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# Superconvergence in Iterated Solutions of Integral Equations

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# SUPERCONVERGENCE IN ITERATED SOLUTIONS OF INTEGRAL EQUATIONS

by

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# ABSTRACT

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In this thesis, we investigate the superconvergence phenomenon of the iterated numerical solutions for the Fredholm integral equations of the second kind as well as a class of nonlinear Hammerstein equations. The term superconvergence was first described in the early 70s in connection with the solution of two-point boundary value problems and other related partial differential equations. Superconvergence in this context was understood to mean that the order of convergence of the numerical solutions arising from the Galerkin as well as the collocation method is higher at the knots than we might expect from the numerical solutions that are obtained by applying a class of piecewise polynomials as approximating functions. The type of superconvergence that we investigate in this thesis is different. We are interested in finding out whether or not we obtain an enhancement in the global rate of convergence when the numerical solutions are iterated through integral operators. A general operator approximation scheme for the second kind linear equation is described that can be used to explain some of the existing superconvergence results. Moreover, a corollary to the general approximation scheme will be given which can be used to establish the superconvergence of the iterated degenerate kernel method for the Fredholm equations of the second kind. We review the iterated Galerkin method for Hammerstein equations and discuss the iterated degenerate kernel method for Hammerstein equations. Also, we investigate the iterated collocation method for Hammerstein and weakly singular Hammerstein equations and its corresponding superconvergence phenomena for the iterated solutions. The type of regularities that the solution of weakly singular Hammerstein equations possess is investigated. Subsequently, we establish the singularity preserving Galerkin method for Hammerstein equations. Finally, the superconvergence results for the iterated solutions corresponding to this method will be described.

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# CHAPTER I
# INTRODUCTION

In this thesis, we investigate the superconvergence phenomenon of the iterated numerical solutions for the Fredholm integral equations of the second kind as well as a class of nonlinear Hammerstein equations. The term superconvergence was first described in the early 70s in connection with the solution of two-point boundary value problems and other related partial differential equations. Superconvergence in this context was understood to mean that the order of convergence of the numerical solutions arising from the Galerkin as well as the collocation method is higher at the knots than we might expect from the numerical solutions that are obtained by applying a class of piecewise polynomials as approximating functions. See references [9], [10], [17], [18], [20], [60], [61], [62], [68], and [69]. The idea of superconvergence that we study here is different and it was originated by Sloan in references [60]-[61]. We now describe the Sloan's iterates and its superconvergence phenomenon in relation to the Fredholm integral equations of the second kind. The equation can be written as

$$y(t) - \int_a^b k(s,t)y(s)ds = f(t), \quad t \in [a,b] \tag{1.1}$$

or if we let

$$Ky(t) = \int_a^b k(s,t)y(s)ds \tag{1.2}$$

then the above equation can be written in operator form as,

$$y - Ky = f. \tag{1.3}$$

The kernel $k$ of the integral operator $K$ is assumed to be well behaved so that $K$ defines a compact operator on some appropriate Banach space, $X$, with $f \in X$. When the Galerkin or collocation methods are applied to approximate the solution $y$ in (1.3) using piecewise polynomials of order $r$, the best results in terms of the order of convergence that we can expect in an appropriate $L_p$ norm is $O(h^r)$ where $h = max(t_{i+1} - t_i)$, for $i = 1, \ldots, n-1$, with $\{t_i\}_{i=1}^n$ the prescribed set of knots.

Both the Galerkin and the collocation methods can be described within the general framework of the projection method. Specifically, let $S_n$ be a finite dimensional subspace of

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a Banach space $X$. $S_n$, for example, may be taken as the space of all piecewise polynomials of order r, the space of all trigonometric polynomials or the space spanned by wavelet basis, etc. In the Galerkin method, we take $X = L_2[a,b]$ and we approximate the solution $y$ in equation (1.3) by $y_n$ from the space $S_n$ by requiring that

$$(y_n - Ky_n - f, \phi_n) = 0, \qquad \text{for all } \phi_n \in S_n \tag{1.4}$$

where $(\cdot,\cdot)$ denotes the usual $L_2$ inner product.

In the collocation method, we take $X = C[a,b]$. Suppose that $\{u_j\}_{j=1}^n$ is a basis for $S_n$ and choose a suitable set of distinct points, $\{t_i\}_{i=1}^n$, $t_i \in [a,b]$, $i = 1,\ldots,n$, so that $\det[u_j(t_i)]_{i,j=1,\ldots,n} \neq 0$. We seek an approximate solution $y_n$ in the form $y_n = \sum_{j=1}^n a_j u_j$ where $\{a_j\}_{j=1}^n$ are defined by requiring that

$$y_n(t_i) - Ky_n(t_i) - f(t_i) = 0 \qquad \text{for } i = 1,\ldots,n. \tag{1.5}$$

The equations (1.4) and (1.5) can be easily seen to be equivalent to:

$$\sum_{j=1}^{n} a_j[(u_j, u_i) - (Ku_j, u_i)] = (f, u_i), \; i = 1,\ldots,n, \tag{1.6}$$

and

$$\sum_{j=1}^{n} a_j[u_j(t_i) - Ku_j(t_i)] = f(t_i), \; i = 1,\ldots,n. \tag{1.7}$$

To see that equations (1.6) and (1.7) are special cases of the general projection scheme, we consider the following. For the Galerkin method, we take the projection $P_n: L_2[a,b] \to S_n$ that is orthogonal, -i.e., with $v \in L_2[a,b]$, $P_n v \in S_n$ is defined from $(P_n v - v, u_i) = 0$, for each $i = 1, 2, \ldots, n$. For the collocation method, $P_n: C[a,b] \to S_n$ is the interpolatory projector. Namely, with $v \in C[a,b]$, $P_n v \in S_n$ is defined from the conditions, $P_n v(t_i) = v(t_i), i = 1,\ldots,n$. The residual function $r_n$ is defined by $r_n = f - (y_n - Ky_n)$. Clearly $r_n$ is nonzero unless the solution $y$ of equation (1.3) happens to belong to the space $S_n$. Now both equations (1.6) and (1.7) can be written as

$$P_n r_n = 0, \tag{1.8}$$

where, of course, the projection $P_n$ in (1.8) is orthogonal and interpolatory for the Galerkin method and the collocation method respectively. Also, since $P_n y_n = y_n$, $y_n \in S_n$, we can express equation (1.8) as

$$y_n - P_n K y_n = P_n f \tag{1.9}$$

This is the classical projection equation. As stated earlier, under some suitable smoothness conditions on $k$ and $f$, if $S_n$ is the space of all piecewise polynomials of degree less than or equal to $r$, then it is expected that

$$\|y_n - y\|_\infty = O(h^r).$$

Now we are ready to describe Sloan's iterate which is the main topic of this thesis.

Let $y_n$ be the solution of (1.9). We define Sloan's iterate by

$$y_n^I = f + Ky_n. \tag{1.10}$$

Both the iterated collocation and the iterated Galerkin methods can be generalized using the projection operators. For $y_n^I$ in equation (1.10), from equation (1.9) we have,

$$y_n = P_n f + P_n K y_n = P_n[f + Ky_n] = P_n y_n^I$$

and

$$y_n^I - KP_n y_n^I = f. \tag{1.11}$$

It is useful in the sequel that we provide at this point a detailed review of the superconvergence phenomenon of Sloan's iterates. The review below is based upon the paper by Graham, Joe and Sloan [22].

For any positive integer $n$, let

$$\Pi_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

be a set of partition points (knots) and for $i = 1, 2, \ldots, n$ set

$$I_i = (x_{i-1}, x_i), \qquad h_i = x_i - x_{i-1}, \qquad h = h(n) = \max_{1 \le i \le n} h_i.$$

We assume that $h \to 0$ as $n \to \infty$. Let $r$ be a positive integer and $\nu$ an integer satisfying $0 \le \nu < r$. Let $S_{r,n}^\nu$ denote the space of splines of order $r$, continuity $\nu$, and knots at $\prod_n$. This means that $y_n \in S_{r,n}^\nu$ if and only if $y_n$ is a piecewise polynomial of degree $\le r - 1$ on each $I_i$ and has $\nu - 1$ continuous derivatives on $(a, b)$. If $\nu = 0$, then there is no continuity requirement at the knots. As in [22], in this case, we take $y_n \in S_{r,n}^0$ to be left continuous at the nonzero knots and right continuous at 0. Denote by $P_n^G$ the orthogonal projection onto $S_{r,n}^\nu$. It is well-known that when $\nu = 0$ or 1,

$$\|P_n^G\|_{L_\infty \to L_\infty} \le c. \tag{1.12}$$

for all $n \in N$ and for all partitions $\prod_n$ [8]. For $\nu > 1$, the projections $\{P_n^G\}$ are also uniformly bounded in every $L_p$ norm $(1 \leq p \leq \infty)$ under the quasiuniform mesh assumption

$$\frac{h}{\min\limits_{1 \leq i \leq n} h_i} \leq c, \qquad \text{for each } n \text{ and some constant } c > 0. \tag{1.13}$$

See [19].

For the collocation method, we denote the interpolatory projector by $P_n^C$. We select the collocation points $\{\tau_{ij}\}_{j=1}^r$ to be the zeros of the $r$th degree Legendre polynomial (the Gaussian quadrature points) on $[-1, 1]$ shifted to the interval $I_i$. $P_n^C g \in S_{r,n}^0$ is defined for all $g \in C[a,b] \dotplus S_{r,n}^0$ (here $C[a,b] \dotplus S_{r,n}^0$ denotes the direct sum of $C[a,b]$ and $S_{r,n}^0$) by

$$P_n^C g(\tau_{ij}) = g(\tau_{ij}), \qquad 1 \leq i \leq n, \qquad 1 \leq j \leq r. \tag{1.14}$$

The uniform boundedness of the projectors $\{P_n^C\}$ follows by noting that $\|P_n^C\|$ is the norm of the Lagrange interpolation operator for polynomial interpolation at the $r$ Gauss-Legendre points, hence from approximation theory, it is uniformly bounded in $n$. For the Galerkin and the collocation methods ($P_n = P_n^G$ or $P_n = P_n^C$ respectively), we have the following fundamental results from [22]. Here we denote the $t$-section of $k$ by $k_t$, -i.e.,

$$k_t(s) \equiv k(t,s). \tag{1.15}$$

**Theorem 1.1** *Assume that $f \in C[a,b]$ and $k_t \in L_1[a,b]$. Also assume that*

$$\lim_{t \to \tau} \|k_t - k_\tau\|_{L_1} = 0, \qquad \textit{for } \tau \in [a,b]. \tag{1.16}$$

*Then in both the Galerkin and the collocation methods, for sufficiently large $n$, we have*

*(i) $y_n$ in (1.9) exists uniquely in $S_{r,n}^\nu$ (with $\nu = 0$ in the collocation case), and $y_n^I$ exists uniquely in $C[a,b]$;*

*(ii) there exist $c > 0$ such that $\inf\limits_{\phi_n \in S_{r,n}^\nu} \|y - \phi_n\|_\infty \leq \|y - y_n\|_\infty \leq c \inf\limits_{\phi_n \in S_{r,n}^\nu} \|y - \phi_n\|_\infty$;*

*(iii) there exist $c_1, c_2 > 0$ such that $c_1\|K(y - P_n y)\|_\infty \leq \|y - y_n^I\|_\infty \leq c_2\|K(y - P_n y)\|_\infty$.*

Before we present the methods of obtaining the superconvergence of the Galerkin and the collocation methods, it is also beneficial to review the following standard results from approximation theory. For $1 \leq p \leq \infty$ and $m$ a nonnegative integer, let $W_p^m = W_p^m(a,b)$

denote the Sobolev space of functions such that $g^{(k)} \in L_p(a,b)$ for $k = 0,\ldots,m$ where $g^{(k)}$ denotes the $k$th derivative of $g$ in the sense of distribution. We define the norm for $W_p^m$ by

$$\|g\|_{W_p^m} = \sum_{k=0}^{m} \|g^{(k)}\|_{L_p}.$$

The following two theorems are described in [22] and they are standard results in approximation theory.

**Theorem 1.2** *Let $0 \le \nu < r$ and let $1 \le p \le \infty$. If $g \in W_p^m$, $m \ge 0$, then for each $n \ge 1$, there exists $\phi_n \in S_{r,n}^{\nu}$ such that*

$$\|g - \phi_n\|_{L_p} \le ch^{m^*}\|g\|_{W_p^m},$$

*where $m^* = \min\{m,r\}$ and $c$ is a constant independent of $h$ and $g$.*

**Theorem 1.3** *Let $l$ be a positive integer.*

*(i) Let $g \in W_1^l$. Then there exists a polynomial $p$ of degree $\le l-1$ such that*

$$\|(g-p)^{(j)}\|_{W_1^l} \le c(b-a)^{l-j}\|g\|_{W_1^l} \qquad 0 \le j \le l.$$

*(ii) Let $g \in W_1^l$. Define $\|g\|_{L_1,I_i}$ as the $L_1$ norm of $g$ restricted to the interval $I_i$. Then for each $n \ge 1$, there exists $\phi_n \in S_{l,n}^0$ with the properties*

*(a)* $\|(g-\phi_n)^{(j)}\|_{L_1,I_i} \le ch_i^{l-j}\|g^{(l)}\|_{L_1,I_i}, \qquad 1 \le i \le n,\ 0 \le j \le l.$

*(b)* $\max_{1\le i\le n}\|\phi_n^{(j)}\|_{\infty,I_i} \le c\|g\|_{W_1^l}, \qquad j \ge 0$, *where $c$ is independent of $n$ and $g$.*

We are now in a position to state the superconvergence results of Sloan's iterate for the Galerkin as well as for the collocation methods. The outline of proofs are also included because they are frequently referred in the sequel and also this will make this thesis as self-contained as possible.

**Theorem 1.4** *(Theorem 4.1 of [22]) Let $y_n^{GI}$ denote the iterated Galerkin solution. Assume that $f \in C[a,b]$ in (1.3) and that (1.16) holds. Suppose $y \in W_p^l$ $(0 \le l)$ and $k_t \in W_q^m$ $(0 \le m)$, with $\|k_t\|_{W_q^m}$ bounded independently of $t$, $p$ and $q$ conjugate indices and $y_n^G \in S_{r,n}^{\nu}$, $0 \le \nu < r$. Let $\delta_1 = \min(l,r)$ and $\delta_2 = \min(m,r)$. Then*

$$\|y - y_n^{GI}\|_\infty = O(h^{\delta_1+\delta_2}).$$

**Proof:** From Theorem 1.1, in order to estimate $\|y - y_n^{GI}\|_\infty$, it is sufficient to estimate $\|K(y - P_n^G y)(t)\|_\infty$. For $t \in [a,b]$, we have

$$\begin{aligned} |K(y - P_n^G y)(t)| &= \left|\int_a^b k_t(s)(y - P_n^G y)(s)ds\right| \\ &= |(k_t, y - P_n^G y)| \\ &= |(k_t - o_n, y - P_n^G y)|, \end{aligned}$$

where $o_n$ is any element of $S_{r,n}^\nu$, and the last step follows from the orthogonality of $P_n^G$. Using Hölder's inequality, we have

$$\begin{aligned} |K(y - P_n^G y)(t)| &\leq \|k_t - o_n\|_{L_q}\|y - P_n^G y\|_{L_p} \\ &= \|k_t - o_n\|_{L_q}\|(I - P_n^G)(y - v_n)\|_{L_p} \\ &\leq \|k_t - o_n\|_{L_q}(1 + \|P_n^G\|_{L_p \to L_p})\|y - v_n\|_{L_p}, \end{aligned}$$

where $v_n$ is any element of $S_{r,n}^\nu$. Two applications of Theorem 1.2 finish the proof. □

**Theorem 1.5** *(Theorem 4.2 of [22]) Let $y_n^{CI}$ denote the iterated collocation solution. Assume that $f \in C[a,b]$ in (1.3) and that (1.16) holds. Suppose $y \in W_1^l$ $(0 < l \leq 2r)$ and $k_t \in W_1^m$ $(0 < m \leq r)$ with $\|k_t\|_{W_1^m}$ bounded independent of $t$, and $y_n^C \in S_{r,n}^0$ is the solution of (1.9) with $P_n = P_n^C$, $r > 0$. Then*

$$\|y - y_n^{CI}\|_\infty = O(h^\gamma), \qquad \textit{where } \gamma = \min\{l, r+m\}.$$

**Proof:** Throughout this proof, $c$ is a generic constant. Using Theorem 1.3, there exists $v_n \in S_{l,n}^0$ such that

$$\sum_{i=1}^n \|(y - v_n)^{(j)}\|_{L_1,I_i} \leq ch^{l-j}\|y\|_{W_1^l}, \qquad 0 \leq j \leq l, \tag{1.17}$$

$$\max_{1\leq i\leq n} \|v_n^{(j)}\|_{\infty,I_i} \leq c\|y\|_{W_1^l}, \qquad j \geq 0. \tag{1.18}$$

Also by Theorem 1.3, for each $t \in [0,1]$, there exists $o_{n,t} \in S_{m,n}^0$ such that

$$\sum_{i=1}^n \|(k_t - o_{n,t}^{(j)})\|_{L_1,I_i} \leq ch^{m-j}\sup_{t'}\|k_{t'}\|_{W_1^m}, \qquad 0 \leq j \leq m, \tag{1.19}$$

$$\max_{1\leq i\leq n} \|o_{n,t}^{(j)}\|_{\infty,I_i} \leq c\sup_{t'}\|k_{t'}\|_{W_1^m}, \qquad j \geq 0. \tag{1.20}$$

As in the previous theorem, we need to estimate $\|K(y - P_n^C y)\|_\infty$. For $t \in [0,1]$ we have

$$\begin{aligned} K(y - P_n^C y)(t) &= (k_t, y - P_n^C y) \\ &= (k_t - \varphi_{n,t}, y - P_n^C y) + (\varphi_{n,t}, (I - P_n^C)(y - v_n)) + \\ &\qquad (\varphi_{n,t}, (I - P_n^C)v_n). \end{aligned}$$

Now we must show that each of the three terms in the last expression is bounded by $ch^\gamma$, uniformly in $t$. For the first term, we obtain, for arbitrary $\xi_n \in S^0_{r,n}$,

$$\begin{aligned} |(k_t - \varphi_{n,t}, y - P_n^C y)| &\le \|k_t - \varphi_{n,t}\|_{L_1}\|(I - P_n^C)(y - \xi_n)\|_{L_\infty} \\ &\le c\|k_t - \varphi_{n,t}\|_{L_1}\|y - \xi_n\|_{L_\infty}, \end{aligned}$$

where the last step follows from (1.12). Now it follows from the Sobolev embedding theorem that $W_1^l \subset W_\infty^{l-1}$, and hence $y \in W_\infty^{l-1}$. With an appropriate choice of $\xi_n$, it follows from Theorem 1.2 and from (1.19) with $j = 0$, that

$$|(k_t - \varphi_{n,t}, y - P_n^C y)| \le h^{m+\min(r,l-1)}K_m\|y\|_{W_\infty^{l-1}},$$

where $K_m = c\sup_{t'}\|k_{t'}\|_{W_1^m}$. Then because $m + \min(r, l-1) \ge \gamma$, it follows that

$$|(k_t - \varphi_{n,t}, y - P_n^C y)| \le ch^\gamma, \tag{1.21}$$

with $c$ independent of $n$ and $t$.

For the second term we obtain using Hölder's inequality and from (1.20),

$$\begin{aligned} |(\varphi_{n,t}, (I - P_n^C)(y - v_n))| &\le cK_m\|(I - P_n^C)(y - v_n)\|_{L_1} \\ &= cK_m \textstyle\sum_{i=1}^n \|(I - P_n^C)(y - v_n)\|_{L_1,I_i}. \end{aligned}$$

But we have

$$\begin{aligned} \|P_n^C(y - v_n)\|_{L_1,I_i} &\le h_i\|P_n^C(y - v_n)\|_{L_\infty,I_i} \\ &\le ch_i\|y - v_n\|_{L_\infty,I_i} \\ &\le c(\|y - v_n\|_{L_1,I_i} + h_i\|(y - v_n)^{(1)}\|_{L_1,I_i}), \end{aligned}$$

where the penultimate step follows from (1.12) and the last step from the following observations. For $g \in W_1^l(I_i)$, $|I_i|$ =the length of $I_i$, for $x, t \in I_i$, we have $g(x) - g(t) = \int_t^x g^{(1)}(s)ds$. By the mean value theorem for integrals, there exists $\sigma \in I_i$ such that

$$\|g\|_{L_1,I_i} = \int_{I_i} |g(s)|ds = |I_i||g(\sigma)|.$$

Putting $t = \sigma$, we have $g(x) = g(\sigma) + \int_\sigma^x g^{(1)}(s)ds$ and hence

$$\begin{aligned} \|g\|_{L_\infty, I_i} &\le |g(\sigma)| + \|g^{(1)}\|_{L_1, I_i} \\ &= |I_i|^{-1}\|g\|_{L_1, I_i} + \|g^{(1)}\|_{L_1, I_i}. \end{aligned}$$

Hence

$$\begin{aligned} |(\varphi_{n,t}, (I - P_n^C)(y - v_n))| &\le \textstyle\sum_{i=1}^n [c_1\|y - v_n\|_{L_1, I_i} + c_2 h_i \|(y - v_n)^{(1)}\|_{L_1, I_i}] \\ &\le ch^l \|y\|_{W_1^l} \le h^\gamma. \end{aligned}$$

where we have used (1.17) with $j = 0, 1$. Finally, to prove the third term is of order $h^\gamma$, we note that

$$(\varphi_{n,t}, (I - P_n^C)v_n) = \sum_{i=1}^n (\varphi_{n,t}, (I - P_n^C)v_n)_{I_i}. \tag{1.22}$$

It is clear that (1.22) is zero if $0 < l \le r$, since we have $P_n^C v_n = v_n$. Therefore we need consider only the case $r < l \le 2r$.

