 that need to be placed in sections two and three, treatment 6 must be placed in four blocks of size one and treatments 7 and 8 must be placed in two blocks of size two and one block of size one in sections two and three. Thus, zero occurrences of treatment 6 and two occurrences of treatments 7 and 8 must be placed in section two. Therefore, the candidate collection that must be placed in section two given the first section one arrangement is

$$\begin{matrix} 7 & 7 & 8 \\ 8 & & \end{matrix} .$$

In a similar manner we can construct section two candidate collections for the remaining eight section one arrangements. The section two candidate collection list and the corresponding section one arrangements are:

1. Section one arrangements 1 and 2

   $\begin{matrix} 7 & 7 & 8 \\ 8 & & \end{matrix}$,

2. Section one arrangements 3 and 4

   $\begin{matrix} 6 & 7 & 8 \\ 8 & & \end{matrix}$ and $\begin{matrix} 6 & 8 & 8 \\ 7 & & \end{matrix}$,

3. Section one arrangements 5 to 7

   7 8 8 , and

4. Section one arrangements 8 and 9

   6 7 8 .

Given a particular section one arrangement and the corresponding section two candidate collection, an exhaustive numerical search of all possible admissible U-BIBD(15,21,5;7)s can be conducted. Once all admissible designs are listed for each arrangement/candidate pair, isomorphic designs can be eliminated by studying valid treatment permutations. For example, consider the first section one arrangement with the corresponding section two candidate collection (collection one). The numerical search results in two U-BIBD(15,21,5;7)s, but under the permutation $\begin{smallmatrix} 4 & 5 \\ 5 & 4 \end{smallmatrix}$, the section one arrangement remains unchanged and one design is transformed into the second design. Thus, there is only one nonisomorphic U-BIBD(15,21,5;7). In general, permutations that, when applied to section one arrangements, leave the arrangement unchanged can be applied to resulting U-BIBD(15,21,5;7)s in order to eliminate isomorphic designs. A second example is arrangement two. Each of the permutations $\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}$ and $\begin{smallmatrix} 7 & 8 \\ 8 & 7 \end{smallmatrix}$ when applied to the arrangement leave it unchanged. An exhaustive numerical search using arrangement two and candidate collection one results in six U-BIBD(15,21,5;7)s, but by applying combinations of the aforementioned permutations, four isomorphic designs can be eliminated from this set.

Exhaustive numerical searches starting with every possible section one arrangement and corresponding section two candidate collection(s) results in 40 U-BIBD(15,21,5;7)s. Carefully applying appropriate permutations to the resulting designs as is described above reduces the list to 28 possibly nonisomorphic U-BIBD(15,21,5;7)s. This completes the two common case.

1.4.3 A- and D-optimal Design

The final step of the search for the A- and D-optimal design in $D(15, 21, 5)$ is an enumeration of the possible completions of the 96 possibly nonisomorphic U-BIBD(15,21,5;6)s and the 28 possibly nonisomorphic U-BIBD(15,21,5;7)s to V-BIBD(15,21,5)s with $\delta_d \leq 4$ and $w \leq 4$. With respectively 42 and 49 total assignments of the remaining treatments still to be made, and in light of the fact that concurrence counts involving any one of treatments 11 to 15 can *not* be constant, this is a nontrivial exercise. The 124 candidate UBIBDs are too numerous to allow an analytic approach analogous to sections 1.4.1 and 1.4.2. However, the list of 124 designs is small enough to bring the completion problem within computational reach. Now an exhaustive blind computer enumeration can be performed by adding the remaining treatments to each U-BIBD in all possible ways, requiring only that $\lambda_{dii'} \in \{\lambda - 1, \lambda, \lambda + 1\}$ for all $i \neq i'$, kicking out the resulting designs violating the restrictions on δ_d and w_d. Among the designs so found, only two distinct discrepancy patterns occur: D7 and D10, each with $\delta_d = w_d = 4$.

This establishes that $\delta = 4$ for $D(15, 21, 5)$, and minimum discrepancy is not achievable in conjunction with minimum deficiency for this class. The optimality values for designs having discrepancy matrix D7 are:

$$\text{A-value} = 2.33631, \quad \text{D-value} = -25.07572, \quad \text{and} \quad \text{E-value} = 0.17857,$$

and the optimality values for designs having discrepancy matrix D10 are:

$$\text{A-value} = 2.33635, \quad \text{D-value} = -25.07565, \quad \text{and} \quad \text{E-value} = 0.18164.$$

An example of a design having discrepancy matrix D7 is in table 1.11, and an example of a design having discrepancy matrix D10 is in table 1.12. Of the two minimum

Table 1.11: An A- and D-optimal Design In $D(15, 21, 5)$

1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4	5	5
2	2	3	4	5	7	8	3	4	5	7	8	4	5	6	8	5	6	7	6	6
3	6	6	10	9	13	10	11	6	10	9	9	7	7	9	12	8	8	9	11	7
4	7	9	13	11	14	11	13	12	12	11	10	10	8	10	14	9	11	12	12	10
5	8	12	14	15	15	12	15	15	14	14	13	11	13	14	15	15	14	13	13	15

Table 1.12: A Design In $D(15, 21, 5)$ Having Discrepancy Matrix D10

1	1	1	1	1	1	1	2	2	2	2	1	3	3	3	3	4	4	4	5	5
2	2	3	4	5	7	8	3	4	5	7	8	4	5	6	8	5	6	7	6	6
3	6	6	12	9	10	11	10	6	9	9	10	7	7	12	9	8	8	9	10	7
4	7	9	13	12	11	14	13	11	11	12	12	11	8	13	11	10	9	10	11	14
5	8	10	14	15	13	15	14	15	13	14	15	12	13	15	14	14	13	15	12	15

discrepancy patterns found, D7 is A- and D-superior and thus, according corollary 1.3.4, produces A- and D-optimal designs.

The A-, D- and E-efficiencies for the design with discrepancy matrix D10 and for Zhang's design from table 1.1 relative to the A- and D-optimal design with discrepancy matrix D7 are provided in table 1.13.

Table 1.13: A-, D-, and E-efficiencies Relative To An A- and D-optimal Design

	D10	Zhang
A-efficiency	0.99998	0.99936
D-efficiency	0.99993	0.99554
E-efficiency	0.98311	0.98395

Are designs in $D(15, 21, 5)$ having discrepancy matrix D7 ϕ_p-better then these two competitors for $p \geq 2$? Is such a design ϕ_p-optimal in D in for any $p \geq 2$? Could it be E-optimal? The first question can be answered by calculating the ϕ_p-values for the three competitors, and the second question can be answered by checking the bounds 1.9 and 1.10 for the ϕ_p-optimality criterion (1.5). We discuss the question of E-optimality in detail in section 1.5.

We have calculated ϕ_p-values and bounds for $1 \leq p \leq 60$. From the calculations and the facts that $\phi_p(C_d) = \left(\sum_{i=1}^{v-1} z_{di}^{-p}\right)^{(1/p)}$ is a monotone decreasing function of p and is bounded below by the E-value of d, $E_d = z_{d1}^{-1}$, we can make three observations concerning ϕ_p optimality in $D(15, 21, 5)$:

1. Designs having discrepancy matrix D7 are ϕ_p-better than designs having dis-

crepancy matrix $D10$ and Zhang's table 1.1 design for all $p \geq 1$.

2. Designs having discrepancy matrix D7 are ϕ_p-better than binary designs that are not an NBBD(2) for $1 \leq p \leq 3$.

3. Designs having discrepancy matrix D7 are ϕ_p-better than nonbinary designs for $1 \leq p \leq 6$.

The first observation follows from the fact that the ϕ_p-values for designs having discrepancy matrix D7 are less than those of the two competitors for $p < 60$, and the ϕ_p-value of these competitors are less than the E-value of D7 at $p = 60$. The others follow from checking (1.9) and (1.10).

1.5 E-optimal Design in $D(15, 21, 5)$

Let $D(v, b, k)$ be an irregular BIBD setting, and, as usual, denote the binary subclass of D by $M(v, b, k)$ and subclass of M containing only equireplicate designs by $M_0(v, b, k)$. Suppose the eigenvalue/vector pairs of the information matrix C_d for a design $d \in D$ are $(z_{d1}, \mathbf{e}_{d1}), (z_{d2}, \mathbf{e}_{d2}), \ldots, (z_{dv}, \mathbf{e}_{dv})$. It follows from the fact $C_d\mathbf{1} = 0$ that $(z_{di}, \mathbf{e}_{di}) = (0, \mathbf{1})$ for some i, say $i = v$. Moreover, since D contains only connected designs, rank $C_d = v - 1$ and $z_{di} > 0$ for all $1 \leq i \leq v - 1$. Therefore, a set of eigenvalue/vector pairs for C_d corresponding to the nonzero eigenvalues are $(z_{d1}, \mathbf{e}_{d1}), (z_{d2}, \mathbf{e}_{d2}), \ldots, (z_{d,v-1}, \mathbf{e}_{d,v-1})$, and $\mathbf{e}_i^T\mathbf{1} = 0$, for all $i = 1, 2, \ldots v - 1$. For notational simplicity, redefine the E-value of $d \in D$ given by (1.6) to be the minimum nonzero eigenvalue of C_d, or

$$E_d = \min_{i<v} z_{di}. \tag{1.21}$$

Then the E-optimal design $d^* \in D$, defined by (1.7), has E-value

$$E_{d^*} = \max_{d \in D} E_d = \max_{d \in D} \min_{i<v} z_{di}. \tag{1.22}$$

In this section we will develop the theory for identifying E-optimal designs in $D(v, b, k)$ and outline a procedure for constructing these designs. Our results will be applied to the setting $(v,b,k) = (15,21,5)$, and finally the surprising E-optimal design in $D(15,21,5)$ will be reported.

Recall from equation (1.12), the information matrix for a design $d \in M_0(v,b,k)$ is

$$C_d = \frac{\lambda v}{k}(I - \frac{1}{v}J) - \frac{1}{k}\Delta_d,$$

where $\Delta_d = (\delta_{dii'})$ is the discrepancy matrix for the design, Δ_d has zero sum rows and columns, and the nonzero elements of Δ_d can be restricted to the first $s \leq v$ rows and columns. Since $\Delta_d \mathbf{1} = 0$, $(0, \mathbf{1})$ is an eigenvalue/vector pair for Δ_d, and any set of $v-1$ vectors mutually orthogonal to $\mathbf{1}$ constitute a set of eigenvectors for Δ_d. Then, if $(u_{di}, \mathbf{e}_{di})$ is an eigenvalue/vector pair of Δ_d, the corresponding eigenvalue of C_d is

$$z_{di} = \frac{\lambda v}{k} - \frac{1}{k}u_{di}. \tag{1.23}$$

Furthermore, if the maximum eigenvalue of the discrepancy matrix Δ_d is

$$U_d = \max_i u_{di},$$

then the E-value for d given by (1.21) becomes

$$E_d = \frac{\lambda v}{k} - \frac{1}{k}U_d, \tag{1.24}$$

establishing a direct relationship between the E-value of a design $d \in M_0(v,b,k)$ and the maximum eigenvalue U_d of the discrepancy matrix Δ_d associated with the design.

The following two lemmas and corollary establish conditions for which a search for the E-optimal design in $D(v,b,k)$ can be restricted to the subclasses $M(v,b,k)$ and $M_0(v,b,k)$.

Lemma 1.5.1 *Let $\bar{d}$ be a binary design in an irregular BIBD setting $D(v,b,k)$ with discrepancy matrix $\Delta_{\bar{d}}$ having maximum eigenvalue $U_{\bar{d}}$. If $U_{\bar{d}} < 2$ then the E-optimal design must be in $M(v,b,k)$.*

Proof Let d be a nonbinary design in an irregular BIBD setting $D(v, b, k)$ with E-value E_d. From the proof of Theorem 3.1 of Jacroux (1980b),

$$E_d \leq \frac{[r(k-1)-2]v}{k(v-1)} \leq \frac{\lambda v - 2}{k}.$$

From equation (1.24), the E-value of an equireplicate design $\bar{d}$ is

$$E_{\bar{d}} = \frac{\lambda v}{k} - \frac{1}{k} U_{\bar{d}}.$$

Design $\bar{d}$ is E-better than nonbinary d if and only if $E_{\bar{d}} > E_d$ which is true if

$$E_{\bar{d}} > \frac{\lambda v - 2}{k},$$

which implies $U_{\bar{d}} < 2$. □

Lemma 1.5.2 *Let d be a nonequireplicate design in an irregular BIBD setting $D(v, b, k)$, and define $\rho_d = \max_i\{r - r_{di}\}$. Let $\bar{d} \in D(v, b, k)$ be an equireplicate design with discrepancy matrix $\Delta_{\bar{d}}$ having maximum eigenvalue $U_{\bar{d}}$. If $U_{\bar{d}} < (k-1)\rho_d$ then $\bar{d}$ is E-better than d.*

Proof If E_d is the E-value of d then, by Theorem 3.1 of Jacroux (1980a),

$$E_d \leq \frac{(r-\rho_d)(k-1)v}{(v-1)k} = \frac{\lambda v}{k}\left[1 - \frac{\rho_d}{r}\right],$$

the equality because $\frac{k-1}{v-1} = \frac{\lambda}{r}$ in a BIBD setting. From equation (1.24), the E-value for equireplicate $\bar{d}$ is

$$E_{\bar{d}} = \frac{\lambda v}{k} - \frac{1}{k} U_{\bar{d}}.$$

Design $\bar{d}$ is E-better than d if and only if $E_{\bar{d}} > E_d$ which is true if

$$U_{\bar{d}} < \frac{v}{v-1}(k-1)\rho_d. \tag{1.25}$$

Inequality (1.25) is satisfied if $U_{\bar{d}} < (k-1)\rho_d$. □

Corollary 1.5.3 *If there exists an equireplicate design $\bar{d} \in D(v,b,k)$ having $\delta_{\bar{d}} \leq 4$ and $\gamma^-_{\bar{d}2} + \gamma^+_{\bar{d}2} \leq 1$, or $(\delta_{\bar{d}}, l_{\bar{d}}) = (5,2)$, then the E-best design in $D(v,b,k)$ must be equireplicate.*

Proof Since nonexistence of a BIBD implies $k \geq 5$ and nonequireplicate designs $d \in M(v,b,k)$ have $\rho_d \geq 1$, we need establish that $U_{\bar{d}} < 4$ for all 51 discrepancy matrices satisfying the conditions of the corollary, which are listed in Appendix A. The corresponding list of $U_{\bar{d}}$-values is given in Appendix B, and the largest value is 3.44949 for D51. □

Corollary 1.5.4 *If there exists a binary, equireplicate design $d \in D(v,b,k)$ with discrepancy matrix Δ_d having maximum eigenvalue $U_d < 2$, then the the E-optimal design must be in $M_0(v,b,k)$.*

If Δ_D is the class of all admissible discrepancy matrices for designs in $M_0(v,b,k)$, that is, the class of all integer-valued square matrices of dimension $s \leq v$ having zeros on the diagonal and zero-sum rows and columns, the expression for the E-value of the E-optimal design $d^* \in M_0$ given by (1.22) is

$$E_{d^*} = \frac{\lambda v}{k} - \frac{1}{k}\min_{\Delta_D} U_d = \frac{\lambda v}{k} - \frac{1}{k}U_{d^*}. \tag{1.26}$$

Solving (1.22) is equivalent to solving

$$U_{d^*} = \min_{\Delta_D}\max_{i} u_{di} \tag{1.27}$$

and using (1.26) to obtain the E-value of the E-optimal design in the class.

Now the fundamental question is: is it possible to solve (1.27) without enumerating all of the admissible discrepancy matrices $\Delta_d \in \Delta_D$? To attack this one must first ask: what is the relationship between E-value U_d, design discrepancy δ_d, concurrence range l_d, and treatment deficiency w? We begin to answer this question by ranking the discrepancy matrices listed in Appendix A by their maximum eigenvalue U_d, from largest to smallest, as shown in Appendix B. It is immediately clear from the list that the E-ranking of a design is not a function of δ_d, l_d, and w alone. For example, designs having discrepancy matrix D1 with discrepancy $\delta_d = 2$, if they exist, are E-inferior to $\delta_d = 3$ designs with discrepancy matrix D2, $\delta_d = 4$ designs with discrepancy matrix D5, and $\delta_d = 5$ designs with discrepancy matrix D13, and the same designs are E-superior to designs with discrepancy matrices D3, D8, and D20 with discrepancies $\delta_d = 3$, $\delta_d = 4$, and $\delta_d = 5$, respectively. Also, designs with discrepancy matrix D18 having discrepancy $\delta_d = 4$ and concurrence range $l_d = 3$ are E-inferior to some designs having discrepancy $\delta_d = 4$ and concurrence range $l_d = 2$ or $l_d = 3$, for example designs having discrepancy matrix D5 or D12, and are E-superior to designs having discrepancy matrix D9 or D48 also with discrepancy $\delta_d = 4$ and concurrence ranges $l_d = 2$ and $l_d = 3$.

Furthermore, suppose in a setting $M_0(v, b, k)$ no design having discrepancy matrix $D2$ exists, but, for some $n \geq 2$, a design having discrepancy matrix $nD2 = I_n \otimes D2$, where $\otimes$ is the kronecker product and I_n is the $n \times n$ identity matrix, exists. Since the eigenvalues for $nD2$ are n copies of the eigenvalues of $D2$, and designs having discrepancy matrix $D2$ are E-better than designs having any of the other 50 discrepancy matrices in Appendix A, then that $\delta_d = 3n \geq 6$ design would be E-better than any design having one of the discrepancy matrices in the list. Therefore, even if the existence question for designs having one of the discrepancy matrices in Appendix A has been completely solved, we then still may not know whether there exists a design with larger discrepancy and/or larger concurrence range that is E-better than the best of these. Clearly we need to investigate the discrepancy matrix/E-value relationship more thoroughly. The following three lemmas will help.

Lemma 1.5.5 *Suppose $d \in M_0(v, b, k)$ has discrepancy matrix $\Delta_d = (\delta_{dii'})$. If U_d is the maximum eigenvalue of Δ_d then*

$$\min_{i \neq i'} \delta_{dii'} \geq -U_d \tag{1.28}$$

and

$$\max_{i \neq i'} \delta_{dii'} \leq \frac{v-2}{v} U_d. \tag{1.29}$$

Proof A design $d \in M_0(v, b, k)$ with discrepancy matrix Δ_d will have information matrix $C_d = (c_{dii'})$ given by (1.12) having E-value E_d. By Proposition 3.2 of Jacroux (1980b), for all $\lambda_{dii'}$, $i \neq i'$,

$$E_d \leq \frac{r(k-1) + \lambda_{dii'}}{k} \tag{1.30}$$

and

$$E_d \leq \frac{[r(k-1) - \lambda_{dii'}]v}{(v-2)k}. \tag{1.31}$$

Since M_0 is a BIBD setting, $r(k-1) = \lambda(v-1)$. Using this expression, the relationship $\lambda_{dii'} = \lambda + \delta_{dii'}$, and by writing E_d in terms of U_d using (1.24), inequality (1.30) may be written as

$$\delta_{dii'} \geq -U_d, \quad \text{for all } i \neq i',$$

and, similarly, inequality (1.31) becomes

$$\delta_{dii'} \leq \frac{v-2}{v} U_d, \quad \text{for all } i \neq i'.$$

Inequalities (1.28) and (1.29) follow immediately. □

Corollary 1.5.6 *Let Δ_d and $\Delta_{\bar{d}}$ be discrepancy matrices for designs $d \neq \bar{d}$ in an irregular BIBD setting $M_0(v, b, k)$. Suppose the maximum eigenvalue of $\Delta_{\bar{d}} = (\delta_{\bar{d}ii'})$ is $U_{\bar{d}}$ and the maximum eigenvalue of $\Delta_d = (\delta_{dii'})$ is U_d. If $\bar{d}$ is E-better than d then*

$$\min_{i \neq i'} \delta_{\bar{d}ii'} > -U_d \tag{1.32}$$

and

$$\max_{i \neq i'} \delta_{\bar{d}ii'} < \frac{v-2}{v} U_d \tag{1.33}$$

Corollary 1.5.6 potentially can significantly limit the discrepancy matrix search for the E-optimal design by bounding the minimum and maximum treatment concurrences of designs that can be E-better than a known design d having discrepancy matrix Δ_d with maximum eigenvalue U_d. For example, if a design having discrepancy matrix D2 or $I_n \otimes D2$ exists, then $U_d = 1.73205$, and the corollary says that the discrepancy matrix of a potentially E-better design can not have an element less than -1 or greater than 1. Consequently, potential E-better designs must have a concurrence range equal to 2. The following two lemmas will lead to corollaries that provide more information about the discrepancy matrices of E-optimal designs, further limiting the number of discrepancy matrices for potentially E-better designs.

Lemma 1.5.7 *Suppose $d \in M_0(v,b,k)$ has discrepancy matrix $\Delta_d = (\delta_{dii'})$ with maximum eigenvalue U_d. Then, for all $m \leq v$,*

$$\sum_{i<i'\leq m}\sum \delta_{dii'} \leq \frac{m(v-m)}{2v} U_d \tag{1.34}$$

Proof A design $d \in M_0(v,b,k)$ with discrepancy matrix Δ_d will have information matrix $C_d = (c_{dii'})$ given by (1.12) having E-value E_d and by Lemma 3.2 (b) of Jacroux (1989), for all $m \leq v$,

$$E_d \leq \frac{v}{m(v-m)} \left(\sum_{i=1}^{m} c_{dii} + \sum_{i=1}^{m} \sum_{\substack{i'=1 \\ i' \neq i}}^{m} c_{dii'} \right). \tag{1.35}$$

Substituting

$$c_{dii} = \frac{\lambda(v-1)}{k} \quad \text{and} \quad c_{dii'} = -\frac{(\lambda + \delta_{dii'})}{k}$$

into (1.35), writing E_d in terms of U_d using (1.24), and solving for $\sum\sum_{1\leq i<i'\leq m} \delta_{dii'}$ yields (1.34). □

Corollary 1.5.8 *Let Δ_d and $\Delta_{\bar{d}}$ be the discrepancy matrices for designs $d \neq \bar{d}$ in $M_0(v,b,k)$. Suppose the maximum eigenvalue of Δ_d is U_d and the maximum* *eigenvalue of $\Delta_{\bar{d}}$ is $U_{\bar{d}}$. If $\bar{d}$ is E-better than d then the elements of every $m \times m$, $m \leq v$, leading minor $\Delta_{\bar{d}11} = (\delta_{\bar{d}ii'})$ of $\Delta_{\bar{d}}$ must satisfy*

$$\sum_{i<i'}\sum \delta_{\bar{d}ii'} \leq \frac{m(v-m)}{2v} U_d. \tag{1.36}$$

Lemma 1.5.9 *Let Δ_d be the discrepancy matrix for a design $d \in M_0(v,b,k)$, and define Δ_{d11} to be the $m \times m$, $m \leq v$, leading minor of Δ_d. Let (u_i, ξ_i), $1 \leq i \leq m$, be the eigenvalue/vector pairs for Δ_{d11}, and write $x_i = \xi_i^T \mathbf{1}$, where $\mathbf{1}_{v\times 1}$ is a vector whose elements are all 1. If U_d is the maximum eigenvalue of Δ_d, then*

$$\max_i \left[\frac{v - x_i^2}{v}\right]^{-1} u_i \leq U_d. \tag{1.37}$$

Proof Since Δ_d has row and column sums of zero,

$$U_d = \max_{\substack{\mathbf{x}^T\mathbf{x}=1 \\ \mathbf{x}^T\mathbf{1}=0}} \mathbf{x}^T \Delta_d \mathbf{x}.$$

Partition Δ_d as

$$\Delta_d = \begin{pmatrix} \Delta_{d11} & \Delta_{d12} \\ \Delta_{d21} & \Delta_{d22} \end{pmatrix}$$

and consider the vector $\mathbf{y}^T = (\mathbf{w}^T, \mathbf{0}^T)$, $\mathbf{w}^T\mathbf{w} = 1$, so that

$$\mathbf{y}^T \Delta_d \mathbf{y} = \mathbf{w}^T \Delta_{d11} \mathbf{w}.$$

Then, provided $\mathbf{w}^T\mathbf{1} = 0$,

$$U_d \geq \mathbf{w}^T \Delta_{d11} \mathbf{w}.$$

If $\mathbf{w}^T\mathbf{1} \neq 0$, consider

$$\begin{aligned} \mathbf{y}^* &= (I - \frac{1}{v}J)\mathbf{y} \\ &= \mathbf{y} - \frac{1}{v}\sum y_i \mathbf{1} \\ &= \mathbf{y} - \frac{1}{v}\sum w_i \mathbf{1} \end{aligned}$$

where $I_{v\times v}$ is the identity matrix and $J_{v\times v}$ is the matrix whose elements are all 1. Then $\mathbf{y}^{*T}\mathbf{1} = 0$ and

$$\begin{aligned}\mathbf{y}^{*T}\mathbf{y}^{*} &= \mathbf{y}^T\mathbf{y} - \frac{2}{v}(\sum w_i)\mathbf{y}^T\mathbf{1} + \frac{1}{v^2}(\sum w_i)^2\mathbf{1}^T\mathbf{1} \\ &= \frac{v-(\sum w_i)^2}{v} \\ &= s, \text{ say}\end{aligned}$$

Then

$$\begin{aligned}U_d \geq \frac{1}{s}\mathbf{y}^{*T}\Delta_d\mathbf{y}^{*} &= \frac{1}{s}(\mathbf{y} - \frac{1}{v}\sum w_i\,\mathbf{1})^T\Delta_d(\mathbf{y} - \frac{1}{v}\sum w_i\,\mathbf{1}) \\ &= \frac{1}{s}\mathbf{y}^T\Delta_d\mathbf{y} \quad (\text{since } \mathbf{1}^T\Delta_d = 0) \\ &= \frac{1}{s}\mathbf{w}^T\Delta_{d11}\mathbf{w} \\ &= \left[\frac{v-(\sum w_i)^2}{v}\right]^{-1}\mathbf{w}^T\Delta_{d11}\mathbf{w}.\end{aligned}$$

Let $\xi_1, \xi_2, \ldots, \xi_m$ be the eigenvectors of Δ_{d11} with eigenvalues $u_{d1}, u_{d2}, \ldots, u_{dm}$, respectively, and suppose $\xi_i^T\mathbf{1} = x_i$, say. Then

$$\begin{aligned}U_d &\geq \max_i\left[\frac{v-x_i^2}{v}\right]^{-1}\xi_i^T\Delta_{d11}\xi_i \\ &= \max_i\left[\frac{v-x_i^2}{v}\right]^{-1}u_{di}. \ \square\end{aligned}$$

Corollary 1.5.10 *Let Δ_d and $\Delta_{\bar d}$ be the discrepancy matrices for designs $d \neq \bar d$ in $M_0(v,b,k)$. Suppose the maximum eigenvalue of Δ_d is U_d, the maximum eigenvalue of $\Delta_{\bar d}$ is $U_{\bar d}$, and $\Delta_{\bar d11}$ is a $m \times m$ leading minor of $\Delta_{\bar d}$ for any $m \leq v$. Let $(u_{\bar d i}, \xi_i)$ be the eigenvalue/vector pairs for $\Delta_{\bar d11}$, and write $x_i = \xi_i^T\mathbf{1}$, where $\mathbf{1}$ is the $m \times 1$ vector whose elements are all 1. If $\bar d$ is E-better than d then*

$$\max_i\left[\frac{v-x_i^2}{v}\right]^{-1}u_{\bar d i} < U_d. \tag{1.38}$$

With corollaries 1.5.6, 1.5.8, and 1.5.10 in hand, given an irregular BIBD setting $D(v,b,k)$, we are ready to outline a procedure for finding the discrepancy matrices $\{\Delta_{\bar d1}, \Delta_{\bar d2}, \ldots, \Delta_{\bar dt}\} \in \Delta_D$, $t \geq 1$, with maximum eigenvalue U_{d^*} given in

(1.27), that is, finding the *E-best discrepancy matrices* in Δ_D. The procedure starts with a discrepancy matrix Δ_d having maximum eigenvalue $U_d < 2.0$ for a design $d \in M_0(v, b, k)$ that is suspected to exist, and, consequently, assumes the search can be limited to designs in $M_0(v, b, k)$. It then enumerates a list of discrepancy matrices $\{\Delta_{\tilde{d}1}, \Delta_{\tilde{d}2}, \ldots, \Delta_{\tilde{d}n}\} \in \Delta_D$ having maximum eigenvalues $\{U_{\tilde{d}1}, U_{\tilde{d}2}, \ldots, U_{\tilde{d}n}\}$ such that $U_{\tilde{d}i} \leq U_d$ for each $i \leq n$, that is, it enumerates a list of *E-better discrepancy matrices* in Δ_D. If no such discrepancy matrix exists, the procedure will establish the fact. The $1 \leq t \leq n$ E-best discrepancy matrices will have maximum eigenvalue U_{d^*} satisfying

$$U_{d^*} = \min\{U_{\tilde{d}1}, U_{\tilde{d}2}, \ldots, U_{\tilde{d}n}, U_d\}. \tag{1.39}$$

The procedure is:

1. Apply conditions (1.32) and (1.33) from corollary 1.5.6 to U_d in order to establish bounds for the maximum and minimum elements of a discrepancy matrix $\Delta_{\tilde{d}} = (\delta_{\tilde{d}ii'})$ that is E-better than Δ_d.

2. Create an exhaustive list of symmetric and nonisomorphic $m \times m$ matrices that could serve as the leading minor for a discrepancy matrix $\Delta_{\tilde{d}}$ that is E-better than Δ_d, for a convenient value of $m \leq v$. Each matrix must satisfy the following requirements:

   (a) All diagonal elements must be equal to zero.

   (b) Each off-diagonal element must satisfy the bounds determined in step 1.

   (c) The elements must satisfy condition (1.36) of corollary 1.5.8.

   (d) If the rows and columns do not sum to zero, then $m < v$.

   We will refer to this list of discrepancy matrices as the ***starter candidate list***, and matrices in this list as ***starter candidates***.

3. Remove starter candidates that do not satisfy condition (1.38) of corollary 1.5.10 (these are determined by computation).

4. For each remaining starter candidate enumerate all nonisomorphic one row and one column extensions to symmetric matrices satisfying conditions (a) - (d) of step 2 and step 3.

5. If any of the extensions have zero sum rows and columns, then they are discrepancy matrices and should be copied to the E-better discrepancy matrix list.

6. If there are no remaining extensions or the extensions are $v \times v$, the search is over.

7. The remaining extensions form a new list of starter candidates. Return to step 4.

Now we have a (hopefully small) list of E-competitive discrepancy matrices $\{\Delta_{\bar{d}1}, \Delta_{\bar{d}2}, \ldots, \Delta_{\bar{d}n}, \Delta_d\}$ and a corresponding list of maximum eigenvalues $\{U_{\bar{d}1}, U_{\bar{d}2}, \ldots, U_{\bar{d}n}, U_d\}$. We are assured that this list is not empty because at minimum it will consist of Δ_d. However, it remains to determine if any corresponding designs can be constructed.

As an aside, if there exists an irregular BIBD setting $M_0(v', b', k')$, $v' \leq v$, discrepancy matrices from the E-competitive discrepancy matrix list can potentially serve as the discrepancy matrix for the E-best design $d' \in D(v', b', k')$ provided their dimension is less than v' and a design $d' \in D(v', b', k')$ having the discrepancy matrix can be constructed.

We now apply the procedure outlined above to the irregular BIBD setting $D(21, 15, 5)$ discussed at the beginning of this chapter. From the A- and D-optimal design search in section 1.4 it was established that the only designs having $\delta_d \leq 4$ and

$l_d = 2$ that exist in the setting have discrepancy matrix D7 or D10 listed in Appendix A. For our search we will conjecture that a design $d \in M_0(v, b, k)$ having the $\delta = 6$ discrepancy matrix $I_2 \otimes D2$ exists, and consequently search for discrepancy matrices $\Delta_{\bar{d}}$ that are E-better than $\Delta_d = D2$ having minimum eigenvalue $U_{\bar{d}} < 1.73205 = U_d$; such a design, if it exists, is E-better than D7 and D10 designs as well as Zhang's design of table 1.1. Now, according to conditions (1.32) and (1.33) from Step 1, the elements of discrepancy matrices for potentially E-better designs must be in the set $\{-1, 0, 1\}$. Thus, we will select our starter candidate list by partitioning the potential E-best discrepancy matrices into three cases according to the number of 1s (hence -1s) allowed to occur in a row of the discrepancy matrix and then by applying the element sum condition (1.36) of Step 2. The cases along with the candidate starter lists described in step two of the search procedure are:

Case 1: Discrepancy matrices with three or more ones in at least one row. Without loss of generality, we assume the first row (and column) of each starter has at least three ones. Therefore, Case 1 starters will have dimension four. Since condition (1.36) requires the sum of the elements below and above the diagonal to be less than or equal to two, the four nonisomorphic structures are:

$$(i)\ \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{matrix} \qquad (ii)\ \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{matrix} \qquad (iii)\ \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{matrix} \qquad (iv)\ \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Case 2: Discrepancy matrices with no more that two ones in the same row and exactly two ones in at least one row. We assume the first row of each starter in this case has exactly two ones, and, consequently, each starter is of dimension three. Then, by condition (1.36), the sum of the elements below and above the diagonal must be less than or equal to two. The two nonisomorphic structures are:

$$\begin{array}{ccc} & (i) & \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{array} \qquad \begin{array}{ccc} & (ii) & \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}$$

Case 3: Discrepancy matrices with a single one in any row. The only possible structure clearly is:

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

For the first search (Case 1), Step 3 of the procedure that applies (1.38) to each starter candidate immediately eliminates 1(iii), 1(iv), and 2(ii). Continuing the procedure with candidates 1(i) and 1(ii) does not result in an E-better discrepancy matrix, and, therefore, discrepancy matrices having three or more ones in any row are eliminated. Case 3 results in one discrepancy matrix, matrix D2. The interesting case is 2(i) for which we will demonstrate the search procedure.

Since Case 2 searches for discrepancy matrices having no more than two 1s (and two -1s) in any row, for the first extension we require a -1 to be placed in the first row. There are three possible extensions, and they are:

Extension		$\max_i \left[\frac{v-x_i^2}{v}\right]^{-1} u_{di}$
$(E1a)$	$\begin{matrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{matrix}$	1.15616
$(E1b)$	$\begin{matrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{matrix}$	1.5557
$(E1c)$	$\begin{matrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{matrix}$	1.1989

Continuing the process using $(E1c)$ as a starter does not lead to any E-better discrepancy matrices; however, each of $(E1a)$ and $(E1b)$ ultimately yields one discrepancy matrix that is E-better than D2. Since the E-best discrepancy matrix results from using $(E1a)$ as a starter, we continue the demonstration by extending matrix $(E1a)$ and, since the first row can not receive any 1s but needs two -1s in order to fulfill the zero-sum row requirement of a discrepancy matrix, without loss of generality, we will require a -1 to be placed in the first row of each extension. Two admissible matrices result, and they are:

	Extension	$\max_i \left[\frac{v-x_i^2}{v}\right]^{-1} u_{di}$
$(E2a)$	$\begin{matrix} 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{matrix}$	1.6180
$(E2b)$	$\begin{matrix} 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \end{matrix}$	1.6411

Attempts to extend matrix $(E2b)$ with the requirement that a 1 be placed in the fifth row results in no admissible matrices. Using matrix $(E2a)$ as a starter and enumerating all extensions having a -1 in the second row results in two admissible matrices. The resulting extensions are:

	Extension	$\max_i \left[\frac{v-x_i^2}{v}\right]^{-1} u_{di}$
$(E3a)$	$\begin{matrix} 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 \end{matrix}$	1.6920
$(E3b)$	$\begin{matrix} 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \end{matrix}$	1.6407

Enumerating all extensions of $(E3b)$ requiring a 1 to be placed in the sixth row does not produce any admissible matrices. The only admissible extension of $(E3a)$ places a -1 in rows three and four and a 1 in rows five and six and produces the 7×7 discrepancy matrix $\Delta_{\bar{d}}$ having $\delta_{\bar{d}} = 7$ and maximum eigenvalue $U_{\bar{d}} = U_{d^\bullet} = 1.6920$ shown in table 1.14. If a V-BIBD d with $w = 5$ having this discrepancy matrix exists, then it will have E-value 5.66160 and be the E-best design in $D(15, 21, 5)$.

Table 1.14: A Discrepancy Matrix With Maximum Eigenvalue 1.6920

0	1	1	−1	−1	0	0
1	0	−1	1	0	−1	0
1	−1	0	0	1	0	−1
−1	1	0	0	0	1	−1
−1	0	1	0	0	−1	1
0	−1	0	1	−1	0	1
0	0	−1	−1	1	1	0

As mentioned above, using $(E1b)$ as a starter also produces a discrepancy matrix. It is the 9×9 matrix shown in table 1.15 having discrepancy 6 and maximum eigenvalue 1.7321. This matrix is E-equivalent to the 12×12 discrepancy matrix $I_2 \otimes D2$ also having discrepancy 6.

Table 1.15: A Discrepancy Matrix With Maximum Eigenvalue 1.7321

0	1	1	−1	0	0	−1	0	0
1	0	−1	0	0	0	0	0	0
1	−1	0	0	0	0	0	0	0
−1	0	0	0	1	1	−1	0	0
0	0	0	1	0	−1	0	0	0
0	0	0	1	−1	0	0	0	0
−1	0	0	−1	0	0	0	1	1
0	0	0	0	0	0	1	0	−1
0	0	0	0	0	0	1	−1	0

The search for the A- and D-optimal design in the previous section enumerated all nonisomorphic U-BIBDs with $w = 7$ and $w = 6$. We can now use these designs to search for an E-optimal design by searching their extensions to V-BIBDs, requiring the finished designs to contain a discrepancy matrix of the form in table 1.14. Doing so produces an E-optimal design having optimality values:

$$\text{A-value} = 2.33830, \quad \text{D-value} = -25.06954, \quad \text{and} \quad \text{E-value} = 5.66160$$

The design is shown in table 1.16. The A-, D-, and E-efficiencies for designs with

Table 1.16: An E-optimal Design In $D(15, 21, 5)$

1	1	2	4	5	2	1	5	1	4	3	2	1	3	4	1	3	3	2	2	1
2	6	3	5	6	4	3	9	2	7	5	6	4	7	8	10	6	4	7	5	5
3	7	8	6	8	9	7	11	6	9	8	11	8	10	12	11	9	6	8	7	9
4	8	9	7	10	10	11	12	10	10	10	12	11	12	13	13	13	13	13	13	14
5	9	11	11	12	12	12	13	14	15	15	15	14	14	14	15	14	15	15	14	15

discrepancy matrices D7 and D10, and for Zhang's design from table 1.1 with respect to the E-optimal design with the discrepancy matrix in table 1.14 are provided in table 1.17.

Table 1.17: A-, D-, and E-efficiencies Relative To An E-optimal Design

	D7	D10	Zhang
A-efficiency	1.00085	1.00083	1.00021
D-efficiency	1.00620	1.00613	1.00172
E-efficiency	0.98912	0.97242	0.97324

We have calculated the ϕ_p-values of designs having discrepancy matrix D7 (that is, A- and D-optimal designs) and of E-optimal designs for $p \leq 100$. From these we conclude that discrepancy D7 designs are ϕ_p-better for $p \leq 38$, and E-optimal designs are ϕ_p-better for all $p \geq 39$ (the ϕ_p-value of E-best designs is less than $1/5.6 = 0.17857$ when $p = 100$).

CHAPTER II

RESOLVABLE DESIGNS WITH TWO BLOCKS PER REPLICATE: GENERAL THEORY

2.1 Introduction

When an incomplete block design is used, it is sometimes necessary to conduct the experiment in stages. For example, consider an industrial experiment to compare the effect of nine, say, combinations of materials used to manufacture an airplane part on the overall weight and strength of the part. Suppose the company conducting the experiment has two machines that manufacture the part, one machine can produce five parts at a time, and the other four. The experiment then consists of a series of "runs" in which each material combination is used one time. Moreover, suppose the machines frequently break down, and, as a result, it may not be possible to complete the desired number of runs. The experimenter is interested in knowing the allocation of the material combinations to the machines in each of the runs that will provide the best weight/strength estimates and comparisons. There are many other examples of similar experimental designs in agricultural trials, see Patterson and Silvey (1980), for example. These types of experiments fall into the category of ***resolvable block designs*** and are the topic of this remainder of this manuscript.

A resolvable block design setting $D(v, r; k_1, k_2, \ldots, k_s)$ with treatment replication r consists of r sets of blocks of sizes $k_1, k_2, \ldots, k_s$, where $\sum k_j = v$. A resolvable design is an assignment of v treatments to the $b = rs$ blocks in such a way that each treatment occurs once in each set, which is consequently called a *replicate*.

An example of a resolvable design in $D(9,4;5,4)$ that can be used for the airplane part experiment described above is shown in table 2.18 with the blocks written as columns. Later we will prove that this design is optimal with respect to many useful optimality criteria.

Table 2.18: A Resolvable Design In $D(9,4;4,5)$

1	6	1	4	1	2	1	3
2	7	2	5	4	3	2	5
3	8	3	8	5	8	4	7
4	9	6	9	6	9	6	9
5		7		7		8	

The origin of the concept of resolvability dates to the literature of the 19th century, for example, "Kirkman's schoolgirl problem" (Kirkman, 1850). A paper by Preece (1982) is an excellent source for many historical references of resolvable designs. Yates provided the first systematic study of resolvable designs when he introduced square *lattice* designs (1936, 1940), and the terms "resolvable design" and "affine resolvable" were introduced by Bose (1942). Yates' lattice designs were extended to *rectangular lattices* by Harshbarger (1946, 1949). Williams (1975) and Patterson and Williams (1976) introduced *α-designs*. Bailey, Monod, and Morgan (1995) discuss a class of designs that were introduced by Bose (1942) called *affine resolvable designs*. In that paper they provide constructions by using orthogonal arrays which were introduced by Rao (1947). A book by John and Williams (1995) and a manuscript by Morgan (1996) provide excellent summaries of these major classes of resolvable designs with references.

Virtually all of the references listed above describe design settings having equal block sizes; not much is known about resolvable designs with unequal block sizes. Two references for such such designs are Patterson and Williams (1976) and Kageyama (1988). Our treatment of resolvable designs will allow for unequal block sizes.