As $\varphi_{n,t}$ is a polynomial of degree $\le m - 1$ on $I_i$, we can write

$$\varphi_{n,t}(s) = \sum_{k=0}^{m-1} \varphi_{n,t}^{(k)}(t_i)(s - t_i)^k / k!, \qquad s \in I_i,$$

where $t_i$ is the midpoint of $I_i$. We then have

$$\begin{aligned} |(\varphi_{n,t}, (I - P_n^C)v_n)_{I_i}| &= |\textstyle\sum_{k=0}^{m-1} \varphi_{n,t}^{(k)}(t_i) \int_{I_i} (s - t_i)^k (I - P_n^C) v_n(s) ds / k!| \\ &\le h_i^{2r+1} \textstyle\sum_{k=0}^{m-1} \|\varphi_{n,t}^{(k)}\|_{L_\infty, I_i} \|v_n^{(2r-k)}\|_{L_\infty, I_i} \\ &\le ch_i^{2r+1}, \end{aligned}$$

where the first inequality follows from Lemma A.2 of [22] (for completeness we state below without proof Lemma A.2) and the final step is a consequence of (1.18) and (1.20). Using (1.22) we see that

$$|(\varphi_{n,t}, (I - P_n^C)v_n)| \le \sum_{i=1}^n ch_i^{2r+1} \le ch^{2r} \le ch^\gamma.$$

□

**Lemma 1.6** *(Lemma A.2 of [22]) Let $r$ be a fixed positive integer, let $j$ be an integer in $r \le j \le 2r$, let $J$ be any bounded open interval, and let $t_J$ be any point in $J$. For $g \in C^j(\bar{J})$, define*

$$E_j(g) = \int_J (s - t_J)^{2r-j} (I - P_J) g(s) ds,$$

*where $P_J$ is the polynomial of degree $\leq r-1$ that coincides with $g$ at the zeros of the $r$th degree Legendre polyniminal shifted to $J$. Then*

$$|E_J(g)| \leq \frac{|J|^{2r+1}}{r!(j-r)!}\|g^{(j)}\|_{\infty,J}.$$

It has been demonstrated that (see refs. [60]-[66]) under mild conditions on $k$ and $f$, $y_n^I$ converges faster globally to $y$ than $y_n$ does to $y$, -i.e., $\|y_n^I - y\| = O(h^\beta)$ with $r < \beta \leq 2r$. The doubling of convergence rate to $2r$ is attained in the case that the kernel $k$ and the forcing term $f$ in (1.3) are at least $r$ times continuously differentiable functions. This observation applies both to the iterated Galerkin method and to the iterated collocation method. Particularly, for the iterated collocation method, superconvergence occurs when the collocation points are the Gaussian points due to the orthogonality of Legendre polynomials. In the cases of the weakly singular Fredholm equations as well as the weakly singular Hammerstein equations, some enhancements in the convergence rates for the Sloan iterates were observed in [34], [52], [37], [70], and [35]. There is one important difference that we must consider between the Galerkin and the collocation methods. Namely, in the collocation method, the sensitivity of the superconvergence to the location of the collocation points must be considered [38] whereas the Galerkin method obviates such considerations.

This thesis is organized as follows. In Chapter 2, a general operator approximation scheme for the second kind linear equation is described that can be used to explain the superconvergence results of Theorems 1.4 and 1.5. Moreover, a corollary will be given that can be used to establish the superconvergence of the iterated degenerate kernel method. Chapters 3,4, and 5 are devoted to a study of Hammerstein equations. Hammerstein equations arise naturally from the study of a class of boundary value problems with certain nonlinear boundary conditions. We review the iterated Galerkin method for Hammerstein equations in Chapter 3. In addition to the review, a discussion of the iterated degenerate kernel method for Hammerstein equations is also included in this chapter. Chapter 4 is devoted to an investigation of the iterated collocation method for Hammerstein equations. The weakly singular Hammerstein equations are also treated in Chapters 3 and 4, and its corresponding superconvergence phenomena for the iterated solutions are described. The type of regularities that the solution of weakly singular Hammerstein equations possess is given in Chapter 5. The result obtained in the chapter extends the result of Cao and Xu in [11]. Subsequently, we establish the singularity preserving Galerkin method for

Hammerstein equations. The superconvergence results for the iterated solutions corresponding to this method will conclude this chapter. In the final chapter, Chapter 6, we state briefly future research areas that are related to the topics encompassed in this thesis.

# CHAPTER II
# THE ITERATED DEGENERATE KERNEL METHOD

## INTRODUCTION

In this chapter, we start by considering the Fredholm integral equation of the second kind given by (1.3). We assume that

$$f \in C[a,b]. \tag{2.1}$$

With $K : C[a,b] \to C[a,b]$, the integral operator defined in (1.2), the compactness of $K$ is guaranteed by assuming (1.16), i.e.

$$\lim_{t \to \tau} \int_a^b |k_t(s) - k_\tau(s)|ds = 0 \qquad \text{for each } \tau \in [a,b]. \tag{2.2}$$

See [21].

In order to establish a general iterated approximation scheme, we assume that $\{K_n\}$ is a sequence of operators converging to $K$ in some operator norm. That is,

$$\|K_n - K\|_{L_p} \to 0 \text{ as } n \to \infty \text{ for some } 1 \leq p \leq \infty. \tag{2.3}$$

For each $n \geq 1$, we assume that we have an equation whose solution approximates the solution $y$ of (1.3)

We denote this approximating equation by

$$y_n = f_n + K_n y_n. \tag{2.4}$$

For example, in the case of the projection method, equation (2.4) is identified by letting $K_n = P_n K$ and $f_n = P_n f$ where $P_n$ is a projection of a Banach space $X$ onto some finite dimensional subspace $X_n$ of $X$. In the case of the degenerate kernel method, $K_n$ denotes the finite rank separable operator, -i.e. $K_n y(t) = \int_a^b \sum_{i=1}^n \sum_{j=1}^n a_{ij}\varphi_i(t)\varphi_j(s)y(s)ds$ where $\{\varphi_i\}_{i=1}^n$ is a linearly independent family of functions defined on $[a,b]$ and $f_n = f$ for each $n \geq 1$. We define the iterated approximation corresponding to (2.4) by

$$y_n^I = f + K y_n. \tag{2.5}$$

As was indicated previously, the iterated approximations for the Galerkin and for the collocation methods exhibit, under suitable smoothness conditions on the kernel $k$ and on the forcing term $f$, global superconvergence. It is shown in this chapter that, a similar superconvergence result can be obtained for the iterated approximations for the degenerate kernel method. Next, we prove the main theorem of this chapter. Known superconvergence results are special cases of this theorem and it can be used to establish the superconvergence of the iterated degenerate kernel method.

**Theorem 2.1** *Consider equation (1.3) in a Banach space $(X, \|\cdot\|)$ where $K$ is a compact linear operator of $X$ into $X$. We assume that 1 is not an eigenvalue of $K$ and that condition (2.3) is satisfied with respect to the norm $\|\cdot\|$. Let $y_n$ and $y_n^I$ satisfy equations (2.4) and (2.5) respectively. Then, for sufficiently large $n$, there exists a constant $c > 0$, independent of $n$, such that*

$$\|y - y_n^I\| \leq c\{\|K - K_n\|^2 + \|K(K - K_n)y_n\| + \|K - K_n\|\,\|f - f_n\| + \|K(f - f_n)\|\}. \quad (2.6)$$

**Proof:** From (1.3) and (2.5),

$$y - y_n^I = K(y - y_n). \quad (2.7)$$

Applying $K$ on both sides of (1.3) and (2.4), we obtain

$$Ky = Kf + K^2 y \quad (2.8)$$

and

$$Ky_n = KK_n y_n + Kf_n. \quad (2.9)$$

It follows from (2.8) and (2.9) that

$$\begin{aligned} K(y - y_n) &= K^2 y - KK_n y_n + K(f - f_n) \\ &= K(Ky - K_n y_n) + K_n(Ky - Ky_n) - K_n(Ky - Ky_n) + K(f - f_n) \\ &= K_n(Ky - Ky_n) + (K - K_n)(Ky - Ky_n) + K(K - K_n)y_n + K(f - f_n). \end{aligned} \quad (2.10)$$

Since $\|K_n - K\| \to 0$ as $n \to \infty$ and $(I - K)^{-1}$ exists by assumption, we conclude [3] that $(I - K_n)^{-1}$ exists and is uniformly bounded for sufficiently large $n$. Therefore,

$$K(y - y_n) = (I - K_n)^{-1}\{(K - K_n)(Ky - Ky_n) + K(K - K_n)y_n + K(f - f_n)\}.$$

Taking the norm on both sides,

$$\begin{aligned} \|K(y - y_n)\| \leq\; & \|(I - K_n)^{-1}\|\{\|K - K_n\|\|K\|\|y - y_n\| + \|K(K - K_n)y_n\| \\ & + \|K(f - f_n)\|\}. \end{aligned} \quad (2.11)$$

Since

$$\begin{aligned} y - y_n &= Ky - K_n y_n + f - f_n \\ &= Ky - K_n y + K_n y - K_n y_n + f - f_n \end{aligned}$$

we obtain

$$(I - K_n)(y - y_n) = Ky - K_n y + f - f_n.$$

or

$$y - y_n = (I - K_n)^{-1}\{(K - K_n)y + f - f_n\}. \tag{2.12}$$

From (2.7), (2.11) and (2.12),

$$\begin{aligned} \|y - y_n^I\| &= \|K(y - y_n)\| \\ &\le c\{\|K - K_n\|\|y - y_n\| + \|K(K - K_n)y_n\| + \|K(f - f_n)\|\} \\ &\le c\{\|K - K_n\|^2 + \|K - K_n\|\|f - f_n\| + \|K(K - K_n)y_n\| \\ &\quad + \|K(f - f_n)\|\}. \end{aligned}$$

This completes the proof. □

A new version of this theorem was recently obtained and is given below. The new theorem does not change the original conclusions presented but provides a simpler expression for the bound on $\|y - y_n^I\|$.

**Theorem 2.2** *Consider equation (1.3) in a Banach space $(X, \|\cdot\|)$ where $K$ is a compact linear operator of $X$ into $X$. We assume that 1 is not an eigenvalue of $K$ and that condition (2.3) is satisfied with respect to the norm $\|\cdot\|$. Let $y_n$ and $y_n^I$ satisfy equations (2.4) and (2.5) respectively. Then, for sufficiently large $n$, there exists a constant $c > 0$, independent of $n$, such that*

$$\|y - y_n^I\| \le c\{\|K(K - K_n)y_n\| + \|K(f - f_n)\|\}. \tag{2.13}$$

**Proof:** From (1.3) and (2.5),

$$y - y_n^I = K(y - y_n). \tag{2.14}$$

Applying $K$ on both sides of (1.3) and (2.4), we obtain

$$Ky = Kf + K^2 y \tag{2.15}$$

and

$$Ky_n = KK_n y_n + Kf_n. \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\begin{aligned} K(y - y_n) &= K^2 y - KK_n y_n + K(f - f_n) \\ &= K(Ky - K_n y_n) + KKy_n - KKy_n + K(f - f_n) \\ &= K(Ky - Ky_n) + K(K - K_n)y_n + K(f - f_n). \end{aligned} \tag{2.17}$$

Since $K$ is a compact linear operator which does not have 1 as an eigenvalue, then $(I - K)^{-1}$ exists and is bounded. Therefore,

$$K(y - y_n) = (I - K)^{-1}\{K(K - K_n)y_n + K(f - f_n)\}.$$

Taking the norm on both sides,

$$\|K(y - y_n)\| \le \|(I - K)^{-1}\|\{\|K(K - K_n)y_n\| + \|K(f - f_n)\|\}. \tag{2.18}$$

From (2.14), and (2.18),

$$\begin{aligned} \|y - y_n^I\| &= \|K(y - y_n)\| \\ &\le c\{\|K(K - K_n)y_n\| + \|K(f - f_n)\|\} \end{aligned}$$

This completes the proof. □

The following corollary is based upon Theorem 2.1.

**Corollary 2.3** *For the iterated approximation scheme (2.5), if $f_n = f$ for all $n$ in (2.4), then*

$$\|y - y_n^I\| \le c\{\|K - K_n\|^2 + \|K(K - K_n)y_n\|\}.$$

Now we note that Theorem 2.1 includes the results of superconvergence of the iterated Galerkin and the iterated collocation schemes. Let $P_n^G$ denote an orthogonal projection (with respect to the standard $L_2$ inner product) onto $S_{r,n}^{\nu}$. In the Galerkin method, equation (2.4) becomes

$$y_n^G - P_n^G K y_n^G = P_n^G f \tag{2.19}$$

-i.e. $K_n = P_n^G K$ and $f_n = P_n^G f$. The corresponding iteration approximation to (2.5) is given by

$$y_n^{GI} = f + K y_n^G. \tag{2.20}$$

If $f \in W_p^m$, $(m \geq 0)$, then from Theorem 1.2, there exists $v_n \in S_{r,n}^{\nu}$ $(0 \leq \nu < r)$ such that

$$\|f - v_n\|_{L_p} \leq ch^{\min(m,r)}\|f\|_{W_p^{\min(m,r)}} \tag{2.21}$$

where $c$ is a constant independent of $n$ (see e.g. [59]). Once again, we use $c$ for a generic constant independent of $n$ below. Under the assumption of the quasiuniform mesh (1.13),

$$\sup_n \|P_n^G\|_{L_p \to L_p} \leq c. \tag{2.22}$$

Since

$$\begin{aligned} \|f - P_n^G f\|_{L_p} &= \|f - v_n + P_n^G v_n - P_n^G f\|_{L_p} \\ &\leq (1 + \|P_n^G\|_{L_p \to L_p})\|f - v_n\|_{L_p}, \end{aligned} \tag{2.23}$$

from (2.21), (2.22) and (2.23), we obtain

$$\|f - P_n^G f\|_{L_p} \leq ch^{\min(m,r)}\|f\|_{W_p^{\min(m,r)}}. \tag{2.24}$$

Now let $\xi(t) = \int_a^b k(t,s)y_n^G(s)ds$. Then, following the argument used in the proof of Theorem 1.4,

$$\begin{aligned} |K(K - K_n)y_n^G(t)| &= |\int_a^b k(t,u)\{\xi(u) - P_n^G\xi(u)\}du| \\ &= |(k_t, \xi - P_n^G\xi)| \\ &= |(k_t - \varphi_n, \xi - P_n^G\xi)| \text{ for every } \varphi_n \in S_{r,n}^{\nu} \\ &\leq \|k_t - \varphi_n\|_{L_q}\|\xi - P_n^G\xi\|_{L_p}, \end{aligned} \tag{2.25}$$

where $\frac{1}{q} + \frac{1}{p} = 1$ with convention that if $p = 1$, then $q = \infty$. In (2.25), we have used the orthogonality in the third equality and the Hölder inequality in the last step. If $k_t \in W_q^m$ with $\|k_t\|_{W_q^m}$ bounded independently of $t$ and if $\xi(t) \in W_p^m$ then from Theorem 1.2 there exists $\varphi_n \in S_{r,n}^{\nu}$ such that $\|k_t - \varphi_n\|_{L_q} \leq ch^{\min(m,r)}\|k_t\|_{W_q^{\min(m,r)}}$. Finally, from (2.25) we obtain

$$\|K(K - K_n)y_n^G\|_\infty \leq ch^{2\min(m,r)}.$$

Similarly, we can show that whenever $f \in W_p^m$,

$$\|K(f - P_n^G f)\|_\infty \leq ch^{2\min(m,r)}$$

and that, with $K_n = P_n K$,

$$\|K - K_n\|_\infty \leq ch^{\min(m,r)}.$$

Using these estimates, we obtain Theorem 1.4 as a corollary. In summary, we have

**Corollary 2.4** *(see Theorem 1.4) Let $y_n^G$ and $y_n^{GI}$ denote the solutions for (2.19) and (2.20) respectively. Suppose that $y \in W_p^m$, $k_t \in W_q^m$ $(m \geq 0)$ with $\|k_t\|_{W_q^m}$ bounded independently of $t$ and that $f, \xi \in W_p^m$ where $\xi(t) \equiv \int_a^b k(t,s)y_n^G(s)ds$. Then*

$$\|y - y_n^{GI}\|_\infty \leq ch^{2\min(m,r)}$$

*where $c$ is independent of $n$.*

For the iterated collocation method we select in the partition $\prod_n$, for each $i$, $\{t_{ij}\}_{j=1}^r$ such that

$$t_{i-1} \leq t_{i1} < t_{i2} < \cdots < t_{ir} \leq t_i.$$

Let $P_n^C$ denote the interpolatory projector of $C[a,b]$ onto $S_{r,n}^\nu$ defined by $P_n^C y(t_{ij}) = y(t_{ij})$ for each $i = 1, \ldots, n$ and $j = 1, 2, \ldots, r$. In the collocation method, equation (2.4) becomes

$$y_n^C - P_n^C K y_n^C = P_n^C f \tag{2.26}$$

-i.e. $K_n = P_n^C K$ and $f_n = P_n^C f$. The corresponding iterated collocation solution is defined by

$$y_n^{CI} = f + K y_n^C. \tag{2.27}$$

As in Corollary 2.4 for the iterated Galerkin method, to see that the iterated collocation method of (2.27) is a special case of Theorem 2.1, we must examine the terms in the right side of (2.6). The second term of (2.6) in this case is analyzed as follows: Let $\chi(t) = \int_a^b k(t,s)y_n^C(s)ds$. Then

$$\begin{aligned} K(K - K_n)y_n^C(t) &= (k_t, \chi - P_n^C \chi) \\ &= (k_t - \varphi_{n,t}, \chi - P_n^C \chi) + (\varphi_{n,t}, (I - P_n^C)(\chi - \upsilon_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n^C)\upsilon_n). \end{aligned} \tag{2.28}$$

where $\varphi_{n,t} \in S_{m,n}^0$ and $\upsilon_n \in S_{l,n}^0$. Now arguing exactly as in the proof of Theorem 1.5, we obtain

$$\|K(K - K_n)y_n^C\|_\infty \leq ch^{\min(l,m+r)}$$

where $c$ is a constant independent of $n$. Additional terms in (2.6) can be bounded similarly.

**Corollary 2.5** *(see Theorem 1.5) Let $y_n^C$ and $y_n^{CI}$ be the solutions of (2.26) and (2.27) respectively. Suppose $f \in C[a,b]$, $y \in W_1^l$ $(0 < l \leq 2r)$ and $k_t \in W_1^m$ $(0 < m \leq r)$, with $\|k_t\|_{W_1^m}$ bounded independently of $t$. Then*

$$\|y - y_n^{CI}\|_\infty \leq ch^{\min(l,r+m)}$$

*where c is independent of n.*

Now we can use Theorem 2.1, Corollary 2.3 in particular, to prove the superconvergence of the iterated degenerate kernel method.

Consider equation (1.1). The degenerate kernel method for approximating the solution of (1.1) requires us to approximate the kernel $k$ by a degenerate kernel whose general form can be described as

$$k_n(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\varphi_i(s)\varphi_j(t) \tag{2.29}$$

where $\{\varphi_i\}_{i=1}^{n}$ is a set of linearly independent functions in an underlying Banach space $X$. The operator $K$ in (1.2) is then approximated by a sequence of finite rank operators

$$K_n y(t) = \int_a^b k_n(t,s)y(s)ds. \tag{2.30}$$

Subsequently an approximate solution $y_n$ is found by solving

$$y_n(t) - \int_a^b k_n(t,s)y_n(s)ds = f(t) \qquad a \le t \le b. \tag{2.31}$$

Equation (2.31) can be written as

$$y_n(t) - \sum_{i=1}^{n}\varphi_i(t)\{\sum_{j=1}^{n}\int_a^b a_{ij}\varphi_j(s)y_n(s)ds\} = f(t) \qquad a \le t \le b.$$

If we put

$$c_i \equiv \sum_{j=1}^{n}\int_a^b a_{ij}\varphi_j(s)y_n(s)ds. \tag{2.32}$$

then $y_n$ can be written as

$$y_n(t) = f(t) + \sum_{i=1}^{n} c_i\varphi_i(t). \tag{2.33}$$

Upon substituting (2.33) into (2.32), we obtain the following $n \times n$ system of linear equations for $c_i$.

$$c_i - \sum_{l=1}^{n} c_l \sum_{j=1}^{n}\int_a^b a_{ij}\varphi_j(s)\varphi_l(s)ds = \sum_{j=1}^{n}\int_a^b a_{ij}\varphi_j(s)f(s)ds \qquad 1 \le i \le n. \tag{2.34}$$

Finally, once these $c_i$'s are found by solving (2.34), equation (2.33) gives the required approximate solution for the degenerate kernel method. Equation (2.31) is written in operator form as

$$y_n - K_n y_n = f \tag{2.35}$$

which is a special form of (2.4) with $f_n = f$ for all $n$. When the degenerate kernel solution $y_n$ is iterated as in (2.5), an interesting question is to ask under what conditions is the superconvergence of the iterates guaranteed. The superconvergence of the degenerate kernel method hinges critically upon the ways that the kernel $k$ is decomposed. We demonstrate two different methods that guarantee the superconvergence of the iterates of the degenerate kernel method.

In the first method, we examine the least-squares approximation. For each positive integer $n$, assume that the partition $\prod_n$ satisfies the quasiuniform condition (1.13). Let $B_1, B_2, \ldots, B_d$ be the $B$-spline basis for $S^{\nu}_{r,n}$ [59], with $d = nr - \nu(n-1)$ the dimension of $S^{\nu}_{r,n}$, and, $r$ and $\nu$ are integers such that $0 \leq \nu < r$. Assume that $k_n(t,s)$ is the least-squares approximation of $k(t,s)$ from the tensor product space $S^{\nu}_{r,n} \otimes S^{\nu}_{r,n}$, -i.e. assume that $a_{ij}$ in (2.29) are such that

$$\int_a^b \int_a^b |k(t,s) - \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij} B_i(s) B_j(t)|^2 ds dt = \min_{b_{ij} \in R} \int_a^b \int_a^b |k(t,s) - \sum_{i=1}^{d}\sum_{j=1}^{d} b_{ij} B_i(s) B_j(t)|^2 ds dt.$$

**Theorem 2.6** *Let $y \in L_2[a,b]$ be the solution of (1.1) and $y_n$ the solution of (2.35) where $k_n$ in (2.29) is defined by the least-squares approximation for $k$ from $S^{\nu}_{r,n} \otimes S^{\nu}_{r,n}$. Assume that $k(t,u) \in W_2^m([a,b] \times [a,b])$, $0 \leq m \leq r$, $k_t(u) y_n(s) \in W_2^l([a,b] \times [a,b])$ for each $n$ and $t \in [a,b]$ and that $\|k_t(u) y_n(s)\|_{L_2}$ is uniformly bounded in $t$, where $0 \leq l \leq r$. Then*

$$\|y - y_n^I\|_{L_2} = O(h^{\eta})$$

*with $\eta = \min\{m + l, 2m\}$.*

**Proof:** Using Corollary 2.3 and noting $\|K - K_n\|_{L_2} = O(h^m)$ [16], we obtain

$$\|y - y_n^I\|_{L_2} = O(h^{2m}) + O(\|K(K - K_n) y_n\|_{L_2}). \tag{2.36}$$

Hence we only need to estimate the order of convergence of $\|K(K - K_n) y_n\|_{L_2}$. Note that

$$\begin{aligned} |K(K - K_n) y_n(t)| &= |\int_a^b k(t,u) \int_a^b [k(u,s) - k_n(u,s)] y_n(s) ds du| \\ &= |\int_a^b \int_a^b k(t,u)[k(u,s) - k_n(u,s)] y_n(s) ds du|. \end{aligned}$$

Let $v_t(u,s) = k(t,u) y_n(s)$ and let $\varphi_n(u,s) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} B_i(u) B_j(s)$ be any element from $S^{\nu}_{r,n} \otimes S^{\nu}_{r,n}$. Then since $k_n$ is the best approximation in $L_2$ norm of $k$ from $S^{\nu}_{r,n} \otimes S^{\nu}_{r,n}$,

$$\int_a^b \int_a^b \varphi_n(u,s)[k(u,s) - k_n(u,s)] ds du = 0.$$

therefore

$$|K(K-K_n)y_n(t)| = \left|\int_a^b \int_a^b [v_t(u,s) - \varphi_n(u,s)][k(u,s) - k_n(u,s)]dsdu\right|.$$

Applying the Cauchy-Schwartz inequality,

$$|K(K-K_n)y_n(t)| \le \|v_t - \varphi_n\|_{L_2}\|k - k_n\|_{L_2}.$$

Noting that $\|k-k_n\|_{L_2} = O(h^m)$ and choosing $\varphi_n$ particularly so that $\|v_t - \varphi_n\|_{L_2} = O(h^l)$, (2.36) proves the desired result. □

The second method that produces superconvergence of the iterates of the degenerate kernel solutions is based upon the idea of approximating the kernel $k$ by interpolation. Let $\xi_1, \xi_2, \ldots, \xi_r$ be the zeros of the $r$th degree Legendre polynomial in $[-1,1]$. We shift these points to each subinterval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$ to obtain $\{\tau_{ij}\}_{j=1}^r$. Denote the interpolation polynomials by $\varphi_{ij}$, -i.e.

$$\varphi_{ij}(\tau_{\alpha\beta}) = \begin{cases} 1 & \text{if } (i,j) = (\alpha,\beta) \\ 0 & \text{if } (i,j) \ne (\alpha,\beta) \end{cases} \tag{2.37}$$

An approximating degenerate kernel $k_n$ is now defined by

$$k_n(s,t) = \sum_{i=1}^{n}\sum_{j=1}^{r}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{r} k(\tau_{ij},\tau_{\alpha\beta})\varphi_{ij}(s)\varphi_{\alpha\beta}(t). \tag{2.38}$$

Let the interpolation projector of $C([a,b]\times[a,b])$ into $S^0_{r,n} \otimes S^0_{r,n}$ be denoted by $P_n$. That is,

$$P_n k(s,t) = k_n(s,t)$$

where $k_n$ is defined in (2.38). The following theorem demonstrates the superconvergence of the iterated degenerate kernel method when the kernel is decomposed by the interpolation.

**Theorem 2.7** *Assume that in equation (1.1), $k(u,s) \in W_1^m([a,b]\times[a,b])$, $0 < m \le r$, and $k_t(u)y_n(s) \in W_1^l([a,b]\times[a,b])$, $0 < l \le r$, for each $t \in [a,b]$ with $\|k_t(\cdot)y_n(\cdot)\|_{W_1^l([a,b]\times[a,b])}$ bounded independent of $t$ and $n$. Then*

$$\|y - y_n^I\|_\infty = O(h^\nu), \qquad \nu = \min\{m+l, 2m\}.$$

**Proof:** As in the proof of Theorem 2.6. we need to estimate the error of $\|K(K-K_n)y_n\|_\infty$. By taking $\varphi_n \in S^0_{l,n} \otimes S^0_{l,n}$ and $\psi_n \in S^0_{m,n} \otimes S^0_{m,n}$, for each $t \in [a,b]$,

$$\begin{aligned} K(K-K_n)y_n(t) &= \int_a^b k(t,u)\int_a^b[k(u,s)-k_n(u,s)]y_n(s)ds\,du \\ &= \int_a^b\int_a^b k(t,u)y_n(s)[k(u,s)-k_n(u,s)]ds\,du \\ &\equiv (k_t(u)y_n(s), k(u,s)-k_n(u,s)) \\ &= (k_t(u)y_n(s)-\varphi_n(u,s), k(u,s)-k_n(u,s)) \\ &\quad +(\varphi_n(u,s),(I-P_n)(k(u,s)-\psi_n(u,s)))+(\varphi_n(u,s),(I-P_n)\psi_n(u,s)) \end{aligned}$$

The rest of the proof follows once again by an argument similar to the one given in the proof of theorem 4.2 of Graham, Joe and Sloan [22]. A straightforward modification is needed to accommodate the bivariate functions. On this point, the reader is referred to the book by Cheney [16] that contains a discussion on various methods of approximating a bivariate function by elements from the tensor product space of finite dimensional univariate functions. □

## NUMERICAL EXAMPLES FOR FREDHOLM EQUATIONS

We present numerical examples for a second kind Fredholm equation using least-squares (Table 2.1) and interpolation (Table 2.2) to approximate $k(s,t)$. Let $k(s,t) = e^{st}$. $f$ is chosen so that the solution is $y(t) = 1$. Then, the computed errors for the least squares method are shown in the following table. The linear spline basis was used in computations.

Table 2.1: Least Squares Results for Fredholm Equations

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .13626032769435e-1 | .80131241576e-4 |
| 3 | .62297093334709e-2 | .15763350088e-4 |
| 4 | .35682049430 72e-2 | .49780036 48e-5 |
| convergence rate ≈ | 1.93 | 4 |

For the interpolation method, using the roots of the second degree Legendre polynomial, we have the following.

Table 2.2: Interpolation Results for Fredholm Equations

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .130858600472291e-1 | .8242888311 3e-4 |
| 3 | .608756295958 8e-2 | .16471687222e-4 |
| 4 | .350188436357 3e-2 | .52318145 55e-5 |
| convergence rate ≈ | 1.9 | 4 |

In these examples, by the conditions in Theorems 2.6 and 2.7 we have that $m = r$ and $l = r$. Thus, both theorems predict a doubling of the convergence rate. As we can see, with the linear spline basis, $r = 2$, the convergence rate for the non-iterated solution is $\approx 2$, while for the iterated solution it is $2r = 4$.

# CHAPTER III
# THE ITERATED GALERKIN METHOD FOR HAMMERSTEIN EQUATIONS

## INTRODUCTION

In this section, we review the Galerkin method and the iterated Galerkin method for Hammerstein equations that were recently developed in [44]. The review given here for the aforementioned paper is extensive since the Galerkin method and the iterated Galerkin method are two fundamental topics and we feel that any thesis that deals with various numerical methods for the Hammerstein equations should contain a discussion on the subject. The Hammerstein equations can be written as

$$x(t) - \int_a^b k(t,s)\psi(s,x(s))ds = f(t). \tag{3.1}$$

We assume throughout, unless stated otherwise, the following conditions on $k$, $f$ and $\psi$:

(i) $\lim_{t\to\tau} \|k_t - k_\tau\|_\infty = 0, \qquad \tau \in [a,b]$;

(ii) $M \equiv \sup_{a\le s\le b} \int_a^b |k(t,s)|dt < \infty$;

(iii) $f \in C[a,b]$;

(iv) $\psi(t,x)$ is continuous in $t \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$, independent of $t$, for which

$$|\psi(t,x_1) - \psi(t,x_2)| \le C_1|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty); \tag{3.2}$$

(v) the partial derivative $\psi^{(0,1)}$ of $\psi$ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$, independent of $t$, such that

$$|\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| \le C_2|x_1 - x_2|, \text{ for all } x_1, x_2 \in (-\infty,\infty); \tag{3.3}$$

(vi) for $x \in C[0,1]$, $\psi(.,x(.))$, $\psi^{(0,1)}(.,x(.)) \in C[a,b]$.

We note that the condition (ii) is a consequence of the condition (i). We listed (ii) because of its use in the sequel. Additional assumptions will be given later as needed. Without loss of generality we will restrict the interval $(a, b)$ to $(0, 1)$.

Results concerning the Galerkin approximation using spline functions for the solutions of equation (3.1) with smooth and weakly singular kernels are presented.

Let $n$ be a positive integer and $\{X_n\}$ be a sequence of finite dimensional subspaces of $C[0, 1]$ such that for any $x \in C[0, 1]$ there exists a sequence $\{x_n\}$, $x_n \in X_n$, for which

$$\|x_n - x\|_\infty \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Let $P_n : L_2[0, 1] \to X_n$ be an orthogonal projection for each $n$. We assume that the projection $P_n$ when restricted to $C[0, 1]$ is uniformly bounded, i.e.

$$P := \sup_n \|P_n|_{C[0,1]}\|_\infty < \infty. \tag{3.5}$$

Then from (3.4) and (3.5), it follows that for each $x \in C[0, 1]$,

$$\|P_n x - x\|_\infty \to 0, \text{ as } n \to \infty. \tag{3.6}$$

Now let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t, s)\psi(s, x(s))ds.$$

With this notation, equation (3.1) takes the following operator form

$$x - K\Psi x = f. \tag{3.7}$$

In many interesting cases, equation (3.1) allows multiple solutions. Hence it is assumed for the remainder of this paper that we are treating a solution $x$ of equation (3.1) that is isolated.

Let $\{\varphi_{nj}\}_{j=1}^n$ be a set of linearly independent functions that spans $X_n$. The Galerkin method is to find

$$x_n = \sum_{j=1}^n b_{nj}\varphi_{nj}$$

that satisfies

$$x_n - P_n K\Psi x_n = P_n f. \tag{3.8}$$

Equivalently one is required to find $b_{nj}$'s that satisfy the system of nonlinear equations described by

$$\sum_{j=1}^n b_{nj} < \varphi_{nj}, \varphi_{ni} > - < \int_0^1 k(t, s)\psi(s, \sum_{j=1}^n b_{nj}\varphi_{nj}(s))ds, \varphi_{ni} > = < f, \varphi_{ni} >, \ 1 \le i \le n. \tag{3.9}$$

where $< \ldots >$ denotes the inner product in $L_2$.

We next estimate the error of the Galerkin approximate solutions to the exact solution. For notational convenience, we introduce operators $\hat{T}$ and $T_n$ by letting

$$\hat{T}x \equiv f + K\Psi x \tag{3.10}$$

and

$$T_n x_n \equiv P_n f + P_n K\Psi x_n \tag{3.11}$$

so that equations (3.7) and (3.8) can be written respectively as $x = \hat{T}x$ and $x_n = T_n x_n$. A proof of the following theorem can be made by directly applying Theorem 2 of Vainikko [71]. The paper of Atkinson and Potra [7] is also useful in this connection.

**Theorem 3.1** *Let $x \in C[0,1]$ be an isolated solution of equation (3.7). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(x)$, where $(K\Psi)'(x)$ denotes the Fréchet derivative of $K\Psi$ at $x$. Then the Galerkin approximation equation (3.8) has a unique solution $x_n \in B(x, \delta)$ for some $\delta > 0$ and for sufficiently large $n$. Moreover, there exists a constant $0 < q < 1$, independent of $n$, such that*

$$\frac{\alpha_n}{1+q} \leq \|x_n - x\|_\infty \leq \frac{\alpha_n}{1-q}, \tag{3.12}$$

*where $\alpha_n \equiv \|(I - T_n'(x))^{-1}(T_n(x) - \hat{T}(x))\|_\infty$. Finally,*

$$E_n(x) \leq \|x_n - x\|_\infty \leq C E_n(x), \tag{3.13}$$

*where $C$ is a constant independent of $n$ and $E_n(x) = \inf_{u \in X_n} \|x - u\|_\infty$.*

For any positive integer $n$, we assume that the partition $\prod_n$ satisfies the quasiuniform mesh condition (1.13).

Using Theorems 1.2 and 3.1 and the inequality (3.13), we obtain the following theorem.

**Theorem 3.2** *Let $x$ be an isolated solution of equation (3.1) and let $x_n$ be the solution of equation (3.8) in a neighborhood of $x$. Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. If $x \in W_\infty^l$ $(0 \leq l \leq r)$, then*

$$\|x - x_n\|_\infty = O(h^l).$$

*If $x \in W_p^l$ $(0 < l \leq r, 1 \leq p < \infty)$, then*

$$\|x - x_n\|_\infty = O(h^{l-1}).$$

We remark that a similar result concerning the Galerkin method for Urysohn equations was obtained by Atkinson and Potra [7]. Hence, Theorem 3.2 may be derived by specializing their result to Hammerstein equations.

In the remaining portion of this section, we investigate the order of convergence of the Galerkin method for Hammerstein equations with weakly singular kernels. For this purpose, we define some necessary notation. For simplicity, we let $[a,b] = [0,1]$. For any $\epsilon \in R$, let $[0,1]_\epsilon = \{t \in [0,1] : t+\epsilon \in [0,1]\}$. Let $\Delta_h$ denote the forward difference operator with step size $h$. For $\alpha > 0$ and $1 \le p \le \infty$, we define the Nikol'skii space $N_p^\alpha[0,1]$ by

$$N_p^\alpha[0,1] = \left\{ x \in L_p[0,1] : |x|_{\alpha,p} = \sup_{h \neq 0} \left( \frac{1}{|h|^{\alpha_0}} \|\Delta_h^2 x^{([\alpha])}\|_{L_p[0,1]_{2h}} \right) < \infty \right\}. \tag{3.14}$$

where $[\alpha]$ is an integer and $0 < \alpha_0 \le 1$ are chosen so that $\alpha = [\alpha] + \alpha_0$. $N_p^\alpha[0,1]$ is a Banach space with the norm $\|x\|_{\alpha,p} = \|x\|_p + |x|_{\alpha,p}$ [24]. We remark that the function $t^{\alpha-1}$ is in $N_1^\alpha[0,1]$ but is not in $N_1^\beta[0,1]$, for any $\beta > \alpha$, and $\log t \in N_1^1[0,1]$. It is known from Graham [24] that

$$N_p^{m+\epsilon}[0,1] \subseteq W_p^m[0,1] \subseteq N_p^m[0,1] \subseteq N_p^{m-\epsilon}[0,1], \tag{3.15}$$

for $m \in N$, $0 < \epsilon < 1$, and $1 \le p \le \infty$; and

$$N_p^\alpha[0,1] \subseteq N_q^\beta[0,1], \tag{3.16}$$

for $\alpha > 0$, $1 \le p \le q \le \infty$ and $\beta = \alpha - (1/p - 1/q) > 0$. We consider Hammerstein equations with kernels given by

$$k(t,s) = m(t,s)\bar{k}(t-s), \quad t,s \in [0,1], \tag{3.17}$$

with $\bar{k} \in N_1^\alpha[0,1]$ for some $0 < \alpha < 1$ and $m \in C^2([0,1] \times [0,1])$, and $\psi$ as defined in the previous section.

When no further conditions are made on the partition $\prod_n$ other than the one given by (1.13), the next theorem gives the best possible order of convergence of the Galerkin approximation to the solution of equation (3.1) with a weakly singular kernel defined by (3.17).

**Theorem 3.3** *Let $x$ be an isolated solution of equation (3.1) with a kernel given by (3.17). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. If $f \in N_1^{\beta+1}[0,1]$ for some $0 < \beta < 1$, then*

$$\|x - x_n\|_\infty = O(h^{\gamma}).$$

*with* $\gamma = \min\{\alpha, \beta\}$.

**Proof:** By Theorem 3.1, we have

$$\|x - x_n\|_\infty \le C \inf_{u \in S^\nu_{r,n}} \|x - u\|_\infty. \tag{3.18}$$

A similar proof to the one given for Theorem 3 (ii) of Graham [24] shows that if $f \in N_1^{\beta+1}[0, 1]$ then $x \in N_1^{\min\{\alpha+1, \beta+1\}}[0, 1] \subseteq N_\infty^{\min\{\alpha, \beta\}}[0, 1]$. In addition, (3.15) implies that $f \in W_1^1[0, 1]$. Hence $f$ is equal to an absolutely continuous function almost everywhere. Without loss of generality, we have $f \in W_1^1[0, 1] \cap C[0, 1]$. It can be shown that $x \in C[0, 1]$. Thus, $x \in N_\infty^\gamma[0, 1] \cap C[0, 1]$. It was proved in Graham [24] that if $\phi \in N_\infty^\eta[0, 1] \cap C[0, 1]$ for some $0 < \eta < 1$, then there exists a spline $v \in S^\nu_{r,n}$ such that $\|\phi - v\|_\infty \le Ch^\eta$ where $C$ is a constant independent of $h$. The result of this theorem follows immediately from (3.18) and the above argument. □

Now we consider a special form of (3.17). Namely we assume

$$k(t, s) = m(t, s) g_\alpha(|t - s|), \tag{3.19}$$

where $m \in C^{\mu+1}([0, 1] \times [0, 1])$ and

$$g_\alpha(s) = \begin{cases} s^{\alpha-1}, & 0 < \alpha < 1, \\ \log s, & \alpha = 1. \end{cases} \tag{3.20}$$

With these kernels, certain regularities of the solutions of (3.1) are known. Let $S$ be a finite set in $[0, 1]$ and we define the function $\omega_S(t) = \inf\{|t - s| : s \in S\}$. A function $x$ is said to be of $Type(\alpha, k, S)$, for $-1 < \alpha < 0$, if

$$|x^{(k)}(t)| \le C[\omega_S(t)]^{\alpha-k} \quad t \notin S,$$

and for $\alpha > 0$, if the above condition holds and $x \in Lip(\alpha)$. Kaneko, Noren and Xu [36] proved that if $f$ is of $Type(\beta, \mu, \{0, 1\})$, then a solution of equation (3.1) is of $Type(\gamma, \mu, \{0, 1\})$, where $\gamma = \min\{\alpha, \beta\}$. In order to recover the optimal rate of convergence of numerical solutions, we define a partition $\prod_n^\gamma$ of $[0, 1]$ corresponding to the regularity of a solution. The knots of this partition $\prod_n^\gamma$ are given by

$$\begin{aligned} t_i &= (1/2)(2i/n)^q, & 0 \le i \le n/2, \\ t_i &= 1 - t_{n-i}, & n/2 < i \le n. \end{aligned} \tag{3.21}$$

where $q = \frac{r}{\gamma}$. Let $S_{r,n}^{\nu,\gamma} = S_r^{\nu}(\prod_n^{\gamma})$, with $r = 1$ and $\nu = 0$, or $r \geq 2$ and $\nu \in \{0,1\}$. The following theorem gives the order of convergence of the Galerkin approximations to the solution of Hammerstein equations with kernels defined by (3.19) and (3.20). It should be noted that the technique of approximating a solution of the type described above by elements from the nonlinear spline space has been used on many occasions in dealing with the weakly singular Fredholm integral equations. For example, Vainikko and Uba [73] describe the collocation method, whereas in Vainikko, Pedas and Uba [74] they describe the Galerkin method. Schneider [56] on the other hand establishes the product-integration method based upon the idea of the nonlinear spline approximation with nonuniform knots.