Cheng and Bailey (1991) have shown that square lattice designs are A-, D-, and E-optimal among the class of binary, equireplicate designs, and Bailey, Monod, and Morgan (1995) proved that affine-resolvable designs are optimal with respect to many optimality criteria, including A-, D-, and E-optimality, using Schur-optimality. Our concern will be A-, E-, Schur-, and type-1 optimality of resolvable designs.

We will restrict our discussion to the subclass of resolvable designs having $s = 2$ blocks per replicate in this document; however, the theoretical framework introduced here can be extended (perhaps with considerable difficulty) to settings having $s > 2$ blocks per replicate. We will leave that investigation for future work. The total number of blocks will be $b = 2r$. The sizes of the two blocks in each replicate may be unequal but will be the same for all replications. The size of the first block of each replicate will be denoted by k_1, the size of the second block by k_2, and, without loss of generality, we will assume $k_1 \geq k_2$. Then $v = k_1 + k_2$, and the *block sizes* vector is $\mathbf{k} = \mathbf{1} \otimes (k_1, k_2)^T$ where $\mathbf{1}$ is the $r \times 1$ vector of 1s and $\otimes$ denotes the Kronecker product. The general setup is pictured in figure 2.2. The number of treatments v and the block sizes will be arbitrary.

Certain classes of optimal resolvable designs with $s = 2$ and $r \geq v$ can be constructed from Balanced Incomplete Block Designs. Suppose $D(v, b, k)$ is a BIBD setting, and let $d \in D$ be a BIBD. It is well known that d is universally optimal (Kiefer, 1975). Let $S = \{1, 2, \ldots, v\}$ be the set containing all of the available treatments for the setting D. A new design, $\bar{d}$, also having b blocks, can be obtained from d by taking each of the b blocks of $\bar{d}$ to be the complement of the corresponding blocks of d. That is, if b_i and $\bar{b}_i$ are the ith blocks of respectively d and $\bar{d}$, then $\bar{b}_i = S \backslash b_i$, $i = 1, 2, \ldots, b$. Design $\bar{d}$ is called the *complement* or *complementary design* of d; it is a BIBD With parameters $\bar{v} = v$, $\bar{b} = b$, $\bar{k} = v - k$, $\bar{r} = b - r$, and $\bar{\lambda} = b - 2r + \lambda$, and are therefore universally optimal (Street and Street, 1987, page 45). Since $b_i \cup \bar{b}_i = S$ for each i, the design $d^* = d \cup \bar{d}$ is a resolvable design with

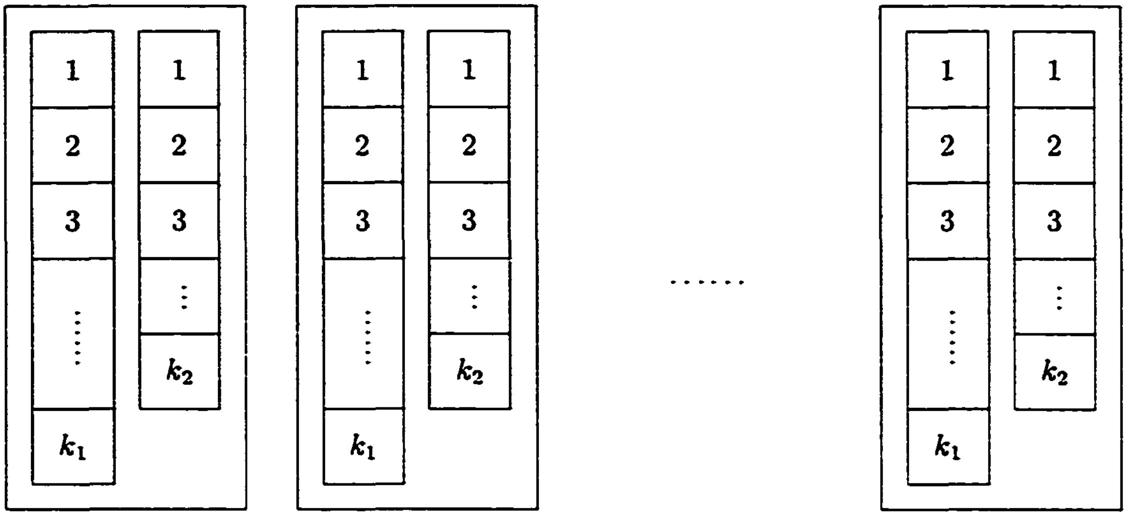


Figure 2.2: Resolvable Design With $s = 2$, Arbitrary r, and $k_1 \geq k_2$

$v^* = v$ treatments in $b^* = 2b$ blocks divided into $r^* = b$ replicates each containing two blocks of sizes $k_1^* = k$ and $k_2^* = v - k$. It follows from Fisher's inequality that $b^* \geq 2v$, or $r^* \geq v$. Furthermore, the information matrix for d^*, which is

$$C_{d^*} = C_d + C_{\bar{d}} = \frac{b(v-2)}{v-1}\left(I - \frac{1}{v}J\right), \tag{2.40}$$

is completely symmetric and of maximal trace, and, therefore, by Kiefer's result (1975), d^* is universally optimal.

For example, a design d in the BIBD setting $D(7,7,4)$ having $r = 4$ and $\lambda = 2$ with the blocks written as columns is

$$\begin{matrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 4 & 3 & 4 & 4 \\ 3 & 5 & 5 & 6 & 6 & 5 & 5 \\ 4 & 6 & 7 & 7 & 7 & 7 & 6. \end{matrix}$$

The complementary design $\bar{d} \in D(7,7,3)$ having $\bar{r} = 3$ and $\bar{\lambda} = 1$ is

$$\begin{matrix} 5 & 3 & 2 & 2 & 1 & 1 & 1 \\ 6 & 4 & 4 & 3 & 4 & 3 & 2 \\ 7 & 7 & 6 & 5 & 5 & 6 & 7, \end{matrix}$$

and the universally optimal resolvable design $d^* = d \cup \bar{d}$ is

$$\begin{array}{cc|cc|cc|cc|cc|cc|cc} 1 & 5 & 1 & 3 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 3 & 1 \\ 2 & 6 & 2 & 4 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 4 & 2 \\ 3 & 7 & 5 & 7 & 5 & 6 & 5 & 5 & 6 & 5 & 5 & 6 & 5 & 7 \\ 4 & & 6 & & 7 & & 7 & & 7 & & 7 & & 6. & \end{array} \tag{2.41}$$

Despite the elegance of constructing resolvable designs using BIBDs and their complements, and the potential for generalizing this technique to irregular BIBD settings or to settings that do not satisfy the necessary conditions for a BIBD by applying some of the ideas of Chapter I or from Morgan and Srivastav (2000), our discussion of resolvable block designs will not utilize this approach. Our concern will be resolvable designs with a small number of replications, and the number of replications in designs constructed using the procedure described above require $r \geq v$ which is a relatively large number of replications. As a result, our optimality analysis will take the more traditional approach of directly working with the information matrix for various design settings. We will make the requirement that $v \geq b$, that is $r \leq \frac{v}{2}$, for reasons that will be apparent shortly. For the remainder of this chapter $D(v, r; k_1, k_2)$ will denote the subclass of binary, connected, and equireplicate block designs that are resolvable and satisfy the conditions described above.

A design $d \in D$ has information matrix

$$C_d = r\,I - N_d\,\mathbf{k}^{-\delta}\,N_d^T \tag{2.42}$$

where I is the identity matrix of order v, $\mathbf{k}^{\delta}$ is the $b \times b$ diagonal matrix whose diagonal elements are the elements of $\mathbf{k}$, $\mathbf{k}^{-\delta}$ is the inverse of $\mathbf{k}^{\delta}$, and N_d is the $v \times b$ incidence matrix. Of concern to us is identifying and constructing the A- and E-optimal designs $d \in D$ for various choices of r, v and (k_1, k_2), requiring calculation of the eigenvalues of the information matrix C_d. This task is simplified by the following manipulation. If the treatments and blocks of design $d \in D$ are interchanged so that treatment i in block j becomes treatment j in block i, then we obtain a design having incidence matrix N_d^T that places b treatments into v blocks of equal size r

with treatment replication vector $\mathbf{k} = \mathbf{1} \otimes (k_1, k_2)^T$. This design is called the *dual* of d and has information matrix

$$C_{\text{dual}} = \mathbf{k}^{\delta} - \frac{1}{r} N_d^T N_d. \tag{2.43}$$

The (j, j')th element of the concurrence matrix $N_d^T N_d$ of the dual design of d indicates the number of treatments simultaneously occurring in blocks j and j', that is, the number of block j and j' *block concurrences*, of the corresponding $d \in D$. The elements of $N_d^T N_d$ are referred to as *block concurrence counts*.

If we multiply C_d by $\frac{1}{r}$, and if we right and left multiply C_{dual} by $\mathbf{k}^{-\delta/2}$, equations (2.42) and (2.43) become

$$\frac{1}{r} C_d = I - \frac{1}{r} N_d \mathbf{k}^{-\delta} N_d^T = C_d^* \tag{2.44}$$

and

$$\mathbf{k}^{-\delta/2} C_{\text{dual}} \mathbf{k}^{-\delta/2} = I - \frac{1}{r} \mathbf{k}^{-\delta/2} N_d^T N_d \mathbf{k}^{-\delta/2} = C_{\text{dual}}^* \tag{2.45}$$

where the $b \times b$ matrix $\mathbf{k}^{-\delta/2}$ is the inverse of $\mathbf{k}^{\delta/2}$, which is the diagonal matrix having the elements of $\sqrt{\mathbf{k}} = \mathbf{1} \otimes (\sqrt{k_1}, \sqrt{k_2})^T$ on the diagonal. Define the $v \times b$ matrix $B_d = N_d \mathbf{k}^{-\delta/2}$ and substitute into (2.44) and (2.45) to obtain

$$C_d^* = I - \frac{1}{r} B_d B_d^T \tag{2.46}$$

and

$$C_{\text{dual}}^* = I - \frac{1}{r} B_d^T B_d. \tag{2.47}$$

Suppose $a_1, a_2, \ldots, a_b$ are the eigenvalues of $B_d^T B_d$, then, since the nonzero eigenvalues of $B_d B_d^T$ and $B_d^T B_d$ are identical, the eigenvalues of $B_d B_d^T$ (for $v \geq b$) are $a_1, a_2, \ldots, a_b$ and $v - b$ copies of 0. Note that $B_d^T B_d \mathbf{k}^{1/2} = r\mathbf{k}^{1/2}$; that is, $a_i = r$ for some i, say $i = b$. Thus, C_{dual}^* has $b - 1$ nonzero eigenvalues $(1 - \frac{1}{r}a_1), (1 - \frac{1}{r}a_2), \ldots, (1 - \frac{1}{r}a_{b-1})$ and one eigenvalue equal to 0, and C_d^* has $b - 1$ nonzero eigenvalues $(1 - \frac{1}{r}a_1), (1 - \frac{1}{r}a_2), \ldots, (1 - \frac{1}{r}a_{b-1})$, one eigenvalue equal to 0, and $v - b$ eigenvalues equal to 1. It follows that the eigenvalues of C_d are $(r-a_1), (r-a_2), \ldots, (r-a_{b-1})$, 0, and $v-b$ copies of r. Therefore, an eigenvalue-based optimality investigation of designs in $D(v,b;k_1,k_2)$ can be performed by restricting our efforts to studying the eigenvalues of C^*_{dual}. Since we will be investigating design settings with a fixed number of blocks b but for a varying number of treatments v, the dimension of C_{dual} will remain constant for all v. Furthermore, working with C_{dual} requires us to focus on block concurrences in the formation of $N_d^T N_d$. This approach will significantly simplify our search for optimal designs in D.

Define the symmetric matrix $A_d = B_d^T B_d = \mathbf{k}^{-\delta/2} N_d^T N_d \mathbf{k}^{-\delta/2}$. Then $C^*_{\text{dual}} = I - \frac{1}{r} A_d$. If $(a_1, \mathbf{x}_1), (a_2, \mathbf{x}_2), \ldots, (a_b, \mathbf{x}_b)$ are the eigenvalue/vector pairs of A_d, its spectral decomposition is

$$A_d = \sum_{i=1}^{b} a_i \, \mathbf{x}_i \, \mathbf{x}_i^T, \tag{2.48}$$

$\mathbf{x}_i^T \mathbf{x}_i = 1$, and $\mathbf{x}_i^T \mathbf{x}_j = 0$ for $i \neq j$. Since $A_d \mathbf{k}^{1/2} = r\mathbf{k}^{1/2}$, then $\left(r, \frac{\mathbf{k}^{1/2}}{\sqrt{(\mathbf{k}^{1/2})^T \mathbf{k}^{1/2}}}\right)$ is one of the eigenvalue/vector pairs, the bth pair say. Note that this eigenvalue corresponds to the eigenvalue equal to zero that is common to C_{dual} and C_d. The bth term of (2.48) is then

$$a_b \mathbf{x}_b \mathbf{x}_b^T = r \frac{\mathbf{k}^{1/2} (\mathbf{k}^{1/2})^T}{(\mathbf{k}^{1/2})^T \mathbf{k}^{1/2}} = \frac{1}{k_1 + k_2} \left[J \otimes \begin{pmatrix} k_1 & \sqrt{k_1 k_2} \\ \sqrt{k_1 k_2} & k_2 \end{pmatrix} \right] \tag{2.49}$$

where J is a $r \times r$ matrix of 1s. Subtracting (2.49) from (2.48) yields the new matrix

$$A_d^* = A_d - \frac{1}{k_1 + k_2} \left[J \otimes \begin{pmatrix} k_1 & \sqrt{k_1 k_2} \\ \sqrt{k_1 k_2} & k_2 \end{pmatrix} \right] = \sum_{i=1}^{b-1} a_i \mathbf{x}_i \mathbf{x}_i^T. \tag{2.50}$$

Clearly, $(a_i, \mathbf{x}_i)$, $1 \leq i < b-1$ are eigenvalue/vector pairs for A_d^*, and for the eigenvector $\mathbf{x}_b$, A_d^* has an eigenvalue of 0. Furthermore, $(a_i, \mathbf{x}_i)$ is an eigenvalue/vector of A_d^* if and only if $(1 - \frac{1}{r} a_i, \mathbf{x}_i)$ is an eigenvalue/vector pair of C_d^* if and only if $r - a_i$ is an eigenvalue of C_d. Therefore, we can obtain all the eigenvalue-based optimality information for any design $d \in D$ using equation (2.50) provided we can construct $N_d^T N_d$ for an arbitrary $d \in D$ in order to obtain an explicit expression for A_d^*.

We will construct the concurrence matrix for a dual design $N_d^T N_d$ by first observing the block concurrences for the blocks of two arbitrary replicates, n and n' say, of a design $d \in D$. Replication n, $1 \leq n \leq r$, contains blocks $2n-1$ and $2n$ which will be denoted by b_{2n-1} and b_{2n}, respectively. Denote the b_{2n-1} and $b_{2n'-1}$ block concurrence counts by $\phi_{nn'}$, and, without loss of generality, assume $1 \leq n \leq n' \leq r$. The remaining $k_1 - \phi_{nn'}$ treatments in $b_{2n'-1}$ are also in b_{2n}. If the k_1 treatments in b_{2n-1} are labeled $1, 2, \ldots, k_1$ and the k_2 treatments in b_{2n} are labeled $k_1+1, k_1+2, \ldots, k_1+k_2$, then, since these labels are arbitrary, we can assume treatments $1, 2, \ldots, \phi_{nn'}$ are in b_{2n-1} and $b_{2n'-1}$, and treatments $k_1+1, k_1+2, \ldots, 2k_1-\phi_{nn'}$ are in b_{2n} and $b_{2n'-1}$. Now, the remaining $k_1-\phi_{nn'}$ treatments in b_{2n-1} that are not in $b_{2n'-1}$, which are treatments $\phi_{nn'}+1, \phi_{nn'}+2, \ldots, k_1$, must also be in $b_{2n'}$, and the $k_2-k_1+\phi_{nn'}$ treatments in b_{2n} that are not in $b_{2n'-1}$, which are treatments $2k_1-\phi_{nn'}+1, 2k_1-\phi_{nn'}+2, \ldots, k_1+k_2$, are in $b_{2n'}$. Refer to figure 2.3 below to see the treatment placements. Thus, once

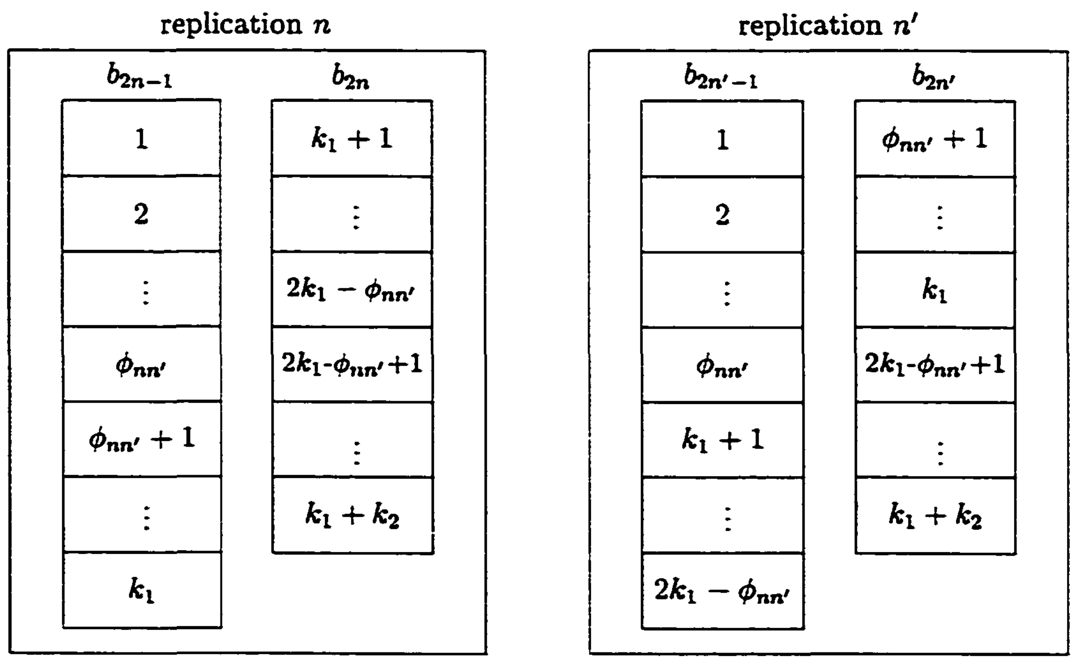


Figure 2.3: Replication n and n' Block Concurrences

the b_{2n-1} and $b_{2n'-1}$ and the b_{2n} and $b_{2n'-1}$ block concurrences are chosen, all of the remaining replication n and n' block concurrences are prescribed; moreover, the block concurrence counts for each pair of blocks is determined once $\phi_{nn'}$ is chosen.

The intersection of rows $2n-1$ and $2n$ of N_d^T with columns $2n'-1$ and $2n'$ of N_d in $N_d^T N_d$, which makes up the submatrix of $N_d^T N_d$ corresponding to the block concurrence counts for the blocks in replications n and n', is

$$\Phi_{nn'} = \begin{pmatrix} \phi_{nn'} & k_1 - \phi_{nn'} \\ k_1 - \phi_{nn'} & k_2 - k_1 + \phi_{nn'} \end{pmatrix}.$$

Note that, since n and n' are arbitrary, the block concurrence submatrix for any two replications will have the same structure, and when $n = n'$

$$\Phi_{nn} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Therefore, $N_d^T N_d$ is

$$\begin{pmatrix} k_1 & 0 & \phi_{12} & k_1 - \phi_{12} & \phi_{13} & k_1 - \phi_{13} & \cdots & \phi_{1r} & k_1 - \phi_{1r} \\ 0 & k_2 & k_1 - \phi_{12} & k_2 - k_1 + \phi_{12} & k_1 - \phi_{12} & k_2 - k_1 + \phi_{12} & \cdots & k_1 - \phi_{1r} & k_2 - k_1 + \phi_{1r} \\ & & k_1 & 0 & \phi_{23} & k_1 - \phi_{23} & & \phi_{2r} & k_1 - \phi_{2r} \\ & & 0 & k_2 & k_1 - \phi_{23} & k_2 - k_1 + \phi_{23} & & k_1 - \phi_{2r} & k_2 - k_1 + \phi_{2r} \\ & & & & k_1 & 0 & & \phi_{3r} & k_1 - \phi_{3r} \\ & & & & 0 & k_2 & & k_1 - \phi_{3r} & k_2 - k_1 + \phi_{3r} \\ & & & & & & & \vdots & \vdots \\ & & & & & & & k_1 & 0 \\ & & & & & & & 0 & k_2 \end{pmatrix}.$$

Which may be written

$$N_d^T N_d = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1r} \\ & \Phi_{22} & & \Phi_{2r} \\ & & & \vdots \\ & & & \Phi_{rr} \end{pmatrix}. \qquad (2.51)$$

Clearly block concurrences will be constrained by a particular choice of $d \in D$. The question is, what are the admissible block concurrences and block concurrence counts? In particular, what range of values can $\phi_{nn'}$, $n \leq n'$ assume? First consider the block concurrences for blocks b_1 and b_2 of replication one with blocks $b_{2n'-1}$ and

$b_{2n'}$ of replication n', $1 \leq n' \leq r$. When $n' > 1$, the b_1 and $b_{2n'-1}$ block concurrence count $\phi_{1n'}$ must be less than or equal to k_1, and the $k_1 - \phi_{1n'}$ block concurrences b_2 has with $b_{2n'-1}$ must be less than or equal to k_2. Therefore, $k_1 - k_2 \leq \phi_{1n'} \leq k_1$ for all $1 < n' \leq r$, and when $n' = 1$, $\phi_{1n'} = k_1$.

Now we will investigate the replication two, containing blocks b_3 and b_4, and replication n' $(2 \leq n' \leq r)$ block concurrences. The $\phi_{2n'}$ treatments common to blocks b_3 and $b_{2n'-1}$ can be divided into two groups: treatments from b_1 and treatments from b_2. The b_3 and $b_{2n'-1}$ block concurrences among treatments from b_1 must be in b_1, b_3, and $b_{2n'-1}$, and, consequently are among the ϕ_{12} treatments from b_3 that are in b_1 and the $\phi_{1n'}$ treatments from $b_{2n'-1}$ that are in b_1. Thus, the number of b_3 and $b_{2n'-1}$ block concurrences with the treatments from b_1 can be no larger than $\min\{\phi_{12}, \phi_{1n'}\}$. Similarly, the b_3 and $b_{2n'-1}$ block concurrences among treatments from b_2 must be in b_2, b_3, and $b_{2n'-1}$, and are among the $k_1 - \phi_{12}$ treatments from b_3 that are in b_2 and the $k_1 - \phi_{1n'}$ treatments from $b_{2n'-1}$ that are in b_2. Thus, the number of b_3 and $b_{2n'-1}$ block concurrences with the treatments in b_2 can be no larger than $\min\{k_1 - \phi_{12}, k_1 - \phi_{1n'}\}$, and $\phi_{2n'} \leq \min\{\phi_{12}, \phi_{1n'}\} + \min\{k_1 - \phi_{12}, k_1 - \phi_{1n'}\}$. Now, if $\phi_{12} + \phi_{1n'} > k_1$ then b_3 and $b_{2n'-1}$ must have at least $(\phi_{12} + \phi_{1n'}) - k_1$ block concurrences from b_1, and if $(k_1 - \phi_{12}) + (k_1 - \phi_{1n'}) > k_2$ then b_3 and $b_{2n'-1}$ must have at least $2k_1 - (\phi_{12} + \phi_{1n'}) - k_2$ block concurrences from b_2. Note that if $\phi_{12} + \phi_{1n'} \leq k_1$ or $(k_1 - \phi_{12}) + (k_1 - \phi_{1n'}) \leq k_2$, then b_3 and $b_{2n'-1}$ need not have any block concurrences among the treatments in b_1 or b_2, respectively. Therefore $\max\{0, (\phi_{12} + \phi_{1n'}) - k_1\} + \max\{0, 2k_1 - (\phi_{12} + \phi_{1n'}) - k_2\} \leq \phi_{2n'} \leq \min\{\phi_{12}, \phi_{1n'}\} + \min\{k_1 - \phi_{12}, k_1 - \phi_{1n'}\}$, $2 \leq n' \leq r$.

We will now generalize the previous discussion to the replication n with replication n', $1 < n \leq n' \leq r$, block concurrences. As in the replication two and n' case above, the $\phi_{nn'}$ b_{2n-1} and $b_{2n'-1}$ block concurrences can be divided into two groups, but now the groups are made up of treatments from b_{2l-1} and treatments from b_{2l}, for arbitrary $1 \leq l < n$. For each l and for the same reasons outlined above in the replication two block concurrence argument replacing b_1 with b_{2l-1} and b_2 with b_{2l}, $\phi_{nn'} \leq \min\{\phi_{ln}, \phi_{ln'}\} + \min\{k_1 - \phi_{ln}, k_1 - \phi_{ln'}\}$, and $\phi_{nn'} \geq \max\{0, (\phi_{ln} + \phi_{ln'}) - k_1\} + \max\{0, 2k_1 - (\phi_{ln} + \phi_{ln'}) - k_2\}$. Then for $2 \leq n \leq n' \leq r$, $\max_{1 \leq l < n}\{\max\{0, (\phi_{ln} + \phi_{ln'}) - k_1\} + \max\{0, 2k_1 - (\phi_{ln} + \phi_{ln'}) - k_2\}\} \leq \phi_{nn'}$ and $\phi_{nn'} \leq \min_{1 \leq l < n}\{\min\{\phi_{ln}, \phi_{ln'}\} + \min\{k_1 - \phi_{ln}, k_1 - \phi_{ln'}\}\}$.

In summary, the block concurrence count for the first block of replication n, b_{2n-1}, and the first block of replication n', $b_{2n'-1}$ must satisfy

$$k_1 - k_2 \leq \phi_{1n'} \leq k_1, \tag{2.52}$$

when $n = 1$ and

$$\begin{aligned} &\max_{1 \leq l < n}\{\max\{0, (\phi_{ln} + \phi_{ln'}) - k_1\} + \max\{0, 2k_1 - (\phi_{ln} + \phi_{ln'}) - k_2\}\} \\ &\quad \leq \phi_{nn'} \leq \min_{1 \leq l < n}\{\min\{\phi_{ln}, \phi_{ln'}\} + \min\{k_1 - \phi_{ln}, k_1 - \phi_{ln'}\}\}, \end{aligned} \tag{2.53}$$

when $2 \leq n \leq n' \leq r$. The remaining block concurrence counts for each pair of blocks of any two replications n and n' which are, $k_1 - \phi_{nn'}$ (twice) and $k_2 - k_1 + \phi_{nn'}$, are expressions involving only the $\phi_{nn'}$s and block sizes k_1 and k_2, and their constraints follow from (2.52) and (2.53). Therefore, the b_{2n-1} and $b_{2n'-1}$ block concurrence, that is, the block concurrence for the first block of replications n with the first block of replication n', once chosen determine a bound for the block concurrence counts for the remaining blocks of replications n and n'. We will assume the $\phi_{nn'}$s satisfy (2.52) and (2.53).

Now that we have derived $N_d^T N_d$ for an arbitrary resolvable design $d \in D$, we are ready to write an explicit expression for $A_d = \mathbf{k}^{-\delta/2} N_d^T N_d \mathbf{k}^{-\delta/2}$ using (2.51). By rewriting $b \times b$, $b = 2r$, diagonal matrix $\mathbf{k}^{-\delta/2}$ as

$$I \otimes \begin{pmatrix} \frac{1}{\sqrt{k_1}} & 0 \\ 0 & \frac{1}{\sqrt{k_2}} \end{pmatrix} = I \otimes \boldsymbol{\kappa}^{-\delta/2}$$

it follows that

$$A = \begin{pmatrix} \kappa^{-\delta/2}\Phi_{11}\kappa^{-\delta/2} & \kappa^{-\delta/2}\Phi_{12}\kappa^{-\delta/2} & \cdots & \kappa^{-\delta/2}\Phi_{1r}\kappa^{-\delta/2} \\ & \kappa^{-\delta/2}\Phi_{22}\kappa^{-\delta/2} & & \kappa^{-\delta/2}\Phi_{2r}\kappa^{-\delta/2} \\ & & & \vdots \\ & & & \kappa^{-\delta/2}\Phi_{rr}\kappa^{-\delta/2} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi^*_{11} & \Phi^*_{12} & \cdots & \Phi^*_{1r} \\ & \Phi^*_{22} & & \Phi^*_{2r} \\ & & & \vdots \\ & & & \Phi^*_{rr} \end{pmatrix}, \tag{2.54}$$

where $\Phi^*_{nn} = I$, the 2×2 identity matrix, and

$$\Phi^*_{nn'} = \begin{pmatrix} \frac{\phi_{nn'}}{k_1} & \frac{k_1 - \phi_{nn'}}{\sqrt{k_1 k_2}} \\ \frac{k_1 - \phi_{nn'}}{\sqrt{k_1 k_2}} & \frac{k_2 - k_1 + \phi_{nn'}}{k_2} \end{pmatrix} \tag{2.55}$$

for $1 \leq n \leq n' \leq r$. A^*_d in (2.50) can now be easily obtained by subtracting

$$\frac{1}{k_1 + k_2} \begin{pmatrix} k_1 & \sqrt{k_1 k_2} \\ \sqrt{k_1 k_2} & k_2 \end{pmatrix} \tag{2.56}$$

from each $\Phi^*_{nn'}$ in (2.54). Since subtracting (2.56) from $\Phi^*_{nn} = I$ yields

$$\frac{1}{k_1 + k_2} \begin{pmatrix} k_2 & -\sqrt{k_1 k_2} \\ -\sqrt{k_1 k_2} & k_1 \end{pmatrix},$$

and subtracting (2.56) from $\Phi^*_{nn'}$ given in (2.55) yields

$$\frac{\phi_{nn'}(k_1 + k_2) - k_1^2}{k_1 k_2 (k_1 + k_2)} \begin{pmatrix} k_2 & -\sqrt{k_1 k_2} \\ -\sqrt{k_1 k_2} & k_1 \end{pmatrix},$$

then

$$A^*_d = \frac{1}{(k_1 + k_2) k_1 k_2} \begin{pmatrix} k_1 k_2 & \phi^*_{12} & \phi^*_{12} & \cdots & \phi^*_{1r} \\ & k_1 k_2 & \phi^*_{23} & & \phi^*_{2r} \\ & & k_1 k_2 & & \phi^*_{3r} \\ & & & & \vdots \\ & & & & k_1 k_2 \end{pmatrix} \otimes \begin{pmatrix} k_2 & -\sqrt{k_1 k_2} \\ -\sqrt{k_1 k_2} & k_1 \end{pmatrix}$$

where

$$\phi^*_{nn'} = \phi_{nn'}(k_1 + k_2) - k_1^2.$$

Since the eigenvalues of $\begin{pmatrix} k_2 & -\sqrt{k_1k_2} \\ -\sqrt{k_1k_2} & k_1 \end{pmatrix}$ are 0 and $k_1 + k_2$, then the $b = 2r$ eigenvalues of A_d^* are r copies of 0 and $\frac{1}{k_1k_2}$ times the r eigenvalues of

$$M_d = \begin{pmatrix} k_1k_2 & \phi_{12}^* & \phi_{13}^* & \cdots & \phi_{1r}^* \\ & k_1k_2 & \phi_{23}^* & & \phi_{2r}^* \\ & & k_1k_2 & & \phi_{3r}^* \\ & & & & \vdots \\ & & & & k_1k_2 \end{pmatrix}. \tag{2.57}$$

Suppose the eigenvalue of M_d are $e_1, e_2, \ldots, e_r$. Then the eigenvalues of A_d are r, $r-1$ copies of 0, and $(\frac{e_1}{k_1k_2}, \frac{e_2}{k_1k_2}, \ldots, \frac{e_r}{k_1k_2})$; the eigenvalues of C_{dual}^* are 0, $r-1$ copies of 1, and $(1 - \frac{e_1}{rk_1k_2}, 1 - \frac{e_2}{rk_1k_2}, \ldots, 1 - \frac{e_r}{rk_1k_2})$; and the eigenvalues of C_d are 0, $v - r - 1$ copies of r, and $(r - \frac{e_1}{k_1k_2}, r - \frac{e_2}{k_1k_2}, \ldots, r - \frac{e_r}{k_1k_2})$. Therefore, an eigenvalue-based optimality analysis of resolvable designs $d \in D(v, r; k_1, k_2)$ can focus on the matrix M_d for the corresponding set of block concurrence counts $\{\phi_{12}, \phi_{13}, \phi_{23}, \ldots, \phi_{r,r-1}\}$. We will use this fact in the following sections in which we discuss resolvable design settings for particular values of r.

2.2 General Results

Let $D(v, r; k_1, k_2)$ be a resolvable design setting with $s = 2$. Given values of k_1, k_2, and r, an experimenter is concerned with knowing the assignment of the treatments to the blocks that will yield the best possible information about the effect of the $v = k_1 + k_2$ treatments, that is, they want to know the optimal design $d \in D$. As we saw in the introduction, there are many different ways in which a design $d \in D$ can be considered optimal, and for each type of optimality to be achieved, a specific optimality criteria must be satisfied. In this chapter we will primarily investigate A- and E-optimality, but will often find much more.

Since designs in D are differentiated from one another by their block concurrences $\{\phi_{12}, \phi_{13}, \phi_{23}, \ldots, \phi_{r,r-1}\}$, our optimality investigation will focus on describing the structure of the matrix given by (2.57),which is

$$M_d = \begin{pmatrix} k_1k_2 & \phi^*_{12} & \phi^*_{13} & \cdots & \phi^*_{1r} \\ & k_1k_2 & \phi^*_{23} & & \phi^*_{2r} \\ & & k_1k_2 & & \phi^*_{3r} \\ & & & & \vdots \\ & & & & k_1k_2 \end{pmatrix},$$

where

$$\phi^*_{nn'} = \phi_{nn'}(k_1 + k_2) - k_1^2,$$

for a design $d \in D$ that is optimal with respect to one or more eigenvalue optimality criterion. For convenience, we will refer to the matrix M_d as the *Optimality Matrix* for the design d.

Suppose the eigenvalues of an optimality matrix M_d are $e_1 \geq e_2 \geq \ldots \geq e_r$, then tr $M_d = \sum_{i=1}^{r} e_i = rk_1k_2$ for any set of treatment concurrences, and the eigenvalues of C_d, which are $0 < z_{d1} \leq z_{d2} \leq \cdots \leq z_{d,v-1}$, in terms of the eigenvalues of M_d, are 0 and

$$z_{di} = \begin{cases} r - \frac{e_i}{k_1k_2} & \text{if } 1 \leq i \leq r \\ r & \text{if } r - 1 \leq i \leq v - 1. \end{cases} \tag{2.58}$$

Now, if $\mathcal{M}$ is the class of all optimality matrices for designs in D, the A-optimal design $d \in D$ with optimality matrix M_d, will have block concurrences that minimize

$$\sum_{i=1}^{r} \left(r - \frac{e_i}{k_1k_2}\right)^{-1},$$

over $M_d \in \mathcal{M}$, and the E-optimal design will maximize the minimum eigenvalue of C_d, that is, maximize

$$\left(r - \frac{e_1}{k_1k_2}\right),$$

over $M_d \in \mathcal{M}$ or, equivalently, minimize e_1 over $M_d \in \mathcal{M}$. The Type-1 optimal design $d \in D$ will be the design that minimizes

$$\sum_{i=1}^{r} f\left(r - \frac{e_i}{k_1k_2}\right) \tag{2.59}$$

over $M_d \in \mathcal{M}$ for all type-1 criteria f.

The following definitions from *Inequalities: Theory of Majorization and Its Applications* by Albert W. Marshall and Ingram Olkin (1979) will prove to be extremely useful for determining when a design $d \in D$ satisfies (2.59). After stating the definitions, we state a theorem, and, afterward, review some of their consequences that provide the link between the definition and Type-1 optimality.

Definition 2.2.1 Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be nonincreasing sequences of real numbers such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If

$$\sum_{i=1}^{l} x_i \leq \sum_{i=1}^{l} y_i, \qquad \text{for all } 1 \leq l \leq n,$$

or, equivalently,

$$\sum_{i=n}^{n-l+1} x_i \geq \sum_{i=n}^{n-l+1} y_i, \qquad \text{for all } 1 \leq l \leq n$$

then $\{y_i\}_{i=1}^n$ is said to *majorize* $\{x_i\}_{i=1}^n$.

Definitions 2.2.2 Suppose the eigenvalues, written in nonincreasing order, of the optimality matrices for designs d and d^* in $D(v, r; k_1, k_2)$ are $\{e_1, e_2, \ldots, e_r\}$ and $\{e_1^*, e_2^*, \ldots, e_r^*\}$, respectively.

1. If $\{e_1, e_2, \ldots, e_r\}$ majorizes $\{e_1^*, e_2^*, \ldots, e_r^*\}$, and the two vectors are not identical, then design d^* is said to be *Schur-better* than design d, and d is said to be *Schur-inferior* to d^*.

2. Design d^* is defined to be *Schur-optimal* if it is Schur-better than every other design in D.

The following theorem is due to Hardy, Littlewood, and Pólya and can be found in Marshall and Olkin (1979, p. 108). It shows why the majorization relationship and Schur-optimality are important.

Theorem 2.2.1 *Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be sequences of real numbers such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. For all continuous real-valued convex functions f,*

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$$

if and only if $\{y_i\}_{i=1}^n$ majorizes $\{x_i\}_{i=1}^n$.

Corollary 2.2.2 *Let d and d^* be in $D(v, r; k_1, k_2)$. If d^* is Schur-better than d, then d^* is superior to d with respect to every type-1 optimality criterion. Thus Schur-optimality implies optimality with respect to every type-1 criterion.*

Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be two nonincreasing sequences of real numbers such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. The following facts about majorization will be used extensively in the subsequent sections.

Fact I: 1. If $x_1 \geq x_2 = x_3 = \cdots = x_n$ and $y_1 \geq x_1$, then $\{y_i\}_{i=1}^n$ majorizes $\{x_i\}_{i=1}^n$.

2. If $x_1 = x_2 = \cdots = x_{n-1} \geq x_n$ and $x_n \geq y_n$, then $\{y_i\}_{i=1}^n$ majorizes $\{x_i\}_{i=1}^n$.

Fact II: Let a and b be real numbers. If $\{y_i\}_{i=1}^n$ majorizes $\{x_i\}_{i=1}^n$ then $\{a - \frac{y_i}{b}\}_{i=1}^n$ majorizes $\{a - \frac{x_i}{b}\}_{i=1}^n$.

Fact III: Let $\{a\}_{i=1}^m$ be a sequence of real numbers. If $\{y_i\}_{i=1}^n$ majorizes $\{x_i\}_{i=1}^n$ then $\{\{y_i\}_{i=1}^n \cup \{a\}_{i=1}^m\}$ majorizes $\{\{y_i\}_{i=1}^n \cup \{a\}_{i=1}^m\}$.

2.3 Equal Concurrences

A resolvable design $d \in D(v, r; k_1, k_2)$ having block concurrence counts $\phi_{12} = \phi_{13} = \phi_{23} = \cdots = \phi_{r-1,r} = \theta$ for some $k_1 - k_2 \leq \theta \leq k_1$ is called an *equal concurrence design* with block concurrences equal to θ, or *ECD(θ)*. The optimality matrix (2.57) for an *ECD(θ)* may be written in the following form

$$M_d = \{k_1k_2 - [\theta(k_1 + k_2) - k_1^2]\}I + [\theta(k_1 + k_2) - k_1^2]J, \tag{2.60}$$

where I is the $r \times r$ identity matrix, and J is the $r \times r$ matrix of ones. The eigenvalues of M_d are $r-1$ copies of

$$k_1k_2 + [k_1^2 - \theta(k_1+k_2)] \tag{2.61}$$

and one copy of

$$k_1k_2 - (r-1)[k_1^2 - \theta(k_1+k_2)]. \tag{2.62}$$

Theorem 2.3.1 *Suppose $D(v,r;k_1,k_2)$ is a resolvable design setting for which $(k_1+k_2) \mid k_1^2$, and define*

$$\theta^* = \frac{k_1^2}{k_1+k_2}.$$

Then ECD(θ^*)s in D are Schur-optimal whenever they exist.

Proof Let $D(v,r;k_1,k_2)$ be a resolvable design setting and suppose $(k_1+k_2) \mid k_1^2$. Since

$$k_1 - k_2 \leq \frac{k_1^2}{k_1+k_2} \leq k_1$$

then $\theta = \theta^*$ is an admissible value for the common treatment concurrences of an *ECD(θ)* in D. Since eigenvalues of the optimality matrix of an *ECD(θ^*)*, which are $k_1k_2 - (r-1)[k_1^2 - \theta^*(k_1+k_2)] = k_1k_2 + [k_1^2 - \theta^*(k_1+k_2)] = k_1k_2$, are identical for $\theta = \theta^*$, then they are majorized by the eigenvalues of every competing design in D that is not an *ECD(θ^*)*. Therefore, *ECD(θ^*)*s are Schur-optimal. □

Theorem 2.3.1 generalizes corollary 3.4 of Bailey, Monod, and Morgan (1995) when $s = 2$, which established that affine-resolvable designs are Schur-optimal. We require only that the first blocks of each replicate have the same block concurrence, and we allow for unequal block sizes. When $k_1 = k_2$ and $2 \mid k_1$, our designs are affine-resolvable designs.

Example Consider the setting $D(9,4;6,3)$. Since $(k_1+k_2) \mid k_1^2$, then $\theta^* = 4$, and if an $ECD(4)$ exists it is Schur-optimal. In fact, an $ECD(4)$ does exist and is shown in table 2.19.

Table 2.19: A Schur-optimal *ECD*(4) in $D(9,4;6,3)$

1	7	1	5	1	3	3	1
2	8	2	6	2	4	4	2
3	9	3	9	5	9	5	9
4		4		6		6	
5		7		7		7	
6		8		8		8	

Is some *ECD* Schur-optimal when $(k_1 + k_2) \nmid k_1^2$? If not, what are the optimal classes of designs for the various optimality criteria? Our subsequent discussion will first focus on optimal *ECD*s when $(k_1 + k_2) \nmid k_1^2$, and then will be extended to include designs that are not *ECD*s. We will leave the existence question for later.