**Theorem 3.4** *Let $x$ be an isolated solution of (3.1) with kernels (3.19) and (3.20) and let $x_n$ be the Galerkin approximation to $x$. Let $m \in C^{\mu+1}([0,1] \times [0,1])$, and $f$ be of Type$(3, \mu, \{0,1\})$. Assume that $\psi \in C^{(0,1)}([0,1]\times(-\infty,\infty))$ for $\mu = 0, 1$ and $\psi \in C^{\mu-1}([0,1]\times(-\infty,\infty))$ for $\mu \geq 2$. We also assume 1 is not an eigenvalue of $(K\Psi)'(x)$. Then*

$$\|x - x_n\|_\infty = O(\frac{1}{n^r}).$$

**Proof:** This follows from Theorem 3.1, the regularity of the solution $x$, and from the results of Rice [53]. □

## THE ITERATED GALERKIN METHOD

In this section, we study the superconvergence of the iterated Galerkin method for the Hammerstein equation (3.1). Generalizing the linear case we first define the iterated scheme. Assume that $x$ is an isolated solution of (3.1). As before, let $P_n$ be the orthogonal projection from $L_2[0,1]$ onto $X_n$ with conditions (3.4) and (3.5) satisfied. Assume that $x_n$ is the unique solution of (3.8) in the sphere $B(x,\delta)$ for some $\delta > 0$. Define

$$x_n^I = f + K\Psi x_n. \tag{3.22}$$

Applying $P_n$ to the both sides of (3.22), we obtain

$$P_n x_n^I = P_n f + P_n K\Psi x_n. \tag{3.23}$$

Comparing (3.23) with (3.8), we see that

$$P_n x_n^I = x_n. \tag{3.24}$$

Upon substituting (3.24) into (3.22), we find that the function $x_n^I$ satisfies the following new Hammerstein equation

$$x_n^I = f + K\Psi P_n x_n^I. \tag{3.25}$$

By letting $S_n \equiv f + K\Psi P_n$, we may rewrite (3.25) as $x_n^I = S_n x_n^I$. We first study the invertibility of the linear operators $I - S_n'(x)$ in the following theorem, which will be used to prove the main results of this section.

**Lemma 3.5** *Let $x \in C[0,1]$ be an isolated solution of (3.1). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. Then for sufficiently large $n$, the operators $I - S_n'(x)$ are invertible and there exists a constant $L > 0$ such that*

$$\|(I - S_n'(x))^{-1}\|_\infty \leq L, \text{ for sufficiently large } n.$$

**Proof:** This follows from an application of the collectively compact operator theory. See [44] for detail. □

For simplicity, from Lemma 3.5 we assume without loss of generality that $I - S_n'(x)$ is invertible for each $n \geq 1$ and

$$L = \sup\{\|(I - S_n'(x))^{-1}\|_\infty : n \geq 1\} < \infty.$$

Throughout the rest of this section, we assume without further mention that $\delta > 0$ satisfies $LC_2MP\delta < 1$ and $\delta_1$ is chosen so that $C_1M\delta_1 \leq \delta$. The following lemma establishes that $x_n^I$ defined in (3.22) is the unique solution of (3.25) in some neighborhood of $x$ and provides an error bound for $x_n^I$ approximating $x$.

**Lemma 3.6** *Let $x \in C[0,1]$ be an isolated solution of equation (3.1) and $x_n$ be the unique solution of (3.8) in the sphere $B(x, \delta_1)$. Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. Then for sufficiently large $n$, $x_n^I$ defined by the iterated scheme (3.22) is the unique solution of (3.25) in the sphere $B(x, \delta)$. Moreover, there exists a constant $0 < q < 1$, independent of $n$, such that*

$$\frac{\beta_n}{1+q} \leq \|x_n^I - x\|_\infty \leq \frac{\beta_n}{1-q},$$

*where $\beta_n = \|(I - S_n'(x))^{-1}[S_n(x) - \hat{T}(x)]\|_\infty$. Finally,*

$$\|x_n^I - x\|_\infty \leq CE_n(x),$$

*where $E_n(x)$ is defined in Theorem 3.1.*

**Proof:** This follows easily using Lemma 2.1 and Theorem 2 of Vainikko [71]. □

One way to ensure the superconvergence of the iterated Galerkin method is to assume

$$\|(K\Psi)'(x)(I-P_n)|_{C[a,b]}\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{3.26}$$

In this case, using the identity (ref. Theorem 2.3 of Atkinson and Potra [7])

$$\begin{aligned}
&(I-(K\Psi)'(x))(x_n^I - x)\\
&= [I-(K\Psi)'(x)(I-P_n)][K\Psi(x_n) - K\Psi(x) - (K\Psi)'(x)(x_n - x)]\\
&-(K\Psi)'(x)(I-P_n)((K\Psi)'(x) - I)(x_n - x),
\end{aligned}$$

we obtain

$$\begin{aligned}
\|x_n^I - x\|_\infty \le\ & \|(I-(K\Psi)'(x))^{-1}\|_\infty \{\|I-(K\Psi)'(x)(I-P_n)\|_\infty\\
&\times \sup_{0\le\theta\le 1} \|(K\Psi)'(x+\theta(x_n - x)) - (K\Psi)'(x)\|_\infty \|x - x_n\|_\infty\\
&+\|(K\Psi)'(x)(I-P_n)((K\Psi)'(x) - I)(x_n - x)\|_\infty\}.
\end{aligned}$$

This with (3.26) gives the superconvergence of $x_n^I$ to $x$. In the next theorem, we establish superconvergence of the iterated Galerkin method in a general setting. In establishing superconvergence of the iterates of the Fredholm equations, many authors assumed the condition $\|K(I-P_n)\| \to 0$ as $n \to \infty$ with $K$ being a compact linear operator (e.g., Theorem 5 of Graham [24] and Theorem 3.1 of Sloan [62]). In our current problem, this is equivalent to assuming condition (3.26). However, the next theorem is proved without assumption (3.26). First, we apply the mean-value theorem to $\psi(s, y)$ to conclude

$$\psi(s, y) = \psi(s, y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))(y - y_0), \tag{3.27}$$

where $\theta := \theta(s, y_0, y)$ with $0 < \theta < 1$. The boundedness of $\theta$ is essential for the proof of the next theorem, although it may depend on $s, y_0, y$. Let

$$g(t, s, y_0, y, \theta) = k(t, s)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)),$$

$$(G_n x)(t) = \int_0^1 g(t, s, P_n x(s), P_n x_n^I(s), \theta)x(s)ds,$$

and $(Gx)(t) = \int_0^1 g_t(s)x(s)ds$, where $g_t(s) = k(t, s)\psi^{(0,1)}(s, x(s))$.

**Theorem 3.7** *Let $x \in C[0, 1]$ be an isolated solution of equation (3.1) and $x_n$ be the unique solution of (3.8) in the sphere $B(x, \delta_1)$. Let $x_n^I$ be defined by the iterated scheme (3.22).*

*Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. Then, for all $1 \le p \le \infty$,*

$$\|x - x_n^I\|_\infty \le C\left\{\|x - P_nx\|_\infty^2 + \sup_{0\le t\le 1}\inf_{u\in X_n}\|k(t,.)\psi^{(0,1)}(.,x(.)) - u\|_q\|x - P_nx\|_p\right\},$$

*where $1/p + 1/q = 1$ and $C$ is a constant independent of $n$.*

**Proof:** Note that from equations (3.1) and (3.25) we have

$$x - x_n^I = K(\Psi x - \Psi P_nx_n^I) = K(\Psi x - \Psi P_nx) + K(\Psi P_nx - \Psi P_nx_n^I). \tag{3.28}$$

Replacing $y$ by $P_nx_n^I$ and $y_0$ by $P_nx$ in equation (3.27), the last term of (3.28) can be written as

$$K(\Psi P_nx - \Psi P_nx_n^I)(t) = (G_nP_n(x - x_n^I))(t).$$

Equation (3.28) now becomes

$$x - x_n^I = K(\Psi x - \Psi P_nx) + G_nP_n(x - x_n^I). \tag{3.29}$$

By using condition (3.2) and the fact that $0 < \theta < 1$, we have, for all $x \in C[0,1]$,

$$\|(G_nx) - (Gx)\|_\infty \le \sup_{0\le t\le 1}\int_0^1 |k(t,s)|ds\|x\|_\infty(\|P_nx - x\|_\infty + \|P_n\|_\infty\|x_n^I - x\|_\infty).$$

Consequently, by assumption (3.4) and Lemma 3.6,

$$\|G_n - G\|_\infty \le M(\|P_nx - x\|_\infty + P\|x_n^I - x\|_\infty) \to 0 \quad \text{as} \quad n \to \infty.$$

That is, $G_n \to G$ in the norm of $C[0,1]$ as $n \to \infty$. Moreover, for each $x \in C[0,1]$,

$$\sup_{0\le t\le 1}|(GP_nx)(t) - (Gx)(t)| = \sup_{0\le t\le 1}|\int_0^1 g_t(s)[P_nx(s) - x(s)]ds| \le MM_1\|P_nx - x\|_\infty,$$

where

$$M_1 = \sup_{0\le t\le 1}|\psi^{(0,1)}(t,x(t))| < +\infty.$$

It follows that $GP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. Again since $P_n$ is uniformly bounded, we have for each $x \in C[0,1]$,

$$\|G_nP_nx - Gx\|_\infty \le \|G_n - G\|_\infty\|P_n\|_\infty\|x\|_\infty + \|GP_nx - Gx\|_\infty.$$

Thus, $G_nP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. By Assumptions ii, v, and vi, we see that there exists a constant $C > 0$ such that for all $n$

$$|\psi^{(0,1)}(s, P_nx(s) + \theta(P_nx_n^I(s) - P_nx(s)))| \le C_2\|P_nx - x\|_\infty + \theta C_2P\|x_n^I - x\|_\infty + M_1 \le C.$$

By a proof similar to that for Lemma 3.5, we can show that $\{G_n P_n\}$ is collectively compact. Since $G = (K\Psi)'(x)$ is compact and $(I-G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I - G_n P_n)^{-1}$ exists and is uniformly bounded for sufficiently large $n$. By (3.29), we have the following estimate

$$\sup_{0\le t\le 1} |(x - x_n^I)(t)| \le C' \sup_{0\le t\le 1} |K(\Psi x - \Psi P_n x)(t)|.$$

Next, we estimate the function $d(t) \equiv |K(\Psi x - \Psi P_n x)(t)|$. Using (3.27) with $y = P_n x$ and $y_0 = x$, we obtain, for $0 < \theta < 1$,

$$d(t) = \left|\int_0^1 g(t, s, x(s), P_n x(s), \theta)(x(s) - P_n x(s))ds\right|.$$

Note that $\int_0^1 u(s)[x(s) - P_n x(s)]ds = 0$, for all $u \in X_n$. Thus, for all $u \in X_n$,

$$\begin{aligned} d(t) &= \left|\int_0^1 [g(t, s, x(s), P_n x(s), \theta) - u(s)](x(s) - P_n x(s))ds\right| \\ &\le \int_0^1 |g(t, s, x(s), P_n x(s), \theta) - g_t(s)|ds\|x - P_n x\|_\infty \\ &\quad + \left|\int_0^1 [g_t(s) - u(s)](x(s) - P_n x(s))ds\right|. \end{aligned}$$

Now, by condition (3.2), we have

$$\int_0^1 |g(t, s, x, P_n x(s), \theta) - g_t(s)|ds \le C_1\theta \int_0^1 |k(t, s)|ds\|x - P_n x\|_\infty \le C_1 M\|x - P_n x\|_\infty.$$

Moreover, for $1/p + 1/q = 1$,

$$\left|\int_0^1 [g_t(s) - u(s)][x(s) - P_n x(s)]ds\right| \le \|g_t - u\|_q\|x - P_n x\|_p.$$

Therefore,

$$d(t) \le C_1 M\|x - P_n x\|_\infty^2 + \|g_t - u\|_q\|x - P_n x\|_p, \quad \text{for all} \quad u \in X_n.$$

Hence the desired result follows. □

In the next two theorems, we consider the case that $X_n = S^\nu_{r,n}$ with $\Pi_n$ an arbitrary partition of $[0, 1]$ satisfying (1.13). First, we consider the case when both the kernels and the solutions of equation (3.1) are smooth.

**Theorem 3.8** *Let $x \in W_p^l$ $(0 < l \le r)$ be an isolated solution of (3.1), $x_n$ be the unique solution of (3.8) in $B(x, \delta_1)$, and $x_n^I$ be defined by the iterated scheme (3.22). Assume that*

1 *is not an eigenvalue of* $(K\Psi)'(x)$. *Assume that for all* $t \in [0,1]$, $k_t(.)\psi^{(0,1)}(.,x(.)) \in W_q^m (0 \le m \le r)$ *and* $\|k_t(.)\psi^{(0,1)}(.,x(.))\|_{W_q^m}$ *is bounded uniformly in* $t$. *Then*

$$\|x - x_n^I\|_\infty = O(h^{\mu+\min\{\mu,\nu\}}),$$

*where* $\mu = \min\{l, r\}$ *and* $\nu = \min\{m, r\}$.

**Proof:** Since the partition $\Pi_n$ of $[0,1]$ satisfies condition (1.13), we conclude that

$$P := \sup_n \|P_n\|_\infty < \infty.$$

Hence,

$$\|x - P_n x\|_p \le \|x - P_n x\|_\infty \le (1+P) \inf_{u\in S_{r,n}^\nu} \|x-u\|_\infty \le C h^\mu.$$

In addition,

$$\sup_{0\le t\le 1} \inf_{u\in S_{r,n}^\nu} \|k_t(.)\psi^{(0,1)}(.,x(.)) - u\|_q \le C h^\nu.$$

The result of this theorem follows from Theorem 3.7 with $X_n = S_{r,n}^\nu$. □

We remark that Theorem 3.8 may be obtained from Theorem 5.2 of Atkinson and Potra [7], Theorem 3.8 being a special case of Atkinson and Potra's theorem to Hammerstein equations.

In the following theorem, we assume that $k(t,s)$ is a kernel given by (3.17), i.e., $k(t,s) = m(t,s)\bar{k}(t-s)$, with $\bar{k} \in N_1^\alpha[0,1]$ for some $0 < \alpha < 1$ and $m \in C^2([0,1]\times[0,1])$. Also, we assume that $S_{r,n}^\nu$ is such that $\nu \ge 1$.

**Theorem 3.9** *Let* $x$ *be an isolated solution of equation (3.1) with kernels given by (3.17),* $x_n$ *be the unique solution of equation (3.8) in* $B(x,\delta_1)$, *and* $x_n^I$ *be defined by iterated scheme (3.22). Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x)$, $f \in N_1^{\beta+1}[0,1]$ *for some* $0 < \beta < 1$, $\psi^{(0,1)}(.,x(.)) \in W_1^1$ *for* $x \in W_1^1$. *If, for each* $v_t \in S_{r,n}^\nu$, $\|v_t(\cdot)\psi^{(0,1)}(\cdot,x(\cdot))\|_{L_1}$ *and* $\|k_t\|_{L_1}$ *are uniformly bounded in* $t$, *then*

$$\|x - x_n^I\|_\infty = O(h^{2\gamma}),$$

*with* $\gamma = \min\{\alpha, \beta\}$.

**Proof:** Following the proof of Theorem 3.8, we have

$$\|x - P_n x\|_\infty \le (1+P) \inf_{u\in S_{r,n}^\nu} \|x-u\|_\infty. \tag{3.30}$$

As stated in the proof of Theorem 3.4, we know that

$$x \in X_\infty^\gamma[0,1] \cap C[0,1] \cap W_1^1. \tag{3.31}$$

Using (3.30) and an argument similar to the one used in the proof of Theorem 3.4, we obtain $\|x - P_n x\|_\infty \leq Ch^\gamma$. Now, by Theorem 4(i) of Graham [24], we find that there exists $v_t \in S_{r,n}^\nu$ such that $\|k_t - v_t\|_{L_1} = O(h^\alpha)$. Since $\nu \geq 1$, it follows that $S_{r,n}^\nu \subset W_1^1$. Thus, $v_t \in W_1^1$. From (3.31), $x \in W_1^1$. This yields that $\psi^{(0,1)}(.,x(.)) \in W_1^1$. Consequently, $v_t(.)\psi^{(0,1)}(.,x(.)) \in W_1^1$. The remark made before Theorem 3.2 implies that there exists $u_t \in S_{r,n}^\nu$ for which

$$\|v_t(.)\psi^{(0,1)}(.,x(.)) - u_t(.)\|_{L_1} = O(h).$$

Therefore,

$$\begin{aligned}
\|g_t - u_t\|_{L_1} &= \int_0^1 |m(t,s)k(t-s)\psi^{(0,1)}(s,x(s)) - u_t(s)|ds \\
&\leq \int_0^1 |m(t,s)k(t-s)\psi^{(0,1)}(s,x(s)) - v_t(s)\psi^{(0,1)}(s,x(s))|ds \\
&\quad + \int_0^1 |v_t(s)\psi^{(0,1)}(s,x(s)) - u_t(s)|ds \\
&\leq \|k_t - v_t\|_{L_1}\|\psi^{(0,1)}(.,x(.))\|_\infty + \|v_t(.)\psi^{(0,1)}(.,x(.)) - u_t\|_{L_1} \\
&= O(h^\alpha) + O(h) = O(h^\alpha).
\end{aligned}$$

Now, applying Theorem 3.7 with $q = 1$, $p = \infty$, and $X_n = S_{r,n}^\nu$, we conclude that

$$\begin{aligned}
\|x - x_n^I\|_\infty &\leq C\left\{\|x - P_n x\|_\infty^2 + \inf_{u \in S_{r,n}^\nu} \|g_t - u_t\|_{L_1}\|x - P_n x\|_\infty\right\} \\
&= O(h^{\alpha+\gamma}) + O(h^{2\gamma}) = O(h^{2\gamma}).
\end{aligned}$$

The proof is complete. □

Next, we apply Theorem 3.7 to equation (3.1) with kernels given by (3.19) and (3.20) and use $X_n = S_{r,n}^{\nu,\gamma}$ as approximate spaces such that $r \geq 2$ and $\nu = 1$. Proofs of the next two theorems are similar to the one given for the previous theorem and we refer the reader to [44] for detail.

**Theorem 3.10** *Let $x$ be an isolated solution of (3.1) with weakly singular kernels given by (3.19) and (3.20). Let $x_n$ be the unique solution of (3.8) in $B(x,\delta_1)$, and $x_n^I$ be defined by the iterated scheme (3.22). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$ and that the* *hypotheses of Theorem 3.4 are satisfied with* $\mu \geq 1$. *Also assume that* $\psi^{(0,1)}(\cdot, x(\cdot))$ *is of* $Type(\alpha, r, \{0, 1\})$ *for* $\alpha > 0$ *whenever* $x$ *is of the same type. Then*

$$\|x - x_n^I\|_\infty = O(\frac{1}{n^{\alpha+r}}).$$

As the last application of Theorem 3.7, we consider equation (3.1) with kernels having singularity at the four corners of the square $[0, 1] \times [0, 1]$, a problem that arises from boundary integration for the harmonic Dirichlet problem in plane domains with corners (see Kress [46]). In the following theorem, we assume $k_s(t) = k(t, s)$ is of $Type(\alpha, \mu, \{0, 1\})$, for $\alpha > 0$, and $k_t(s) = k(t, s)$ is of $Type(\alpha, \mu, \{0, 1\})$, for $\alpha > -1$, e.g., $k(t, s) = m(t, s)\sqrt{t}$, and $k(t, s) = m(t, s)\frac{1}{\sqrt{1-s}}$, etc., with $m(t, s)$ smooth, and assume $f$ is of $Type(\beta, \mu, \{0, 1\})$, for $\alpha, \beta > 0$ and a positive integer $\mu$. It is not difficult to prove that an isolated solution $x$, of the corresponding equation (3.1), is of $Type(\gamma, \mu, \{0, 1\})$, where $\gamma = \min\{\alpha, \beta\}$ if $\alpha > 0$ and $\gamma = \min\{\alpha + 1, \beta\}$ if $-1 < \alpha < 0$ by modifying the proofs of theorems in Kaneko, Noren and Xu [36]. We again let $q = \frac{r}{\gamma}$ and define the Galerkin subspace $S_{r,n}^{\nu,\gamma}$ with $r = 1$ and $\nu = 0$, and $r \geq 2$ and $\nu \in \{0, 1\}$, where partition $\Pi_n^\gamma$ is defined as in (3.21). The following theorem describes the order of convergence of the Galerkin approximation $x_n$ and that of superconvergence of the iterated Galerkin approximation $x_n^I$.

**Theorem 3.11** *Let* $x$ *be an isolated solution of (3.1) with kernels of the type defined in the paragraph preceding this theorem. Let* $x_n$ *be the unique solution of (3.8) in* $B(x, \delta_1)$, *and* $x_n^I$ *be defined by the iterated scheme (3.22). Assume that* 1 *is not an eigenvalue of* $(K\Psi)'(x)$ *and that* $f$ *is of* $Type(\beta, r, \{0, 1\})$. *Also assume that* $\psi^{(0,1)}(\cdot, x(\cdot))$ *is of* $Type(\gamma, r, \{0, 1\})$ *whenever* $x$ *is of the same type. Then,*

$$\|x - x_n\|_\infty = O(\frac{1}{n^r}),$$

*and*

$$\|x - x_n^I\|_\infty = O(\frac{1}{n^{2r}}).$$

## ITERATED DEGENERATE KERNEL METHOD FOR HAMMERSTEIN EQUATIONS

A study of the degenerate kernel method for Hammerstein equations was made by Kaneko and Xu [41]. A brief outline of the method is described below for convenience. As in the Fredholm equation case, the kernel $k$ in (3.1) is replaced by $k_n$ of (2.29). The equation that one must solve is the following:

$$y_n(t) - \int_a^b k_n(t,s)\psi(s, y_n(s))\,ds = f(t), \qquad a \leq t \leq b. \tag{3.32}$$

Following analogously the development made in (2.33) and (2.34), with

$$c_i \equiv \sum_{j=1}^{n} \int_a^b a_{ij}\varphi_j(s)\psi(s, y_n(s))ds, \tag{3.33}$$

$y_n$ can be written as

$$y_n(t) = f(t) + \sum_{i=1}^{n} c_i\varphi_i(t). \tag{3.34}$$

Substituting (3.34) into (3.33), we obtain the following $n$ nonlinear equations in $n$ unknowns $c_1, \cdots, c_n$,

$$c_i = \sum_{j=1}^{n} \int_a^b a_{ij}\varphi_j(s)\psi(s, f(s) + \sum_{l=1}^{n} c_l\varphi_l(s))ds, \qquad 1 \leq i \leq n. \tag{3.35}$$

As before

$$K\Psi y(t) \equiv \int_a^b k(t,s)\psi(s, y(s))ds$$

so that (3.1) becomes

$$y - K\Psi y = f. \tag{3.36}$$

Similarly we write equation (3.32) as

$$y_n - K_n\Psi y_n = f \tag{3.37}$$

The iterated solution $y_n^I$ is now obtained by

$$y_n^I = f + K\Psi y_n. \tag{3.38}$$

The Fréchet derivative of $K\Psi$ at $\varphi_0 \in C[a,b]$ is denoted and defined by

$$(K\Psi)'(\varphi_0)(\varphi)(t) = \int_a^b k(t,s)\psi^{(0,1)}(s, \varphi_0(s))\varphi(s)ds$$

for $\varphi \in C[a,b]$ and $v^{(0,1)}$ denoting the first partial derivative of $v$ with respect to the second variable. The following theorem describes the superconvergence phenomenon of $y_n^I$ to $y$. Here we assume that the decomposition of the kernel in (2.29) is done by the interpolation scheme of the previous section. The case for the least-squares approximation is similar.

**Theorem 3.12** *Assume $y \in C[a,b]$ is an isolated solution in equation (3.1). $k(u,s) \in W_1^m([a,b] \times [a,b])$, $0 < m \leq r$, and $\eta_{t,n}(u,s) \equiv k_t(u)v(s,y_n(s))$ and $\eta_{t,n}(u,s) \in W_1^l([a,b] \times [a,b])$, for each $n$ and $t \in [a,b]$, $0 < l \leq 2r$, where $y_n$ is the solution of (3.37). Assume also that 1 is not an eigenvalue of $(K\Psi)'(y)$ and that $\|\eta_{t,n}\|_{W_1^l}$ is uniformly bounded in $t$ and $n$. Then*

$$\|y - y_n^I\|_\infty = O(h^\nu), \qquad \nu = \min\{2m, l\}.$$

**Proof:** From (3.36) and (3.37),

$$y - y_n^I = K\Psi y - K\Psi y_n. \tag{3.39}$$

Now

$$\begin{aligned} K\Psi y - K\Psi y_n &= K\Psi(f + K\Psi y) - K\Psi(f + K_n\Psi y_n) \\ &= (K\Psi)'(\theta(n)(f + K_n\Psi y_n) + (1-\theta(n))(f + K\Psi y))(K\Psi y - K_n\Psi y_n) \\ &\qquad \text{for some} \quad 0 \leq \theta(n) \leq 1 \\ &= K_{\theta(n)}(K\Psi y - K_n\Psi y_n + (K\Psi y - K\Psi y_n) - (K\Psi y - K\Psi y_n)), \end{aligned}$$

where $K_{\theta(n)} \equiv (K\Psi)'(\theta(n)(f + K_n\Psi y_n) + (1-\theta(n))(f + K\Psi y))$. Since $K$ is compact, $(K\Psi)'(y)$ is also compact [50]. Also since the solutions $y_n$ of degenerate kernel method converge to the solution $y$ of (3.1) [41], $\{K_{\theta(n)}\}$ converges in operator norm to $(K\Psi)'(y)$. From this, along with the fact that 1 is not an eigenvalue of $(K\Psi)'(y)$, an application of theorem 10.1 [47] yields that $(I - K_{\theta(n)})^{-1}$ exists and uniformly bounded for sufficiently large $n$. Hence we obtain

$$K\Psi y - K\Psi y_n = (I - K_{\theta(n)})^{-1} K_{\theta(n)}(K - K_n)\Psi y_n. \tag{3.40}$$

Combining (3.39) and (3.40), and taking the norm on both sides, we obtain

$$\|y - y_n^I\|_\infty \leq c\|(K - K_n)\Psi y_n\|_\infty.$$

for some constant $c$ independent of $n$. Now using the assumptions on $k$ and $\eta_l$ and arguing as in the proof of Theorem 2.7 , we obtain the desired result. □

Finally we consider a computational problem associated with (3.35). It is customary that the system of nonlinear equations (3.35) is solved by an iterative scheme. For example, the fixed point iteration scheme for (3.35) is to generate $\{c_i^{(k)}\}_{i=1}^n$ for $k \geq 1$ with a given initial vector $\{c_i^{(0)}\}_{i=1}^n$ by

$$c_i^{(k+1)} = \sum_{j=1}^n \int_a^b a_{ij}\varphi_j(s)v(s, f(s) + \sum_{l=1}^n c_l^{(k)}\varphi_l(s))ds, \qquad 1 \leq i \leq n. \tag{3.41}$$

In this scheme, at each step $k$ of iteration, the integrals in (3.41) must be computed since the integrands contain the different values of $c_i^{(k)}$. To circumvent this difficulty, we propose the following device whose idea was originally discussed in [48]. We let

$$z_n(t) = v(t, y_n(t)) \tag{3.42}$$

where $y_n$ is defined in (3.34). We have, assuming that $k_n$ takes the form of (2.29),

$$z_n(t) = v(t, f(t) + \sum_{i=1}^n a_{ij}\varphi_i(t)\int_a^b \sum_{j=1}^n \varphi_j(s)z_n(s)ds). \tag{3.43}$$

Equation (3.43) can be solved by the collocation-type scheme that was developed by Kumar and Sloan [48]. Namely let $\{\eta_i\}_{i=1}^n$ be $n$ functions defined on $[a, b]$ and let $\{t_j\}_{j=1}^n$ be $n$ distinct points for which

$$\det(\eta_i(t_j)) \neq 0. \tag{3.44}$$

The element $z_n$ in (3.42) is now approximated in the form $\sum_{j=1}^n \alpha_j\eta_j$. The $\alpha_j$'s can be found by solving the following nonlinear equations. Note that the constants $\alpha_j$'s are moved out of the integrals. This makes the repeated computations of the integrals unnecessary when the following system of nonlinear equations is to be solved by an iterated scheme.

$$\sum_{j=1}^n \alpha_j\eta_j(t_k) = v(t_k, f(t_k) + \sum_{i=1}^n a_{ij}\varphi_i(t_k)\sum_{l=1}^n \alpha_l\int_a^b \sum_{j=1}^n \varphi_j(s)\eta_l(s)ds), \tag{3.45}$$

for $1 \leq k \leq n$. If we denote $A \equiv [\eta_j(t_i)]$ and the right side of (3.45) by $v_i(\bar{\alpha})$, then with $\bar{v}(\bar{\alpha}) \equiv (v_i(\bar{\alpha}))$ and $\bar{\alpha}^{(k)} \equiv (\alpha_i^{(k)})$, (3.45) may be solved by the fixed point iteration scheme that can be described as

$$\bar{\alpha}^{(k)} = A^{-1}\bar{v}(\bar{\alpha}^{(k-1)}). \tag{3.46}$$

## NUMERICAL EXAMPLES FOR HAMMERSTEIN EQUATIONS

Here we present numerical examples for a Hammerstein equation using least-squares (Table 3.1) and interpolation (Table 3.2) to approximate $k(s,t)$. Let $k(s,t) = e^{st}$, $v(s,t) = cos(s+t)$, and $f$ is chosen so that $y(t) = 1$. Then, the computed errors for the least squares method are shown in the following table. The linear spline basis was used in computations.

Table 3.1: Least Squares Results for Hammerstein Equations

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .28059448920008e-2 | .56676667568e-5 |
| 3 | .12905495465556e-2 | .12129557404e-5 |
| 4 | .741545558372e-3 | .39390349993e-6 |
| convergence rate ≈ | 1.92 | 3.85 |

For the interpolation method, using the roots of the second order Legendre polynomial for interploation points, we obtained the following.

Table 3.2: Interpolation Results for Hammerstein Equations

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .27550396054150e-2 | .36150374187 6e-4 |
| 3 | .1272147104832e-2 | .70508042364e-5 |
| 4 | .7306199305565e-3 | .22199115879e-5 |
| convergence rate ≈ | 1.92 | 4 |

# CHAPTER IV
# THE ITERATED COLLOCATION METHOD FOR HAMMERSTEIN EQUATIONS

## INTRODUCTION

In this chapter, the collocation method for Hammerstein equations is presented. Some material from approximation theory is also reviewed to make the presentation more self-contained. We let $[a, b] = [0, 1]$ for convenience in this chapter. We consider the following Hammerstein equation

$$x(t) - \int_0^1 k(t, s)\psi(s, x(s))ds = f(t), \quad 0 \le t \le 1, \tag{4.1}$$

where $k$, $f$ and $\psi$ are known functions and $x$ is the function to be determined. We will assume the conditions (i)-(vi) stated in the beginning of Chapter 3.

We let

$$(K\Psi)(x)(t) \equiv \int_0^1 k(t, s)\psi(s, x(s))ds.$$

With this notation, equation (4.1) takes the following operator form

$$x - K\Psi x = f. \tag{4.2}$$

For the collocation method, we are interested in $S_{r,n}^{\nu}$ with $\nu = 0$ or 1. That is, the space of piecewise polynomials with no continuity at the knots or the space of continuous piecewise polynomials with no continuity requirement on the derivatives at the knots. We assume that the sequence of partitions $\Pi_n$ of $[0, 1]$ satisfies the quasiuniform mesh condition (1.13).

In many cases, equation (4.1) possesses multiple solutions (see e.g. [41]). Hence, it is assumed for the remainder of this paper that we treat an isolated solution $x$ of (4.1). Let $I_i = (t_{i-1}, t_i)$ for each $i = 1, \ldots, n$. Then for $\nu = 0$, we let $\tau_{i1}, \tau_{i2}, \ldots, \tau_{ir}$ be the Gaussian points (the zeros of the $r$th degree Legendre polynomial on $[-1, 1]$) shifted to the interval $I_i$. We define

$$G_0 = \{\tau_{ij} : 1 \le i \le n, 1 \le j \le r\}. \tag{4.3}$$

The points in $G_0$ give rise to the piecewise collocation method where no continuity between polynomials is assumed. This is the approach taken by Graham, Joe and Sloan [22]. Joe [37], on the other hand, considered the continuous piecewise polynomial collocation method.

His method corresponds with taking $\nu = 1$. Here we define the set $G_1$ of collocation points to be the set consisting of the knots along with the Lobatto points (the zeros of the first derivative of the $r-1$th degree Legendre polynomial) shifted to the interval $I_i$. Namely, let $\xi_{r-1} = 1$ and for $1 \le l \le r-2$ $(r \ge 3)$, let $\xi_l$ denotes the $l$th Lobatto point. If $|I_i|$ denotes the length of $I_i$, then $G_1$ contains

$$\tau_{(i-1)(r-1)+l+1} = \frac{1}{2}(t_{i-1} + t_i + |I_i|\xi_l), \qquad 1 \le i \le n,\ 1 \le l \le r-1, \text{ and } \tau_1 = t_0 = 0. \quad (4.4)$$

The analysis for the discontinuous polynomial collocation method [22] and that of the continuous polynomial collocation method [37] are very similar. We therefore confine ourselves in this thesis to developing the discontinuous collocation method for Hammerstein equations that is analogous to the method of [22]. An obvious extension to the continuous piecewise collocation method will be left to the reader. It is noted that, in the case of continuous polynomial collocation method using the Lobatto points, one can bring via the iterated collocation scheme the order of convergence from $r$ up to $2r-2$. This is due to the fact that the $r$th degree Legendre polynomial on $[-1, 1]$ is orthogonal to polynomials of degree $\le r-1$ whereas the polynomial $(t-1)(t+1)G^{(1)}_{r-1}(t)$ is only orthogonal to polynomials of degree $\le r-3$ where $G^{(1)}_{r-1}(t)$ is the first derivative of the $r-1$ degree Legendre polynomial. Define the interpolatory projection $P_n$ from $C[0, 1] \dotplus S^{\nu}_r(\Pi_n)$ to $S^{\nu}_r(\Pi_n)$ by requiring that, for $x \in C[0, 1] \dotplus S^{\nu}_r(\Pi_n)$,

$$P_n x(\tau_{ij}) = x(\tau_{ij}), \qquad \text{for all } \tau_{ij} \in G_0. \quad (4.5)$$

Then we have, for $x \in C[0, 1] \dotplus S^{\nu}_r(\Pi_n)$

$$P_n x \to x, \qquad \text{as } n \to \infty \quad (4.6)$$

and consequently

$$\sup_n \|P_n\| < c. \quad (4.7)$$

The collocation equation corresponding to (4.2) can be written as

$$x_n - P_n K\Psi x_n = P_n f \quad (4.8)$$

where $x_n \in S^{\nu}_r(\Pi_n)$. Now we let

$$\hat{T}x \equiv f + K\Psi x$$

and

$$T_n x_n \equiv P_n f + P_n K \Psi x_n$$

so that equations (4.2) and (4.8) can be written respectively as $x = \hat{T}x$ and $x_n = T_n x_n$. Now we can see that Theorems 3.1 and 3.2 apply to the collocation case.

When the kernel $k$ is of weakly singular type, see equations (3.19) and (3.20), then the solution $x$ of equation (4.2) does not, in general, belong to $W_p^m$. It was proved by Kaneko, Noren and Xu [36] that if $f$ is of $Type(\beta, \mu, \{0, 1\})$, then a solution of equation (4.1) with the kernel defined by (3.19) is of $Type(\gamma, \mu, \{0, 1\})$, where $\gamma = \min\{\alpha, \beta\}$. The optimal rate of convergence of the collocation solution $x_n$ to $x$ can be recovered by selecting the knots that are defined by

$$\begin{aligned} t_i &= (1/2)(2i/n)^q, & 0 \le i \le n/2, \\ t_i &= 1 - t_{n-i}, & n/2 < i \le n, \end{aligned} \tag{4.9}$$

where $q = r/\gamma$ denotes the index of singularity. Details can be found in [37].

## THE ITERATED COLLOCATION METHOD

The faster convergence of the iterated Galerkin method for the Fredholm integral equations of the second kind compared to the Galerkin method was first observed by Sloan in [60] and [61]. On the other hand, the superconvergence of the iterated collocation method was studied in [22] and [37]. Given the equation of the second kind

$$x - Kx = f, \tag{4.10}$$

where $K$ is a compact operator on $X \equiv C[0, 1]$ and $x, f \in X$, the collocation approximation $x_n$ is the solution of the following projection equation

$$x_n - P_n K x_n = P_n f. \tag{4.11}$$

Here $P_n$ is the interpolatory projection of (4.5). The iterated collocation method obtains a solution $x_n^I$ by

$$x_n^I = f + K x_n. \tag{4.12}$$

Under the assumption of

$$\|KP_n - K\| \to 0 \qquad \text{as } n \to \infty \tag{4.13}$$

it can be shown that

$$\|x - x_n^I\| \le \|(I - KP_n)^{-1}\| \|K(x - P_n x)\|. \tag{4.14}$$

The assumption (4.13) is satisfied if $X = L_2$ and $P_n$ is the orthogonal projection satisfying $\|P_n g - g\| \to 0$ for all $g$ in the closure of the range of the adjoint $K^*$ of $K$ since in this case $\|KP_n - K\| = \|P_n K^* - K^*\|$. Hence the superconvergence of the iterated Galerkin method for the Fredholm equations of the second kind (4.10) can be established rather easily by (4.14). The results of Sloan et al [22] were recently generalized to the iterated Galerkin method for Hammerstein equations by Kaneko and Xu [44]. The main theorem of [44], Theorem 3.3, that guarantees the superconvergence of the iterates was proved by making use of the collectively compact operator theory.

The purpose of this section is to study the superconvergence of the iterated collocation method. For the collocation solution $x_n$ of (4.8), we define

$$x_n^I = f + K\Psi x_n. \tag{4.15}$$

A standard argument shows that $x_n^I$ satisfies

$$x_n^I = f + K\Psi P_n x_n^I. \tag{4.16}$$

We denote the right side of (4.16) by $S_n x_n^I$, namely

$$S_n x_n^I \equiv f + K\Psi P_n x_n^I. \tag{4.17}$$

Both Lemmas 3.5 and 3.6 are applicable. Following the development made in [44], we let

$$\psi(s, y) = \psi(s, y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))(y - y_0), \tag{4.18}$$

where $\theta := \theta(s, y_0, y)$ with $0 < \theta < 1$. Also let

$$g(t, s, y_0, y, \theta) = k(t, s)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)),$$

$$(G_n x)(t) = \int_0^1 g(t, s, P_n x(s), P_n x_n^I(s), \theta) x(s) ds,$$

and $(Gx)(t) = \int_0^1 g_t(s)x(s)ds$, where $g_t(s) = k(t, s)\psi^{(0,1)}(s, x(s))$. Now we are ready to state and prove our main theorem of this chapter. The proof is a combination of the idea used in [44] (Theorem 3.3) and the one used in [22] (Theorem 4.2).

**Theorem 4.1** *Let $x \in C[0,1]$ be an isolated solution of equation (4.2) and $x_n$ be the unique solution of (4.8) in the sphere $B(x,\delta_1)$. Let $x_n^I$ be defined by the iterated scheme (4.16). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. Assume that $x \in W_1^l$ $(0 < l \leq 2r)$ and $g_t \in W_1^m$ $(0 < m \leq r)$ with $\|g_t\|_{W_1^m}$ bounded independently of $t$. Then*

$$\|x - x_n^I\|_\infty = O(h^\gamma), \qquad where\ \gamma = min\{l, r+m\}.$$

**Proof:** From equations (4.2) and (4.17), we obtain

$$x - x_n^I = K(\Psi x - \Psi P_n x_n^I) = K(\Psi x - \Psi P_n x) + K(\Psi P_n x - \Psi P_n x_n^I). \tag{4.19}$$

Using (4.18), the last term of (4.19) can be written as

$$K(\Psi P_n x - \Psi P_n x_n^I)(t) = (G_n P_n(x - x_n^I))(t).$$

Equation (4.19) then becomes

$$x - x_n^I = K(\Psi x - \Psi P_n x) + G_n P_n(x - x_n^I). \tag{4.20}$$

Using the Lipschitz condition (3.3) imposed on $\psi^{(0,1)}$, for $x \in C[0,1]$,

$$\|(G_n x) - (Gx)\|_\infty \leq C_2 \sup_{0\leq t\leq 1} \int_0^1 |k(t,s)|ds\|x\|_\infty(\|P_n x - x\|_\infty + \|P_n\|_\infty\|x_n^I - x\|_\infty).$$

This shows that

$$\|G_n - G\|_\infty \leq MC_2(\|P_n x - x\|_\infty + c\|x_n^I - x\|_\infty) \to 0 \quad \text{as} \quad n \to \infty.$$

Also, for each $x \in C[0,1]$,

$$\sup_{0\leq t\leq 1} |(GP_n x)(t) - (Gx)(t)| = \sup_{0\leq t\leq 1} |\int_0^1 g_t(s)[P_n x(s) - x(s)]ds| \leq MM_1\|P_n x - x\|_\infty.$$

where

$$M_1 = \sup_{0\leq t\leq 1} |\psi^{(0,1)}(t, x(t))| < +\infty.$$

It follows that $GP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. Again since $P_n$ is uniformly bounded, we have for each $x \in C[0,1]$,

$$\|G_n P_n x - Gx\|_\infty \leq \|G_n - G\|_\infty\|P_n\|_\infty\|x\|_\infty + \|GP_n x - Gx\|_\infty.$$

Thus, $G_nP_n \to G$ pointwise in $C[0,1]$ as $n \to \infty$. By Assumptions (ii), (v), and (vi), we see that there exists a constant $C > 0$ such that for all $n$

$$|v^{(0,1)}(s, P_nx(s) + \theta(P_nx_n^I(s) - P_nx(s)))| \le C_2\|P_nx - x\|_\infty + \theta C_2P\|x_n^I - x\|_\infty + M_1 \le C.$$

This implies that $\{G_nP_n\}$ is a family of collectively compact operators [2]. Since $G = (K\Psi)'(x)$ is compact and $(I-G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I - G_nP_n)^{-1}$ exists and is uniformly bounded for sufficiently large $n$. Now using (4.20), we see that

$$\|x - x_n^I\|_\infty \le C\|K(\Psi x - \Psi P_nx)\|.$$

Hence we need to estimate $\|K(\Psi x - \Psi P_nx)\|$. The following four inequalities are known (Theorem 4.2 [22]). Let $v_n \in S_l^0(\Pi_n)$ be such that

$$\sum_{i=1}^{n} \|(x - v_n)^{(j)}\|_{W_1^m(I_i)} \le ch^{l-j}\|x\|_{W_1^l}, \qquad 0 \le j \le l. \tag{4.21}$$

$$\max_{1\le i\le n} \|v_n^{(j)}\|_{W_\infty^m(I_i)} \le c\|x\|_{W_1^l}, \qquad j \ge 0. \tag{4.22}$$

Also for each $t \in [0,1]$, there exists $\varphi_{n,t} \in S_m^0(\Pi_n)$ such that

$$\sum_{i=1}^{n} \|(g_t - \varphi_{n,t})^{(j)}\|_{W_1^m(I_i)} \le ch^{m-j}K_m, \qquad 0 \le j \le m. \tag{4.23}$$

$$\max_{1\le i\le n} \|\varphi_{n,t}^{(j)}\|_{W_\infty^m(I_i)} \le cK_m, \qquad j \ge 0. \tag{4.24}$$

where $K_m = \sup_{0\le t\le 1} \|k_t\|_{W_1^m} < \infty$. Now for $t \in [0,1]$ we have

$$\begin{aligned} K(\Psi x - \Psi P_nx)(t) &= (g_t - \varphi_{n,t}, x - P_nx) + (\varphi_{n,t}, (I - P_n)(x - v_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n)v_n). \end{aligned} \tag{4.25}$$

Using equations (4.21)-(4.24) along with the arguments from Theorem 1.5 we can show that each of the three terms is bounded by $ch^\gamma$ uniformly in $t$. This completes our proof. □

One way to establish the superconvergence of the iterated collocation method for the Fredholm equation is to assume (4.13). In the context of the present discussion, (4.13) is equivalent to assuming

$$\|(K\Psi)'(x)(I - P_n)|_{C[a,b]}\|_\infty \to 0 \qquad \text{as } n \to \infty. \tag{4.26}$$

Theorem 4.1 was thus proved under weaker assumptions. The idea used to prove Theorem 4.1 originates from [6] (section 6) in which the superconvergence of the iterated collocation method for the Fredholm equations was established by showing that $\{KP_n\}$ is a family of collectively compact operators.