Define the ***block concurrence parameter***

$$\bar{\theta} = \text{int}\left(\frac{k_1^2}{k_1 + k_2}\right). \tag{2.63}$$

Note that

$$0 \leq \frac{k_1^2}{k_1 + k_2} - \bar{\theta} < 1,$$

or, with $v = k_1 + k_2$,

$$0 \leq \gamma < v$$

where γ, the ***block irregularity,*** is defined by

$$\gamma = k_1^2 - v\bar{\theta}.$$

The irregularity is zero if and only if $(k_1 + k_2) \mid k_1^2$.

Relative to $\bar{\theta}$, designs in $D(v, r; k_1, k_2)$ fall into four categories:

1. *ECD(θ)*s having $\theta = \bar{\theta}$, or *ECD($\bar{\theta}$)*s.

2. *ECD(θ)*s having $\theta = \bar{\theta} + 1$, or *ECD($\bar{\theta} + 1$)*s.

3. Designs having $\phi_{ii'} \in \{\bar{\theta},\ \bar{\theta}+1\}$ for all $1 \leq i \neq i' \leq r$, with at least one $\phi_{ii'} = \bar{\theta}$ and at least one $\phi_{jj'} = \bar{\theta} + 1$, $1 \leq j \neq j' \leq r$, and $i \neq j$ or $i' \neq j'$.

4. Designs having $\phi_{ii'} < \bar{\theta}$ or $\phi_{ii'} > \bar{\theta}+1$ for at least one $1 \leq i \neq i' \leq r$.

Designs falling into the first category are *ECD(θ^*)*s when $(k_1+k_2) \mid k_1^2$. Designs falling into the third or fourth categories will be referred to as *nearly equal concurrence designs* or *NECDs*, and *unequal concurrence designs*, or *UECDs*, respectively. We will first investigate *ECD($\bar{\theta}$)*s and *ECD($\bar{\theta}+1$)*s.

Define the block concurrence *discrepancy matrix* $\Delta_d = (\delta_{dii'})$, where

$$\delta_{dii'} = \begin{cases} \phi_{ii'} - \bar{\theta} & \text{if } i \neq i' \\ 0 & \text{if } i = i'. \end{cases}$$

For each $1 \leq i \neq i' \leq r$, the off-diagonal elements of Δ_d, $\delta_{dii'}$, will be referred to as block concurrence *discrepancies*. The block concurrence discrepancies and the block concurrence discrepancy matrix are denoted using the same notation as pairwise concurrence discrepancies and the discrepancy matrix in Chapter 1. They both measure the total departure from symmetry of a design, but they are not the same. In Chapter 1, symmetry implies treatment concurrence balance; however, for the remainder of our discussion, symmetry will refer to block concurrence balance.

Define the symbol $p = k_1k_2$ for the product of the block sizes. The optimality matrix (2.57) can now be written

$$M_d = pI - \gamma(J - I) + v\Delta_d. \tag{2.64}$$

Note that for *ECD($\bar{\theta}$)*s, since $\phi_{ii'} = \bar{\theta}$ for each $1 \leq i \neq i' \leq r$ and $\Delta_d = 0$,

$$M_d = (p+\gamma)I - \gamma J$$

and the eigenvalues of M_d are $r-1$ copies of

$$\xi_1(\gamma) = p + \gamma$$

and one copy of

$$\xi_2(\gamma) = p - (r-1)\gamma,$$

and $\xi_1(\gamma) \geq \xi_2(\gamma)$. For *ECD*($\bar{\theta}+1$)s, $\phi_{ii'} = \bar{\theta}+1$ and $\Delta_d = (J-I)$,

$$M_d = [p-(v-\gamma)]I + (v-\gamma)J$$

and the eigenvalues of M_d are $r-1$ copies of

$$\xi_1(\gamma - v) = p - (v-\gamma)$$

and one copy of

$$\xi_2(\gamma - v) = p + (r-1)(v-\gamma),$$

and $\xi_2(\gamma - v) \geq \xi_1(\gamma - v)$.

The following theorem due to Cheng (1978, Theorem 2.3) will be used to establish ϕ_f-optimality for certain classes of *ECD*($\bar{\theta}$)s.

Theorem 2.3.2 *If there exists a design $\bar{d} \in M(v,b,k)$ such that*

(i) $C_{\bar{d}}$ has two distinct eigenvalues $z_{\bar{d}1} = z_{\bar{d}2} = \ldots = z_{\bar{d},v-2} \leq z_{\bar{d},v-1}$.

(ii) $\bar{d}$ minimizes $\sum_{i=1}^{v-1} z_{\bar{d}i}$ over M_d,

then $\bar{d}$ is ϕ_f-optimal for all type-1 criteria with $\lim_{x\to 0^+} f(x) = \infty$.

Theorem 2.3.3 *When $0 \leq \gamma \leq \frac{v}{2}$, ECD($\bar{\theta}$)s minimize tr C_d^2, uniquely so if $\gamma < \frac{v}{2}$. Consequently, ECD($\bar{\theta}$)s are ϕ_f-optimal in $D(v,r;k_1,k_2)$ for all type-1 criteria with* $\lim_{x\to 0^+} f(x) = \infty$.

Proof Let M_d be the optimality matrix for $d \in D(v,r;k_1,k_2)$, and recall that tr $M_d = pr$ and $(M_d)_{ii'} = (v\delta_{dii'} - \gamma)$. If $e_1 \geq e_2 \geq \cdots \geq e_r > 0$ are the eigenvalues of M_d then

$$\begin{aligned} \text{tr } C_d^2 &= (v-r-1)r^2 + \sum_{i=1}^{r}\left(r - \frac{e_i}{p}\right)^2 \\ &= (v-1)r^2 - \frac{2r}{p}\text{tr } M_d + \frac{1}{p^2}\text{tr } M_d^2 \\ &= (v-3)r^2 + r + \frac{2}{p^2}\sum\sum_{i<i'}(v\delta_{dii'} - \gamma)^2, \end{aligned}$$

so that tr C_d^2 is minimized by designs that minimize $\sum\sum_{i<i'}(v\delta_{dii'}-\gamma)^2$. Since $\delta_{dii'}$ is integral, the unique minimum of tr C_d^2 on $0 \le \gamma < \frac{v}{2}$ is at $\delta_{dii'} \equiv 0$. For $\gamma = \frac{v}{2}$, any values $\delta_{dii'} \in \{0,1\}$ minimize tr C_d^2.

The eigenvalues of the information matrix for a design in $D(v,r;k_1,k_2)$ are $0 < z_{d1} \le z_{d2} \le \cdots \le z_{dr}$ and $v-r-1$ copies of r, and $\sum_{i=1}^r z_{di} = r(r-1)$ is constant for all designs in D. For *ECD($\bar{\theta}$)*s, $z_{di} = r - \frac{\xi_1(\gamma)}{p}$, $1 \le i \le r-1$ and $z_{dr} = r - \frac{\xi_2(\gamma)}{p}$, and when $0 \le \gamma \le \frac{v}{2}$ they minimize $\sum_{i=1}^r z_{di}^2$. Thus these eigenvalues satisfy the conditions of Theorem 2.3.2. □

Corollary 2.3.4 *When $\gamma = \frac{v}{2}$, if the eigenvalues of the information matrix for a NECD are identical to the eigenvalues of an ECD($\bar{\theta}$), then the NECD is ϕ_f-optimal in D and ϕ_f-equivalent to ECD($\bar{\theta}$)s for all type-1 criteria with* $\lim_{x\to 0^+} f(x) = \infty$.

In the remainder of this document we will take the phrase "type-1 optimal" to mean ϕ_f-optimal for all type-1 criteria f with $\lim_{x\to 0^+} f(x) = \infty$.

Now define the F-criterion as the value of the largest eigenvalue of C_d that is not constrained by the setting to equal r, that is,

$$\phi_F(C_d) = z_{dr}.$$

Although not a member of the type-1 family, this criterion can be important in establishing Schur-optimality. Since $z_{dr} = r - \frac{e_r}{p}$, minimizing $\phi_F(C_d)$ over D is equivalent to maximizing e_r over $\mathcal{M}$. Here is another easily established fact about *ECD($\bar{\theta}$)*s.

Theorem 2.3.5 *An ECD($\bar{\theta}$) is Schur-better than a competitor with a different set of eigenvalues if and only if it is F-equivalent or better than that competitor. Consequently, ECD($\bar{\theta}$)s are Schur-optimal if and only if they are F-optimal.*

Proof Let $d \in D(v,b;k_1,k_2)$ be an *ECD($\bar{\theta}$)*. Then the eigenvalues of the optimality matrix for d are $r-1$ copies of $\xi_1(\gamma)$ and one copy of $\xi_2(\gamma)$, and $\xi_1(\gamma) \ge \xi_2(\gamma)$.

Suppose the optimality matrix for a competing design $\bar{d} \in D$ that is not an *ECD*($\bar{\theta}$) has eigenvalues $e_1 \geq e_2 \geq \cdots \geq e_r$. Now, the *ECD*($\bar{\theta}$) is F-equivalent or better than $\bar{d}$ if and only if $e_r \leq \xi_2(\gamma)$, which is a necessary and sufficient condition for the eigenvalues of the information matrix for $\bar{d}$ to majorize the eigenvalues of the information matrix for the *ECD*($\bar{\theta}$) . □

A result of similar flavor holds for *ECD*($\bar{\theta}+1$)s using the E-criterion. As pointed out by Kunert (1985, page 385), facts 1-3 of section 2.2 says that *ECD*($\bar{\theta}+1$)s are Schur-best whenever they are E-optimal. We state this as:

Theorem 2.3.6 *An ECD($\bar{\theta}+1$) is Schur-better than a competitor with a different set of eigenvalues if and only if it is E-equivalent or better than that competitor. Consequently, ECD($\bar{\theta}+1$)s are Schur-optimal if and only if they are E-optimal.*

Proof Let $d^* \in D(v, b; k_1, k_2)$ be an *ECD*($\bar{\theta}+1$). Then the eigenvalues of the optimality matrix for d^* are $r-1$ copies of $\xi_1(v-\gamma)$ and one copy of $\xi_2(v-\gamma)$, and $\xi_2(v-\gamma) \geq \xi_1(v-\gamma)$. Suppose the optimality matrix for a competing design $d \in D$ that is not an *ECD*($\bar{\theta}+1$) has eigenvalues $e_1 \geq e_2 \geq \cdots \geq e_r$. Now, the *ECD*($\bar{\theta}+1$) is E-equivalent or better than d if and only if $e_1 \geq \xi_2(v-\gamma)$, which is a necessary and sufficient condition for the eigenvalues of the information matrix for d to majorize the eigenvalues of the information matrix for the *ECD*($\bar{\theta}+1$) . □

Corollary 2.3.7 *ECD($\bar{\theta}$)s are Schur-better than ECD($\bar{\theta}+1$)s if and only if*

$$\gamma \leq \frac{1}{r}v,$$

and ECD($\bar{\theta}+1$)s are Schur-better than ECD($\bar{\theta}$)s if and only if

$$\gamma \geq \frac{r-1}{r}v.$$

Proof Note that the eigenvalues of *ECD*($\bar{\theta}$)s and *ECD*($\bar{\theta}+1$)s are never identical. *ECD*($\bar{\theta}$)s are F-equivalent or better than *ECD*($\bar{\theta}+1$)s if and only if $\xi_1(\gamma-v) \leq \xi_2(\gamma)$

which is equivalent to $\gamma \geq \frac{1}{r}v$. *ECD*($\bar{\theta}+1$)s are E-equivalent or better than *ECD*($\bar{\theta}$)s if and only if $\xi_1(\gamma) \geq \xi_2(v-\gamma)$ which is equivalent to $\gamma \leq \frac{r-1}{r}v$. □

Corollary 2.3.8 *ECD($\bar{\theta}$)s are E-better than ECD($\bar{\theta}+1$)s if and only if*

$$\gamma < \frac{r-1}{r}v$$

and ECD($\bar{\theta}$)s and ECD($\bar{\theta}+1$)s are E-equivalent when

$$\gamma = \frac{r-1}{r}v.$$

Lemma 2.3.9 *Suppose a design $d \in D(v,r;k_1,k_2)$ has optimality matrix M_d and concurrence discrepancy matrix $\Delta_d = (\delta_{dii'})$, and suppose the maximum eigenvalue of M_d is e_1 and the minimum eigenvalue of M_d is e_r. If $\delta_{d12} \leq 0$ then*

$$e_1 \geq p+\gamma-v\delta_{d12} \quad \text{and} \quad e_r \leq p-\gamma+v\delta_{d12}.$$

If $\delta_{d12} \geq 0$ then

$$e_1 \geq p-\gamma+v\delta_{d12} \quad \text{and} \quad e_r \leq p+\gamma-v\delta_{d12}.$$

Proof The leading 2×2 minor of M_d, which is $M_{d11} = (p+\gamma-v\delta_{d12})I-(\gamma-v\delta_{d12})J$, has eigenvalues

$$p+\gamma-v\delta_{d12} \quad \text{and} \quad p-\gamma+v\delta_{d12}.$$

A Sturmian Separation Theorem (Rao, 1973, page 64) provides the bounds. □

Corollary 2.3.10 *Suppose $d \in D(v,b;k_1,k_2)$ is a UECD with $\delta_{dii'} \leq -\alpha$ for at least one $1 \leq i \neq i' \leq r$, and for some integer $\alpha \geq 1$. ECD($\bar{\theta}$)s are Schur-better than d if*

$$\gamma \leq \frac{\alpha}{r-2}v,$$

and ECD($\bar{\theta}+1$)s are Schur better than d if

$$\gamma \geq \frac{r-\alpha-1}{r}v.$$

Proof Let $d \in D$ be a *UECD* as described in the lemma and let e_1 and e_r be the maximum and minimum eigenvalues, respectively, of the optimality matrix for d. For a proper labeling of the design replications, $\delta_{d12} \leq -\alpha$. Then from lemma 2.3.9, $e_1 \geq p+\gamma-\alpha v$, and $p-\gamma+\alpha v \geq e_r$. By Theorem 2.3.5, an *ECD($\bar{\theta}$)* is Schur-better than d if $\xi_2(\gamma) \geq p-\gamma+v\alpha \geq e_r$, or

$$\gamma \leq \frac{\alpha}{r-2}v.$$

By Theorem 2.3.6, an *ECD($\bar{\theta}$+1)* is Schur-better than d if $e_1 \geq p+\gamma-v\alpha \geq \xi_2(\gamma-v)$, or

$$\gamma \geq \frac{r-\alpha-1}{r}v.$$

□

Corollary 2.3.11 *When $r \leq 4$, all UECDs with $\delta_{dii'} \leq -1$ for some $1 \leq i \neq i' \leq r$ are Schur-inferior to an ECD, and when $r = 5$ or 6, UECDs with $\delta_{dii'} \leq -2$ for some $1 \leq i \neq i' \leq r$ are Schur-inferior to an ECD .*

Corollary 2.3.12 *Suppose $d \in D(v, b; k_1, k_2)$ is a UECD with $\delta_{dii'} \geq \alpha$ for at least one $1 \leq i \neq i' \leq r$, and for some integer $\alpha \geq 2$. ECD($\bar{\theta}$)s are Schur-better than d if*

$$\gamma \leq \frac{\alpha}{r}v,$$

and ECD($\bar{\theta}$ + 1)s are Schur better than d if

$$\gamma \geq \frac{r-\alpha-1}{r-2}v.$$

Proof Let $d \in D$ be a *UECD* as described in the lemma, and let e_1 and e_r be the maximum and minimum eigenvalue, respectively, of the optimality matrix for d. For a proper labeling of the design replications, $\delta_{d12} \geq \alpha \geq 2$. Then from lemma 2.3.9, $e_1 \geq p-\gamma+\alpha v$, and $p+\gamma-\alpha v \geq e_r$. By Theorem 2.3.5, an *ECD($\bar{\theta}$)* is Schur-better than d if $\xi_2(\gamma) \geq p+\gamma-\alpha v \geq e_r$ or

$$\gamma \leq \frac{\alpha}{r}v.$$

By Theorem 2.3.6, an *ECD*($\bar{\theta}+1$) is Schur-better than d if $e_1 \geq p-\gamma+\alpha v \geq \xi_2(\gamma-v)$ or

$$\gamma \geq \frac{r-\alpha-1}{r-2}v.$$

□

Corollary 2.3.13 *When $r \leq 4$, all UECDs with $\delta_{dii'} \geq 2$ for some $1 \leq i \neq i' \leq r$ are Schur-inferior to an ECD, and when $r = 5$ or 6, UECDs with $\delta_{dii'} \geq 3$ for some $1 \leq i \neq i' \leq r$ are Schur-inferior to an ECD .*

Corollaries 2.3.11 and 2.3.13 say that optimal designs in settings $D(v,r;k_1,k_2)$ with $r \leq 4$ must be an *ECD*($\bar{\theta}$) , an *ECD*($\bar{\theta}+1$) , or an *NECD* , and optimal designs in settings with $r = 5$ or 6 must have block concurrence discrepancies $\delta_{dii'} \in \{-1,0,1,2\}$ for all $1 \leq i \neq i' \leq r$. Now we will show that *UECD*s are always E-inferior to an *ECD*($\bar{\theta}$), and *ECD*($\bar{\theta}$)s are E-optimal when $0 \leq \gamma \leq \frac{v}{2}$.

Corollary 2.3.14 *For all $r \geq 2$ and $0 \leq \gamma < v$, ECD($\bar{\theta}$)s are E-better than UECDs.*

Proof Suppose $d \in D(v,r;k_1,k_2)$ is an *UECD*, and $\delta_{dii'} \leq -\alpha$ for some $1 \leq i \neq i' \leq r$ and integer $\alpha \geq 1$. Than, for a proper labeling of the design replications, $\delta_{d12'} \leq -\alpha$, and $e_1 \geq p+\gamma-v\delta_{d12} > \xi_1(\gamma)$, and *ECD*($\bar{\theta}$)s are E-better than d. Now suppose $\delta_{dii'} \geq \alpha$ for some $1 \leq i \neq i' \leq r$ and integer $\alpha \geq 2$. Then, for a proper labeling of the design replications, $\delta_{d12} \geq \alpha$ and $e_1 \geq p-\gamma+v\delta_{d12} > \xi_1(\gamma)$ and *ECD*($\bar{\theta}$)s are E-better than d. □

Corollary 2.3.15 *When $0 \leq \gamma \leq \frac{v}{2}$, ECD($\bar{\theta}$)s are E-optimal, uniquely so when $\gamma \neq \frac{v}{2}$.*

Proof By corollary 2.3.8, *ECD*($\bar{\theta}$)s are E-equivalent or better than *ECD*($\bar{\theta}+1$)s when $\gamma \leq \frac{v}{2}$, E-equivalent only when $r = 2$ and $\gamma = \frac{v}{2}$. *ECD*($\bar{\theta}$)s are always E-better than *UECD*s by corollary 2.3.14. The maximum eigenvalue of the optimality matrix for *ECD($\bar{\theta}$)s* in any resolvable design setting $D(v,r;k_1,k_2)$ is $\xi_1(\gamma) = p+\gamma$, and with a proper labeling of the replications, the optimality matrix of a *NECD* has $\delta_{d12} = 1$. Then, from 2.3.9, $z_1 \geq p + (v-\gamma)$. Since $p+(v-\gamma) > \xi_1(\gamma)$ when $0 < \gamma < \frac{v}{2}$, and $p+(v-\gamma) = \xi_1(\gamma)$ when $\gamma = \frac{v}{2}$, the result follows. □

The next lemma provides bounds for the maximum and minimum eigenvalues of the optimality matrix in terms of the eigenvalues derived from the block concurrence discrepancy matrix for the design.

Lemma 2.3.16 *Suppose e_1 and e_r are the maximum and minimum eigenvalues, respectively, of the optimality matrix M_d for $d \in D(v,r;k_1,k_2)$. If u_1 and u_r are the maximum and minimum eigenvalues of $\Delta_{d0} = P^T\Delta_d P$, where $P = (I - \frac{1}{r}J)$ and Δ_d is the block concurrence discrepancy matrix, then*

$$e_1 \geq p + \gamma + vu_1$$

provided $u_1 > 0$, and

$$e_r \leq p + \gamma + vu_r.$$

Proof

$$\begin{aligned} e_1 &= \max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T M_d \mathbf{x} \qquad (2.65)\\ &= \max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T[(p+\gamma)I - \gamma J + v\Delta_d]\mathbf{x} \\ &\geq \max_{\substack{\mathbf{x}^T\mathbf{x}=1\\ \mathbf{x}^T\mathbf{1}=0}} \mathbf{x}^T[(p+\gamma)I - \gamma J + v\Delta_d]\mathbf{x} \\ &= p+\gamma+v\max_{\substack{\mathbf{x}^T\mathbf{x}=1\\ \mathbf{x}^T\mathbf{1}=0}} \mathbf{x}^T\Delta_d\mathbf{x} \\ &= p+\gamma+v\max_{\substack{\mathbf{x}^T\mathbf{x}=1\\ \mathbf{x}^T\mathbf{1}=0}} \mathbf{x}^T P^T\Delta_d P\mathbf{x} \\ &= p+\gamma+v\max_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T P^T\Delta_d P\mathbf{x} \qquad (2.66)\\ &= p+\gamma+vu_1. \end{aligned}$$

Equality (2.66) holds since $u_1 > 0$, $\mathbf{1}^T P^T \Delta_d P \mathbf{1} = 0$, and $P^T \Delta_d P \mathbf{1} = 0\mathbf{1}$ (that is, $\mathbf{1}$ is an eigenvector of $P^T \Delta_d P$ with eigenvalue 0). Likewise we find

$$\begin{aligned} e_r &= \min_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T M_d \mathbf{x} \\ &= \min_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T[(p+\gamma)I - \gamma J + v\Delta_d]\mathbf{x} \\ &\leq p + \gamma + v \min_{\substack{\mathbf{x}^T\mathbf{x}=1 \\ \mathbf{x}^T\mathbf{1}=0}} \mathbf{x}^T P^T \Delta_d P \mathbf{x} \\ &= p + \gamma + v u_r \end{aligned} \tag{2.67}$$

Equality (2.67) is true provided $u_r < 0$, for similar reasons to above. If $u_r > 0$, the bound still holds, since

$$e_r \leq \frac{\mathrm{tr}(M_d)}{r} = p \leq p + \gamma + v u_r. \ \square$$

We end this section with a corollary that provides conditions for when a design $d \in D$ is Schur-inferior to an *ECD* and for when d is E-inferior to an *ECD*($\bar{\theta}$) .

Corollary 2.3.17 *Let $d \in D(v, r; k_1, k_2)$ be a resolvable design with optimality matrix M_d, whose eigenvalues are not identical to those of an ECD($\bar{\theta}$) or an ECD($\bar{\theta}+1$). Let u_1 and u_r be the maximum and minimum eigenvalues, respectively, of $\Delta_{d0} = P^T \Delta_d P$, $P = (I - \frac{1}{r}J)$. If*

$$\gamma < -\frac{u_r}{r} v \tag{2.68}$$

then ECD($\bar{\theta}$)s are Schur-better than d. If $u_1 > 0$ and

$$\gamma > \left(\frac{r - u_1 - 1}{r}\right) v \tag{2.69}$$

then ECD($\bar{\theta}+1$)s are Schur-better than d. Furthermore, if

$$u_1 > 0 \tag{2.70}$$

then ECD($\bar{\theta}$)s are E-better, but not necessarily Schur-better, than d.

Proof The result follows immediately from Theorems 2.3.5 and 2.3.6 and lemma 2.3.16. $\square$

2.4 Special Cases: $(k_1 - k_2) \leq 2$

In this section we will investigate the three important special cases of k_1 and k_2 being equal or nearly so: that when $k_2 = k_1$, that when $k_2 = k_1 - 1$, and that when $k_2 = k_1 - 2$. For each case, results that follow immediately from the theory earlier in this chapter are reported. If we write $k_2 = k_1 - n$, then $(k_1 - k_2) \leq 2$ says that $n = 0$, 1, or 2, and for any n

$$\frac{k_1^2}{k_1 + k_2} = \frac{k_1^2}{2k_1 - n} = \frac{k_1}{2} + \frac{n}{4} + \frac{n^2}{4(2k_1 - n)}. \tag{2.71}$$

Recall that $\bar{\theta}$ is the integer part of (2.71), and $\gamma = k_1^2 - v\bar{\theta}$.

Lemma 2.4.1 *When $k_1 = k_2$, if $2 \mid k_1$ then $\gamma = 0$, and if $2 \not| \, k_1$ then $\gamma = \frac{v}{2}$.*

Proof When $k_1 = k_2$, $n = 0$ and (2.71) becomes $\frac{k_1^2}{k_1+k_2} = \frac{k_1}{2}$, and the result clearly follows. □

Corollary 2.4.2 *Let $k_1 = k_2$.*

(i) If $2 \mid k_1$ then $(k_1 + k_2) \mid k_1^2$ and ECD(θ^*)s are Schur-optimal.

(ii) If $2 \not| \, k_1$ then ECD($\bar{\theta}$)s are E- and type-1 optimal.

When $k_1 = k_2$ and $v = 2 \mid k_1$, the resulting design is an affine-resolvable design since every pair of blocks from different replicates have block concurrence $\theta^* = \frac{k_1}{2}$. Bailey, Monod, and Morgan (corollary 3.4, 1995) proved that affine-resolvable designs are Schur-optimal. For $2 \not| \, k_1$, the result is from Theorem 2.3.3 and corollary 2.3.15. The optimality need not be uniquely so.

Lemma 2.4.3 *When $k_1 - k_2 = 1$, if $2 \mid k_1$ then $\gamma = \frac{k_1}{2}$, and if $2 \not| \, k_1$ then $\gamma = \frac{3k_1 - 1}{2}$.*

Proof When $k_1 - k_2 = 1$ then $n = 1$ and the last term on the right hand side of (2.71) becomes $\frac{n^2}{4(2k_1 - n)} = \frac{1}{4(2k_1 - 1)}$. Then

$$\bar{\theta} = \begin{cases} \frac{k_1}{2} & \text{if } 2 \mid k_1 \\ \frac{k_1 - 1}{2} & \text{if } 2 \not| \, k_1 \end{cases},$$

and, since $v = 2k_1 - 1$, $\frac{v}{4(2k_1-1)} = \frac{1}{4}$, and

$$\gamma = \begin{cases} \frac{k_1}{2} & \text{if } 2 \mid k_1 \\ \frac{3k_1-1}{2} & \text{if } 2 \nmid k_1 \end{cases}.$$

□

Corollary 2.4.4 *Let $k_1 - k_2 = 1$.*

(i) If $2 \mid k_1$, then $\frac{v}{4} < \gamma < \frac{v}{3}$, and ECD($\bar{\theta}$)s are E- and type-1 optimal.

(ii) If $2 \nmid k_1$, then $\frac{3v}{4} < \gamma \leq \frac{4v}{5}$.

Lemma 2.4.5 *When $k_1 - k_2 = 2$, if $2 \mid k_1$ then $\gamma = k_1$, and if $2 \nmid k_1$ then $\gamma = 1$.*

Proof When $k_1 - k_2 = 2$ then $n = 2$ and the last term on the right hand side of (2.71) becomes $\frac{n^2}{4(2k_1-n)} = \frac{1}{2(k_1-1)}$. Then

$$\bar{\theta} = \begin{cases} \frac{k_1}{2} & \text{if } 2 \mid k_1 \\ \frac{k_1+1}{2} & \text{if } 2 \nmid k_1 \end{cases},$$

and, since $v = 2(k_1 - 1)$, $\frac{v}{2(2k_1-1)} = 1$, and

$$\gamma = \begin{cases} k_1 & \text{if } 2 \mid k_1 \\ 1 & \text{if } 2 \nmid k_1 \end{cases}.$$

□

Corollary 2.4.6 *Let $k_1 - k_2 = 2$. Then $k_1 \geq k_2 \geq 2$ implies $k_1 \geq 4$, and*

(i) If $k_1 = 4$, then $\gamma = \frac{2v}{3}$.

(ii) If $k_1 = 6$, then $\gamma = \frac{3v}{5}$.

(iii) If $2 \mid k_1$ and $k_1 \geq 8$, then $\frac{v}{2} < \gamma < \frac{3v}{5}$.

(iv) If $2 \nmid k_1$, then $0 < \gamma \leq \frac{v}{6}$, and ECD($\bar{\theta}$)s are E- and type-1 optimal.

CHAPTER III

APPLICATION: OPTIMAL RESOLVABLE DESIGNS WITH UP TO FIVE REPLICATES AND TWO BLOCKS PER REPLICATE

3.1 Introduction

Optimality in resolvable designs settings $D(v, r; k_1, k_2)$ for $2 \leq r \leq 5$ will be investigated in this chapter. As stated in Chapter II, the primary goal is to determine A- and E-optimal designs, though often we can do much more. If the E-optimal design is not unique, the Schur-best of the E-optimal designs, or the *(E,S)-optimal design* will be identified.

Definition 3.1.1 A design d in a class of designs D is said to be *(E,S)-optimal* if

(i) d is E-optimal, and

(ii) among all E-optimal designs in D, d is Schur-optimal.

We review some important facts from section 2.3 concerning Schur- and type-1 optimality in $D(v, r; k_1, k_2)$ before commencing our eigenvalue optimality discussion.

1. When $(k_1 + k_2) \mid k_1^2$, *ECD*(θ^*)s with

$$\theta^* = \frac{k_1^2}{k_1 + k_2}$$

are Schur-optimal whenever they exist.

2. When $0 \leq \gamma \leq \frac{v}{2}$, *ECD($\bar{\theta}$)*s with

$$\bar{\theta} = \text{int}\left(\frac{k_1^2}{k_1 + k_2}\right)$$

are type-1 and E-optimal, uniquely so when $\gamma < \frac{v}{2}$, whenever they exist.

3. When $r \leq 4$, *UECD*s are Schur-inferior to an *ECD($\bar{\theta}$)* or an *ECD($\bar{\theta}$ + 1)* whenever the *ECD*s exist.

4. When $r = 5$, *UECD*s having at least one $\delta_{dii'} \leq -2$ or at least one $\delta_{dii'} \geq 3$ are Schur-inferior to an *ECD($\bar{\theta}$)* or an *ECD($\bar{\theta}$ + 1)* whenever the *ECD*s exist.

Therefore, in the sequal we will restrict our attention to *ECD*s and *NECD*s, when $r \leq 4$, or *ECD*s, *NECD*s, and *UECD*s having $-1 \leq \delta_{dii'} \leq 2$, when $r = 5$. From fact 2 it follows immediately that

Corollary 3.1.1 *When* $0 \leq \gamma \leq \frac{v}{2}$, *ECD($\bar{\theta}$)s are (E,S)-optimal, uniquely so when* $\gamma < \frac{v}{2}$.

By lemma 2.3.15, when $\gamma = \frac{v}{2}$, *ECD($\bar{\theta}$)*s are Schur-optimal but may not be uniquely so. Therefore, *ECD($\bar{\theta}$)*s are not uniquely (E,S)-optimal when $\gamma = \frac{v}{2}$ only when a competing design that is not an *ECD($\bar{\theta}$)* has identical eigenvalues to the *ECD($\bar{\theta}$)*.

The eigenvalues of the optimality matrix M_d of designs in resolvable design settings $D(v, b; k_1, k_2)$ can be directly used to determine the Schur-, E-, and (E,S)-optimal designs. Establishing A-optimality requires working with the eigenvalues of the information matrix C_d of the the designs; however, we can still restrict our efforts to working with the eigenvalues of M_d in A-optimality investigations, as shown next.

Recall that if $z_1, z_2, \ldots, z_{v-1}$ are the nonzero eigenvalues of the information matrix C_d for a resolvable design $d \in D(v, r; k_1, k_2)$, then the A-value for the design is

$$\sum_{i=1}^{v-1} z_i^{-1}, \tag{3.72}$$

and the A-optimal design minimizes (3.72). Furthermore, if $e_1 \geq e_1 \geq \cdots \geq e_r$ are the eigenvalues of the the optimality matrix M_d of a design $d \in D$, then the eigenvalue of the information matrix C_d corresponding to each e_i, $1 \leq i \leq r$, is $z_i = r - \frac{e_i}{p}$. Moreover, the eigenvalues of the information matrices for resolvable designs in D are 0, $v - r - 1$ copies of r, and $r - \frac{e_1}{p} \leq r - \frac{e_2}{p} \leq \cdots \leq r - \frac{e_r}{p}$. Thus, the class of designs that minimizes

$$\sum_{i=1}^{r} \left(r - \frac{e_i}{p}\right)^{-1} \tag{3.73}$$

will also minimize (3.72), and, therefore, will be A-optimal.

The following three facts concerning bounds on $\frac{p}{v}$, and a lemma relating intervals of γ to ranges of values of k_2 for fixed values of k_1 and $\bar{\theta}$, will be needed to establish results on A-optimality.

Fact 3.1.2 *If $k_1 \geq k_2 \geq 2$, then*

$$\frac{k_1 k_2}{k_1 + k_2} \geq 1$$

Fact 3.1.3 *If*

(i) $k_1 \geq k_2 \geq 4$, *or*

(ii) $k_2 = 3$ *and* $k_1 \geq 6$ *then*

$$\frac{k_1 k_2}{k_1 + k_2} \geq 2.$$

Fact 3.1.4 *If*

(i) $k_1 \geq k_2 \geq 5$,

(ii) $k_2 = 4$ *and* $k_1 \geq 7$, *or*

(iii) $k_2 = 3$ *and* $k_1 \geq 15$

then

$$\frac{k_1 k_2}{k_1 + k_2} \geq \frac{5}{2}.$$

Lemma 3.1.5 *Suppose* $k_1 \geq k_2 = n \geq 2$ *for a given integer* n, *and let* $x = int\left(\frac{n^2}{k_1+n}\right)$. *For any real numbers* $0 \leq \alpha \leq \beta \leq 1$,

$$\alpha v \leq \gamma \leq \beta v$$

if and only if

$$\frac{n^2 - n(\beta + x)}{\beta + x} \leq k_1 \leq \frac{n^2 - n(\alpha + x)}{\alpha + x}.$$

Proof If $k_1 \geq k_2 = n$, then

$$\frac{k_1^2}{k_1 + k_2} = \frac{k_1^2}{k_1 + n} = k_1 - n + \frac{n^2}{k_1 + n}.$$

If we define

$$x = \mathrm{int}\left(\frac{n^2}{k_1 + n}\right)$$

then $\bar{\theta} = k_1 - n + x$ and $\gamma = n^2 - x(k_1 + n)$. Now, for any real numbers $0 \leq \alpha \leq \beta \leq 1$, $\gamma \geq \alpha v$ if and only if

$$k_1 \leq \frac{n^2 - n(\alpha + x)}{\alpha + x},$$

and $\gamma \leq \beta v$ if and only if

$$k_1 \geq \frac{n^2 - n(\beta + x)}{\beta + x}.$$

□

The following bounds will be useful to the constructions.

Lemma 3.1.6 *Let* k_1 *and* k_2 *be two integers satisfying* $3 \leq k_1$ *and* $2 \leq k_2 \leq k_1$, *and let* $\bar{\theta}$ *be as defined by (2.63). Then,*

$$\bar{\theta} + 2 \leq k_1 \leq \begin{cases} 2\bar{\theta} + 1 & \text{if } k_1 \text{ odd} \\ 2\bar{\theta} & \text{if } k_1 \text{ even.} \end{cases} \tag{3.74}$$

Proof For a resolvable block design setting $D(v, r; k_1, k_2)$, write

$$\frac{k_1^2}{k_1 + k_2} = k_1 - k_2 + \frac{k_2^2}{k_1 + k_2}. \tag{3.75}$$

For a fixed value of $k_1 \geq 3$, since (3.75) is a decreasing function of k_2, $2 \leq k_2 \leq k_1$, then

$$\frac{k_1}{2} \leq \frac{k_1^2}{k_1 + k_2} \leq k_1 - 2 + \frac{4}{k_1 + 2}. \tag{3.76}$$

Since $\frac{4}{k_1+2} < 1$ for all $k_1 \geq 3$ then, by taking the integer part of each term in (3.76), we have

$$\left.\begin{array}{ll} \frac{k_1 - 1}{2} & \text{if } k_1 \text{ odd} \\ \frac{k_1}{2} & \text{if } k_1 \text{ even} \end{array}\right\} \leq \bar{\theta} \leq k_1 - 2. \tag{3.77}$$

Rewriting (3.77) in terms of k_1 yields (3.74).

Corollary 3.1.7 *Let $3 \leq k_1$ and $2 \leq k_2 \leq k_1$ be integers, and let $\bar{\theta}$ be given by (2.63). Then, $2k_1 - \bar{\theta} \leq k_1 + k_2$.*

Lemma 3.1.8 *Let k_1 and k_2 be two integers satisfying $3 \leq k_1$ and $2 \leq k_2 \leq k_1$, and let $\bar{\theta}$ be as defined by (2.63). Then the following inequalities hold:*

1. *If $k_1 = 2\bar{\theta}$ then $k_1 - 3 \leq k_2 \leq k_1$.*

2. *if $k_1 = 2\bar{\theta} + 1$ then $k_1 - 1 \leq k_2 \leq k_1$*

Proof

1. Let $k_1 = 2\bar{\theta}$. Then

$$k_1 = 2 \text{ int} \left(\frac{k_1^2}{k_1 + k_2} \right)$$

if and only if

$$\frac{k_1}{2} \leq k_1 - \frac{k_1 k_2}{k_1 + k_2} < \frac{k_1}{2} - 1$$

if and only if

$$k_1 - 4 + \frac{8}{k_1 + 2} = \frac{k_1(k_1 - 2)}{k_1 + 2} < k_2 \leq k_1$$

then

$$k_1 - 3 \leq k_2 \leq k_1.$$

2. Similarly, if $k_1 = 2\bar{\theta} + 1$ then

$$k_1 - 2\,\text{int}\left(\frac{k_1^2}{k_1 + k_2}\right) + 1$$

if and only if

$$k_1 - 2 + \frac{2}{k_1 + 1} < k_2 \leq k_1$$

then

$$k_1 - 2 \leq k_2 \leq k_1. \square$$

3.2 Resolvable Designs With Two Replicates

3.2.1 Schur-optimality

For two replicates M_d has two eigenvalues, as given in section 2.3. It follows from lemma 2.3.9 that the eigenvalues of any design that is not an *ECD* majorize the eigenvalues of at least one of *ECD*($\bar{\theta}$) and *ECD*($\bar{\theta} + 1$). Thus only *ECD*s need to be considered in this section. The *ECD*s are:

ECD($\bar{\theta}$): The optimality matrix for *ECD*($\bar{\theta}$)s is $M_d = pI - \gamma(J - I)$. The eigenvalues of M_d are

$$\begin{aligned} \xi_1(\gamma) &= p + \gamma \\ \xi_2(\gamma) &= p - \gamma, \end{aligned}$$

and they satisfy

$$\xi_1(\gamma) > \xi_2(\gamma).$$

ECD($\bar{\theta} + 1$): The optimality matrix for *ECD*($\bar{\theta}$+1)s is $M_d = pI - \gamma(J - I) + v(J - I)$. The eigenvalues of M_d are

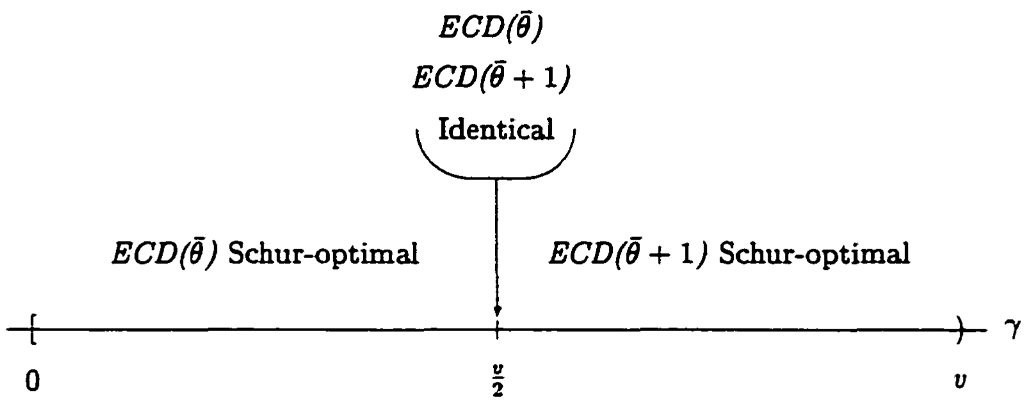
$$\begin{aligned} \xi_1(\gamma - v) &= p - (v - \gamma) \\ \xi_2(\gamma) &= p + (v - \gamma), \end{aligned}$$

and they satisfy

$$\xi_2(\gamma - v) > \xi_1(\gamma - v).$$

Corollary 2.3.7 of Lemmas 2.3.5 and 2.3.6 establish conditions for when *ECD*($\bar{\theta}$)s are Schur-better than *ECD*($\bar{\theta}+1$)s and for when *ECD*($\bar{\theta}+1$)s are Schur-better than *ECD*($\bar{\theta}$)s; see table 3.20.

Table 3.20: Schur-optimal Designs In $D(v, 2; k_1, k_2)$



3.2.2 Special Cases: $(k_1 - k_2) \leq 2$

We now apply the optimality results from section 3.2.1 to the three special cases described in section 2.4.

Corollary 3.2.1 *Suppose $k_1 = k_2$ and $r = 2$. Then*

(i) If $2 \mid k_1$ then $\gamma = 0$, and ECD(θ^*)s exist and are Schur-optimal.

(ii) If $2 \not| k_1$ then $\gamma = \frac{v}{2}$, and ECD($\bar{\theta}$)s and ECD($\bar{\theta}+1$)s are identical and Schur-optimal.