Finally in this section, we investigate the superconvergence of the iterated collocation method for weakly singular Hammerstein equations. Specifically, we consider equation (4.2) with kernel given by (3.19) and (3.20). An enhancement in the rate of convergence is given in the following theorem.

**Theorem 4.2** *Let $x \in C[0,1]$ be an isolated solution of equation (4.2) and $x_n$ be the unique solution of (4.8) in the sphere $B(x, \delta_1)$ with kernel defined by (3.19) and (3.20) and knots defined by (4.9). Let $x_n^I$ be defined by the iterated scheme (4.16). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$ and that $\psi^{(0,1)}(\cdot, x(\cdot))$ is of $Type(\alpha, r, \{0,1\})$ for $\alpha > 0$ whenever $x$ is of the same type. Then*

$$\|x - x_n^I\|_\infty = O(h^{r+\alpha}).$$

**Proof:** We follow the proof of Theorem 4.1 exactly the same way to (4.25), which is

$$\begin{aligned} K(\Psi x - \Psi P_n x)(t) &= (g_t - \varphi_{n,t}, x - P_n x) + (\varphi_{n,t}, (I - P_n)(x - v_n)) \\ &\quad + (\varphi_{n,t}, (I - P_n)v_n). \end{aligned}$$

The difference in superconvergence arises from the degree to which we may bound the first term. As in Kaneko and Xu [44] (Theorem 3.6), using an argument similar to [41], it can be proved that there exists $u \in S_r^\nu(\Pi_n)$ with knots $\Pi_n$ given by (4.9) such that $\|g_t - u\|_{L_1} = O(h^\alpha)$. Here $h = \max_{1 \le i \le n}\{x_i - x_{i-1}\}$. Then

$$\begin{aligned} |(g_t - \varphi_{n,t}, x - P_n x)| &\le \|g_t - \varphi_{n,t}\|_{L_1} \|x - P_n x\|_\infty \\ &= O(h^{\alpha + r}). \end{aligned}$$

The rest of proof follows once again in the same way as described in Theorem 1.5.□

## THE DISCRETE COLLOCATION METHOD FOR WEAKLY SINGULAR HAMMERSTEIN EQUATIONS

Several papers have been written on the subject of the discrete collocation method. Joe [32] gave an analysis of discrete collocation method for second kind Fredholm integral equations. A discrete collocation-type method for Hammerstein equations was described by Kumar in [49]. Most recently Atkinson and Flores [5] put together the general analysis of the discrete collocation methods for nonlinear integral equations. In this section, we describe a discrete collocation method for weakly singular Hammerstein equations. In the aforementioned papers [32, 49, 5], their discussions are primarily concerned with integral equations with smooth kernels. Even though, in principle, an analysis for the discrete collocation method for weakly singular Hammerstein equations is similar to the one given in [5], we feel that a detailed discussion on some specific points pertinent to weakly singular equations, -e.g., a selection of a particular quadrature scheme and a convergence analysis etc. will be of great interest to practitioners. Our convergence analysis of the discrete collocation method presented in this section is different from the one given in [5] in that it is based upon Theorem 2 of Vainikko [71]. The idea of the quadrature used here was recently developed by Kaneko and Xu [42] and a complete Fortran program based on the idea was developed by Kaneko and Padilla [39]. A particular case of the quadrature schemes developed in [44] is concerned with an approximation of the integral

$$I(f) = \int_0^1 f(s)ds. \tag{4.27}$$

where $f \in Type(\alpha, 2r, S)$ with $\alpha > -1$. For simplicity of demonstration, we assume $S = \{0\}$. We define $q = \frac{2r+1}{\alpha+1}$ and a partition

$$\pi_n: s_0 = 0, s_1 = n^{-q}, s_j = j^q s_1, \qquad j = 2, 3, \ldots, n. \tag{4.28}$$

Now we construct a piecewise polynomial $S_r$ of degree $r-1$ by the following rule: $S_r(s) = 0$, $s \in [s_0, s_1)$ and $S_r(s)$ is the Lagrange polynomial of degree $r-1$ interpolating $f$ at $\{u_j^i\}_{j=1}^r$ for $s \in [s_i, s_{i+1})$, $i = 1, 2, \ldots, n-2$ and for $x \in [s_{n-1}, s_n]$. Here $\{u_j^i\}_{j=1}^r$ denote the zeros of the $r$th degree Legendre polynomial transformed into $[s_i, s_{i+1})$. Our approximation process consists of two stages. First, $I(f)$ is approximated by

$$\hat{I}(f) = \int_{s_1}^1 f(s)ds = \sum_{i=1}^{n-1} \int_{s_i}^{s_{i+1}} f(s)ds. \tag{4.29}$$

Second, $\hat{I}(f)$ is approximated by $\hat{I}(S_r) = \int_{s_1}^1 S_r(s)ds$. A computation of $\hat{I}(S_r)$ can be accomplished as follows: let $\theta: [s_i, s_{i+1}] \to [-1, 1]$ be defined by $\theta = \frac{2s-(s_{i+1}+s_i)}{s_{i+1}-s_i}$ so that

$$\hat{I}(f) = \int_{-1}^1 F_f(\theta)d\theta \tag{4.30}$$

where

$$F_f(\theta) = \sum_{i=1}^{n-1} \frac{1}{2}(s_{i+1} - s_i) f(\frac{1}{2}(s_{i+1} - s_i)\theta + \frac{1}{2}(s_{i+1} + s_i)).$$

If $\{\tau_i : i = 1, 2, \ldots, r\}$ denotes the zeros of the Legendre polynomial of degree $r$, then

$$S_r(s) = \sum_{i=1}^{r} F_f(\tau_i) l_i(s)$$

with $l_i(s)$ the fundamental Lagrange polynomial of degree $r-1$ so that

$$\hat{I}(S_r) = \sum_{i=1}^{r} w_i F_f(\tau_i), \qquad \text{where } w_i = \int_{-1}^{1} l_i(s)ds. \tag{4.31}$$

It was proved in [41] that

$$|I(f) - \hat{I}(S_r)| = O(n^{-2r}). \tag{4.32}$$

In this section, we examine equation (4.1) with the kernel $k$ defined by (3.19) and (3.20). When the knots are selected according to (4.9), as stated earlier, it was shown in [37] that the solution $x_n$ of the collocation equation (4.8) converges to the solution $x$ of (4.1) in the rate that is optimal to the degree of polynomials used. Specifically, $x_n$ must be found by solving

$$x_n(u_j^i) - \int_0^1 g_\alpha(|u_j^i - s|) m(u_j^i, s) v(s, x_n(s)) ds = f(u_j^i) \tag{4.33}$$

where $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, r$.

The discrete collocation method for equation (4.1) is obtained when the integral in (4.33) is replaced by a numerical quadrature given in (4.31). Let $k_{ij}(s) \equiv g_\alpha(|u_j^i - s|) m(u_j^i, s)$. Then

$$\begin{aligned} \int_0^1 g_\alpha(|u_j^i - s|) m(u_j^i, s) v(s, x_n(s)) ds &= \int_0^1 k_{ij}(s) v(s, x_n(s)) ds \\ &= \int_0^{u_j^i} + \int_{u_j^i}^1 k_{ij}(s) v(s, x_n(s)) ds. \end{aligned} \tag{4.34}$$

The integrals in the last expression of (4.34) represent two weakly singular integrals which can be approximated to within $O(n^{-2r})$ order of accuracy by (4.31) by transforming them to $[-1, 1]$ and selecting the points in (4.28) appropriately.

Writing (4.33) as

$$P_n x_n - P_n K \Psi x_n = P_n f, \tag{4.35}$$

we consider the approximation $\tilde{x}_n$ to $x_n$ defined as the solution of

$$\tilde{x}_n = Q_n \tilde{x}_n \equiv P_n K_n \Psi \tilde{x}_n + P_n f, \tag{4.36}$$

where $K_n$ is the discrete collocation approximation to the integrals in (4.34) described above.

We will use Theorem 2 of [71] to find a unique solution to (4.36) in some $\delta$ neighborhood of $x_n$, where $n$ is sufficiently large. Clearly, $Q_n'(x) = P_n K_n \Psi'(x)$, where $\Psi'(x)[y](s) = \psi^{(0,1)}(s, x(s))y(s)$. For sufficiently large $n$, (4.35) has a unique solution in some $\delta$ neighborhood of $x$. To see that $I - Q_n'(x_n)$ is continuously invertible with $\{(I - Q_n'(x_n))^{-1}\}_{n=N}^{\infty}$ uniformly bounded, it is enough to observe that $\{Q_n'(x_n)\}_{n=1}^{\infty}$ is collectively compact, and to do this we will show that

$$| Q_n'(x_n)[x](t) - Q_n'(x_n)[x](t') | = | P_n K_n \Psi'(x_n)x(t) - P_n K_n \Psi'(x_n)x(t') | \to 0 \tag{4.37}$$

as $t \to t'$, for each $x \in C[0,1]$, [2]. Here $N$ is some sufficiently large number.

If we show (4.37), then part (a) of Theorem 2 [71] is also verified. In order to verify part (b) of Theorem 2 [71], we only need to establish (because of the uniform boundedness of $\{(I - Q_n(x_n))^{-1}\}_{n=N}^{\infty}$) that

$$\| Q_n'(x) - Q_n'(x_n) \|_{\infty} \leq L \| x - x_n \|_{\infty} \leq L\delta. \tag{4.38}$$

for some constant $L$, and

$$\| Q_n(x_n) - T_n(x_n) \| \to 0 \text{ as } n \to \infty. \tag{4.39}$$

Once this is done, Theorem 2 [71] applies yielding a unique solution $\tilde{x}_n$ in some neighborhood of $x_n$ (for sufficiently large $n$) and

$$\| x_n - \tilde{x}_n \| \leq L\bar{\alpha}_n \leq L \| Q_n(x_n) - T_n(x_n) \|_{\infty} . \tag{4.40}$$

(Here and throughout the remainder of the section, $L$ denotes a generic constant, the exact value of which may differ at each occurrence.) This inequality will be used to obtain the order of convergence.

Considering (4.37), the right hand side is bounded by $T_1 + T_2 + T_3$, where

$$\begin{aligned} T_1 &= | P_n K_n \Psi'(x_n)x(t) - P_n K \Psi'(x_n)x(t) |, \\ T_2 &= | P_n K \Psi'(x_n)x(t) - P_n K \Psi'(x_n)x(t') |, \\ T_3 &= | P_n K \Psi'(x_n)x(t') - P_n K_n \Psi'(x_n)x(t') | . \end{aligned}$$

Let $\epsilon > 0$. Since $\{P_n\}_{n=1}^{\infty}$ is uniformly bounded, $T_1 + T_3 < \frac{2\epsilon}{3}$ by applying (4.32) with $f(s) = \psi^{(0,1)}(s, x_n(s))x(s)$ and letting $n$ be sufficiently large. For $T_2$ we have $T_2 \leq M \int_0^1 | k(t,s) - k(t',s) | \, ds \leq M(S_1 + S_2)$.

where

$$S_1 = \int_0^1 g_\alpha(|s-t|)\,|m(t,s) - m(t',s)|\,ds$$

and

$$S_2 = \int_0^1 |g_\alpha(|t-s|) - g_\alpha(|t'-s|)|\,|m(t',s)|\,ds.$$

but

$$\begin{aligned} S_1 &\le \sup_{0\le s\le 1} |m(t,s) - m(t',s)| \int_0^1 g_\alpha(|t-s|)ds \\ &\le L \sup_{0\le s\le 1} |m(t,s) - m(t',s)| \to 0 \text{ as } t \to t'. \end{aligned}$$

and

$$\begin{aligned} S_2 &\le L \textstyle\int_0^1 |g_\alpha(|t-s|) - g_\alpha(|t'-s|)|\,ds \\ &= \tfrac{L}{\gamma}\{|t^\alpha - (t')^\alpha| + |(1-t)^\alpha - (1-t')^\alpha| + \tfrac{4}{2^\alpha}|t-t'|^\alpha\} \\ &\to 0 \text{ as } t \to t'. \end{aligned}$$

Hence (4.37) holds. For (4.38).

$$\| Q_n'(x) - Q_n'(x_n) \|_\infty = \| P_n K_n(\Psi'(x) - \Psi'(x_n)) \| \le MC \| x - x_n \| \le M\delta = q < 1$$

for $\delta$ sufficiently small. Note that we have used the uniform boundedness of $\{P_n\}, \{K_n\}$ and because $\Psi^{(0,1)}(s, y(s))$ is locally Lipschitz, so is the operator $\Psi' : C[0,1] \to B(C[0,1], C[0,1])$ (the space of bounded linear operators from $C[0,1]$ into $C[0,1]$).

For (4.39), we have

$$\| Q_n(x_n) - T_n(x_n) \|_\infty = \| P_n(K_n\Psi x_n - K\Psi x_n) \| \le L \| (K_n - K)\Psi(x_n) \le L(R_1 + R_2 + R_3) \tag{4.41}$$

where

$$R_1 = \| K_n\Psi(x_n) - K_n\Psi(x) \|,\ R_2 = \| K_n\Psi(x) - K\Psi(x) \|,\ R_3 = \| K\Psi(x) - K\Psi(x_n) \|. \tag{4.42}$$

But

$$R_1 \le L \| \Psi(x_n) - \Psi(x) \| \le C_1 L \| x_n - x \| \tag{4.43}$$

because $\Psi$ is a Lipschitz operator and $\{K_n\}$ is uniformly bounded, and also

$$R_3 \le M \| \Psi(x) - \Psi(x_n) \| \le C_1 M \| x_n - x \|. \tag{4.44}$$

Finally,

$$R_2 = O(n^{-2r}) \tag{4.45}$$

by (4.32) using $f(s) = \Psi(x, x(s))$.

Thus Vainikko's Theorem yields a unique solution $\tilde{x}_n$ for $n$ sufficiently large and (4.40) holds. Now (4.40) and (4.41) - (4.45) show that

$$\| x_n - \tilde{x}_n \| = O(n^{-\beta}) \tag{4.46}$$

where $\beta$ is the minimum of $2r$ and the order of convergence of $\| x - x_n \|$. We summarize the results obtained above in the following theorem:

**Theorem 4.3** *Let $x$ be an isolated solution of equation (4.2) and let $x_n$ be the solution of equation (4.8) in a neighborhood of $x$. Moreover, let $\tilde{x}_n$ be the solution of (4.36). Assume that 1 is not an eigenvalue of $(K\Psi)'(x)$. If $x \in W^l_\infty$, then*

$$\|x - \tilde{x}_n\|_\infty = O(h^\mu),$$

*where $\mu = \min\{l, r\}$. If $x \in W^l_p$ $(1 \leq p < \infty)$, then*

$$\|x - \tilde{x}_n\|_\infty = O(h^\nu),$$

*where $\nu = \min\{l - 1, r\}$.*

## NUMERICAL EXAMPLES

In this section we present three numerical examples (Tables 4.1 - 4.3). Let $k(s,t) = e^{s-t}$ and $\Psi(s, x(s)) = cos(s+x(s))$. The spline coefficients were obtained using a Newton-Raphson algorithm. Also, the Gauss-type quadrature algorithm described in [42] is used to calculate all integrations. The computed errors for the solution and the iterated solution are shown in the following table.

For the second example, let $k(s,t) = log(|s-t|)$ and $\Psi(s, x(s)) = cos(s + x(s))$. The computed errors for the solution and iterated solution of the weakly singular integral are shown in the following table.

For the third example, let $k(s,t) = \frac{1}{\sqrt{|s-t|}}$, $\Psi(s, x(s)) = cos(s + x(s))$, and $x(t) = \cos(t)$. The computed errors for the solution and iterated solution of the weakly singular integral are shown in the following table.

Table 4.1: Smooth Kernel Collocation Results

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .153571593748756e-1 | .286029074365e-4 |
| 3 | .71758714356116e-2 | .47721991441e-5 |
| 4 | .41291276625525e-2 | .14180649575e-5 |
| 5 | .26770046422053e-2 | .5636996160e-6 |
| convergence rate ≈ | 2 | 4 |

Table 4.2: Log Kernel Collocation Results

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | .157961272540103e-1 | .24257900549439e-2 |
| 3 | .71150661058771e-2 | .7663852778203e-3 |
| 4 | .41192622669880e-2 | .3210258989686e-3 |
| 5 | .25982238843077e-2 | .1770978040470e-3 |
| convergence rate ≈ | 2 | 3 |

Table 4.3: Sqrt$^{-1}$ Kernel Collocation Results

| | Errors | |
|---|---|---|
| $n$ | non-iterated | iterated |
| 2 | 0.01540556116740788 | 0.005968844100471715 |
| 3 | 0.00722550448387438 | 0.00256622209942683 |
| 4 | 0.00416092487581254 | 0.001371170616411344 |
| 5 | 0.00269785684908008 | 0.000835161756464808 |
| convergence rate ≈ | 2 | 2.2 |

# CHAPTER V
# THE SINGULARITY PRESERVING METHOD

## INTRODUCTION

In this chapter, we are concerned with the problem of approximating the solutions of weakly singular Hammerstein equations (4.1) with logarithmic kernel by the Galerkin method that preserves the singularity of the exact solution. Namely we establish a method that generates an approximate solution in terms of a collection of basis functions some of which are comprised of singular elements that reflect the characteristics of the singularity of the exact solution. The idea of the method originates in the recent paper by Cao and Xu [11]. Cao and Xu studied the characteristics of the singularities that are pertinent to the solutions of the weakly singular Fredholm equations of the second kind. It is well documented (see, e.g. [58],[54],[25],[72]) that the solutions of the weakly singular Fredholm equations (1.1) exhibit, in general, mild singularities even in the case of a smooth forcing term $f$. Here by "mild" singularities, we mean singularities in derivatives. The papers of Richter [54] and Graham [25] contain singularity expansions of the solutions of equation (1.1) with kernel given by (3.19) and (3.20) in the case of $m(s,t) \equiv 1$. The results of Graham were recently generalized by Cao and Xu for weakly singular Fredholm equations. Information concerning the type of singularities that solutions have is useful when solving equation (1.1) numerically. In order to approximate functions with mild singularities, many investigators utilized the theorem of Rice [53] that gives an optimal order of approximation to such functions. Based upon this idea of approximating the solutions by splines defined on nonuniform knots, the collocation method, the Galerkin method and the product-integration method were established for equation (1.1) with weakly singular kernels (3.19) by Vainikko and Uba [73], by Graham [25] and by Schneider [57] respectively. A modified collocation method was introduced in [43] which also uses the idea of Rice. Recently there has been some considerable interest in the study of the weakly singular Hammerstein equation. A study on the regularities of the solution of (4.1) is reported in [36], extending the results of [58]. Subsequently, Kaneko, Noren and Xu used the regularity results to establish the collocation method for weakly singular Hammerstein equations in [37]. The approximate solutions provided by these methods are in the form of piecewise polynomials that are not always satisfactory as a tool for approximating functions with singularities. This observation is quite evident in the areas of finite element analysis. Hughes and Akin [30] list several problems (e.g. 'upwind' finite elements for treating convection operators [29],[31],[27]; boundary-layer elements [1] etc.) in which the finite element shape functions are constructed to include polynomials as well as singular functions. Singular shape functions are introduced to the set of basis functions through asymptotic analysis on the solution of the problem that is being considered. It should be pointed out that the analysis involved in the aforementioned papers on the finite element method is centered around the collocation method. The problems such as the choice for the extra collocation points for singular basis elements or the rate of convergence are not addressed in these papers. It should be pointed out that the location of additional collocation points for singular basis elements is critical in determining the rate of convergence of numerical solutions. A detailed discussion on this subject can be found in [38]. A singularity preserving collocation method, because of the reasons mentioned above, seems to be more difficult to establish.

In this chapter, a singularity expansion for the solution of equation (4.1) with logarithmic kernel is given. This extends the results in [36] and [11]. Only the logarithmic kernel is considered here because of its important application to obtaining numerical solution of a Dirichlet problem with nonlinear boundary condition as described in Concluding Remarks. It is a routine matter, however, to establish, following the ensuing argument, a singularity expansion for the solution of (4.1) with an algebraic singularity. The chapter is organized as follows: first we study the regularity property of the solution of (4.1) and establish its singularity expansion. The results obtained there generalize the results of [11] and [36]. Secondly, the singularity expansion is then utilized to achieve the singularity preserving Galerkin method for equation (4.1). Finally, the iterated singularity preserving Galerkin method is discussed.

## SINGULARITY EXPANSION FOR WEAKLY SINGULAR HAMMERSTEIN EQUATIONS

In this section, we consider the following Hammerstein equation with logarithmic singularity,

$$y(s) - \int_0^1 \log|s-t| m(s,t) \psi(t, y(t)) dt = f(s), \qquad 0 \le s \le 1 \tag{5.1}$$

(see (4.1) also). We let

$$K\Psi y(s) \equiv \int_0^1 \log|s-t| m(s,t) v(t, y(t)) dt. \tag{5.2}$$

Then equation (5.1) can be written in operator form as

$$y - K\Psi y = f. \tag{5.3}$$

Let $H^n$ denote the Sobolev space. $H^n[0,1] = \{w : w^{(n)} \in L_2[0,1]\}$, equipped with the norm $\|w\|_{H^n} = \left(\sum_{i=0}^n \|w^{(i)}\|_{L_2}^2\right)^{1/2}$ where $w^{(i)}$ describes the $i$th generalized derivative of $w$. We also let $W = W_n$ be the linear space spanned by the functions $s^i \log^j s$, $(1-s)^i \log^j(1-s)$: $i, j = 1, 2, \ldots, n-1$. Throughout this chapter, we assume the following conditions:

$$\begin{cases} m \in C^{2n}([0,1] \times [0,1]), & n \geq 1. \\ m \in C^1([0,1] \times [0,1]), & n = 0. \end{cases} \tag{5.4}$$

$$v \in C^{2n+1}(R \times R) \tag{5.5}$$

$$f \in W + H^n. \tag{5.6}$$

We define

$$Ky(s) \equiv \int_0^1 \log|s-t| m(s,t) y(t) dt. \tag{5.7}$$

First we quote the following result (lemma 4.4(2)) from [11].

**Lemma 5.1** *Let $u_1(s) = s^p \log^q s$, and $u_2(s) = (1-s)^p \log^q(1-s)$, for some integers $p, q \geq 1$ and let $f \in H^{n-1}$. Assume that $m \in C^{n+1}([0,1] \times [0,1])$. Then, there exist $v_n \in H^n$ and constants $\{b_k\}, \{d_j\}, \{c_{ij}\} \in R$ such that.*

$$(Kf)(s) = \sum_{j=1}^{n-1} \left[b_j s^j \log s + d_j (1-s)^j \log(1-s)\right] + v_n(s).$$

$$(Ku_1)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} s^j (\log s)^i + \sum_{j=q+1}^{n-1} d_j (1-s)^j \log(1-s) + v_n(s).$$

*and*

$$(Ku_2)(s) = \sum_{j=p+1}^{n-1} \sum_{i=1}^{q+1} c_{ij} (1-s)^j (\log(1-s))^i + \sum_{j=q+1}^{n-1} d_j s^j \log s + v_n(s).$$

**Lemma 5.2** *If $u_1(s) = s^p \log^q s$, $u_2(s) = (1-s)^r \log^l(1-s)$, for some integers $p, q, r, l \geq 1$ are integers, then $u_1 u_2 \in W \dotplus H^n$.*

**Proof:** Expand $u_1$ in series about $s = 1$ and $u_2$ about $s = 0$ :

$$\begin{aligned} u_1(s) &= \textstyle\sum_{i=0}^{n-1} b_i(1-s)^i + f_1(s), \qquad & u_2(s) &= \textstyle\sum_{i=0}^{n-1} a_i s^i + f_2(s), \\ &\equiv P_1(s) + f_1(s) & &\equiv P_2(s) + f_2(s) \end{aligned}$$

where $f_1^{(k)}(s) = O((1-s)^{n-k})$ near $s = 1$, $f_1$ is analytic at $s = 1$, and $f_1^{(k)} \sim u_1^{(k)}(s) - P_1^{(k)}(0)$ as $s \to 0+$; $f_2^{(k)}(s) = O(s^{n-k})$ near $s = 0$, $f_2$ is analytic at $s = 0$, and $f_2^{(k)}(s) \sim u_2^{(k)}(s) - P_2^{(k)}(1)$ as $s \to 1^-$.

Now $u_1 u_2 = P_1 P_2 + P_1 f_2 + P_2 f_1 + f_1 f_2$. Clearly $P_1 P_2$ is in $H^n$. For $f_1 f_2$, we have

$$\frac{d^n}{ds^n}(f_1(s) f_2(s)) = \sum_{i=0}^{n} \binom{n}{i} f_1^{(i)}(s) f_2^{(n-i)}(s).$$

Each term $f_1^{(i)}(s) f_2^{(n-i)}(s)$, $i = 0, 1, ..., n$ satisfies

$$f_1^{(i)}(s) f_2^{(n-i)}(s) = O(f_1^{(i)}(s) s^i) = O([u_1^{(i)}(s) - P_1^{(i)}(0)] s^i) \to 0$$

as $s \to 0+$.

Similarly

$f_1^{(i)}(s) f_2^{(n-i)}(s) \to 0$ as $s \to 1^-$. Thus $f_1 f_2 \in C^n \subseteq H^n$. For $f_1 P_2$ we have $f_1(s) P_2(s) = (u_1(s) - P_1(s)) P_2(s) = u_1(s) P_2(s) - P_1(s) P_2(s)$. Since $P_2$ is a polynomial, $u_1 \in W$, it is easy to see that $u_1 P_2 \in W \dotplus H^n$ (see [[11], (4.7)]). So $f_1 P_2 \in H^n$. Similarly $f_2 P_1 \in W \dotplus H^n$, and Lemma 5.2 has been verified. □

**Lemma 5.3** *A product of an $H^n$ function with a function in $W$ is in $H^n \dotplus W$.*

**Proof:** Let $g \in H^n$ and let $u_1$ and $u_2$ be defined as before prior to Lemma 5.1. For $g u_1$ we write

$$\begin{aligned} u_1(s) g(s) &= \textstyle\sum_{i=0}^{n-1} \frac{g^{(i)}(0)}{i!} s^{i+p} \log^q s + \frac{s^p \log^q s}{(n-1)!} \int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma \\ &\equiv T_1 + T_2. \end{aligned}$$

Since $T_1 \in W \dotplus H^n$, we turn to $T_2$ and write

$$\begin{aligned} \frac{d^n T_2}{ds^n} &= \tfrac{1}{(n-1)!} \textstyle\sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s] \frac{d^{n-k}}{ds^{n-k}}[\int_0^s g^{(n)}(\sigma)(s-\sigma)^{n-1} d\sigma] \\ &= \tfrac{1}{(n-1)!} \textstyle\sum_{k=1}^{n} \binom{n}{k} \frac{d^k}{ds^k}[s^p \log^q s][(n-1)...k] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma \\ &\quad + s^p \log^q s g^{(n)}(s). \end{aligned}$$

But $s^p \log^q s \in L^\infty$, $g^{(n)} \in L_2[0,1]$ so $(s^p \log^q s)g^{(n)}(s) \in L^2$.

For the terms

$$b_n(s) \equiv \frac{d^k}{ds^k}[s^p \log^q s] \int_0^s g^{(n)}(\sigma)(s-\sigma)^{k-1} d\sigma$$

we have, for some constant $M$ and nonnegative integer $\alpha$

$$\begin{aligned} | b_n(s) | &\leq M \tfrac{(-\log s)^\alpha}{s^{k-1}} \int_0^s | g^{(n)}(\sigma) | s^{k-1} d\sigma \\ &= Ms(-\log s)^\alpha \tfrac{1}{s} \int_0^s | g^{(n)}(\sigma) | d\sigma. \end{aligned}$$

But $g^{(n)} \in L_2[0,1]$, so by Hardy's inequality [55] (p. 72) $\frac{1}{s}\int_0^s | g^{(n)}(\sigma) | d\sigma \in L_2[0,1]$. Since $s(-\log s)^\alpha \in L^\infty$ it follows that $b_n \in L_2[0,1]$. Hence $\frac{d^n T_2}{ds^n} \in L_2[0,1]$, or $T_2 \in H^n$. This proves $gu_1 \in W \dotplus H^n$.

The case for $gu_2 \in W \dotplus H^n$ is similar. □

Finally we need the following:

**Lemma 5.4** *The operator $K\Psi$ maps $W \dotplus H^n$ into $W \dotplus H^{n+1}$.*

**Proof:** Let $y = w + h$, $w \in W$, $h \in H^n$. We use Taylor's theorem in the form

$$v(t,x) = \sum_{k=0}^{n} \frac{1}{k!} v^{(k)}(t,a)(x-a)^k + \frac{1}{n!}\int_a^x (x-\sigma)^n v^{(n+1)}(t,\sigma) d\sigma. \tag{5.8}$$

Letting $x = y(s)$ and $a = h(s)$ allows us to write

$$\begin{aligned} (K\Psi)(y)(t) &= \textstyle\sum_{k=0}^n \frac{1}{k!} \int_0^1 \log | t-s | m(t,s) v^{(k)}(s,h(s)) w(s)^k ds \\ &\quad + \tfrac{1}{n!} \int_0^1 \log | t-s | m(t,s) \int_{h(s)}^{y(s)} v^{(n+1)}(s,\sigma)(y(s)-\sigma)^n d\sigma ds \\ &\equiv \textstyle\sum_{k=0}^n \frac{1}{k!} A_k(t) + \frac{1}{n!} B(t). \end{aligned} \tag{5.9}$$

By (3), $v^{(k)}(s,h(s)) \in H^n$, $k = 0,1,\ldots,n$, and by expanding $w(s)^k$ with the multinomial expansion, it is clear that $w(s)^k$ is a sum of terms in $W$ as well as terms of the form $as^p \log^q s(1-s)^r \log^u(1-s)$, $p,q,r,u \geq 1$ are integers. The constant, $a$, depends on $p,q,r$ and $u$. Since $v^{(k)}(h(s)) \in H^n$ and $w(s)^k \in W \dotplus H^n$, $k = 0,1,\ldots,n$, it follows from Lemma 5.3 that

$$v^{(k)}(s,h(s))w(s)^k \in W \dotplus H^n. \tag{5.10}$$

By Lemma 5.1 and (5.10), we have

$$A_k \in W \dotplus H^{n+1}. \tag{5.11}$$

For $B(t)$, if we prove that

$$F(s) \equiv \int_{h(s)}^{y(s)} v^{(n+1)}(s,\sigma)(y(s)-\sigma)^n d\sigma \in W \dotplus H^n.$$

then, also by Lemma 5.1, $B(t) = K[F](t)$ will be in $W \dotplus H^{n+1}$. This will complete the proof of this lemma. First of all, suppose $n \geq 1$. We write

$$F'(s) = -v^{(n+1)}(s, h(s))w(s)^n h'(s).$$

Since $h \in H^n$, $v \in C^{2n+1}$, $v^{(n+1)}(s, h(s)) \in H^n$. By Lemmas 2 and 3, $-v^{(n+1)}(s, h(s))w(s)^n \in H^n \dotplus W$. Since $h' \in H^{n-1}$, it follows that $-v^{(n+1)}(s, h(s))w(s)^n h'(s) \in H^{n-1} \dotplus W$ (Lemma 5.2). Since $F' \in H^{n-1} \dotplus W$ it is clear that $F \in H^n \dotplus W$. Second, let $n = 0$. Then $F(s) = \int_{h(s)}^{y(s)} v'(s, \sigma)d\sigma = v(s, y(s)) - v(s, h(s)) \in L_2[0, 1] \subseteq W \dotplus H^0$.
Thus

$$B \in W \dotplus H^n. \tag{5.12}$$

By (5.9), (5.11) and (5.12), it follows that $K\Psi$ maps $W \dotplus H^n$ into $W \dotplus H^{n+1}$. □

Using the lemmas which we proved above, we obtain the following main result of this section.

**Theorem 5.5** *Suppose the conditions (5.4)-(5.6) hold and $y$ is an isolated solution of (5.1). Then there are constants $a_{ij}$ and $b_{ij}$, for $i, j = 1, 2, \ldots, n-1$, and there is a function $v_n$ in $H^n$ such that*

$$y(t) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}[a_{ij}t^i \log^j t + b_{ij}(1-t)^i \log^j(1-t)] + v_n(t). \tag{5.13}$$

**Proof:** For $n = 0$, this follows from Lemma 5.4 with $n = 0$. Assume that the result holds for $n = k$, that is, if $f \in H^k \dotplus W_k$, then (5.13) holds with $n = k$. Say $y = w_k + v_k$, where $v_k \in H^k$, $w_k = \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}[a_{ij}t^i \log^j t + b_{ij}(1-t)^i \log^j(1-t)]$.

Now consider the case $n = k+1$ and suppose $f \in H^{k+1} \dotplus W_{k+1}$.

Since $y = w_k + v_k$ we write $y = K\Psi y + f = K\Psi(w_k + v_k) + f$. From Lemma 5.1, $K\Psi(w_k + v_k) \in W_{k+1} \dotplus H^{k+1}$. The proof is complete. □

## SINGULARITY PRESERVING GALERKIN METHOD

In this section, we establish the singularity preserving Galerkin method for equation (5.1). First we recall the definition of the space of spline functions of order $n$. Define the partition of $[0, 1]$ as

$$\Pi_{k+1} : 0 = t_0 < t_1 < \ldots < t_k = 1.$$

Let

$$h = \max_{1\le i\le k}(t_i - t_{i-1}).$$

and assume $h \to 0$ as $k \to \infty$. It is well known that the dimension of $S^{\nu}_{n,k}$ is $d = nk - \nu(k-1)$. $S^{\nu}_{n,k}$ is spanned by a basis consisting of $B$-splines $\{B_i\}_{i=1}^{d}$. We let

$$V_h^n \equiv W \dotplus S^{\nu}_{n,k} \tag{5.14}$$

and denote the orthogonal projection of $L_2[0,1]$ onto $V_h^n$ by $P_h^G$. The singularity preserving Galerkin method for approximating the solution of equation (5.3) requires the solution $y_h \in V_h^n$ to satisfy the following equation:

$$y_h - P_h^G K\Psi y_h = P_h^G f. \tag{5.15}$$

More specifically, we need to find $y_n$ in the form

$$y_h(s) = \sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \sum_{i=1}^{d} \gamma_i B_i(s) \tag{5.16}$$

where $\{\alpha_{ij}, \beta_{ij}\}_{i,j=1}^{n-1}$ and $\{\gamma_i\}_{i=1}^{d}$ are found by solving the following system of nonlinear equations:

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, s^p \log^q s) + \sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), s^p \log^q s) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i (B_i, s^p \log^q s) - (K\Psi(\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i B_i), s^p \log^q s) \qquad = (f, s^p \log^q s) \qquad p, q = 1, 2, \ldots, n-1
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, (1-s)^p \log^q(1-s)) + \\
&\textstyle\sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), (1-s)^p \log^q(1-s)) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i (B_i, (1-s)^p \log^q(1-s)) - \\
&\textstyle(K\Psi(\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} \beta_{ij}(1-s)^i \log^j(1-s) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i B_i), (1-s)^p \log^q(1-s)) \qquad = (f, (1-s)^p \log^q(1-s)) \qquad p, q = 1, 2, \ldots, n-1
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_{i,j=1}^{n-1} \alpha_{ij}(s^i \log^j s, B_p) + \sum_{i,j=1}^{n-1} \beta_{ij}((1-s)^i \log^j(1-s), B_p) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i (B_i, B_p) - (K\Psi(\sum_{i,j=1}^{n-1} \alpha_{ij} s^i \log^j s + \sum_{i,j=1}^{n-1} b_{i,j}(1-s)^i \log^j(1-s) + \\
&\textstyle\sum_{i=1}^{d} \gamma_i B_i), B_p) \qquad = (f, B_p) \qquad p = 1, 2, \ldots, d
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the usual inner product defined on $L_2[0,1]$. Now let $P_h$ be the orthogonal projection of $L_2[0,1]$ onto $S^{\nu}_{n,k}$. Then we have

$$P_h v \to v \qquad \text{as } h \to 0 \qquad \text{for all } v \in L_2[0,1]. \tag{5.17}$$

Recall that if $g \in H^n$, $n \geq 0$, then for each $h > 0$, there exists $\phi_h \in S^{\nu}_{n,k}$ such that

$$\|g - \phi_h\|_{L_2} \leq Ch^n \|g\|_{H^n}, \tag{5.18}$$

where $C > 0$ is a constant independent of $h$. (Theorem 1.2). By virtue of the fact that $P_h u$ is the best $L_2$ approximation of $u$ from $S^{\nu}_{n,k}$, we see immediately that

$$\|P_h u - u\|_{L_2} \leq \|u - \phi_h\|_{L_2} \leq Ch^n \|u\|_{H^n}, \text{ for all } u \in H^n. \tag{5.19}$$

The following lemma from [11] is useful in the sequel.

**Lemma 5.6** *Let $X$ be a Banach space. Suppose that $U_1$ and $U_2$ are two subspaces of $X$ with $U_1 \subseteq U_2$. Assume that $P_1 : X \to U_1$ and $P_2 : X \to U_2$ are linear operators. If $P_2$ is a projection, then*

$$\|x - P_2 x\|_X \leq (1 + \|P_2\|_X)\|x - P_1 x\|_X \text{ for all } x \in X.$$

For convenience, we introduce operators $\hat{T}$ and $T_h$ by letting

$$\hat{T}y \equiv f + K\Psi y \tag{5.20}$$

and

$$T_h y_n \equiv P_h^G f + P_h^G K\Psi y_n \tag{5.21}$$

so that equations (5.1) and (5.15) can be written respectively as $y = \hat{T}y$ and $y_n = T_h y_n$. The following theorem guarantees the existence of a solution of the singularity preserving Galerkin method (5.15) and describes the accuracy of its approximation.

**Theorem 5.7** *Let $y \in L_2[0,1]$ be an isolated solution of equation (5.1). Assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(y)$, where $(K\Psi)'(y)$ denotes the Fréchet derivative of $K\Psi$ at $y$. Then the singularity preserving Galerkin approximation equation (5.15) has a unique solution $y_h$ such that $\|y - y_h\|_{L_2} < \delta$ for some $\delta > 0$ and for all $0 < h < h_0$ for some $h_0 > 0$. Moreover, there exists a constant $0 < q < 1$, independent of $h$, such that*

$$\frac{\alpha_h}{1+q} \leq \|y - y_h\|_{L_2} \leq \frac{\alpha_h}{1-q}. \tag{5.22}$$

*where* $\alpha_h \equiv \|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_{L_2}$. *Finally, if* $y = w + v$ *with* $w \in W$ *and* $v \in H^n$, *then*

$$\|y - y_h\|_{L_2} \le C'h^n \|v\|_{H^n}, \qquad \text{whenever } 0 < h < h_0. \tag{5.23}$$

*where* $C' > 0$ *is a constant independent of* $h$.

**Proof:** The existence of a unique solution $y_h$ of equation (5.15) in the disk of radius $\delta$ about $y$ and the inequalities in (5.20) can be proved using Theorem 2 of Vainikko [71]. A detailed discussion on this application can be found in [37]. To get (5.23), first we note from Lemma 3.1, for $v \in L_2[0, 1]$,

$$\|P_h^G v - v\|_{L_2} \le (1 + \|P_h^G\|_{L_2})\|P_h v - v\|_{L_2}. \tag{5.24}$$

By assumption, $(I - (K\Psi)'(y))^{-1}$ exists. By (5.17), Theorem 3.1 and since $(K\Psi)'(y)$ is a compact linear operator, $\|P_n^G(K\Psi)'(y) - (K\Psi)'(y)\|_\infty \to 0$ as $n \to \infty$. Hence $(I - P_n^G(K\Psi)'(y))^{-1} = (I - T_h'(y))^{-1}$ exists and uniformly bounded in $\|\cdot\|_{L_2}$ norm. Now, from (5.22),

$$\begin{aligned} \|y - y_n\|_{L_2} &\le \tfrac{\alpha_h}{1-q} \\ &= \tfrac{1}{1-q}\|(I - T_h'(y))^{-1}(T_h(y) - \hat{T}(y))\|_{L_2} \\ &\le C'\|P_h^G K\Psi y - K\Psi y + P_h^G f - f\|_{L_2} \\ &= C'\|P_h^G y - y\|_{L_2}. \end{aligned} \tag{5.25}$$

where $C'$ is independent of $h$. Using the uniform boundedness of $\{P_h^G\}$, (5.19), (5.24) and (5.25), we obtain

$$\|y - y_h\|_{L_2} \le C'h^n\|v\|_{H^n}.$$

□

## THE ITERATED SINGULARITY PRESERVING GALERKIN METHOD

In this section, the superconvergence of the iterated singularity preserving Galerkin method is discussed. Throughout this section, the conditions (5.4), (5.5) and (5.6) are maintained. The discussion of this section depends heavily upon the recent paper by Kaneko and Xu [44] so that only the points of distinct differences are explained.