Proof The optimality results follow immeditely from lemma 2.4.1 and the Schur-optimality discussion of section 3.2.1. When $k_1 = k_2$, $r = 2$, and $2 \not| k_1$, the first blocks of the two replicates of any *ECD*($\bar{\theta}$) will have $\bar{\theta} = \frac{k_1-1}{2}$ concurrences. The second blocks of the two replicats of the *ECD*($\bar{\theta}$) will then have $\bar{\theta}+1 = \frac{k_1+1}{2}$ concurrences. By exchanging the two blocks of each replicate, the *ECD*($\bar{\theta}$) becomes an *ECD*($\bar{\theta}+1$) . Therefore, the *ECD*($\bar{\theta}$) and *ECD*($\bar{\theta}+1$) are the same design. □

Corollary 3.2.2 *Suppose* $k_2 = k_1 - 1$ *and* $r = 2$. *Then*

(i) If $2 \mid k_1$ *then* $\frac{v}{4} < \gamma < \frac{v}{3}$, *and ECD*($\bar{\theta}$)*s are Schur-optimal.*

(ii) If $2 \not\mid k_1$ *then* $\frac{3v}{4} < \gamma < \frac{4v}{5}$, *and ECD*($\bar{\theta}+1$)*s are Schur-optimal.*

Corollary 3.2.3 *Suppose* $k_2 = k_1 - 2$ *and* $r = 2$. *Then*

(i) If $2 \mid k_1$ *then* $\frac{v}{2} < \gamma < \frac{2v}{3}$, *and ECD*($\bar{\theta}+1$)*s are Schur-optimal.*

(ii) If $2 \not\mid k_1$ *then* $0 < \gamma < \frac{v}{6}$, *and ECD*($\bar{\theta}$)*s are Schur-optimal.*

3.2.3 Construction of Optimal Designs in $D(v, 2; k_1, k_2)$

In this section constructions for *ECD*s are provided. The common block concurrence θ^*, $\bar{\theta}$, or $\bar{\theta}+1$ is denoted by L so that the constructions are valid for *ECD*(θ^*)s, *ECD*($\bar{\theta}$)s, and *ECD*($\bar{\theta}+1$)s, respectively. Since all v treatments appear once in each replicate, only first-block treatment assignments need be given. The constructions are:

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots L\} \cup \{k_1+1 \ldots 2k_1 - L\}$

3.2.4 Examples

We conclude this section by providing some examples of optimal resolvable designs in $D(v, 2; k_1, k_2)$ when $(k_1 - k_2) \leq 2$. First we construct designs for the two cases when $k_1 = k_2$.

Example Suppose $k_1 = k_2 = 4$. Then, according to corollary 3.2.1 the Schur-optimal design is an *ECD*(θ^*). Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta} = 2$ yields the first block of each replicate. Adding the remaining four treatments to the second block produces a Schur-optimal *ECD*(θ^*) which is:

$$\begin{array}{cc|cc} 1 & 5 & 1 & 3 \\ 2 & 6 & 2 & 4 \\ 3 & 7 & 5 & 7 \\ 4 & 8 & 6 & 8 \end{array}.$$

Example Consider the case where $k_1 = k_2 = 5$. Then, according to corollary 3.2.1 *ECD*($\bar{\theta}$)s and *ECD*($\bar{\theta}+1$)s are identical and Schur-optimal. Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta} = 2$ yields a Schur-optimal *ECD*($\bar{\theta}$) which is:

$$\begin{array}{cc|cc} 1 & 6 & 1 & 3 \\ 2 & 7 & 2 & 4 \\ 3 & 8 & 6 & 5 \\ 4 & 9 & 7 & 9 \\ 5 & 10 & 8 & 10 \end{array}.$$

Now we investigate the two cases when $k_1 - k_2 = 1$.

Example Consider the setting such that $k_1 = 6$ and $k_2 = 5$. By corollary 3.2.2, the Schur-optimal design is an *ECD*($\bar{\theta}$). Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta} = 3$ produces a Schur-optimal *ECD*($\bar{\theta}$) which is:

$$\begin{array}{cc|cc} 1 & 7 & 1 & 4 \\ 2 & 8 & 2 & 5 \\ 3 & 9 & 3 & 6 \\ 4 & 10 & 7 & 10 \\ 5 & 11 & 8 & 11 \\ 6 & & 9 & \end{array}.$$

Example Suppose $k_1 = 5$ and $k_2 = 4$. By corollary 3.2.2, the Schur-optimal design is the *ECD*($\bar{\theta}+1$). Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta}+1 = 3$ produces a Schur-optimal *ECD*($\bar{\theta}+1$) which is:

$$\begin{array}{cc|cc} 1 & 6 & 1 & 4 \\ 2 & 7 & 2 & 5 \\ 3 & 8 & 3 & 8 \\ 4 & 9 & 6 & 9 \\ 5 & & 7 & \end{array}.$$

Now we investigate the two cases when $k_1 - k_2 = 2$.

Example Consider the setting such that $k_1 = 5$ and $k_2 = 3$. By corollary 3.2.3, the Schur-optimal design is the *ECD*($\bar{\theta}$). Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta} = 3$ yields a Schur-optimal resolvable design which is

$$\begin{array}{cc|cc} 1 & 6 & 1 & 4 \\ 2 & 7 & 2 & 5 \\ 3 & 8 & 3 & 8 \\ 4 & & 6 & \\ 5 & & 7 & \end{array}.$$

Example Suppose $k_1 = 6$ and $k_2 = 4$. By corollary 3.2.3, the Schur-optimal design is the *ECD*($\bar{\theta} + 1$). Applying the *ECD* construction from section 3.2.3 with $L = \bar{\theta} + 1 = 3$ yields a Schur-optimal resolvable design which is

$$\begin{array}{cc|cc} 1 & 7 & 1 & 5 \\ 2 & 8 & 2 & 6 \\ 3 & 9 & 3 & 9 \\ 4 & 10 & 4 & 10 \\ 5 & & 7 & \\ 6 & & 8 & \end{array}.$$

3.3 Resolvable Designs With Three Replicates

3.3.1 Introduction

In this section we will study optimality for the resolvable design setting $D(v, 3; k_1, k_2)$. From section 2.3 we have:

ECD($\bar{\theta}$): The optimality matrix for *ECD*($\bar{\theta}$)s is $M_d = pI - \gamma(J - I)$. The eigenvalues of M_d are

$$\begin{aligned} \xi_1(\gamma) &= p + \gamma \qquad \text{(2 copies)} \\ \xi_2(\gamma) &= p - 2\gamma, \end{aligned}$$

and they satisfy

$$\xi_1(\gamma) = \xi_1(\gamma) > \xi_2(\gamma).$$

ECD($\bar{\theta}+1$): The optimality matrix for *ECD($\bar{\theta}$+1)s* is $M_d = pI - \gamma(J-I) + v(J-I)$. The eigenvalues of M_d are

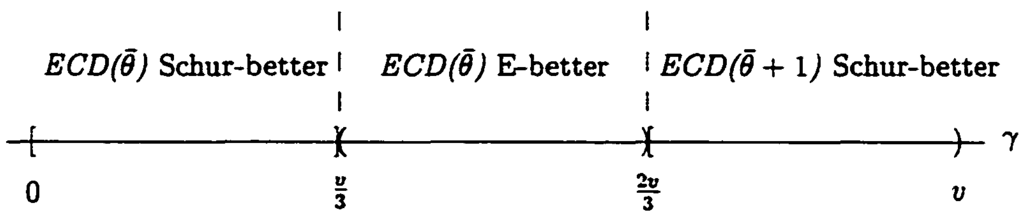
$$\xi_1(\gamma - v) = p - (v - \gamma) \qquad \text{(2 copies)}$$
$$\xi_2(\gamma) = p + 2(v - \gamma),$$

and they satisfy

$$\xi_2(\gamma - v) > \xi_1(\gamma - v) = \xi_1(\gamma - v).$$

Corollaries 2.3.7 and 2.3.8 of Lemmas 2.3.5 and 2.3.6 establish conditions for when *ECD($\bar{\theta}$)s* are E-better then or Schur-better than *ECD($\bar{\theta}$+1)s* and for when *ECD($\bar{\theta}$+1)s* E-better and Schur-better than *ECD($\bar{\theta}$)s*; see table 3.21.

Table 3.21: E- and Schur-comparisons Of *ECDs* In $D(v,3;k_1,k_2)$



Corollaries 2.3.11 and 2.3.13 eliminate *UECDs* from consideration. Conditions for Schur- and E-optimality of *NECDs* or *ECDs* can be established using lemma 2.3.17 and by direct eigenvalue comparisons. The optimality matrix M_d (in order to apply lemma 2.3.17) or the concurrence discrepancy matrix Δ_d must be derived for competing *NECDs*. Recall that *NECDs* have block concurrences discrepancies $\delta_{dii'} \in \{0,1\}$ for all $1 \leq i \neq i' \leq 4$ and have at least one block concurrence discrepcnsy equal to 0 and at least one equal to 1. There are two cases of nonisomorphic *NECDs*; their block concurrence patterns, $\{\delta_{d12}, \delta_{d13}, \delta_{d23}\}$, are listed in table 3.22 and the corresponding block concurrence discrepancy matrices are shown in table 3.23.

Table 3.22: Block Concurrence Discrepancies For *NECDs* In $D(v, 3; k_1, k_2)$

Case	δ_{d12}	δ_{d13}	δ_{d23}
I	1	0	0
II	1	1	0

Table 3.23: Concurrence Discrepancy Matrices for *NECDs* In $D(v, 3; k_1, k_2)$

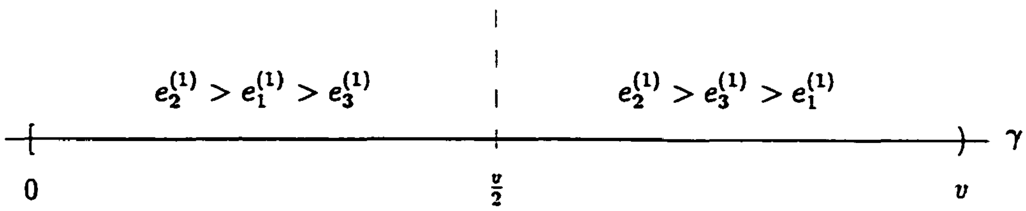
$$\Delta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Delta_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Using the concurrence discrepancy matrices for the two cases of *NECDs*, we begin our eigenvalue optimality investigation by deriving explicit expressions for the eigenvalues of the optimality matrices for each case of *NECDs*. The eigenvalues and their ordering over the admissible region are given below.

Case I: The optimality matrix for Case I *NECDs* is $M_1 = pI - \gamma(J - I) + v\Delta_1$, and the eigenvalues of M_1 are

$$\begin{aligned} e_1^{(1)} &= p - (v - \gamma), \\ e_2^{(1)} &= p + \frac{v-\gamma}{2} + \frac{1}{2}\sqrt{8\gamma^2 + (v-\gamma)^2}, \\ e_3^{(1)} &= p + \frac{v-\gamma}{2} - \frac{1}{2}\sqrt{8\gamma^2 + (v-\gamma)^2}, \end{aligned}$$

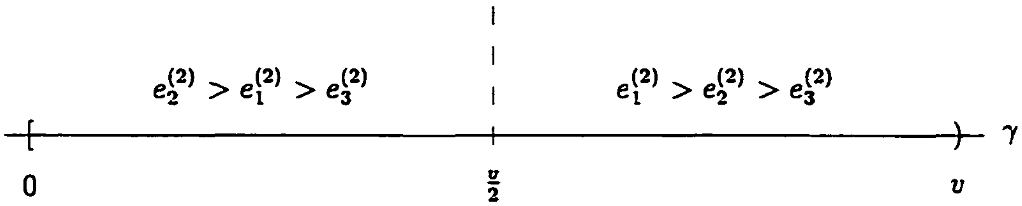
and they satisfy



Case II: The optimality matrix for Case II *NECDs* is $M_2 = pI - \gamma(J - I) + v\Delta_2$, and the eigenvalues of M_2 are

$$\begin{aligned} e_1^{(2)} &= p+\gamma, \\ e_2^{(2)} &= p-\frac{\gamma}{2}+\frac{1}{2}\sqrt{8(v-\gamma)^2+\gamma^2}, \\ e_3^{(2)} &= p-\frac{\gamma}{2}-\frac{1}{2}\sqrt{8(v-\gamma)^2+\gamma^2}, \end{aligned}$$

and they satisfy



3.3.2 (E,S)-optimal Designs in $D(v,3;k_1,k_2)$

Before we determine the E-optimal designs in $D(v,3;k_1,k_2)$ we will make Schur comparisons of Case I designs with *ECD($\bar{\theta}$)*s and *ECD($\bar{\theta}+1$)*s in order to eliminate it as an optimality competitor.

Lemma 3.3.1 *When $0 \le \gamma \le \frac{v}{2}$, ECD($\bar{\theta}$)s are Schur-better than Case I designs.*

Proof By Theorem 2.3.5, *ECD($\bar{\theta}$)*s are Schur-better than Case I designs if they are F-better. When $0 \le \gamma \le \frac{v}{2}$, *ECD*s are F-better than Case I designs since $\xi_2(\gamma) > e_3^{(1)}$. □

Lemma 3.3.2 *When $\frac{v}{2} < \gamma < v$, ECD($\bar{\theta}+1$)s are Schur-better than Case I designs.*

Proof By Theorem 2.3.6, *ECD($\bar{\theta}+1$)*s are Schur-better than Case I designs if they are E-better. When $\frac{v}{2} < \gamma < v$, *ECD($\bar{\theta}+1$)*s are E-better than Case I designs since $\xi_1(\gamma - v) < e_2^{(1)}$. □

Now we will establish that Case II designs are (E,S)-optimal when $\frac{v}{2} < \gamma < \frac{2v}{3}$

Lemma 3.3.3 *When $\frac{v}{2} \leq \gamma < v$, ECD($\bar{\theta}$)s and Case II designs are E-equivalent.*

Proof When $\frac{v}{2} \leq \gamma < v$ the largest eigenvalue of the optimality matrix for Case II designs is $e_1^{(2)} = \xi_1(\gamma)$. □

Lemma 3.3.4 *When $\frac{v}{2} < \gamma < v$, Case II designs are Schur-better than ECD($\bar{\theta}$)s.*

Proof The eigenvalues of *ECD($\bar{\theta}$)*s are $\xi_1(\gamma) = \xi_1(\gamma) > \xi_2(\gamma)$, and when $\frac{v}{2} < \gamma < v$, the eigenvalues of Case II designs are $e_1^{(2)} > e_2^{(2)} > e_3^{(2)}$. Since $\xi_1(\gamma) = e_1^{(2)} > e_2^{(2)}$, then the eigenvalues of the optimality matrix for *ECD($\bar{\theta}$)*s majorize the eigenvalues of the optimality matrix for Case II designs. □

Lemma 3.3.5 *When $\frac{2v}{3} \leq \gamma < v$, ECD($\bar{\theta}+1$)are Schur-better than Case II designs.*

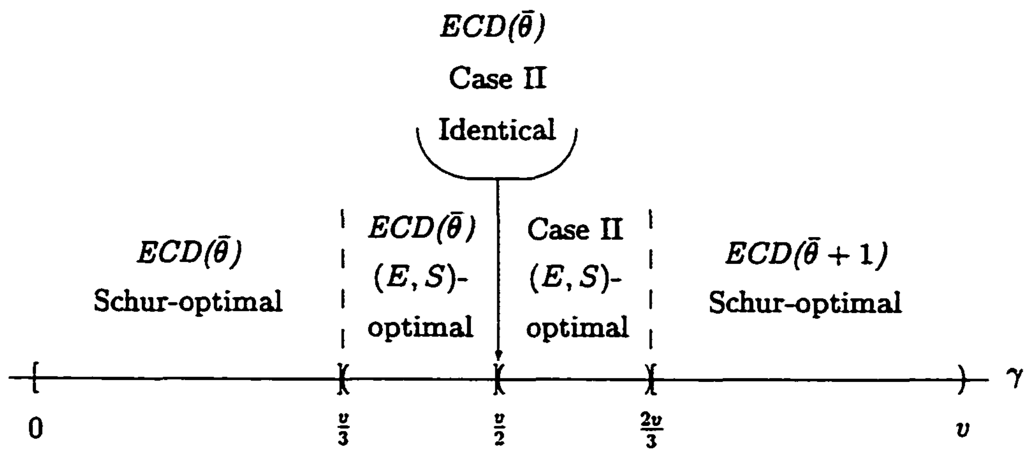
Proof By Theorem 2.3.6, *ECD($\bar{\theta}+1$)*s are Schur-better than Case II designs if they are E-better. When $\frac{2v}{3} < \gamma < v$, *ECD($\bar{\theta}+1$)*s are E-better than Case II designs since $e_1^{(2)} > \xi_2(\gamma - v)$. When $\gamma = \frac{2v}{3}$, since $e_1^{(2)} = \xi_2(\gamma - v)$ and $e_2^{(2)} > \xi(\gamma - v)$, the eigenvalues of the optimality matrix for Case II designs majorize the eigenvalues of the optimality matrix for *ECD($\bar{\theta}+1$)*s. □

Therefore, for all values of $0 \leq \gamma < v$ there is either a unique Schur-optimal design or a unique (E,S)-optimal design. See table 3.24.

3.3.3 A-optimal Design

The lemmas of section 3.3.2 establish that *ECD($\bar{\theta}$)*s are uniquely A-optimal when $0 \leq \gamma < \frac{v}{2}$, *ECD($\bar{\theta}$)*s and Case II designs are identically A-optimal when $\gamma = \frac{v}{2}$, and *ECD($\bar{\theta}+1$)*s are uniquely A-optimal when $\frac{2v}{3} < \gamma < v$; however, on the interval $\frac{v}{2} < \gamma < \frac{2v}{3}$, *ECD($\bar{\theta}+1$)*s and Case II designs are A-optimal candidates. In order to find the design that minimizes (3.73) we need the expressions for the eigenvalues of the information matrices of the competing designs in terms of the eigenvalues of the optimality matrices. These are given below.

Table 3.24: (E,S)- and Schur-optimal Designs In $D(v, 3; k_1, k_2)$



ECD($\bar{\theta} + 1$):

$$z_1^{(\bar{\theta}+1)} = \frac{2p + (v - \gamma)}{p} \quad \text{(2 copies)}$$

$$z_2^{(\bar{\theta}+1)} = \frac{2[p - (v - \gamma)]}{p}$$

Case II:

$$z_1^{(2)} = \frac{2p - \gamma}{p}$$

$$z_2^{(2)} = \frac{4p + \gamma - \sqrt{8(v - \gamma)^2 + \gamma^2}}{2p}$$

$$z_3^{(2)} = \frac{4p + \gamma + \sqrt{8(v - \gamma)^2 + \gamma^2}}{2p}$$

Lemma 3.3.6 *When $\frac{3v}{5} \le \gamma < \frac{2v}{3}$, ECD($\bar{\theta} + 1$)s are A-better than Case II designs. When $\frac{v}{2} < \gamma < \frac{3v}{5}$, ECD($\bar{\theta} + 1$)s are A-better than Case II designs if and only if*

$$-\gamma^3 - 2(p - 2v)\gamma^2 + (8p^2 + 6vp - 5v^2)\gamma - 2v(2p^2 + 2pv - v^2) > 0. \tag{3.78}$$

Proof When $\frac{v}{2} < \gamma < \frac{2v}{3}$, ECD($\bar{\theta} + 1$)s are A-better than Case II designs if and only if $\frac{2}{z_1^{(\bar{\theta}+1)}} + \frac{1}{z_2^{(\bar{\theta}+1)}} < \frac{1}{z_1^{(2)}} + \frac{1}{z_2^{(2)}} + \frac{1}{z_3^{(2)}}$ which holds if and only if condition (3.78) is satisfied. On the interval $\frac{3v}{5} \le \gamma < \frac{2v}{3}$, a lower bound for the left hand side of (3.78), obtained by substituting $\gamma = \frac{2v}{3}$ into the negative terms and $\gamma = \frac{3v}{5}$ into the positive terms, is

$$\frac{2}{675v^3}\left[270\left(\frac{p}{v}\right)^2 - 435\left(\frac{p}{v}\right) - 64\right]. \tag{3.79}$$

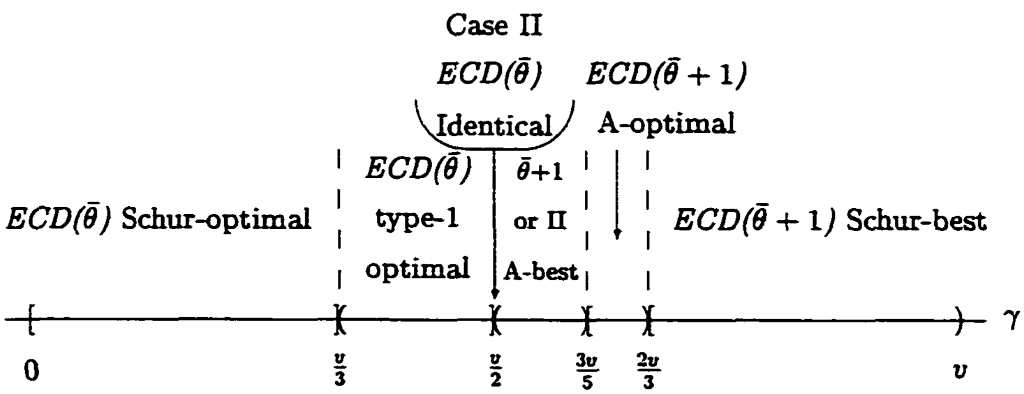
Setting (3.79) equal to zero and solving for $\frac{p}{v}$ yields

$$\frac{p}{v} = \left(\frac{145-\sqrt{28705}}{180}, \frac{145+\sqrt{28705}}{180}\right).$$

Since $\frac{3}{2} \leq \frac{145+\sqrt{28705}}{180} \leq \frac{7}{4}$, and when $\frac{p}{v} = 2$, (3.79) is greater than zero, (3.78) is satisfied on $\frac{3v}{5} \leq \gamma < \frac{2v}{3}$ whenever $\frac{p}{v} > \frac{7}{4}$, and, by fact 3.1.3, this inequality holds when $k_1 \geq k_2 \geq 4$ or $k_2 = 3$ and $k_1 \geq 6$. Thus, (3.78) may not be satisfied when $k_2 \geq k_1 = 2$ or $5 \geq k_1 \geq k_2 = 3$. By corollary 3.1.5, on $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$, $k_2 = 2$ if and only if $k_1 = 4$ and $k_2 = 3$ if and only if $k_1 = 3, 4$ or 5. Since (3.78) is satisfied when $(k_1, k_2) = (4, 2)$, (3,3), (4,3), and (5,3), *ECD*($\bar{\theta}+1$)s are A-better than Case II designs on the interval. □

A summary of the A-best analysis is given in table 3.25 below.

Table 3.25: A-, Type-1, and Schur-optimal Designs in $D(v, 3; k_1, k_2)$



We have found that the A-optimal design in $D(v, 3; k_1, k_2)$ is uniquely an *ECD*($\bar{\theta}$) when $0 \leq \gamma < v$ and uniquely an *ECD*($\bar{\theta}+1$) when $\frac{3v}{5} \leq \gamma < v$. When $\gamma = \frac{v}{2}$ the optimality matrix for $ECD(\bar{\theta})$and Case II designs have identical eigenvalues, and the $ECD(\bar{\theta})$and Case II designs are A-optimal. When $\frac{v}{2} < \gamma < \frac{3v}{5}$ the A-optimal design can either be an $ECD(\bar{\theta}+1)$ or a Case II design, and condition (3.78) must be checked in order to determine if the A-optimal design is an $ECD(\bar{\theta}+1)$ or a Case II design. Table 3.26 lists the parameters k_1, k_2, and γ for ten A-optimal $ECD(\bar{\theta}+1)$s and Case II designs.

3.3.4 Special Cases: $(k_1 - k_2) \leq 2$

We will now apply the optimality results from sections 3.3.2 and 3.3.3 to the three special cases described in section 2.4.

Corollary 3.3.7 *Suppose $k_1 = k_2$ and $r = 3$. Then*

(i) If $2 \mid k_1$ then $\gamma = 0$, and $ECD(\theta^*)$s exist and are Schur-optimal.

(ii) If $2 \nmid k_1$ then $\gamma = \frac{v}{2}$, and $ECD(\bar{\theta})$s and Case II are identical and $(E,S)-$ and ϕ_f-optimal.

Corollary 3.3.8 *Suppose $k_2 = k_1 - 1$ and $r = 3$. Then*

(i) If $2 \mid k_1$ then $\frac{v}{4} < \gamma < \frac{v}{3}$, and $ECD(\bar{\theta})$s are Schur-optimal.

(ii) If $2 \nmid k_1$ then $\frac{3v}{4} < \gamma < \frac{4v}{5}$, and $ECD(\bar{\theta}+1)$s are Schur-optimal.

Corollary 3.3.9 *Suppose $k_2 = k_1 - 2$ and $r = 3$. Then*

(i) If $k_1 = 4$ then $\gamma = \frac{2v}{3}$, and $ECD(\bar{\theta}+1)$s are Schur-optimal.

(ii) If $k_1 = 6$ then $\gamma = \frac{3v}{5}$, Case II designs are (E,S)-optimal, and $ECD(\bar{\theta}+1)$s are A-optimal.

(iii) If $2 \mid k_1$ and $k_1 \geq 8$ then $\frac{v}{2} < \gamma < \frac{3v}{5}$, Case II designs are (E,S)-optimal, and either an $ECD(\bar{\theta}+1)$ or a Case II design is A-optimal.

(iv) If $2 \nmid k_1$ then $0 < \gamma < \frac{v}{6}$, and $ECD(\bar{\theta})$s are Schur-optimal.

Table 3.26: Parameters for A-optimal Designs In $D(v,3,k_1,k_2)$ When $\frac{v}{2} < \gamma < \frac{3v}{5}$

ECD($\bar{\theta}+1$) A-optimal

k_1	k_2	$\frac{\gamma}{v}$	*ECD($\bar{\theta}+1$)* A-value	Case II A-value
8	6	.57	1.51261	1.51398
10	8	.56	1.50794	1.50851
11	5	.56	1.51309	1.51400
12	7	.58	1.50718	1.50845
12	10	.55	1.50545	1.50575
13	3	.56	1.52702	1.52739
14	12	.54	1.50398	1.50415
16	14	.53	1.50303	1.50313
17	6	.57	1.50762	1.50848
17	8	.56	1.50513	1.50570

Case II A-optimal

k_1	k_2	$\frac{\gamma}{v}$	*ECD($\bar{\theta}+1$)* A-value	Case II A-value
5	2	.57	1.58385	1.57738
14	3	.53	1.53069	1.52746
14	9	.52	1.50600	1.50584
26	4	.53	1.51471	1.51413
27	4	.52	1.51571	1.51427
27	20	.51	1.50139	1.50137
29	12	.51	1.50255	1.50249
34	8	.52	1.50422	1.50419
42	5	.53	1.50874	1.50862
43	5	.52	1.50912	1.50868

3.3.5 Construction of Optimal Designs in $D(v,3;k_1,k_2)$

ECD Constructions

Let L be the common *ECD* treatment concurrence. Then for *ECD($\bar{\theta}$)*s, $L=\bar{\theta}$, and for *ECD($\bar{\theta}+1$)*s, $L=\bar{\theta}+1$.

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots L\} \cup \{k_1+1 \ldots 2k_1 - L\}$

Block 1 of Replicate 3:

(i) $k_1 < 2L$:

$$\{1 \ldots 2L - k_1\} \cup \{L+1 \ldots 2k_1 - L\}$$

(ii) $k_1 \geq 2L$:

$$\{L+1 \ldots 2L\} \cup \{k_1+1 \ldots k_1+L\} \cup \{2k_1 - L + \overset{\text{if } k_1=2L+1}{1 \ldots} 3(k_1 - L)\}$$

Case II Constructions

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}+1\} \cup \{k_1+1 \ldots 2k_1 - (\bar{\theta}+1)\}$

Block 1 of Replicate 3: $\{1 \ldots 2(\bar{\theta}+1) - k_1\} \cup \{\bar{\theta}+2 \ldots k_1\} \cup$
$\{k_1 + \overset{\text{if } k_1-L-2>0}{1 \ldots} 2k_1 - \bar{\theta} - 2\} \cup \{2k_1 - \bar{\theta}\}$

3.3.6 Examples of Optimal Resolvable Designs in $D(v, 3; k_1, k_2)$

We will conclude this section by providing some examples of resolvable designs in $D(v, 3; k_1, k_2)$ for various interesting $k_1 \geq 3$ and $2 \leq k_2 \leq k_1$. First we will construct designs for the two cases when $k_1 = k_2$.

Example Suppose $k_1 = k_2 = 4$. Then, according to corollary 3.3.7 the the Schur-optimal design is an *ECD(θ^*)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 2$ yields a Schur-optimal *ECD(θ^*)* which is:

1	5	1	3	3	1
2	6	2	4	4	2
3	7	5	7	5	7
4	8	6	8	6	8

Example Consider the case where $k_1 = k_2 = 5$. Then, according to corollary 3.3.7 the *ECD($\bar{\theta}$)*s and Case II designs are (E,S)- and type-1 optimal. Applying the *ECD* construction given above with $L = \bar{\theta} = 2$ produces an (E,S)- and type-1 optimal *ECD($\bar{\theta}$)* which is:

1	6	1	3	3	1
2	7	2	4	4	2
3	8	6	5	6	5
4	9	7	9	7	6
5	10	8	10	9	10

.

Now we will investigate the two cases when $k_1 - k_2 = 1$.

Example Consider the setting such that $k_1 = 6$ and $k_2 = 5$. By corollary 3.3.8, the Schur-optimal design is an *ECD($\bar{\theta}$)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 3$ produces a Schur-optimal *ECD* which is:

1	7	1	4	4	1
2	8	2	5	5	2
3	9	3	6	6	3
4	10	7	10	7	10
5	11	8	11	8	11
6	12	9	12	9	12

.

Example Suppose $k_1 = 5$ and $k_2 = 4$. By corollary 3.3.8, the Schur-optimal design is an *ECD($\bar{\theta} + 1$)*. Applying the *ECD* construction given above with $L = \bar{\theta} + 1 = 3$ yields a Schur-optimal *ECD($\bar{\theta} + 1$)* which is:

1	6	1	4	1	2
2	7	2	5	4	3
3	8	3	8	5	8
4	9	6	9	6	9
5	10	7	10	7	10

.

Now we will investigate the two cases when $k_1 - k_2 = 2$.

Example Consider the setting such that $k_1 = 8$ and $k_2 = 6$. By corollary 3.3.9, the Case II design is (E,S)-optimal, and the A-optimal design is either the Case II design or the *ECD($\bar{\theta}+1$)*. Checking condition (3.78) establishes that the *ECD($\bar{\theta}+1$)*

is A-optimal. Applying the Case II construction given above with $\bar{\theta} = 4$ yields an (E,S)-optimal Case II design which is:

1	9	1	6	1	3
2	10	2	7	2	4
3	11	3	8	6	5
4	12	4	12	7	11
5	13	5	13	8	13
6	14	9	14	9	14
7		10		10	
8		11		12	

Applying the $ECD(\bar{\theta}+1)$ construction given above with $L = \bar{\theta} + 1 = 5$ produces an A-optimal $ECD(\bar{\theta}+1)$ which is:

1	9	1	6	1	3
2	10	2	7	2	4
3	11	3	8	6	5
4	12	4	12	7	12
5	13	5	13	8	13
6	14	9	14	9	14
7		10		10	
8		11		11	

Example Suppose $k_1 = 5$ and $k_2 = 3$. By corollary 3.3.9, the Schur-optimal design is an $ECD(\bar{\theta})$. Applying the ECD construction given above with $L = \bar{\theta} = 3$ yields a Schur-optimal $ECD(\bar{\theta})$ which is:

1	6	1	4	1	2
2	7	2	5	4	3
3	8	3	8	5	8
4		6		6	
5		7		7	

3.4 Resolvable Designs With Four Replicates

3.4.1 Introduction

In this section we study optimality for the resolvable design setting $D(v, 4; k_1, k_2)$. From section 2.3 we have:

ECD($\bar{\theta}$): The optimality matrix for *ECD($\bar{\theta}$)*s is $M_d = pI - \gamma(J-I)$. The eigenvalues of M_d are

$$\xi_1(\gamma) = p + \gamma \qquad \text{(3 copies)}$$
$$\xi_2(\gamma) = p - 3\gamma,$$

and they satisfy

$$\xi_1(\gamma) = \xi_1(\gamma) = \xi_1(\gamma) > \xi_2(\gamma).$$

ECD($\bar{\theta}$ + 1): The optimality matrix for *ECD($\bar{\theta}$+1)*s is $M_d = pI - \gamma(J-I) + v(J-I)$. The eigenvalues of M_d are

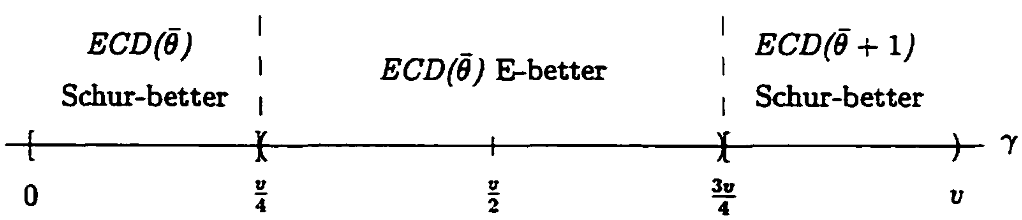
$$\xi_1(\gamma - v) = p - (v - \gamma) \qquad \text{(3 copies)}$$
$$\xi_2(\gamma) = p + 3(v - \gamma),$$

and they satisfy

$$\xi_2(\gamma - v) > \xi_1(\gamma - v) = \xi_1(\gamma - v) = \xi_1(\gamma - v).$$

Theorem 2.3.3, lemma 2.3.7, and corollary 2.3.8 establish conditions for when *ECD($\bar{\theta}$)*s are E-better or Schur-better than *ECD($\bar{\theta}$+1)*s and for when *ECD($\bar{\theta}$+1)*s are E-better and Schur-better than *ECD($\bar{\theta}$)*s; see table 3.27.

Table 3.27: E- and Schur-comparisons Of *ECD*s In $D(v, 4; k_1, k_2)$



As with all $r \leq 4$, corollaries 2.3.11 and 2.3.13 eliminate *UECD*s as optimality competitors. Conditions for Schur- and E-optimality of *NECD*s or *ECD*s can be established using lemma 2.3.17 and by direct eigenvalue comparisons. The optimality matrix M_d (in order to apply lemma 2.3.17) or the concurrence discrepancy matrix Δ_d must be derived for competing *NECDs*. Recall that *NECDs* have block concurrence discrepancies $\delta_{dii'} \in \{0,1\}$ for all $1 \leq i \neq i' \leq 4$ and have at least one block concurrence discrepancy equal to 0 and at least one equal to 1. There are nine cases of nonisomorphic *NECDs*; their block concurrence discrepancies, $\{\delta_{d12}, \delta_{d13}, \delta_{d23}, \delta_{d14}, \delta_{d24}, \delta_{d34}\}$ are listed in table 3.28 and the corresponding block concurrence discrepancy matrices are shown in table 3.29.

Table 3.28: Block Concurrence Discrepancies For *NECDs* In $D(v,4;k_1,k_2)$

Case	δ_{d12}	δ_{d13}	δ_{d23}	δ_{d14}	δ_{d24}	δ_{d34}
I	0	1	1	1	1	1
II	0	1	1	1	1	0
III	0	0	1	1	1	1
IV	0	0	1	0	1	1
V	0	0	0	1	1	1
VI	0	0	1	1	0	1
VII	0	0	1	1	0	0
VIII	0	0	0	0	1	1
IX	0	0	0	0	0	1

Using the concurrence discrepancy matrices for the nine cases of *NECDs*, we begin our eigenvalue optimality investigation with the following application of corollary 2.3.17.

Corollary 3.4.1 *Let $d \in D(v,r;k_1,k_2)$ be an NECD having optimality matrix $M_d = pI - \gamma(I-J) + v\Delta_d$, and let u_1 and u_r be the maximum and minimum eigenvalues, respectively, of $\Delta_{d0} = P^T\Delta_d P$, where $P = (I - \frac{1}{4}J)$. If*

$$\gamma < -\frac{u_r}{4}v$$

then ECD($\bar{\theta}$)s are Schur-better than d. If $u_1 > 0$ and

$$\gamma > \left(\frac{3-u_1}{4}\right)v,$$

Table 3.29: Concurrence Discrepancy Matrices For *NECDs* In $D(v,4;k_1,k_2)$

Case I: $\Delta_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Case II: $\Delta_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

Case III: $\Delta_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Case IV: $\Delta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Case V: $\Delta_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

Case VI: $\Delta_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Case VII: $\Delta_7 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Case VIII: $\Delta_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Case IX: $\Delta_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

then ECD($\bar{\theta}+1$)s are Schur-better than d. Furthermore, if

$$u_1 > 0 \tag{3.80}$$

then ECD($\bar{\theta}$)s are E-better, but not necessarily Schur-better, than d.

We now use these tools to eliminate as many designs as possible. For each *NECD*, condition (3.80) was calculated with results given in table 3.30. Immediately we see all cases except Cases I, II, and V are E-inferior to *ECD($\bar{\theta}$)*s. Values of γ for which *ECD($\bar{\theta}$)*s or *ECD($\bar{\theta}$+1)*s are Schur-better than *NECDs* having any of the concurrence discrepancy matrices listed in table 3.29 have been determined using corollary 3.4.1 and are also listed in table 3.30. We also know by Theorem 2.3.3 and corollary

Table 3.30: Corollary 3.4.1 Results In $D(v, 4; k_1, k_2)$

Case	$-\frac{u_r}{4}$	$\frac{3-u_1}{4}$	u_1
I	.375	.750	.000
II	.500	.750	.000
III	.342	.658	.366
IV	.250	.625	.500
V	.375	.750	.000
VI	.405	.595	.618
VII	.250	.500	1.000
$VIII$	.342	.658	.366
IX	.250	.625	.500

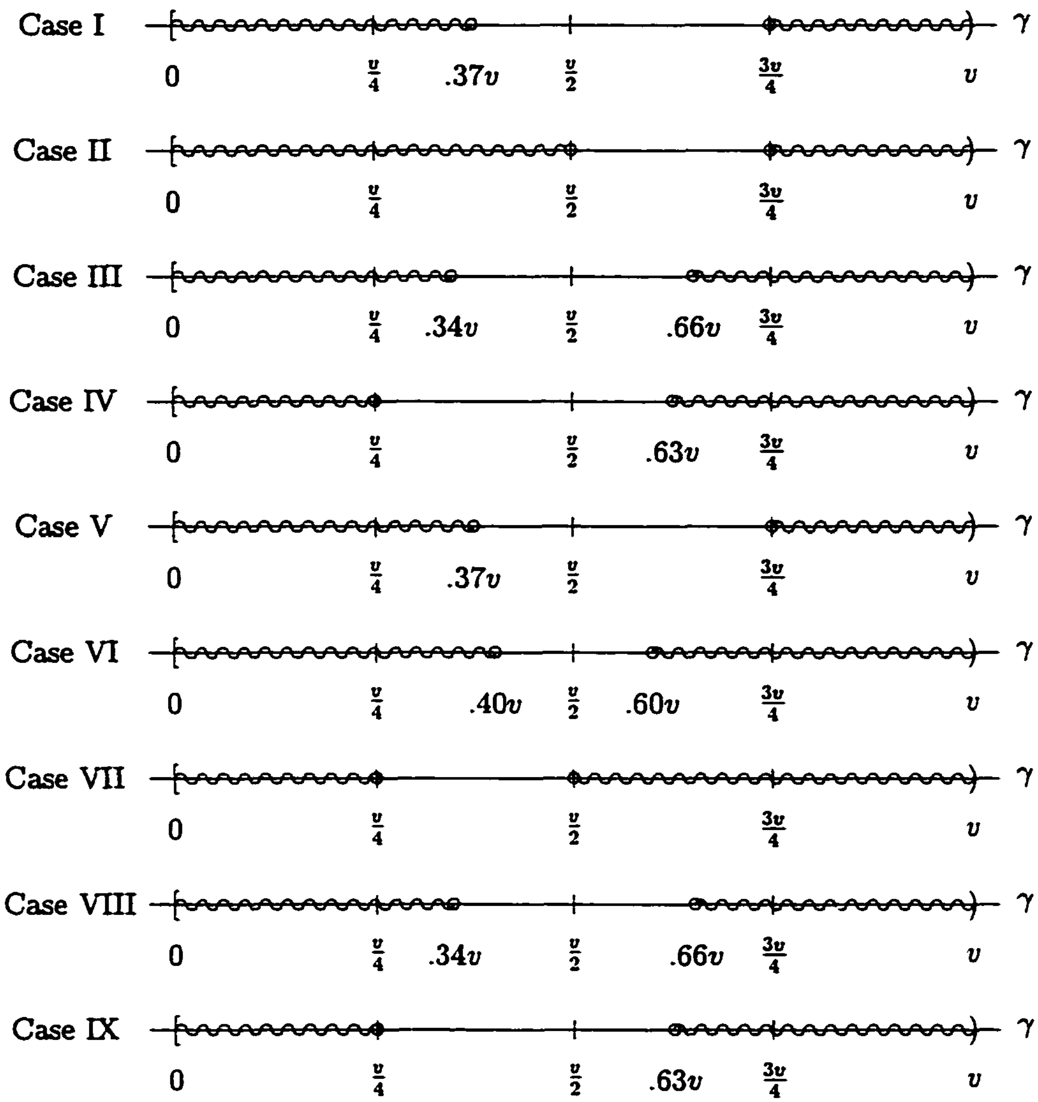
2.3.15 that *ECD($\bar{\theta}$)*s are type-1 and E-optimal on $0 \leq \gamma \leq \frac{v}{2}$, which is sufficient on this range for our primary goals of A-optimality. Here we get stronger optimality for a subset of $0 \leq \gamma < \frac{v}{2}$. Note that on $\frac{3v}{4} < \gamma < v$, *ECD($\bar{\theta}$ + 1)*s are uniquely Schur-optimal.

The majorization results we have so far are summarized in table 3.31 which shows, for each of the nine cases, the range for which each *NECD* majorizes an *ECD*. We see that Case VII designs are Schur-inferior to *ECD($\bar{\theta}$ + 1)*s when $\frac{v}{2} < \gamma < v$. Thus Case VII designs are type-1 inferior to *ECD*s over the entire interval. Case VII is the only case that is completely eliminated from type-1 optimality contention, so in order to proceed, we must make direct eigenvalue comparisons. We need explicit expressions for the eigenvalues of the optimality matrices for each of the remaining eight *NECD* competitors when possible. The eigenvalues and their ordering over the admissible region are given below.

Case I: The optimality matrix for Case I *NECD*s is $M_1 = pI - \gamma(J - I) + v\Delta_1$, and the eigenvalues of M_1 are

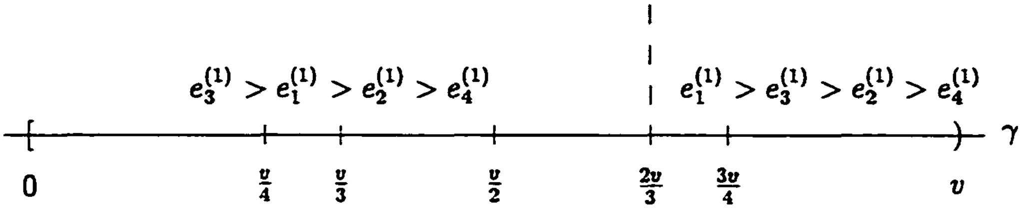
$$\begin{aligned} e_1^{(1)} &= p + \gamma \\ e_2^{(1)} &= p - (v - \gamma) \end{aligned}$$

Table 3.31: Majorization Intervals For *NECDs* In $D(v, 4; k_1, k_2)$



$$e_3^{(1)} = p + (v - \gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$$
$$e_4^{(1)} = p + (v - \gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2},$$

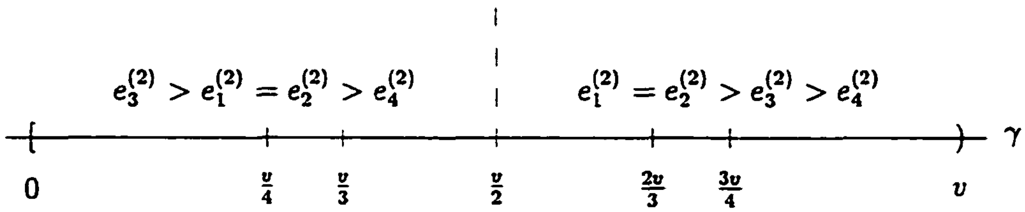
and they satisfy



Case II: The optimality matrix for Case II *NECDs* is $M_2 = pI - \gamma(J - I) + v\Delta_2$, and the eigenvalues of M_2 are

$$\begin{aligned} e_1^{(2)} &= p + \gamma \\ e_2^{(2)} &= p + \gamma \\ e_3^{(2)} &= p + 3(v - \gamma) - v \\ e_4^{(2)} &= p + \gamma - 2v, \end{aligned}$$

and they satisfy



Case III: The optimality matrix for Case III *NECDs* is $M_3 = pI - \gamma(J - I) + v\Delta_3$. Three of the eigenvalues of M_3 can not be expressed in closed form. The fourth eigenvalue is

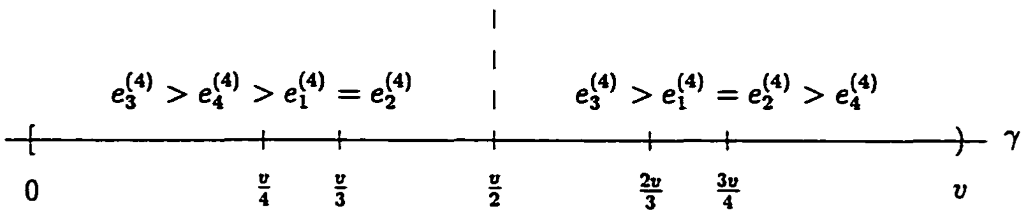
$$e^{(3)} = p - (v - \gamma).$$

Case IV: The optimality matrix for Case IV *NECDs* is $M_4 = pI - \gamma(J - I) + v\Delta_4$, and the eigenvalues of M_4 are

$$\begin{aligned} e_1^{(4)} &= p - (v - \gamma) \\ e_2^{(4)} &= p - (v - \gamma) \end{aligned}$$

$$\begin{aligned} e_3^{(4)} &= p + (v-\gamma) + \sqrt{(v-\gamma)^2 + 3\gamma^2} \\ e_4^{(4)} &= p + (v-\gamma) - \sqrt{(v-\gamma)^2 + 3\gamma^2}, \end{aligned}$$

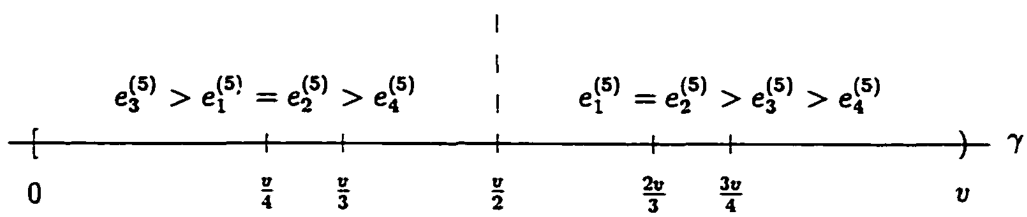
and they satisfy



Case V: The optimality matrix for Case V *NECDs* is $M_5 = pI - \gamma(J-I) + v\Delta_5$, and the eigenvalues of M_5 are

$$\begin{aligned} e_1^{(5)} &= p + \gamma \\ e_2^{(5)} &= p + \gamma \\ e_3^{(5)} &= p - \gamma + \sqrt{3(v-\gamma)^2 + \gamma^2} \\ e_4^{(5)} &= p - \gamma - \sqrt{3(v-\gamma)^2 + \gamma^2}, \end{aligned}$$

and they satisfy

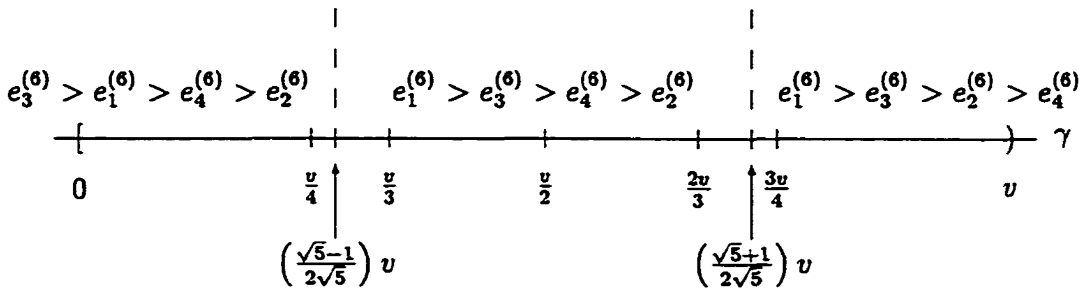


Case VI: The optimality matrix for Case VI *NECDs* is $M_6 = pI - \gamma(J-I) + v\Delta_6$, and the eigenvalues of M_6 are

$$\begin{aligned} e_1^{(6)} &= p - (v-\gamma) + \left(\frac{1+\sqrt{5}}{2}\right) v \\ e_2^{(6)} &= p - (v-\gamma) + \left(\frac{1-\sqrt{5}}{2}\right) v \end{aligned}$$

$$e_3^{(6)} = p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{4(v-2\gamma)^2 + v^2}$$
$$e_4^{(6)} = p + (v-\gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{4(v-2\gamma)^2 + v^2},$$

and they satisfy



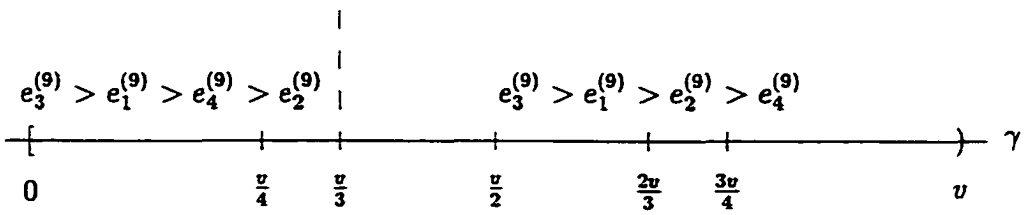
Case VIII: The optimality matrix for Case VIII *NECDs* is $M_8 = pI - \gamma(J-I) + v\Delta_8$. Three of the eigenvalues of M_8 can not be expressed in closed form. The fourth eigenvalue is

$$e^{(8)} = p + \gamma.$$

Case IX: The optimality matrix for Case IX *NECDs* is $M_9 = pI - \gamma(J-I) + v\Delta_9$, and the eigenvalues of M_9 are

$$e_1^{(9)} = p + \gamma$$
$$e_2^{(9)} = p - (v-\gamma)$$
$$e_3^{(9)} = p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16\gamma^2 + v^2}$$
$$e_4^{(9)} = p + (v-\gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{16\gamma^2 + v^2},$$

and they satisfy



We conclude this section with a lemma that uses the explicit expressions for the eigenvalues of the optimality matrices for the nine cases of *NECDs* and corollary 2.3.4 to determine Schur-optimality when $\gamma = \frac{v}{2}$.

Lemma 3.4.2 *When $\gamma = \frac{v}{2}$, ECD($\bar{\theta}$)s, Case II and Case V designs are Schur-optimal.*

Proof Since all cases of *NECDs* except for Cases I, II, and V are E-inferior to *ECD($\bar{\theta}$)s* when $\gamma = \frac{v}{2}$, the optimality matrices for these cases are the only ones that can potentially have eigenvalues that are identical to the eigenvalues of the optimality matrix for *ECD($\bar{\theta}$)s*. Putting $\gamma = \frac{v}{2}$ into the eigenvalue expressions for these three cases gives the result. □

3.4.2 (E,S)-Optimal Designs in $D(v, 4; k_1, k_2)$

Corollary 3.4.1 established that the only *NECDs* that can be E-optimal in the resolvable design setting $D(v, r; k_1, k_2)$ are Cases I, II, and V designs. E-optimality will now be investigated in detail, but first we will review a few useful optimality results from above.

1. *ECD($\bar{\theta}$)s* are E-optimal when $0 \leq \gamma \leq \frac{v}{2}$, uniquely so when $\gamma < \frac{v}{2}$.
2. The optimality matrices for *ECD($\bar{\theta}$)*, Case II, and V designs have identical eigenvalues when $\gamma = \frac{v}{2}$
3. *ECD($\bar{\theta}+1$)s* are Schur-optimal when $\frac{3v}{4} \leq \gamma < v$.

E-optimality is solved for $0 \leq \gamma < \frac{v}{2}$ and $\frac{3v}{4} < \gamma < v$; *ECD($\bar{\theta}$)s*, Case II, and V designs are E-equivalent when $\gamma = \frac{v}{2}$; and Case I, II, and V designs may be E-optimal on $\frac{v}{2} < \gamma \leq \frac{3v}{4}$. In this section we will find the E-optimal designs on $\frac{v}{2} < \gamma \leq \frac{3v}{4}$, and if more than one design is E-optimal on a subinterval of $\frac{v}{2} \leq \gamma \leq \frac{3v}{4}$, then the *(E,S)-optimal design* will be identified, see definition 3.1.1. Based on the conclusions above we can state

Corollary 3.4.3 *When $0 \le \gamma \le \frac{v}{2}$, ECD($\bar{\theta}$)s are (E,S)-optimal, uniquely so when $\gamma < \frac{v}{2}$. When $\frac{3v}{4} < \gamma < v$, ECD($\bar{\theta}+1$)s are (E,S)-optimal.*

The following lemma establishes exactly when Cases I, II, and V are E-optimal.

Lemma 3.4.4

1. *ECD($\bar{\theta}$), Case II, and V designs are E-equivalent and E-better than Case I designs when $\frac{v}{2} \le \gamma < \frac{2v}{3}$.*

2. *When $\frac{2v}{3} \le \gamma < \frac{3v}{4}$, ECD($\bar{\theta}$)s, Case I, II, and V designs are E-equivalent.*

3. *When $\gamma = \frac{3v}{4}$, ECD($\bar{\theta}$)s, ECD($\bar{\theta}+1$)s, Case I, II, and V designs are E-equivalent.*

Proof The maximum eigenvalue of *ECD($\bar{\theta}$)*s is $\xi_1(\gamma) = p + \gamma$, and the maximum eigenvalue of *ECD($\bar{\theta}+1$)*s is $\xi_2(\gamma - v) = p - (v - \gamma)$. On the interval $\frac{v}{2} \le \gamma \le \frac{3v}{4}$, the maximum eigenvalue of Case II and V designs is $e_1^{(2)} = e_1^{(5)} = p + \gamma = \xi_1(\gamma)$; therefore, *ECD($\bar{\theta}$)*s, Case II, and V designs are E-equivalent. On $\frac{v}{2} \le \gamma < \frac{2v}{3}$, Case I designs are E-inferior to *ECD($\bar{\theta}$)*s, Case II, and V designs since they have maximum eigenvalue $e_3^{(1)} = p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2} > \xi_1(\gamma) = e_1^{(2)} = e_1^{(4)}$. However, when $\frac{2v}{3} \le \gamma \le \frac{3v}{4}$, the maximum eigenvalue of Case I designs is $e_1^{(1)} = p + \gamma$ which is identical to the maximum eigenvalues of *ECD($\bar{\theta}$)*s, Case I, and V designs, and Case I is E-equivalent to *ECD($\bar{\theta}$)*s, Case II, and V designs. When $\gamma = \frac{3v}{4}$, $\xi_2(\gamma - v) = \xi_1(\gamma)$, and *ECD($\bar{\theta}$)*s, *ECD($\bar{\theta}+1$)*s, Case I, II, and V are E-equivalent. □

Now Schur comparisons of the E-optimal designs can be made.

Lemma 3.4.5 *Case II designs are Schur-better than ECD($\bar{\theta}$)s when $\frac{v}{2} < \gamma < v$.*

Proof When $\frac{v}{2} < \gamma < v$, the largest two eigenvalues of Case II designs, which are $e_1^{(2)} = e_2^{(2)} = p + \gamma = \xi_1(\gamma)$, are identical to each other and to the largest two eigenvalues of *ECD($\bar{\theta}$)s*. Since the third largest Case II eigenvalue $e_3^{(2)}$ is less than $e_1^{(2)} = e_2^{(2)} = \xi_1(\gamma)$ when $\frac{v}{2} < \gamma < 0$, then the eigenvalues of *ECD($\bar{\theta}$)s* majorize the eigenvalues of Case II designs on the interval, and, therefore, Case II designs are Schur-better. □

Lemma 3.4.6 *When $\frac{v}{2} < \gamma < v$, Case II is Schur-better than Case V.*

Proof When $\frac{v}{2} < \gamma < v$, the largest two eigenvalues of Case V designs, $e_1^{(5)} = e_2^{(5)} = p + \gamma$, are identical to each other and identical to $e_1^{(2)} = e_2^{(2)}$, the largest two eigenvalues of Case II designs. It is then necessary and sufficient for the eigenvalues of Case V designs to majorize the eigenvalues of Case II designs that the third largest eigenvalue of Case V designs be greater than or equal to the third largest eigenvalue of Case II designs, or $e_3^{(5)} \geq e_3^{(2)}$. This inequality is true if and only if $p - \gamma + \sqrt{3(v-\gamma)^2 + \gamma^2} \geq p + 3(v - \gamma) - v$ which is true if and only if $\gamma \geq \frac{v}{2}$. Therefore, Case II is Schur-better. □

Lemma 3.4.7 *When $\frac{2v}{3} \leq \gamma \leq v$, Case I designs are Schur-better than Case II designs.*

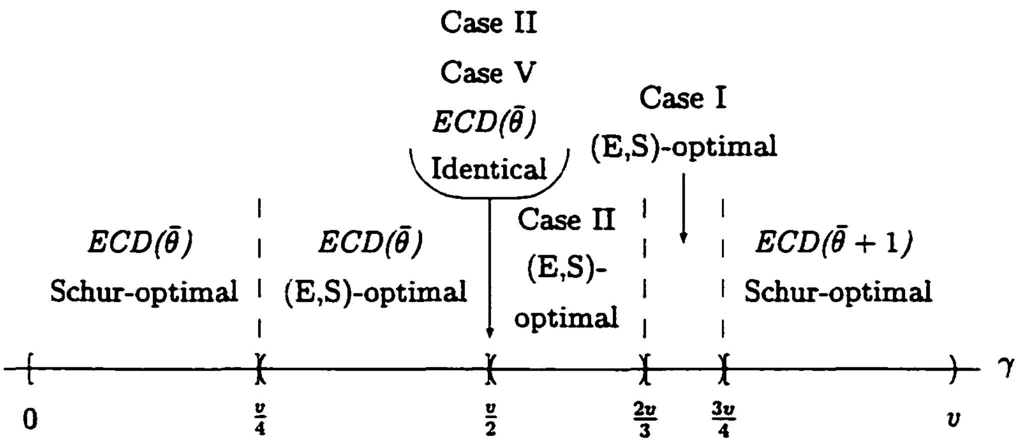
Proof When $\frac{2v}{3} \leq \gamma \leq v$, Case I and Case II designs have the same maximum eigenvalue, which is $e_1^{(1)} = e_1^{(2)} = p + \gamma$. In order to establish the result, we will show that the remaining three eigenvalues of Case II designs $e_2^{(2)} > e_3^{(2)} > e_4^{(2)}$ majorize the remaining three eigenvalues of Case I designs $e_3^{(1)} > e_2^{(1)} > e_4^{(1)}$. Since $e_2^{(2)} = e_1^{(1)} > e_3^{(1)}$, we have the result if and only if $e_4^{(2)} \leq e_4^{(1)}$. This inequality holds if and only if $p + \gamma - 2v \leq p + (v - \gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$ which is true if and only if $8v(v - \gamma) \geq 0$, that is when $\gamma \leq v$. Therefore, Case I designs are Schur-better than Case II designs. □

Lemma 3.4.8 *When $\gamma = \frac{3v}{4}$, ECD($\bar{\theta} + 1$)s are Schur-better than Case I designs.*

Proof When $\gamma = \frac{3v}{4}$, since the largest eigenvalue of Case I designs is equal to the largest eigenvalue of *ECD*$(\bar{\theta}+1)$s, then $p + \gamma = e_1^{(1)} = \xi_2(\gamma - v) = p + 3(v - \gamma)$. Since $e_3^{(1)} > e_2^{(1)} = \xi_1(\gamma - v) > e_4^{(1)}$ then the eigenvalues of Case I designs majorize the eigenvalues of *ECD*$(\bar{\theta}+1)$s. Therefore *ECD*$(\bar{\theta}+1)$s are Schur-better. □

Lemmas 3.4.4, 3.4.5, 3.4.6, and 3.4.7 guarantee that for $0 \leq \gamma < v$ and $\gamma \neq \frac{v}{2}$, there is a unique Schur-best design among the E-best designs, and when $\gamma = \frac{v}{2}$ three classes of designs, *ECD*$(\bar{\theta})$s, Case II, and Case V, have identical eigenvalues and are Schur-best. The (E,S)-optimality breakdown is shown in table 3.32.

Table 3.32: (E,S)- and Schur-optimal Designs In $D(v, 4; k_1, k_2)$



3.4.3 Schur-Optimality in $D(v, 4; k_1, k_2)$

From corollary 3.4.1, we know that *ECD*$(\bar{\theta})$s are Schur-optimal when $0 \leq \gamma \leq \frac{v}{4}$ and *ECD*$(\bar{\theta}+1)$s are Schur-optimal when $\frac{3v}{4} < \gamma < v$, and from Theorem 2.3.3, we know *ECD*$(\bar{\theta})$s are type-1 optimal when $\frac{v}{4} < \gamma \leq \frac{v}{2}$. Now we will focus our attention on A-optimality, and along the way, establish some Schur-orderings. Before fully restricting to A-optimality in section 3.4.4, we will use the explicit expressions for the eigenvalues of the *ECD*s and the five remaining cases of *NECD* competitors to identify subregions of $\frac{v}{2} < \gamma \leq \frac{3v}{4}$ on which various cases are Schur-inferior to other cases. In essence, we will use the eigenvalue expressions to obtain a more accurate version of table 3.31. Recall that Case V and VII designs are Schur-inferior to Case II designs and $ECD(\bar{\theta}+1)$s, respectively, and we do not know the eigenvalues for Cases III and VIII.

Lemma 3.4.9 *When $\frac{v}{2} < \gamma < v$, $ECD(\bar{\theta}+1)$s are Schur-better than Case IV designs, and when $\gamma = \frac{v}{2}$, $ECD(\bar{\theta}+1)$s and Case IV designs have identical eigenvalues.*

Proof In order for the eigenvalues of Case IV design to majorize the eigenvalues of $ECD(\bar{\theta}+1)$s, it is necessary and sufficient for the largest Case IV eigenvalue, which is $e_3^{(4)} = p + (v-\gamma) + \sqrt{(v-\gamma)^2 + 3\gamma^2}$ when $\frac{v}{2} \leq \gamma < v$, to be greater than or equal to the largest $ECD(\bar{\theta}+1)$ eigenvalue $\xi_2(\gamma - v) = p + 3(v-\gamma)$, which is true if and only if $\gamma \geq \frac{v}{2}$. When $\gamma = \frac{v}{2}$, $e_3^{(4)} = \xi_2(\gamma - v)$, and, since the second and third largest eigenvalues of Case IV designs are identical to the three smallest eigenvalues of $ECD(\bar{\theta}+1)$s, Case IV and $ECD(\bar{\theta}+1)$s have identical eigenvalues. □

Lemma 3.4.10 *When $\left(\frac{7-\sqrt{5}}{8}\right)v \leq \gamma \leq v$, $ECD(\bar{\theta}+1)$s are Schur-better than Case VI designs.*

Proof The eigenvalues of Case VI designs majorize the eigenvalues of $ECD(\bar{\theta}+1)$s when the largest Case VI eigenvalue is greater than the unique largest $ECD(\bar{\theta}+1)$ eigenvalue $\xi_2(\gamma - v) = p + 3(v-\gamma)$. When $\left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)v \leq \gamma \leq v$, the largest Case VI eigenvalue is $e_1^{(6)} = p - (v-\gamma) + \left(\frac{1+\sqrt{5}}{2}\right)v$, and $e_1^{(6)} \geq \xi_2(\gamma - v)$ if and only if $\left(\frac{7-\sqrt{5}}{8}\right)v \leq \gamma \leq v$. When $\gamma = \left(\frac{7-\sqrt{5}}{8}\right)$, since the four Case VI eigenvalues are unique, the $ECD(\bar{\theta}+1)$ and Case VI eigenvalues are not identical. Therefore, $ECD(\bar{\theta}+1)$s are Schur-better than Case VI eigenvalues when $\left(\frac{7-\sqrt{5}}{8}\right)v \leq \gamma \leq v$. □

Lemma 3.4.11 *When $\frac{v}{2} < \gamma < \frac{3v}{4}$, Case I designs are Schur-better than Case IX designs, and when $\gamma = \frac{v}{2}$, Case I and Case IX designs have identical eigenvalues.*

Proof On the interval $\frac{v}{2} \le \gamma < v$, the ranking of Case IX eigenvalues is consistently $e_3^{(9)} > e_1^{(9)} > e_2^{(9)} > e_4^{(9)}$, and the third largest Case IX and Case I eigenvalues are identically $e_2^{(9)} = e_2^{(1)} = p-(v-\gamma)$. On $\frac{v}{2} \le \gamma \le \frac{3v}{4}$, the largest two Case I eigenvalues are $e_3^{(1)} > e_1^{(1)} = e_1^{(9)}$, and Case IX eigenvalues majorize Case I eigenvalues if and only if $e_3^{(9)} \ge e_3^{(1)}$, if and only if $p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16\gamma^2 + v^2} \ge p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$, if and only if $\gamma \ge \frac{v}{2}$. □

Since it is possible to express in closed form only one of the eigenvalues of the optimality matrices for Case III designs and Case VIII designs, we will derive bounds for their maximum and minimum eigenvalues in order to eliminate them from optimality contention. As usual, let e_1 and e_r be the maximum and minimum eigenvalues of an optimality matrix M_d, respectively. Then

$$e_1 = \max_{\mathbf{x}^T\mathbf{x}=1} = \mathbf{x}^T M_d \mathbf{x} \ge \mathbf{x}^{*T} M_d \mathbf{x}^* \tag{3.81}$$

and

$$e_r = \min_{\mathbf{x}^T\mathbf{x}=1} \mathbf{x}^T M_d \mathbf{x} \le \mathbf{x}^{*T} M_d \mathbf{x}^* \tag{3.82}$$

where $\mathbf{x}^*$ is any fixed, normalized vector. Typically we take, for a fixed value of $\gamma = \gamma^*$, $\mathbf{x}^*$ to be an eigenvector of $M_d^* = (p+\gamma^*)I - \gamma^* J + v\Delta_d$. Bounds obtained using this procedure are used in the next two lemmas.

Lemma 3.4.12 *When $\frac{v}{2} < \gamma \le \frac{2v}{3}$, Case I designs are Schur-better than Case III designs*

Proof Since the known eigenvalue of the optimality matrix for Case III designs $e^{(3)} = p - (v-\gamma)$, is identical to one of the eigenvalues of the optimality matrix for Case I designs, $e_1^{(1)}$, we need to show that the remaining three Case III eigenvalues majorize the remaining three Case I eigenvalues on the interval. When $\frac{v}{2} < \gamma \le \frac{2v}{3}$, the maximum Case I eigenvalue is $e_3^{(1)} = p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$ and the minimum Case I eigenvalue is $e_4^{(1)} = p + (v-\gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$. If the maximum and minimum Case III eigenvalues are $e_1^{(3)}$ and $e_4^{(3)}$, respectively, then, since there are only three eigenvalues per case to compare, Case III eigenvalues majorize Case I eigenvalues when $e_1^{(3)} \geq e_3^{(1)}$ and $e_4^{(3)} \leq e_4^{(1)}$.

When $\gamma^* = \frac{v}{2}$,

$$M_3^* = \begin{pmatrix} p & -\gamma & -\gamma & -\gamma+v \\ -\gamma & p & -\gamma+v & -\gamma+v \\ -\gamma & -\gamma+v & p & -\gamma+v \\ -\gamma+v & -\gamma+v & -\gamma+v & p \end{pmatrix},$$

and the normalized eigenvectors of M_3^* are

$$\begin{aligned} \mathbf{x}_1^* &= \frac{1}{2\sqrt{5-\sqrt{5}}}(1-\sqrt{5}, 2, 2, -1+\sqrt{5})^T \\ \mathbf{x}_2^* &= \frac{1}{2\sqrt{5+\sqrt{5}}}(1+\sqrt{5}, 2, 2, -1-\sqrt{5})^T \\ \mathbf{x}_3^* &= \frac{1}{\sqrt{2}}(0, -1, 1, 0)^T \\ \mathbf{x}_4^* &= \frac{1}{\sqrt{2}}(1, 0, 0, 1)^T. \end{aligned}$$

Now, if $X^* = (\mathbf{x}_1^* \mid \mathbf{x}_2^* \mid \mathbf{x}_3^* \mid \mathbf{x}_4^*)$, then

$$X^{*T} M_3 X^* = \begin{pmatrix} \mathbf{x}_1^{*T} M_3 \mathbf{x}_1^* \\ \mathbf{x}_2^{*T} M_3 \mathbf{x}_2^* \\ \mathbf{x}_3^{*T} M_3 \mathbf{x}_3^* \\ \mathbf{x}_4^{*T} M_3 \mathbf{x}_4^* \end{pmatrix} = \begin{pmatrix} p + \frac{\sqrt{5}}{5}(3v-\gamma) \\ p - \frac{\sqrt{5}}{5}(3v-\gamma) \\ p-(v-\gamma) \\ p+(v-\gamma) \end{pmatrix}. \tag{3.83}$$

The first two components of the vector on the right had side of (3.83) serve as the bounds defined by (3.81) and (3.82), respectively, that is, $e_1^{(3)} \geq p + \frac{\sqrt{5}}{5}(3v-\gamma)$ and $e_4^{(3)} \leq p - \frac{\sqrt{5}}{5}(3v-\gamma)$ for all $\frac{v}{2} < \gamma \leq \frac{3v}{4}$. Since

$$p + \frac{\sqrt{5}}{5}(3v-\gamma) \geq e_3^{(1)}$$

and

$$p - \frac{\sqrt{5}}{5}(3v-\gamma) \leq e_4^{(1)},$$

Case I is Schur-better than Case III on the interval. □

Lemma 3.4.13 *When $\frac{v}{2} < \gamma \leq \frac{2v}{3}$ Case I designs are Schur-better than Case VIII designs*

Proof Since the known eigenvalue of the optimality matrix for Case VIII designs $e^{(8)} = p + \gamma$, is identical to one of the eigenvalues of the optimality matrix for Case I designs, $e_2^{(1)}$, we need to show that the remaining three Case VIII eigenvalues majorize the remaining three Case I eigenvalues on the interval. When $\frac{v}{2} < \gamma \leq \frac{2v}{3}$, the maximum Case I eigenvalue is $e_3^{(1)} = p + (v-\gamma) - \frac{v}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$ and the minimum Case I eigenvalue is $e_4^{(1)} = p + (v-\gamma) - \frac{v}{2} - \frac{1}{2}\sqrt{16(v-\gamma)^2 + v^2}$. If the maximum and minimum Case VIII eigenvalues are $e_1^{(8)}$ and $e_4^{(8)}$, respectively, then, since there are only three eigenvalues per case to compare, Case VIII eigenvalues majorize Case I eigenvalues when $e_1^{(8)} \geq e_3^{(1)}$ and $e_4^{(8)} \leq e_4^{(1)}$. When $\gamma^* = \frac{v}{2}$,

$$M_8^* = \begin{pmatrix} p & -\gamma & -\gamma & -\gamma \\ -\gamma & p & -\gamma & -\gamma+v \\ -\gamma & -\gamma & p & -\gamma+v \\ -\gamma & -\gamma+v & -\gamma+v & p \end{pmatrix},$$

and the normalized eigenvectors of M_8^* are

$$\begin{aligned} \mathbf{x}_1^* &= \frac{1}{2\sqrt{5-\sqrt{5}}}\,(-2, -1+\sqrt{5}, -1+\sqrt{5}, 2)^T \\ \mathbf{x}_2^* &= \frac{1}{2\sqrt{5+\sqrt{5}}}\,(-2, -1-\sqrt{5}, -1-\sqrt{5}, 2)^T \\ \mathbf{x}_3^* &= \frac{1}{\sqrt{2}}\,(0, -1, 1, 0)^T \\ \mathbf{x}_4^* &= \frac{1}{\sqrt{2}}\,(1, 0, 0, 1)^T. \end{aligned}$$

Now, if $X^* = (\mathbf{x}_1^* \mid \mathbf{x}_2^* \mid \mathbf{x}_3^* \mid \mathbf{x}_4^*)$, then

$$X^{*^T} M_8 X^* = \begin{pmatrix} \mathbf{x}_1^{*T} M_8 \mathbf{x}_1^* \\ \mathbf{x}_2^{*T} M_8 \mathbf{x}_2^* \\ \mathbf{x}_3^{*T} M_8 \mathbf{x}_3^* \\ \mathbf{x}_4^{*T} M_8 \mathbf{x}_4^* \end{pmatrix} = \begin{pmatrix} p + \frac{\sqrt{5}}{5}(2v+\gamma) \\ p - \frac{\sqrt{5}}{5}(2v+\gamma) \\ p+\gamma \\ p-\gamma \end{pmatrix}. \tag{3.84}$$

The first two components of the vector on the right hand side of (3.84) will serve as the bounds defined by (3.81) and (3.82), respectively, that is, $e_1^{(8)} \geq p + \frac{\sqrt{5}}{5}(2v+\gamma)$

and $e_4^{(8)} \leq p - \frac{\sqrt{5}}{5}(2v+\gamma)$ for all $\frac{v}{2} < \gamma \leq \frac{3v}{4}$. Since

$$p + \frac{\sqrt{5}}{5}(2v+\gamma) \geq e_3^{(1)}$$

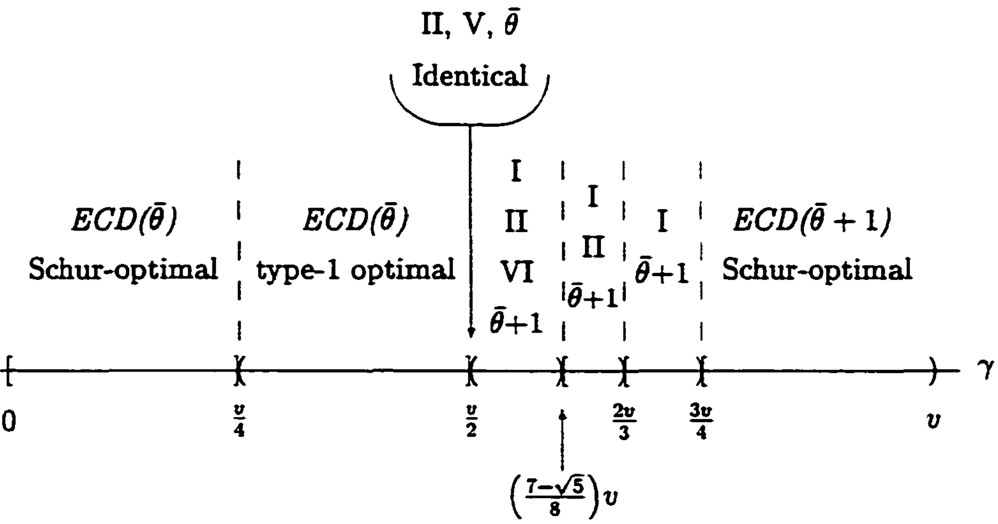
and

$$p - \frac{\sqrt{5}}{5}(2v+\gamma) \leq e_4^{(1)},$$

Case I is Schur-better than Case VIII on the interval. □

The results of the majorization analysis are summarized in table 3.33 in which, for subintervals of $0 \leq \gamma < v$, the cases not ruled out by majorization are listed. For example, when $\frac{v}{2} < \gamma \leq \frac{(7-\sqrt{5})}{8}v$, the A-best design is either an *ECD*($\bar{\theta}+1$), Case I, II, or VI design.

Table 3.33: Remaining Optimality Candidates in $D(v, 4; k_1, k_2)$



3.4.4 A-optimality in $D(v, 4; k_1, k_2)$

Now that we have eliminated as many designs as possible using majorization, eigenvalue optimality investigations must focus on specific functions of the eigenvalues of the information matrices for the remaining design competitors. In this section, we will find the A-optimal design(s) in $D(v, 4; k_1, k_2)$ on the interval $\frac{v}{2} < \gamma < \frac{3v}{4}$ by directly comparing the A-values of the designs that were not eliminated by majorization on the subinterval.

There are four classes of designs, *ECD*($\bar{\theta}+1$)*s*, and three classes of *NECDs*, that can potentially be optimal on the interval. Each class along with the interval on which the designs in the class are optimality competitors and the eigenvalues of the information matrix for the designs are listed below.

ECD($\bar{\theta}+1$): $\frac{v}{2} < \gamma < v$,

$$\begin{aligned} z_1^{(\bar{\theta}+1)} &= \frac{3p+(v-\gamma)}{p} \quad \text{(3 copies)} \\ z_2^{(\bar{\theta}+1)} &= \frac{3[p-(v-\gamma)]}{p} \end{aligned}$$

Case I: $\frac{v}{2} < \gamma \leq \frac{3v}{4}$,

$$\begin{aligned} z_1^{(1)} &= \frac{3p+(v-\gamma)}{p} \\ z_2^{(1)} &= \frac{3p-\gamma}{p} \\ z_3^{(1)} &= \frac{6p-2(v-\gamma)+v-\sqrt{16(v-\gamma)^2+v^2}}{2p} \\ z_4^{(1)} &= \frac{6p-(v-\gamma)+v+\sqrt{16(v-\gamma)^2+v^2}}{2p} \end{aligned}$$

Case II: $\frac{v}{2} \leq \gamma \leq \frac{2v}{3}$,

$$\begin{aligned} z_1^{(2)} &= \frac{3p-\gamma}{p} \\ z_2^{(2)} &= \frac{3p-\gamma}{p} \\ z_3^{(2)} &= \frac{3p-3(v-\gamma)+v}{p} \\ z_4^{(2)} &= \frac{3p-\gamma+2v}{p} \end{aligned}$$

Case VI: $\frac{v}{2} < \gamma \leq \left(\frac{7-\sqrt{5}}{8}\right)v$,

$$z_1^{(6)} = \frac{6p+2(v-\gamma)-(1+\sqrt{5})v}{2p}$$

$$\begin{aligned} z_2^{(6)} &= \frac{6p + 2(v-\gamma) - (1-\sqrt{5})v}{2p} \\ z_3^{(6)} &= \frac{6p - 2(v-\gamma) + v - \sqrt{4(v-2\gamma)^2 + v^2}}{2p} \\ z_4^{(6)} &= \frac{6p - 2(v-\gamma) + v + \sqrt{4(v-2\gamma)^2 + v^2}}{2p} \end{aligned}$$

Now we will make A-value comparisons for the competitors on $\frac{v}{2} < \gamma < \frac{3v}{4}$.