Let $y_0$ be an isolated solution of (5.1). Assume that $y_n$ is the unique solution of (5.15) in the sphere $\|y_0 - y\|_{L_2} \le \delta$, for some $\delta > 0$. Define

$$y_h^I = f + K\Psi y_h. \tag{5.26}$$

Applying $P_h^G$ to both sides of (5.26), we obtain

$$P_h^G y_h^I = P_h^G f + P_h^G K\Psi y_h. \tag{5.27}$$

Comparing (5.27) with (5.15),

$$P_h^G y_h^I = y_h. \tag{5.28}$$

Substitution of (5.28) into (5.26) yields that $y_h^I$ satisfies the following Hammerstein equation.

$$y_h^I = f + K\Psi P_h^G y_h^I. \tag{5.29}$$

The theorem of Kaneko and Xu [44] (Theorem 3.3), with only very minor modification can be written in the following form.

**Theorem 5.8** *Let $y_0 \in C[0,1]$ be an isolated solution of equation (2.1) and $y_n$ be the unique solution of (2.5) in the sphere $B(y_0,\delta)$. Let $y_h^I$ be defined by the iterated scheme (4.1). Assume that 1 is not an eigenvalue of $(K\Psi)'(y_0)$. Then, for all $1 \le p \le \infty$,*

$$\|y_0 - y_h^I\|_\infty \le C\left\{\|y_0 - P_h^G y_0\|_\infty^2 + \sup_{0\le t\le 1}\inf_{u\in V_h^n}\|k(t,.)\psi^{(0,1)}(.,y_0(.)) - u\|_q \|y_0 - P_h^G y_0\|_p\right\},$$

*where $1/p + 1/q = 1$ and $C$ is a constant independent of $h$.*

As a corollary, we obtain the main result of the section. First, we introduce some notations. Applying the mean-value theorem to $\psi(s,y)$ to get

$$\psi(s,y) = \psi(s,y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))$$

where $\theta \equiv \theta(s,y_0,y)$ with $0 < \theta < 1$ and $\psi^{(0,1)}$ denotes the partial derivative of $\psi$ with respect to the second variable. Also

$$k(s,t) \equiv \log(|s-t|)m(s,t)$$

and

$$g(s,t,y_0,y,\theta) \equiv k(s,t)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)).$$

**Theorem 5.9** *Assume the hypotheses of the previous theorem. Assume also that (5.4)-(5.6) hold. Then*

$$\|y_0 - y_h^I\|_\infty = O(h^{n+1}).$$

**Proof:** First of all, for each $u \in V_h^n$,

$$\|y_0 - P_h^G y_0\|_\infty \le \|y_0 - u\|_\infty + \|P_h^G u - P_h^G y_0\|_\infty \le (1 + P)\|y_0 - u\|_\infty, \tag{5.30}$$

where $P \equiv \sup_{h>0} P_h^G < \infty$. Since $y_0 = w + v$ for some $w \in W$ and $v \in H^n$, we let $u = w + u^*$, where $u^* \in S_{n,k}^\nu$. We obtain $\|y_0 - u\|_\infty = \|v - u^*\|_\infty$. With (5.30) this yields

$$\|y_0 - P_h^G y_0\|_\infty \le (1 + P) \inf_{u^* \in S_{n,k}^\nu} \|v - u^*\|_\infty \le C h^n. \tag{5.31}$$

The last inequality follows from (3.5). Secondly, by [12], [Theorem 4 (i)], there exists $v_t \in S_{n,k}^\nu$ such that $\|k_t - v_t\|_{L_1} = O(h)$. Since $\nu \ge 1$, $S_{n,k}^\nu = S_h^{n,\nu} \subseteq H^1$, so $v_t \in H^1$.

Since $y_0 \in W \dot{+} H^n$ it follows that $\psi^{(0,1)}(\cdot, y_0(\cdot)) \in W \dot{+} H^{n-1}$, by expanding $\psi^{(0,1)}(\cdot, y_0(\cdot))$ in Taylor series centered at $v$ (recall $y_0 = w + v, v \in H^n$) and using (2.10) and (2.12). Consequently, $v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) \in W \dot{+} H^{n-1}$. Say $v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) = a_t + b_t$, where $a_t \in W$ and $b_t \in H^{n-1}$. Now there exists $u_t \in S_{n,k}^\nu$ such that $\|u_t - b_t\|_{L_1} = O(h^{n-1})$ and

$$\|g_t - u_t - a_t\|_{L_1} \le \|k_t - v_t\|_{L_1}\|\psi^{(0,1)}(\cdot, y_0(\cdot))\|_\infty + \|v_t(\cdot)\psi^{(0,1)}(\cdot, y_0(\cdot)) - u_t - a_t\|,$$

$$= O(h) + O(h^{n-1}) = O(h),$$

provided $n \ge 2$. Now we apply Theorem 4.1 to get

$$\|y_0 - y_n^I\|_\infty = O(h^{2n}) + O(h^{n+1}) = O(h^{n+1}).$$

## NUMERICAL EXAMPLE

Let $m(s,t) = 1$, $g(|s-t|) = \log(|s-t|)$ and $\psi(s,t) = \cos(s+t)$ in equation (4.1). We assume $f$ in such a way that $x(t) = \sin t + t\log t$ is the solution. Using splines of order 2 we approximate the solution of the Hammerstein equation with

$$y_0(t) = \sum_{i=1}^{4} \gamma_i B_i$$

and

$$y_1(t) = \sum_{i=1}^{4} \gamma_i B_i + \alpha t \log t + \beta(1-t)\log(1-t) \tag{5.32}$$

$y_0$ represents the numerical solution that uses only the spline basis elements whereas $y_1$ represents the current scheme. $y_0$ is computed for comparison. The computed errors for the spline-only solution and the singularity preserving solution are shown in Table 5.1.

Table 5.1: Singularity Preserving Method Results

| | Errors | |
|---|---|---|
| $n$ | $y_0$ | $y_1$ |
| 2 | .032756 | .004002 |
| 3 | .018526 | .001945 |
| 4 | .012246 | .001147 |
| convergence rate ≈ | 1.4 | 1.8 |

Notice that the convergence rate for $y_0$ is lower due to the logarithmic singularity in the kernel and due to the use of the uniform partition of [0, 1]. The use of nonuniform partition to obtain the optimal rate of convergence of numerical solution was recently established in [44] for the Galerkin method. It should be pointed out that, as the number of partition points increases, the distribution of these nonuniform points become extremely skewed toward the end points of the interval. This will cause a sensitivity in numerical computations, frequently requiring computations in double precision. An introduction of the singular elements in the basis and working with the uniform partition points will eliminate this problem. The coefficients in (5.32) were obtained by solving the set of nonlinear equations of Section 3 (immediately following (5.16)) using the Newton-Raphson algorithm. Also, the Gauss-type quadrature algorithm described in [42] is used to calculate all integrals.

# CHAPTER VI
# CONCLUDING REMARKS

In this thesis, we investigated the superconvergence of the iterated solutions of several different numerical schemes for the Fredholm equations of the second kind as well as for the class of nonlinear Hammerstein equations. The superconvergence result established for the iterated degenerate kernel scheme is new even in the case of the Fredholm equations. It should be noted that, in order to double the rate of convergence of a numerical scheme such as the collocation method, we must in general double the order of the polynomials to be used resulting in more expensive computational cost. The iterated shemes provide us with an inexpensive alternative to achieve the same goal of accelerating the convergence rates.

One of the important areas to which the iterated methods discussed here can be applied is the area of boundary integral equations. As an example, consider the following elliptic boundary value problem:

$$\begin{aligned} &\Delta u(P) = 0, && P \in D \\ &\tfrac{\partial u(P)}{\partial n_P} = -cu(P) + f(P), && P \in \Gamma \equiv \partial D. \end{aligned} \tag{6.1}$$

where $D$ is a bounded simply connected open region in $R^2$ with a smooth boundary $\Gamma$. In equation (6.1), $n_P$ denotes the exterior unit normal to $\Gamma$ at $P$. $f$ is continuous on $\Gamma$ and $c$ is a positive constant. The function $u$ is to be determined. We assume $u \in C^2(D) \cap C^1(\bar{D})$.

It is well-known that using Green's representation formula for harmonic functions, the function $u$ satisfies

$$u(P) = \frac{1}{2\pi} \int_\Gamma u(Q) \frac{\partial}{\partial n_Q} \log|P - Q| d\Gamma(Q) - \frac{1}{2\pi} \int_\Gamma \frac{\partial u(Q)}{\partial n_Q} \log|P - Q| d\Gamma(Q) \tag{6.2}$$

for all $P \in D$. Moving the point $P$ to a point on $\Gamma$ and using the boundary condition in (6.1), we obtain the following boundary integral equation.

$$\begin{aligned} u(P) - \tfrac{1}{\pi} \textstyle\int_\Gamma & \tfrac{\partial u(Q)}{\partial n_Q} \log|P - Q| d\Gamma(Q) - \tfrac{c}{\pi} \textstyle\int_\Gamma u(Q) \log|P - Q| d\Gamma(Q) \\ &= -\tfrac{1}{\pi} \textstyle\int_\Gamma f(Q) \log|P - Q| d\Gamma(Q). \qquad P \in \Gamma. \end{aligned} \tag{6.3}$$

We have now concentrated all the information on $u$ to the boundary $\Gamma$. One of the primary advantages, of course, of dealing with the boundary integral equations by transforming the original boundary value problem is that we have reduced the dimensionality of the problem by one. Now once $u$ is computed along $\Gamma$ from equation (6.3), equation (6.2) now yields the value $u(P)$ for all $P \in D$. Any numerical method can be applied to approximate the solution of (6.3) and subsequently the order of approximation can be enhanced by the iteration process. A reduction in computational cost to achieve the enhancement can be seen in a more pronounced way when the elliptic problem is proposed in a higher dimensional space due to its exponential growth in the number of unknowns involved. In this connection, we note as a future research topic an application of wavelet bases to the boundary integral equations. Wavelet bases give rise to sparse linear systems that result in the reduction of the computational cost. It is also interesting to consider the iterated numerical methods described in this thesis in connection with wavelet bases.

Another interesting application of the iterated scheme is the following. When superconvergence of the iterated solutions of a certain numerical scheme is known to exist, then the residual of the numerical solution can be used as an estimator of the error of the numerical solution. For example, if $y_n$ denotes the approximation to equation (4.2), the error of the approximation is

$$\epsilon_n \equiv y - y_n \tag{6.4}$$

and the residual is defined by

$$\delta_n \equiv f - (y_n - K\Psi y_n). \tag{6.5}$$

Now

$$\begin{aligned} \delta_n &= f - (y_n - K\Psi y_n) \\ &= (y - K\Psi y) - (y_n - K\Psi y_n) \\ &= (y - y_n) - (K\Psi)'(\eta_n)(y - y_n) \\ &= (I - (K\Psi)'(\eta_n))\epsilon_n, \end{aligned} \tag{6.6}$$

where $\eta_n$ is between $y$ and $y_n$. Also note in particular from (6.6) that

$$(K\Psi)'(\eta_n)(\epsilon_n) = K\Psi(y) - K\Psi(y_n). \tag{6.7}$$

Now

$$\begin{aligned} (I - (K\Psi)'(\eta_n)P_n)(K\Psi)'(\eta_n)\epsilon_n &= (I - (K\Psi)'(\eta_n)P_n)(K\Psi(y) - K\Psi(y_n)) \\ &= (K\Psi(y) - K\Psi(y_n)) - (K\Psi)'(\eta_n)(P_n K\Psi(y) - P_n K\Psi(y_n)) \\ &= (K\Psi)'(\eta_n)\epsilon_n - (K\Psi)'(\eta_n)P_n\epsilon_n \\ &= (K\Psi)'(\eta_n)(I - P_n)\epsilon_n. \end{aligned}$$

In the third line we made use of

$$P_n K\Psi(y) - P_n K\Psi(y_n) = P_n(y - f) - (y_n - P_n f) = P_n(y - y_n) = P_n \epsilon_n.$$

Now we assume that 1 is not an eigenvalue of $(K\Psi)'(y)$ so that $(I - (K\Psi)'(y))^{-1}$ exists. Also assume that $\iota^{(0,1)}(t, y)$ is continuous in $y$ and uniformly continuous in $t$. Then $(K\Psi)'(y)$ is continuous as a function of $y$ in the space of all bounded linear operators $B(C[0, 1], C[0, 1])$. Since $\eta_n$ lies between $y_n$ and $y$, $\eta_n \to y$ as $n \to \infty$. It follows that $(I - (K\Psi)'(\eta_n))P_n$ converges to $(I - (K\Psi)'(y))$ in the space $B(C[0, 1], C[0, 1])$. Therefore, $((I - (K\Psi)'(\eta_n))P_n)^{-1}$ exists and uniformly bounded for all sufficiently large $n$. An $\epsilon/3$ argument also shows that $\lim_{h\to 0} \|(K\Psi)'(\eta_n)(I - P_n)\|_{L_2} = 0$. Hence

$$(K\Psi)'(\eta_n)\epsilon_n = (I - (K\Psi)'(\eta_n)P_n)^{-1}(K\Psi)'(\eta_n)(I - P_n)\epsilon_n. \tag{6.8}$$

and

$$\|(K\Psi)'(\eta_n)\epsilon_n\|_{L_2} \leq \mu(h)\|\epsilon_n\|_{L_2} \tag{6.9}$$

where $\mu(h) \equiv \|(I - (K\Psi)'(\eta_n)P_n)^{-1}\|_{L_2}\|(K\Psi)'(\eta_n)(I - P_n)\|_{L_2} \to 0$ as $h \to 0$ or equivalently $n \to \infty$. From (6.9) and (6.6),

$$\|\delta_n\|_{L_2} = \|\epsilon_n - (K\Psi)'(\eta_n)\epsilon_n\|_{L_2} \leq (1 + \mu(h))\|\epsilon_n\|_{L_2}. \tag{6.10}$$

This equation states that the residual can be used as an estimator for the actual error. What is interesting at this point is to observe that superconvergence of the iterates can be used as a sufficient condition for (6.10) to occur. To see this, denote the iterates by

$$y_n^I = f + K\Psi(y_n). \tag{6.11}$$

Then with $\epsilon_n^I = y - y_n^I$,

$$\begin{aligned} \delta_n &= f - (y_n - K\Psi y_n) \\ &= y_n^I - y_n \\ &= \epsilon_n - \epsilon_n^I. \end{aligned}$$

From this, we obtain

$$\left|1 - \frac{\|\epsilon_n^I\|}{\|\epsilon_n\|}\right| \leq \frac{\|\delta_n\|}{\|\epsilon_n\|} \leq 1 + \frac{\|\epsilon_n^I\|}{\|\epsilon_n\|}. \tag{6.12}$$

Namely, the superconvergence of the iterates,-i.e.,

$$\lim_{n\to\infty} \frac{\|\epsilon_n^I\|}{\|\epsilon_n\|} = 0$$

gives a sufficient condition for the inequality (6.10) to occur. We note here that (6.10) was proved without reference to the superconvergence of the iterates. Because of (6.12), the results presented in (6.9) and (6.10) can be obtained by demonstrating the superconvergence of the iterates for the Galerkin solution for Hammerstein equation under the condition $\lim_{h\to 0}\|(K\Psi)'(\eta_n)(I-P_n)\| = 0$ which was taken earlier. In this case,

$$\begin{aligned}
&(I-(K\Psi)'(y))(y_n^I - y)\\
&= [I-(K\Psi)'(y)(I-P_n)][K\Psi(y_n) - K\Psi(y) - (K\Psi)'(y)(y_n - y)]\\
&-(K\Psi)'(y)(I-P_n)((K\Psi)'(y) - I)(y_n - y),
\end{aligned}$$

we obtain

$$\begin{aligned}
\|y_n^I - y\| \quad \le \quad & \|(I-(K\Psi)'(y))^{-1}\| \, \{\|I-(K\Psi)'(y)(I-P_n)\| \\
& \times \sup_{0\le\theta\le 1} \|(K\Psi)'(y+\theta(y_n - y)) - (K\Psi)'(y)\| \|y - y_n\| \\
& + \|(K\Psi)'(y)(I-P_n)((K\Psi)'(y) - I)(y_n - y)\| \, \}.
\end{aligned}$$

In any case, we demonstrated the fact that, when the superconvergence of the iterated solutions is guaranteed, an error of the numerical solution is estimated by the size of the residual. Of course, the residual is an easily computable quantity whereas the actual error is not in most of the practical problems.

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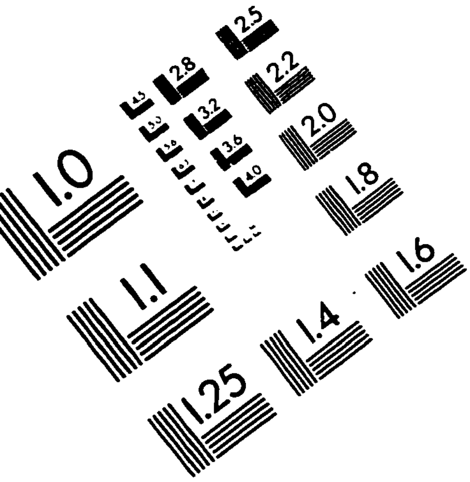
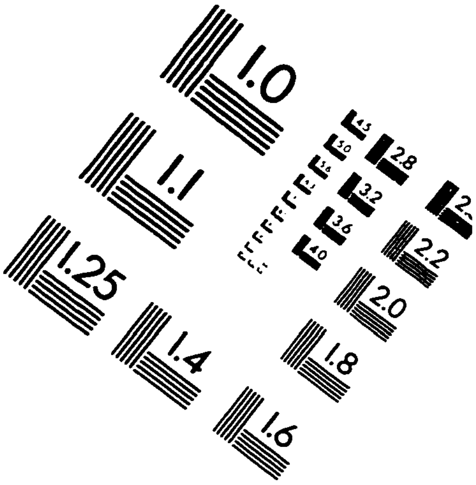
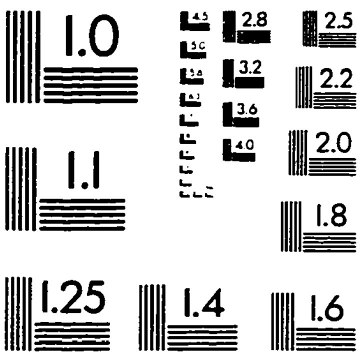
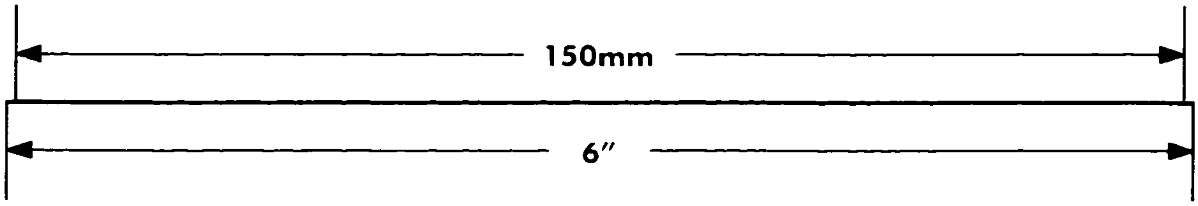
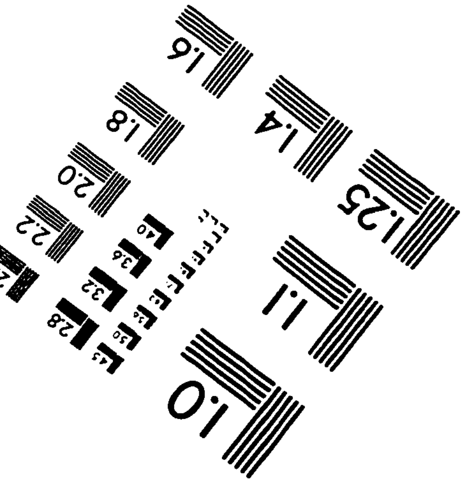
# IMAGE EVALUATION TEST TARGET (QA-3)

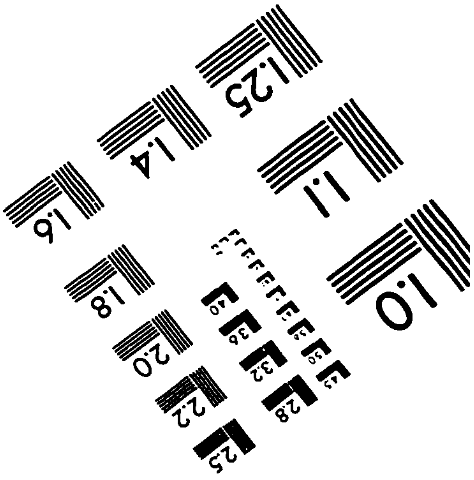








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