Lemma 3.4.14 *When $\frac{v}{2} < \gamma \leq \left(\frac{7-\sqrt{5}}{8}\right) v$, Case II designs are A-better than Case VI designs.*

Proof Case II designs are A-better than Case VI designs if and only if

$$\frac{1}{z_1^{(2)}} + \frac{1}{z_2^{(2)}} + \frac{1}{z_3^{(2)}} + \frac{1}{z_4^{(2)}} \leq \frac{1}{z_1^{(6)}} + \frac{1}{z_2^{(6)}} + \frac{1}{z_3^{(6)}} + \frac{1}{z_4^{(6)}}$$

if and only if

$$\begin{aligned} &6(2p+v)\gamma^4 + 12v^2(3p+2v)\gamma^2 + 18p^2(9p^2+2v^2)\gamma + 9p^2(18p^2+15pv+4v^2) - \\ &\quad [2\gamma^5 + v(16v+27p)\gamma^3 + 9p^2(12p+v)\gamma^2 + 2v^3(9p+8v)\gamma + \\ &\quad 9p^2v(9p^2+2v^2)] \geq 0 \end{aligned} \tag{3.85}$$

A lower bound for the left hand side of (3.85) on $\frac{v}{2} \leq \gamma \leq \left(\frac{7-\sqrt{5}}{8}\right) v$ can be obtained by substituting $\gamma = \frac{v}{2}$ into the positive terms and $\gamma = \left(\frac{7-\sqrt{5}}{8}\right) v$ into the negative terms. Doing so yields

$$\begin{aligned} &v^3[-24192(1-\sqrt{5})\left(\frac{p}{v}\right)^3 - \\ &\quad 288(27-7\sqrt{5})\left(\frac{p}{v}\right)^2 - 16(1896-657\sqrt{5})\left(\frac{p}{v}\right) - (20225-7817\sqrt{5})]. \end{aligned} \tag{3.86}$$

If we can show that the lower bound (3.86) is greater than zero when $\frac{p}{v} \geq x$ for some real number $x < 1$ then the result follows from corollary (3.1.2). Consider the function

$$f(x) = -24192(1-\sqrt{5})x^3 - 288(27-7\sqrt{5})x^2 - 16(1896-657\sqrt{5})x - (20225-7817\sqrt{5}).$$

Clearly the lower bound (3.86) is greater than zero for all values of $\frac{p}{v} = x$ for which $f(x) > 0$. The derivative of $f(x)$ is

$$f'(x) = -72576(1-\sqrt{5})x^2 - 576(27-7\sqrt{5})x - 16(1896-657\sqrt{5}),$$

and $f'(x) = 0$ if and only if

$$x = \frac{-9\left[(27-7\sqrt{5}) \mp 2\sqrt{-17890+8841\sqrt{5}}\right]}{2268(1-\sqrt{5})}.$$

Since

$$-\frac{1}{2} < \frac{-9\left[(27-7\sqrt{5}) - 2\sqrt{-17890+8841\sqrt{5}}\right]}{2268(1-\sqrt{5})} < 0$$
$$< \frac{-9\left[(27-7\sqrt{5}) + 2\sqrt{-17890+8841\sqrt{5}}\right]}{2268(1-\sqrt{5})} < \frac{2}{3},$$

$f'\left(\frac{2}{3}\right) > 0$, and $f\left(\frac{2}{3}\right) > 0$, then $f(x) > 0$ for all $x \geq \frac{2}{3}$. Therefore, Case II designs are A-better than Case VI designs on $\frac{v}{2} \leq \gamma \leq \left(\frac{7-\sqrt{5}}{8}\right)v$. □

Lemma 3.4.15 *When $\frac{3v}{5} \leq \gamma \leq \frac{3v}{4}$ ECD($\bar{\theta}+1$)s are A-better than Case I designs, and when $\frac{v}{2} \leq \gamma < \frac{3v}{5}$, ECD($\bar{\theta}+1$)s are A-better than Case I designs provided*

$$-2\gamma^3 + 10v\gamma^2 + (18p^2 + 9pv - 14v^2)\gamma - 3v(3p^2 - 3pv - 2v^2) \geq 0 \qquad (3.87)$$

Proof *ECD($\bar{\theta}+1$)s* are A-better than Case I designs if and only if

$$\frac{3}{z_1^{(\bar{\theta}+1)}} + \frac{1}{z_2^{(\bar{\theta}+1)}} \leq \frac{1}{z_1^{(1)}} + \frac{1}{z_2^{(1)}} + \frac{1}{z_3^{(1)}} + \frac{1}{z_4^{(1)}}$$

if and only if

$$-2\gamma^3 + 10v\gamma^2 + (18p^2 + 9pv - 14v^2)\gamma - 3v(3p^2 + 3pv - 2v^2) \geq 0,$$

which is (3.87). On $\frac{3v}{5} \leq \gamma \leq \frac{3v}{4}$, the left hand side of (3.87) is bounded below by

$$\frac{9}{160v}[32\left(\frac{p}{v}\right)^2 - 64\left(\frac{p}{v}\right) - 31], \qquad (3.88)$$

which is obtained by substituting $\gamma = \frac{3v}{5}$ into the positive terms and $\gamma = \frac{3v}{4}$ into the negative terms. Setting the bound (3.88) equal to zero and solving for $\frac{p}{v}$ yields

$$\frac{p}{v} = \frac{8 \mp 3\sqrt{14}}{8}.$$

Since

$$-\frac{1}{2} < \frac{8 - 3\sqrt{14}}{8} < 0 < \frac{8 + 3\sqrt{14}}{8} < \frac{5}{2},$$

and (3.88) is greater than zero when $\frac{p}{v} \geq \frac{5}{2}$, (3.87) is satisfied on $\frac{3v}{5} \leq \gamma \leq \frac{3v}{4}$ when $\frac{p}{v} \geq \frac{5}{2}$, and, by corollary 3.1.4, this inequality holds if $k_1 \geq k_2 \geq 5$, $k_2 = 4$ and $k_1 \geq 7$, or $k_2 = 3$ and $k_1 \geq 15$. Thus (3.87) may not be satisfied when $k_1 \geq k_2 = 2$, $14 \geq k_1 \geq k_2 = 3$, or $6 \geq k_1 \geq k_2 = 4$. By lemma 3.1.2, when $\frac{3v}{5} \leq \gamma \leq \frac{3v}{4}$, $k_2 = 2$ if and only if $k_1 = 4$, $14 \geq k_1 \geq k_2 = 3$ if and only if $k_1 = 9, 10, 11$, or 12, and $6 \geq k_1 \geq k_2 = 4$ if and only if $k_1 = 6$. Since condition (3.87) is satisfied when $(k_1, k_2) = (4,2), (9,3), (10,3), (11,3), (12,3)$, and (6,4), then *ECD*($\bar{\theta}+1$)s are A-better than Case I designs on the interval. □

Lemma 3.4.16 *When $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$ ECD($\bar{\theta}+1$)s are A-better than Case II designs. When $\frac{v}{2} \leq \gamma \leq \frac{3v}{5}$, ECD($\bar{\theta}+1$)s are A-better than Case II designs provided*

$$-2\gamma^3 + 12v\gamma^2 + (18p^2 + 15pv - 16v^2)\gamma - 3v(3p^2 + 4pv - 2v^2) \geq 0. \tag{3.89}$$

Proof *ECD*($\bar{\theta}+1$)s are A-better than Case II designs if and only if

$$\frac{3}{z_1^{(\bar{\theta}+1)}} + \frac{1}{z_2^{(\bar{\theta}+1)}} \leq \frac{1}{z_1^{(2)}} + \frac{1}{z_2^{(2)}} + \frac{1}{z_3^{(2)}} + \frac{1}{z_4^{(2)}}$$

if and only if

$$-2\gamma^3 + 12v\gamma^2 + (18p^2 + 15pv - 16v^2)\gamma - 3v(3p^2 + 4pv - 2v^2) \geq 0$$

which is (3.89). On $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$, the left hand side of (3.89) is bounded from below by

$$\frac{1}{675v}[1215\left(\frac{p}{v}\right)^2 - 2025\left(\frac{p}{v}\right) - 634], \tag{3.90}$$

which results from substituting $\gamma = \frac{2v}{5}$ into the positive terms and $\gamma = \frac{2v}{3}$ into the negative terms. Since the bound (3.90) is equal to zero if and only if

$$\frac{p}{v} = \frac{225 \mp \sqrt{88665}}{270},$$

$$\frac{3}{10} < \frac{225 - \sqrt{88665}}{270} < 0 < \frac{225 + \sqrt{88665}}{270} < 1.95,$$

and (3.90) is greater than zero when $\frac{p}{v} = 2$, then (3.89) is satisfied on $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$ when $\frac{p}{v} \geq 2$. By fact 3.1.3, this inequality holds when $k_1 \geq k_2 \geq 4$ or $k_2 = 3$ and $k_1 \geq 6$. Thus, (3.89) may not be satisfied when $k_2 \geq k_1 = 2$ or $5 \geq k_1 \geq k_2 = 3$. On $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$, (k_1, k_2) does not take on the values (3,3), (4,3), or (5,3), and by corollary 3.1.5, $k_2 = 2$ if and only if $k_1 = 4$. Since (3.89) is satisfied when $(k_1, k_2) = (4, 2)$, then *ECD*$(\bar{\theta} + 1)$s are A-better than Case II designs on the interval. □

Lemma 3.4.17 *When* $\frac{3v}{5} \leq \gamma \leq \frac{2v}{3}$, *Case I designs are A-better than Case II designs, and when* $\frac{v}{2} \leq \gamma < \frac{3v}{5}$, *Case I designs are A-better than Case II designs provided*

$$2\gamma^4 - 2(3p + 8v)\gamma^3 - (18p^2 - 21pv - 34v^2)\gamma^2 + \\ 2(27p^3 + 45p^2v - 6pv^2 - 14v^3)\gamma - v(27p^3 + 54p^2v - 8v^3) \geq 0. \tag{3.91}$$

Proof Case I designs are A-better than Case II designs if and only if

$$\frac{1}{z_1^{(1)}} + \frac{1}{z_2^{(1)}} + \frac{1}{z_3^{(1)}} + \frac{1}{z_4^{(1)}} \leq \frac{1}{z_1^{(2)}} + \frac{1}{z_2^{(2)}} + \frac{1}{z_3^{(2)}} + \frac{1}{z_4^{(2)}}$$

if and only if

$$2\gamma^4 - 2(3p + 8v)\gamma^3 - (18p^2 - 21pv - 34v^2)\gamma^2 + \\ 2(27p^3 + 45p^2v - 6pv^2 - 14v^3)\gamma - v(27p^3 + 54p^2v - 8v^3) \geq 0.$$

which is (3.91). On $\frac{3v}{5} \leq \frac{2v}{3}$ (3.91) is bounded from below by

$$\frac{1}{16875v^3}[91125\left(\frac{p}{v}\right)^3 - 135000\left(\frac{p}{v}\right)^2 - 37425\left(\frac{p}{v}\right) - 49076] \tag{3.92}$$

which results from substituting $\gamma = \frac{3v}{5}$ into the positive terms and $\gamma = \frac{2v}{3}$ into the negative terms. We will now show that the bound (3.92) is greater than zero on $\frac{3v}{5} \le \gamma \le \frac{2v}{3}$ when $\frac{p}{v} \ge 2$ by using the function

$$f(x) = 91125x^3 - 135000x^2 - 37425x - 49076.$$

since the bound is greater than or equal to zero when $\frac{p}{v} = x$ for values of x such that $f(x) \ge 0$. Since $f'(x) = 0$ if and only if

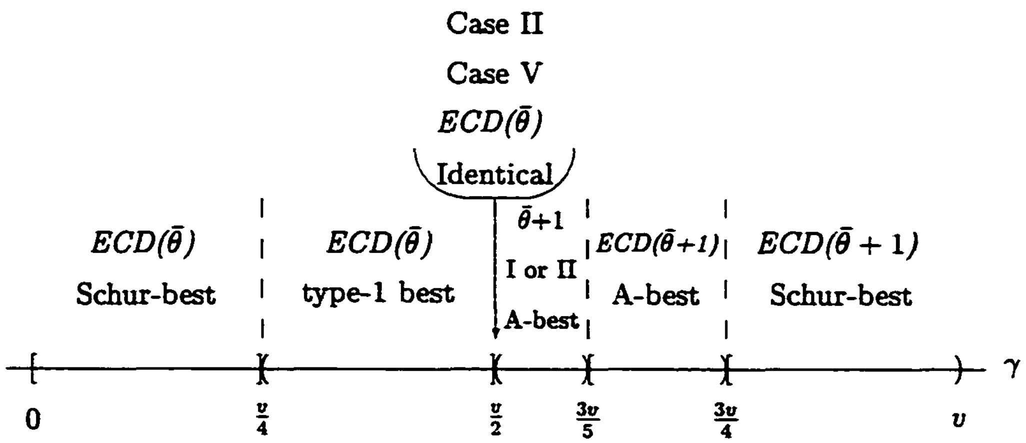
$$x = \frac{400 \mp \sqrt{62455}}{405},$$

$$-1.2 < \frac{400 - \sqrt{62455}}{405} < 0 < \frac{400 + \sqrt{62455}}{405} < 1.3,$$

and $f(2) = 65074$, then (3.92) is greater than zero and (3.91) is satisfied when $\frac{p}{v} \ge 2$. From the proof of lemma 3.4.16 we know the only pair (k_1, k_2) for which $\frac{p}{v} \not\ge 2$ on $\frac{3v}{4} \le \gamma \le \frac{2v}{3}$ is (4,2), and it is easy to see that (3.91) is satisfied when $(k_1, k_2) = (4, 2)$. Therefore, Case I designs are A-better than Case II designs on $\frac{3v}{5} \le \gamma \le \frac{2v}{3}$. □

A summary of the A-best analysis is given in table 3.34 below.

Table 3.34: A-, Type-1, and Schur-optimal Designs In $D(v, 4; k_1, k_2)$



Note that the A-best design is uniquely an $ECD(\bar{\theta})$ when $0 \le \gamma < \frac{v}{2}$ and uniquely an $ECD(\bar{\theta}+1)$ when $\frac{3v}{4} \le \gamma < v$. When $\gamma = \frac{v}{2}$ the eigenvalues for $ECD(\bar{\theta})$s, Case II and Case V designs are identical, and the same designs are A-best. On the interval $\frac{v}{2} < \gamma < \frac{3v}{5}$, the A-best design can be either an *ECD($\bar{\theta}$ + 1)*, a Case I, or a Case II design; conditions (3.87), (3.89), and (3.91) must be checked in order to determine the A-best design. For $10,000 \geq k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$ with $\frac{v}{2} < \gamma < \frac{3v}{5}$ the designs were ranked by their A-value with the following results:

Table 3.35: A-optimal Design Counts In $D(v, 4; k_1, k_2)$ When $\frac{v}{2} < \gamma \leq \frac{3v}{5}$

A-optimal	interval	count
ECD(θ + 1)	$.5v < \gamma < .60v$	5,027,032
Case I	$.5v < \gamma \leq .53v$	77
Case II	$.5v < \gamma \leq .57v$	18,034

Case II A-optimal, *ECD($\bar{\theta}$ + 1)* second best

k_1	k_2	$\frac{\gamma}{v}$	*ECD($\bar{\theta}$ + 1)* A-value	Case I A-value	Case II A-value
332	41	0.5067	1.33341528681224	1.33341528616835	1.33341529455844
615	61	0.5044	1.33336898681649	1.33336898657864	1.33336898804561
1026	85	0.5032	1.33335121489144	1.33335121481050	1.33335121515767
1589	113	0.5024	1.33334325422629	1.33334325419645	1.33334325429825
2328	145	0.5018	1.33333926800729	1.33333926799523	1.33333926803025
3267	181	0.5015	1.33333709653531	1.33333709653001	1.33333709654363
4430	221	0.5012	1.33333583303917	1.33333583303667	1.33333583304252
5841	265	0.5010	1.33333505784887	1.33333505784761	1.33333505785033
7524	313	0.5008	1.33333456108502	1.33333456108436	1.33333456108571
9503	365	0.5007	1.33333423094228	1.33333423094191	1.33333423094262

			Case I A-optimal, Case II second best		
				A-value	
k_1	k_2	$\frac{\gamma}{v}$	$ECD(\bar{\theta}+1)$	Case I	Case II
85	7	0.5326	1.33577654097824	1.33576851757101	1.33576872031938
113	8	0.5289	1.33518039792047	1.33517611041577	1.33517651436696
145	9	0.5260	1.33477950084018	1.33477701479240	1.33477739857049
181	10	0.5236	1.33449673022044	1.33449519480483	1.33449551298767
221	11	0.5216	1.33428971292312	1.33428871579687	1.33428896905532
265	12	0.5199	1.33413355144468	1.33413287687614	1.33413307629157
313	13	0.5184	1.33401281884187	1.33401234675116	1.33401250391211
365	14	0.5172	1.33391753469016	1.33391719472945	1.33391731931784
421	15	0.5161	1.33384100659906	1.33384075574521	1.33384085529994
481	16	0.5151	1.33377860909054	1.33377842004996	1.33377850029529
545	17	0.5142	1.33372706130054	1.33372691620282	1.33372698145427
613	18	0.5135	1.33368398324226	1.33368387006051	1.33368392357517
685	19	0.5128	1.33364761415738	1.33364752459940	1.33364756884820
761	20	0.5122	1.33361662859149	1.33361655681632	1.33361659368708
841	21	0.5116	1.33359001324405	1.33358995505806	1.33358998600468
925	22	0.5111	1.33356698266158	1.33356693500184	1.33356696115336

3.4.5 Special Cases: $(k_1 - k_2) \leq 2$

We now apply the optimality results from sections 3.4.2 and 3.4.4 to the three special cases described in section 2.4.

Corollary 3.4.18 *Suppose $k_1 = k_2$ and $r = 4$. Then*

(i) If $2 \mid k_1$ then $\gamma = 0$, and ECD(θ^*)s exist and are Schur-optimal.

(ii) If $2 \nmid k_1$ then $\gamma = \frac{v}{2}$, and ECD($\bar{\theta}$)s, Case II, and V are identical and $(E, S)-$ and type-1

Corollary 3.4.19 *Suppose $k_2 = k_1 - 1$ and $r = 4$. Then*

(i) If $2 \mid k_1$ then $\frac{v}{4} < \gamma < \frac{v}{3}$, and ECD($\bar{\theta}$)s are (E,S)- and type-1 optimal.

(ii) If $2 \nmid k_1$ then $\frac{3v}{4} < \gamma < \frac{4v}{5}$, and ECD($\bar{\theta}+1$)s are Schur-optimal.

Corollary 3.4.20 *Suppose $k_2 = k_1 - 2$ and $r = 4$. Then*

(i) If $k_1 = 4$ then $\gamma = \frac{2v}{3}$, Case I designs are (E,S)-optimal, and ECD($\bar{\theta} + 1$)s are A-optimal.

(ii) If $k_1 = 6$ then $\gamma = \frac{3v}{5}$, Case II designs are (E,S)-optimal, and ECD($\bar{\theta} + 1$)s are A-optimal.

(iii) If $2 \mid k_1$ and $k_1 \geq 8$ then $\frac{v}{2} < \gamma < \frac{3v}{5}$, Case II designs are (E,S)-optimal, and either an ECD($\bar{\theta} + 1$) , a Case I, or a Case II design is A-optimal.

(iv) If $2 \nmid k_1$ then $0 < \gamma < \frac{v}{6}$, and ECD($\bar{\theta}$)s are Schur-optimal.

3.4.6 Construction of Optimal Designs in $D(v, 4; k_1, k_2)$

The A-, (E,S)-, type-1, and Schur-optimal resolvable designs in $D(v, 4; k_1, k_2)$, $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$, are *ECD($\bar{\theta}$)* , *ECD($\bar{\theta}$+1)* , Case I, and Case II designs depending on the value of $0 \leq \gamma < v$. In particular, when $0 \leq \gamma < \frac{v}{2}$, *ECD($\bar{\theta}$)*s are type-1 optimal; when $\gamma = \frac{v}{2}$, *ECD($\bar{\theta}$)*s, Case II, and Case V designs are type-1 equivalent and type-1 optimal; when $\frac{3v}{4} \leq \gamma < v$, *ECD($\bar{\theta}$ + 1)*s are Schur-optimal; and when $\frac{v}{2} < \gamma < \frac{3v}{4}$, A- and (E,S)-optimal designs are *ECD($\bar{\theta}$ + 1)*s, Case I designs, and Case II designs. Furthermore, in the previous section we determined that, when $k_1 - k_2 \leq 1$, A-, (E,S)-, and Schur-optimal designs are *ECD*s and when $k_1 - k_2 = 2$, A- and (E,S)-optimal designs can be *ECD*, Case I, and Case II designs. However, we have yet to address the question of if and when the theoretically optimal designs exist, and if they do, provide a means for finding the optimal design. In this section we will determine constructions for *ECD*s, Case I, and Case II designs. The constructions for *ECD*s will be described in such a way that they will be valid for *ECD(θ^*)*s, *ECD($\bar{\theta}$)*s and *ECD($\bar{\theta}$ + 1)*s.

Now we are ready to provide constructions for the first block of each replicate for values of k_1 in the interval given by (3.74).

Construction of $ECD(\bar{\theta})$s

Let L be the common ECD treatment concurrence. Then for $ECD(\bar{\theta})$s, $L = \bar{\theta}$, and for $ECD(\bar{\theta}+1)$s, $L = \bar{\theta}+1$. Not that when $k_1 \geq 2L$, then by 3.74, $L = \bar{\theta}$.

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots L\} \cup \{k_1+1 \ldots 2k_1 - L\}$

Block 1 of Replicate 3:

(i) $k_1 < 2L$ or ($k_1 = 2L$ and L even):

$\{1 \ldots \overset{\text{if } k_1<2L}{2L - k_1}\} \cup \{L+1 \ldots 2k_1 - L\}$

(ii) $k_1 = 2L$ and L odd:

$\{1 \ldots \frac{L-1}{2}\} \cup \{L+1 \ldots \frac{3L+1}{2}\} \cup \{k_1+1 \ldots k_1 + \frac{L+1}{2}\} \cup$
$\{2k_1+1-L \ldots 2k_1 - \frac{L+1}{2}\}$

(iii) $k_1 = 2L+1$:

$\{L+1 \ldots 2L\} \cup \{k_1+1 \ldots k_1+L\} \cup \{2k_1+1-L \ldots 3(k_1-L)\}$

Block 1 of Replicate 4:

(i) $L+1 \leq k_1 \leq \frac{3}{2}L$

$\{1 \ldots \overset{\text{if } 2k_1<3L}{3L - 2k_1}\} \cup \{2L+1-k_1 \ldots 2k_1 - L\}$

(ii) $\frac{3}{2}L < k_1 \leq 2L$ and L even:

$\{2L+1-k_1 \ldots \frac{5}{2}L - k_1\} \cup \{L+1 \ldots \frac{3}{2}L\} \cup$
$\{k_1+1 \ldots k_1 + \frac{L}{2}\} \cup \{2k_1+1-L \ldots 3k_1 - \frac{5}{2}L\}$

(iii) $\frac{3}{2}L < k_1 < 2L$ and L odd:

$\{1\} \cup \{2L+1-k_1 \ldots \frac{5L-1}{2} - k_1\} \cup \{L+1 \ldots \frac{3L-1}{2}\} \cup$
$\{k_1+1 \ldots k_1 + \frac{L-1}{2}\} \cup \{2k_1+1-L \ldots 3k_1 - \frac{5L-1}{2}\}$

(iv) $k_1 = 2L$ and L odd:

$\{1\} \cup \{L+1 \ldots \frac{3L-1}{2}\} \cup \{k_1 - \frac{L-3}{2} \ldots k_1\} \cup$
$\{k_1+1 \ldots k_1 + \frac{L-1}{2}\} \cup \{2k_1 - \frac{3}{2}(L-1) \ldots 2k_1 - L\} \cup$
$\{3k_1 - \frac{5L-1}{2}\}$

(v) $2L < k_1 < 3L$:

$\{1 \ldots \overset{\text{if } k_1 \le 3L-2}{1 + \text{int}\left(\frac{3L-k_1-2}{2}\right)}\} \cup \{L+1 \ldots L+1+\text{int}\left(\frac{3L-k_1-1}{2}\right)\} \cup$
$\{2L+1 \ldots k_1\} \cup \{k_1+1 \ldots k_1+1+\text{int}\left(\frac{3L-k_1-1}{2}\right)\} \cup$
$\{k_1+L+1 \ldots 2k_1-L\} \cup \{2k_1+1-L \ldots \overset{\text{if } L-3-2\,\text{int}\left(\frac{3L-k_1-1}{2}\right) \ge 0}{2k_1 - 2 - 2\,\text{int}\left(\frac{3L-k_1-1}{2}\right)}\}$
$\cup \{3(k_1-L)+1 \ldots 3(k_1-L)+1+\text{int}\left(\frac{3L-k_1-1}{2}\right)\}$

(vi) $k_1 = 3L$:

$\{2L+1 \ldots 3L\} \cup \{k_1+L+1 \ldots k_1+2L\} \cup$
$\{2k_1+1-L \ldots 2k_1\}$

Construction of Case I Designs

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}\} \cup \{k_1+1 \ldots 2k_1 - \bar{\theta}\}$

Block 1 of Replicate 3: $\{1 \ldots 2(\bar{\theta}+1) - k_1\} \cup \{\bar{\theta}+1 \ldots k_1 - 1\} \cup$
$\{k_1+1 \ldots 2k_1 - (\bar{\theta}+1)\}$

Block 1 of Replicate 4:

(i) $\bar{\theta}+2 \le k_1 \le \frac{3}{2}(\bar{\theta}+1)$:

$\{1 \ldots 3\bar{\theta}+4-2k_1\} \cup \{\overset{\text{if } k_1 > \bar{\theta}+2}{2\bar{\theta}+3-k_1} \ldots \bar{\theta}\} \cup \{\bar{\theta}+1 \ldots k_1-1\} \cup$
$\{k_1+1 \ldots \overset{\text{if } k_1 > \bar{\theta}+2}{2k_1 - (\bar{\theta}+2)}\} \cup \{2k_1 - \bar{\theta}\}$

(ii) $\frac{3}{2}(\bar{\theta}+1) < k_1 \le 2\bar{\theta}+1$ and $\bar{\theta}$ even

$\{1\} \cup \{\overset{\text{if } \bar{\theta}-2>0}{2\bar{\theta}+3-k_1} \ldots \frac{5}{2}\bar{\theta}+1-k_1\} \cup \{\bar{\theta}+1 \ldots \frac{3}{2}\bar{\theta}\} \cup$
$\{k_1+1 \ldots k_1 + \frac{\bar{\theta}}{2}\} \cup \{k_1\} \cup \{2k_1 - \bar{\theta} \ldots 3k_1 - 2 - \frac{5}{2}\bar{\theta}\}$

(iii) $\frac{3}{2}(\bar{\theta}+1) < k_1 \le 2\bar{\theta}+1$ and $\bar{\theta}$ odd

$$\{\overset{\text{if } \bar{\theta}>1}{2\bar{\theta}+3-k_1} \ldots \tfrac{5\bar{\theta}+3}{2}-k_1\} \cup \{\bar{\theta}+1 \ldots \tfrac{3\bar{\theta}+1}{2}\} \cup \{k_1\} \cup$$
$$\{k_1+1 \ldots k_1+\tfrac{\bar{\theta}+1}{2}\} \cup \{2k_1-\bar{\theta} \ldots 3k_1-\tfrac{5}{2}(\bar{\theta}+1)\}$$

Construction of Case II Designs

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}\} \cup \{k_1+1 \ldots 2k_1-\bar{\theta}\}$

Block 1 of Replicate 3: $\{1 \ldots 2(\bar{\theta}+1)-k_1\} \cup \{\bar{\theta}+1 \ldots k_1-1\} \cup$
$\{k_1+1 \ldots 2k_1-(\bar{\theta}+1)\}$

Block 1 of Replicate 4:

(i) $\bar{\theta}+2 \le k_1 \le \frac{3}{2}(\bar{\theta}+1)$:

$$\{1 \ldots 3\bar{\theta}+4-2k_1\} \cup \{\overset{\text{if } k_1>\bar{\theta}+2}{2\bar{\theta}+3-k_1 \ldots \bar{\theta}}\} \cup \{\overset{\text{if } k_1>\bar{\theta}+2}{\bar{\theta}+1 \ldots k_1-2}\} \cup$$
$$\{k_1\} \cup \{\overset{\text{if } k_1>\bar{\theta}+2}{k_1+1 \ldots 2k_1-(\bar{\theta}+2)}\} \cup \{2k_1-\theta\}$$

(ii) $\frac{3}{2}(\bar{\theta}+1) < k_1 \le 2\bar{\theta}+1$ and $\bar{\theta}$ even

$$\{2\bar{\theta}+3-k_1 \ldots \tfrac{5}{2}\bar{\theta}+2-k_1\} \cup \{\bar{\theta}+1 \ldots \tfrac{3}{2}\bar{\theta}\} \cup \{k_1\} \cup$$
$$\{k_1+1 \ldots k_1+\tfrac{\bar{\theta}}{2}\} \cup \{2k_1-\bar{\theta} \ldots 3k_1-2-\tfrac{5}{2}\bar{\theta}\}$$

(iii) $\frac{3}{2}(\bar{\theta}+1) < k_1 \le 2\bar{\theta}+1$ and $\bar{\theta}$ odd

$$\{2\bar{\theta}+3-k_1 \ldots \tfrac{5}{2}(\bar{\theta}+1)-k_1\} \cup \{\bar{\theta}+1 \ldots \tfrac{3\bar{\theta}+1}{2}\} \cup$$
$$\{\overset{\text{if } \bar{\theta}+1>0}{k_1+1 \ldots k_1+\tfrac{\bar{\theta}-1}{2}}\} \cup \{2k_1-\bar{\theta} \ldots 3k_1-\tfrac{5\bar{\theta}+3}{2}\}$$

3.4.7 Examples of Resolvable Designs in $D(v, 4; k_1, k_2)$

We will now use the constructions of the previous section to provide some examples of resolvable designs in $D(v, 4; k_1, k_2)$ for various interesting $k_1 \ge 3$ and $2 \le k_2 \le k_1$. First we construct designs for the two cases when $k_1 = k_2$.

Example Suppose $k_1 = k_2 = 8$. Then, according to corollary 2.4.2 the the Schur-optimal design is an *ECD(θ^*)*. Applying the *ECD* construction given above with

$L = \bar{\theta} = 4$, and using condition (i) for block 1 of replicarte 3 and condition (ii) for block 1 of replicate 4 yields a Schur-optimal *ECD(θ^*)* which is:

1	9	1	5	5	1	1	3
2	10	2	6	6	2	2	4
3	11	3	7	7	3	5	7
4	12	4	8	8	4	6	8
5	13	9	13	9	13	9	11
6	14	10	14	10	14	10	12
7	15	11	15	11	15	13	15
8	16	12	16	12	16	14	16

Example Consider the case where $k_1 = k_2 = 3$. Then, according to corollary 2.4.2 the (E,S)- and typt-1 optimal design is an *ECD($\bar{\theta}$)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 1$, condition (iii) for block 1 of replicate 3, and condition (vi) for block 1 of replicate 4 produces an (E,S)- and A-optimal resolvable *ECD($\bar{\theta}$)* which is:

1	4	1	2	2	1	3	1
2	5	4	3	4	3	5	2
3	6	5	6	6	5	6	4

Now we investigate the two cases when $k_1 - k_2 = 1$.

Example Consider the setting such that $k_1 = 6$ and $k_2 = 5$. By corollary 2.4.4, the (E,S)- and type-1 optimal design is an *ECD($\bar{\theta}$)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 3$ using condition (i) for block 1 or replicate 3 and condition (ii) for block 1 or replicate 4 yields an (E,S)- and type-1 optimal *ECD($\bar{\theta}$)* which is:

1	7	1	4	1	2	1	2
2	8	2	5	4	3	4	3
3	9	3	6	5	6	6	5
4	10	7	10	7	9	7	8
5	11	8	11	8	11	9	10
6		9		10		11	

Example Suppose $k_1 = 5$ and $k_2 = 4$. By corollary 2.4.4, the Schur-optimal design is an *ECD($\bar{\theta} + 1$)*. Applying the *ECD* construction given above with $L = \bar{\theta} + 1 = 3$ using condition (i) for block 1 of replicate 3 and condition (iii) for block 1 or replicate

4 produces the a Schur-optimal $ECD(\bar{\theta}+1)$ which is:

1 6 2 7 3 8 4 9 5	1 4 2 5 3 8 6 9 7	1 2 4 3 5 8 6 9 7	1 3 2 5 4 7 6 9 8

Finally, for our last example we investigate a setting for which the (E,S)-optimal and A-optimal designs are not the same.

Example Consider the setting for which $k_1 = 12$ and $k_2 = 7$. For this setting $\bar{\theta} = 7$ and $\gamma = .58v$, and since $\frac{v}{2} < \gamma < \frac{3v}{5}$, the (E,S)-optimal design is a Case II design and the A-optimal design may be an $ECD(\bar{\theta}+1)$, Case I, or a Case II design. In order to determine the A-optimal design, the optimality conditions (3.87), (3.89), and (3.91) must be checked, and in doing so, we observe that all three conditions are positive (81488, 92508, and 27236404, respectively). Thus, $ECD(\bar{\theta}+1)$s are A-better than both Case I and Case II designs, and Case I designs are A-better than Case II designs which means an $ECD(\bar{\theta}+1)$is A-optimal.

Applying the Case II construction for $\bar{\theta} = 7$ using condition (iii) for block 1 or replicate 4 yields an (E,S)-optimal Case II design which is:

1 13 2 14 3 15 4 16 5 17 6 18 7 19 8 20 9 10 11 12	1 8 2 9 3 10 4 11 5 12 6 18 7 19 13 20 14 15 16 17	1 5 2 6 3 7 4 12 8 17 9 18 10 19 11 20 13 14 15 16	1 2 5 3 6 4 7 11 8 16 9 18 10 19 12 20 13 14 15 17

The E-value for this design is 2.86 and the A-value is 1.3383. Since 2.86 > 2.71 Case II is E-better than the $ECD(\bar{\theta}+1)$, and since 1.3377 < 1.3383 the $ECD(\bar{\theta}+1)$ is A-better than the Case II design.

Applying the *ECD* construction with $L = \bar{\theta} + 1 = 8$ using condition (i) for block 1 if replicate 3 and condition (i) for block 1 of replicate 4 yields an A-optimal *ECD*($\bar{\theta}+1$) which is:

1	13	1	9	1	5	5	1
2	14	2	10	2	6	6	2
3	15	3	11	3	7	7	3
4	16	4	12	4	8	8	4
5	17	5	17	9	17	9	17
6	18	6	18	10	18	10	18
7	19	7	19	11	19	11	19
8	20	8	20	12	20	12	10
9		13		13		13	
10		14		14		14	
11		15		15		15	
12		16		16		16	

The E-value for this design is 2.71 and the A-value is 1.3377.

3.5 Resolvable Designs With Five Replicates

3.5.1 Introduction

In this section we study optimality for the the resolvable design setting $D(v, 5; k_1, k_2)$. We will determine (E,S)-optimal designs, and the A-optimal designs in some special cases. We also exploit the majorization theory of Chapter II in so far as possible. From section 2.3 we have:

ECD($\bar{\theta}$): The optimality matrix for *ECD*($\bar{\theta}$)s is $M_d = pI - \gamma(J-I)$. The eigenvalues of M_d are

$$\begin{aligned} \xi_1(\gamma) &= p+\gamma \qquad \text{(4 copies)} \\ \xi_2(\gamma) &= p-4\gamma, \end{aligned}$$

$$\xi_1(\gamma) = \xi_1(\gamma) = \xi_1(\gamma) > \xi_2(\gamma)$$

ECD($\bar{\theta}+1$): The optimality matrix for *ECD*($\bar{\theta}$+1)s is $M_d = pI - \gamma(J-I) + v(J-I)$.

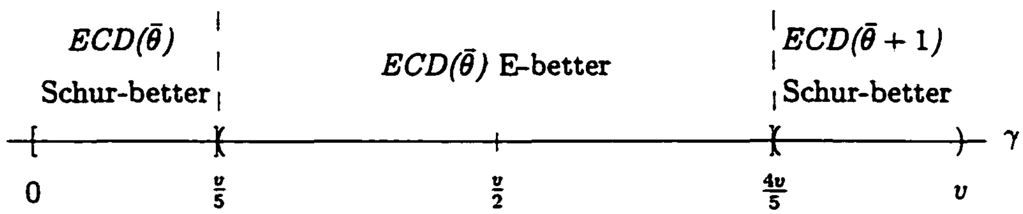
The eigenvalues of M_d are

$$\xi_1(\gamma - v) = p - (v - \gamma) \qquad \text{(4 copies)}$$
$$\xi_2(\gamma - v) = p + 4(v - \gamma),$$

$$\xi_2(\gamma - v) > \xi_1(\gamma - v) = \xi_1(\gamma - v) = \xi_1(\gamma - v)$$

Theorem 2.3.3, lemma 2.3.7, and corollary 2.3.8 establish conditions for when *ECD*($\bar{\theta}$)s are E-better or Schur-better than *ECD*($\bar{\theta}+1$)s and for when *ECD*($\bar{\theta}+1$)s are E-better and Schur-better than *ECD*($\bar{\theta}$)s; see table 3.36.

Table 3.36: E- and Schur-comparisons Of *ECD*s In $D(v, 5; k_1, k_2)$



Conditions for Schur- and E-optimality of *NECD*s or *ECD*s can be established using lemma 2.3.17 and by direct eigenvalue comparisons. The optimality matrix M_d (in order to apply lemma 2.3.17) or the concurrence discrepancy matrix Δ_d must be derived for competing *NECD*s. Recall that *NECD*s have block concurrences $\phi_{ii'} \in \{\bar{\theta}, \bar{\theta}+1\}$ for all $1 \le i \ne i' \le 4$ and have at least one block concurrence equal to $\bar{\theta}$ and at least one equal to $\bar{\theta}+1$. There are 32 cases of nonisomorphic *NECD*s; their block concurrence patterns, $\{\phi_{12}, \phi_{13}, \phi_{14}, \phi_{15}, \phi_{23}, \phi_{24}, \phi_{25}, \phi_{34}, \phi_{35}, \phi_{45}\}$ are listed in table 3.37 and the corresponding block concurrence discrepancy matrices are shown in table 3.38.

Table 3.37: Block Concurrence Discrepancies For *NECD* In $D(v, 5; k_1, k_2)$

Case	δ_{d12}	δ_{d13}	δ_{d14}	δ_{d15}	δ_{d23}	δ_{d24}	δ_{d25}	δ_{d34}	δ_{d35}	δ_{d45}
I	1	0	0	0	0	0	0	0	0	0
II	1	1	0	0	0	0	0	0	0	0
III	1	0	0	0	0	0	0	1	0	0
IV	1	1	1	0	0	0	0	0	0	0
V	1	1	0	0	1	0	0	0	0	0
VI	1	0	0	0	1	0	0	1	0	0
VII	1	1	0	0	0	0	0	0	0	1
$VIII$	1	1	1	1	0	0	0	0	0	0
IX	1	1	1	0	1	0	0	0	0	0
X	1	1	1	0	0	0	0	0	0	1
XI	1	1	0	0	0	0	0	1	0	1
XII	1	0	1	0	1	0	0	1	0	0
$XIII$	1	1	0	0	1	0	0	0	0	1
XIV	1	1	1	1	1	0	0	0	0	0
XV	1	1	1	0	1	0	0	1	0	0
XVI	1	1	1	0	1	0	0	0	1	0
$XVII$	1	1	1	0	1	0	0	0	0	1
$XVIII$	1	1	0	0	1	0	1	0	1	0
XIX	1	0	0	1	1	0	0	1	0	1
XX	1	1	1	1	1	0	0	1	0	0
XXI	1	1	1	1	1	0	0	0	0	1
$XXII$	1	1	1	0	1	1	0	1	0	0
$XXIII$	1	1	1	0	1	1	0	0	1	0
$XXIV$	1	1	1	0	1	0	1	0	1	0
XXV	1	1	1	0	0	0	1	0	1	1
$XXVI$	1	1	1	1	1	0	0	1	0	1
$XXVII$	1	1	1	1	1	1	0	1	0	0
$XXVIII$	1	1	1	1	1	1	1	0	0	0
$XXIX$	1	1	1	0	1	1	0	0	1	1
XXX	1	1	1	1	1	1	1	1	0	0
$XXXI$	1	1	1	1	1	1	1	0	1	0
$XXXII$	1	1	1	1	1	1	1	1	1	0

Table 3.38: Concurrence Discrepancy Matrices For *NECDs* In $D(v, 5; k_1, k_2)$

$$\Delta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_8 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_9 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_{10} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Delta_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_{11} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Delta_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_{12} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_{13} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Delta_7 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \Delta_{14} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 3.38: Continued

$$\Delta_{15} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&0&1&0&0\\ 0&0&0&0&0 \end{pmatrix} \qquad \Delta_{22} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&1&0\\ 1&1&0&1&0\\ 1&1&1&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\Delta_{16} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&0&0\\ 1&1&0&0&1\\ 1&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix} \qquad \Delta_{23} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&1&0\\ 1&1&0&0&1\\ 1&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix}$$

$$\Delta_{17} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&0&0\\ 1&1&0&0&0\\ 1&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix} \qquad \Delta_{24} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&1&0&1\\ 1&1&0&0&1\\ 1&0&0&0&0\\ 0&1&1&0&0 \end{pmatrix}$$

$$\Delta_{18} = \begin{pmatrix} 0&1&1&0&0\\ 1&0&1&0&1\\ 1&1&0&0&1\\ 0&0&0&0&0\\ 0&1&1&0&0 \end{pmatrix} \qquad \Delta_{25} = \begin{pmatrix} 0&1&1&1&0\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 0&1&1&1&0 \end{pmatrix}$$

$$\Delta_{19} = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix} \qquad \Delta_{26} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix}$$

$$\Delta_{20} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&0&0\\ 1&1&0&1&0\\ 1&0&1&0&0\\ 1&0&0&0&0 \end{pmatrix} \qquad \Delta_{27} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&1&0\\ 1&1&0&1&0\\ 1&1&1&0&0\\ 1&0&0&0&0 \end{pmatrix}$$

$$\Delta_{21} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&0&0\\ 1&1&0&0&0\\ 1&0&0&0&1\\ 1&0&0&1&0 \end{pmatrix} \qquad \Delta_{28} = \begin{pmatrix} 0&1&1&1&1\\ 1&0&1&1&1\\ 1&1&0&0&0\\ 1&1&0&0&0\\ 1&1&0&0&0 \end{pmatrix}$$

Table 3.38: Continued

$$\Delta_{29} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \Delta_{31} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\Delta_{30} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta_{32} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Using the concurrence discrepancy matrices for the 32 cases of *NECDs*, we begin our eigenvalue optimality investigation by applying the following corollary of lemma 2.3.17.

Corollary 3.5.1 *Let $d \in D(v, 5; k_1, k_2)$ be an NECD having optimality matrix $M_d = pI - \gamma(I - J) + v\Delta$, and let u_1 and u_r be the maximum and minimum eigenvalues, respectively, of $P^T \Delta P$, where $P = (I - \frac{1}{5}J)$. If*

$$\gamma < -\frac{u_r}{5}v$$

then ECD($\bar{\theta}$)s are Schur-better than d. If $u_1 > 0$ and

$$\gamma > \left(\frac{4 - u_1}{5}\right) v,$$

then ECD($\bar{\theta} + 1$)s are Schur-better than d. Furthermore, if

$$u_1 > 0 \tag{3.93}$$

then ECD($\bar{\theta}$)s are E-better, but not necessarily Schur-better, than d.

We now use these tools to eliminate as many designs as possible. For each *NECD*, condition (3.93) was calculated with results given in column four of table 3.39. We see all cases except Cases VIII, XXV, XXVIII, and XXXII are E-inferior to *ECD($\bar{\theta}$)s*.

Table 3.39: Corollary 3.5.1 Results In $D(v, 5; k_1, k_2)$

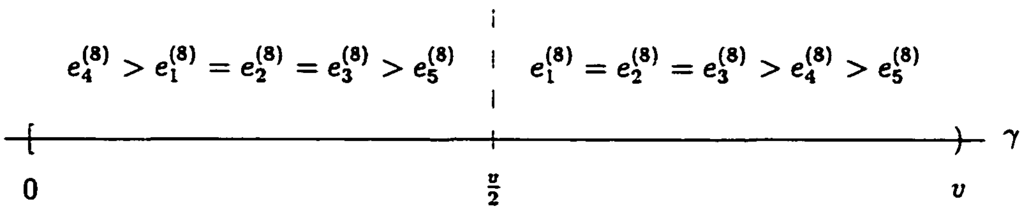
Case	$-\frac{u_r}{5}$	$\frac{4-u_l}{5}$	u_l
I	0.200	0.680	0.600
II	0.276	0.684	0.580
III	0.200	0.600	1.000
IV	0.316	0.724	0.380
V	0.200	0.640	0.800
VI	0.324	0.676	0.618
VII	0.274	0.566	1.169
VIII	0.320	0.800	0.000
IX	0.282	0.689	0.554
X	0.359	0.645	0.773
XI	0.343	0.600	1.000
XII	0.400	0.720	0.400
XIII	0.200	0.520	1.400
XIV	0.305	0.695	0.525
XV	0.305	0.695	0.525
XVI	0.324	0.676	0.618
XVII	0.334	0.579	1.104
XVIII	0.305	0.695	0.525
XIX	0.324	0.676	0.618
XX	0.311	0.718	0.410
XXI	0.280	0.600	1.000
XXII	0.200	0.680	0.600
XXIII	0.355	0.641	0.797
XXIV	0.355	0.641	0.797
XXV	0.480	0.800	0.000
XXVI	0.324	0.676	0.618
XXVII	0.276	0.684	0.580
XXVIII	0.360	0.800	0.000
XXIX	0.434	0.726	0.369
XXX	0.316	0.724	0.380
XXXI	0.316	0.724	0.380
XXXII	0.320	0.800	0.000

Values of γ for which *ECD($\bar{\theta}$)*s or *ECD($\bar{\theta}+1$)*s are Schur-better than *NECD*s having any of the concurrence discrepancy matrices listed in table 3.38 have been determined using corollary 3.5.1 and are also listed in table 3.39. *ECD($\bar{\theta}$)*are uniquely Schur-optimal on $0 \leq \gamma < \frac{v}{5}$, and *ECD($\bar{\theta}+1$)*are uniquely Schur-optimal on $\frac{4v}{5} < \gamma < v$.

Since none of the four remaining *NECDs* cases are completely eliminated from (E,S)-optimality contention, in order to proceed we must make direct eigenvalue comparisons; consequently, we need explicit expressions for the eigenvalues of the optimality matrices. The eigenvalues and their ordering over the admissible region are given below.

Case VIII: The optimality matrix for Case VIII *NECDs* is $M_8 = pI - \gamma(J - I) + v\Delta_8$, and the eigenvalues of M_8 are

$$\begin{aligned}
e_1^{(8)} &= p + \gamma \\
e_2^{(8)} &= p + \gamma \\
e_3^{(8)} &= p + \gamma \\
e_4^{(8)} &= p - \frac{3\gamma}{2} + \frac{1}{2}\sqrt{16(v-\gamma)^2 + 9\gamma^2} \\
e_5^{(8)} &= p - \frac{3\gamma}{2} - \frac{1}{2}\sqrt{16(v-\gamma)^2 + 9\gamma^2}.
\end{aligned}$$



Case XXV: The optimality matrix for Case XXV *NECDs* is $M_{25} = pI - \gamma(J - I) + v\Delta_{25}$, and the eigenvalues of M_{25} are

$$\begin{aligned}
e_1^{(25)} &= p + \gamma \\
e_2^{(25)} &= p + \gamma \\
e_3^{(25)} &= p + \gamma \\
e_4^{(25)} &= p - \frac{3\gamma}{2} + \frac{1}{2}\sqrt{24(v-\gamma)^2 + \gamma^2} \\
e_5^{(25)} &= p - \frac{3\gamma}{2} - \frac{1}{2}\sqrt{24(v-\gamma)^2 + \gamma^2}
\end{aligned}$$

$e_4^{(25)} > e_1^{(25)} = e_2^{(25)} = e_3^{(25)} > e_5^{(25)}$ | $e_1^{(25)} = e_2^{(25)} = e_3^{(25)} > e_4^{(25)} > e_5^{(25)}$

0 — $\frac{v}{2}$ — v $\quad \gamma$

Case XXVIII: The optimality matrix for Case XXVIII *NECDs* is $M_{28} = pI - \gamma(J - I) + v\Delta_{28}$, and the eigenvalues of M_{28} are

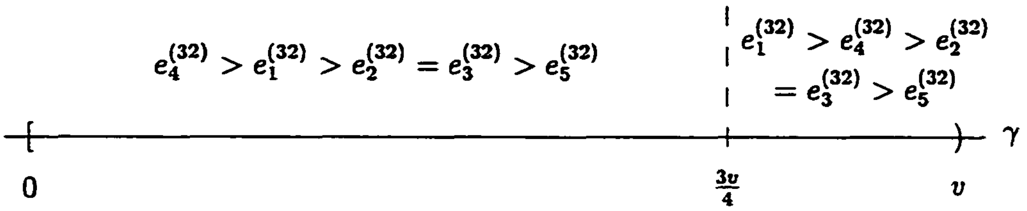
$$\begin{aligned}
e_1^{(28)} &= p + \gamma \\
e_2^{(28)} &= p + \gamma \\
e_3^{(28)} &= p - (v - \gamma) \\
e_4^{(28)} &= p - \frac{(3\gamma - v)}{2} + \frac{1}{2}\sqrt{(v + \gamma)^2 + 24(v - \gamma)^2} \\
e_5^{(28)} &= p - \frac{(3\gamma - v)}{2} - \frac{1}{2}\sqrt{(v + \gamma)^2 + 24(v - \gamma)^2}
\end{aligned}$$

$e_4^{(28)} > e_1^{(28)} = e_2^{(28)} > e_3^{(28)} > e_5^{(28)}$ | $e_1^{(28)} = e_2^{(28)} > e_4^{(28)} > e_3^{(28)} > e_5^{(28)}$

0 — $\frac{2v}{3}$ — v $\quad \gamma$

Case XXXII: The optimality matrix for Case XXXII *NECDs* is $M_{32} = pI - \gamma(J - I) + v\Delta_{32}$, and the eigenvalues of M_{32} are

$$\begin{aligned}
e_1^{(32)} &= p + \gamma \\
e_2^{(32)} &= p - (v - \gamma) \\
e_3^{(32)} &= p - (v - \gamma) \\
e_4^{(32)} &= p + \frac{2v - 3\gamma}{2} + \frac{1}{2}\sqrt{(2v - \gamma)^2 + 24(v - \gamma)^2} \\
e_5^{(32)} &= p + \frac{2v - 3\gamma}{2} - \frac{1}{2}\sqrt{(2v - \gamma)^2 + 24(v - \gamma)^2}
\end{aligned}$$



We conclude this section by settling the case $v = \frac{v}{2}$.

Lemma 3.5.2 *When $\gamma = \frac{v}{2}$, ECD($\bar{\theta}$)s, Case VIII and Case XXV designs are type-1 equivalent.*

Proof Since all cases of *NECDs* except for Cases VIII, XXV, XXVIII, and XXXII are E-inferior to *ECD($\bar{\theta}$)s*, when $\gamma = \frac{v}{2}$, the optimality matrices for these cases are the only optimality matrices that can potentially have eigenvalues that are identical to the eigenvalues of the optimality matrix for *ECD($\bar{\theta}$)s* and, therefore, be type-1 equivalent to *ECD($\bar{\theta}$)s*. When $\gamma = \frac{v}{2}$, it is easy to prove that the eigenvalues of the optimality matrices for *ECD($\bar{\theta}$)s*, Case VIII, and XXV designs are identical, and the eigenvalues of the optimality matrix for Case XXVIII and XXXII designs are not identical to those of *ECD($\bar{\theta}$)s* using the explicit expressions for the eigenvalues. □

3.5.2 (E,S)-Optimal Designs in $D(v, 5; k_1, k_2)$

In section 3.5.1 we proved that the only *NECDs* that can be E-optimal in a resolvable design setting $D(v, 4; k_1, k_2)$ are Cases VIII, XXV, XXVIII, and XXXII. Before investigating E-optimality in detail we will review a few useful optimality results from above.

1. *ECD($\bar{\theta}$)s* are uniquely Schur-optimal when $0 \leq \gamma < \frac{v}{5}$,
2. *ECD($\bar{\theta}$)*, Case VIII, and XXV designs are type-1 equivalent when $\gamma = \frac{v}{2}$
3. *ECD($\bar{\theta}$)s* and *ECD($\bar{\theta}+1$)s* are E-equivalent when $\gamma = \frac{4v}{5}$.

4. *ECD($\bar{\theta}+1$)s* are uniquely Schur-optimal when $\frac{4v}{5} < \gamma < v$.

Thus, all *UECDs* are E-inferior to an *ECD*; on $0 \leq \gamma < \frac{v}{2}$, *ECD($\bar{\theta}$)s* are E-optimal; *ECD($\bar{\theta}$)s*, Case VIII, and XXV designs are E-equivalent when $\gamma = \frac{v}{2}$; Case VIII, XXV, XXVIII, and XXII designs can be E-optimal on $\frac{v}{2} < \gamma \leq \frac{4v}{5}$; and *ECD($\bar{\theta}+1$)s* are E-optimal when $\frac{4v}{5} < \gamma < v$. In this section we will find the E-optimal designs on $\frac{v}{2} < \gamma \leq \frac{4v}{5}$, and when the E-optimal design is not unique, the (E,S)-optimal design will be identified.

Lemma 3.5.3

1. *ECD($\bar{\theta}$), Case VIII, and XXV designs are E-equivalent and E-better than Case XXVIII and Case XXXII designs when $\frac{v}{2} \leq \gamma < \frac{2v}{3}$.*

2. *When $\frac{2v}{3} \leq \gamma < \frac{3v}{4}$, ECD($\bar{\theta}$)s, Case VIII, XXV, and XXVIII designs are E-equivalent and E-better than Case XXXII designs.*

3. *When $\frac{3v}{4} \leq \gamma < \frac{4v}{5}$, ECD($\bar{\theta}$)s, Case VIII, XXV, XXVIII, and XXXII designs are E-equivalent.*

4. *When $\gamma = \frac{4v}{5}$, ECD($\bar{\theta}$)s, ECD($\bar{\theta}+1$)s, Case VIII, XXV, XXVIII, and XXXII designs are E-equivalent.*

Proof The maximum eigenvalue of the optimality matrix for *ECD($\bar{\theta}$)s* is $\xi_1(\gamma) = p + \gamma$, and the maximum eigenvalue of the optimality matrix for *ECD($\bar{\theta}+1$)s* is $\xi_2(\gamma - v) = p - (v - \gamma)$. On the interval $\frac{v}{2} \leq \gamma < v$, the maximum eigenvalue of the optimality matrices for Case VIII and XXV designs is $e_1^{(8)} = e_1^{(25)} = \xi_1(\gamma)$; therefore, *ECD($\bar{\theta}$)s*, Case VIII, and XXV designs are E-equivalent on the interval, and *ECD($\bar{\theta}$)s*, Case VIII, XXV, and *ECD($\bar{\theta}+1$)s* are E-equivalent when $\gamma = \frac{4v}{5}$. On the interval $\frac{v}{2} < \gamma < \frac{2v}{3}$, the maximum eigenvalue of the optimality matrix for Case XXVIII is $e_4^{(28)} > \xi_1(\gamma)$, and on $\frac{2v}{3} \leq \gamma < v$ the maximum eigenvalue of the optimality matrix for

Case XXVIII designs is $e_1^{(28)} = \xi_1(\gamma)$. Thus, when $\frac{v}{2} < \gamma < \frac{2v}{3}$, Case XXVIII designs are E-inferior to *ECD($\bar{\theta}$)*s, when $\frac{2v}{3} \leq \gamma < \frac{4v}{5}$ Case XXVIII designs are E-equivalent to *ECD($\bar{\theta}$)*s, and when $\gamma = \frac{4v}{5}$ Case XXVIII designs are E-equivalent to *ECD($\bar{\theta}$)*s and *ECD($\bar{\theta}+1$)*s. On the interval $\frac{v}{2} < \gamma < \frac{3v}{4}$ the maximum eigenvalue of the optimality matrix for Case XXXII designs is $e_4^{(32)} > \xi_1(\gamma)$, and when $\frac{3v}{4} \leq \gamma < v$ the maximum eigenvalue of the optimality matrix for Case XXXII designs is $e_1^{(32)} = \xi_1(\gamma)$. Therefore, Case XXXII designs are E-inferior to *ECD($\bar{\theta}$)*s when $\frac{v}{2} < \gamma < \frac{3v}{4}$, Case XXXII designs are E-equivalent to *ECD($\bar{\theta}$)*s when $\frac{3v}{4} \leq \gamma < \frac{4v}{5}$, and Case XXXII designs are E-equivalent to *ECD($\bar{\theta}$)*s and *ECD($\bar{\theta}+1$)*s when $\gamma = \frac{4v}{5}$. □

Now Schur comparisons of the E-optimal designs can be made.

Lemma 3.5.4 *Case XXV designs are Schur-better than Case VIII when $\frac{v}{2} < \gamma < v$.*

Proof When $\frac{v}{2} < \gamma < v$, the eigenvalues of the optimality matrix for Case VIII designs are $e_1^{(8)} = e_2^{(8)} = e_3^{(8)} > e_4^{(8)} > e_5^{(8)}$ and the eigenvalues of the optimality matrix for Case XXV designs are $e_1^{(25)} = e_2^{(25)} = e_3^{(25)} > e_4^{(25)} > e_5^{(25)}$. Since $e_1^{(8)} = e_2^{(8)} = e_3^{(8)} = e_1^{(25)} = e_2^{(25)} = e_3^{(25)}$ and $e_4^{(8)} \geq e_5^{(8)}$ then the eigenvalues of the optimality matrix for Case VIII designs majorize the eigenvalues of the optimality matrix for Case XXV designs. □

Lemma 3.5.5 *Case XXVIII designs are Schur-better than Case XXV when $\frac{2v}{3} \leq \gamma < v$.*

Proof When $\frac{2v}{3} \leq \gamma < v$ the eigenvalues of the optimality matrix for Case XXV designs are $e_1^{(25)} = e_2^{(25)} = e_3^{(25)} > e_4^{(25)} > e_5^{(25)}$, and the eigenvalues of the optimality matrix for Case XXVIII designs are $e_1^{(28)} = e_2^{(28)} \geq e_4^{(28)} > e_3^{(28)} > e_5^{(28)}$. Since $e_1^{(25)} = e_2^{(25)} = e_3^{(25)} = e_1^{(28)} = e_2^{(28)} \geq e_4^{(28)}$ and $e_5^{(25)} < e_5^{(28)}$, then the eigenvalues of the optimality matrix for Case XXV designs majorize the eigenvalues of the optimality matrix for Case XXVIII designs. □

Lemma 3.5.6 *Case XXXII designs are Schur-better than Case XXVIII when* $\frac{3v}{4} < \gamma < v$.

Proof When $\frac{3v}{4} \leq \gamma < v$ the eigenvalues of the optimality matrix for Case XXVIII designs are $e_1^{(28)} = e_2^{(28)} \geq e_4^{(28)} > e_3^{(28)} > e_5^{(28)}$, and the eigenvalues of the optimality matrix for Case XXXII designs are $e_1^{(32)} \geq e_4^{(32)} > e_2^{(32)} = e_3^{(32)} > e_5^{(32)}$. Since $e_1^{(28)} = e_2^{(28)} = e_1^{(32)} \geq e_4^{(32)}$ and $e_4^{(28)} > e_3^{(28)} = e_2^{(32)} = e_3^{(32)}$ then the eigenvalues of the optimality matrix for Case XXVIII designs majorize the eigenvalues of the optimality matrix for Case XXXII designs. □

Lemma 3.5.7 *ECD($\bar{\theta}+1$)s are Schur-better than Case XXXII when* $\gamma = \frac{4v}{5}$.

Proof When $\gamma = \frac{4v}{5}$ the eigenvalues of the optimality matrix for Case XXXII designs are $e_1^{(32)} \geq e_4^{(32)} > e_2^{(32)} = e_3^{(32)} > e_5^{(32)}$, and the eigenvalues of *ECD($\bar{\theta}+1$)s* are $\xi_2(\gamma - v) > \xi_1(\gamma - v) = \xi_1(\gamma - v) = \xi_1(\gamma - v) = \xi_1(\gamma - v)$. Since $e_1^{(32)} = \xi_2(\gamma - v)$ and $e_4^{(32)} > \xi_1(\gamma - v) = e_2^{(32)} = e_3^{(32)} > e_5^{(32)}$ then the eigenvalues of the optimality matrix for Case XXXII designs majorize the eigenvalues for *ECD($\bar{\theta}+1$)s*. □

Lemmas 3.5.3, 3.5.4, 3.5.5, 3.5.6, and 3.5.7 guarantee that for $\frac{v}{2} < \gamma \leq \frac{4v}{5}$ there is a unique Schur-best design among the E-best designs, and when $\gamma = \frac{v}{2}$ three classes of designs, *ECD($\bar{\theta}$)s*, Case VIII, and XXV, have identical eigenvalues and are Schur-best. The (E,S)-optimality breakdown is shown in table 3.40.

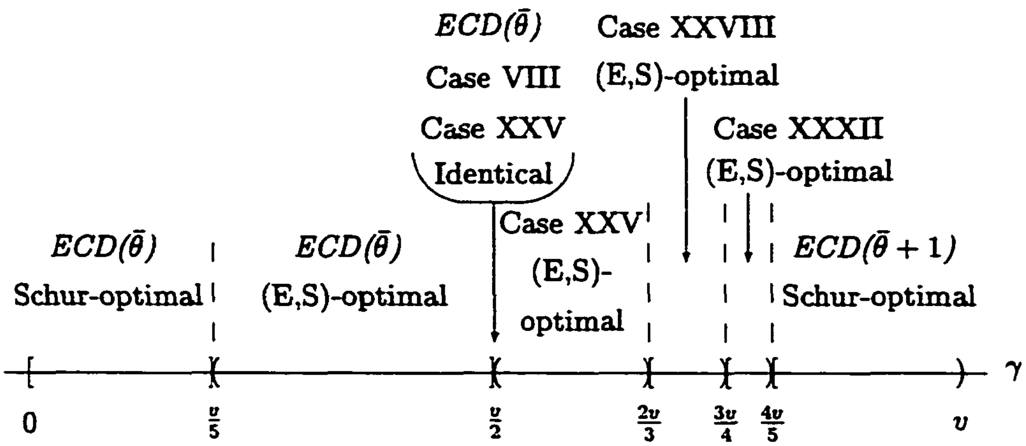
3.5.3 Special Cases: $(k_1 - k_2) \leq 2$

We will now apply the optimality results in the setting $D(v, 5; k_1, k_2)$ from section 3.5.2 to the three special cases when $(k_1 - k_2) \leq 2$ described in section 2.4.

Corollary 3.5.8 *Suppose* $k_1 = k_2$ *and* $r = 5$. *Then*

(i) If $2 \mid k_1$ *then* $\gamma = 0$, *and ECD(θ^*)s exist and are Schur-optimal.*

Table 3.40: (E,S)- and Schur-optimal Designs In $D(v, 5; k_1, k_2)$



(ii) If $2 \nmid k_1$ then $\gamma = \frac{v}{2}$, and ECD($\bar{\theta}$)s, Case VIII, and XXV are type-1 and (E,S)-optimal

Corollary 3.5.9 *Suppose $k_2 = k_1 - 1$ and $r = 5$. Then*

(i) If $2 \mid k_1$ then $\frac{v}{4} < \gamma < \frac{v}{3}$, and ECD($\bar{\theta}$)s are type-1 and (E,S)-optimal.

(ii) If $2 \nmid k_1$ then $\frac{3v}{4} < \gamma < \frac{4v}{5}$, and Case XXII is (E,S)-optimal.

By corollary 2.3.17, when $\frac{3v}{4} < \gamma < \frac{4v}{5}$, the optimality candidates are Case VIII, XXV, XXVIII, XXXII, and ECD($\bar{\theta}+1$)s, see table 3.39. On the interval, Cases VIII, XXV, and XXVIII were eliminated by majorization in section 3.5.2, leaving only Case XXXII and ECD($\bar{\theta}+1$)s as optimality candidates. We will state an A-optimality result for corollary 3.5.9 after proving the following lemma.

Lemma 3.5.10 *When $\frac{3v}{4} < \gamma < \frac{4v}{5}$, ECD($\bar{\theta}+1$)s are A-better than Case XXXII designs.*

Proof Recall that if e_i, $i = 1, 2, \ldots, 5$ is an eigenvalues of the optimality matrix for a design $d \in D(v, 5; k_1, k_2)$, then $5 - \frac{e_i}{p}$ is a corresponding eigenvalue of the information matrix of d, and the A-value of the design in terms of the eigenvalues of the optimality matirx is $\sum_{i=1}^{5} \frac{p}{5p-e_i}$. Since $e_2^{(32)} = e_3^{(32)} = \xi_1(\gamma - v)$, then *ECD($\bar{\theta}+1$)s* are A-better than Case XXXII designs if and only if

$$\frac{2p}{5p-\xi_1(\gamma-v)} + \frac{p}{5p-\xi_2(\gamma-v)} < \frac{p}{5p-e_1^{(32)}} + \frac{p}{5p-e_4^{(32)}} + \frac{p}{5p-e_5^{(32)}}. \quad (3.94)$$

Substituting the closed form expressions for the eigenvalues of *ECD($\bar{\theta}+1$)s* and Case XXXII designs from section 3.5.1 into (3.94) yields

$$-3\gamma^3 + 2(2p+9v)\gamma^2 + (32p^2 + 12pv - 27v^2)\gamma - 4v(4p^2 + 4pv - 3v^2) > 0. \quad (3.95)$$

A lower bound for the left hand side of (3.95) on the interval $\frac{3v}{4} \le \gamma \le \frac{4v}{5}$ obtained by substituting $\gamma = \frac{3v}{4}$ into the negative terms and $\gamma = \frac{4v}{5}$ into the postitive terms is

$$p^2 v \left[8\left(\frac{p}{v}\right)^2 - \frac{19}{4}\left(\frac{p}{v}\right) - \frac{1011}{1000} \right]. \quad (3.96)$$

Setting (3.96) equal to zero and solving for $\frac{p}{v}$ yields

$$\frac{p}{v} = \frac{475 \mp \sqrt{3461145}}{1600}.$$

Since $\frac{475-\sqrt{3461145}}{1600} < 0 < \frac{475+\sqrt{3461145}}{1600} < 1.5$, and when $\frac{p}{v} = 2$, (3.96) is greater than zero, then (3.95) is satisfied whenever $\frac{p}{v} \ge 2$. By fact 3.1.3, this inequality holds when $k_1 \ge k_2 \ge 4$ or when $k_2 = 3$ and $k_1 \ge 6$. Thus, (3.89) may not be satisfied when $k_2 \ge k_1 = 2$ or $5 \ge k_1 \ge k_2 = 3$. On $\frac{3v}{5} \le \gamma \le \frac{2v}{3}$, (k_1, k_2) does not take on the values (3,3), (4,3), or (5,3), and by corollary 3.1.5, $k_2 = 2$ if and only if $k_1 = 3$. Since (3.89) is satisfied when $(k_1, k_2) = (3, 2)$, then *ECD($\bar{\theta}+1$)s* are A-better than Case XXXII designs on the interval. □

Corollary 3.5.11 *Suppose $k_2 = k_1 - 1$, $r = 5$, and $2 \nmid k_1$. Then $\frac{3v}{4} < \gamma < \frac{4v}{5}$, Case XXXII is (E,S)-optimal, and ECD($\bar{\theta}+1$)s are A-optimal.*

Corollary 3.5.12 *Suppose $k_2 = k_1 - 2$ and $r = 5$. Then*

(i) If $k_1 = 4$ then $\gamma = \frac{2v}{3}$, Case XXVIII designs are (E,S)-optimal.

(ii) If $2 \mid k_1$ *and* $k_1 \geq 6$ *then* $\frac{v}{2} < \gamma \leq \frac{3v}{5}$*, and Case XXV designs are (E,S)-optimal.*

(iii) If $2 \nmid k_1$ *then* $0 < \gamma < \frac{v}{6}$*, and ECD*($\bar{\theta}$)*s are uniquely Schur-optimal (hence (E,S)-optimal).*

3.5.4 Construction of Optimal Designs in $D(v, 5; k_1, k_2)$

The (E,S)- and Schur-optimal resolvable designs in $D(v, 5; k_1, k_2)$, $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$, are *ECD*($\bar{\theta}$) , *ECD*($\bar{\theta}+1$) , Case XXV, Case XXVIII, and Case XXXII depending on the value of $0 \leq \gamma < v$. Now we will provide constructions for these optimal deisngs designs. The constructions for *ECD*s will be described in such a way that they will be valid for *ECD*(θ^*)s, *ECD*($\bar{\theta}$)s and *ECD*($\bar{\theta}+1$)s. For brevity, treatment arrangements for the first block of each replicate only are given.

Construction of $ECD(\bar{\theta})$s

Let L be the common *ECD* treatment concurrence. When $\gamma \leq \frac{v}{2}$, $L = \bar{\theta}$, and the design is an *ECD*($\bar{\theta}$), and when $\gamma > \frac{v}{2}$, $L = \bar{\theta}+1$, and the design is an *ECD*($\bar{\theta}+1$).

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots L\} \cup \{k_1+1 \ldots 2k_1 - L\}$

I. If $k_1 \leq 4/3L$

Block 1 of Replicate 3: $\{1 \ldots 2L - k_1\} \cup \{L+1 \ldots 2k_1 - L\}$

Block 1 of Replicate 4: $\{1 \ldots 3L - 2k_1\} \cup \{2L - k_1 + 1 \ldots 2k_1 - L\}$

Block 1 of Replicate 5: $\{3L - 2k_1 + 1 \ldots 2k_1 - L\}$

If $k_1 < 4L/3$

Block 1 of Replicate 5: $\{1 \ldots 4L - 3k_1\}$

II. If $(4/3L < k_1 < 3/2L$ and $k_1 > 7)$ or $(k_1 = 3/2L$, $k_1 \geq 7$ and $k_1 \neq 15)$

Let $x = int\left(\frac{k_1 - L}{3}\right)$

Block 1 of Replicate 3: $\{1 \ldots 2L - k_1 + x\} \cup \{L + 1 \ldots k_1 - x\} \cup$
$\{k_1 + 1 \ldots 2k_1 - L - x\} \cup \{2k_1 - L + 1 \ldots 2k_1 - L + x\}$

Block 1 of Replicate 4: $\{1 \ldots 3L{-}2k_1{+}2x\} \cup \{2L{-}k_1{+}x{+}1 \ldots L\} \cup$
$\{L + 1 \ldots k_1 - x\} \cup \{k_1 + x + 1 \ldots 2k_1 - L\} \cup$
$\{2k_1 - L + 1 \ldots 2k_1 - L + x\}$

Block 1 of Replicate 5: $\{3L - 2k_1 + 2x + 1 \ldots 2L - k_1\} \cup$
$\{2L - k_1 + x + 1 \ldots L\} \cup \{L + x + 1 \ldots k_1\} \cup$
$\{k_1 + 1 \ldots 2k_1 - L - x\} \cup \{2k_1 - L + 1 \ldots 2k_1 - L + x\}$

If $4L - 3k_1 + 4x > 0$

Block 1 of Replicate 5: $\cup \{1 \ldots 4L - 3k_1 + 4x\}$

III. If $(3/2L < k_1 < 2L)$, $(k_1 = 3/2L$ and $k_1 = 6)$ or $(k_1 = 3/2L$ and $k_1 = 15)$

A. If $2 \mid (k_1 - L)$

Block 1 of Replicate 3: $\{1 \ldots \frac{3L-k_1}{2}\} \cup \{L + 1 \ldots \frac{k_1+L}{2}\} \cup$
$\{k_1 + 1 \ldots \frac{3k_1-L}{2}\} \cup \{2k_1 - L + 1 \ldots \frac{5k_1-3L}{2}\}$

Block 1 of Replicate 4: $\{1 \ldots \frac{3L-k_1}{2}\} \cup \{\frac{k_1+L+2}{2} \ldots k_1\} \cup$
$\{\frac{3k_1-L+2}{2} \ldots 2k_1 - L\} \cup \{2k_1 - L + 1 \ldots \frac{5k_1-3L}{2}\}$

1. If $4 \mid (k_1 - L)$

Block 1 of Replicate 5: $\{1 \ldots 2L - k_1\} \cup \{\frac{3L-k_1+2}{2} \ldots L\} \cup$
$\{L + 1 \ldots \frac{k_1+3L}{4}\} \cup \{\frac{k_1+L+2}{2} \ldots \frac{3k_1+L}{4}\} \cup$
$\{k_1 + 1 \ldots \frac{5k_1-L}{4}\} \cup \{\frac{3k_1-L+2}{2} \ldots \frac{7k_1-3L}{4}\} \cup$
$\{2k_1 - L + 1 \ldots \frac{5k_1-3L}{2}\}$

2. If $4 \nmid (k_1 - L)$

Block 1 of Replicate 5: $\{1 \ldots 2L-k_1\} \cup \{\frac{3L-k_1+2}{2} \ldots L\} \cup$
$\overset{\text{if } k_1\geq L+3}{\{L+1 \ldots \frac{k_1+3L-2}{4}\}} \cup \{\frac{k_1+L+2}{2} \ldots \frac{3k_1+L+2}{4}\} \cup$
$\{k_1+1 \ldots \frac{5k_1-L+2}{4}\} \cup \overset{\text{if } k_1\geq L+3}{\{\frac{3k_1-L+2}{2} \ldots \frac{7k_1-3L-2}{4}\}} \cup$
$\{2k_1-L+1 \ldots \frac{5k_1-3L}{2}\}$

B. If $2 \not| \ (k_1 - L)$

Block 1 of Replicate 3: $\{1 \ldots \frac{3L-k_1-1}{2}\} \cup \{L+1 \ldots \frac{k_1+L+1}{2}\} \cup$
$\{k_1+1 \ldots \frac{3k_1-L+1}{2}\} \cup \{2k_1-L+1 \ldots \frac{5k_1-3L-1}{2}\}$

Block 1 of Replicate 4: $\{1 \ldots 2L-k_1\} \cup \{\frac{3L-k_1+1}{2} \ldots L-1\} \cup$
$\{L+1\} \cup \{\frac{k_1+L+3}{2} \ldots k_1\} \cup \{k_1+2 \ldots \frac{3k_1-L+3}{2}\} \cup$
$\{2k_1-L+1 \ldots \frac{5k_1-3L-1}{2}\}$

Block 1 of Replicate 5: $\{1 \ldots 2L-k_1\} \cup \overset{\text{if } k_1\geq L+5}{\{\frac{3L-k_1+1}{2} \ldots L-2\}} \cup$
$\{L\} \cup \{L+2 \ldots \frac{k_1+L+3}{2}\} \cup \{k_1+1\} \cup$
$\{\frac{3k_1-L+3}{2} \ldots 2k_1-L\} \cup \{2k_1-L+1 \ldots \frac{5k_1-3L-1}{2}\}$

IV. If $k_1 = 2L$

A. If $2 \mid L$

Block 1 of Replicate 3: $\{L+1 \ldots k_1\} \cup \{k_1+1 \ldots 2k_1-L\}$

Block 1 of Replicate 4: $\{1 \ldots L/2\} \cup \{L+1 \ldots 3/2L\} \cup$
$\{k_1+1 \ldots k_1+L/2\} \cup \{2k_1-L+1 \ldots 2k_1-L/2\}$

Block 1 of Replicate 5: $\{1 \ldots L/2\} \cup \{3/2L+1 \ldots k_1\} \cup$
$\{k_1+1/2L+1 \ldots 2k_1-L\} \cup \{2k_1-L+1 \ldots 2k_1-1/2L\}$

B. If $2 \not| \ L$

Block 1 of Replicate 3: $\{1\} \cup \{L+1 \ldots k_1-1\} \cup$
$\{k_1+1 \ldots 2k_1-L-1\} \cup \{2k_1-L+1\}$

Block 1 of Replicate 4: $\{2 \ldots \frac{L+1}{2}\} \cup \{L+1 \ldots \frac{3L-1}{2}\} \cup$
$\{k_1 \ldots \frac{2K_1+L-1}{2}\} \cup \{2k_1-L \ldots \frac{4k_1-L-1}{2}\}$

Block 1 of Replicate 5: $\{\overset{\text{if } L>3}{2 \ldots \frac{L-1}{2}}\} \cup \{\frac{L+3}{2}\} \cup \{\frac{3L+1}{2} \ldots k_1\} \cup \{\frac{2k_1+L+1}{2} \ldots \frac{4k_1-L-1}{2}\}$

V. If $k_1 = 2L+1$

Block 1 of Replicate 3: $\{L+1 \ldots 2L\} \cup \{k_1+1 \ldots k_1+L\} \cup \{2k_1-L+1\}$

A. If $2 \mid L$ and $k_1 > 5$

Block 1 of Replicate 4: $\{1 \ldots \frac{L-2}{2}\} \cup \{L+1 \ldots 3/2L\} \cup \{k_1 \ldots \frac{2k_1+L}{2}\} \cup \{2k_1-L\} \cup \{2k_1-L+2 \ldots \frac{4k_1-L+2}{2}\}$

Block 1 of Replicate 5: $\{\overset{\text{if } L>4}{1 \ldots \frac{L-4}{2}}\} \cup \{L/2\} \cup \{\frac{3L+2}{2} \ldots k_1\} \cup \{\frac{2k_1+L+2}{2} \ldots 2k_1-L\} \cup \{2k_1-L+2 \ldots \frac{4k_1-L+2}{2}\}$

B. If $2 \not| L$ and $k_1 > 7$

Block 1 of Replicate 4: $\{1 \ldots \frac{L-1}{2}\} \cup \{L+1 \ldots \frac{3L-1}{2}\} \cup \{k_1 \ldots \frac{2k_1+L-1}{2}\} \cup \{2k_1-L \ldots \frac{4k_1-L+1}{2}\}$

Block 1 of Replicate 5: $\{\overset{\text{if } L>5}{1 \ldots \frac{L-4}{2}}\} \cup \{\frac{L+2}{2} \ldots \frac{L+4}{2}\} \cup \{\frac{3L+1}{2} \ldots k_1-2\} \cup \{k_1\} \cup \{\frac{2k_1+L+1}{2} \ldots 2k_1-L-2\} \cup \{2k_1-L \ldots \frac{4k_1-L+1}{2}\}$

The *ECD* constructions given above are valid for all $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$ except for the following seven (k_1, k_2) pairs:

Pair	k_1	k_2	$\bar{\theta}$	$\underset{v}{I}$
1	3	2	2	.80
2	3	3	1	.50
3	5	5	2	.50
4	6	2	4	.50
5	7	2	5	.44
6	7	3	4	.90
7	7	7	3	.50

Constructions do not exist for pairs 1, 2, 4, and 5; however, valid constructions exist for the remaining three (3, 6, and 7). The first block of each replicate (written in columns) of these designs are:

Pair 3:

	$(k_1, k_2) = (5, 5)$			
1	1	3	3	1
2	2	4	5	4
3	6	6	6	8
4	7	7	8	9
5	8	9	10	10

Pair 6:

	$(k_1, k_2) = (7, 3)$			
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	6	4	4
5	5	7	6	7
6	8	8	8	9
7	9	9	10	10

Pair 7:

	$(k_1, k_2) = (7, 7)$			
1	1	4	1	2
2	2	5	4	5
3	3	6	7	7
4	8	8	8	9
5	9	9	11	11
6	10	10	12	12
7	11	12	13	14

Construction of Case XXV Designs

Since Case XXV designs are (E,S)-optimal on $\frac{v}{2} < \gamma < \frac{2v}{3}$, then the following constructions are valid for values of (k_1, k_2) that produce a value of γ in the interval.

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}\} \cup \{k_1 + 1 \ldots 2k_1 - \bar{\theta}\}$

Block 1 of Replicate 3: $\{1 \ldots 2\bar{\theta} - k_1 + 2\} \cup \{\bar{\theta} + 1 \ldots k_1 - 1\} \cup \{k_1 + 1 \ldots 2k_1 - \bar{\theta} - 1\}$

I. If $k_1 < 3/2\bar{\theta}$

Block 1 of Replicate 4: $\{1 \ldots 2k_2 - 4k_1 + 5\bar{\theta} + 4\} \cup \{2\bar{\theta} - k_1 + 3 \ldots k_1 - k_2\} \cup \{\bar{\theta} + 1 \ldots 2k_1 - k_2 - \bar{\theta} - 2\} \cup \{k_1 \ldots 3k_1 - k_2 - 2\bar{\theta} - 2\} \cup \{2k_1 - \bar{\theta} \ldots k_2 + k_1\}$

A. If $k_1 < k_2 + \bar{\theta}$

Block 1 of Replicate 5: $\{1 \ldots 4k_2 - 7k_1 + 8\bar{\theta} + 6\} \cup \{2k_2 - 4k_1 + 5\bar{\theta} + 5 \ldots k_1 - k_2\} \cup \{k_2 - k_1 + 2\bar{\theta} + 2 \ldots k_1\} \cup \{k_2 + \bar{\theta} + 2 \ldots k_1 + k_2\}$

B. If $k_1 = k_2 + \bar{\theta}$

Block 1 of Replicate 5: $\{1 \ldots 4\bar{\theta} - 3k_1 + 6\} \cup \{3\bar{\theta} - 2k_1 + 5 \ldots \bar{\theta}\} \cup \{k_2 - k_1 + 2\bar{\theta} + 2 \ldots k_1\} \cup \{k_2 + \bar{\theta} + 2 \ldots k_1 + k_2\}$

C. If $k_1 > k_2 + \bar{\theta}$

Block 1 of Replicate 5: $\{k_2 - k_1 + 2\bar{\theta} + 2 \ldots k_1\} \cup \{k_2 + \bar{\theta} + 2 \ldots k_1 + k_2\}$

II. If $3/2\bar{\theta} \leq k_1 < 2\bar{\theta} + 1$

A. If $2 \mid (k_1 - \bar{\theta} - 1)$

Block 1 of Replicate 4: $\{1 \ldots 2\bar{\theta} - k_1 + 1\} \cup \{2\bar{\theta} - k_1 + 3 \ldots \frac{3\bar{\theta} - k_1 + 5}{2}\} \cup \{\bar{\theta} + 1 \ldots \frac{k_1 + \bar{\theta} - 1}{2}\} \cup \{k_1 + 1 \ldots \frac{3k_1 - \bar{\theta} - 1}{2}\} \cup \{2k_1 - \bar{\theta} + 1 \ldots \frac{5k_1 - 3\bar{\theta} - 1}{2}\}$

Block 1 of Replicate 5: $\{1 \ldots \overset{\text{if } k_1<2\bar{\theta}}{2\bar{\theta} - k_1}\} \cup$

$\{2\bar{\theta} - k_1 + 2 \ldots \frac{3\bar{\theta}-k_1+5}{2}\} \cup \{\frac{k_1+\bar{\theta}+1}{2} \ldots k_1 - 1\} \cup$

$\{\frac{3k_1-\bar{\theta}+1}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup \{2k_1 - \bar{\theta} + 1 \ldots \frac{5k_1-3\bar{\theta}-1}{2}\}$

B. If $2 \not| \, (k_1 - \bar{\theta} - 1)$

Block 1 of Replicate 4: $\{1 \ldots \overset{\text{if } k_1<2\bar{\theta}}{2\bar{\theta} - k_1}\} \cup$

$\{2\bar{\theta} - k_1 + 3 \ldots \frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\bar{\theta} + 1 \ldots \frac{\bar{\theta}+k_1}{2}\} \cup$

$\{k_1 + 1 \ldots \frac{3k_1-\bar{\theta}}{2}\} \cup \{2k_1 - \bar{\theta} + 1 \ldots \frac{5k_1-3\bar{\theta}-2}{2}\}$

1. If $3/2\bar{\theta} \leq k_1 < 2\bar{\theta} - 1$

Block 1 of Replicate 5: $\{1 \ldots \overset{\text{if } k_1<2\bar{\theta}-2}{2\bar{\theta} - k_1 - 2}\} \cup$

$\{2\bar{\theta} - k_1 + 1 \ldots \frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\frac{\bar{\theta}+k_1}{2} \ldots k_1 - 1\} \cup$

$\{\frac{3k_1-\bar{\theta}}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup \{2k_1 - \bar{\theta} + 1 \ldots \frac{5k_1-3\bar{\theta}-2}{2}\}$

2. If $k_1 = 2\bar{\theta} - 1$

Block 1 of Replicate 5: $\{3 \ldots \frac{3\bar{\theta}-k_1+4}{2}\} \cup \{\frac{3\bar{\theta}-k_1+8}{2}\} \cup$

$\{\frac{\bar{\theta}+k_1}{2} \ldots k_1 - 1\} \cup \{\frac{3k_1-\bar{\theta}}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup$

$\{2k_1 - \bar{\theta} + 1 \ldots \frac{5k_1-3\bar{\theta}-2}{2}\}$

3. If $k_1 \geq 2\bar{\theta}$

Block 1 of Replicate 5: $\{3 \ldots \frac{\bar{\theta}+6}{2}\} \cup \{\frac{\bar{\theta}+k_1}{2} \ldots k_1 - 1\} \cup$

$\{\frac{3k_1-\bar{\theta}}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup \{2k_1 - \bar{\theta} + \overset{\text{if } \bar{\theta}>6}{1} \ldots \frac{4k_1-\bar{\theta}-6}{2}\} \cup$

$\{\frac{5k_1-3\bar{\theta}}{2} \ldots \frac{5k_1-3\bar{\theta}+2}{2}\}$

The Case XXV constructions given above are valid for all $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$ such that $\frac{v}{2} < \gamma < \frac{2v}{3}$ except for the following four (k_1, k_2) pairs:

Pair	k_1	k_2	$\bar{\theta}$	$\frac{\gamma}{v}$
1	5	2	3	.57
2	6	4	3	.60
3	8	6	4	.57
4	11	5	7	.56

Constructions do not exist for the fist pair; however, valid constructions exist for the remaining three (pairs 2, 3 and 4). The first block of each replicate (written in columns) of these designs are:

Pair 3:

$(k_1, k_2) = (6, 4)$

1	1	1	1	2
2	2	2	3	3
3	3	4	4	5
4	7	5	6	6
5	8	7	7	8
6	9	8	9	9

Pair 6:

$(k_1, k_2) = (8, 6)$

1	1	1	1	1
2	2	2	3	3
3	3	5	4	4
4	4	6	5	7
5	9	7	6	8
6	10	9	11	9
7	11	10	12	10
8	12	11	13	13

Pair 7:

$(k_1, k_2) = (11, 5)$

1	1	1	1	1
2	2	2	2	4
3	3	3	3	5
4	4	4	6	6
5	5	5	7	7
6	6	8	8	9
7	7	9	9	10
8	12	10	11	11
9	13	12	12	13
10	14	13	13	14
11	15	14	15	15

Construction of Case XXVIII Designs

Since Case XXVIII designs are (E,S)-optimal on $\frac{2v}{3} \le \gamma < \frac{3v}{4}$, then the following constructions are valid for values of (k_1, k_2) that produce a value of γ in the interval.

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}+1\} \cup \{k_1+1 \ldots 2k_1-\bar{\theta}-1\}$

Block 1 of Replicate 3: $\{1 \ldots 2\bar{\theta}-k_1+2\} \cup \{\bar{\theta}+2 \ldots k_1\} \cup$
$\{k_1+1 \ldots 2k_1-\bar{\theta}-1\}$

I. If $k_1 < 3/2\bar{\theta}$

Block 1 of Replicate 4: $\{1 \ldots 2k_2-4k_1+5\bar{\theta}+4\} \cup$
$\{2\bar{\theta}-k_1+3 \ldots k_1-k_2+1\} \cup \{\bar{\theta}+2 \ldots 2k_1-k_2-\bar{\theta}-1\} \cup$
$\{k_1+1 \ldots 3k_1-k_2-2\bar{\theta}-2\} \cup \{2k_1-\bar{\theta} \ldots k_1+k_2\}$

Block 1 of Replicate 5: $\{1 \ldots 4k_2-7k_1+8\bar{\theta}+6\} \cup$
$\{2k_2-4k_1+5\bar{\theta}+5 \ldots 2\bar{\theta}-k_1+2\} \cup \{2\bar{\theta}-k_1+3 \ldots k_1-k_2+1\} \cup$
$\{k_2-k_1+2\bar{\theta}+3 \ldots k_1\} \cup \{k_2+\bar{\theta}+2 \ldots 2k_1-\bar{\theta}-1\} \cup$
$\{2k_1-\bar{\theta} \ldots k_1+k_1\}$

II. If $k_1 \ge 3/2\bar{\theta}$ and $k_1 \ne 13$

A. If $2 \mid (k_1-\bar{\theta}-1)$

Block 1 of Replicate 4: $\{1 \ldots 2\bar{\theta}-k_1+1\} \cup$
$\{2\bar{\theta}-k_1+3 \ldots \frac{3\bar{\theta}-k_1+5}{2}\} \cup \{\bar{\theta}+2 \ldots \frac{k_1+\bar{\theta}+1}{2}\} \cup$
$\{k_1+1 \ldots \frac{3k_1-\bar{\theta}-1}{2}\} \cup \{2k_1-\bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-3}{2}\}$

Block 1 of Replicate 5: $\{1 \ldots 2\bar{\theta}-k_1\} \cup$
$\{2\bar{\theta}-k_1+2 \ldots \frac{3\bar{\theta}-k_1+5}{2}\} \cup \{\frac{k_1+\bar{\theta}+3}{2} \ldots k_1\} \cup$
$\{\frac{3k_1-\bar{\theta}+1}{2} \ldots 2k_1-\bar{\theta}-1\} \cup \{2k_1-\bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-3}{2}\}$

B. If $2 \not| \ (k_1 - \bar{\theta} - 1)$

Block 1 of Replicate 4: $\{1 \ldots 2\bar{\theta} - k_1\} \cup$

$\{2\bar{\theta} - k_1 + 3 \ldots \frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\bar{\theta} + 2 \ldots \frac{\bar{\theta}+k_1+2}{2}\} \cup$

$\{k_1 + 1 \ldots \frac{3k_1-\bar{\theta}}{2}\} \cup \{2k_1 - \bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-4}{2}\}$

Block 1 of Replicate 5: $\{1 \ldots \overset{\text{if } k_1 \le 2\bar{\theta}-2}{2\bar{\theta} - k_1 - 2}\} \cup$

$\{2\bar{\theta} - k_1 + 1 \ldots \frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\frac{\bar{\theta}+k_1+2}{2} \ldots k_1\} \cup$

$\{\frac{3k_1-\bar{\theta}}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup \{2k_1 - \bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-4}{2}\}$

The Case XXVIII constructions given are valid for all $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$ such that $\frac{2v}{3} \leq \gamma < \frac{3v}{4}$ except for $(k_1, k_2, \bar{\theta}) = (4, 2, 1)$ and (13, 9,7). A construction for $(k_1, k_2) = (4, 2)$ does not exist; however, there does exist a vaild construction for $(k_1, k_2) = (13, 9)$ which is:

	$(k_1, k_2) = (13, 9)$			
1	1	1	1	2
2	2	2	4	4
3	3	3	5	5
4	4	9	6	6
5	5	10	7	7
6	6	11	9	11
7	7	12	10	12
8	8	13	11	13
9	14	14	14	16
10	15	15	15	17
11	16	16	16	18
12	17	17	19	19
13	18	18	20	21

Construction of Case XXXII Designs

Since Case XXXII designs are (E,S)-optimal on $\frac{3v}{4} \leq \gamma < \frac{4v}{5}$, then the following constructions are valid for values of (k_1, k_2) that produce a value of γ in the interval.

Block 1 of Replicate 1: $\{1 \ldots k_1\}$

Block 1 of Replicate 2: $\{1 \ldots \bar{\theta}+1\} \cup \{k_1+1 \ldots 2k_1-\bar{\theta}-1\}$

Block 1 of Replicate 3: $\{1 \ldots 2\bar{\theta}-k_1+2\} \cup \{\bar{\theta}+2 \ldots k_1\} \cup$
$\{k_1+1 \ldots 2k_1-\bar{\theta}-1\}$

I. If $k_1 < 3/2\bar{\theta}$

Block 1 of Replicate 4: $\{1 \ldots 2k_2-4k_1+5\bar{\theta}+5\} \cup$
$\{2\bar{\theta}-k_1+3 \ldots k_1-k_2\} \cup \{\bar{\theta}+2 \ldots 2k_1-k_2-\bar{\theta}-1\} \cup$
$\{k_1+1 \ldots 3k_1-k_2-2\bar{\theta}-2\} \cup \{2k_1-\bar{\theta} \ldots k_1+k_2\}$

A. If $k_1 \leq \frac{4k_2+8\bar{\theta}+7}{7}$

Block 1 of Replicate 5: $\{1 \ldots 4k_2-7k_1+8\bar{\theta}+8\} \cup$
$\underbrace{\{2k_2-4k_1+5\bar{\theta}+6 \ldots k_1-k_2-1\}}^{\text{if } \frac{3k_1+5\bar{\theta}+7}{5} \leq k_1 \leq \frac{4k_2+8\bar{\theta}+7}{7}} \cup \{k_1-k_2+1\} \cup$
$\{k_2-k_1+2\bar{\theta}+3 \ldots k_1\} \cup \{k_2+\bar{\theta}+2 \ldots k_1+k_2\}$

B. If $k_1 > \frac{4k_2+8\bar{\theta}+7}{7}$

Block 1 of Replicate 5:
$\{2k_2-4k_1+5\bar{\theta}+6 \ldots 4k_2-8k_1+10\bar{\theta}+10\} \cup$
$\{2\bar{\theta}-k_1+3 \ldots 3k_2-6k_1+8\bar{\theta}+7\} \cup$
$\{k_1-k_2+1 \ldots 8k_1-5k_2-8\bar{\theta}-7\} \cup \{k_2-k_1+2\bar{\theta}+3 \ldots k_1\} \cup$
$\{k_2+\bar{\theta}+2 \ldots k_1+k_2\}$

II. If $k_1 \geq 3/2\bar{\theta}$ and $k_1 \neq 7$

A. If $2 \mid (k_1-\bar{\theta}-1)$

Block 1 of Replicate 4: $\{1 \ldots \frac{3\bar{\theta}-k_1+3}{2}\} \cup \{\bar{\theta}+2 \ldots \frac{k_1+\bar{\theta}+1}{2}\} \cup$
$\{k_1+1 \ldots \frac{3k_1-\bar{\theta}-1}{2}\} \cup \{2k_1-\bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-3}{2}\}$

Block 1 of Replicate 5: $\{1 \ldots \frac{3\bar{\theta}-k_1+1}{2}\} \cup \{\frac{3\bar{\theta}-k_1+5}{2}\} \cup$
$\{\frac{k_1+\bar{\theta}+3}{2} \ldots k_1\} \cup \{\frac{3k_1-\bar{\theta}+1}{2} \ldots 2k_1-\bar{\theta}-1\} \cup$
$\{2k_1-\bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-3}{2}\}$

B. If $2 \not| (k_1 - \bar{\theta} - 1)$

Block 1 of Replicate 4: $\{1 \ldots \overset{\text{if } k_1 \leq 2\bar{\theta}}{2\bar{\theta} - k_1 + 1}\} \cup$

$\{2\bar{\theta} - k_1 + 3 \ldots \frac{3\bar{\theta}-k_1+4}{2}\} \cup \{\bar{\theta} + 2 \ldots \frac{\bar{\theta}+k_1+2}{2}\} \cup$

$\{k_1 + 1 \ldots \frac{3k_1-\bar{\theta}}{2}\} \cup \{2k_1 - \bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-4}{2}\}$

1. If $k_1 < 2\bar{\theta}$

Block 1 of Replicate 5: $\{1 \ldots 2\bar{\theta} - k_1\} \cup \{2\bar{\theta} - k_1 + 2\} \cup$

$\{2\bar{\theta} - k_1 + 3 \ldots \frac{3\bar{\theta}-k_1+2}{2}\} \cup \{\frac{3\bar{\theta}-k_1+6}{2}\}$

2. If $k_1 = 2\bar{\theta}$

Block 1 of Replicate 5: $\{1\} \cup \{2\bar{\theta} - k_1 + 3 \ldots \frac{7\bar{\theta}-3k_1}{2}\} \cup$

$\{\frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\frac{3\bar{\theta}-k_1+8}{2}\}$

3. If $k_1 = 2\bar{\theta} + 1$

Block 1 of Replicate 5: $\{2\bar{\theta} - k_1 + 3 \ldots \frac{7\bar{\theta}-3k_1+2}{2}\} \cup$

$\{\frac{3\bar{\theta}-k_1+6}{2}\} \cup \{\frac{3\bar{\theta}-k_1+8}{2}\}$

Block 1 of Replicate 5: {blocks from 1 to 3 above} $\cup$

$\{\frac{\bar{\theta}+k_1+2}{2} \ldots k_1\} \cup \{\frac{3k_1-\bar{\theta}}{2} \ldots 2k_1 - \bar{\theta} - 1\} \cup$

$\{2k_1 - \bar{\theta} \ldots \frac{5k_1-3\bar{\theta}-4}{2}\}$

The Case XXXII constructions given above are valid for all $k_1 \geq 3$ and $k_1 \geq k_2 \geq 2$ such that $\frac{3v}{4} \leq \gamma < \frac{4v}{5}$ except for the pair $(k_1, k_2, \bar{\theta}) = (7, 6, 3)$. The vaild construction for $(k_1, k_2) = (7, 6)$ is:

$(k_1, k_2) = (7, 6)$

1	1	1	2	2
2	2	5	3	4
3	3	6	5	6
4	4	7	6	7
5	8	8	8	9
6	9	9	9	10
7	10	10	11	12

3.5.5 Examples of Optimal Resolvable Designs in $D(v, 5; k_1, k_2)$

We conclude this chapter by providing some examples of resolvable designs in $D(v, 5; k_1, k_2)$ and for various interesting $k_1 \geq 3$ and $2 \leq k_2 \leq k_1$. First we construct designs for the two cases when $k_1 = k_2$.

Example Suppose $k_1 = k_2 = 8$. Then, according to corollary 2.4.2 the the Schur-optimal design is an *ECD(θ^*)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 4$, yields a Schur-optimal *ECD(θ^*)* which is:

1	9	1	5	5	1	1	3	1	3
2	10	2	6	6	2	2	4	2	4
3	11	3	7	7	3	5	7	7	5
4	12	4	8	8	4	6	8	8	6
5	13	9	13	9	13	9	11	11	9
6	14	10	14	10	14	10	12	12	10
7	15	11	15	11	15	13	15	13	15
8	16	12	16	12	16	14	16	14	16

Example Consider the case where $k_1 = k_2 = 11$. Then, according to corollary 2.4.2 the (E,S)- and type-1 optimal design is an *ECD($\bar{\theta}$)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 5$ produces an (E,S)- and type-1 optimal design which is:

1	12	1	6	6	1	1	3	3	1
2	13	2	7	7	2	2	4	4	2
3	14	3	8	8	3	6	5	8	5
4	15	4	9	9	4	7	8	9	6
5	16	5	10	10	5	11	9	11	7
6	17	12	11	12	11	12	10	14	10
7	18	13	18	13	17	13	14	15	12
8	19	14	19	14	19	17	15	17	13
9	20	15	20	15	20	18	16	18	16
10	21	16	21	16	21	19	21	19	21
11	22	17	22	18	22	20	22	20	22

Now we investigate the two cases when $k_1 - k_2 = 1$.

Example Consider the setting such that $k_1 = 6$ and $k_2 = 5$. By corollary 2.4.4, the (E,S)- and type-1 optimal design is an *ECD($\bar{\theta}$)*. Applying the *ECD* construction given above with $L = \bar{\theta} = 3$ yields an (E,S)- and type-1 optimal design which is:

1	7	1	4	1	2	2	1	3	1
2	8	2	5	4	3	4	3	5	2
3	9	3	6	5	6	6	5	6	4
4	10	7	10	7	9	7	8	8	7.
5	11	8	11	8	11	9	11	9	11
6		9		10		10		10	

Example Suppose $k_1 = 13$ and $k_2 = 12$. By corollary 3.5.11, the (E,S)-optimal design is a Case XXXII design, and the A-optimal design is an *ECD*$(\bar{\theta}+1)$. Applying the Case XXXII construction given above with $\bar{\theta} = 6$ produces an (E,S)-optimal design which is:

1	14	1	8	1	2	1	5	1	4
2	15	2	9	8	3	2	6	2	6
3	16	3	10	9	4	3	7	3	7
4	17	4	11	10	5	4	11	5	8
5	18	5	12	11	6	8	12	11	9
6	19	6	13	12	7	9	13	12	10
7	20	7	20	13	20	10	17	13	14.
8	21	14	21	14	21	14	18	17	15
9	22	15	22	15	22	15	19	18	16
10	23	16	23	16	23	16	23	19	23
11	24	17	24	17	24	20	24	20	24
12	25	18	25	18	25	21	25	21	25
13		19		19		22		22	

Applying the *ECD* construction given above with $L = \bar{\theta} + 1 = 7$ produces an A-optimal design which is:

1	14	1	8	1	5	1	5	1	2
2	15	2	9	2	6	2	6	5	3
3	16	3	10	3	7	3	7	6	4
4	17	4	11	4	11	4	8	7	9
5	18	5	12	8	12	11	9	8	10
6	19	6	13	9	13	12	10	11	13
7	20	7	20	10	17	13	14	12	16.
8	21	14	21	14	18	17	15	14	18
9	22	15	22	15	19	18	16	15	19
10	23	16	23	16	23	19	23	17	23
11	24	17	24	20	24	20	24	20	24
12	25	18	25	21	25	21	25	21	25
13		19		22		22		22	

Four our final example we investigate a setting for which $k_1 - k_2 = 2$.

Example Suppose $k_1 = 12$ and $k_2 = 10$. Then by corollary 3.5.12, the (E,S)-optimal design is a Case XXV design. Applying the Case XXV construction for $\bar{\theta} = 6$ yields an (E,S)-optimal design which is:

1	13	1	7	1	3	3	1	3	1
2	14	2	8	2	4	4	2	4	2
3	15	3	9	7	5	5	10	5	7
4	16	4	10	8	6	6	11	6	8
5	17	5	11	9	12	7	12	9	12
6	18	6	12	10	18	8	16	10	13
7	19	13	19	11	19	9	17	11	14
8	20	14	20	13	20	13	18	15	18
9	21	15	21	14	21	14	21	16	19
10	22	16	22	15	22	15	22	17	20
11		17		16		19		21	
12		18		17		20		22	

3.6 Robustness of Optimal Designs

As was mentioned in the airplane part manufacturing example of section 2.1, an important question regarding optimal resolvable designs with r replications is whether optimality holds if fewer than r replicates of the experiment are completed. That is, is optimality of a resolvable design in $D(v, r; k_1, k_2)$ robust to the loss of an arbitrary replicate. With the optimality results of the previous few sections in hand, we are now ready to investigate robustness, but first we need the following definition.

Definition 3.6.1 Let d be a resolvable design in $D(v, r; k_1, k_2)$. A design $d^* \in D^*(v, r^*; k_1, k_2)$, $r^* < r$, is said to be a *subdesign* of d if the r^* replicates of d^* are also replicates of d.

Recall that the optimal resolvable design in $D(v, r, k_1, k_2)$ depends on the location of $\gamma = k_1^2 - \bar{\theta}v$ in the interval $0 \leq \gamma < v$. The value of γ does not depend on the number of replicates r; however, subintervals of $0 \leq \gamma < v$ on which various classes of designs are optimal does depend on r, see tables 3.20, 3.24, 3.25, 3.32, 3.34, and 3.40.

The intervals on which the optimality of *ECDs* is robust to the loss of replicates for the various criteria are established by the following two lemmas.

Lemma 3.6.1 *Let $D(v,r;k_1,k_2)$ be a resolvable design setting such that $0 \le \gamma \le \frac{v}{2}$, and let $d \in D$ be an ECD($\bar{\theta}$). If $d^* \in D^*(v,r^*;k_1,k_2)$, is any subdesign of d, then d^* is an ECD($\bar{\theta}$) and is type-1 and (E,S)-optimal.*

Proof Since all subdesigns of an *ECD* clearly are necessarily also an *ECD* , then d^* is an*ECD($\bar{\theta}$)*. By corollaries 2.3.4 and 2.3.15, *ECD($\bar{\theta}$)s* are at least type-1 and (E,S)-optimal for all r when $0 \le \gamma \le \frac{v}{2}$. □

Lemma 3.6.2 *Let $D(v,r;k_1,k_2)$ be a resolvable design setting , and let $d \in D$ be a Schur-optimal ECD. If $d^* \in D^*(v,r^*;k_1,k_2)$, is any subdesign of d, then d^* is an ECD and the following are true about the Schur-optimality of d^*.*

1. *If $r = 5$, and $0 \le \gamma \le \frac{v}{5}$ or $\frac{4v}{5} \le \gamma < v$, then d^* is Schur-optimal.*
2. *If $r = 4$, and $0 \le \gamma \le \frac{v}{4}$ or $\frac{3v}{4} \le \gamma < v$, then d^* is Schur-optimal.*
3. *If $r = 3$, and $0 \le \gamma \le \frac{v}{3}$ or $\frac{2v}{3} \le \gamma < v$, then $d \in D(v,r^*;k_1,k_2)$ is Schur-optimal.*

Proof Corollary 2.3.17 provides the subintervals of $0 \le \gamma < v$ on which *ECDs* are Schur-optimal. □

When $\frac{v}{2} < \gamma < \frac{4v}{5}$, regions of the interval on which various resolvable designs are optimal are determined by the design replication r. A robustness argument for these values of γ must involve direct comparisons of optimal designs for different values of r.

Lemma 3.6.3 *Let $D(v, 3; k_1, k_2)$ be a resolvable design setting, and suppose $\frac{v}{2} < \gamma < \frac{2v}{3}$. If $d \in D$ is an (E,S)-optimal Case II design, then 2/3 of the possible subdesigns $d^* \in D^*(v, 2; k_1, k_2)$ of d are (E,S)-optimal and the remaining 1/3 are not.*

Proof The discrepancy matrix for the (E,S)-optimal Case II design $d \in D(v, 3; k_1, k_2)$ is

$$\Delta_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Removing row column two or three from Δ_2 produces the discrepancy matrix for a Schur-optimal *ECD*$(\bar{\theta} + 1)$ in $D^*(v, 2; k_1, k_2)$. Removing row and column one from Δ_2 produces the discrepancy matrix for an *ECD*$(\bar{\theta})$ in D^* which is not optimal on the interval. Therefore, two of the three subdesigns are Schur-optimal. □

Lemma 3.6.4 *Let $D(v, 3; k_1, k_2)$ be a resolvable design setting, suppose $\frac{v}{2} < \gamma < \frac{2v}{3}$, and let $d \in D$ be an A-optimal design. If $d^* \in D^*(v, 2; k_1, k_2)$ is a subdesign of d then the following are true.*

1. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, and d is an A-optimal ECD$(\bar{\theta} + 1)$, then d^* is Schur-optimal.*
2. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, and d is an A-optimal Case II design, then 2/3 of the possible d^* are Schur-optimal and 1/3 are not optimal.*
3. *If $\frac{3v}{5} < \gamma < \frac{2v}{3}$, then d^* is A-optimal*

Proof If $\frac{v}{2} < \gamma < \frac{3v}{5}$ and the A-optimal $d \in D(v, 3; k_1, k_2)$ is an *ECD*$(\bar{\theta} + 1)$, then a subdesign $d^* \in D^*(v, 2; k_1, k_2)$ is a Schur-optimal *ECD*$(\bar{\theta} + 1)$. If $\frac{v}{2} < \gamma < \frac{3v}{5}$ and the A-optimal $d \in D$ is a Case II design, then it was established in the previous lemma that 2/3 of the subdesigns $d^* \in D^*$ are Schur-optimal *ECD*$(\bar{\theta}+1)$s and 1/3 of the subdesigns d^* are *ECD*$(\bar{\theta})$s and are not optimal. If $\frac{3v}{5} < \gamma < \frac{2v}{3}$, the A-optimal design $d \in D$ is an *ECD*$(\bar{\theta} + 1)$, and any subdesign $d^* \in D^*$ is a Schur-optimal *ECD*$(\bar{\theta} + 1)$. □

Lemma 3.6.5 *Let $D(v,4;k_1,k_2)$ be a resolvable design setting, suppose $d \in D$ is an (E,S)-optimal design, and let $d^* \in D^*(v,3;k_1,k_2)$ be a subdesign of d.*

1. *If $\frac{v}{2} < \gamma < \frac{2v}{3}$, then d^* is (E,S)-optimal.*

2. *If $\frac{2v}{3} < \gamma < \frac{3v}{4}$, then 1/2 of the subdesigns d^* of d are Schur-optimal and the remaining 1/2 are not optimal.*

Proof When $\frac{v}{2} < \gamma < \frac{2v}{3}$, Case II designs in $D(v,4;k_1,k_2)$ are (E,S)-optimal and have discrepancy matrix

$$\Delta_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Removing any one of the four rows and columns from Δ_2 produces the discrepancy matrix for an (E,S)-optimal Case II design $d^* \in D^*(v,3;k_1,k_2)$.

When $\frac{2v}{3} < \gamma < \frac{3v}{4}$, Case I designs in D are (E,S)-optimal and have discrepancy matrix

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Removing row and column one or two from Δ_1 produces the discrepancy matrix of a Schur-optimal *ECD*($\bar{\theta}+1$) in D, and removing row and column three or four produces the discrepancy matrix of a Case II design d^* which is not optimal on the interval. □

Lemma 3.6.6 *Let $d \in D(v,4;k_1,k_2)$ be an A-optimal resolvable design, suppose $\frac{v}{2} < \gamma < \frac{3v}{4}$, and let $d^* \in D^*(v,3;k_1,k_2)$ be a subdesign of d.*

1. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, an ECD($\bar{\theta}+1$) is A-optimal in D, and an ECD($\bar{\theta}+1$) is A-optimal in D^*, then d^* is always A-optimal.*

2. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, an ECD($\bar{\theta}+1$) is A-optimal in D, and a Case II design is A-optimal in D^*, then d^* is never optimal.*

3. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, a Case I design is A-optimal in D, and an ECD($\bar{\theta}+1$) is A-optimal in D^*, then 1/2 of the possible d^* are A-optimal and 1/2 are not optimal.*

4. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, a Case I design is A-optimal in D, and a Case II design is A-optimal in D^*, then 1/2 of the possible d^* are A-optimal and 1/2 are not optimal.*

5. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, a Case II design is A-optimal in D, and an ECD($\bar{\theta}+1$) is A-optimal in D^*, then d^* is never optimal.*

6. *If $\frac{v}{2} < \gamma < \frac{3v}{5}$, a Case II design is A-optimal in D, and a Case II design is A-optimal in D^*, then d^* is always A-optimal.*

7. *If $\frac{3v}{5} < \gamma < \frac{3v}{4}$, then d^* is always Schur-optimal.*

Proof Since all subdesigns of ECD($\bar{\theta}+1$)s are ECD($\bar{\theta}+1$)s, then 1, 2, and 7 follow immediately, and 3, 4, 5, and 6 follow from the previous lemma. □

Lemma 3.6.7 *Let $D(v, 5; k_1, k_2)$ be a resolvable design setting, suppose $d \in D$ is an (E,S)-optimal design, and let $d^* \in D^*(v, 4; k_1, k_2)$ be a subdesign of d.*

1. *If $\frac{v}{2} < \gamma < \frac{2v}{3}$, then 3/5 of the possible d^* are (E,S)-optimal and the remaining 2/5 are E-optimal.*

2. *If $\frac{2v}{3} \le \gamma < \frac{3v}{4}$, then 3/5 of the possible d^* are (E,S)-optimal and the remaining 2/5 are E-optimal.*

3. *If $\frac{3v}{4} \le \gamma < \frac{4v}{5}$, then 2/5 of the possible d^* are Schur-optimal and the remaining 3/5 are not optimal.*

Proof When $\frac{v}{2} < \gamma < \frac{2v}{3}$, Case XXV designs in $D(v,5;k_1,k_2)$ are (E,S)-optimal and have discrepancy matrix

$$\Delta_{25} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Removing row and column two, three, or four from Δ_{25} produces the discrepancy matrix of an (E,S)-optimal Case II design $d^* \in D^*(v,4;k_1,k_2)$, and removing row and column one or five from Δ_{25} produces the discrepancy matrix of an E-optimal Case V design d^*.

When $\frac{2v}{3} < \gamma < \frac{3v}{4}$, Case XXVIII designs in D are (E,S)-optimal and have discrepancy matrix

$$\Delta_{28} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Removing row and column three, four, or five from Δ_{28} produces the discrepancy matrix of an (E,S)-optimal Case I design $d^* \in D^*$, and removing row and column one or two produces the discrepancy matrix for a E-optimal Case V design d^*.

When $\frac{3v}{4} < \gamma < \frac{4v}{5}$, Case XXXII designs in D are (E,S)-optimal and have discrepancy matrix

$$\Delta_{32} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Removing row and column three or four from Δ_{32} produces the discrepancy matrix for a Schur-optimal $ECD(\bar{\theta}+1)$ in D^*, and removing row and column one, two, or three produces the discrepancy matrix for a Case I design d^* which is not optimal on the interval. □

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APPENDIX A

DISCREPANCY MATRICES

D1

$$\begin{matrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{matrix}$$

$\delta = 2,\ l = 2,\ w = 2$

D2

$$\begin{matrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{matrix}$$

$\delta = 3,\ l = 2,\ w = 3$

D3

$$\begin{matrix} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{matrix}$$

$\delta = 3,\ l = 2,\ w = 3$

D4

$$\begin{matrix} 0 & -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{matrix}$$

$\delta = 4,\ l = 2,\ w = 3$

D5

$$\begin{matrix} 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{matrix}$$

$\delta = 4,\ l = 2,\ w = 4$

D6

0	−1	1	0	0	0	0	0
−1	0	0	0	1	0	0	0
1	0	0	−1	0	0	0	0
0	0	−1	0	0	0	1	0
0	1	0	0	0	−1	0	0
0	0	0	0	−1	0	0	1
0	0	0	1	0	0	0	−1
0	0	0	0	0	1	−1	0

$\delta = 4,\ l = 2,\ w = 4$

D7

0	−1	1	0	0	0	0	0
−1	0	0	1	0	0	0	0
1	0	0	−1	0	0	0	0
0	1	−1	0	0	0	0	0
0	0	0	0	0	−1	1	0
0	0	0	0	−1	0	0	1
0	0	0	0	1	0	0	−1
0	0	0	0	0	1	−1	0

$\delta = 4,\ l = 2,\ w = 4$

D8

0	−1	−1	1	1	0	0
−1	0	0	0	0	1	0
−1	0	0	0	0	0	1
1	0	0	0	0	−1	0
1	0	0	0	0	0	−1
0	1	0	−1	0	0	0
0	0	1	0	−1	0	0

$\delta = 4,\ l = 2,\ w = 3$

D9

0	−1	−1	1	1	0
−1	0	0	0	0	1
−1	0	0	0	0	1
1	0	0	0	0	−1
1	0	0	0	0	−1
0	1	1	−1	−1	0

$\delta = 4,\ l = 2,\ w = 2$

D10

0	−1	−1	1	1	0	0
−1	0	1	0	0	0	0
−1	1	0	0	0	0	0
1	0	0	0	0	−1	0
1	0	0	0	0	0	−1
0	0	0	−1	0	0	1
0	0	0	0	−1	1	0

$\delta = 4,\ l = 2,\ w = 4$

D11

0	−1	−1	1	1	0
−1	0	1	−1	0	1
−1	1	0	0	0	0
1	−1	0	0	0	0
1	0	0	0	0	−1
0	1	0	0	−1	0

$\delta = 4,\ l = 2,\ w = 3$

D12

0	−2	1	1	0
−2	0	1	0	1
1	1	0	−1	−1
1	0	−1	0	0
0	1	−1	0	0

$\delta = 4,\ l = 3,\ w = 3$

D13

0	1	1	−1	−1	0	0
1	0	−1	0	1	−1	0
1	−1	0	1	0	0	−1
−1	0	1	0	0	0	0
−1	1	0	0	0	0	0
0	−1	0	0	0	0	1
0	0	−1	0	0	1	0

$\delta = 5,\ l = 2,\ w = 4$

D14

0	−1	−1	1	1	0	0	0
−1	0	0	0	0	−1	1	1
−1	0	0	0	0	1	0	0
1	0	0	0	−1	0	0	0
1	0	0	−1	0	0	0	0
0	−1	1	0	0	0	0	0
0	1	0	0	0	0	0	−1
0	1	0	0	0	0	−1	0

$\delta = 5,\ l = 2,\ w = 5$

D15

0	1	1	−1	−1	0	0	0
1	0	−1	1	0	−1	0	0
1	−1	0	0	0	0	0	0
−1	1	0	0	0	0	0	0
−1	0	0	0	0	0	1	0
0	−1	0	0	0	0	0	1
0	0	0	0	1	0	0	−1
0	0	0	0	0	1	−1	0

$\delta = 5,\ l = 2,\ w = 4$

D16

0	−1	−1	1	1	0	0
−1	0	1	0	0	1	−1
−1	1	0	0	0	−1	1
1	0	0	0	−1	0	0
1	0	0	−1	0	0	0
0	1	−1	0	0	0	0
0	−1	1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D17

0	1	1	−1	−1	0
1	0	−1	1	0	−1
1	−1	0	−1	1	0
−1	1	−1	0	0	1
−1	0	1	0	0	0
0	−1	0	1	0	0

$\delta = 5,\ l = 2,\ w = 4$

D18

0	−2	1	1	0	0
−2	0	0	0	1	1
1	0	0	−1	0	0
1	0	−1	0	0	0
0	1	0	0	0	−1
0	1	0	0	−1	0

$\delta = 4,\ l = 3,\ w = 4$

D19

0	1	1	1	−1	−1	−1
1	0	−1	−1	1	0	0
1	−1	0	0	0	0	0
1	−1	0	0	0	0	0
−1	1	0	0	0	0	0
−1	0	0	0	0	0	1
−1	0	0	0	0	1	0

$\delta = 5,\ l = 2,\ w = 3$

D20

0	1	1	−1	−1	0	0	0	0
1	0	−1	0	0	0	0	0	0
1	−1	0	0	0	0	0	0	0
−1	0	0	0	0	1	0	0	0
−1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	−1	0
0	0	0	0	1	0	0	0	−1
0	0	0	0	0	−1	0	0	1
0	0	0	0	0	0	−1	1	0

$\delta = 5,\ l = 2,\ w = 5$

D21

0	1	1	−1	−1	0	0	0
1	0	0	1	0	−1	−1	0
1	0	0	0	0	0	0	−1
−1	1	0	0	0	0	0	0
−1	0	0	0	0	1	0	0
0	−1	0	0	1	0	0	0
0	−1	0	0	0	0	0	1
0	0	−1	0	0	0	1	0

$\delta = 5,\ l = 2,\ w = 4$

D22

0	1	1	−1	−1	0	0
1	0	−1	0	0	−1	1
1	−1	0	1	0	0	−1
−1	0	1	0	0	0	0
−1	0	0	0	0	1	0
0	−1	0	0	1	0	0
0	1	−1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D23

0	1	0	0	0	0	0	0	0	−1
1	0	−1	0	0	0	0	0	0	0
0	−1	0	1	0	0	0	0	0	0
0	0	1	0	−1	0	0	0	0	0
0	0	0	−1	0	1	0	0	0	0
0	0	0	0	1	0	−1	0	0	0
0	0	0	0	0	−1	0	1	0	0
0	0	0	0	0	0	1	0	−1	0
0	0	0	0	0	0	0	−1	0	1
−1	0	0	0	0	0	0	0	1	0

$\delta = 5,\ l = 2,\ w = 5$

D24

0	1	−1	0	0	0	0	0	0	0
1	0	0	−1	0	0	0	0	0	0
−1	0	0	1	0	0	0	0	0	0
0	−1	1	0	0	0	0	0	0	0
0	0	0	0	0	1	−1	0	0	0
0	0	0	0	1	0	0	−1	0	0
0	0	0	0	−1	0	0	0	1	0
0	0	0	0	0	−1	0	0	0	1
0	0	0	0	0	0	1	0	0	−1
0	0	0	0	0	0	0	1	−1	0

$\delta = 5,\ l = 2,\ w = 5$

D25

0	1	1	−1	−1	0	0	0	0
1	0	0	0	0	−1	0	0	0
1	0	0	0	0	0	−1	0	0
−1	0	0	0	0	0	0	1	0
−1	0	0	0	0	0	0	0	1
0	−1	0	0	0	0	1	0	0
0	0	−1	0	0	1	0	0	0
0	0	0	1	0	0	0	0	−1
0	0	0	0	1	0	0	−1	0

$\delta = 5,\ l = 2,\ w = 5$

D26

0	1	1	−1	−1	0	0	0	0
1	0	−1	0	0	0	0	0	0
1	−1	0	0	0	0	0	0	0
−1	0	0	0	1	0	0	0	0
−1	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	−1	0
0	0	0	0	0	1	0	0	−1
0	0	0	0	0	−1	0	0	1
0	0	0	0	0	0	−1	1	0

$\delta = 5,\ l = 2,\ w = 5$

D27

0	1	1	−1	−1	0	0	0	0
1	0	0	0	0	−1	0	0	0
1	0	0	0	0	0	−1	0	0
−1	0	0	0	0	1	0	0	0
−1	0	0	0	0	0	0	1	0
0	−1	0	1	0	0	0	0	0
0	0	−1	0	0	0	0	0	1
0	0	0	0	1	0	0	0	−1
0	0	0	0	0	0	1	−1	0

$\delta = 5,\ l = 2,\ w = 4$

D28

0	1	1	−1	−1	0	0	0
1	0	0	1	0	−1	−1	0
1	0	0	0	0	0	0	−1
−1	1	0	0	0	0	0	0
−1	0	0	0	0	0	0	1
0	−1	0	0	0	0	1	0
0	−1	0	0	0	1	0	0
0	0	−1	0	1	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D29

0	1	1	−1	−1
1	0	−1	1	−1
1	−1	0	−1	1
−1	1	−1	0	1
−1	−1	1	1	0

$\delta = 5,\ l = 2,\ w = 5$

D30

0	0	0	0	−1	−1	1	1
0	0	−1	1	1	0	−1	0
0	−1	0	0	0	1	0	0
0	1	0	0	0	0	0	−1
−1	1	0	0	0	0	0	0
−1	0	1	0	0	0	0	0
1	−1	0	0	0	0	0	0
1	0	0	−1	0	0	0	0

$\delta = 5, l = 2, w = 4$

D31

0	0	0	0	−1	−1	1	1
0	0	−1	−1	1	1	0	0
0	−1	0	1	0	0	0	0
0	−1	1	0	0	0	0	0
−1	1	0	0	0	0	0	0
−1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	−1
1	0	0	0	0	0	−1	0

$\delta = 5, l = 2, w = 4$

D32

0	1	1	−1	−1	0	0	0
1	0	0	0	0	1	−1	−1
1	0	0	0	0	−1	0	0
−1	0	0	0	0	0	1	0
−1	0	0	0	0	0	0	1
0	1	−1	0	0	0	0	0
0	−1	0	1	0	0	0	0
0	−1	0	0	1	0	0	0

$\delta = 5, l = 2, w = 4$

D33

0	−1	−1	1	1	0	0	0
−1	0	0	0	0	−1	1	1
−1	0	0	0	0	1	0	0
1	0	0	0	0	0	−1	0
1	0	0	0	0	0	0	−1
0	−1	1	0	0	0	0	0
0	1	0	−1	0	0	0	0
0	1	0	0	−1	0	0	0

$\delta = 5, l = 2, w = 4$

D34

0	−1	−1	1	1	0	0	0
−1	0	0	−1	0	1	1	0
−1	0	0	0	0	0	0	1
1	−1	0	0	0	0	0	0
1	0	0	0	0	0	0	−1
0	1	0	0	0	0	−1	0
0	1	0	0	0	−1	0	0
0	0	1	0	−1	0	0	0

$\delta = 5, l = 2, w = 4$

D35

0	0	0	0	−1	−1	1	1
0	0	−1	0	1	1	−1	0
0	−1	0	1	0	0	0	0
0	0	1	0	0	0	0	−1
−1	1	0	0	0	0	0	0
−1	1	0	0	0	0	0	0
1	−1	0	0	0	0	0	0
1	0	0	−1	0	0	0	0

$\delta = 5, l = 2, w = 3$

D36

0	1	1	1	−1	−1	−1	0
1	0	−1	0	0	0	0	0
1	−1	0	0	0	0	0	0
1	0	0	0	0	0	0	−1
−1	0	0	0	0	1	0	0
−1	0	0	0	1	0	0	0
−1	0	0	0	0	0	0	1
0	0	0	−1	0	0	1	0

$\delta = 5,\ l = 2,\ w = 4$

D37

0	1	−1	−1	1	0	0
1	0	0	0	−1	1	−1
−1	0	0	1	0	−1	1
−1	0	1	0	0	0	0
1	−1	0	0	0	0	0
0	1	−1	0	0	0	0
0	−1	1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 3$

D38

0	1	0	1	−1	−1	0
1	0	−1	−1	0	0	1
0	−1	0	0	1	1	−1
1	−1	0	0	0	0	0
−1	0	1	0	0	0	0
−1	0	1	0	0	0	0
0	1	−1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 3$

D39

0	−2	1	1	0	0
−2	0	0	0	1	1
1	0	0	0	−1	0
1	0	0	0	0	−1
0	1	−1	0	0	0
0	1	0	−1	0	0

$\delta = 4,\ l = 3,\ w = 3$

D40

0	2	−1	−1	0	0
2	0	0	0	−1	−1
−1	0	0	0	1	0
−1	0	0	0	0	1
0	−1	1	0	0	0
0	−1	0	1	0	0

$\delta = 4,\ l = 3,\ w = 3$

D41

0	1	1	−1	−1	0	0	0	0
1	0	0	0	0	−1	0	0	0
1	0	0	0	0	0	−1	0	0
−1	0	0	0	1	0	0	0	0
−1	0	0	1	0	0	0	0	0
0	−1	0	0	0	0	0	1	0
0	0	−1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	−1
0	0	0	0	0	0	1	−1	0

$\delta = 5,\ l = 2,\ w = 4$

D42

0	−1	−1	1	1	0	0	0
−1	0	0	−1	0	1	1	0
−1	0	0	0	0	0	0	1
1	−1	0	0	0	0	0	0
1	0	0	0	0	−1	0	0
0	1	0	0	−1	0	0	0
0	1	0	0	0	0	0	−1
0	0	1	0	0	0	−1	0

$\delta = 5,\ l = 2,\ w = 4$

D43

0	−1	−1	1	1	0	0
−1	0	1	0	0	1	−1
−1	1	0	−1	0	0	1
1	0	−1	0	0	0	0
1	0	0	0	0	−1	0
0	1	0	0	−1	0	0
0	−1	1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D44

0	1	1	−1	−1	0	0	0
1	0	0	0	0	1	−1	−1
1	0	0	0	0	−1	0	0
−1	0	0	0	1	0	0	0
−1	0	0	1	0	0	0	0
0	1	−1	0	0	0	0	0
0	−1	0	0	0	0	0	1
0	−1	0	0	0	0	1	0

$\delta = 5,\ l = 2,\ w = 5$

D45

0	1	1	−1	−1	0	0
1	0	−1	0	0	−1	1
1	−1	0	0	0	1	−1
−1	0	0	0	1	0	0
−1	0	0	1	0	0	0
0	−1	1	0	0	0	0
0	1	−1	0	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D46

0	−1	−1	1	1	0	0	0
−1	0	1	−1	0	1	0	0
−1	1	0	0	0	0	0	0
1	−1	0	0	0	0	0	0
1	0	0	0	0	0	−1	0
0	1	0	0	0	0	0	−1
0	0	0	0	−1	0	0	1
0	0	0	0	0	−1	1	0

$\delta = 5,\ l = 2,\ w = 4$

D47

0	1	1	−1	−1	0
1	0	−1	−1	0	1
1	−1	0	1	0	−1
−1	−1	1	0	1	0
−1	0	0	1	0	0
0	1	−1	0	0	0

$\delta = 5,\ l = 2,\ w = 4$

D48

0	2	−1	−1	0	0
2	0	0	0	−1	−1
−1	0	0	1	0	0
−1	0	1	0	0	0
0	−1	0	0	0	1
0	−1	0	0	1	0

$\delta = 4,\ l = 3,\ w = 4$

D49

0	−1	−1	−1	1	1	1
−1	0	1	1	−1	0	0
−1	1	0	0	0	0	0
−1	1	0	0	0	0	0
1	−1	0	0	0	0	0
1	0	0	0	0	0	−1
1	0	0	0	0	−1	0

$\delta = 5,\ l = 2,\ w = 3$

D50

0	−1	−1	1	1	0	0
−1	0	1	0	−1	1	0
−1	1	0	−1	0	0	1
1	0	−1	0	0	0	0
1	−1	0	0	0	0	0
0	1	0	0	0	0	−1
0	0	1	0	0	−1	0

$\delta = 5,\ l = 2,\ w = 4$

D51

0	2	−1	−1	0
2	0	−1	0	−1
−1	−1	0	1	1
−1	0	1	0	0
0	−1	1	0	0

$\delta = 4,\ l = 3,\ w = 3$

APPENDIX B

DISCREPANCY MATRICES RANKED BY MAXIMUM EIGENVALUE

rank	Matrix	δ_d	l_d	w	U_d
1	D2	3	2	4	1.73205
2	D13	5	2	4	1.87939
3	D23	5	2	5	1.90211
4	D5	4	2	4	1.93543
5	D1	2	2	2	2.00000
6	D7	4	2	4	2.00000
7	D14	5	2	3	2.00000
8	D15	5	2	3	2.00000
9	D24	5	2	5	2.00000
10	D6	4	2	4	2.00000
11	D4	4	2	3	2.00000
12	D20	5	2	5	2.13452
13	D16	5	2	4	2.23607
14	D3	3	2	3	2.23607
15	D26	5	2	5	2.23607
16	D29	5	2	5	2.23607
17	D17	5	2	4	2.29240
18	D25	5	2	5	2.30278
19	D27	5	2	5	2.35829
20	D21	5	2	3	2.37720
21	D28	5	2	3	2.37951
22	D12	4	3	3	2.41421
23	D41	5	2	5	2.42534
24	D8	4	2	3	2.44949
25	D22	5	2	4	2.45585
26	D10	4	2	4	2.47283
27	D30	5	2	3	2.52434
28	D19	5	2	3	2.52543
29	D32	5	2	3	2.56155
30	D33	5	2	3	2.56155

rank	Matrix	δ_d	l_d	w	U_d
31	D44	5	2	3	2.56155
32	D45	5	2	4	2.56155
33	D31	5	2	3	2.56155
34	D18	4	3	4	2.56155
35	D34	5	2	3	2.61050
36	D35	5	2	3	2.64575
37	D42	5	2	3	2.69963
38	D36	5	2	4	2.71519
39	D37	5	2	3	2.79793
40	D38	5	2	3	2.79793
41	D46	5	2	3	2.81361
42	D9	4	2	2	2.82843
43	D43	5	2	4	2.85323
44	D47	5	2	4	2.89511
45	D11	4	2	3	2.90321
46	D48	4	3	4	3.00000
47	D39	4	3	3	3.00000
48	D40	4	3	3	3.00000
49	D50	5	2	4	3.04892
50	D49	5	2	3	3.15633
51	D51	4	3	3	3.44949

VITA

Brian Henry Reck was born in Las Vegas, NV on January 5, 1968. In 1991 he earned a Bachelor of Science degree in mathematics from the University of Redlands. After spending a year working as a Resident Director at the University of Redlands, Brian began his graduate studies at Old Dominion University. He earned a Master of Science in Computational and Applied Mathematics at Old Dominion University in 1995. He will obtain a Doctorate in Statistics in August 2002.

Brian has worked as an instructor at Old Dominion University and Averett University and as a statistical consultant at the Center for Pediatric Research and Eastern Virginia Medical School while attending Old Dominion University. With the completion of his Ph.D., Brian hopes to obtain a tenure-track job at a research university.

PUBLICATIONS:

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